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On the classical limit and the infrared problem for non-relativistic fermions interacting with the electromagnetic field

by

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ABSTRACT. — The classical limit and the infrared divergence problem for a non-relativistic charged quantum particle interacting with the quantized electromagnetic field are analized. Overall three momentum conservation is taken into account. A unitary transformation associated to the coherent state corresponding to a particle surrounded by a cloud of soft photons is performed upon the hamiltonian and the particle-field states. The transformed state representing a moving dressed quantum particle and its energy are given by the Brillouin-Wigner perturbation theory. It is shown formally that the quantum energy approaches the classical value for the energy of the classical particle interacting with the classical electromagnetic field as the Planck constant $\hbar$ goes to zero. Moreover, all Feynman diagrams contributing to this quantum energy are infrared finite, without needing add diagrams of the same order in the electric charge to obtain the infrared finiteness. Those properties justify the usefulness of the unitary transformation. The Compton effect in the forward direction is studied using dressed charged particle states after the unitary transformation has been performed. The quantum cross section approaches the classical limit (Thomson's formula) as $\hbar \to 0$, and the Feynman diagrams are free of infrared divergences.

RÉSUMÉ. — La limite classique et le problème infrarouge pour une particule quantique non-relativiste en interaction avec le champ électromagnétique quantifié sont étudiés en supposant la conservation de l'impulsion totale. Une transformation unitaire (associée à l'état cohérent qui correspond à une particule et un nuage de photons à très petite énergie) est appliquée à l'hamiltonien et aux états physiques. L'état transformé, qui représente une particule quantique habillée en mouvement, et l'énergie associée sont donnés par la théorie de Brillouin et Wigner. On démontre formellement que l'énergie quantique tend vers la valeur classique de l'énergie de la particule classique en interaction avec un champ électromagnétique classique, lorsque la constante de Planck $\hbar$ tend vers zéro. Tous les diagrammes de Feynman qui contribuent à l'énergie quantique sont libres de divergences infrarouges et il n'est pas nécessaire de faire des sommations sur des diagrammes du même ordre dans la charge électrique pour obtenir la finitude infrarouge. Ces résultats montrent l'intérêt de la transformation unitaire. La diffusion vers l'avant d'un photon par une particule habillée est étudiée au moyen des états obtenus au moyen de la transformation unitaire. On voit que la section efficace quantique tend vers la limite classique (formule de Thomson) lorsque $\hbar \to 0$, et que les diagrammes de Feynman n'ont pas divergences infrarouges.

1. INTRODUCTION

As Rohrlich [*1*] has pointed out, the infrared behaviour and the classical limit are two of the main problems that Quantum Electrodynamics still presents. They are not independent, but are deeply related to each other [*2*]; for instance, Thomson's cross section can be recovered by taking either the low-frequency limit or the limit $\hbar \to 0$.

The first consistent treatments of the infrared problem are due to Bloch, Nordsick and others [*3*]. In particular, Kibble [*4*] and Chung [*5*] proposed that charged particles carry coherent states of radiation with them, thus contributing to solve the difficulties pointed out by Dollard [*6*]. Since then, many contributions have been made along the same line: Fadeev and Kulish [*7*], Zwanzinger [*8*], Papanicolau [*9*], Krauss *et al.* [*10*], Harada and Kubo [*11*], Korthals and De Rafael [*12*] and Dahmen *et al.* [*13*], to quote some. Bialynicki-Birula [*14*] has studied the classical limit: to do this, he has also used coherent states.

In the study of non-relativistic particles interacting with massive scalar bosons, Gross [*15*] and Nelson [*16*] have made use of a unitary transformation deeply related to coherent states in order to solve ultraviolet divergence problems and prove the selfadjointness of the hamiltonian. A mathematical study of the infrared problem for a non-relativistic particle interacting with a quantized massless scalar field was made by Frohlich [*17*]. Using the unitary transformation of Gross and Nelson, a rigorous solution for the dressed particle was constructed in [*18*]. On the other hand, the classical limit for a non-relativistic particle interacting with scalar bosons and the infrared problem in the massless case was studied in [*19*]. In this paper, based upon [*20*], the more difficult problem of one non-relativistic charged quantum particle interacting with the quantized electromagnetic (EM) field will be treated. Specifically, we shall concentrate in the classical limit and the infrared problem for an isolated dressed particle and the Compton effect. Our study takes into account three momentum conservation.

The methods to be used and the main results are the following. In section 2 we summarize a specific solution for the classical equations of motion, which will be useful in next sections. In section 3, we outline some standard facts about Non-relativistic Quantum Electrodynamics and try to construct the dressed one-particle state, as well as its energy, through the Brillouin-Wigner perturbation theory. We find that each individual Feynman diagram contributing to the quantum energy is infrared finite, but it behaves oddly as $\hbar \to 0$ if the quantum particle is not at rest. In order to solve this problem, a unitary transformation, which generalizes for Non-relativistic Quantum Electrodynamics those previously used for scalar bosons [*15, 16*] and is closely related to the coherent state accompanying the particle, is introduced in sections 4 and 5.

In section 5, by extending Nelson's and Fröhlich's arguments [*16, 17*], we show that the transformed hamiltonian is a well defined selfadjoint operator, construct formally the state of the dressed charged particle and its energy by using the Brillouin-Wigner perturbative expansion, and obtain the associated new Feynman rules. A net interesting consequence of introducing the unitary transformation is that each individual contribution associated to the new Feynman diagrams goes to zero in the classical limit (as $\hbar \to 0$), so that both the quantum unperturbed and total energies are seen to approach formally the classical energy (obtained in section 2); this result is physically natural and would have been difficult to obtain within the framework of the canonical formalism, as discussed in section 3. Moreover, each individual Feynman diagram contributing to the quantum energy is infrared finite.

In section 6, by using the new hamiltonian and the dressed charged particle state constructed in section 5, we study the elastic scattering of a photon by a non-relativistic charged particle (Compton effect) in the forward direction. We show formally that the contribution associated to each individual Feynman diagram is infrared finite, and, using the optical theorem, so is the total quantum cross section. The latter has a well defined classical limit as $\hbar \to 0$; in particular, the classical (Thomson's) cross section is obtained as $\hbar \to 0$, to lowest order in the electric charge.

2. CLASSICAL CHARACTERIZATION OF THE MODEL

By assumption, the classical hamiltonian for a particle of charge q and mass m in an EM field $\vec{A}(\vec{x}, t)$, in the Coulomb gauge $\vec{\nabla}\vec{A}(\vec{x}, t) = 0$, is given by [21, 22]

$$H = \frac{1}{2m}\left[\vec{p} - \frac{q}{c}\vec{A}(\vec{x}, t)\right]^2 + 2\sum_{\lambda=1,2}\int d^3\vec{k}\left(\frac{\omega}{c}\right)^2 |a(\vec{k}, \lambda, t)|^2 + U. \quad (2.1)$$

In this expression, $a(\vec{k}, \lambda, t)$ and $a^*(\vec{k}, \lambda, t)$ are the classical amplitudes of the field

$$\vec{A}(\vec{x}, t) = \sum_{\lambda=1,2}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\vec{\varepsilon}(\vec{k}, \lambda)[c(\vec{k})a(\vec{k}, \lambda, t)e^{i\vec{k}\vec{x}} + c^*(\vec{k})a^*(\vec{k}, \lambda, t)e^{-i\vec{k}\vec{x}}], \quad (2.2)$$

where $\vec{\varepsilon}(\vec{k}, 1)$, $\vec{\varepsilon}(\vec{k}, 2)$ and $\vec{\varepsilon}(\vec{k}, 3) \equiv \hat{k} \equiv \vec{k}/|\vec{k}|$ form a linear polarization basis, and $c(\vec{k})$ and $c^*(\vec{k})$ are cut-off factors defined as the Fourier transforms of the charge density measured in units of q:

$$c(\vec{k}) = \int d^3\vec{x}\rho(\vec{x})e^{i\vec{k}\vec{x}}, \qquad \int\rho(\vec{x})d^3\vec{x} = 1.$$

The U on the right-hand side (r. h. s.) of eq. (2.1) is one half of the interaction of the charge density with itself:

$$U = \frac{q^2}{8\pi}\int d^3\vec{x}\int d^3\vec{x}'\frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (2.3)$$

The reason to consider charge densities for point particles has a quantum origin: such particles have associated wave packets of certain width.

The total momentum $\vec{\Pi}$ of the system is given by

$$\vec{\Pi} = \vec{p} + 2\sum_{\lambda=1,2}\int d^3\vec{k}\,\frac{\vec{k}\omega}{c^2}|a(\vec{k}, \lambda, t)|^2 \quad (2.4)$$

The hamiltonian (2.1) yields the following equations of motion:

$$m\vec{v} + \frac{q}{c}\vec{A}\Big|_{\vec{x}=\vec{x}_t} = \vec{p}, \qquad m\dot{\vec{v}} = q\left(\frac{\vec{v}}{c}\wedge\vec{B} + \vec{E}\right)\Big|_{\vec{x}=\vec{x}_t} \quad (2.5)$$

$$\dot{a}(\vec{k}, \lambda, t) = -i\omega a(\vec{k}, \lambda, t) + i\frac{qc}{2\omega}\frac{c^*(\vec{k})}{(2\pi)^{3/2}}\vec{v}\,\vec{\varepsilon}(\vec{k}, \lambda)e^{-i\vec{k}\vec{x}_t}, \quad (2.6)$$

together with the complex conjugate of eq. (2.6). Here, $\vec{x}_t$ denotes the position in which the particle is at time t.

In what follows, we will consider a particle moving at constant velocity $\vec{v}_0$. In this case, $\vec{x}_t = \vec{v}_0 t$, and the solution of eq. (2.6) is given by

$$a(\vec{k}, \lambda, t) = a(\vec{k}, \lambda, 0)e^{-i\omega t} - \frac{qc}{2\omega}\frac{c^*(\vec{k})}{(2\pi)^{3/2}}\frac{\vec{v}_0\vec{\varepsilon}(\vec{k}, \lambda)}{\omega - \vec{k}\vec{v}_0}(e^{-i\omega t} - e^{-i\vec{k}\vec{v}_0 t}). \quad (2.7)$$

The first term on the r. h. s. of eq. (2.7) is associated to the radiation emitted by the free EM field. The second one gives a correction to its amplitude, due to the presence of a charged particle. Finally, the third one corresponds to the radiation emitted by the particle and gives rise to the self-interaction, (notice that its frequency is $\vec{k}\vec{v}_0$, which differs from $\omega = kc$). Therefore, to calculate the energy and momentum of the particle, we choose the initial condition

$$a(\vec{k}, \lambda, 0) = \frac{qc}{2\omega}\frac{c^*(\vec{k})}{(2\pi)^{3/2}}\frac{\vec{v}_0\vec{\varepsilon}(\vec{k}, \lambda)}{\omega - \vec{k}\vec{v}_0},$$

so

$$a(\vec{k}, \lambda, t) = \frac{qc}{2\omega}\frac{c^*(\vec{k})}{(2\pi)^{3/2}}\frac{\vec{v}_0\vec{\varepsilon}(\vec{k}, \lambda)}{\omega - \vec{k}\vec{v}_0}e^{-i\vec{k}\vec{v}_0 t}. \quad (2.8)$$

Substituting eq. (2.8) into eqs. (2.1) and (2.4) we have

$$\mathrm{E} = \frac{1}{2}mv_0^2 + \mathscr{E}_{\text{class.}} + \mathrm{U}, \qquad \mathscr{E}_{\text{class.}} \equiv \frac{q^2}{2}\sum_{\lambda=1,2}\int d^3\vec{k}\,\frac{|c(\vec{k})|^2}{(2\pi)^3}\frac{[\vec{v}_0\vec{\varepsilon}(\vec{k}, \lambda)]^2}{(\omega - \vec{k}\vec{v}_0)^2}, \quad (2.9)$$

$$\vec{\pi} = m\vec{v}_0 + \frac{q^2}{c^2}\sum_{\lambda=1,2}\int d^3\vec{k}\,\frac{|c(\vec{k})|^2}{(2\pi)^3}\frac{[\vec{v}_0\vec{\varepsilon}(\vec{k}, \lambda)]\vec{\varepsilon}(\vec{k}, \lambda)}{\omega(\omega - \vec{k}\vec{v}_0)}$$

$$+ \frac{q^2}{2}\sum_{\lambda=1,2}\int d^3\vec{k}\,\vec{k}\,\frac{|c(\vec{k})|^2}{(2\pi)^3}\frac{[\vec{\varepsilon}(\vec{k}, \lambda)\vec{v}_0]^2}{\omega(\omega - \vec{k}\vec{v}_0)}. \quad (2.10)$$

For a cut-off factor having spherical symmetry ($c(\vec{k}) = c(k)$, $k \equiv |\vec{k}|$) and small velocities ($v_0/c \ll 1$), eqs. (2.9) and (2.10) take the form

$$\mathrm{E} = \frac{1}{2}(m + \delta m)v_0^2 + \mathrm{U} + 0\left(\frac{v_0^4}{c^4}\right)$$
$$\vec{\Pi} = (m + \delta m)\vec{v}_0 + 0\left(\frac{\vec{v}_0^3}{c^3}\right)$$

where

$$\delta m = \frac{4}{3}\frac{\mathrm{U}}{c^2} = \frac{4}{3}\left(\frac{q}{2\pi c}\right)^2\int_0^\infty |c(k)|^2 dk.$$

3. QUANTIZATION AND DIFFICULTIES OF THE CLASSICAL LIMIT IN THE CONVENTIONAL FORMALISM

Canonical quantization transforms the classical hamiltonian (2.1) into [21]

$$H = H_0 + H_I, \tag{3.1}$$

$$H_0 = \frac{\vec{P}^2}{2m} + \sum_{\lambda=1,2} \int d^3\vec{k}\hbar\omega a^+(\vec{k},\lambda)a(\vec{k},\lambda) + U, \tag{3.2}$$

$$H_1 = -\frac{q}{mc}[\vec{P}\vec{A}^{(-)} + \vec{A}^{(+)}\vec{P}] + \frac{q^2}{2mc^2}[\vec{A}^{(-)2} + \vec{A}^{(+)2} + 2\vec{A}^{(+)}\vec{A}^{(-)}], \tag{3.3}$$

where

$$\vec{A}^{(+)} = \sum_{\lambda=1,2} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} c \sqrt{\frac{\hbar}{2\omega}} \vec{\varepsilon}(\vec{k},\lambda)c^*(\vec{k})a^\dagger(\vec{k},\lambda)e^{-i\vec{k}\vec{x}}, \quad \vec{A}^{(-)} = [\vec{A}^{(+)}]^\dagger, \tag{3.4}$$

and $\vec{A} = \vec{A}^{(+)} + \vec{A}^{(-)}$ is the EM field in racionalized Heavyside-Lorentz units. All the operators are expressed in the Schrödinger picture, in which we will work. The annihilation and creation operators satisfy the commutations rules

$$[a(\vec{k},\lambda), a(\vec{k}',\lambda')] = [a^\dagger(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = 0,$$
$$[a(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k} - \vec{k}'). \tag{3.5}$$

The total momentum is

$$\vec{\Pi} = \vec{P} + \sum_{\lambda=1,2} \int d^3\vec{k}\hbar\vec{k}a^\dagger(\vec{k},\lambda)a(\vec{k},\lambda) \tag{3.6}$$

and it commutes with the hamiltonian (3.1). Therefore, there exist common eigenstates of $\vec{\Pi}$ and H.

Let $\mathcal{H}_{\vec{\pi}}$ be the subspace (of the Hilbert space for the system) formed by all the states $|\Psi\rangle$ such that $(\vec{\Pi} - \vec{\pi})|\Psi\rangle = 0$. The set of states

$$|\Psi(\vec{q}_{\vec{\pi}}; \vec{k}_1\lambda_1, \ldots, \vec{k}_n\lambda_n)\rangle = \psi(\vec{q}_{\vec{\pi}}) \otimes \left[\frac{1}{\sqrt{n!}} a^\dagger(\vec{k}_1,\lambda_1)\ldots a^\dagger(\vec{k}_n,\lambda_n)|0\rangle\right] \tag{3.7}$$

where $\vec{q}_{\vec{\pi}} = \vec{\pi} - \hbar\Sigma_{i=1}^n \vec{k}_i$, constitute a basis of $\mathcal{H}_{\vec{\pi}}$ with respect to the restricted scalar product

$$\langle \Psi(\vec{q}_{\vec{\pi}}; \vec{k}'_1\lambda'_1, \ldots, \vec{k}'_{n'}\lambda'_{n'}) | \Psi(\vec{q}_{\vec{\pi}}; \vec{k}_1\lambda_1, \ldots, \vec{k}_n\lambda_n)\rangle =$$
$$= \frac{\delta_{nn'}}{n!} \sum_{\nu(1)\ldots\nu(n)} \delta^{(3)}(\vec{k}'_{\nu(1)} - \vec{k}_1)\delta_{\lambda'_{\nu(1)}\lambda_1} \ldots \delta^{(3)}(\vec{k}'_{\nu(n)} - \vec{k}_n)\delta_{\lambda'_{\nu(n)}\lambda_n}. \tag{3.8}$$

In eq. (3.8) the sum $\sum_{\nu(1)\dots\nu(n)}$ is extended over the $n!$ permutations $(\nu(1), \dots, \nu(n))$ of $(1, \dots, n)$. Notice that: *i*) the ordinary scalar product of the same states in the full Hilbert space equals the restricted one (3.8) times a volume divergent factor $\delta^{(3)}(\vec{0})$, and, *ii*) the states (3.7) are eigenstates of H_0.

We are interested in the ground state $| \Psi_+(\vec{\pi}) >$ of H (the dressed charged particle) with energy $E = E(\vec{\pi})$ and momentum $\vec{\pi}$:

$$(H - E) | \Psi_+(\vec{\pi}) > = 0, \qquad (\vec{\Pi} - \vec{\pi}) | \Psi_+(\vec{\pi}) > = 0. \tag{3.9}$$

We shall try to construct it from $| \Psi(\vec{\pi}) >$, regarding (3.7) as unperturbed states and H_I as a perturbation. The Brillouin-Wigner approach leads to [23]

$$| \Psi_+(\vec{\pi}) > = | \Psi(\vec{\pi}) > + Q_{\vec{\pi}} G_0(E) H_I | \Psi_+(\vec{\pi}) > , \tag{3.10}$$

$$E = \frac{\pi^2}{2m} + U + \langle \Psi(\vec{\pi}) | H_I | \Psi_+(\vec{\pi}) \rangle \equiv M(E), \tag{3.11}$$

where

$$Q_{\vec{\pi}} \equiv \mathbb{1} - P_{\vec{\pi}}, \qquad P_{\vec{\pi}} \equiv | \Psi(\vec{\pi}) > < \Psi(\vec{\pi}) |, \qquad G_0(z) \equiv (z - H_0)^{-1}. \tag{3.12}$$

Notice that eq. (3.11), i. e., $E = M(E)$, is an equation whose unknown is the energy E of the particle in its ground state. Our task consists in finding the solution of such equation. We solve it in a formal way; that is, we suppose the existence of the solution and construct it iteratively using eq. (3.10), and, finally, we study its properties in the classical and infrared limits.

Let us suppose that the equation $E = M(E)$ has a unique solution E which belongs to the spectrum of H. We want to study its classical limit, this is, its limit when $\hbar \to 0$. To do this we write the equation $E = M(E)$ as

$$E = \frac{\pi^2}{2m} + U + \left\langle \Psi(\vec{\pi}) | H_I \sum_{n=0}^{\infty} [Q_{\vec{\pi}} G_0(E) H_I]^n | \Psi(\vec{\pi}) \right\rangle. \tag{3.13}$$

In eq. (3.13), the series on the r. h. s. is the sum of the contributions of all the Feynman diagrams. The Feynman rules associated to this Brillouin-Wigner expansion for H_I are:

1.*a*) absorption of one photon $(\vec{k}, \lambda)$, fig. 3.1 :

$$-\frac{q}{m} \frac{c(\vec{k})}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega}} \, [\vec{\varepsilon}(\vec{k}, \lambda) \vec{p}]$$

1.*b*) emission of one photon $(\vec{k}, \lambda)$, fig. 3.2:

$$-\frac{q}{m} \frac{c^*(\vec{k})}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega}} \, [\vec{p}\, \vec{\varepsilon}(\vec{k}, \lambda)]$$

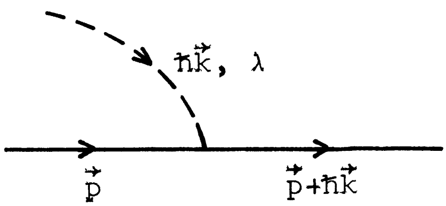


FIG. 3-1.

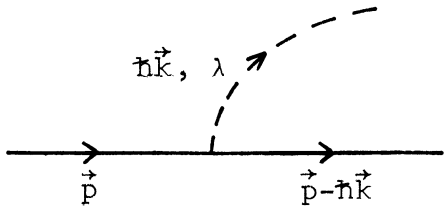


FIG. 3-2.

1.*c*) absorption of one photon $(\vec{k}, \lambda)$ and emission of another $(\vec{k}', \lambda')$, fig. 3.3:

$$\frac{q^2}{m} \frac{c(\vec{k})}{(2\pi)^{3/2}} \frac{c^*(\vec{k}')}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega}} \sqrt{\frac{\hbar}{2\omega'}} \left[\vec{\varepsilon}(\vec{k}, \lambda)\vec{\varepsilon}(\vec{k}', \lambda')\right]$$

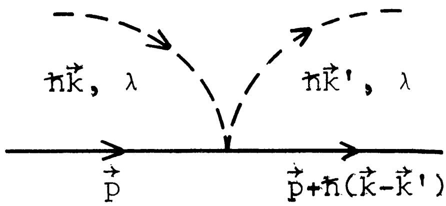


FIG. 3-3.

1.*d*) absorption of two photons $(\vec{k}, \lambda)$ and $(\vec{k}', \lambda')$, fig. 3.4:

$$\frac{q^2}{m} \frac{c(\vec{k})}{(2\pi)^{3/2}} \frac{c(\vec{k}')}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega}} \sqrt{\frac{\hbar}{2\omega'}} \left[\vec{\varepsilon}(\vec{k}, \lambda)\vec{\varepsilon}(\vec{k}', \lambda')\right]$$

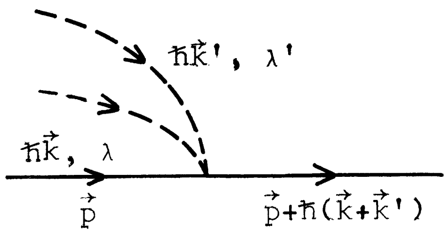


FIG. 3-4.

1.*e*) emission of two photons $(\vec{k}, \lambda)$ and $(\vec{k}', \lambda')$, fig. 3.5:

$$\frac{q^2}{m} \frac{c^*(\vec{k})}{(2\pi)^{3/2}} \frac{c^*(\vec{k}')}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega}} \sqrt{\frac{\hbar}{2\omega'}} [\vec{\varepsilon}(\vec{k}, \lambda)\vec{\varepsilon}(\vec{k}', \lambda')]$$

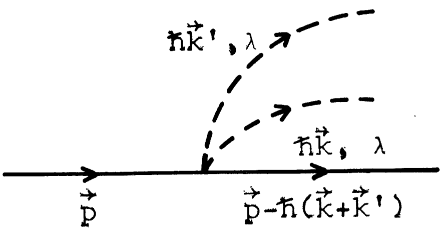


FIG. 3-5.

2) propagator, fig. 3.6:

$$\frac{1}{E - \dfrac{\left(\vec{p} - \hbar \sum_{i=1}^{n} \vec{k}_i\right)^2}{2m} - \hbar \sum_{i=1}^{n} \omega_i - U}$$

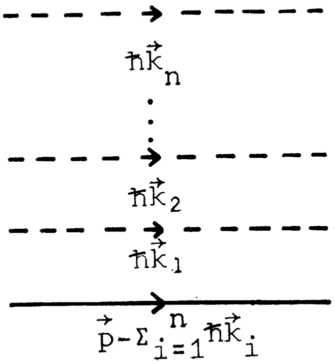


FIG. 3-6.

For instance, the diagram of fig. 3.7 gives a contribution

$$\frac{q^6}{4m^3} \sum_{\lambda_1,\lambda_2,\lambda_3=1,2} \int d^3\vec{k}_1 \int d^3\vec{k}_2 \int d^3\vec{k}_3 \frac{|c(\vec{k}_1)|^2}{(2\pi)^3} \frac{|c(\vec{k}_2)|^2}{(2\pi)^3} \frac{|c(\vec{k}_3)|^2}{(2\pi)^3} \frac{\hbar^3}{8\omega_1\omega_2\omega_3} \times$$

$$\times \frac{[\vec{\varepsilon}(\vec{k}_1, \lambda_1)\vec{\varepsilon}(\vec{k}_2, \lambda_2)][\vec{\varepsilon}(\vec{k}_2, \lambda_2)\vec{\varepsilon}(\vec{k}_3, \lambda_3)][\vec{\varepsilon}(\vec{k}_3, \lambda_3)\vec{\varepsilon}(\vec{k}_1, \lambda_1)]}{\left\{ E - \dfrac{[\vec{\pi} - \hbar(\vec{k}_1 + \vec{k}_2)]^2}{2m} - \hbar(\omega_1 + \omega_2) - U \right\} \left\{ E - \dfrac{[\vec{\pi} - \hbar(\vec{k}_1 + \vec{k}_3)]^2}{2m} - \hbar(\omega_1 + \omega_3) - U \right\}}, \quad (3.14)$$

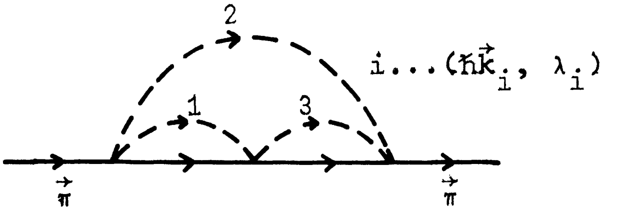


FIG. 3-7.

which comes from the term $\langle \Psi(\vec{\pi}) \mid [Q_{\vec{\pi}} G_0(E) H_I]^3 \mid \Psi(\vec{\pi}) \rangle$ on the r. h. s. of eq. (3.13).

We shall study the classical limit of each diagram. We have constructed $\mid \Psi_+(\vec{\pi}) \rangle$ from $\mid \Psi(\vec{\pi}) \rangle$, for which

$$\langle \Psi(\vec{\pi}) \mid \vec{\Pi} \mid \Psi(\vec{\pi}) \rangle = \langle \Psi \mid (\vec{\pi}) \mid \vec{P} \mid \Psi(\vec{\pi}) \rangle = m \langle \Psi(\vec{\pi}) \mid \vec{V} \mid \Psi(\vec{\pi}) \rangle ,$$

where $\vec{V}$ is the velocity operator. Defining $\vec{v}_0 \equiv \langle \Psi(\vec{\pi}) \mid \vec{V} \mid \Psi(\vec{\pi}) \rangle$, we have $\vec{\pi} = m\vec{v}_0$. Now, we distinguish two cases:

i) $\vec{v}_0 = 0$ (equivalently, $\vec{\pi} = 0$). Then, vertices 1.*a*) and 1.*b*) are identically zero when the canonical momentum $\vec{p}$ equals $\vec{\pi}$, and, for

$$\vec{p} = \vec{\pi} - \hbar \Sigma_{i=1}^{l} \vec{k}_i,$$

they behave as $\hbar^{3/2}$ when $\hbar \to 0$, since we have an $\hbar^{1/2}$ from $\sqrt{\hbar/2\omega}$ and another $\hbar$ from $\hbar \Sigma_{i=1}^{l} \vec{k}_i$. Therefore, if there is any vertex 1.*a*) or 1.*b*), it will go with $\hbar^{3/2}$. Vertices 1.*c*)-1.*e*) behave as $\hbar$ for $\hbar \to 0$. We suppose that

$$E \to E_{\text{class.}}(\vec{v}_0 = 0) = U \quad \text{when} \quad \hbar \to 0 . \tag{3.15}$$

Each diagram containing n vertices has $n-1$ propagators. So: *a*) if $E \to U$ for $\hbar \to 0$ as $\hbar^a$ with $a \geqslant 1$, the propagator behaves as $\hbar^{-1}$ because of the presence of $\hbar \Sigma_{i=1}^{n} \omega_i$ in its denominator, and all diagrams tend to zero, at least, as $\hbar$; *b*) if, for $\hbar \to 0$, $E \to U$ as $\hbar^b$ with $b < 1$, the propagator behaves as $\hbar^{-b}$, which makes diagrams approach zero, at least, as $\hbar^{n-(n-1)b}$ $(b < 1)$. Therefore, all diagrams tend to vanish in the limit $\hbar \to 0$ independently of the way in which $E \to U$. Then, the r. h. s. of eq. (3.13) tends to U when $\hbar \to 0$, in agreement with eq. (3.15), which gives a consistent behaviour in the classical limit.

ii) $\vec{v}_0 \neq 0$ (equivalently, $\vec{\pi} \neq 0$). In this case, vertices 1.*a*) and 1.*b*) behave as $\hbar^{1/2}$ when $\hbar \to 0$ and 1.*c*)-1.*e*) as $\hbar$. Now, we suppose that

$$E \to E_{\text{class.}}(\vec{v}_0 \neq 0) = \frac{1}{2} m v_0^2 + \mathscr{E}_{\text{class.}} + U \quad \text{if} \quad \hbar \to 0 \tag{3.16}$$

which makes the propagator tend to $\mathscr{E}_{\text{class.}}^{-1}$ for $\hbar \to 0$. Then, any diagram containing n vertices tends to zero, at least, as $\hbar^{n/2}$ in the limit $\hbar \to 0$. From this, it follows that the r. h. s. of eq. (3.13) approaches

$$\frac{\pi^2}{2m} + \mathrm{U} = \frac{1}{2} m v_0^2 + \mathrm{U}$$

when $\hbar \to 0$, in contradiction with eq. (3.16).

In other words, we want to study the classical limit of the solution E of equation $\mathrm{E} = \mathrm{M(E)}$. To do this, we suppose that E tends to its classical value and analyse the limit $\hbar \to 0$ of the r. h. s. of eq. (3.13). For $\vec{v}_0 = 0$ we get that it tends to the classical limit of E, so eq. (3.13) holds in the limit $\hbar \to 0$. However, for $\vec{v}_0 \neq 0$ the r. h. s. of eq. (3.13) approaches something different from the classical value of E, so eq. (3.13) doesn't hold in the classical limit. In this sense, the above formal manipulations to study the limit $\hbar \to 0$ are not correct for $\vec{v}_0 \neq 0$ and we say that the solution E of $\mathrm{E} = \mathrm{M(E)}$ has a bad behaviour in the classical limit. In next sections we shall perform a unitary transformation in order to solve this difficulty and get the desired behaviour.

Notice that all diagrams are free of infrared divergences. For instance, the contribution (3.14) of the diagram of fig. 3.7 behaves as

$$\int d^3\vec{k}_1 \int d^3\vec{k}_2 \int d^3\vec{k}_3 \left[k_1 k_2 k_3 (k_1 + k_2)(k_1 + k_3)\right]^{-1}$$

in the infrared limit, and such integral is convergent in this limit.

To see the infrared finiteness of each individual Feynman diagram, we first observe that, in the low-frequency limit: *i*) vertices 1.*a*) and 1.*b*) go like $k^{-1/2}$ and 1.*c*)-1.*e*) like $(kk')^{-1/2}$, and, *ii*) if $\mathrm{E} = \frac{\pi^2}{2m} + \mathrm{U}$, the propagator behaves as $(\Sigma_{i=1}^{n} k_i)^{-1}$, and in any other case it tends to something finite. Now, the existence of infrared divergences would amount to say that the denominator of the integrand of the integral which gives the contribution of a diagram tends to zero faster than the numerator. But the latter is $d^3\vec{k}_1 d^3\vec{k}_2 \ldots d^3\vec{k}_l$, so all the terms of the denominator must be, at least, of order two in every k_i $(i = 1, 2, \ldots, l)$ except in one k_i, in which the order must be three. More explicitly, we must have

$$\int d^3\vec{k}_1 \int d^3\vec{k}_2 \ldots \int d^3\vec{k}_l \,|\, c(\vec{k}_1) \,|^2 \,|\, c(\vec{k}_2) \,|^2 \ldots |c(\vec{k}_l)\,|^2 \times$$
$$\times \frac{1}{k_1 k_2 \ldots k_l} \frac{1}{\sum\limits_{i_1,\ldots,i_l} c_{i_1 i_2 \ldots i_l} k_1^{m_{i_1}} k_2^{m_{i_2}} \ldots k_l^{m_{i_l}}}, \qquad (3.17)$$

where $c_{i_1 i_2 \ldots i_l}$ are linear coefficients and $m_{i_1}, m_{i_2}, \ldots, m_{i_l}$ verify the minimal condition $m_{i_1}, m_{i_2}, \ldots, m_{i_l} \geqslant 1$ but one of them that must be greater or equal than two. In eq. (3.17) we have only written those parts that are relevant in the analysis of the infrared divergences.

Let us see some examples of how the integral (3.17) can diverge. For $l = 1$ it reduces to

$$\int d^3\vec{k}_1 \frac{|c(\vec{k}_1)|^2}{k_1 k_1^m}, \tag{3.18}$$

which is not well defined if $m \geqslant 2$. For $l = 2$ there are three posibilities: *i*)

$$\int d^3\vec{k}_1 \int d^3\vec{k}_2 \frac{|c(\vec{k}_1)|^2 |c(\vec{k}_2)|^2}{k_1 k_2 k_1^{m_1} k_2^{m_2}}, \tag{3.19}$$

which is infrared divergent if $m_1 \geqslant 2$ and $m_2 \geqslant 1$, *ii*)

$$\int d^3\vec{k}_1 \int d^3\vec{k}_2 \frac{|c(\vec{k}_1)|^2 |c(\vec{k}_2)|^2}{k_1 k_2 (k_1 + k_2)^m}, \tag{3.20}$$

which has a bad behaviour in the low-frequency limit when $m \geqslant 4$, and, *iii*)

$$\int d^3\vec{k}_1 \int d^3\vec{k}_2 \frac{|c(\vec{k}_1)|^2 |c(\vec{k}_2)|^2}{k_1 k_2 (k_1 + k_2)^m k_1^n}, \tag{3.21}$$

which diverges for $m \geqslant 2$ and $n \geqslant 1$ or $m \geqslant 1$ and $n \geqslant 2$. But the integrals (3.18) with $m = 2$, (3.19) with $m_1 = 2$ and $m_2 = 1$, (3.20) with $m = 4$ and (3.21) with $m = 2$ and $n = 1$ correspond to the diagrams of figs. 3.8,

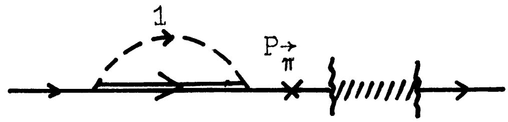


FIG. 3-8.

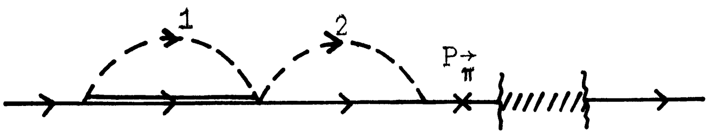


FIG. 3-9.

3.9, 3.10 and 3.11, respectively, (fig. 3.12 stands for any diagram between A and B). Notice that all these diagrams are generated by terms in which the projector $P_{\vec{\pi}}$ appears; for instance, the first of them comes from

$$\langle \Psi(\vec{\pi}) | H_I Q_{\vec{\pi}} G_0(E) H_I P_{\vec{\pi}} H_I \ldots H_I | \Psi(\vec{\pi}) \rangle .$$

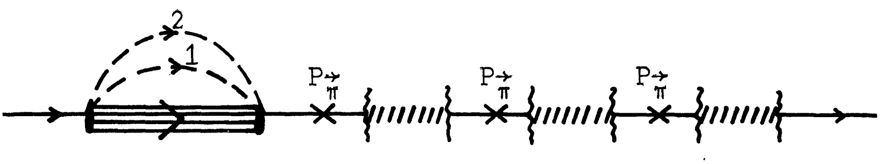


FIG. 3-10.

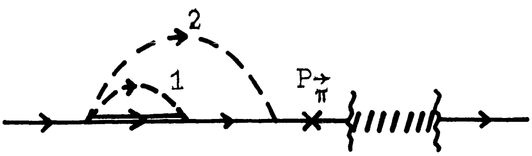


FIG. 3-11.

FIG. 3-12.

But the Brillouin-Wigner expansion, series on the r. h. s. of eq. (3.13), doesn't contain any term with a projector $P_{\vec{\pi}}$. More generally, for the integral (3.17) to be divergent one needs, at least, one projector $P_{\vec{\pi}}$ in the term of the series which gives rise to the integral, and this is impossible in the Brillouin-Wigner expansion.

We emphasize that the key point is the use of the Brillouin-Wigner perturbative series. It can be seen that in the Schrödinger-Rayleigh perturbation theory there are diagrams with infrared divergences [*20*].

4. UNITARY TRANSFORMATION

Let us assume with Chung [*5*] and Fadeev and Kulish [*7*] that the charged particle carries a coherent state, that is, instead of considering the bare state $|\Psi(\vec{\pi})\rangle$ we will consider the state

$$|\tilde{\Psi}(\vec{\pi})\rangle \equiv e^{W}|\Psi(\vec{\pi})\rangle, \tag{4.1}$$

where

$$W = \sum_{\lambda=1,2}\int d^3\vec{k}\,[z(\vec{k},\lambda)a^{\dagger}(\vec{k},\lambda)e^{-i\vec{k}\vec{x}} - z^{*}(\vec{k},\lambda)a(\vec{k},\lambda)e^{i\vec{k}\vec{x}}] \tag{4.2}$$

and $z(\vec{k},\lambda)$ is to be determined. The coherency of $|\tilde{\Psi}(\vec{\pi})\rangle$ tells us that

$$a(\vec{k},\lambda)|\tilde{\Psi}(\vec{\pi})\rangle = z(\vec{k},\lambda)e^{-i\vec{k}\vec{x}}|\tilde{\Psi}(\vec{\pi})\rangle \tag{4.3}$$

Of course the momentum $\vec{\pi}$ of the particle is the same, i. e.,

$$(\vec{\Pi} - \vec{\pi}) \mid \tilde{\Psi}(\vec{\pi}) \rangle = 0 , \tag{4.4}$$

as follows from eqs. (4.1)-(4.3) and

$$(\vec{\Pi} - \vec{\pi}) \mid \Psi(\vec{\pi}) \rangle = 0 .$$

To fix $z(\vec{k}, \lambda)$ we can apply the variational principle. To do this, we first evaluate $\langle \mathrm{H} \rangle_{\tilde{\Psi}(\vec{\pi})}$:

$$\langle \mathrm{H} \rangle_{\tilde{\Psi}(\vec{\pi})} = \frac{1}{2} m \vec{v}_0^2 + \sum_{\lambda=1,2} \int d^3\vec{k} \left(\frac{\hbar^2 k^2}{2m} + \hbar\omega \right) \mid z(\vec{k}, \lambda) \mid^2 + \mathrm{U} ,$$

where $\vec{v}_0 \equiv \langle \tilde{\Psi}(\vec{\pi}) \mid \vec{\mathrm{V}} \mid \tilde{\Psi}(\vec{\pi}) \rangle$. Then, we equal to zero its derivative with respect to $z^*(\vec{k}, \lambda)$:

$$0 = - \vec{v}_0 \left[\hbar \vec{k} z(\vec{k}, \lambda) + q \frac{c^*(\vec{k})}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega}} \vec{\varepsilon}(\vec{k}, \lambda) \right] + \left(\frac{\hbar^2 k^2}{2m} + \hbar\omega \right) z(\vec{k}, \lambda) .$$

And, finally, we solve this algebraic equation for $z(\vec{k}, \lambda)$:

$$z(\vec{k}, \lambda) = \frac{c^*(\vec{k})}{(2\pi)^{3/2}} \frac{q}{\sqrt{2\hbar\omega}} \frac{\vec{\varepsilon}(\vec{k}, \lambda) \vec{v}_0}{\omega + \dfrac{\hbar k^2}{2m} - \vec{k} \vec{v}_0} \tag{4.5}$$

We could have also reached the same result by imposing Heisenberg equations of motion for annihilation and creation operators and taking mean values for the state $\mid \tilde{\Psi}(\vec{\pi}) \rangle_2$ [20].

Notice that; *i*) $z(\vec{k}, \lambda) \exp(-i\vec{k}\vec{x}) = \langle a(\vec{k}, \lambda) \rangle_{\tilde{\Psi}(\vec{\pi})}$, so (4.5) is the analogue of eq. (2.8), and, *ii*) the term $\hbar k^2/2m$ of the denominator of eq. (4.5) has a quantum origin.

Making use of eqs. (3.6) and (4.2)-(4.5) we get the relation between $\vec{v}_0$ and $\vec{\pi}$:

$$\vec{\pi} = m\vec{v}_0 + \vec{\Gamma} + \vec{\mathrm{K}} , \tag{4.6}$$

where

$$\vec{\Gamma} \equiv \frac{q}{c} \langle \vec{\mathrm{A}} \rangle_{\tilde{\Psi}(\vec{\pi})} = 2q \sum_{\lambda=1,2} \int d^3\vec{k} \sqrt{\frac{\hbar}{2\omega}} \frac{c(\vec{k})}{(2\pi)^{3/2}} z(\vec{k}, \lambda) \vec{\varepsilon}(\vec{k}, \lambda)$$

$$= q^2 \int d^3\vec{k} \frac{\mid c(\vec{k}) \mid^2}{(2\pi)^3} \frac{\vec{v}_0 - (\hat{k}\vec{v}_0)\hat{k}}{\omega\left(\omega + \dfrac{\hbar k^2}{2m} - \vec{k}\vec{v}_0 \right)} , \tag{4.7}$$

$$\vec{\mathrm{K}} = \sum_{\lambda=1,2} \int d^3\vec{k} \hbar \vec{k} \mid z(\vec{k}, \lambda) \mid^2 = q^2 \int d^3\vec{k} \frac{\mid c(\vec{k}) \mid^2}{(2\pi)^3} \frac{[v_0^2 - (\hat{k}\vec{v}_0)]\hat{k}}{2\left(\omega + \dfrac{\hbar k^2}{2m} - \vec{k}\vec{v}_0 \right)^2} . \tag{4.8}$$

We would like to use the coherent state $|\tilde{\Psi}(\vec{\pi})\rangle$ to describe the « free » particle. We do this by absorbing the exponential factor e^{W} in the hamiltonian as a unitary transformation, and this is possible because

$$(e^{W})^{\dagger} = (e^{W})^{-1},$$

as follows from eq. (4.2).

5. SELFADJOINTNESS OF THE NEW HAMILTONIAN AND CORRECT CLASSICAL LIMIT

According to section 4 we define the following unitary transformation

$$0' = e^{-W}0e^{W} \qquad |\Psi'\rangle = e^{-W}|\Psi\rangle, \tag{5.1}$$

where primes stand for the transformed operators and states, and W is given by eqs. (4.2) and (4.5).

To calculate the transformed operators we use the formula

$$e^{-W}0e^{W} = 0 + [0, W] + \frac{1}{2!}[[0, W], W] + \frac{1}{3!}[[[0, W], W], W] + \ldots \tag{5.2}$$

Proceeding in this way we get that the total momentum $\vec{\Pi}$ transforms into itself, $\vec{\Pi}' = \vec{\Pi}$, so the subspace $\mathscr{H}_{\vec{\pi}}$ transforms into itself.

The transformed hamiltonian is

$$H'_0 = H' + H'_I \tag{5.3}$$

with

$$H'_0 = \frac{(\vec{P} - \vec{\Gamma} - \vec{K})^2}{2m} + \sum_{\lambda=1,2} \int d^3\vec{k}\hbar\omega a^{\dagger}(\vec{k}, \lambda)a(\vec{k}, \lambda) + \mathscr{E}_{\text{quan.}} + U, \tag{5.4}$$

$$\begin{aligned} H'_I = \sum_{\lambda=1,2} \int d^3\vec{k} \Bigg\{ \Bigg[\left(\hbar\omega + \frac{\hbar^2 k^2}{2m}\right) z(\vec{k}, \lambda) \\ - \frac{1}{m} f(\vec{k}, \lambda)(\vec{P} - \vec{\Gamma} - \vec{K}) \Bigg] a^{+}(\vec{k}, \lambda)e^{-ikx} + \text{h. c.} \Bigg\} + \\ + \sum_{\lambda,\lambda'=1,2} \int d^3\vec{k} \int d^3\vec{k}' \left[\frac{1}{2m} \vec{f}(\vec{k}, \lambda)\vec{f}(\vec{k}', \lambda')a^{+}(\vec{k}, \lambda)a^{+}(\vec{k}', \lambda')e^{-i(\vec{k}+\vec{k}')\vec{x}} + \text{h.c.} \right] \\ + \sum_{\lambda,\lambda'=1,2} \int d^3\vec{k} \int d^3\vec{k}' \frac{1}{m} \vec{f}(\vec{k}, \lambda)\vec{f}^{*}(\vec{k}', \lambda')a^{+}(\vec{k}, \lambda)a(\vec{k}', \lambda')e^{-i(\vec{k}-\vec{k}')\vec{x}}. \end{aligned} \tag{5.5}$$

$\vec{\Gamma}$ and $\vec{K}$ are given by eqs. (4.7) and (4.8), and

$$\vec{f}(\vec{k},\lambda) = \hbar\vec{k}z(\vec{k},\lambda) + q\frac{c^*(\vec{k})}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{2\omega}}\,\vec{\varepsilon}(\vec{k},\lambda)$$
$$= q\frac{c^*(\vec{k})}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{2\omega}}\left\{\vec{k}\frac{[\vec{\varepsilon}(\vec{k},\lambda)\vec{v}_0]}{\omega+\frac{\hbar k^2}{2m}-\vec{k}\vec{v}_0}+\vec{\varepsilon}(\vec{k},\lambda)\right\}, \quad (5.6)$$

$$\mathscr{E}_{\text{quan.}} = \sum_{\lambda=1,2}\int d^3\vec{k}\left(\frac{\hbar^2k^2}{2m}+\hbar\omega\right)|\,z(\vec{k},\lambda)\,|^2$$
$$= \frac{q^2}{2}\int d^3\vec{k}\left(\frac{\hbar k^2}{2m}+\omega\right)\frac{|\,c(\vec{k})\,|^2}{(2\pi)^3}\frac{v_0^2-(\hat{k}\vec{v}_0)^2}{\omega\left(\omega+\frac{\hbar k^2}{2m}-\vec{k}\vec{v}_0\right)^2}. \quad (5.7)$$

By generalizing Nelson's arguments, Fröhlich [17] stated in a sketchly way that the hamiltonian for a massless scalar boson field interacting with a non-relativistic fermion is selfadjoint. For completeness, we shall outline the proof of the selfadjointness of H′, eqs. (5.3)-(5.5). Following the arguments of Nelson [16], for all $|\,\Psi\,\rangle\,\varepsilon\mathscr{H}_{\vec{\pi}}$ of finite norm and in the domain of $H_0'^{1/2}$, we find that

$$\left|\langle\,\Psi,\sum_{\lambda=1,2}\int d^3\vec{k}\left[-\frac{1}{m}\vec{f}(\vec{k},\lambda)(\vec{P}-\vec{\Gamma}-\vec{K})a^\dagger(\vec{k},\lambda)e^{-i\vec{k}\vec{x}}+\text{h. c.}\right]\Psi\,\rangle_{\vec{\pi}}\right|$$
$$\leqslant\frac{4q}{m}c_1'\,\|\,H_0'^{1/2}\Psi\,\|_{\vec{\pi}}^2, \quad (5.8)$$

$$\left|\langle\,\Psi,\sum_{\lambda,\lambda'=1,2}\int d^3\vec{k}\int d^3\vec{k}'\frac{1}{m}\vec{f}(\vec{k},\lambda)\vec{f}^*(\vec{k}',\lambda')a^\dagger(\vec{k},\lambda)a(\vec{k}',\lambda')e^{-i(\vec{k}-\vec{k}')\vec{x}}\Psi\,\rangle_{\vec{\pi}}\right|$$
$$\leqslant\frac{q^2}{m}c_1'^2\,\|\,H_0'^{1/2}\Psi\,\|_{\vec{\pi}}^2, \quad (5.9)$$

where the subindex $\vec{\pi}$ refers to norms in $\mathscr{H}_{\vec{\pi}}$ and

$$c_1'^2 \equiv \sum_{\lambda=1,2}\int d^3\vec{k}\frac{\vec{f}^*(\vec{k},\lambda)\vec{f}(\vec{k},\lambda)}{q^2\hbar\omega}$$
$$= \int d^3\vec{k}\frac{|\,c(\vec{k})\,|^2}{\omega^2(2\pi)^3}\left[1+\frac{k^2}{2}\frac{v_0^2-(\hat{k}\vec{v}_0)^2}{\left(\omega+\frac{\hbar k^2}{2m}-\vec{k}\vec{v}_0\right)^2}\right]. \quad (5.10)$$

From the Schwartz inequality,

$$\left\|\sum_{\lambda=1,2}\int d^3\vec{k}g(\vec{k},\lambda)a(\vec{k},\lambda)e^{i\vec{k}\vec{x}}\Psi\right\|_{\vec{\pi}}^2 \leqslant \sum_{\lambda=1,2}\int d^3\vec{k}\frac{g(\vec{k},\lambda)g^*(\vec{k},\lambda)}{\hbar\omega}\,\|\,H_0'^{1/2}\Psi\,\|_{\vec{\pi}}^2$$
$$(5.11)$$

with $g(\vec{k}, \lambda)$ either a scalar or vector function, and

$$2 \| H_0'^{1/2}\Psi \|_{\vec{\pi}} \| \Psi \|_{\vec{\pi}} \leqslant \alpha \| H_0'^{1/2}\Psi \|_{\vec{\pi}}^2 + \frac{1}{\alpha} \| \Psi \|_{\vec{\pi}}^2$$

for all $\alpha > 0$, it follows that

$$\left| \langle \Psi, \sum_{\lambda=1,2} \int d^3\vec{k} \left[\left(\hbar\omega + \frac{\hbar^2 k^2}{2m} \right) z(\vec{k}, \lambda) a^\dagger(\vec{k}, \lambda) e^{-i\vec{k}\vec{x}} + \text{h. c.} \right] \Psi \rangle_{\vec{\pi}} \right|$$
$$\leqslant q\alpha_1 c_3' \| H_0'^{1/2}\Psi \|_{\vec{\pi}}^2 + \frac{qc_3'}{\alpha_1} \| \Psi \|_{\vec{\pi}}^2, \quad (5.12)$$

$$c_3'^2 \equiv \sum_{\lambda=1,2} \int d^3\vec{k} \left(\hbar\omega + \frac{\hbar^2 k^2}{2m} \right) \frac{|z(\vec{k}, \lambda)|^2}{q^2 \hbar\omega}$$
$$= \int d^3\vec{k} \left(1 + \frac{\hbar}{2mc} \right) \frac{|c(\vec{k})|^2}{(2\pi)^3} \frac{v_0^2 - (\hat{k}\vec{v}_0)^2}{\left(\omega + \frac{\hbar k^2}{2m} - \vec{k}\vec{v}_0 \right)^2}, \quad (5.13)$$

$$\left| \langle \Psi, \sum_{\lambda,\lambda'=1,2} \int d^3\vec{k} \int d^3\vec{k}' \frac{1}{2m} [\vec{f}(\vec{k}, \lambda) f(\vec{k}', \lambda') a^\dagger(\vec{k}, \lambda) a^\dagger(\vec{k}', \lambda') e^{-i(\vec{k}+\vec{k}')\vec{x}} \right.$$
$$\left. + \text{h.c.}] \Psi \rangle_{\vec{\pi}} \right| \leqslant \frac{q^2}{m} c_1'(c_1' + \alpha_2 c_2' \hbar^{1/2}) \| H_0'^{1/2}\Psi \|_{\vec{\pi}}^2 + \frac{q^2}{m} \frac{c_1' c_2'}{\alpha_2} \hbar^{1/2} \| \Psi \|_{\vec{\pi}}^2, \quad (5.14)$$

$$c_2'^2 \equiv \sum_{\lambda=1,2} \int d^3\vec{k} \frac{\vec{f}(\vec{k}, \lambda) \vec{f}^*(\vec{k}, \lambda)}{\hbar q^2} = \int d^3\vec{k} \frac{|c(\vec{k})|^2}{\omega (2\pi)^3} \left[1 + \frac{k^2}{2} \frac{v_0^2 - (\hat{k}\vec{v}_0)^2}{\left(\omega + \frac{\hbar k^2}{2m} - \vec{k}\vec{v}_0 \right)^2} \right].$$
$$(5.15)$$

Notice that $c_1'^2$, $c_2'^2$ and $c_3'^2$ are infrared finite and depend on $\vec{v}_0$. For $\vec{v}_0 = 0$ they reduce to

$$c_1^2 = \int d^3\vec{k} \frac{|c(\vec{k})|^2}{2\omega^2 (2\pi)^3}, \qquad c_2^2 \equiv \int d^3\vec{k} \frac{|c(\vec{k})|^2}{\omega (2\pi)^3}, \qquad c_3^2 = 0,$$

which are the bounds before introducing the unitary transformation, as can be easily proved. This agrees with the fact that for $\vec{v}_0 = 0$, e^W reduces to the unit operator.

Eqs. (5.8)-(5.15) imply that for any $\alpha_1, \alpha_2 > 0$,

$$|\langle \Psi, H_1' \Psi \rangle_{\vec{\pi}}| \leqslant e_1 \| H_0'^{1/2}\Psi \|_{\vec{\pi}}^2 + e_2 \| \Psi \|_{\vec{\pi}}^2,$$

where

$$e_1 \equiv \frac{q^2}{m} c_1'(2c_1' + \alpha_2 \alpha_1 \hbar^{1/2}) + q\left(\frac{4c_1'}{m} + \alpha_1 c_3' \right), \qquad e_2 \equiv \frac{q^2}{m} \frac{c_1' c_2'}{\alpha_2} \hbar^{1/2} + q \frac{c_3'}{\alpha_1}.$$

Notice that $e_2 > 0$ and that there exist values of α_1, α_2 and q which satisfy $e_1 < 1$. Then, the KLMN theorem [*24*] stablishes that H$'$ is a well defined selfadjoint operator (since H'_0 is selfadjoint).

Once more, we are interested in the ground state $|\Psi'_+(\vec{\pi})\rangle$ of H$'$ (the dressed charged particle) of energy $\mathrm{E} = \mathrm{E}(\vec{\pi})$ and momentum $\vec{\pi}$:

$$(\mathrm{H}' - \mathrm{E})\,|\,\Psi'_+(\vec{\pi})\rangle = 0 \qquad (\vec{\Pi} - \vec{\pi})\,|\,\Psi'_+(\vec{\pi})\rangle = 0\,. \tag{5.16}$$

We shall proceed in the same way as in section 3, this is, we shall regard states (3.7) as unperturbed, H'_I as a perturbation and try to construct $|\Psi'_+(\vec{\pi})\rangle$ from $|\Psi(\vec{\pi})\rangle$ using the Brillouin-Wigner expansion:

$$|\,\Psi'_+(\vec{\pi})\rangle = |\,\Psi(\vec{\pi})\rangle + \mathrm{Q}_{\vec{\pi}}\mathrm{G}'_0(\mathrm{E})\mathrm{H}'_\mathrm{I}\,|\,\Psi'_+(\vec{\pi})\rangle\,, \tag{5.17}$$

$$\mathrm{E} = \frac{1}{2}mv_0^2 + \mathscr{E}_{\mathrm{quan.}} + \mathrm{U} + \langle\,\Psi(\vec{\pi})\,|\,\mathrm{H}'_\mathrm{I}\,|\,\Psi'_+(\vec{\pi})\rangle \equiv \mathrm{M}'(\mathrm{E})\,, \tag{5.18}$$

where

$$\mathrm{Q}_{\vec{\pi}} \equiv \mathbb{1} - \mathrm{P}_{\vec{\pi}}, \qquad \mathrm{P}_{\vec{\pi}} \equiv |\,\Psi(\vec{\pi})\rangle\langle\,\Psi(\vec{\pi})\,|, \qquad \mathrm{G}'_0(z) = (z - \mathrm{H}'_0)^{-1}\,. \tag{5.19}$$

Let us suppose that the equation $\mathrm{E} = \mathrm{M}'(\mathrm{E})$ has a unique solution E which belongs to the spectrum of H$'$. We shall study the behaviour of E when $\hbar \to 0$. For doing it, we write the equation $\mathrm{E} = \mathrm{M}'(\mathrm{E})$ as

$$\mathrm{E} = \frac{1}{2}mv_0^2 + \mathscr{E}_{\mathrm{quan.}} + \mathrm{U} + \left\langle\,\Psi(\vec{\pi})\,\middle|\,\mathrm{H}'_\mathrm{I}\sum_{n=0}^{\infty}[\mathrm{Q}_{\vec{\pi}}\mathrm{G}'_0(\mathrm{E})\mathrm{H}'_\mathrm{I}]^n\,\middle|\,\Psi(\vec{\pi})\right\rangle. \tag{5.20}$$

The series on the r. h. s. of eq. (5.20) is the sum of the contributions of all Feynman diagrams. These contributions are easy to evaluate using the Feynman rules associated to the Brillouin-Wigner expansion for H'_I:

1.*a*) absorption of one photon $(\vec{k}, \lambda)$, fig 5.1:

$$\left(\hbar\omega + \frac{\hbar^2k^2}{2m}\right)z(\vec{k}, \lambda) - \frac{1}{m}\vec{f}(\vec{k}, \lambda)\vec{p}_c$$

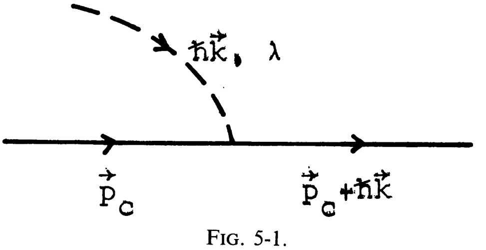


FIG. 5-1.

1.*b*) emission of one photon $(\vec{k}, \lambda)$, fig. 5.2:

$$\left(\hbar\omega + \frac{\hbar^2k^2}{2m}\right)z^*(\vec{k}, \lambda) - \frac{1}{m}\,\vec{p}_c\vec{f}^*(\vec{k}, \lambda)$$

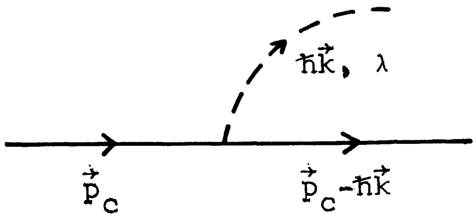


FIG. 5-2.

1.*c*) absorption of one photon $(\vec{k}, \lambda)$ and emission of another one $(\vec{k}, \lambda')$. fig. 5.3:

$$\frac{1}{m}\vec{f}^*(\vec{k}, \lambda)\vec{f}(\vec{k}', \lambda')$$

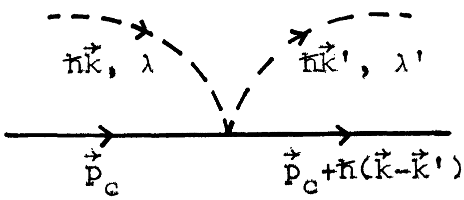


FIG. 5-3.

1.*d*) absorption of two photons $(\vec{k}, \lambda)$ and $(\vec{k}', \lambda')$, fig. 5.4:

$$\frac{1}{2m}\vec{f}^*(\vec{k}, \lambda)\vec{f}^*(\vec{k}', \lambda')$$

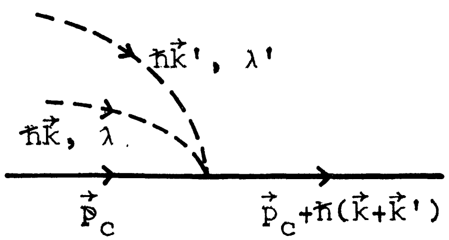


FIG. 5-4.

1.*e*) emission of two photons $(\vec{k}, \lambda)$ and $(\vec{k}', \lambda')$, fig. 5.5:

$$\frac{1}{2m}\vec{f}(\vec{k}, \lambda)\vec{f}(\vec{k}', \lambda')$$

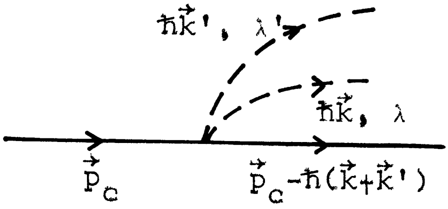
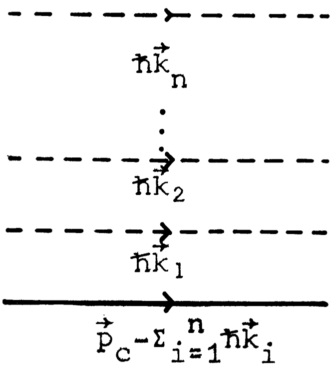


FIG. 5-5.

2) propagator, fig. 5.6:

$$\frac{1}{E - \dfrac{\left(\vec{p}_c - \hbar\sum_{i=1}^{n}\vec{k}_i\right)^2}{2m} - \hbar\sum_{i=1}^{n}\omega_i - \mathscr{E}_{\text{quan.}} - U}$$



where $\vec{p}_c$ stands for the kinetic momentum, being $\vec{P} - \frac{q}{c}\vec{A}$ the kinetic momentum operator. For instance, the contribution of the diagram of fig. 5.7 is

$$\frac{q^6}{4m^3}\sum_{\lambda_1,\lambda_2,\lambda_3=1,2}\int d^3\vec{k}_1\int d^3\vec{k}_2\int d^3\vec{k}_3\,[\vec{f}(\vec{k}_1,\lambda_1)\vec{f}(\vec{k}_2,\lambda_2)]$$
$$[\vec{f}^*(\vec{k}_2,\lambda_2)\vec{f}(\vec{k}_3,\lambda_3)][\vec{f}^*(\vec{k}_3,\lambda_3)\vec{f}^*(\vec{k}_1,\lambda_1)]\times$$
$$\times\frac{1}{\left\{E - \dfrac{[m\vec{v}_0 - \hbar(\vec{k}_1+\vec{k}_2)]^2}{2m} - \hbar(\omega_1+\omega_2) - \mathscr{E}_{\text{quan.}} - U\right\}\left\{E - \dfrac{[m\vec{v}_0 - \hbar(\vec{k}_1+\vec{k}_3)]^2}{2m} - \hbar(\omega_1+\omega_3) - \mathscr{E}_{\text{quan.}} - U\right\}} \quad (5.21)$$

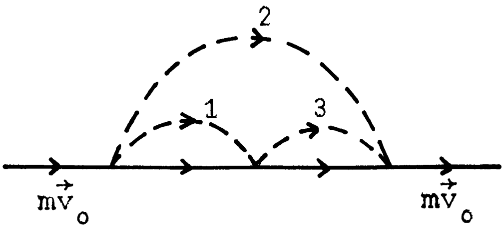


FIG. 5-7.

In order to study the classical limit of each Feynman diagram, we first analyse the behaviour of vertices and propagators in the limit $\hbar \to 0$. The kinetic momentum is, in general, $\vec{p}_c = m\vec{v}_0 - \hbar\Sigma_{i=1}^{l}\vec{k}_i$. From this and eq. (5.5) it follows that vertices 1.a) and 1.b) behaves as

$$\left(\hbar\omega + \frac{\hbar^2k^2}{2m}\right)z(\vec{k}, \lambda) - \frac{1}{m}\vec{f}(\vec{k}, \lambda)\vec{p}_c = \frac{1}{m}\vec{f}(\vec{k}, \lambda)\hbar\sum_{i=1}^{l}\vec{k}_i \underset{\hbar\to 0}{\sim} \hbar^{3/2}$$

$$\left(\hbar\omega + \frac{\hbar^2k^2}{2m}\right)z^*(\vec{k}, \lambda) - \frac{1}{m}\vec{p}_c\vec{f}^*(\vec{k}, \lambda) = \frac{1}{m}\vec{f}^*(\vec{k}, \lambda)\hbar\sum_{i=1}^{l}\vec{k}_i \underset{\hbar\to 0}{\sim} \hbar^{3/2}$$

for any $\vec{v}_0$. Moreover, this implies that if $\vec{p}_c = m\vec{v}_0$, vertices 1.a) and 1.b) give null contribution, so diagrams of the type of figs. 5.8 and 5.9 don't contribute. Eq. (5.5) implies that, for all $\vec{v}_0$, vertices 1.c)-1.e) approach zero as $\hbar$ when $\hbar \to 0$. For the propagator we suppose that

$$\mathrm{E} \to \mathrm{E}_{\text{class.}} = \frac{1}{2}mv_0^2 + \mathscr{E}_{\text{class.}} + \mathrm{U} \qquad \text{for} \quad \hbar \to 0. \qquad (5.22)$$

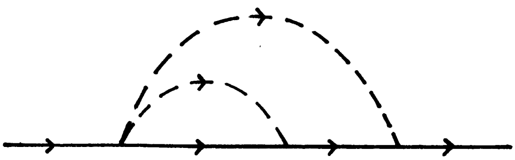


FIG. 5-8.

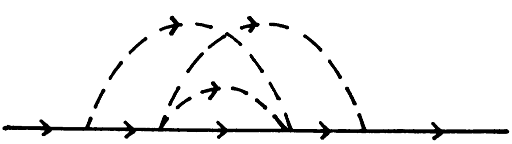


FIG. 5-9.

Now, each diagram containing n vertices has $n-1$ propagators, so, by the same kind of arguments as in section 3, its contribution tends to zero, at least, as $\hbar$ when $\hbar \to 0$ independently of the way in which $E \to E_{\text{class.}}$ for $\hbar \to 0$. From this and the fact that $\mathscr{E}_{\text{quan.}}$ approaches $\mathscr{E}_{\text{class.}}$ if $\hbar \to 0$ it follows that the r. h. s. of eq. (5.20) tends to $\frac{1}{2} m v_0^2 + \mathscr{E}_{\text{class.}} + U$, in agreement with eq. (5.22).

We see that in this new formalism the classical limit of the quantum energy is correct. Notice that for $\vec{v}_0 = 0$, $e^W = 1$, which explains why the correct classical limit for the case $\vec{v}_0 = 0$ was already obtained in section 3.

With respect to the infrared limit, all diagrams are free of divergences. The reason is that, in the low-frequency limit: *i*) vertices behaves in the same way that those of section 3, and, *ii*) if $E = \frac{1}{2} m v_0^2 + \mathscr{E}_{\text{quan.}} + U$, the propagator goes like $(\Sigma_{i=1}^{n} k_i)^{-1}$, and in any other case it tends to something finite, so all what was said at the end of section 3 is valid here. Observe that $\mathscr{E}_{\text{quan.}}$ is infrared finite.

Once we have seen that the classical limit is the correct one, we can construct the state that describes the dressed charged particle. It will be given by the formal solution of eq. (5.17):

$$\left| \Psi'_{+}(\vec{\pi}) \right\rangle = \sum_{n=0}^{\infty} \left[Q_{\vec{\pi}} \frac{1}{E - H'_0} H'_I \right]^n \left| \Psi(\vec{\pi}) \right\rangle \tag{5.23}$$

Notice that after the unitary transformation has been performed we have variational properties. In fact, the variational principle ensures

$$\langle \Psi(\vec{\pi}) \,|\, H' \,|\, \Psi(\vec{\pi}) \rangle = \frac{1}{2} m v_0^2 + \mathscr{E}_{\text{quan.}} + U \geqslant E\,,$$

which is one more argument in favour of the unitary transformation.

6. FORWARD COMPTON EFFECT

Let us consider the Compton effect or scattering of a photon by a charged particle. We shall study the forward case, in which the initial and final velocities of the charged particle coincide, i. e., $\vec{v}_{0,1} = \vec{v}_{0,2} \equiv \vec{v}_0$, which implies $|\Psi'_{1+}\rangle = |\Psi'_{2+}\rangle \equiv |\Psi'_{+}\rangle$, $H'_1 = H'_2 \equiv H'$ and $E_1 = E_2 \equiv E$.

By extending Wick's idea for the scattering of massive mesons by static nucleons [*25*], the scattering solution $|\Psi_{\pm}^{(\cdot)}\rangle$ must consist of two parts: a free photon plus a dressed charged particle, $a^{\dagger}(\vec{k}, \lambda)\,|\,\Psi_{\pm}\rangle$, and a scattering contribution $|\,\chi_{\pm}\rangle$. In Schrödinger picture this is

$$|\,\Psi_{\pm}^{(\cdot)}\rangle = a^{+}(\vec{k}, \lambda)\,|\,\Psi_{\pm}\rangle + |\,\chi_{\pm}\rangle\,. \tag{6.1}$$

Notice that $|\Psi_\pm\rangle$ is the state of the dressed charged particle defined as

$$|\Psi_\pm\rangle = e^{W}|\Psi'_\pm\rangle, \tag{6.2}$$

with $|\Psi'_\pm\rangle$ the state given by eq. (5.23). The state $|\Psi^{(1)}_\pm\rangle$ satisfies

$$[H' - E^{(\cdot)}]|\Psi^{(\cdot)}_\pm\rangle = 0, \qquad E^{(\cdot)} = E + \hbar\omega. \tag{6.3}$$

The matrix element for the scattering operator is given by

$$S_{21} = \langle \Psi^{(1)}_{2-} | \Psi^{(1)}_{1+}\rangle, \tag{6.4}$$

where subindices 1 and 2 stand for initial and final states.

Introducing in eq. (6.4) the unit operator as $e^{W}e^{-W}$ we have

$$S_{21} = \langle \Psi^{(\cdot)}_{2-} | e^{W_2}e^{-W_2}e^{W_1}e^{-W_1} | \Psi^{(1)}_{1+}\rangle = \langle \Psi'^{(1)}_{2-} | e^{-W_2}e^{W_1} | \Psi'^{(1)}_{1+}\rangle, \tag{6.5}$$

$|\Psi'^{(\cdot)}_\pm\rangle$ being the transformed states of $|\Psi^{(\cdot)}_\pm\rangle$ under the unitary transformation (5.1). To calculate them explicitly we use eq. (5.2) with $0 = a^\dagger(\vec{k}, \lambda)$ and eq. (6.2):

$$|\Psi'^{(\cdot)}_\pm\rangle = e^{-W}|\Psi^{(\cdot)}_\pm\rangle = a^\dagger(\vec{k}, \lambda)|\Psi'_\pm\rangle + |\chi'_\pm\rangle, \tag{6.6}$$

where we have defined

$$|\chi'_\pm\rangle \equiv z(\vec{k}, \lambda)|\Psi'_\pm\rangle + e^{-W}|\chi_\pm\rangle. \tag{6.7}$$

From eq. (6.3) it follows that

$$[H' - E^{(\cdot)}]|\Psi'^{(\cdot)}_\pm\rangle = 0. \tag{6.8}$$

Now, eqs. (6.6), (6.8), (5.3), (5.4) and (5.16) yield

$$|\Psi'^{(\cdot)}_\pm\rangle = a^\dagger(\vec{k}, \lambda)|\Psi'_\pm\rangle + \frac{1}{E^{(\cdot)} - H' \pm i0}[H'_I, a^\dagger(\vec{k}, \lambda)]|\Psi'_\pm\rangle. \tag{6.9}$$

Notice that $|\Psi'_+\rangle = |\Psi'_-\rangle$ since $|\Psi'_+\rangle$ is stationary, eq. (5.16).

For the forward case, eq. (6.5) reduces to

$$S_{21}\,(\text{forward}) = \langle \Psi'^{(1)}_- | \Psi'^{(1)}_+\rangle, \tag{6.10}$$

or, equivalently,

$$\begin{aligned} S_{21}\,(\text{forward}) = {} & \langle \Psi'^{(1)}_{2-} | \Psi'^{(1)}_{1-}\rangle \\ & - \langle \Psi'^{(1)}_{2-} | 2\pi i\delta(E^{(1)}_1 - H')[H'_I, a^\dagger(\vec{k}_1, \lambda_1)] | \Psi'_+\rangle, \end{aligned} \tag{6.11}$$

where we have used that

$$\frac{1}{x \pm i0} = \mathrm{PV}\frac{1}{x} \mp i\pi\delta(x). \tag{6.12}$$

Since $H'|\Psi'^{(1)}_{2-}\rangle = E^{(1)}_2|\Psi'^{(1)}_{2-}\rangle$, $E^{(1)}_2 = E + \hbar\omega$, eq. (6.11) can be written as

$$S_{21}\,(\text{forward}) = \delta^{(3)}(\hbar\vec{k}_2 - \hbar\vec{k}_1)\delta_{\lambda_1\lambda_2} - 2\pi i\delta(\hbar\omega_2 - \hbar\omega_1)T_{21}\,(\text{forward}), \tag{6.13}$$

where T_{21} (forward) is given by

$$T_{21}\,(\text{forward}) = \langle \Psi'^{(1)}_{2-} \mid [H'_I, a^\dagger(\vec{k}, \lambda)] \mid \Psi'_+ \rangle . \tag{6.14}$$

Now, using eqs. (5.3) and (5.4) it is easy to prove

$$a(\vec{k}, \lambda) \mid \Psi'_+ \rangle = \frac{1}{H' - E + \hbar\omega} [H'_I, a(\vec{k}, \lambda)] \mid \Psi'_+ \rangle ,$$

that, together with eq. (6.9), gives for eq. (6.14) the form

$$\begin{aligned} T_{21}\,(\text{forward}) &= \langle \Psi'_+ \mid [a(\vec{k}_2, \lambda_2), [H'_I, a^\dagger(\vec{k}_1, \lambda_1)]] \mid \Psi'_+ \rangle + \\ &+ \langle \Psi'_+ \mid \left\{ [H'_I, a^\dagger(\vec{k}_2, \lambda_2)]^\dagger \frac{1}{E + \hbar\omega_1 - H' + i0} [H'_I, a^\dagger(\vec{k}_1, \lambda_1)] \right. \\ &\left. - [H'_I, a^\dagger(\vec{k}_1, \lambda_1)] \frac{1}{E - \hbar\omega_2 - H'} [H'_I, a(\vec{k}_2, \lambda_2)] \right\} \mid \Psi'_+ \rangle . \end{aligned} \tag{6.15}$$

To calculate T_{21} (forward) to any order in the perturbation H'_I we use eqs. (6.15), (5.23) and

$$\frac{1}{z - H'} = \frac{1}{z - H'_0} \sum_{n=0}^{\infty} \left(H'_I \frac{1}{z - H'_0} \right)^n . \tag{6.16}$$

In order to make explicit calculations it is necessary to evaluate the commutators appearing in eq. (6.8):

$$\begin{aligned} &[H'_I, a^\dagger(\vec{k}_1, \lambda_1)] = - [H'_I, a(\vec{k}_1, \lambda_1)]^+ = \\ &= \left[\left(\hbar\omega_1 + \frac{\hbar^2 k_1^2}{2m} \right) z^*(\vec{k}_1, \lambda_1) - \frac{1}{m} f^*(\vec{k}_1, \lambda_1)(\vec{P} - \vec{\Gamma} - \vec{K}) \right] e^{i\vec{k}_1\vec{x}} + \\ &+ \sum_{\lambda=1,2} \int d^3\vec{k} \frac{1}{m} [f(\vec{k}, \lambda) a^\dagger(\vec{k}, \lambda) e^{-i\vec{k}\vec{x}} + \text{h. c.}] f^*(\vec{k}_1, \lambda_1) e^{i\vec{k}_1\vec{x}} \end{aligned} \tag{6.17}$$

and

$$[a(\vec{k}_2, \lambda_2), [H'_I, a^\dagger(\vec{k}_1, \lambda_1)]] = \frac{1}{m} f(\vec{k}_2, \lambda_2) f^*(\vec{k}_1, \lambda_1) e^{i(\vec{k}_1 - \vec{k}_2)\vec{x}} , \tag{6.18}$$

$\vec{f}(\vec{k}, \lambda)$ being given in eq. (5.6).

The Feynman rules are those of section 5, adding to the propagator $+i0$ when it is needed, (i. e., to scattering propagators of diagrams in which the absorption of the photon $(\vec{k}_1, \lambda_1)$ takes place before the emission of the photon $(\vec{k}_2, \lambda_2)$).

We are interested in the classical limit and the infrared problem. To study them we shall use the optical theorem [25]

$$\text{Im}\, T_{21}\,(\text{forward}) = - \frac{\hbar}{2} F\sigma$$

where F is the flux of incoming particles. In our case, they are photons, for which $F = c(2\pi)^{-3}$, so

$$\mathrm{Im}\ T_{21}\ (\text{forward}) = -\frac{\hbar}{2}\frac{c}{(2\pi)^3}\sigma \tag{6.19}$$

Using the Feynman rules of section 5, we find that all scattering diagrams tend to zero for $\hbar \to 0$ at least as $\hbar$. Thus, T_{21} approaches zero, at least, as $\hbar$ in the classical limit. Now, taking into account that on the r. h. s. of eq. (6.19) there is an $\hbar$, we reach the result that the cross section holds finite when $\hbar \to 0$. Notice that the unitary transformation performed in section 5 ensures us the necessary behaviour of the diagrams in the classical limit.

Furthermore, we can recover Thomson's cross section

$$\sigma = \frac{8\pi}{3}r_0^2, \qquad r_0 \equiv \frac{q^2}{4\pi mc^2}, \tag{6.20}$$

the charged particle being initially at rest. According to eq. (6.19), only the imaginary part of T_{21} will contribute to the cross section. This implies that only the second term on the r. h. s. of eq. (6.15) will contribute. Moreover, Thomson's formula (6.20) is of order four in q, so from the diagrams coming from the second term on the r. h. s. of eq. (6.15) we will only consider those whose contributions to T_{21} are of order four or smaller in the electric charge; they are the diagrams of figs. 6.1-6.4. Using the Feynman rules we have that the diagrams of figs. 6.1, 6.3 and 6.4 give

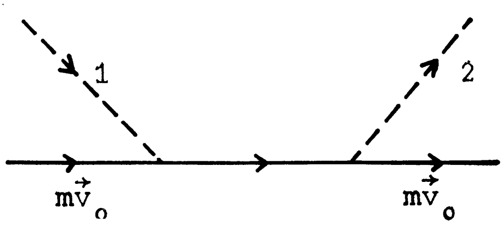


FIG. 6-1.

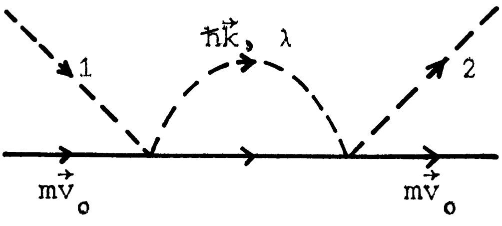


FIG. 6-2.

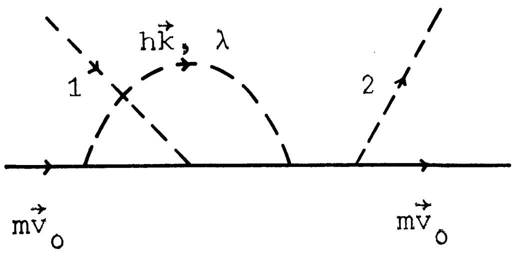


FIG. 6-3.

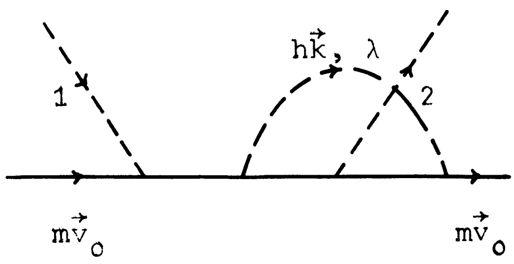


FIG. 6-4.

null contribution and that the contribution of the diagram of fig. 6.3 is given by

$$\frac{1}{m^2}\sum_{\lambda=1,2}\int d^3\vec{k}\,[f^*(\vec{k}_1,\lambda_1)f(\vec{k},\lambda)]$$
$$\frac{1}{\mathrm{E}-\dfrac{[m\vec{v}_0-\hbar(\vec{k}_1-\vec{k}_1)]^2}{2m}-\hbar(\omega-{}_1)-\mathscr{E}_{\text{quan.}}-\mathrm{U}}\,[f^*(\vec{k},\lambda)f(\vec{k}_2,\lambda_2)] \quad (6.21)$$

Now, remembering we are in the forward case ($\vec{k}_1=\vec{k}_2$, $\lambda_1=\lambda_1$) with zero velocity ($\vec{v}_0 = 0$), and making use of eq. (5.6), eq. (6.21) reduces to

$$\frac{q^4}{m^2}\frac{|\,c(\vec{k}_1)\,|^2}{(2\pi)^3}\frac{\hbar}{2\omega_1}\sum_{\lambda=1,2}$$
$$\int d^3\vec{k}\,\frac{|\,c(\vec{k})\,|^2}{(2\pi)^3}\frac{\hbar}{2\omega}\,[\vec{\varepsilon}(\vec{k}_1,\lambda_1\,\vec{\varepsilon}(\vec{k},\lambda)]^2\,\frac{1}{\mathrm{E}-\dfrac{[\hbar(\vec{k}_1-\vec{k})]^2}{2m}-\hbar(\omega-\omega_1)-\mathscr{E}_{\text{quan.}}-\mathrm{U}}$$

Taking its imaginary part we get

$$\mathrm{Im}\,\mathrm{T}_{21}^{(q^4)}\,(\text{forward}) = -\,\pi\frac{q^4}{m^2}\frac{|\,c(\vec{k}_1)\,|^2}{(2\pi)^3}\frac{\hbar}{2\omega_1}\sum_{\lambda=1,2}$$
$$\int d^3\vec{k}\,\frac{|c(\vec{k})|^2}{(2\pi)^3}\frac{\hbar}{2\omega}[\vec{\varepsilon}(\vec{k}_1,\lambda_1)\vec{\varepsilon}(\vec{k},\ \lambda)]^2\delta\left(\mathrm{E}-\frac{\hbar^2(\vec{k}-\vec{k}_1)^2}{2m}-\hbar(\omega-\omega_1)-\mathscr{E}_{\text{quan.}}-\mathrm{U}\right)$$

For charged particles with mass sufficiently large we can neglect $\hbar^2(\vec{k}_1 - \vec{k})^2/2m$. Then, after integrating in spherical coordinates and omitting the cut-off factors, eq. (6.22) takes the form

$$\operatorname{Im} T_{21}^{(q^4)}(\text{forward}) = -\pi \frac{q^4}{m^2} \frac{\vec{\varepsilon}^2(k_1, \lambda_1)}{3(2\pi)^5 c^3} \frac{E - U + \hbar\omega_1}{\omega_1} \qquad (6.23)$$

We know from section 5 that $E = E_0 + \Delta E$, where ΔE stands for the corrections due to the dressing. But these corrections go with powers of q greater than four, so up to order four in q we can neglect them. In particular, if $\vec{v}_0 = 0$, $E_0 = U$, and then, the imaginary part of T_{21} (forward), up to order q^4, is given by

$$\operatorname{Im} T_{21}^{(q^4)}(\text{forward}) = -\frac{1}{2} \sum_{\lambda_1 = 1,2} \pi \frac{q^4}{m^2} \frac{\vec{\varepsilon}^2(\vec{k}_1, \lambda_1)}{3(2\pi)^5 c^3} \hbar \qquad (6.24)$$

The sum $\frac{1}{2}\Sigma_{\lambda_1 = 1,2}$ comes from the fact that we are interested in the unpolarized cross section. From eqs. (6.19) and (6.24) we get eq. (6.20), as we wanted to prove.

With respect to the infrared problem we can say that since each diagram with n vertices has $n-1$ propagators and the low-frequency limit of vertices and propagators is the same as in section 5, all diagrams are infrared finite. Then, T_{21} (forward) is free of infrared divergences, and eq. (6.19) ensures us that so is the total (quantum) cross section.

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