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Invariant functions of vector field realizations of Lie algebras and some applications to representation theory and dynamical systems

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Abstract. Realizations of Lie algebras by means vector fields associated to a linear representation and their corresponding invariant functions are inspected from the perspective of the embedding problem of Lie algebras and the branching rules for the subduction of representations. Some applications concerning the construction of dynamical systems with prescribed Lie algebras of point symmetries and the symmetry breaking with respect to embeddings of algebras into $\mathfrak{so}(N)$ are discussed.

To the memory of Sigitas Movšauskas

1. Introduction

The literature of realizations of Lie algebras as vector fields on manifolds is extensive, having many ramifications in different disciplines, ranging from the theory of differential equations and differential geometric problems to cohomology theories as well a s integrable systems and General Relativity (see e.g. [1, 2, 3, 4, 5] and references therein). Among the pioneering contributions to the field, we may refer to the work of Dickson on differential equations from the Lie group perspective [6], an approach that revitalized Lie's classical theory and motivated later studies on the Lie symmetry method [7, 8] from both the mathematical and physical perspective [9]. Special types of realizations, corresponding to the coadjoint representation of Lie algebras, have been shown to be an indispensable tool in the modelization and interpretation of physical phenomena [10, 11], such as the symmetry groups of physical systems, and constitutes nowadays a well-established research subject. A relevant research line that combines the analytical and geometrical properties of Lie groups is given by those systems of differential equations admitting a nonlinear superposition principle [12, 13, 14], an approach that has provided new tools for the elucidation of physical properties at the classical and quantum level.

In this work we analyze some features of realizations of Lie algebras by vector fields from the perspective of the representation theory and its relation to the branching rules of representations for embeddings of (simple) Lie algebras. It turns out that the analysis of the invariants of a given realization can provide useful information concerning the embedding of a Lie algebra $\mathfrak{g}'$ into another algebra $\mathfrak{g}$, allowing to determine the corresponding conjugacy class within $\mathfrak{g}$.

This can further be used to decide whether a subalgebra in a given realization corresponds to an irreducible embedding, or either determine the multiplicity of the trivial representation in the branching rules. An an application to second-order dynamical systems, it is shown that invariants of realizations can be used for the construction of (non-conservative) Lagrangian systems having an exact prescribed Lie algebra of Lie point and/or Noether symmetries.

Unless otherwise stated, any Lie algebra $\mathfrak{g}$ and any representation is considered over the field $\mathbb{R}$ of real numbers. The Einstein summation convention is used.

2. Realizations of Lie algebras by vector fields

Let $\{X_1, \cdots, X_r\}$ be a basis of a Lie algebra $\mathfrak{g}$ and consider a realization by vector fields in n-coordinates, i.e., a map $\Phi : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^n)$ such that $\Phi(X_\alpha) \neq 0$ holds for all α and satisfying the property $\Phi[X_\alpha, X_\beta] = [\Phi(X_\alpha), \Phi(X_\beta)]$ for all α, β. In matrix terms, the map Φ can be described as

$$(\Phi(X_\alpha))_\alpha = A_\Phi(\mathfrak{g}) \frac{\partial}{\partial \mathbf{x}}, \tag{1}$$

with $A_\Phi(\mathfrak{g}) = \left(\xi_i^j(\mathbf{x})\right) \in M_{r\times n}(C^\infty(\mathbb{R}^n))$ being a the functional matrix that specifies the components of $\mathbf{X}_\alpha = \Phi(X_\alpha)$ in the coordinates $\left\{x^1, \cdots, x^n\right\}$. We recall that $\{\mathbf{X}_1, \cdots, \mathbf{X}_s\}$ are linearly independent as first-order differential operators if a dependence relation

$$\lambda^1(\mathbf{x})\mathbf{X}_1 + \cdots + \lambda^s(\mathbf{x})\mathbf{X}_s = 0 \tag{2}$$

with $\lambda^1(\mathbf{x}) \in C^\infty(\mathbb{R}^n)$ implies that $\lambda^1(\mathbf{x}) = \cdots = \lambda^s(\mathbf{x}) = 0$. In this context, it is important to observe that albeit the generators $X_1, \ldots, X_r$ are independent, this does generally not imply that the vector fields $\mathbf{X}_1, \cdots, \mathbf{X}_r$ are linearly independent.[1] The number of linearly independent vector fields is given by the rank $r_\Phi(\mathfrak{g})$ of the coefficient matrix $A_\Phi(\mathfrak{g})$, from which $r_\Phi(\mathfrak{g}) \leq \dim \mathfrak{g} = r$ follows. As obviously the maximal possible number of linearly independent vector fields is given by the number of independent variables n, in any case the bound $r_\Phi(\mathfrak{g}) \leq \min(\dim \mathfrak{g}, n)$ holds. Dependence relations of the type (2) are necessarily given whenever the inequality $n < \dim \mathfrak{g}$ holds.

Invariant functions of a realization $\Phi : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^n)$ correspond to elements $F(\mathbf{x}) \in C^\infty(\mathbb{R}^n)$ such that the linear first-order system of PDEs

$$A_\Phi(\mathfrak{g}) \frac{\partial F}{\partial \mathbf{x}} = \begin{pmatrix} \xi_1^1 & \cdots & \xi_1^n \\ \vdots & & \vdots \\ \xi_r^1 & \cdots & \xi_r^n \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial x^1} \\ \vdots \\ \frac{\partial F}{\partial x^n} \end{pmatrix} = 0 \tag{3}$$

is satisfied. As the vector fields span a finite-dimensional Lie algebra, the system (3) is always complete, and can thus be reduced to its Jacobian form [15]. A particularly relevant case is given when $A_\Phi(\mathfrak{g})$ is the coefficient matrix of the realization associated to the coadjoint representation of the Lie algebra $\mathfrak{g}$, where these functions coincide with the (generalized) Casimir invariants of $\mathfrak{g}$ [10]. In the general case, if the matrix $A_\Phi(\mathfrak{g})$ has rank $r_\Phi(\mathfrak{g})$, the solution of (3) is given in terms of $\chi_0 = n - r_\Phi(\mathfrak{g})$ independent functions J_k that can be taken as the fundamental invariants. We observe that if J is a solution of (3), then for any vector field $\mathbf{X}_\alpha \in \Phi(\mathfrak{g})$ the condition

$$\mathbf{X}_\alpha(J) = \xi_\alpha^l(\mathbf{x}) \frac{\partial J}{\partial x^l} = \xi_\alpha^l(\mathbf{x}) \frac{\partial J}{\partial x^k} dx^k \left(\frac{\partial}{\partial x^l}\right) = dJ(\mathbf{X}_\alpha) = 0 \tag{4}$$

[1] While for $\mathfrak{g}$ a dependence relation is always expressed by scalars, for $\Phi(\mathfrak{g})$ dependence relations are given in terms of functions.

is satisfied, showing that the total differential dJ of the scalar field J annihilates the realization [2].

Assuming that the rank of $A_{\Phi}(\mathfrak{g})$ is $k < n$, we can suppose without loss of generality that, after a reordering of the indices, the vector fields $\mathbf{X}_i$ with $1 \leq i \leq k$ are linearly independent. For the remaining generators we thus have dependence relations

$$\mathbf{X}_{k+s} = \lambda_s^1(\mathbf{x})\mathbf{X}_1 + \cdots \lambda_s^k(\mathbf{x})\mathbf{X}_k, \ 1 \leq s \leq r-k. \tag{5}$$

In these conditions, we can find linearly independent differential 1-forms $\omega^j \in \Omega^1(\mathbb{R}^n)$ such that $\omega^i(\mathbf{X}_j) = \delta_i^j$ for $1 \leq i, j \leq k$ and $\omega^i(\mathbf{X}_{k+s}) = \lambda_s^k(\mathbf{x})$ for $1 \leq s \leq r-k$. As $k < n$, the set $\left\{\omega^1, \cdots, \omega^k\right\}$ can be augmented with $(n-k)$ 1-forms θ^l $(1 \leq l \leq n-k)$ such that $\theta^l(\mathbf{X}_j) = 0$ for $1 \leq j \leq k$ and $1 \leq l \leq n-k$, implying that θ^l belongs to the annihilator $\mathrm{Ann}(\Phi(\mathfrak{g}))$ of the realization $\Phi(\mathfrak{g})$ [2]. Hence, if J is an invariant of the realization, then necessarily $dJ \in \mathrm{Ann}(\Phi(\mathfrak{g}))$. As the vector fields span the Lie algebra $\mathfrak{g}$, it follows that $\mathrm{Ann}(\Phi(\mathfrak{g}))$ is a differential ideal, meaning that for each $\theta \in \mathrm{Ann}(\Phi(\mathfrak{g}))$, the exterior derivative $d\theta$ also belongs to $\mathrm{Ann}(\Phi(\mathfrak{g}))$ [1]. We remark that this reformulation in terms of differential forms can be seen as an algebraic approach to find the solutions of the system (3), up to the determination of integrating factors. Indeed, if $\theta \in \mathrm{Ann}(\Phi(\mathfrak{g}))$, we can ask whether there exists some function $J \in C^{\infty}(\mathbf{R}^n)$ such that $dJ = \theta$. Although θ is not necessarily an exact 1-form, we can (at least locally, under certain natural assumptions) find functions f, J' such that $\theta = f dJ'$. The condition to be satisfied for this to happen (see e.g. [1]) is that

$$\theta \wedge d\theta = 0 \tag{6}$$

holds. This result can be easily extended to a (local) basis of independent one-forms ω^i $(1 \leq i \leq \chi_0)$ in the annihilator $\mathrm{Ann}(\Phi(\mathfrak{g}))$ of the realization, requiring that for the relative volume form $\Omega = \omega^1 \wedge \cdots \wedge \omega^{\chi_0}$, the constraints

$$d\omega^i \wedge \Omega = 0, \ 1 \leq i \leq \chi_0 \tag{7}$$

are satisfied. As a consequence, for any invariant J of the realization (3), it follows that the total differential is a combination of the ω^i:

$$dJ = \varphi_j(\mathbf{x})\omega^i, \ \varphi(\mathbf{x}) \in C^{\infty}(\mathbb{R}^n). \tag{8}$$

Letting $\omega = \omega_i(\mathbf{x})\, dx^i \in \Omega^1(\mathbb{R}^n)$, the action of ω on the (linearly independent) vector fields $\mathbf{X}_{\alpha}$ leads to relations of the type

$$\omega(\mathbf{X}_{\alpha}) = \omega_i(\mathbf{x})\, \xi_{\alpha}^i(\mathbf{x}) = 0, \ 1 \leq \alpha \leq r_{\Phi}(\mathfrak{g}) \tag{9}$$

that can formally be solved with respect to $\chi_0 = n - r_{\Phi}(\mathfrak{g})$ components $\omega_1, \cdots, \omega_{\chi_0}$ of ω, so that the latter is expressed as

$$\omega = \sum_{k=1}^{\chi_0} \omega_k(\mathbf{x})\, dx^k + \sum_{l=\chi_0+1}^{n} \Psi_l\left(\omega_1, \cdots, \omega_{\chi_0}, \xi_{\alpha}^1, \cdots, \xi_{\alpha}^n\right) dx^l. \tag{10}$$

Evaluating now the exterior derivative $d\omega = 0$ provides us with a system of conditions on the components ω_k that eventually allow us to derive independent scalar fields $\{\Theta_1(\mathbf{x}), \cdots, \Theta_{\chi_0}(\mathbf{x})\}$ such that the relations

$$\omega_k = \sum_a^{\chi_0} \frac{\partial \Theta_a}{\partial x^k}, \ \ \Psi_l = \sum_a^{\chi_0} \frac{\partial \Theta_a}{\partial x^l}, \ \ 1 \leq k \leq \chi_0, \ \chi_0 + 1 \leq l \leq n \tag{11}$$

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are simultaneously satisfied. It then follows that $\omega = d\Theta_1 + \cdots + d\Theta_{\chi_0}$ is a total differential, and the solutions $\{\Theta_1(\mathbf{x}), \cdots, \Theta_{\chi_0}(\mathbf{x})\}$ can be taken as a set of fundamental invariants of the system. Although this procedure is certainly not simpler than solving directly the first-order system (3) by the usual methods [15], it gives some hints on the structure of the invariants, specially for nonlinear realizations.

As an illustrating example, consider the five-dimensional (solvable) Lie algebra $\mathfrak{g}$ generated by the vector fields in $\mathbb{R}^5$:

$$\begin{aligned}&\mathbf{X}_1 = x^1\frac{\partial}{\partial x^5}, \quad \mathbf{X}_2 = \left(x^1 + x^2\right)\frac{\partial}{\partial x^5}, \quad \mathbf{X}_3 = \left(x^2 + x^3\right)\frac{\partial}{\partial x^5}, \quad \mathbf{X}_4 = \left(x^3 + x^4\right)\frac{\partial}{\partial x^5},\\ &\mathbf{X}_5 = x^1\frac{\partial}{\partial x^1} + \left(x^1 + x^2\right)\frac{\partial}{\partial x^2} + \left(x^2 + x^3\right)\frac{\partial}{\partial x^3} + \left(x^3 + x^4\right)\frac{\partial}{\partial x^4}.\end{aligned}$$

If $\omega = \omega_i(\mathbf{x})dx^i \in \Omega^1\left(\mathbb{R}^5\right)$ belongs to the annihilator, the condition $\omega\left(\mathbf{X}_1\right) = 0$ implies that $\omega_5\left(\mathbf{x}\right) = 0$. A dependence condition relating the remaining components $\omega_i(\mathbf{x})$ is obtained from the identity $\omega\left(\mathbf{X}_5\right) = 0$:

$$x^1\omega_1\left(x\right) + \left(x^1 + x^2\right)\omega_2\left(x\right) + \left(x^2 + x^3\right)\omega_3\left(x\right) + \left(x^3 + x^4\right)\omega_4\left(x\right) = 0.$$

As at most three of these functions can be chosen freely, the realization admits three invariants (alternatively, the rank of the matrix $A_\Phi\left(\mathfrak{g}\right)$ is two). Taking e.g. $\omega_3\left(\mathbf{x}\right) = \omega_4\left(\mathbf{x}\right) = 0$ gives the 1-form

$$\theta_1 = -\frac{\left(x^1 + x^2\right)\omega_2\left(x\right)}{x^1}dx^1 + \omega_2\left(x\right)dx^2.$$

Now θ_1 is closed ($d\theta_1 = 0$) whenever $\omega_2\left(\mathbf{x}\right) = \frac{1}{x^1}F\left(\frac{x^2}{x^1} - \ln x^1\right)$ holds. The simplest solution can thus be taken as $\omega_2\left(\mathbf{x}\right) = \exp\left(-\frac{x^2}{x^1}\right)$. A routine computation shows that

$$\theta_1 = \exp\left(-\frac{x^2}{x^1}\right)\left(-\frac{\left(x^1 + x^2\right)}{x^1}dx^1 + dx^2\right) = d\left(x^1\exp\left(-\frac{x^2}{x^1}\right)\right),$$

from which we conclude that θ_1 is an exact form. In analogous manner, the remaining fundamental invariants

$$J_2 = \frac{2x^1x^3 - \left(x^2\right)^2}{\left(x^1\right)^2}, \; J_3 = \frac{3\left(x^1\right)^2x^4 - 3x^1x^2x^3 + \left(x^2\right)^3}{\left(x^1\right)^3}$$

can be easily found starting from the choices

$$\omega_1(\mathbf{x}) = \frac{2((x^2)^2 - x^1x^3)}{(x^1)^3}, \quad \omega_2(\mathbf{x}) = -\frac{2x^2}{(x^1)^2}, \quad \omega_3(\mathbf{x}) = \frac{2}{x^1}, \quad \omega_4(\mathbf{x}) = 0$$

and

$$\omega_1(\mathbf{x}) = \frac{6x^1x^2x^3 - 3(x^2)^3 - 2(x^1)^2x^4}{(x^1)^4}, \; \omega_2(\mathbf{x}) = \frac{3((x^2)^2 - x^1x^3)}{(x^1)^3}, \; \omega_3(\mathbf{x}) = -\frac{3x^2}{(x^1)^2}, \; \omega_4(\mathbf{x}) = \frac{2}{x^1},$$

respectively.

2.1. First-order systems and realizations

As already observed, vector field realizations of Lie algebras have important applications in the analysis of systems of ordinary differential equations, such as the study of superposition formulae [12, 13]. A first-order system

$$\dot{x}^i = F^i\left(t, \mathbf{x}\right), \quad 1 \leq i \leq n \tag{12}$$

is said to have a fundamental system of solutions if the general solution can be written in terms of m independent particular solutions $\{\mathbf{y}_1, \cdots, \mathbf{y}_m\}$ and n constants $\{C_1, \cdots, C_n\}$:

$$x^i = \varphi^i\left(\mathbf{y}_1, \cdots, \mathbf{y}_m, C_1, \cdots, C_n\right). \tag{13}$$

An expression of the latter type is called a (nonlinear) superposition formula or principle. A classical criterion due to Lie [12] establishes that such a principle exists whenever the independent variable t and the dependent coordinates x^i can be separated by means of a realization $\Phi : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^n)$ such that the identity

$$\dot{x}^i = F^i\left(t, \mathbf{x}\right) = \varphi_1(t)\xi_1^i\left(\mathbf{x}\right) + \cdots + \varphi_r(t)\xi_r^i\left(\mathbf{x}\right) \tag{14}$$

holds for each index, where $\mathbf{X}_\alpha = \xi_\alpha^l\left(\mathbf{x}\right)\frac{\partial}{\partial x^l}$ with $1 \leq \alpha \leq r = \dim \mathfrak{g}$ form a basis of $\mathfrak{g}$. A system (12) can be thought of as a (nonautonomous) dynamical system in phase space (albeit not necessarily Hamiltonian), for which reason the notion of cyclic coordinates arises at once. If for some $i_0 \in \{1, \cdots, n\}$ the constraint $\xi_\alpha^{i_0} = 0$ is satisfied for $1 \leq \alpha \leq r$, then clearly $\dot{x}^{i_0} = 0$ and x^{i_0} can be seen as a cyclic coordinate in the classical sense, implying that the system can be reduced by one degree of freedom, a fact that translates to the realization of the Lie algebra:

$$\widetilde{\mathbf{X}}_\alpha = \sum_{i=1}^{i_0-1}\xi_\alpha^i\left(\mathbf{x}\right)\frac{\partial}{\partial x^i} + \sum_{i_0+1}^{r}\xi_\alpha^i\left(\mathbf{x}\right)\frac{\partial}{\partial x^i},\ 1 \leq \alpha \leq r. \tag{15}$$

If $\Phi : \mathfrak{g} \to \mathfrak{X}\left(\mathbb{R}^n\right)$ is a realization of $\mathfrak{g}$ of rank $r_\Phi\left(\mathfrak{g}\right)$ and J is an invariant, then it follows at once from (14) that

$$\dot{J} = \dot{x}^i\frac{\partial J}{\partial x^i} = \sum_{s=1}^{r}\varphi_s(t)\xi_s^i(\mathbf{x})\frac{\partial J}{\partial x^i} = 0, \tag{16}$$

a fact that enables us to consider a new reference $\left\{y^1, \cdots, y^n\right\}$ in which $y^n = J$ is a (generalized) cyclic coordinate, as well as to reduce the system by one degree of freedom. Clearly, the maximal number of such cyclic coordinates is given by χ_0.

Under specific circumstances, it may be useful to simplify a realization normalizing one of the vector fields, i.e., introducing a reference $\left\{y^1, \cdots, y^n\right\}$ in which $\widetilde{\mathbf{X}}_1 = \frac{\partial}{\partial y^1}$ holds. Clearly, the new coordinates $\left\{y^1, \cdots, y^n\right\}$ are obtained as solutions of the system

$$\mathbf{X}_1\left(y^k\right) = \xi_1^1\left(\widetilde{\mathbf{x}}\right)\frac{\partial y^k}{\partial x^1} + \cdots + \xi_1^n\left(\widetilde{\mathbf{x}}\right)\frac{\partial y^k}{\partial x^1} = \delta_1^k, \tag{17}$$

with $\widetilde{\mathbf{x}} = \left(x^1, \cdots, x^n\right)$ and $2 \leq k \leq n$. This implies that the coordinates $\left\{y^2, \cdots, y^n\right\}$ are invariant functions of the differential operator $\mathbf{X}_1$, while y^1 is obtained as a particular solution of an inhomogeneous equation.[2]

[2] Depending on the solutions chosen, standard forms can correspond to non-equivalent realizations of the Lie algebra $\mathfrak{g}$. This may alter the physical interpretation of the symmetry generators.

3. Linear representations and embeddings

In this section, we analyze some properties of realizations related to linear representations of Lie algebras and its consequences for branching rules of representations in embedding problems $\mathfrak{g}' \subset \mathfrak{g}$.

Realizations of a Lie algebra $\mathfrak{g}$ by vector fields having linear components

$$\mathbf{X}_\alpha = a^\alpha_{jk} x^j \frac{\partial}{\partial x^k},\ 1 \leq \alpha \leq r, \tag{18}$$

naturally correspond to a (faithful) representation F of $\mathfrak{g}$, the matrix elements of which are obtained by matrix transposition:

$$F\left(\mathbf{X}_\alpha\right) = A_{\mathbf{X}_\alpha};\ \left(A_{\mathbf{X}_\alpha}\right)_{jk} = a^\alpha_{kj},\ 1 \leq j, k \leq n. \tag{19}$$

In this context, we convene to call a realization $\Phi : \mathfrak{g} \to \mathfrak{X}\left(\mathbb{R}^n\right)$ nonlinear if it is not equivalent, by a change of coordinates, to a realization arising from a linear representation of $\mathfrak{g}$. It follows that if the Lie algebra $\mathfrak{g}$ does not have a faithful n-dimensional real representation, any realization in n coordinates $\Phi : \mathfrak{g} \to \mathfrak{X}\left(\mathbb{R}^n\right)$ is necessarily nonlinear.

Consider an embedding $f : \mathfrak{g}' \longrightarrow \mathfrak{g}$ of a (simple) Lie algebra $\mathfrak{g}'$ into a (simple) Lie algebra $\mathfrak{g}$. Any embedding of Lie algebras determines an integer factor j_f given by the relation

$$\big(f(X), f(X') = j_f \left(X, X'\right), \tag{20}$$

where $(.,.)$ is the usual scalar product in $\mathfrak{g}$ obtained from the root system. The scalar j_f is generally called the embedding index of $\mathfrak{g}'$ in the Lie algebra $\mathfrak{g}$ [16, 17]. Given disjoint subalgebras $\mathfrak{g}'_j$ of $\mathfrak{g}$, the direct sum of the subalgebras defines an embedding $f = \sum f_j$, the index of which is simply the sum of the various indices j_{f_j}. Further, for reduction chains $\mathfrak{g} \supset \mathfrak{g}' \supset \mathfrak{g}''$, the index of the last algebra in $\mathfrak{g}$ is the product of the corresponding indices of the chain members [18]. Given an embedding $f : \mathfrak{g}' \to \mathfrak{g}$ and a linear representation F of $\mathfrak{g}$, the embedding index is determined by the formula:

$$j_f = \frac{l_{fF}}{l_F}, \tag{21}$$

where l_F and l_{fF} denote the index of F and the subduced representation fF on the subalgebra, respectively. We recall that the index l_F of any representation is obtained from the quadratic Casimir operator C_2 of $\mathfrak{g}$ by means of the formula

$$l_F = \frac{\dim F}{\dim \mathfrak{g}} C_2(F). \tag{22}$$

Embeddings (and hence the index) are characterized by the branching rule obtained from the lowest dimensional (irreducible) representation of $\mathfrak{g}$, and have been tabulated for all simple Lie algebras (see e.g. [17, 19]).

Let $\Phi : \mathfrak{g} \to \mathfrak{X}\left(\mathbb{R}^n\right)$ be a realization and $\mathfrak{g}' \subset \mathfrak{g}$ a subalgebra. The restriction $\Phi|_{\mathfrak{g}'} : \mathfrak{g}' \to \mathfrak{X}\left(\mathbb{R}^n\right)$ provides us with a realization of $\mathfrak{g}'$ such that $r_{\Phi|_{\mathfrak{g}'}}\left(\mathfrak{g}'\right) \leq r_\Phi\left(\mathfrak{g}\right)$. Now, as we are dealing with a subalgebra, the subduced realization may admit cyclic coordinates, arising from invariant functions specifically related to the embedding $\mathfrak{g}' \subset \mathfrak{g}$. In this context, we remark that the missing label problem corresponds to a special case of subduced realizations, associated to the coadjoint representation of Lie algebras [20, 21, 23].

Proposition 1 *Let $\mathfrak{g}' \subset \mathfrak{g}$ be a proper subalgebra and $\Phi(\mathfrak{g})$ the realization associated to the representation Γ of $\mathfrak{g}$. Let $\Gamma \downarrow \Gamma_1' \oplus \cdots \oplus \Gamma_l'$ be the branching rule as sum of representations of $\mathfrak{g}'$. If the multiplicity $\mathrm{mult}_{\Gamma_0'}(\Gamma)$ of the trivial representation Γ_0' of $\mathfrak{g}'$ in Γ is k, then the subduced realization $\Phi|_{\mathfrak{g}'} : \mathfrak{g}' \to \mathfrak{X}(\mathbb{R}^n)$ admits k linear invariants.*

Suppose that $\Gamma \downarrow \Gamma_1' \oplus \cdots \oplus \Gamma_l'$ contains k copies of the trivial representation Γ_0' of $\mathfrak{g}'$. Reordering the coordinates if necessary, for any generator $X \in \mathfrak{g}'$, the representation matrix $A_F(X)$ decomposes as a block matrix with respect to the branching rule of Γ:

$$A_\Gamma(X) = \begin{pmatrix} A_{\Gamma_1'}(X) & & & \\ & A_{\Gamma_2'}(X) & & \\ & & \ddots & \\ & & & A^k_{\Gamma_0'}(X) \end{pmatrix}, \tag{23}$$

and it follows from (18) that the vector field $\mathbf{X}$ associated to X is given by

$$\mathbf{X} = \mathbf{x}\, A_\Gamma(X) \frac{\partial}{\partial \mathbf{x}}. \tag{24}$$

Now take the basis of 1-forms $\left\{dx^1, \cdots, dx^n\right\}$ dual to the coordinates. It is immediate to verify that $dx^j(\mathbf{X}) = 0$ holds for $j = n-k+1, \cdots, n$, showing that $\left\{x^{n-k+1}, \cdots, x^n\right\}$ are invariants of the subduced realization, from which the assertion follows.

It is clear that any invariant of a realization of $\mathfrak{g}$ is also an invariant for the subduced realization associated to the embedding $\mathfrak{g}' \subset \mathfrak{g}$. Additionally it may happen, depending on the particular branching rule, that such an invariant splits up as a sum of the invariants of the subalgebra, hence leading to independent invariants in $\mathfrak{g}'$. In this frame, it is worthy to be mentioned that the computation of invariants of realizations associated to linear representations is deeply connected with the problem of determining the generalized Casimir invariants of inhomogeneous Lie algebras [24].

The converse of the preceding result also holds, and can be used as an indirect procedure to determine the embedding of a Lie algebra into another:

Proposition 2 *Let $\mathfrak{g}' \subset \mathfrak{g}$ be a proper subalgebra and Γ a representation of $\mathfrak{g}$. If the subduced realization of $\mathfrak{g}'$ by vector fields possesses k linear invariants, then the branching rule of Γ contains k copies of the trivial representation of $\mathfrak{g}'$.*

We indicate that the latter result, the proof of which follows at once reversing the argumentation in Proposition 1, can be used to distinguish non-conjugate subalgebras of a given Lie algebra, as well as to decide whether a Lie algebra $\mathfrak{g}' \subset \mathfrak{g}$ in a given realization results from the restriction of a realization of $\mathfrak{g}$ with the same number of coordinates.

To exemplify this situation, consider in $\mathfrak{X}(\mathbb{R}^3)$ the Lie algebra $\mathfrak{g}$ generated by the vector fields $\mathbf{X}_1 = x\frac{\partial}{\partial z}$, $\mathbf{X}_2 = y\frac{\partial}{\partial x}$ and $\mathbf{X}_3 = z\frac{\partial}{\partial y}$. A routine computation shows that $\mathfrak{g} \simeq \mathfrak{sl}(3,\mathbb{R})$ and that the realization arises from the fundamental three-dimensional representation $\Gamma = [1,0]$. Now consider the subalgebras generated by the vector fields

$$\mathbf{Y}_1 = x\frac{\partial}{\partial z},\ \mathbf{Y}_2 = z\frac{\partial}{\partial x},\ \mathbf{Y}_3 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}; \mathbf{Z}_1 = x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x},\ \mathbf{Z}_2 = y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y},\ \mathbf{Z}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

The Lie algebras generated by the $\mathbf{Y}_i$ respectively $\mathbf{Z}_i$ are both isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, as can be easily verified. It follows at once that the $\mathfrak{sl}(2,\mathbb{R})$-copy generated by the $\mathbf{Y}_i$ possesses the linear invariant y, indicating that the representation Γ branches as the direct sum of the irreducible 2-dimensional and the trivial representation of $\mathfrak{sl}(2,\mathbb{R})$. On the contrary, for the copy generated by the vector fields $\mathbf{Z}_i$ there is no linear invariant, and the subduced representation remains necessarily irreducible. As conjugate subalgebras must give rise to the same branching rules [18], we conclude that the two copies of $\mathfrak{sl}(2,\mathbb{R})$ are not conjugate, and in particular nonequivalent as $\mathfrak{sl}(2,\mathbb{R})$-subalgebras of vector fields in $\mathbb{R}^3$.

In a general frame, the detailed analysis of the branching rules of semisimple Lie algebras (see e.g. [17]) indicates additional patterns that can be potentially useful for the obtainment and characterization of invariants of (linear) realizations. Two of such properties, that still have to be justified for arbitrary representations of simple Lie algebras of rank $l \geq 8$ and their corresponding real forms of representations, can be enumerated as follows:

(i) If the Lie algebra $\mathfrak{g}'$ is irreducibly embedded into $\mathfrak{g}$ and $\Phi_\Gamma : \mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{R}^n)$ is a realization associated to the lowest dimensional irreducible representation Γ of $\mathfrak{g}$ and $\dim \Gamma < \dim \mathfrak{g}'$, then $r_{\Gamma'}(\mathfrak{g}') = r_\Gamma(\mathfrak{g})$ and both realizations have the same invariants.

(ii) The rank of a realization of $\mathfrak{g}$ associated to a reducible representation $\Gamma \oplus \Gamma^*$ with $\dim \Gamma < \frac{1}{2} \dim \mathfrak{g}$ and Γ^* the dual representation satisfies the relation $r(\mathfrak{g}) \leq 2\, r_\Gamma(\mathfrak{g}) - 1$.

3.1. Subduced realizations of $\mathfrak{so}(7)$

In order to illustrate some of the preceding properties, we analyze the compact simple exceptional Lie algebra $G_{2(-14)}$ as a subalgebra of the compact orthogonal algebra $\mathfrak{so}(7)$ in some detail. Restricting the adjoint representation $[0,1,0]$ of $\mathfrak{so}(7)$ to G_2, we obtain the branching rule

$$[0,1,0] \downarrow [1,0] \oplus [0,1]\,, \tag{25}$$

where $[1,0]$ is the adjoint representation of G_2 and $[0,1]$ is defining 7-dimensional representation. Actually, this representation characterizes the embedding $G_{2(-14)} \subset \mathfrak{so}(7)$, that has index $j_f = 1$. From the realization of $\mathfrak{so}(7)$ associated to the fundamental 7-dimensional representation $[1,0,0]$, given by the vector fields $\mathbf{E}_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$ $(i \neq j)$ the restriction to $G_{2(-14)}$ provides the following realization as vector fields in $\mathbb{R}^7$:

$$\begin{array}{rcl}
\mathbf{X}_1 &=& -x^7\frac{\partial}{\partial x^1} + x^4\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^4} + x^1\frac{\partial}{\partial x^7},\\
\mathbf{X}_2 &=& x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} - x^7\frac{\partial}{\partial x^4} + x^4\frac{\partial}{\partial x^7},\\
\mathbf{X}_4 &=& x^6\frac{\partial}{\partial x^1} + x^4\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^4} - x^1\frac{\partial}{\partial x^6},\\
\mathbf{X}_{11} &=& -\sqrt{3}x^6\frac{\partial}{\partial x^1} - 2\sqrt{3}x^3\frac{\partial}{\partial x^2} + \sqrt{3}x^4\frac{\partial}{\partial x^3} - \sqrt{3}x^3\frac{\partial}{\partial x^4} + 2\sqrt{3}x^2\frac{\partial}{\partial x^5} + \sqrt{3}x^1\frac{\partial}{\partial x^6},\\
\mathbf{X}_{13} &=& -2\sqrt{3}x^5\frac{\partial}{\partial x^1} + \sqrt{3}x^6\frac{\partial}{\partial x^2} + \sqrt{3}x^7\frac{\partial}{\partial x^3} + 2\sqrt{3}x^1\frac{\partial}{\partial x^5} - \sqrt{3}x^2\frac{\partial}{\partial x^6} - \sqrt{3}x^3\frac{\partial}{\partial x^7}.
\end{array} \tag{26}$$

The remaining vector fields corresponding to generators of $G_{2(-14)}$ follow from the commutators:

$$\begin{array}{c}
[\mathbf{X}_1,\mathbf{X}_2] = \frac{1}{2}\mathbf{X}_3,\ [\mathbf{X}_1,\mathbf{X}_4] = \mathbf{X}_7,\ [\mathbf{X}_1,\mathbf{X}_{11}] = \mathbf{X}_{14},\ [\mathbf{X}_1,\mathbf{X}_{13}] = -\mathbf{X}_{12},\ [\mathbf{X}_2,\mathbf{X}_4] = \mathbf{X}_6,\\
[\mathbf{X}_4,\mathbf{X}_{13}] = \mathbf{X}_{10},\ [\mathbf{X}_1,\mathbf{X}_6] = \mathbf{X}_5,\ [\mathbf{X}_4,\mathbf{X}_{14}] = -\mathbf{X}_9,\ [\mathbf{X}_6,\mathbf{X}_7] - [\mathbf{X}_4,\mathbf{X}_5] = \frac{2}{\sqrt{3}}\mathbf{X}_8.
\end{array} \tag{27}$$

It hence suffices to analyze the properties of the symmetry generators $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_4, \mathbf{X}_{11}$ and $\mathbf{X}_{13}$, as all the remaining elements of the basis are expressible as commutators of these five elements. As $\dim G_{2(-14)} > 7$, the fact that the reduction $[1,0,0] \downarrow [0,1]$ is irreducible implies that the rank of the realization matrix is preserved, according to the property (i) above. Clearly there is only one invariant function $F(\mathbf{x})$, given by the quadratic form $Q_2 = \sum_{k=1}^{7} \left(x^i\right)^2$. Introducing Q_2 as a new coordinate allows us to consider it as cyclic, and hence the realization can be reduced to a genuinely nonlinear one in $\mathbb{R}^6$, as $[0,1]$ is the lowest dimensional faithful representation of $G_{2(-14)}$. Further, considering the embedding $\mathfrak{su}(3) \subset G_{2(-14)}$, where the subalgebra is generated by $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_4\}$ and their commutators, the branching rule is given by $[0,1] \downarrow (1,0) \oplus (0,1) \oplus (0,0)$, where the representation $(1,0) \oplus (0,1)$ is reducible as a complex representation, but irreducible as a real one. As a copy of the trivial representation of $\mathfrak{su}(3)$ appears, we have an additional coordinate that is cyclic. Hence the realization in $\mathbb{R}^7$ of $\mathfrak{su}(3)$ possesses two invariants, one being Q_2, as well as x^5 (observe that this provides an example of the property (ii) in the preceding paragraph). With respect to a new set of coordinates $\left\{y^1, \cdots, y^5\right\} = \left\{x^1, \cdots, x^4, x^6\right\}$, we write the realization of $\mathfrak{su}(3)$ in $\mathfrak{X}\left(\mathbb{R}^5\right)$ as

$$\begin{array}{rcllll} \mathbf{X}_1 & = & -J_0 \frac{\partial}{\partial y^1} & +y^4 \frac{\partial}{\partial y^2} & & -y^2 \frac{\partial}{\partial y^4} & \\ \mathbf{X}_2 & = & y^2 \frac{\partial}{\partial y^1} & -y^1 \frac{\partial}{\partial y^2} & & -J_0 \frac{\partial}{\partial y^4} & \\ \mathbf{X}_4 & = & y^5 \frac{\partial}{\partial y^1} & & +y^4 \frac{\partial}{\partial y^3} & -y^3 \frac{\partial}{\partial y^4} & -y^1 \frac{\partial}{\partial y^5} \end{array}, \tag{28}$$

where $J_0 = \sqrt{\sum_{k=1}^{5} \left(y^k\right)^2 + \lambda}$ for some constant λ and the remaining vector fields are obtained from (27). Finally, if we extend the chain to $\mathfrak{so}(3) \subset \mathfrak{su}(3) \subset G_{2(-14)}$, the resulting real representation for $\mathfrak{so}(3)$ is $R^{II}_{1/2} \oplus \Gamma^3_0$, with invariants $J_1 = \sqrt{(y^1)^2 + (y^2)^2 + (y^4)^2 + \lambda}$ and three linear ones.

On the other hand let the realization of $\mathfrak{so}(3)$ as subalgebra of $\mathfrak{so}(7)$ be given by

$$\begin{array}{rl} \mathbf{X}_1 = & -\sqrt{\frac{3}{2}} x^4 \frac{\partial}{\partial x^1} + \sqrt{\frac{3}{2}} x^3 \frac{\partial}{\partial x^2} - \left(\sqrt{\frac{3}{2}} x^2 + \sqrt{\frac{5}{2}} x^6\right) \frac{\partial}{\partial x^3} + \sqrt{\frac{3}{2}} x^1 \frac{\partial}{\partial x^4} \\ & +\sqrt{\frac{5}{2}} x^5 \frac{\partial}{\partial x^4} - \sqrt{\frac{5}{2}} x^4 \frac{\partial}{\partial x^5} + \sqrt{\frac{5}{2}} x^3 \frac{\partial}{\partial x^6} - \sqrt{6} x^7 \frac{\partial}{\partial x^6} + \sqrt{6} x^6 \frac{\partial}{\partial x^7}, \\ \mathbf{X}_2 = & -\sqrt{\frac{3}{2}} x^3 \frac{\partial}{\partial x^1} - \sqrt{\frac{3}{2}} x^4 \frac{\partial}{\partial x^2} + \left(+\sqrt{\frac{3}{2}} x^1 - \sqrt{\frac{5}{2}} x^5\right) \frac{\partial}{\partial x^3} + \sqrt{\frac{3}{2}} x^2 \frac{\partial}{\partial x^4} \\ & -\sqrt{\frac{5}{2}} x^6 \frac{\partial}{\partial x^4} + \sqrt{\frac{5}{2}} x^3 \frac{\partial}{\partial x^5} + \sqrt{6} x^7 \frac{\partial}{\partial x^5} + \sqrt{\frac{5}{2}} x^4 \frac{\partial}{\partial x^6} - \sqrt{6} x^5 \frac{\partial}{\partial x^7}, \\ \mathbf{X}_2 = & 3x^2 \frac{\partial}{\partial x^1} - 3x^1 \frac{\partial}{\partial x^2} + 2x^4 \frac{\partial}{\partial x^3} - 2x^3 \frac{\partial}{\partial x^4} + x^6 \frac{\partial}{\partial x^5} - x^5 \frac{\partial}{\partial x^6}. \end{array} \tag{29}$$

It is clear that no linear invariants of the realization can exist as the rank of the realization matrix is three. By the preceding results, we conclude that the branching rule associated to $[1,0,0]$ does not contain a copy of the trivial representation of $\mathfrak{so}(3)$. As the latter is not a maximal subalgebra of $\mathfrak{so}(7)$, it follows that $\mathfrak{so}(3)$ must be contained in either $G_{2(-14)}$, $\mathfrak{su}(4)$ or $\mathfrak{su}(2)^3$, that correspond to the semisimple maximal subalgebras of $\mathfrak{so}(7)$. From the branching rules (see e.g. [17]) it follows that the only possibility to provide the irreducible representation associated to (29) is given by $G_{2(-14)}$, from which we deduce that $\mathfrak{so}(3)$ is irreducibly embedded into $G_{2(-14)}$ with embedding index $j_f = 28$.

4. The Lie algebra $\mathfrak{so}(3)$ as a principal subalgebra of $\mathfrak{so}(2J+1)$

As is well known, for any integer J the $\mathfrak{so}(3)$-representation R^I_J of dimension $2J+1$ and highest weight $2J$ is of the first class (that is, its complexification remains irreducible, see [25]). Therefore, $\mathfrak{so}(3)$ can be represented as a subalgebra of the compact Lie algebra $\mathfrak{so}(2J+1)$. Indeed, as follows from the structure theory, for $J \neq 3$ the Lie algebra $\mathfrak{so}(3)$ is irreducibly embedded into $\mathfrak{so}(2J+1)$ as a three-dimensional principal subalgebra, meaning in particular that the branching rule for the defining representation $\left[1, 0^{(J-1)}\right]$ of $\mathfrak{so}(2J+1)$ remains irreducible when restricted to $\mathfrak{so}(3)$.[3] The explicit construction of representation matrices for the real

[3] The exceptional case $J = 3$ corresponds to the chain $\mathfrak{so}(3) \subset G_{2(-14)} \subset \mathfrak{so}(7)$ that has been crucial in atomic spectroscopy [10].

irreducible representations of $\mathfrak{so}(3)$ was developed in [26], for which reason we merely recall the construction. For any $J \geq 1$ define the scalars

$$a_l = \sqrt{\frac{2l\,J - l\,(l-1)}{4}},\ 1 \leq l \leq J-1;\ \nu_0 = \sqrt{\frac{J\,(J+1)}{2}}. \tag{30}$$

Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_{2J+1}\}$ denote a basis of the representation space of the real representation R_J^I. Further let $\left[\frac{n}{2}\right]$ denote the integer part of $\frac{n}{2}$. Then the matrix elements are given by

$$\begin{aligned}\langle \mathbf{e}^k | R_J^I\,(X_1)\,|\mathbf{e}_l\rangle &= \left(\frac{1+(-1)^{k-1}}{2}\right)\left(\delta_{k+3}^l\, a_{([\frac{k+1}{2}])} + \delta_k^{l+1} a_{([\frac{k-1}{2}])}\right) - (a_J + \nu_0) \times \\ &\quad \left(\delta_{2J+1}^l \delta_k^{2J} - \delta_{2J}^l \delta_k^{2J+1}\right) - \left(\frac{1+(-1)^k}{2}\right)\left(\delta_{k+1}^l a_{([\frac{k}{2}])} + \delta_k^{l+3} a_{([\frac{k-2}{2}])}\right), \\ \langle \mathbf{e}^k | R_J^I\,(X_2)\,|\mathbf{e}_l\rangle &= \delta_{k+2}^l\, a_{([\frac{k+1}{2}])} - \delta_k^{l+2} a_{([\frac{k-1}{2}])} - (a_J + \nu_0)\left(\delta_{2J+1}^l \delta_k^{2J-1} - \delta_{2J-1}^l \delta_k^{2J+1}\right), \\ \langle \mathbf{e}^k | R_J^I\,(X_3)\,|\mathbf{e}_l\rangle &= \frac{\left(1+(-1)^k\right)\delta_k^{l+1}\,(2J+2-k) + \left((-1)^k - 1\right)\delta_l^{k+1}\,(2J+1-k)}{4},\end{aligned} \tag{31}$$

where $1 \leq k, l \leq 2J+1$. The matrices $R_J^I\,(X_k)$ constructed with these values are clearly skew-symmetric, and thus belong to $\mathfrak{so}\,(2J+1)$, showing that the linear map

$$\varphi_J : \mathfrak{so}\,(3) \to \mathfrak{so}\,(2J+1)\,; \quad X_k \mapsto R_J^I\,(X_k) \tag{32}$$

defines a Lie algebra homomorphism and an irreducible embedding. The case $J = 1$ is somewhat degenerate, as it corresponds to the adjoint representation of $\mathfrak{so}(3)$, thus possessing the quadratic Casimir operator as only invariant. Denoting for $J \geq 2$

$$\langle \mathbf{e}^k | R_J^I\,(X_\alpha)\,|\mathbf{e}_l\rangle = \left(R_J^I\,(X_\alpha)\right)_{k,l},\ \alpha = 1,2,3, \tag{33}$$

it follows at once that the vector fields corresponding to each representation R_J^I are given by

$$\mathbf{X}_\alpha = x^k \left(R_J^I\,(X_\alpha)\right)_{k,l} \frac{\partial}{\partial x^l},\ \alpha = 1,2,3. \tag{34}$$

It can be easily verified that for any $j \in \{1, \cdots, 2J+1\}$ there is always an α such that the condition

$$dx^j\,(\mathbf{X}_\alpha) = \left(R_J^I\,(X_\alpha)\right)_{k,j} \neq 0, \tag{35}$$

is satisfied, showing that there are no linear invariants. On the other hand, due to the embedding $\mathfrak{so}(3) \subset \mathfrak{so}(2J+1)$, the quadratic form $Q = \sum_{p=1}^{2J+1} (x^p)^2$ is an invariant. In this case, for any $J \geq 2$, the rank of the coefficient matrix is always three, and thus the realization possesses exactly $2J-2$ fundamental invariants. The explicit obtainment of the invariants of these realizations, either in the semi-algebraic formulation developed in an earlier section, or solving the corresponding systems of PDEs, presents considerable difficulties, and is currently not explicitly solved. Without going into the computation details, that shall be given elsewhere, we remark that the invariants of realizations associated to the real representations R_j^I are divided into two cases, according to the parity of J:

(i) If $J = 2q \geq 2$ is even, then the fundamental invariants of (34) can be chosen to have degrees $d = 2, 3, \cdots, 2J-1$.

(ii) If $J=2q+1\geq 1$ is odd, then any invariant of the realization (34) has even degree.

It may be observed that for half-integer values, the real irreducible representations $R^{II}_{J/2}$ of $\mathfrak{so}(3)$ corresponding to the real form of the representations $D_{J/2}$ with highest weight J are of the second class, meaning that their complexification is reducible as a complex representation [25]. Although these representations define an embedding $\mathfrak{so}(3)\subset\mathfrak{so}(4J+2)$, it is not embedded as a maximal subalgebra [18]. In this situation, the pertinent Lie algebras to be analyzed with respect to the embedding are the symplectic algebras [27, 28]. As for half-integer values $J/2$ the scalar J is necessarily odd, the corresponding invariants of the realization are all of even degree, in analogy to the integer case.

5. Symmetries of second-order dynamical (Lagrangian) systems

Invariants of realizations can also be used directly for the construction and symmetry analysis of second-order dynamical systems, either as Lie-point or Noether symmetries [4, 29]. Here, the representation theory can further be applied to study the symmetry-breaking phenomena of systems with a given (maximal) symmetry by means of additional potential terms, as well as to guarantee that a given realization of a Lie algebra corresponds to the maximal symmetry of a system.

Among the various methods to compute symmetries of differential equations, those of most common use are the direct prolongation method and the formulation in terms of differential operators [7]. Any system of n second-order ordinary differential equations

$$\ddot{\mathbf{x}}=\omega^{\alpha}\left(t,\mathbf{x}\right),\ 1\leq\alpha\leq n, \tag{36}$$

can be rewritten in equivalent form as the partial differential equation

$$\mathbf{A}f=\left(\frac{\partial}{\partial t}+\dot{x}^{i}\frac{\partial}{\partial x^{i}}+\omega^{i}\left(t,\mathbf{x}\right)\frac{\partial}{\partial\dot{x}^{i}}\right)f=0. \tag{37}$$

In this context, a vector field $X=\zeta\left(t,\mathbf{x}\right)\frac{\partial}{\partial t}+\eta_i\left(t,\mathbf{x}\right)\frac{\partial}{\partial x^{i}}\in\mathfrak{X}\left(\mathbb{R}^{n+1}\right)$ is a Lie point symmetry of the system (36) whenever the prolonged vector field $\dot{X}=X+\dot{\eta}_i\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial\dot{x}^{i}}$ satisfies the commutator

$$\left[\dot{X},\mathbf{A}\right]=-\frac{d\zeta}{dt}\mathbf{A}, \tag{38}$$

where $\dot{\eta}_i=-\frac{d\zeta}{dt}\dot{x}_i+\frac{d\eta_i}{dt}$. The advantage of this reformulation is that the prolongation of the symmetry generator X is already contained in the commutator, the symmetry condition being given by the coefficients in $\frac{\partial}{\partial\dot{x}^{i}}$. Whenever the system arises from a variational principle, a point symmetry that satisfies the condition

$$\dot{\mathbf{X}}\left(L\right)+\mathbf{A}\left(\zeta\right)-\mathbf{A}\left(V\right)=0 \tag{39}$$

is called a Noether symmetry, where $V\left(t,\mathbf{x}\right)$ a function independent on $\dot{\mathbf{x}}$ and L is an admissible Lagrangian of the system. By some abuse of notation, we shall call a symmetry satisfying (39) with $\zeta=V=0$ a variational symmetry. Noether symmetries can alternatively be deduced from a constant of the motion, i.e., a function $J\left(\dot{q}_j,q_j,t\right)$ such that

$$\frac{dJ}{dt}=\frac{\partial J}{\partial t}+\dot{x}^{i}\frac{\partial J}{\partial x^{i}}+\ddot{x}^{i}\frac{\partial J}{\partial\dot{x}^{i}}=\mathbf{A}\left(J\right)=0. \tag{40}$$

Any first integral of (36) determines (up to a gauge term) a Noether symmetry

$$X_J = \zeta\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial t} + \eta^{\alpha}\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial x^{\alpha}} + \dot{\eta}^{\alpha}\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial \dot{q}^{\alpha}}, \tag{41}$$

the components of which are recovered using the relations

$$\frac{\partial^2 L}{\partial \dot{x}^{\alpha}\partial \dot{x}^{\beta}}\left(\eta_{\alpha} - \dot{x}^{\alpha}\zeta\right) + \frac{dJ}{d\dot{x}^{\beta}} = 0. \tag{42}$$

As vector field, X_J is always a point symmetry if the components do not depend on the $\dot{\mathbf{x}}$.[4]

Proposition 3 *Let $\Phi : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^n)$ be a realization of a Lie algebra $\mathfrak{g}$ associated to a linear skew-symmetric representation Γ and J an invariant of the realization. Then $\Phi(\mathfrak{g})$ can be realized as a Lie algebra of Lie point symmetries of the second-order dynamical system with equations of the motion*

$$\ddot{x}^i - \varphi(t)\frac{\partial J}{\partial x^i} = 0,\ 1 \leq i \leq n \tag{43}$$

arising from the regular Lagrangian $L = \frac{1}{2}\sum_{k=1}^{n}(\dot{x}^i)^2 + \varphi(t)J(\mathbf{x})$.

The equations of the motion, expressed as a vector field, are in this case

$$\mathbf{A} = \frac{\partial}{\partial t} + \dot{x}^i\frac{\partial}{\partial x^i} + \varphi\left(t\right)\frac{\partial J}{\partial x^i}\frac{\partial}{\partial \dot{x}^i},$$

while the first prolongation of $\mathbf{X}$ is given by

$$\dot{\mathbf{X}}_{\alpha} = \xi_{\alpha}\frac{\partial}{\partial x^j} + \dot{x}^i\frac{\partial \xi_{\alpha}^j}{\partial x^i}\frac{\partial}{\partial \dot{x}^j}.$$

Evaluating the commutator (38) gives

$$\left[\dot{\mathbf{X}}_{\alpha},\mathbf{A}\right] = \left(\dot{x}^i\frac{\partial \xi_{\alpha}^j}{\partial x^i} - \dot{\xi}_{\alpha}^j\right)\frac{\partial}{\partial x^j} + \left(\dot{x}^i\frac{\partial \dot{\xi}_{\alpha}^j}{\partial x^i} + \varphi\left(t\right)\frac{\partial J}{\partial x^i}\frac{\partial \dot{\xi}_{\alpha}^i}{\partial \dot{x}^j} - \varphi\left(t\right)\xi_{\alpha}^i\frac{\partial^2 J}{\partial x^i\partial x^j}\right)\frac{\partial}{\partial \dot{x}^j}. \tag{44}$$

The term in $\frac{\partial}{\partial x^j}$ is obviously zero by the definition of $\dot{\xi}_{\alpha}^j$, hence the symmetry condition reduces to

$$\dot{x}^i\frac{\partial \dot{\xi}_{\alpha}^j}{\partial x^i} + \varphi\left(t\right)\frac{\partial J}{\partial x^i}\frac{\partial \dot{\xi}_{\alpha}^i}{\partial \dot{x}^j} - \varphi\left(t\right)\xi_{\alpha}^i\frac{\partial^2 J}{\partial x^i\partial x^j} = 0,\ 1 \leq j \leq n \tag{45}$$

where

$$\frac{\partial \dot{\xi}_{\alpha}^j}{\partial x^i} = \dot{x}^l\frac{\partial^2 \xi_{\alpha}^j}{\partial x^i\partial x^l},\ \ \frac{\partial \dot{\xi}_{\alpha}^j}{\partial \dot{x}^i} = \frac{\partial \xi_{\alpha}^j}{\partial x^i} = -\frac{\partial \xi_{\alpha}^i}{\partial x^j},\ 1 \leq i,j \leq n.$$

Inserting these expression in (45) gives the equation

$$\dot{x}^i\left(\dot{x}^l\frac{\partial^2 \xi_{\alpha}^j}{\partial x^i\partial x^l}\right) + \varphi\left(t\right)\left(\frac{\partial J}{\partial x^i}\frac{\partial \xi_{\alpha}^j}{\partial x^i} - \xi_{\alpha}^i\frac{\partial^2 J}{\partial x^i\partial x^j}\right) = \dot{x}^i\dot{x}^l\frac{\partial^2 \xi_{\alpha}^j}{\partial x^i\partial x^l} - \varphi\left(t\right)\frac{\partial}{\partial x^j}\left(\xi_{\alpha}^i\frac{\partial J}{\partial x^i}\right). \tag{46}$$

As J is an invariant of the realization, the last term is zero, while the first one vanishes whenever the component functions ξ_{α}^j are linear. The latter requirement is fulfilled because the realization

[4] Noether symmetries that are not point symmetries are sometimes called dynamical symmetries, like in [7], although there is no such restriction in Noether's original approach [30].

arises from a skew-symmetric linear representation of $\mathfrak{g}$, so that $\left[\dot{\mathbf{X}}_\alpha, \mathbf{A}\right] = 0$ holds and we conclude that $\mathbf{X}_\alpha$ is a point symmetry of the system.

As the result does not depend on the particular structure of the invariant, it allows us to construct dynamical systems with a given Lie algebra of point symmetries. We observe that the approach is valid for both conservative and non-conservative Lagrangian systems, as the function $\varphi(t)$ in the potential does not explicitly intervene in the commutator. The procedure can be easily extended to subalgebras. If $\mathfrak{g}' \subset \mathfrak{g}$ is a subalgebra and $\{J_1, \cdots, J_k\}$ are invariants of the subduced realization of $\mathfrak{g}'$, then for generic functions $\{\varphi_1(t), \cdots, \varphi_k(t)\}$ the system

$$\ddot{x}^i = \sum_{l=1}^{k} \varphi_l(t) \frac{\partial J_l}{\partial x^i} \; 1 \leq j \leq n \tag{47}$$

turns out to have exactly a Lie-point symmetry algebra isomorphic to $\mathfrak{g}'$. Starting from the system associated to an invariant of $\mathfrak{g}$, the restriction to subalgebras $\mathfrak{g}'$ and the addition of potential terms corresponding to invariants of the latter (and not of the former) can be seen as a recursive procedure of breaking the symmetry of a system. In connection with the branching rules, it also allows to determine whether a given subalgebra can be realized as a maximal symmetry algebra or not.

Let us illustrate this fact considering once more the compact Lie algebra $\mathfrak{so}(7)$ in the fundamental 7-dimensional representation $\Gamma = [100]$ and the quadratic invariant $J = \left(x^1\right)^2 + \cdots + \left(x^7\right)^2$ of the realization. As the rank of the realization remains irreducible when restricted to $G_{2(-14)}$, it turns out that a system of the type

$$\ddot{x}^i = \varphi(t) \frac{\partial F(J)}{\partial x^i}, \tag{48}$$

where $F(J)$ is an arbitrary function of J, is always $G_{2(-14)}$-invariant, but there is no possibility to break the maximal symmetry of $\mathfrak{so}(7)$ to the subalgebra, as both have the same invariant functions for the realization associated to the corresponding representations. A dimensional reduction of the symmetry algebra would require an additional term invariant by the generators of $G_{2(-14)}$, but not invariant by all of $\mathfrak{so}(7)$. Such a symmetry-breaking can be obtained, for example, taking the principal $\mathfrak{so}(3)$-subalgebra of index $j_f = 28$, as in this case, although the embedding is still irreducible, there are 3 invariants of the subduced realization. As commented before, the fundamental invariants can be chosen as J, as well as a fourth and sixth order invariants J_2 and J_3 respectively, with J_2 being explicitly given by:

$$\begin{aligned} J_2 = \; & \left(x^1\right)^4 + \left(x^2\right)^4 - \tfrac{3}{4}\left(\left(x^3\right)^4 + \left(x^4\right)^4\right) + \tfrac{3}{5}\left(\left(x^5\right)^4 + \left(x^6\right)^4 + \frac{\left(x^7\right)^4}{10}\right) + \\ & \left(4\left(\left(x^1\right)^2 + \left(x^2\right)^2\right) + 7\left(x^7\right)^2\right)\left(\left(x^5\right)^2 + \left(x^6\right)^2\right) + 2\left(x^1x^2\right)^2 - 3\left(x^3x^4\right)^2 \\ & + 2\left(\left(x^1\right)^2 + \left(x^2\right)^2\right)\left(\left(x^3\right)^2 + \left(x^4\right)^2\right) + 4\sqrt{\tfrac{3}{5}}\left(x^2\left(x^6\right)^3 - x^1\left(x^5\right)^3\right) \\ & + 2\sqrt{15}\left(x^1x^5 - x^2x^6\right)\left(\left(x^3\right)^2 - \left(x^4\right)^2\right) + 5\left(\left(x^3\right)^2 + \left(x^4\right)^2\right)\left(x^7\right)^2 \\ & + 4\sqrt{\tfrac{3}{5}}\left(\left(x^5\right)^2\left(x^3x^7 - 3x^2x^6\right) - \left(x^6\right)^2\left(x^3x^7 - 3x^1x^5\right)\right) + \tfrac{6}{5}\left(x^5\right)^2\left(x^6\right)^2 \\ & + \tfrac{1}{5}\left(x^7\right)^2\left(\left(x^5\right)^2 + \left(x^6\right)^2\right) + 4\sqrt{15}x^3x^4\left(x^1x^6 + x^2x^5\right) \\ & - 12x^2x^7\left(x^4x^5 + x^3x^6\right) - 12x^1x^7\left(x^3x^5 - x^4x^6\right) + 8\sqrt{\tfrac{3}{5}}x^4x^5x^6x^7. \end{aligned}$$

A routine computation shows that the equations of the motion

$$\ddot{x}^i = \varphi(t) \frac{\partial F(J)}{\partial x^i} + \psi(t) \frac{\partial G(J_2)}{\partial x^i} \tag{49}$$

are $\mathfrak{so}(3)$-invariant, and that no other generator of $G_{2(-14)}$ possesses J_2 as an invariant. For generic functions $\varphi(t)$ and $\psi(t)$, it is straightforward to verify that no other point symmetry, specifically with components in $\frac{\partial}{\partial t}$, can exist.

We finally comment how, for realizations related to orthogonal representations, the dynamical systems (48) having potentials associated to invariants of the realization are not only point symmetries, but also special Noether symmetries, namely variational symmetries.

Proposition 4 *Let $\Phi_\Gamma : \mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{R}^n)$ be the subduced realization associated to the embedding $\mathfrak{g} \subset \mathfrak{so}(N)$. Let $\{J_1, \cdots, J_{\chi_0}\}$ be invariants of the realization. Then the generators $\mathbf{X}_\alpha$ of $\Phi(\mathfrak{g})$ are variational symmetries of the Lagrangian*

$$L = \frac{1}{2}\sum_{k=1}^{n}\left(\dot{x}^i\right)^2 - \sum_{k=1}^{\chi_0}\varphi_k(t)\, J_k. \tag{50}$$

In order to satisfy the Noether symmetry condition, the equation (39) must be satisfied identically. Evaluating it we obtain

$$\dot{\mathbf{X}}(L) - \mathbf{A}(\zeta) - \mathbf{A}(V) = -\varphi_k(t)\,\xi_\alpha^j(\mathbf{x})\frac{\partial J_k}{\partial x^j} + \dot{x}^l\frac{\partial \xi_\alpha^j}{\partial x^l}\dot{x}^j - \frac{\partial V}{\partial t} - \dot{x}^l\frac{\partial V}{\partial x^l}. \tag{51}$$

The first term automatically vanishes as J_k is an invariant of the realization for any k. The terms linear in $\dot{\mathbf{x}}$ imply that V is at most a time function, hence implying that V must be a constant in order to fulfill (51). Finally, the quadratic term in $\dot{\mathbf{x}}$ can be rewritten as

$$\sum_{k=1}^{n}\left(\dot{x}^k\right)^2\frac{\partial \xi_\alpha^k}{\partial x^k} + \sum_{l<j}\dot{x}^l\dot{x}^j\left(\frac{\partial \xi_\alpha^j}{\partial x^l} + \frac{\partial \xi_\alpha^l}{\partial x^j}\right) \tag{52}$$

As the realization arises from the restriction of a representation of $\mathfrak{so}(N)$, the representation matrices are skew-symmetric, and hence the vector fields satisfy the constraints

$$\frac{\partial \xi_\alpha^k}{\partial x^k} = 0,\ \frac{\partial \xi_\alpha^j}{\partial x^l} + \frac{\partial \xi_\alpha^l}{\partial x^j} = 0,\ 1 \leq \alpha \leq \dim\mathfrak{g}, \tag{53}$$

showing that (52) and hence (51) vanishes.

As an explicit example leading to systems of this type we may consider the vector fields in formula (29). For generic choices of the functions $\varphi(t)$ and $\psi(t)$ the Lie point and Noether symmetry algebras of the system determined by the equations of motion (49) coincide and are isomorphic to $\mathfrak{so}(3)$.

6. Concluding remarks

The approach to realizations by vector fields of Lie algebras by means of linear representations and their invariants has been shown to exhibit some interesting connections with the branching rules for the embedding problem of Lie algebras, such as the properties of irreducibly embedded subalgebras or the multiplicity of representations by subduction. This analysis allows to distinguish between nonconjugate subalgebras of a given Lie algebra by means of the invariants of a given realization, without knowing a priori the embedding index or the corresponding branching rule. Some properties that hold at least for low ranks have been recognized, and their validity in the arbitrary case could be of potential use in the context of a further systematization of computational criteria for branching rules, as well as for the Clebsch-Gordan problem. On the other hand, the study of the orders and structure of the invariants of a realization suggests, in some sense, a natural extension of the analysis of generalized Casimir invariants in Lie algebras and the associated missing label problem, a relevant problem in applications, where the exact implications with the invariant theory of inhomogeneous Lie algebras has still to be studied in detail.

Some applications to the construction of second-order dynamical systems with prescribed symmetry, as well as a procedure to break the symmetry to that of a proper subalgebra, have been indicated. An interesting question that arises in this context is whether similar statements concerning second-order dynamical systems and potentials determined by the invariants of a nonlinear realization can be established, and to what extent these criteria are further related with the representation theory, e.g. by means of projections. Within this frame, a problem that is worthy to be mentioned is the possibility of extending the constructions to symmetries explicitly depending on the velocities, generalizing the method to Noether symmetries that are no more point symmetries of a system. Even if such an ansatz only succeeds partially, any progress in this direction would be of interest for the structural analysis of the constants of the motion of dynamical systems.

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