

Three representations of the fractional p -Laplacian: semigroup, extension and Balakrishnan formulas

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Abstract

We introduce three representation formulas for the fractional p -Laplace operator in the whole range of parameters $0 < s < 1$ and $1 < p < \infty$. Note that for $p \neq 2$ this is a nonlinear operator. The first representation is based on a splitting procedure that combines a renormalized nonlinearity with the linear heat semigroup. The second adapts the nonlinearity to the Caffarelli-Silvestre linear extension technique. The third one is the corresponding nonlinear version of the Balakrishnan formula. We also discuss the correct choice of the constant of the fractional p -Laplace operator in order to have continuous dependence as $p \rightarrow 2$ and $s \rightarrow 0^+, 1^-$.

A number of consequences and proposals are derived. Thus, we propose a natural spectral-type operator in domains, different from the standard restriction of the fractional p -Laplace operator acting on the whole space. We also propose numerical schemes, a new definition of the fractional p -Laplacian on manifolds, as well as alternative characterizations of the $W^{s,p}(\mathbb{R}^n)$ seminorms.

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1 Introduction

In recent years, nonlocal differential operators and equations have attracted an increasing amount of attention. In particular, the fractional Laplacian is defined for suitable functions $u(x)$, $x \in \mathbb{R}^n$, as

$$(-\Delta)^s u(x) = c_{n,s} \text{PV} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad (1.1)$$

where $s \in (0, 1)$, $c_{n,s} > 0$ is a normalization constant and PV denotes that the integral is taken in the principal value sense, cf. the classical references [34, 48]. It appears (but for the constant factor) as the subdifferential of the fractional energy

$$\mathcal{J}_{2,s}(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy, \quad (1.2)$$

that is related to the definition of the standard norm of the space $W^{s,2}(\mathbb{R}^n)$. Besides, there are several known representation formulas that have proven to be quite convenient in tackling problems involving fractional operators like (1.1), see the list in [33] where a rather thorough discussion is made. Three of those representations are quite relevant and will be re-examined here in a nonlinear context. They are the so called semigroup (or Bochner's subordination) formula; the Caffarelli-Silvestre extension problem, that allows to write the nonlocal operator as in terms of a local problem in one extra dimension; and Balakrishnan's formula.

(i) A first representation was given by S. Bochner (see [10]). In modern form, it says that for a smooth enough function u the following formula holds

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta}[u](x) - u(x)) \frac{dt}{t^{1+s}}, \quad (1.3)$$

where $e^{t\Delta}$ denotes the heat semigroup, i. e., $e^{t\Delta}[u](x) := w(x, t)$ where w is the solution of

$$\begin{cases} \partial_t w(x, t) - \Delta w(x, t) = 0, & x \in \mathbb{R}^n, t > 0, \\ w(x, 0) = u(x), & x \in \mathbb{R}^n. \end{cases}$$

(ii) On the other hand, the fractional Laplacian can be written in local terms since L. Caffarelli and L. Silvestre showed in [19] that the extension problem

$$\begin{cases} \Delta_x v + \frac{1-2s}{y} v_y + v_{yy} = 0 & x \in \mathbb{R}^n, y > 0, \\ v(x, 0) = u(x) & x \in \mathbb{R}^n, \end{cases}$$

allows to compute the fractional Laplacian $(-\Delta)^s u$ as a weighted Neumann limit of v

$$(-\Delta)^s u(x) = \frac{4^s \Gamma(s)}{\Gamma(-s)} \lim_{z \rightarrow 0} \frac{v(x, y) - v(x, 0)}{y^{2s}}. \quad (1.4)$$

(iii) A further representation for the fractional Laplacian is given by A. V. Balakrishnan's formula (see [3]), that reads

$$(-\Delta)^s u(x) = \frac{\sin(\pi s)}{\pi} \int_0^\infty (-\Delta)(t - \Delta)^{-1}[u](x) \frac{dt}{t^{1-s}}. \quad (1.5)$$

The three representations have proved to be very useful both in theory and applications, and they have given rise to a wide literature.

Aim and main results. The aim of this paper is to present analogous representations in the non-linear context of the fractional p -Laplacian, and comment on their usefulness in different applications. We first recall that the fractional p -Laplacian is usually defined as follows

$$(-\Delta)_p^s u(x) = C_1 \text{PV} \int_{\mathbb{R}^d} \frac{\Phi_p(u(x) - u(y))}{|x - y|^{n+sp}} dy, \quad (1.6)$$

where $p \in (1, \infty)$ and $s \in (0, 1)$, $\Phi_p(t) = |t|^{p-2}t$, and $C_1 = C_1(n, s, p) > 0$ is a constant. The fractional p -Laplacian is the subdifferential of a convex functional called the (s, p) -energy (or Gagliardo seminorm)

$$\mathcal{J}_{p,s}(u) = \frac{C_1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy. \quad (1.7)$$

defined for those functions in $L^2(\mathbb{R}^n)$ with finite (s, p) -energy, see e.g. [42, 52]. We also recall that this operator has a dense domain in $L^2(\mathbb{R}^n)$ containing smooth enough functions. Actually, $(-\Delta)_p^s u(x)$ is well-defined in the pointwise sense for smooth functions $C_c^\infty(\mathbb{R}^n)$ (assuming also that the gradient does not vanish when p lies in the range $p \in (1, \frac{2}{2-s})$ (see, e.g., Section 3 in [31]).

Our main result is the introduction of three equivalent representation formulas for $(-\Delta)_p^s$, which retain the flavour of the linear case. First, we show that the fractional p -Laplacian can be represented by the formula

$$(-\Delta)_p^s u(x) = C_2 \int_0^{+\infty} e^{t\Delta} [\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1+\frac{sp}{2}}}, \quad (1.8)$$

where $C_2 > 0$. The precise study of this nonlinear representation occupies Section 2, see Theorem 2.1.

Second, we consider an extension operator $E_{s,p}$ related to a local problem in $\mathbb{R}^n \times (0, \infty)$, and such that the following representation holds:

$$(-\Delta)_p^s u(x) = C_3 \lim_{y \rightarrow 0} \frac{E_{s,p} [\Phi_p(u(x) - u(\cdot))](x, y)}{y^{sp}}, \quad (1.9)$$

where $C_3 > 0$. The precise study of this nonlinear representation occupies Section 3, see Theorem 3.1.

Lastly, the fractional p -Laplacian is also equivalently given by the following nonlinear Balakrishnan's formula

$$(-\Delta)_p^s u(x) = C_4 \int_0^\infty \Delta(t - \Delta)^{-1} [\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1-\frac{sp}{2}}}, \quad (1.10)$$

where $C_4 > 0$. This will be studied in Section 4, see Theorem 4.1.

The choice of the constant C_1 is indifferent in most cases and can be fixed as 1, but a precise value $C_1(n, s, p)$ can be important in writing some exact formulas or in the limit cases $s \rightarrow 0^+$, $s \rightarrow 1^-$, or $p \rightarrow 2$. In Section 5 we will discuss the choice of this constant and also give an explicit relation with C_2 , C_3 , and C_4 , which turn out to be dimensionally free after a suitable choice of C_1 .

Representation in bounded domains. As a first consequence of the above results, we remark that the first representation of the fractional p -Laplacian naturally suggests a version of the operator defined in bounded domains Ω of \mathbb{R}^n . Given $u : \Omega \rightarrow \mathbb{R}$ we define a new operator

$$(-\Delta_\Omega)_p^s u(x) := C_2 \int_0^{+\infty} e^{t\Delta_\Omega} [\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1+\frac{sp}{2}}}, \quad (1.11)$$

where $e^{t\Delta_\Omega}$ denotes the heat semigroup in Ω with Dirichlet boundary conditions, i.e. $w(x, t) := e^{t\Delta_\Omega}[f](x)$ is the solution of

$$\begin{cases} w_t(x, t) - \Delta w(x, t) = 0 & x \in \Omega, t > 0, \\ w(x, t) = 0 & x \in \partial\Omega, t > 0, \\ w(x, 0) = f(x) & x \in \Omega. \end{cases}$$

This operator is defined for any $p > 1$ and generalises the Spectral Fractional Laplacian, which is the well-known case $p = 2$. In [Section 6](#) we will show this operator is well-defined and different from the restriction of $(-\Delta)_p^s$. Furthermore, we suggest a generalisation for manifolds that we do not develop in full detail. Analogous variants can be constructed via [\(1.9\)](#) and [\(1.10\)](#).

Further extensions and comments. [Section 7](#) addresses the topic of numerical analysis for the fractional p -Laplacian starting from our representation formulas. [Section 8](#) is devoted to comment on some extensions and applications of our results. In the [Section 9](#), we present some comments and open problems that we find interesting as well as a brief intuition about our results. Finally, we include an appendix with technical results.

Comments on related literature. Some references for stationary boundary value problems involving the fractional p -Laplace operator that we study here: Ishii and Nakamura [\[30\]](#) treat the first boundary value problem; Lindgren and Lindqvist [\[36, 37\]](#) establish the fractional eigenvalues, Perron’s Method and Wiener’s theorem. Regularity was studied in [\[14, 15, 16, 29, 32, 35\]](#) among others. The equivalence of weak and viscosity solutions in boundary value problems involving the fractional p -Laplacian has been recently treated in [\[31\]](#) in the homogeneous case and [\[5\]](#) in the non-homogeneous case. The parabolic problem has been studied in bounded domains: Mazón, Rossi, and Toledo [\[42\]](#), Puhst [\[47\]](#), and Vázquez [\[54\]](#); and also in the whole space \mathbb{R}^n by M. Strömquist [\[50\]](#) and Vázquez [\[52\]](#).

We will make specific comments on the semigroup formula, the extension problem and the Balakrishnan’s formula in the corresponding sections. Other applications, in particular numerical studies are mentioned in [Section 7](#) and [Section 8](#).

The literature contains works on other types of operators that are also known as “nonlocal or fractional p -Laplacian operators”. We are not treating such versions in our paper since they lead to quite different tools and results. For instance, in the book by Andreu-Vailló, Mazón, Rossi, Toledo, [\[1\]](#), the integro-differential operator is defined on the basis of integrable kernels. On the other hand, Bjorland-Caffarelli-Figalli [\[8, 7\]](#) introduce a kind of normalized (game theoretical) fractional infinity Laplacian as well as a fractional p -Laplace operator in non-divergence form (see [\[39, 40\]](#) for the local version). They are the nonlocal counterpart of the normalized standard p -Laplacian in the viscosity sense. Finally, Chasseigne and Jakobsen [\[20\]](#) introduced a related non-normalized nonlocal p -Laplacian in the viscosity sense.

As a final comment, we recall that the standard p -Laplace operator ($s = 1$) has been widely studied both in the stationary and evolution equations. Let us mention some classical monographs by DiBenedetto [\[26\]](#), Lindqvist [\[38\]](#), that contain further references. The monograph [\[53, Section 11\]](#) contains a summary of the evolution results.

2 Semigroup representation

We present an interesting representation of the operator using the heat semigroup. Let us use the notation $e^{t\Delta}$ for this semigroup, i. e., given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we denote $w(x, t) = e^{t\Delta}[f](x)$ to the solution of

$$\begin{cases} \partial_t w(x, t) - \Delta w(x, t) = 0, & x \in \mathbb{R}^n, t > 0, \\ w(x, 0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

The following is our first main result.

Theorem 2.1. *Let $s \in (0, 1)$ and $p \in (1, +\infty)$. Assume $u \in C_b^2(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. If $p \in (1, \frac{2}{2-s})$ assume additionally that $\nabla u(x_0) \neq 0$. Then we have the following representation of the fractional p -Laplacian*

$$(-\Delta)_p^s u(x_0) = C_2(n, s, p) \int_0^{+\infty} e^{t\Delta} [\Phi_p(u(x_0) - u(\cdot))](x_0) \frac{dt}{t^{1+\frac{sp}{2}}}, \quad (2.1)$$

where $C_2(n, s, p) = \frac{\pi^{\frac{n}{2}} C_1(n, s, p)}{2^{sp} \Gamma(\frac{n+sp}{2})}$.

Note that, when $p = 2$, we have that

$$e^{t\Delta}[\Phi_p(u(x) - u(\cdot))](x) = e^{t\Delta}[u(x) - u(\cdot)](x) = u(x) - e^{t\Delta}[u](x).$$

so that, a suitable choice of $C_1(n, s, p)$ allows to recover Bochner's formula (1.3) from (2.1). Notice that $\Gamma(-s) < 0$ while $C_2 > 0$.

Our proof is based on the scaling properties of the representation formula

$$e^{t\Delta}[f](x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) dy.$$

We use here it in combination with the nonlinearity, plus a desingularization step that is done by subtraction inside the argument of the nonlinearity. In this way we obtain the *splitting representation* (2.1) of the fractional p -Laplacian.

Proof of Theorem 2.1. Let $\varepsilon > 0$ and consider the bounded function $g_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \rightarrow 0$ given by $g_\varepsilon(x, y) = \Phi_p(u(x) - u(y))\chi_{|x-y|>\varepsilon}$. Fix $x \in \mathbb{R}^n$ and consider the heat semigroup associated to $g_\varepsilon(x, \cdot)$, i.e.

$$e^{t\Delta}[g_\varepsilon(x, \cdot)](z) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|z-y|^2}{4t}} g_\varepsilon(x, y) dy.$$

Taking $z = x$ we have

$$e^{t\Delta}[g_\varepsilon(x, \cdot)](x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g_\varepsilon(x, y) dy.$$

Let now $\delta > 0$ and integrate the quantity above with respect the singular measure $dt/t^{1+\frac{sp}{2}}$ in the truncated domain $(\delta, +\infty)$ (we do this to avoid with integrability issues around the origin due to the singular behaviour of the measure) to obtain

$$\begin{aligned} \int_\delta^\infty e^{t\Delta}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1+\frac{sp}{2}}} &= \int_\delta^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g_\varepsilon(x, y) dy \right) \frac{dt}{t^{1+\frac{sp}{2}}} \\ &= \int_{\mathbb{R}^n} \left(\int_\delta^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \frac{dt}{t^{1+\frac{sp}{2}}} \right) g_\varepsilon(x, y) dy \end{aligned}$$

where we have used Fubini's theorem to change the order of integration. We use now the change of variables $t = |x - y|^2/(4\tau)$ to get

$$\begin{aligned} \int_\delta^\infty e^{t\Delta}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1+\frac{sp}{2}}} &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(\int_{\frac{|x-y|^2}{4}\delta}^\infty \frac{\tau^{\frac{n}{2}}}{|x-y|^n} e^{-\tau} \frac{(4\tau)^{1+\frac{sp}{2}}}{|x-y|^{2+sp}} \frac{|x-y|^2}{4\tau^2} d\tau \right) g_\varepsilon(x, y) dy \\ &= \frac{2^{sp}}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(\int_{\frac{|x-y|^2}{4}\delta}^\infty e^{-\tau} \frac{d\tau}{\tau^{1-\frac{n+sp}{2}}} \right) \frac{g_\varepsilon(x, y)}{|x-y|^{n+sp}} dy. \end{aligned}$$

Note that, since $g_\varepsilon(x, y) = 0$ if $|x - y| < \varepsilon$, we can consider the above expression to be integrated in $\mathbb{R}^n \setminus B_\varepsilon(x)$. Now, we want to let $\delta \rightarrow 0^+$ to get

$$\begin{aligned} \int_0^\infty e^{t\Delta}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1+\frac{sp}{2}}} &= \frac{2^{sp}}{\pi^{\frac{n}{2}}} \left(\int_0^\infty e^{-\tau} \tau^{-1+\frac{n+sp}{2}} d\tau \right) \int_{\mathbb{R}^n} \frac{g_\varepsilon(x, y)}{|x-y|^{n+sp}} dy \\ &= \frac{2^{sp} \Gamma(\frac{n+sp}{2})}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{g_\varepsilon(x, y)}{|x-y|^{n+sp}} dy \end{aligned}$$

Step 1: Let either $p \in [2, \infty)$ or $p \in (1, 2)$ and $\nabla u(x) \neq 0$.

Passing to the limit in the right hand side is trivial since the integrand converges pointwise and monotonically to $e^{-\tau} \tau^{-1 + \frac{n+sp}{2}}$ which is an integrable function in the domain $(0, +\infty)$, which yields to the definition of the Γ function. To pass to the limit in the left hand side, we estimate

$$\begin{aligned} |e^{t\Delta}[g_\varepsilon(x, \cdot)](x)| &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left| \int_{|x-y|>\varepsilon} e^{-\frac{|x-y|^2}{4t}} \Phi_p(u(x) - u(y)) dy \right| \lesssim \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4t}} |z|^p dy \\ &= \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{r^2}{4}} t^{\frac{p}{2}} r^p t^{\frac{n}{2}} dr = t^{\frac{p}{2}} \int_{\mathbb{R}^n} e^{-\frac{r^2}{4}} r^p dr \lesssim t^{\frac{p}{2}}. \end{aligned}$$

where we have used either [Lemma A.1](#) or [Lemma A.2](#) with $K(z) = e^{-\frac{|z|^2}{4t}} \chi_{|z|>\varepsilon}(z)$ and the change of variables $z = t^{\frac{1}{2}}r$ i.e. $dz = t^{\frac{n}{2}} dr$. Then

$$\int_\delta^1 |e^{t\Delta}[g_\varepsilon(x, \cdot)](x)| \frac{dt}{t^{1+\frac{sp}{2}}} \lesssim \int_\delta^1 t^{\frac{p}{2}} \frac{dt}{t^{1+\frac{sp}{2}}} = \int_\delta^1 \frac{dt}{t^{1-\frac{(1-s)p}{2}}} \leq \int_0^1 \frac{dt}{t^{1-\frac{(1-s)p}{2}}} < +\infty$$

where the bound comes from the fact that $(1-s)p/2 > 0$ since $s \in (0, 1)$.

Now we want to take limits as $\varepsilon \rightarrow 0$. In the right hand side, we use either [Lemma A.1](#) or [Lemma A.2](#) with $K(z) = |z|^{-n-sp} \chi_{|z|<\varepsilon}(z)$ to get

$$\begin{aligned} \left| \text{P.V.} \int_{\mathbb{R}^n} \frac{\Phi_p(u(x) - u(y))}{|x-y|^{n+sp}} dy - \text{P.V.} \int_{\mathbb{R}^n} \frac{g_\varepsilon(x, y)}{|x-y|^{n+sp}} dy \right| &\leq \left| \text{P.V.} \int_{|x-y|<\varepsilon} \frac{\Phi_p(u(x) - u(y))}{|x-y|^{n+sp}} dy \right| \\ &\lesssim \int_{|z|<\varepsilon} \frac{|z|^p}{|z|^{n+sp}} dz \\ &\lesssim \varepsilon^{(1-s)p} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \end{aligned}$$

While in the left hand side we use either [Lemma A.1](#) or [Lemma A.2](#) with $K(z) = e^{-\frac{|z|^2}{4t}} \chi_{|z|<\varepsilon}(z)$ to estimate for $t \in (0, 1)$,

$$\begin{aligned} \left| (e^{t\Delta}[\Phi_p(u(x) - u(\cdot))](x) - e^{t\Delta}[g_\varepsilon(x, \cdot)](x)) \right| &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left| \int_{|x-y|<\varepsilon} e^{-\frac{|x-y|^2}{4t}} \Phi_p(u(x) - u(y)) dy \right| \\ &\lesssim \frac{1}{t^{\frac{n}{2}}} \int_{|z|<\varepsilon} e^{-\frac{|z|^2}{4t}} |z|^p dy \\ &\lesssim t^{\frac{p}{2}} \int_{|r|<\varepsilon t^{-\frac{1}{2}}} e^{-\frac{r^2}{4}} r^p dr \end{aligned}$$

and for $t \geq 1$,

$$\begin{aligned} |e^{t\Delta}[\Phi_p(u(x) - u(\cdot))](x) - e^{t\Delta}[g_\varepsilon(x, \cdot)](x)| &\leq \sup_{|x-y|<\varepsilon} |\Phi_p(u(x) - u(y))| \\ &\lesssim \sup_{|x-y|<\varepsilon} |u(x) - u(y)|^{\min\{1, p-1\}} \leq \varepsilon^{\min\{1, p-1\}} \end{aligned}$$

Then

$$\begin{aligned} \left| \int_0^{+\infty} e^{t\Delta}[\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1+\frac{sp}{2}}} - \int_0^{+\infty} e^{t\Delta}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1+\frac{sp}{2}}} \right| & \\ \leq \int_0^1 \left(\int_{|r|<\varepsilon t^{-\frac{1}{2}}} e^{-\frac{r^2}{4}} r^p dr \right) \frac{dt}{t^{1-\frac{(1-s)p}{2}}} + \varepsilon^{\min\{1, p-1\}} \int_1^{+\infty} \frac{dt}{t^{1+\frac{sp}{2}}}. & \quad (2.2) \end{aligned}$$

Clearly, the second term in the last estimate goes to zero as $\varepsilon \rightarrow 0$. For the first term, let

$$f_\varepsilon(t) := \int_{|r| < \varepsilon t^{-\frac{1}{2}}} e^{-\frac{r^2}{4}} r^p dr.$$

For the first term, clearly $f_\varepsilon \rightarrow 0$ a.e in $t \in [0, \infty)$,

$$|f_\varepsilon(t)| \leq F(t) \equiv \int_{\mathbb{R}^n} e^{-\frac{r^2}{4}} r^p dr$$

and

$$\int_0^1 F(t) \frac{dt}{t^{1-\frac{(1-s)p}{2}}} = \int_{\mathbb{R}^n} e^{-\frac{r^2}{4}} r^p dr \int_0^1 \frac{dt}{t^{1-\frac{(1-s)p}{2}}} < +\infty.$$

so by the Dominated Convergence Theorem this term also converges to zero as $\varepsilon \rightarrow 0$.

Step 2: Now let $p \in (\frac{2}{2-s}, 2)$. We can use [Lemma A.3](#) to get

$$|e^{t\Delta}[g_\varepsilon(x, \cdot)](x)| \lesssim t^{p-1}.$$

so that

$$\int_\delta^1 |e^{t\Delta}[g_\varepsilon(x, \cdot)](x)| \frac{dt}{t^{1+\frac{sp}{2}}} \lesssim \int_0^1 t^{p-1} \frac{dt}{t^{1+\frac{sp}{2}}} < +\infty$$

which is finite since $p-1 > \frac{sp}{2}$ since $p > \frac{2}{2-s}$. That allow us to pass to the limit as $\delta \rightarrow 0$. The limit as $\varepsilon \rightarrow 0$ follows also as in Step 1 one using [Lemma A.3](#) instead of [Lemma A.1](#). \square

3 The new extension problem

The aim of this section is to write corresponding formulas in the present nonlinear setting following the ideas introduced in the linear setting by Caffarelli and Silvestre [\[19\]](#).

For $s \in (0, 1)$ and $p \in (1, +\infty)$ consider the extension problem

$$\begin{cases} \Delta_x U(x, y) + \frac{1-sp}{y} U_y(x, y) + U_{yy}(x, y) = 0 & x \in \mathbb{R}^n, y > 0, \\ U(x, 0) = f(x) & x \in \mathbb{R}^n. \end{cases} \quad (3.1)$$

which extends the usual theory where $p = 2$. We will show that a classical solution of this problem can be recovered by convolution with an explicit Poisson kernel

$$U(x, y) = \int_{\mathbb{R}^n} P(x - \xi, y) f(\xi) d\xi, \quad \text{where} \quad P(x, y) = \frac{\Gamma(\frac{n+sp}{2})}{\pi^{\frac{n}{2}} \Gamma(\frac{sp}{2})} \frac{y^{sp}}{(|x|^2 + y^2)^{\frac{n+sp}{2}}}. \quad (3.2)$$

Finally, let us define the extension operator $E_{s,p}[f] := U$ through formula [\(3.2\)](#). With this extension operator we can give a representation formula for the (s, p) -Laplacian

Theorem 3.1. *Let $s \in (0, 1)$ and $p \in (1, +\infty)$. Assume $u \in C_b^2(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. If $p \in (1, \frac{2}{2-s})$ assume additionally that $\nabla u(x_0) \neq 0$. We have the following representation of the fractional p -Laplacian:*

$$(-\Delta)_p^s u(x_0) = C_3(n, s, p) \lim_{y \rightarrow 0} \frac{E_{s,p} \left[\Phi_p(u(x_0) - u(\cdot)) \right](x_0, y)}{y^{sp}} \quad (3.3)$$

where the constant is given by $C_3(n, s, p) = \frac{\pi^{\frac{n}{2}} \Gamma(\frac{sp}{2})}{\Gamma(\frac{n+sp}{2})} C_1(n, s, p)$.

Actually, for $p = 2$ we have that

$$\frac{E_{s,p}[\Phi_p(u(x) - u(\cdot))](x, y)}{y^{sp}} = \frac{E_{s,2}[u(x) - u(\cdot)](x, y)}{y^{2s}} = \frac{E_{s,2}[u](x, 0) - E_{s,2}[u](x, y)}{y^{2s}}$$

recovering the classical result of [19] for the fractional Laplacian.

Remark 3.2. Notice that, by L'Hôpital's rule

$$(-\Delta)_p^s u(x_0) = \frac{C_3(n, s, p)}{sp} \lim_{y \rightarrow 0} y^{1-sp} \partial_y (E_{s,p}[\Phi_p(u(x_0) - u(\cdot))](x_0, y)).$$

In order to prove **Theorem 3.1** we need an important lemma that gives an alternative representation of the extension operator based on the one by Stinga and Torrea ([49]) for fractional powers of general linear operators.

Lemma 3.3. *Let $s \in (0, 1)$ and $p \in (1, +\infty)$. If $f \in C_b(\mathbb{R}^n)$ then $E_{s,p}[f]$ is a classical solution of (3.1). Furthermore, it can be alternatively represented by*

$$E_{s,p}[f](x, y) = \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \int_0^\infty e^{t\Delta}[f](x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\frac{sp}{2}}}. \quad (3.4)$$

Proof. We proceed in several steps.

Step 1. We will check here that the pointwise formulas (3.4) and (3.2) are well defined for $y > 0$.

First, for (3.4), we use the explicit representation of $e^{t\Delta}$, and the fact that f is bounded

$$\begin{aligned} \left| \int_0^\infty e^{t\Delta}[f](x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\frac{sp}{2}}} \right| &= \left| \int_0^\infty e^{-\frac{y^2}{4t}} \left(\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-\xi|^2}{4t}} f(\xi) d\xi \right) \frac{dt}{t^{1+\frac{sp}{2}}} \right| \\ &\lesssim \int_0^\infty e^{-\frac{y^2}{4t}} \left(\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|\xi|^2}{4t}} d\xi \right) \frac{dt}{t^{1+\frac{sp}{2}}} \leq \int_0^\infty e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\frac{sp}{2}}}. \end{aligned}$$

The last term is clearly finite since $sp/2 > 0$ and $e^{-\frac{y^2}{4t}}/t^\alpha \rightarrow 0$ as $t \rightarrow 0$ for any $\alpha \in \mathbb{R}$ as long as $y > 0$.

Second, for (3.2),

$$\left| \int_{\mathbb{R}^n} \frac{f(\xi)}{(|x-\xi|^2 + y^2)^{\frac{n+sp}{2}}} d\xi \right| \lesssim \int_{\mathbb{R}^n} \frac{d\xi}{(|\xi|^2 + y^2)^{\frac{n+sp}{2}}}$$

which again is finite since $K(\xi) := (|\xi|^2 + y^2)^{-\frac{n+2s}{2}}$ is bounded as long as $y > 0$ and $K(\xi) \leq |\xi|^{-(n+2s)}$ which is an integrable function in $\mathbb{R}^n \setminus B_1$.

Step 2. Now we check that both pointwise formulas are equivalent. Notice that, by Step 1, all the integrals below are well defined, and they allow us to do Fubini to change the order of integration. Then

$$\begin{aligned} \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \int_0^\infty e^{t\Delta}[f](x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\frac{sp}{2}}} &= \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-\xi|^2 + y^2}{4t}} f(\xi) d\xi \frac{dt}{t^{1+\frac{sp}{2}}} \\ &= \int_{\mathbb{R}^n} \left(\frac{y^{sp}}{2^{n+sp}\pi^{\frac{n}{2}}\Gamma(\frac{sp}{2})} \int_0^\infty e^{-\frac{|x-\xi|^2 + y^2}{4t}} \frac{dt}{t^{1+\frac{n+sp}{2}}} \right) f(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \left(\frac{y^{sp}}{\pi^{\frac{n}{2}}\Gamma(\frac{sp}{2})(|x-\xi|^2 + y^2)^{\frac{n+sp}{2}}} \int_0^\infty e^{-\tau} \tau^{-1+\frac{n+sp}{2}} d\tau \right) f(\xi) d\xi \\ &= \int_{\mathbb{R}^n} P(x-\xi, y) f(\xi) d\xi. \end{aligned}$$

Step 3. Now we check that formulas (3.4) and (3.2) satisfy (3.1). We start with the boundary condition. To do so, it is sufficient to show that

$$P(y, z) \rightarrow \delta_0(y) \quad \text{as } z \rightarrow 0^+.$$

For $y \neq 0$ it is easy to check from (3.2) that we have

$$P(y, z) \rightarrow 0 \quad \text{as } z \rightarrow 0^+.$$

It is also standard to get

$$\int_{\mathbb{R}^n} P(x, y) dx = \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \int_0^\infty e^{\frac{y^2}{4t}} \left(\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} dx \right) \frac{dt}{t^{1+\frac{sp}{2}}} = \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \int_0^\infty e^{\frac{y^2}{4t}} \frac{dt}{t^{1+\frac{sp}{2}}} = 1.$$

Finally, it is easy to compute (see for example [19]) that for all $y > 0$ we have

$$\Delta_x P + \frac{1-sp}{y} \frac{\partial P}{\partial y} + \frac{\partial^2 P}{\partial y^2} = 0.$$

We have that U is a pointwise solution of the extension equation. □

Proof of Theorem 3.1. We need to check (3.3). For $y > 0$ we have, from Lemma 3.3, the following identity

$$\begin{aligned} C_3 \frac{E_{s,p} \left[\Phi_p(u(x_0) - u(\cdot)) \right] (x_0, y)}{y^{sp}} &= C_3 \int_{\mathbb{R}^n} \frac{P(x_0 - \xi, y)}{y^{sp}} \Phi_p(u(x_0) - u(\xi)) d\xi \\ &= C_1 \int_{\mathbb{R}^n} \frac{\Phi_p(u(x_0) - u(\xi))}{(|x_0 - \xi|^2 + y^2)^{\frac{n+sp}{2}}} d\xi. \end{aligned}$$

Then,

$$\begin{aligned} &\left| (-\Delta)_p^s u(x) - C_3 \frac{E_{s,p} \left[\Phi_p(u(x_0) - u(\cdot)) \right] (x_0, y)}{y^{sp}} \right| \\ &= C_1 \left| \int_{\mathbb{R}^n} \Phi_p(u(x_0) - u(\xi)) \left(\frac{1}{|x_0 - \xi|^{n+sp}} - \frac{1}{(|x_0 - \xi|^2 + y^2)^{\frac{n+sp}{2}}} \right) d\xi \right|. \end{aligned}$$

Let us denote

$$K_y(\eta) = \left(\frac{1}{|\eta|^{n+sp}} - \frac{1}{(|\eta|^2 + y^2)^{\frac{n+sp}{2}}} \right)$$

and estimate

$$\begin{aligned} &\left| (-\Delta)_p^s u(x) - C_3 \frac{E_{s,p} \left[\Phi_p(u(x_0) - u(\cdot)) \right] (x_0, y)}{y^{sp}} \right| \\ &\lesssim \left| \int_{|x_0 - \xi| \leq 1} \Phi_p(u(x_0) - u(\xi)) K_y(x_0 - \xi) d\xi \right| + \left| \int_{|x_0 - \xi| > 1} K_y(x_0 - \xi) d\xi \right|. \end{aligned}$$

Note that the last term goes to zero by the Monotone Convergence Theorem since $K_y(x - \xi) \rightarrow 0$ as $y \rightarrow 0$ in a monotone way. Now, from either Lemma A.1, Lemma A.2 and Lemma A.3 we get

$$\left| \int_{|\eta| \leq 1} \Phi_p(u(x_0) - u(\xi)) K_y(x_0 - \xi) d\xi \right| \leq \int_{|\eta| \leq 1} K_y(\xi) |\xi|^\alpha d\xi,$$

for α either p or $2p - 2$ depending on the range of p . This term again goes to zero as $y \rightarrow 0$ by the Monotone Convergence Theorem. □

4 Nonlinear Balakrishnan's formula

The aim of this section is to prove the corresponding Balakrishnan-type formula for $p \neq 2$.

Theorem 4.1. *Let $s \in (0, 1)$ and $p \in (1, +\infty)$. Assume $u \in C_b^2(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. If $p \in (1, \frac{2}{2-s})$ assume additionally that $\nabla u(x_0) \neq 0$. We have the following representation of the fractional p -Laplacian:*

$$(-\Delta)_p^s u(x_0) = C_4(n, s, p) \int_0^\infty \Delta(t - \Delta)^{-1} [\Phi_p(u(x_0) - u(\cdot))](x_0) \frac{dt}{t^{1-\frac{sp}{2}}}, \quad (4.1)$$

where the constant is given by $C_4(n, s, p) = \frac{\pi^{\frac{n}{2}}}{2^{sp} \Gamma(\frac{2+sp}{2}) \Gamma(\frac{n+sp}{2})} C_1(n, s, p)$.

Notice that, when $p = 2$ we have that

$$\Delta(t - \Delta)^{-1} [\Phi_p(u(x_0) - u(\cdot))] = \Delta(t - \Delta)^{-1} [u(x_0)] - \Delta(t - \Delta)^{-1} [u(\cdot)] = -\Delta(t - \Delta)^{-1} [u].$$

Hence, we recover the usual Balakrishnan's formula.

To show (4.1) in [Theorem 4.1](#) we need some preliminary results concerning the representation formulas of $(t - \Delta)^{-1}$ and $\Delta(t - \Delta)^{-1}$.

Lemma 4.2. *The following representation formulas hold for all $t \geq 0$:*

(a) $(t - \Delta)^{-1} f = K_t * f$ with $K_t(x) = \int_0^\infty e^{-t\tau} G(\tau, x) d\tau$ where G is the Gaussian Kernel, i.e.,

$$G(\tau, x) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\tau}}$$

(b) $\Delta(t - \Delta)^{-1} f = R_t * f - f$ where $R_t(x) = tK_t(x)$. Moreover, $R_t(x) = t^{\frac{n}{2}} W(t^{\frac{1}{2}}|x|)$ with $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$W(\rho) = \frac{\rho^{2-n}}{(4\pi)^{\frac{n}{2}}} \int_0^\infty e^{-\rho^2 w} w^{-\frac{n}{2}} e^{-\frac{1}{4w}} dw \quad (4.2)$$

Proof. First, taking the Fourier transform

$$\begin{aligned} \mathcal{F}[(t - \Delta)^{-1} f](\xi) &= (t + |\xi|^2)^{-1} \mathcal{F}[f](\xi) = \left(\int_0^\infty e^{-t\tau} e^{-|\xi|^2 \tau} d\tau \right) \mathcal{F}[f](\xi) \\ &= \mathcal{F} \left[\left(\int_0^\infty e^{-t\tau} \mathcal{F}^{-1}[e^{-|\cdot|^2 \tau}] d\tau \right) \star f \right](\xi). \end{aligned}$$

Since $\mathcal{F}[G(t, \cdot)](\xi) = e^{-|\xi|^2 t}$ we recover (a). As shown in [\[41, Section 3.2\]](#) we have

$$tK_t - \Delta K_t = \delta_0.$$

The intuition for (b) is clear and it comes from the fact that $\partial_t G - \Delta G = \delta_0$. We can formally write

$$\Delta K_t = \int_0^\infty e^{-t\tau} \Delta G d\tau = \int_0^\infty e^{-t\tau} \partial_t G d\tau = e^{-t\tau} G \Big|_0^\infty + t \int_0^\infty e^{-t\tau} G d\tau = 0 - \delta_0 + tK_t$$

Hence,

$$\Delta(t - \Delta)^{-1} f = \Delta(K_t \star f) = (\Delta K_t) \star f = (R_t - \delta_0) \star f = R_t \star f - f.$$

We check (4.2) now. First, we note that $G(\tau, x) = \tau^{-\frac{n}{2}} F(\tau^{-\frac{1}{2}}|x|)$ with $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$F(\rho) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{\rho^2}{4}}.$$

Then,

$$\begin{aligned} K_t(x) &= t \int_0^\infty e^{-t\tau} \tau^{-\frac{n}{2}} F(\tau^{-\frac{1}{2}}|x|) d\tau = t|x|^{2-n} \int_0^\infty e^{-t|x|^2 w} w^{-\frac{n}{2}} F(w^{-\frac{1}{2}}) dw \\ &= t^{\frac{n}{2}} (t^{\frac{1}{2}}|x|)^{2-n} \int_0^\infty e^{-(t^{\frac{1}{2}}|x|)^2 w} w^{-\frac{n}{2}} F(w^{-\frac{1}{2}}) dw. \quad \square \end{aligned}$$

We are now ready to prove [Theorem 4.1](#).

Proof of Theorem 4.1. We proceed in several steps.

Step 1: We will check here that given $f \in L^\infty(\mathbb{R}^n)$, $\alpha < 0$ and a finite $M > 0$ we have that

$$\int_0^M |\Delta(t - \Delta)^{-1} f(x)| \frac{dt}{t^{1-\alpha}} < +\infty.$$

From [Lemma 4.2](#) we have

$$|\Delta(t - \Delta)^{-1} f(x)| = |R_t * f(x)| + |f(x)| \leq (\|R_t\|_{L^1(\mathbb{R}^n)} + 1) \|f\|_{L^\infty(\mathbb{R}^n)} = 2\|f\|_{L^\infty(\mathbb{R}^n)}$$

where we have used the fact that $\|G(\tau, |\cdot|)\|_{L^1(\mathbb{R}^n)} = 1$, which implies

$$\|R_t\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(t \int_0^\infty e^{-t\tau} G(\tau, |x|) d\tau \right) dx = t \int_0^\infty e^{-t\tau} d\tau = 1.$$

From this observation, we trivially get

$$\int_0^M |\Delta(t - \Delta)^{-1} f(x)| \frac{dt}{t^{1-\alpha}} < 2 \frac{M^\alpha}{\alpha} \|f\|_{L^\infty(\mathbb{R}^n)} < +\infty.$$

Step 2: Now, given $\varepsilon > 0$, we consider the bounded function $g_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ given by $g_\varepsilon(x, y) = \Phi_p(u(x) - u(y)) \chi_{|x-y|>\varepsilon}$. Note that $g_\varepsilon(x, x) = 0$. Here we will check that the following identity holds.

$$\int_0^\infty \Delta(t - \Delta)^{-1} [g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1-\frac{sp}{2}}} = C \int_{|x-y|>\varepsilon} \frac{\Phi_p(u(x) - u(y))}{|x-y|^{n+sp}} dy. \quad (4.3)$$

where $C = \frac{2^{sp}}{\pi^{\frac{n}{2}}} \Gamma(1 + \frac{sp}{2}) \Gamma(\frac{n+sp}{2}) > 0$.

Step 2 (a): First we check that

$$\int_0^M |\Delta(t - \Delta)^{-1} [g_\varepsilon(x, \cdot)](x)| \frac{dt}{t^{1-\frac{sp}{2}}} < +\infty \quad (4.4)$$

uniformly in M . In this way, the Dominated convergence theorem we get

$$\int_0^M \Delta(t - \Delta)^{-1} [g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1-\frac{sp}{2}}} \xrightarrow{M \rightarrow \infty} \int_0^\infty \Delta(t - \Delta)^{-1} [g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1-\frac{sp}{2}}} < +\infty. \quad (4.5)$$

To check (4.4), we assume first that $p \geq 2$, and use the explicit representation of $\Delta(t - \Delta)^{-1}$ given in [Lemma 4.2](#) together with [Lemma A.1](#) to get

$$\begin{aligned} & \int_0^M |\Delta(t - \Delta)^{-1} [g_\varepsilon(x, \cdot)](x)| \frac{dt}{t^{1-\frac{sp}{2}}} \\ &= \int_0^1 |\Delta(t - \Delta)^{-1} [g_\varepsilon(x, \cdot)](x)| \frac{dt}{t^{1-\frac{sp}{2}}} + \int_1^M \left| \int_{|x-y|>\varepsilon} R_t(x-y) \Phi_p(u(x) - u(y)) \right| \frac{dt}{t^{1-\frac{sp}{2}}} \\ &= \frac{1}{\alpha} \|g_\varepsilon(x, \cdot)\|_{L^\infty(\mathbb{R}^n)} + C \int_1^\infty \left(\int_{|z|>\varepsilon} R_t(z) |z|^p dz \right) \frac{dt}{t^{1-\frac{sp}{2}}}. \end{aligned}$$

We use now the explicit form of R_t to get

$$\begin{aligned} \int_1^\infty \left(\int_{|z|>\varepsilon} R_t(z) |z|^p dz \right) \frac{dt}{t^{1-\frac{sp}{2}}} &= \int_1^\infty \left(\int_{|z|>\varepsilon} t^{\frac{n}{2}} W(t^{\frac{1}{2}}|z|) |z|^p dz \right) \frac{dt}{t^{1-\frac{sp}{2}}} \\ &= \int_1^\infty \left(\int_{|y|>\varepsilon t^{\frac{1}{2}}} W(|y|) |y|^p t^{-\frac{p}{2}} dy \right) \frac{dt}{t^{1-\frac{sp}{2}}} \\ &\leq \left(\int_{|y|>\varepsilon} W(|y|) |y|^p dy \right) \left(\int_1^\infty \frac{dt}{t^{1+\frac{(1-s)p}{2}}} \right). \end{aligned}$$

It is rather standard to check that the last two integrals are finite. Thus, (4.5) is proved for $p \geq 2$. When $p \in (1, 2)$ we use [Lemma A.2](#) and [Lemma A.3](#) instead and the proof follows in the same way.

Step 2 (b): We proceed now by direct computations:

$$\begin{aligned} \int_0^M \Delta(t - \Delta)^{-1} [g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1-\frac{sp}{2}}} &= \int_0^M \int_{|x-y|>\varepsilon} R_t(x-y) \Phi_p(u(x) - u(y)) dy \frac{dt}{t^{1-\frac{sp}{2}}} \\ &= \int_{|x-y|>\varepsilon} \Phi_p(u(x) - u(y)) \left(\int_0^M R_t(x-y) \frac{dt}{t^{1-\frac{sp}{2}}} \right) dy \quad (4.6) \\ &= \int_{|x-y|>\varepsilon} \frac{\Phi_p(u(x) - u(y))}{|x-y|^{n+sp}} \left(\int_0^{M|x-y|^2} W(\tau^{\frac{1}{2}}) \frac{d\tau}{(\tau^{\frac{1}{2}})^{2-n-sp}} \right) dy. \end{aligned}$$

For any $y \neq x$

$$\int_0^{M|x-y|^2} W(\tau^{\frac{1}{2}}) \frac{d\tau}{(\tau^{\frac{1}{2}})^{2-n-sp}} \xrightarrow{M \rightarrow \infty} \int_0^\infty W(\tau^{\frac{1}{2}}) \frac{d\tau}{(\tau^{\frac{1}{2}})^{2-n-sp}} =: C.$$

We just need to check that C is finite which will ensure we can pass to the limit in the last term of (4.6) (we can also do it in the first by Step 2(a)) to get (4.3) by the Dominated Convergence Theorem since

$$\int_{|x-y|>\varepsilon} \frac{|\Phi_p(u(x) - u(y))|}{|x-y|^{n+sp}} \left(\int_0^{M|x-y|^2} W(\tau^{\frac{1}{2}}) \frac{d\tau}{(\tau^{\frac{1}{2}})^{2-n-sp}} \right) dy \leq C \int_{|x-y|>\varepsilon} \frac{|\Phi_p(u(x) - u(y))|}{|x-y|^{n+sp}} dy < +\infty.$$

Let us now compute C explicitly:

$$\begin{aligned} C &= \int_0^\infty \left(\frac{\tau^{1-\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}} \int_0^\infty e^{-\tau w} w^{-\frac{n}{2}} e^{-\frac{1}{4w}} dw \right) \frac{d\tau}{(\tau^{\frac{1}{2}})^{2-n-sp}} \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty w^{-\frac{n}{2}} e^{-\frac{1}{4w}} \left(\int_0^\infty e^{-\tau w} \tau^{\frac{sp}{2}} d\tau \right) dw \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty w^{-1-\frac{n}{2}-\frac{sp}{2}} e^{-\frac{1}{4w}} \left(\int_0^\infty e^{-\eta w} \eta^{\frac{sp}{2}} d\eta \right) dw \\ &= \Gamma\left(1 + \frac{sp}{2}\right) \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty w^{-1-\frac{n}{2}-\frac{sp}{2}} e^{-\frac{1}{4w}} dw \\ &= \Gamma\left(1 + \frac{sp}{2}\right) \frac{4^{\frac{sp}{2}}}{\pi^{\frac{n}{2}}} \int_0^\infty \rho^{-1+\frac{n}{2}+\frac{sp}{2}} e^{-\rho} d\rho \\ &= \frac{2^{sp}}{\pi^{\frac{n}{2}}} \Gamma\left(1 + \frac{sp}{2}\right) \Gamma\left(\frac{n+sp}{2}\right) \end{aligned}$$

Clearly $C < +\infty$, which finish the proof of Step 2.

Step 3. The only thing left is to pass to the limit as $\varepsilon \rightarrow 0$ in (4.3). The limit of the right hand side term was already shown in Step 1 of the proof of [Theorem 2.1](#). For the left hand side, when

$p \geq 2$, we have by [Lemma A.1](#) (we can proceed analogously with [Lemma A.2](#) and [Lemma A.3](#) when $p \in (1, 2)$) we compute directly

$$\begin{aligned} & \left| \int_0^\infty \Delta(t - \Delta)^{-1}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1-\frac{sp}{2}}} - \int_0^\infty \Delta(t - \Delta)^{-1}[\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1-\frac{sp}{2}}} \right| \\ & \leq \int_0^\infty \left| \int_{|x-y|<\varepsilon} \Phi_p(u(x) - u(y)) R_t(x-y) dy \right| \frac{dt}{t^{1-\frac{sp}{2}}} \\ & \leq 2\Phi_p(\|u\|_{L^\infty(\mathbb{R}^n)}) \int_0^1 \left(\int_{|z|<\varepsilon} R_t(z) dz \right) \frac{dt}{t^{1-\frac{sp}{2}}} + C \int_1^\infty \left(\int_{|z|<\varepsilon} R_t(z) |z|^p dz \right) \frac{dt}{t^{1-\frac{sp}{2}}}. \end{aligned}$$

Clearly the left integral in the last equation goes to 0 as $\varepsilon \rightarrow 0$ since $f_\varepsilon(t) := \int_{|z|<\varepsilon} R_t(z) dz \rightarrow 0$ pointwise as $\varepsilon \rightarrow 0$ due to the fact that $\|R_t\|_{L^1(\mathbb{R}^n)} = 1$. For the right integral, we have

$$\int_1^\infty \left(\int_{|z|<\varepsilon} R_t(z) |z|^p dz \right) \frac{dt}{t^{1-\frac{sp}{2}}} = \int_1^\infty \left(\int_{|y|<\varepsilon t^{\frac{1}{2}}} W(|y|) |y|^p dy \right) \frac{dt}{t^{1+\frac{(1-s)p}{2}}}$$

which again goes to 0 as $\varepsilon \rightarrow 0$ since $\int_{|y|<\varepsilon t^{\frac{1}{2}}} W(|y|) |y|^p dy \rightarrow 0$ a.e. in t as $\varepsilon \rightarrow 0$. This finishes the proof. \square

5 The choice of a suitable constant

The choice of the constant $C_1 > 0$ in [\(1.6\)](#) is indifferent in most cases and can be put to 1, but a precise value $C_1(n, s, p)$ can be important in writing some exact formulas or in the limit cases $s \rightarrow 0^+$, $s \rightarrow 1^-$, or $p \rightarrow 2$. Thus, in the linear case $p = 2$ the natural value is fixed by the assumption that the Fourier transform of the operator is a power multiplier, $\mathcal{F}[(-\Delta)^s u] = |\xi|^{2s} \mathcal{F}[u]$, cf. [\[48\]](#). Working out the details, the literature points out the value

$$c_{n,s} = \frac{2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}} |\Gamma(-s)|} = s(1-s) \frac{2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}} \Gamma(2-s)}. \quad (5.1)$$

However, this choice may not seem to be complete when $p \neq 2$. One might expect the choice of C_1 to depend on p in such a way that the expected limits are recovered:

$$(-\Delta)_p^s \phi \xrightarrow{s \rightarrow 1^-} -\Delta_p \phi(x) \quad \text{for all } p \in [2, \infty) \quad \text{and} \quad (-\Delta)_p^s \phi \xrightarrow{p \rightarrow 2^+} (-\Delta)^s \phi(x) \quad \text{for all } s \in (0, 1). \quad (5.2)$$

The choice [\(5.1\)](#) satisfies the second limit, but not the first one. We propose the following constant.

Lemma 5.1. *Consider the constant*

$$C_1(n, s, p) = \frac{\frac{sp}{2}(1-s)2^{2s-1}}{\pi^{\frac{n-1}{2}}} \frac{\Gamma\left(\frac{n+sp}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma(2-s)}$$

in [\(1.6\)](#). Then, for $\phi \in C_b^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ such that $\nabla \phi(x) \neq 0$ we have that [\(5.2\)](#) holds.

Proof. Consider the operator

$$\mathcal{L}_p^s[\phi](x) = \text{PV} \int_{\mathbb{R}^d} \frac{\Phi_p(u(x) - u(y))}{|x-y|^{n+sp}} dy$$

We know from [\[17, Theorem 2.8\]](#) that

$$(1-s)\mathcal{L}_p^s[\phi](x) \xrightarrow{s \rightarrow 1^-} -\frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{p+1}{2}\right)}{p \Gamma\left(\frac{n+p}{2}\right)} \Delta_p \phi(x).$$

Then,

$$\begin{aligned} C_1(n, s, p) \mathcal{L}_p^s[\phi](x) &= \left(\frac{\frac{sp}{2} 2^{2s-1}}{\pi^{\frac{n-1}{2}}} \frac{\Gamma(\frac{n+sp}{2})}{\Gamma(\frac{p+1}{2})\Gamma(2-s)} \right) ((1-s) \mathcal{L}_p^s[\phi](x)) \\ &\xrightarrow{s \rightarrow 1^-} \left(\frac{\frac{p}{2} 2}{\pi^{\frac{n-1}{2}}} \frac{\Gamma(\frac{n+p}{2})}{\Gamma(\frac{p+1}{2})\Gamma(1)} \right) \left(-\frac{\pi^{\frac{n-1}{2}}}{p} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{n+p}{2})} \Delta_p \phi(x) \right) = -\Delta_p \phi(x). \end{aligned}$$

where we have used the fact that $\Gamma(1) = 1$. This proves the first limit in (5.2). For the second limit, we just need to check that $C_1(n, s, 2)$ coincides with $c_{n,s}$ given in (5.1). This is just a simple computation

$$\begin{aligned} C_1(n, s, 2) &= \frac{s(1-s)2^{2s-1}}{\pi^{\frac{n-1}{2}}} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{3}{2})\Gamma(2-s)} \\ &= \frac{s(1-s)2^{2s}}{\pi^{\frac{n}{2}}} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(2-s)} \end{aligned}$$

where we have used the fact that $\Gamma(1/2) = \sqrt{\pi}$ and $z\Gamma(z) = \Gamma(z+1)$. This finished the proof. \square

Remark 5.2. With the choice of $C_1(n, s, p)$ given by Lemma 5.1 we have that

$$C_2(s, p) := C_1(n, s, p) \frac{\pi^{\frac{n}{2}}}{2^{sp}\Gamma(\frac{n+sp}{2})} = \pi^{\frac{1}{2}} \frac{p}{\Gamma(\frac{p+1}{2})} \frac{1}{|\Gamma(-s)|} 2^{s(2-p)-2},$$

$$C_3(s, p) = C_1(n, s, p) \frac{\pi^{\frac{n}{2}} \Gamma(\frac{sp}{2})}{\Gamma(\frac{n+sp}{2})} = \pi^{\frac{1}{2}} \frac{p}{\Gamma(\frac{p+1}{2})} \frac{\Gamma(\frac{sp}{2})}{|\Gamma(-s)|} 2^{s-2}.$$

and

$$C_4(s, p) = C_1(n, s, p) \frac{\pi^{\frac{n}{2}}}{2^{sp}\Gamma(\frac{2+sp}{2})\Gamma(\frac{n+sp}{2})} = \pi^{\frac{1}{2}} \frac{p}{(\frac{p+1}{2})} \frac{1}{|\Gamma(-s)|\Gamma(\frac{2+sp}{2})} 2^{s(2-p)-2}$$

An important fact is that these constants are independent of the dimension. It is also standard to check that

$$C_2(s, 2) = \frac{1}{|\Gamma(-s)|}, \quad C_3(s, 2) = \frac{\Gamma(s)}{|\Gamma(-s)|} 4^s, \quad \text{and} \quad C_4(s, 2) = \frac{1}{|\Gamma(-s)|\Gamma(1+s)} = \frac{\sin(s\pi)}{\pi}$$

which recover the classical constants for the semigroup formula and the extension problem (see for example [33]).

6 A new spectral-type fractional p -Laplacian on domains

There are many well known applications and interesting remarks of the semigroup formula and the extension problem for the fractional Laplacian. After the results that we have shown in this paper, a number of applications can be naturally proposed in the context of the fractional p -Laplacian. Here is a first application.

Let Ω be a smooth domain (bounded or otherwise). The semigroup representation (1.8) motivates the definition of a new operator using formula (1.11), where we replaced $-\Delta$ by $-\Delta_\Omega$, the Laplacian with homogeneous Dirichlet boundary conditions. It is clear that $(-\Delta_\Omega)_p^s$ coincides with $(-\Delta)_p^s$ if $\Omega = \mathbb{R}^n$, and one can even expect some ‘‘continuous’’ dependence on Ω .

This is, to our knowledge, a new example of nonlocal operator when $\Omega \neq \mathbb{R}^n$. We will show below it does not coincide with $(-\Delta)_p^s$ applied to the extension of u by 0 outside Ω . On the other hand, when $p = 2$ we recover the usual Spectral Fractional Laplacian, which is simply the usual spectral power of the Dirichlet Laplacian in a bounded domain as a compact operator; we refer the reader to [18, 49, 12] for modern references.

First, we show it is well-defined for functions defined exclusively in Ω , at least when they are smooth. This will allow to consider a boundary value problem with boundary conditions in $\partial\Omega$, without taking the set $\mathbb{R}^n \setminus \Omega$ into account.

Theorem 6.1. *Let $s \in (0, 1)$ and assume that $u \in C^2(\overline{\Omega})$, where Ω is a smooth subdomain on \mathbb{R}^n . If $p \geq \frac{2}{2-s}$, the following formula*

$$(-\Delta_\Omega)_p^s u(x) := C_2 \int_0^{+\infty} e^{t\Delta_\Omega} [\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1+\frac{sp}{2}}} \quad (6.1)$$

is well-defined at every $x_0 \in \Omega$. If p is close to 1, more precisely $p \in (1, \frac{2}{2-s})$, we need to assume additionally that $\nabla u(x_0) \neq 0$.

Proof. Since Ω is smooth, we can extend u to a $C_c^2(\mathbb{R}^n)$, via the tubular neighbourhood theorem. We will still denote the extended function by u . The time integral in formula (1.11) could be singular at both ends, $t = 0, \infty$, hence we must show the convergence of the integral at either end. The integral over the interval $[1, \infty)$ is not a problem, since $\|e^{t\Delta_\Omega} f\|_{L^\infty(\Omega)} \leq C e^{-\lambda_1 t} \|f\|_{L^\infty(\Omega)}$.

Let us show that there exists

$$\lim_{m \rightarrow \infty} \int_{1/m}^1 e^{t\Delta_\Omega} [\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1+\frac{sp}{2}}}$$

We do this by following the proof done above with $(-\Delta)_p^s$, with suitable modifications. For convenience, we write the semigroup $e^{t\Delta_\Omega}$ in its kernel form K_Ω , and compare with the heat kernel of the whole space $K_{\mathbb{R}^n}$. We define $g(x, y) = \Phi_p(u(x) - u(y))$. Then,

$$e^{t\Delta_\Omega} [\Phi_p(u(x) - u(\cdot))](x) = \int_\Omega K_\Omega(t, x, y) g(x, y) dy.$$

We write

$$\begin{aligned} \int_{1/m}^1 \int_\Omega K_\Omega(t, x, y) g(x, y) dy \frac{dt}{t^{1+\frac{sp}{2}}} &= \int_{1/m}^1 \int_{\mathbb{R}^n} K_{\mathbb{R}^n}(t, x, y) g(x, y) dy \frac{dt}{t^{1+\frac{sp}{2}}} \\ &\quad - \int_{1/m}^1 \int_{\mathbb{R}^n \setminus \Omega} K_{\mathbb{R}^n}(t, x, y) g(x, y) dy \frac{dt}{t^{1+\frac{sp}{2}}} \\ &\quad - \int_{1/m}^1 \int_\Omega (K_{\mathbb{R}^n}(t, x, y) - K_\Omega(t, x, y)) g(x, y) dy \frac{dt}{t^{1+\frac{sp}{2}}}. \end{aligned} \quad (6.2)$$

For the first term on the right-hand side we can compute the limit, which is well-defined by our knowledge of $(-\Delta)_p^s$, notice that by (2.2) in the proof [Theorem 2.1](#) we have

$$(-\Delta)_p^s u(x) = \lim_{m \rightarrow \infty} \int_{1/m}^\infty \int_{\mathbb{R}^n} K_{\mathbb{R}^n}(t, x, y) g(x, y) dy \frac{dt}{t^{1+\frac{sp}{2}}}.$$

For the other two terms in (6.2), we will show the convergence by checking they are Cauchy sequences. To this effect, we write for $k > m > 0$ that

$$\int_{1/m}^1 f(t) dt - \int_{1/k}^1 f(t) dt = \int_{1/m}^{1/k} f(t) dt.$$

We need to show that the two remaining terms give Cauchy sequences, so we take $k > m > 0$. The

second term is easy to estimate

$$\begin{aligned}
\left| \int_{1/k}^{1/m} \int_{\mathbb{R}^n \setminus \Omega} K_{\mathbb{R}^n}(t, x, y) g(x, y) \frac{dt}{t^{1+\frac{sp}{2}}} \right| &\leq \|g(x, \cdot)\|_{L^\infty(\Omega)} \int_{1/k}^{1/m} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\text{dist}(x, \partial\Omega)}^{\infty} e^{-\frac{r^2}{4t}} r^{n-1} dr \frac{dt}{t^{1+\frac{sp}{2}}} \\
&= \|g(x, \cdot)\|_{L^\infty(\Omega)} \int_{1/k}^{1/m} \int_{C/t}^{\infty} e^{-1/\rho} p(\rho) d\rho q(t) dt \\
&\leq C \|g(x, \cdot)\|_{L^\infty(\Omega)} \int_0^{1/m} e^{-D/t} q(t) dt,
\end{aligned}$$

where p and q are bounded by positive and negative powers and $C, D > 0$. Since $e^{-Dt}q(t) \in L^1(0, 1)$, the second term of the right-hand side of (6.2) is a Cauchy sequence.

For the last term of (6.2), using the comparison principle we have that

$$0 \leq K_{\mathbb{R}^n}(t, x, y) - K_{\Omega}(t, x, y) \leq \max_{z \in \partial\Omega} K_{\mathbb{R}^n}(t, x, z) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\text{dist}(x, \partial\Omega)^2}{4t}}.$$

Thus,

$$\int_{1/k}^{1/m} \left| \int_{\Omega} (K_{\mathbb{R}^n}(t, x, y) - K_{\Omega}(t, x, y)) g(x, y) dy \right| \frac{dt}{t^{1+\frac{sp}{2}}} \leq \|g(x, \cdot)\|_{L^\infty(\Omega)} \int_0^{1/m} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\text{dist}(x, \partial\Omega)^2}{4t}} \frac{dt}{t^{1+\frac{sp}{2}}}.$$

Hence, the last term of (6.2) is also a Cauchy sequence. \square

Remark 6.2. This operator is indeed different from the usual one on domains, which uses the formula of $(-\Delta)_p^s$ restricted to functions u defined in \mathbb{R}^n such that $u = 0$ outside Ω (see, for example, [42, 54]). Let Ω be fixed and assume without loss of generality that $0 \in \Omega$. Let $B_R(0) \subset \Omega$ and u a smooth function that is positive, radially symmetric, and strictly decreasing along the radius and 0 outside of B_R . Then, with the notation of the proof of Theorem 6.1

$$\begin{aligned}
(-\Delta)_p^s u(0) - (-\Delta_{\Omega})_p^s u(0) &= \int_0^{\infty} \int_{\Omega} (K_{\mathbb{R}^n}(t, 0, y) - K_{\Omega}(t, 0, y)) \Phi_p(u(0) - u(y)) dy \frac{dt}{t^{1+\frac{sp}{2}}} \\
&\quad + \int_0^{\infty} \int_{\mathbb{R}^n \setminus \Omega} K_{\mathbb{R}^n}(t, x, y) \Phi_p(u(0) - u(y)) dy \frac{dt}{t^{1+\frac{sp}{2}}} > 0.
\end{aligned}$$

This happens because both integrands are nonnegative functions, and $K_{\mathbb{R}^n}(t, x, y) - K_{\Omega}(t, x, y) \geq 0$ for all $x, y \in \Omega$.

Remark 6.3. We can similarly define operators via the extension problem and Balakrishnan's formulas. For example,

$$(-\Delta_{\Omega})_p^s u(x_0) = C_4 \int_0^{\infty} \Delta(t - \Delta_{\Omega})^{-1} [\Phi_p(u(x_0) - u(\cdot))](x_0) \frac{dt}{t^{1-\frac{sp}{2}}}.$$

It is an interesting open problem to check whether these definitions are equivalent. Another formula can be written via the extension.

7 Numerical applications

Among the many possible applications of our representations, we want to comment next on the construction of finite-difference methods for the fractional p -Laplacian. We recall that monotone finite-difference discretizations for the fractional Laplacian (case $p = 2$) have been established in the papers [21] (in \mathbb{R}) and [24] (in \mathbb{R}^n) based on the semigroup representation. The advantage of these types of discretizations is that they have quadratic a error order, independently of the fractional power of the

operator (see [24]). This is a great advantage compared to other finite difference discretizations in the literature (see discussion in [24, 25]).

Similar ideas could be used in the case of the fractional p -Laplacian using the results of **Theorem 2.1**. Let us explain the process at a formal level. Let $h > 0$ and $\{e_i\}_{i=1}^N$ be standard basis of \mathbb{R}^n . Consider the discrete Laplacian given by

$$\Delta_h[u](x) = \sum_{i=1}^d \frac{u(x + he_i) + u(x - he_i) - 2u(x)}{h^2}. \quad (7.1)$$

The already-classical theory shows that this is an ‘‘asymptotically good’’ approximation of Δ (in the sense that it produces convergent and stable schemes in many settings). In accordance to our representation ideas, we are led to define the following discrete operator

$$(-\Delta_h)_p^s[u](x) := C_2 \int_0^\infty e^{t\Delta_h} [\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1+\frac{sp}{2}}}. \quad (7.2)$$

by simply replacing Δ by Δ_h in (1.8). In this setting, $e^{t\Delta_h}[f]$ represents the uniquely-defined solution of $\partial_t w = \Delta_h w$ such that $w(x, 0) = f(x)$. We will call it the *discrete fractional p -Laplacian*.

At least formally, we have an alternative representation of $(-\Delta_h)_p^s$ given in a finite-difference form similar to (7.1). We follow here the presentation of [24]. Let $z_\beta := \beta h$ for $\beta := (\beta_1, \dots, \beta_N) \in \mathbb{Z}^n$ and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$. We have that

$$e^{t\Delta_h}[\psi](x) = \sum_{\beta \in \mathbb{Z}^d} \psi(x + z_\beta) G(\beta, \frac{t}{h^2})$$

where $G(\beta, t) = e^{-2nt} \prod_{i=1}^n I_{|\beta_i|}(2t)$ and I_m is the modified Bessel function of first kind with order $m \in \mathbb{N}$. From (7.2) we get, at least formally,

$$\begin{aligned} (-\Delta_h)_p^s u(x) &= C_2 \int_0^\infty \left(\sum_{\beta \neq 0} \Phi_p(u(x) - u(x + z_\beta)) G(\beta, \frac{t}{h^2}) \right) \frac{dt}{t^{1+\frac{sp}{2}}} \\ &= \sum_{\beta \neq 0} \Phi_p(u(x) - u(x + z_\beta)) C_2 \int_0^\infty G(\beta, \frac{t}{h^2}) \frac{dt}{t^{1+\frac{sp}{2}}}. \end{aligned}$$

Finally, the change of variables $\tau = t/h^2$ allows to write

$$(-\Delta_h)_p^s u(x) = \sum_{\beta \neq 0} \Phi_p(u(x) - u(x + z_\beta)) K_{\beta,h} \quad \text{with} \quad K_{\beta,h} = \frac{C_2}{h^{sp}} \int_0^\infty G(\beta, t) \frac{dt}{t^{1+\frac{sp}{2}}}. \quad (7.3)$$

This process is rigorous when $sp < 2$. However, the weights $K_{\beta,h}$ are not well defined when $sp \geq 2$: for example when $n = 1$, we note that

$$G(1, t) = e^{-2nt} I_1(2t) \sim t \quad \text{for} \quad t \ll 1,$$

which is only enough to show the convergence of the integral in the mentioned range. However, when $sp \geq 2$, it is easy to show that formula (7.2) fails to have a finite value at a point x of global strict maximum of u . An idea to solve this problem is introducing an extra discretization parameter $\delta > 0$ and alternatively define

$$\begin{aligned} (-\Delta_{h,\delta})_p^s &:= C_2 \int_\delta^\infty (e^{t\Delta} [\Phi_p(u(x) - u(\cdot))])(x) \frac{dt}{t^{1+\frac{sp}{2}}} \\ &= \sum_{\beta \neq 0} \Phi_p(u(x) - u(x + z_\beta)) K_{\beta,h,\delta} \quad \text{with} \quad K_{\beta,h,\delta} = \frac{C_2}{h^{sp}} \int_{\delta h^2}^\infty G_h(\beta, t) \frac{dt}{t^{1+\frac{sp}{2}}} \end{aligned}$$

In this way, all the integral are finite and all the steps are well justified rigorously in the whole range $p \in (1, \infty)$ and $s \in (0, 1)$. One should make $\delta \rightarrow 0$ as $h \rightarrow 0$ to show that the discretization is consistent in some proper sense.

Remark 7.1. A different finite difference discretization of the form of (7.3) for nonlocal p -Laplacian type operators (in the sense of [1]) has been obtained in [46]. They are based on obtaining quadrature rules for a singular integral definition like (1.1). In both cases the obtained discretizations are monotone, which yields good properties of the associated schemes.

Remark 7.2. Balakrishnan's formula has been the starting point for one of the most popular numerical schemes for the fractional Laplacian. The scheme combines a quadrature for the time integral and a numerical scheme for a sequence of the spatial operators $\Delta(t - \Delta)^{-1}$. One of the main advantages of this idea is the fact that, contrary to others, it is suited for parallel computing. We refer to [13] as the basic reference to the topic.

Remark 7.3. The extension problem has also been successfully used by the numerical analysis community in the context of the fractional Laplacian. We refer the reader to [23] for a finite difference treatment in the context of nonlocal porous medium equations and to [43] for a finite element one in the context of boundary value problems.

8 Other applications

We collect here a series of directions in which our previous ideas can be developed.

8.1 Equivalent representations of the $W^{s,p}(\mathbb{R}^n)$ seminorm

We can define an equivalent way of defining the (s, p) -energy (1.7). More precisely, consider the Gagliardo semi-norm

$$[u]_{W^{s,p}(\mathbb{R}^n)} := \left(C_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

It is standard to check that we have the following alternative definition

$$[u]_{W^{s,p}(\mathbb{R}^n)} = \left(C_2 \int_{\mathbb{R}^n} \int_0^{+\infty} e^{t\Delta} [|u(x) - u(\cdot)|^p](x) \frac{dt}{t^{1+\frac{sp}{2}}} dx \right)^{\frac{1}{p}}.$$

This formulation was introduced in [51]. We also refer to [27] for a modern reference on the topic in the context of Carnot groups.

The corresponding version with Balakrishnan's formula, is

$$[u]_{W^{s,p}(\mathbb{R}^n)} = \left(C_4 \int_{\mathbb{R}^n} \int_0^{+\infty} \Delta(t - \Delta)^{-1} [|u(x) - u(\cdot)|^p](x) \frac{dt}{t^{1-\frac{sp}{2}}} dx \right)^{\frac{1}{p}}.$$

We have not found this expression in the literature. A similar formula can be constructed via the extension, but it comes as an awkward product of gradients.

8.2 Spectral (s, p) -Laplacian on manifolds

Let (M, g) be a Riemannian manifold. Some authors (see, e.g. [28]) have used the (s, p) -fractional Laplacian on manifolds given by

$$(-\Delta)_{M,p}^s u(x) = C_3(n, s, p) \text{PV} \int_M \frac{\Phi_p(u(x) - u(y))}{d_g(x, y)^{n+sp}} dy,$$

where g is a Riemannian metric on M . This a regional-type operator. Following our approach for bounded domain, we propose also the spectral-type operator

$$(-\Delta_M)_p^s u(x) = C_1(n, s, p) \int_0^{+\infty} e^{-t\Delta_g} \Phi_p(u(x) - u(\cdot)) \frac{dt}{t^{1+sp}}$$

where Δ_g is the Laplace-Beltrami operator on M . The change in sign with respect to the other formulas is due to the convention in Geometry, where Δ_g is the positive operator.

The idea of dealing with this kind of operators has been considered in the case of the fractional versions of the heat equation and porous medium equations. The extension method was used by Banica et al. [4] for the fractional heat equation on hyperbolic space or non compact manifolds. The spectral approach was used quite recently by Berchio et al. [6] for the fractional porous medium equation posed in hyperbolic space.

8.3 Representation via a general parametric family of operators

Following the philosophy of Section 2 and Section 4 we can think of more general representation formulas for the fractional p -Laplacian. Consider a parametric operator family $\{T_t\}_{t \geq 0}$ defined via convolution with measure kernels $K(t, \cdot)$, i.e.,

$$T_t f(x) = \int_{\mathbb{R}^n} K(t, x - y) f(y) dy,$$

At minimum, we need to assume that

$$\int_0^\infty K(t, y) \frac{dt}{t^{1+\alpha}} = \frac{C_K}{|y|^{n+sp}} \quad \text{for all } y \in \mathbb{R}^n.$$

for some finite constant $C_K \neq 0$. Then, we can expect that the following formula holds

$$(-\Delta)_p^s u(x) = C_T \int_0^{+\infty} T_t [\Phi_p(u(x) - u(\cdot))] (x) \frac{dt}{t^{1+\alpha}} \quad (8.1)$$

with $C_T = C_1(n, s, p)/C_K$. These formulas have a particularly nice form if K is self-similar, like for example

$$K(t, y) = t^{-\beta} F(t^{-\gamma}|y|) \quad \text{with } \gamma \neq 0 \quad \text{and} \quad \frac{\beta + \alpha}{\gamma} = n + sp,$$

so that

$$C_K := \frac{1}{\gamma} \int_0^\infty F(r) r^{n+sp-1} dr.$$

In some settings, our proofs of the semigroup and the Balakhrisnan's formulas can be repeated. However, if K changes sign and it is not L^1 , some estimates are likely more difficult. This can be the case, for example, when working with $T_t = (-\Delta)^k (t - \Delta)^m$.

Remark 8.1. Note that when we use the heat semigroup, $T_t = e^{t\Delta}$, we have

$$K(t, y) = t^{-\frac{n}{2}} F(t^{-\frac{1}{2}}|y|) \quad \text{with} \quad F(r) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{r^2}{4}}.$$

Here, $\gamma = 1/2$, and we must take $\alpha = sp/2$. It works for all s and p , see (2.1).

Remark 8.2. Another option is to use a fractional heat semigroup, $T_t = e^{-t(-\Delta)^\sigma}$, with $\sigma \in (0, 1)$. Then the representation (8.1) is correct on the condition that $sp < 2\sigma$.

Let us sketch the proof. It is well known, see for instance [9, 11], that the fractional semigroup has a kernel K with scaling parameters $\beta = n/2\sigma$ and $\gamma = 1/2\sigma$, and a profile F that decays like $F(r) \sim r^{-(n+2\sigma)}$ as $r \rightarrow \infty$. It follows that we must take $\alpha = sp/2\sigma$. In view of the decay of F , the integral for C_K is convergent only if $sp < 2\sigma$, and then the technique applies. On the contrary, the fat tail of F is incompatible with this representation if $sp \geq 2$ because then we cannot find a $\sigma < 1$ such that C_K is finite.

9 Final comments and open problems

Intuition behind our results: splitting the formula for the fractional p -Laplacian. An idea behind the representation formula (1.8) is the following simple principle: the fractional p -Laplacian (which is both a nonlinear and nonlocal operator) of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ can be formally seen a composition of a linear nonlocal operator acting on n variables of a certain function $v : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ built from Φ_p and u . Indeed, let $v(x, y) = \Phi_p(u(y) - u(x))$. Notice that $v(x, x) = 0$. At least formally, for $sp < 2$, and for convenient functions, we have that

$$(-\Delta)_p^s u(x) = C_1 \int_{\mathbb{R}^n} \frac{v(x, x) - v(x, y)}{|x - y|^{n+sp}} dy = \tilde{C}_1 (-\Delta)^{\frac{sp}{2}} [v(x, \cdot)](x).$$

This shows a splitting between the linear part $(-\Delta)^{\frac{sp}{2}}$ and the nonlinear part Φ_p . For $sp \geq 2$ one should define the operator

$$\mathcal{L}^{\frac{sp}{2}}[\psi](x) = C_1 \text{PV} \int_{|y|>0} \frac{\psi(x) - \psi(y)}{|x - y|^{n+sp}} dy.$$

which does not coincide with $(-\Delta)^{\frac{sp}{2}}$. Naturally, this very singular integral is only defined for special ψ and x . A key part of our arguments are based on showing that, even though the operator $\mathcal{L}^{\frac{sp}{2}}$ is hardly ever defined, the composition works. The scaling properties of the corresponding kernels and integrals in t or z will be crucial. This splitting idea is also shared by the two other alternative representations (1.9) and (1.10).

Other representation formulas: For the case $p = 2$, other representation formulas are known (see, e.g. [33]). It seems likely that some of them can be adapted for the case $p \neq 2$, but we have not tried them in this paper.

Uniqueness of the extension problem (3.1) for $sp \geq 2$. For $sp < 2$ the uniqueness of energy solutions of (3.1) is shown in [19]. However, for $sp \geq 2$ the diffusion becomes too singular at $y = 0$ to work properly in the energy setting. It is an open in which sense of solution $E_{s,p}[f]$ is the unique solution of the boundary value problem.

Numerical methods for our new spectral-type fractional p -Laplacian on domains. Following the ideas of [22] one can implement numerical discretization of the fractional p -Laplacian on bounded domains proposed in Section 6.

Regularity and well-posedness The extension techniques have been extensively used in the theory of regularity of nonlocal problems involving linear operators, i.e the case $p = 2$ of the theory presented in this paper. We refer for example to the seminal work of Athanasopoulos and Caffarelli [2] in the context of fractional porous medium and Stefan type problems. It has also been used to prove well-posedness and properties of such type of equations (see [44, 45]).

We believe that our theory could also help to develop similar results for equations involving the fractional p -Laplacian.

Appendix A Technical lemmas

Lemma A.1. *Let $u \in C_b^2(\mathbb{R}^d)$ and $\Phi_p(t) = |t|^{p-2}t$ for $p \geq 2$. Let $K : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be such that $K(z) = K(-z)$ and $\int K(z)|z|^p dz < +\infty$. Then*

$$\left| \text{P.V.} \int_{|x-y|>0} K(x-y)\Phi_p(u(x) - u(y)) dy \right| \leq (p-1) \|\nabla u\|_{L^\infty(\mathbb{R}^n)}^{p-2} \|D^2 u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^d} K(z)|z|^p dz.$$

Proof. By symmetry

$$\begin{aligned}
& \left| \text{PV} \int_{|x-y|>0} K(x-y) \Phi_p(u(x) - u(y)) \, dy \right| \\
&= \left| \text{PV} \int_{|z|>0} K(z) \Phi_p(u(x) - u(x+z)) \, dz \right| \\
&= \frac{1}{2} \left| \text{PV} \int_{|z|>0} K(z) (\Phi_p(u(x) - u(x+z)) + \Phi_p(u(x) - u(x-z))) \, dz \right| \\
&\leq \frac{1}{2} \text{PV} \int_{|z|>0} K(z) |\Phi_p(u(x) - u(x+z)) - \Phi_p(-u(x) + u(x-z))| \, dz
\end{aligned}$$

Now, since $p \geq 2$, we have for some $|\xi| \leq \sup_{x,z} |u(x) - u(x+z)| \leq \|\nabla u\|_{L^\infty(\mathbb{R}^n)} |z|$

$$\begin{aligned}
|\Phi_p(u(x) - u(x+z)) - \Phi_p(-u(x) + u(x-z))| &\leq |\Phi_p'(\xi)| |u(x) - u(x+z) - (-u(x) + u(x-z))| \\
&= (p-1) |\xi|^{p-2} |2u(x) - u(x+z) - u(x-z)| \\
&\leq (p-1) \|\nabla u\|_{L^\infty(\mathbb{R}^n)}^{p-2} |z|^{p-2} \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |z|^2 \\
&= (p-1) \|\nabla u\|_{L^\infty(\mathbb{R}^n)}^{p-2} \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |z|^p.
\end{aligned}$$

and the proof is done. \square

Lemma A.2. Let $u \in C_b^2(\mathbb{R}^d)$ and $\Phi_p(t) = |t|^{p-2}t$ for $p \in (1, 2)$. Let $K : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be such that $K(z) = \kappa(|z|)$ and $\int K(z) |z|^p \, dz < +\infty$. If $\nabla u(x) \neq 0$, then

$$\left| \text{PV} \int_{|x-y|>0} K(x-y) \Phi_p(u(x) - u(y)) \, dy \right| \leq c \int_{\mathbb{R}^d} K(z) |z|^p \, dz.$$

for some constant c depending on p, n and $\|u\|_{C^2(\mathbb{R}^n)}$.

Proof. We follow in the proof of Lemma 3.6 in [31] to show that

$$\begin{aligned}
& \left| \text{PV} \int_{|x-y|>0} K(x-y) \Phi_p(u(x) - u(y)) \, dy \right| \\
&\lesssim \int_{|z|>0} (|\nabla u(x) \cdot z| + \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |z|^2)^{p-2} \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |z|^2 K(z) \, dz
\end{aligned}$$

Changing to polar coordinates and using Lemma 3.5 in [31] we get

$$\begin{aligned}
& \left| \text{PV} \int_{|x-y|>0} K(x-y) \Phi_p(u(x) - u(y)) \, dy \right| \\
&\lesssim \|D^2 u\|_{L^\infty(\mathbb{R}^n)} \int_0^\infty r^2 \kappa(r) \left(\int_{S^n} (|\nabla u(x) \cdot \omega| r + \|D^2 u\|_{L^\infty(\mathbb{R}^n)} r^2)^{p-2} \, d\omega \right) \, dr \\
&= \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |\nabla u(x)|^{p-2} \int_0^\infty r^p \kappa(r) \left(\int_{S^n} \left(\left| \frac{\nabla u(x)}{|\nabla u(x)|} \cdot \omega \right| + \frac{\|D^2 u\|_{L^\infty(\mathbb{R}^n)}}{|\nabla u(x)|} r \right)^{p-2} \, d\omega \right) \, dr \\
&\lesssim \int_0^\infty r^p \kappa(r) \left(1 + \frac{\|D^2 u\|_{L^\infty(\mathbb{R}^n)}}{|\nabla u(x)|} r \right)^{p-2} \, dr \\
&\lesssim \int_0^\infty r^p \kappa(r) \, dr
\end{aligned}$$

where in the last inequality we have used the fact that $f(t) = |t|^{p-2}$ is a decreasing function for $p < 2$. Changing back to regular coordinates, we get the desired result. \square

Lemma A.3. *Let $u \in C_b^2(\mathbb{R}^d)$ and $\Phi_p(t) = |t|^{p-2}t$ for $p \in (1, 2)$. Let $K : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be such that $K(z) = \kappa(|z|)$ and $\int K(z)|z|^{2p-2} dz < +\infty$. Then*

$$\left| \text{PV} \int_{|x-y|>0} K(x-y)\Phi_p(u(x) - u(y)) dy \right| \leq c \int_{\mathbb{R}^d} K(z)|z|^{2p-2} dz.$$

for some constant c depending on p, n and $\|u\|_{C^2(\mathbb{R}^n)}$.

Proof. We follow in the proof of Lema 3.6 [31] to show that

$$\begin{aligned} & \left| \text{PV} \int_{|x-y|>0} K(x-y)\Phi_p(u(x) - u(y)) dy \right| \\ & \lesssim \int_{|z|>0} (|\nabla u(x) \cdot z| + \|D^2 u\|_{L^\infty(\mathbb{R}^n)}|z|^2)^{p-2} \|D^2 u\|_{L^\infty(\mathbb{R}^n)}|z|^2 K(z) dz \\ & \lesssim \|D^2 u\|_{L^\infty(\mathbb{R}^n)}^{p-1} \int_{|z|>0} |z|^{2p-2} K(z) dz \end{aligned}$$

where in the last inequality we have used the fact that $f(t) = |t|^{p-2}$ is a decreasing function for $p < 2$. \square

Remark A.4. If $K(z) = |z|^{-n-sp} \chi_{|z| \leq 1}(z)$ then Lemma A.3 applies for $p > \frac{2}{2-s}$ since

$$\int K(z)|z|^{2p-2} dz \lesssim \int \frac{dz}{|z|^{n+sp+2-2p}}$$

that is finite if $sp + 2 - 2p < 0$, that is $p > \frac{2}{2-s}$.

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References

- [1] F. ANDREU-VAILLO, J. M. MAZÓN, J. D. ROSSI, and J. J. TOLEDO-MELERO. *Nonlocal diffusion problems*. Vol. 165. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010, pp. xvi+256.
- [2] I. ATHANASOPOULOS and L. A. CAFFARELLI. Continuity of the temperature in boundary heat control problems. *Adv. Math.* 224.1 (2010), pp. 293–315.
- [3] A. V. BALAKRISHNAN. Fractional powers of closed operators and the semigroups generated by them. *Pacific Journal of Mathematics* 10.2 (1960), pp. 419–437.

- [4] V. BANICA, M. d. M. GONZÁLEZ, and M. SÁEZ. Some constructions for the fractional Laplacian on noncompact manifolds. *Rev. Mat. Iberoam.* 31.2 (2015), pp. 681–712.
- [5] B. BARRIOS and M. MEDINA. Equivalence of weak and viscosity solutions in fractional non-homogeneous problems. *arXiv preprint arXiv:2006.08384* (2020).
- [6] E. BERTHIO, M. BONFORTE, D. GANGULY, and G. GRILLO. The fractional porous medium equation on the hyperbolic space. *Calc. Var. Partial Differential Equations* 59.5 (2020), p. 169.
- [7] C. BJORLAND, L. CAFFARELLI, and A. FIGALLI. Non-local gradient dependent operators. *Adv. Math.* 230.4-6 (2012), pp. 1859–1894.
- [8] C. BJORLAND, L. CAFFARELLI, and A. FIGALLI. Nonlocal tug-of-war and the infinity fractional Laplacian. *Comm. Pure Appl. Math.* 65.3 (2012), pp. 337–380.
- [9] R. M. BLUMENTHAL and R. K. GETTOOR. Some theorems on stable processes. *Trans. Amer. Math. Soc.* 95 (1960), pp. 263–273.
- [10] S. BOCHNER. Diffusion Equation and Stochastic Processes. *Proceedings of the National Academy of Sciences* 35.7 (1949), pp. 368–370.
- [11] M. BONFORTE, Y. SIRE, and J. L. VÁZQUEZ. Optimal existence and uniqueness theory for the fractional heat equation. *Nonlinear Anal.* 153 (2017), pp. 142–168.
- [12] M. BONFORTE and J. L. VÁZQUEZ. A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains. *Arch. Ration. Mech. Anal.* 218.1 (2015), pp. 317–362.
- [13] A. BONITO and J. E. PASCIAK. Numerical approximation of fractional powers of elliptic operators. *Math. Comp.* 84.295 (2015), pp. 2083–2110.
- [14] L. BRASCO and E. LINDGREN. Higher Sobolev regularity for the fractional p -Laplace equation in the superquadratic case. *Adv. Math.* 304 (2017), pp. 300–354.
- [15] L. BRASCO, E. LINDGREN, and A. SCHIKORRA. Higher Hölder regularity for the fractional p -Laplacian in the superquadratic case. *Adv. Math.* 338 (2018), pp. 782–846.
- [16] L. BRASCO and A. SALORT. A note on homogeneous Sobolev spaces of fractional order. *Annali di Matematica Pura ed Applicata* 198.4 (2019), pp. 1295–1330.
- [17] C. BUCUR and M. SQUASSINA. An asymptotic expansion for the fractional p -Laplacian and gradient dependent nonlocal operators (2020), pp. 1–24.
- [18] X. CABRÉ and J. TAN. Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.* 224.5 (2010), pp. 2052–2093.
- [19] L. CAFFARELLI and L. SILVESTRE. An extension problem related to the fractional laplacian. *Communications in Partial Differential Equations* 32.8 (2007), pp. 1245–1260.
- [20] E. CHASSEIGNE and E. R. JAKOBSEN. On nonlocal quasilinear equations and their local limits. *J. Differential Equations* 262.6 (2017), pp. 3759–3804.
- [21] O. CIAURRI, L. RONCAL, P. R. STINGA, J. L. TORREA, and J. L. VARONA. Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications. *Adv. Math.* 330 (2018), pp. 688–738.
- [22] N. CUSIMANO, F. del TESO, L. GERARDO-GIORDA, and G. PAGNINI. Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions. *SIAM J. Numer. Anal.* 56.3 (2018), pp. 1243–1272.
- [23] F. DEL TESO. Finite difference method for a fractional porous medium equation. *Calcolo* 51.4 (2014), pp. 615–638.
- [24] F. DEL TESO, J. ENDAL, and E. R. JAKOBSEN. Robust numerical methods for nonlocal (and local) equations of porous medium type. Part II: Schemes and experiments. *SIAM Journal on Numerical Analysis* 56.6 (2018), pp. 3611–3647.

- [25] F. DEL TESO, J. ENDAL, and E. R. JAKOBSEN. Robust numerical methods for nonlocal (and local) equations of porous medium type. Part I: Theory. *SIAM J. Numer. Anal.* 57.5 (2019), pp. 2266–2299.
- [26] E. DIBENEDETTO. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993, pp. xvi+387.
- [27] N. GAROFALO and G. TRALLI. A Bourgain-Brezis-Mironescu-Dávila theorem in groups of Heisenberg type (2020).
- [28] L. GUO, B. ZHANG, and Y. ZHANG. Fractional p -Laplacian equations on Riemannian manifolds. *Electron. J. Differ. Equations* 2018.156 (2018), pp. 1–17.
- [29] A. IANNIZZOTTO, S. MOSCONI, and M. SQUASSINA. Global Hölder regularity for the fractional p -Laplacian. *Rev. Mat. Iberoam.* 32.4 (2016), pp. 1353–1392.
- [30] H. ISHII and G. NAKAMURA. A class of integral equations and approximation of p -Laplace equations. *Calc. Var. Partial Differential Equations* 37.3-4 (2010), pp. 485–522.
- [31] J. KORVENPÄÄ, T. KUUSI, and E. LINDGREN. Equivalence of solutions to fractional p -Laplace type equations. *Journal des Mathématiques Pures et Appliquées* 132 (2019), pp. 1–26.
- [32] T. KUUSI, G. MINGIONE, and Y. SIRE. Nonlocal equations with measure data. *Comm. Math. Phys.* 337.3 (2015), pp. 1317–1368.
- [33] M. KWAŚNICKI. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.* 20.1 (2017), pp. 7–51.
- [34] N. S. LANDKOF. *Osnovy sovremennoĭ teorii potentsiala*. Izdat. “Nauka”, Moscow, 1966, p. 515.
- [35] E. LINDGREN. Hölder estimates for viscosity solutions of equations of fractional p -Laplace type. *NoDEA Nonlinear Differential Equations Appl.* 23.5 (2016), Art. 55, 18.
- [36] E. LINDGREN and P. LINDQVIST. Fractional eigenvalues. *Calc. Var. Partial Differential Equations* 49.1-2 (2014), pp. 795–826.
- [37] E. LINDGREN and P. LINDQVIST. Perron’s method and Wiener’s theorem for a nonlocal equation. *Potential Anal.* 46.4 (2017), pp. 705–737.
- [38] P. LINDQVIST. *Notes on the p -Laplace equation*. Vol. 102. Report. University of Jyväskylä Department of Mathematics and Statistics. University of Jyväskylä, Jyväskylä, 2006, pp. ii+80.
- [39] J. J. MANFREDI, M. PARVIAINEN, and J. D. ROSSI. An asymptotic mean value characterization for p -harmonic functions. *Proc. Amer. Math. Soc.* 138.3 (2010), pp. 881–889.
- [40] J. J. MANFREDI, M. PARVIAINEN, and J. D. ROSSI. Dynamic programming principle for tug-of-war games with noise. *ESAIM Control Optim. Calc. Var.* 18.1 (2012), pp. 81–90.
- [41] C. MARTÍNEZ CARRACEDO and M. SANZ ALIX. *The theory of fractional powers of operators*. Vol. 187. North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 2001, pp. xii+365.
- [42] J. M. MAZÓN, J. D. ROSSI, and J. TOLEDO. Fractional p -Laplacian evolution equations. *J. des Math. Pures Appl.* 105.6 (2016), pp. 810–844.
- [43] R. H. NOCHETTO, E. OTÁROLA, and A. J. SALGADO. A PDE approach to fractional diffusion in general domains: a priori error analysis. *Found. Comput. Math.* 15.3 (2015), pp. 733–791.
- [44] A. de PABLO, F. QUIRÓS, A. RODRÍGUEZ, and J. L. VÁZQUEZ. A fractional porous medium equation. *Adv. Math.* 226.2 (2011), pp. 1378–1409.
- [45] A. de PABLO, F. QUIRÓS, A. RODRÍGUEZ, and J. L. VÁZQUEZ. A general fractional porous medium equation. *Comm. Pure Appl. Math.* 65.9 (2012), pp. 1242–1284.
- [46] M. PÉREZ-LLANOS and J. D. ROSSI. Numerical approximations for a nonlocal evolution equation. *SIAM J. Numer. Anal.* 49.5 (2011), pp. 2103–2123.

- [47] D. PUHST. On the evolutionary fractional p -Laplacian. *Appl. Math. Res. Express. AMRX* 2 (2015), pp. 253–273.
- [48] E. M. STEIN. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970, pp. xiv+290.
- [49] P. R. STINGA and J. L. TORREA. Extension problem and Harnack’s inequality for some fractional operators. *Communications in Partial Differential Equations* 35.11 (2010), pp. 2092–2122.
- [50] M. STRÖMQVIST. Local boundedness of solutions to non-local parabolic equations modeled on the fractional p -Laplacian. *J. Differential Equations* 266.12 (2019), pp. 7948–7979.
- [51] M. H. TAIBLESON. On the theory of Lipschitz spaces of distributions on Euclidean n -space. I. Principal properties. *J. Math. Mech.* 13 (1964), pp. 407–479.
- [52] J. L. VÁZQUEZ. The evolution fractional p -Laplacian equation in \mathbb{R}^N . Fundamental solution and asymptotic behaviour. *Nonlinear Anal.* 199 (2020), pp. 112034, 32.
- [53] J. L. VÁZQUEZ. *Smoothing and decay estimates for nonlinear diffusion equations*. Vol. 33. Oxford Lecture Series in Mathematics and its Applications. Equations of porous medium type. Oxford University Press, Oxford, 2006, pp. xiv+234.
- [54] J. L. VÁZQUEZ. The Dirichlet problem for the fractional p -Laplacian evolution equation. *J. Differ. Equ.* 260.7 (2016), pp. 6038–6056.