C^1-FINE APPROXIMATION OF FUNCTIONS ON BANACH SPACES WITH UNCONDITIONAL BASIS

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Abstract

We show that if X is a Banach space having an unconditional basis and a C^p-smooth Lipschitz bump function, then for every C^1-smooth function f from X into a Banach space Y, and for every continuous function $\varepsilon : X \to (0, \infty)$, there exists a C^p-smooth function $g : X \to Y$ such that $\|f - g\| \leqslant \varepsilon$ and $\|f' - g'\| \leqslant \varepsilon$.

1. Introduction

Given a Fréchet smooth function f between Banach spaces, we consider in this note the problem of uniformly approximating both f and its derivative by functions with a higher order of differentiability. More generally, if $f : X \to Y$ is a C^k-smooth function between Banach spaces, and $\varepsilon : X \to (0, \infty)$ a continuous map, then we say that f *is* C^k*-fine approximated* by a C^p-smooth function $g : X \to Y$, where $p > k$, if $\left\| f^{(i)}(x) - g^{(i)}(x) \right\| < \varepsilon(x)$ holds for $i = 0, 1, \ldots, k$ on X (where the superscripts (i) on f and g represent the ith Fréchet derivatives). The finite-dimensional case was satisfactorily solved in the classical paper of Whitney [**11**]. The infinite-dimensional setting has proven to be more difficult, and henceforth in this paper all spaces are taken to be infinite-dimensional.

The question of C^0-fine approximation, that is, uniform approximation of continuous functions by smooth functions, has been well investigated over the last several decades and usually relies on the use of smooth partitions of unity. For a survey of some results in this direction see [**2**, Chapter VIII; **5**]. The problem of C^k-fine approximation when $k > 0$ is much less understood and not generally amenable to a solution by partitions of unity. One of the reasons why the standard partitions of unity argument fails to give (even when the identity map is concerned!) a fine approximation by

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smooth functions in infinite-dimensional spaces is that we cannot find a common bound for all the derivatives of the functions of the family composing the partition of unity.

The most fundamental work in this direction has been by Moulis [**10**]. Variations on Moulis's results can be found in [**7**], although there is a gap in the proof of the generalization of [**10**, Theorem 2] claimed in [**8**] and announced in [**6**]. Indeed, in [**6**, **7**] Heble makes a (correct) proof for the C^k-fine approximation of C^k-smooth maps by C^∞ maps on a dense subset D of X, and then he claims to show that in fact $D = X$, but this last part of the proof is wrong. In [**8**], he claims that he can extend the result in [**7**] from D to all of X; this proof is also flawed and it is not clear at all how one could mend it.

In fact, to our knowledge the only complete results on C^k-fine approximation in infinite-dimensional Banach spaces X when $k > 0$ is the work of Moulis, which considers the case where $X = l_p$ for $p \in (1, \infty)$, or $X = c_0$.

The main result of our note is to extend [**10**, Theorem 1] on C^1-fine approximation by C^α-smooth functions in l_p or c_0 to any Banach space which admits an unconditional Schauder basis and a Lipschitz, C^α-smooth bump function. This generalization is sufficient to allow for a characterization of Banach spaces in which C^1-fine approximation by smoother functions is possible within the class of Banach spaces with unconditional bases which admit a C^1-smooth bump function.

The notation we employ is standard, with X, Y, etc. denoting Banach spaces, and X^*, Y^*, etc. their (continuous) duals. The collection of all continuous, linear maps between Banach spaces X and Y is denoted by $\mathcal{L}(X, Y)$. Smoothness in this note is meant in the Fréchet sense. A C^p*-smooth bump function* on X is a C^p-smooth, real-valued function on X with bounded, non-empty support. Most additional notation is explained as it is introduced in the sequel. For any unexplained terms we refer the reader to [**2**, **3**].

2. Main results

THEOREM 1 *Let X be a Banach space with unconditional basis, and Y be an arbitrary Banach space. Assume that X has a C^p-smooth, Lipschitz bump function. Let G be an open subset of X. Then, for every C^1-smooth function $f : G \to Y$ and for every continuous function $\varepsilon : G \to (0, \infty)$, there exists a C^p-smooth function $g : G \to Y$ such that $\| f(x) - g(x)\|_Y \leqslant \varepsilon(x)$ and $\| f'(x) - g'(x)\|_{\mathcal{L}(X,Y)} \leqslant \varepsilon(x)$ for $x \in G$.*

Here, as throughout the paper, $p \in \mathbb{N} \cup \{\infty\}$, $p \geqslant 1$. We will say that the map g is a C^1-fine approximation of f. As noted in the Introduction, this result provides a characterization, within the class of Banach spaces possessing unconditional bases and C^1 -smooth bump functions, of those spaces in which C^1-fine approximation by smoother functions occurs. Specifically we have the following.

COROLLARY 2 *Let X be a Banach space with an unconditional basis and a C^1-smooth bump function, $G \subseteq X$ an open set, and Y a Banach space. The following statements are equivalent:*

(1) *X has a C^p-smooth Lipschitz bump function;*

(2) *every C^1-smooth function $f : G \to Y$ can be C^1-finely approximated by C^p-smooth functions $g : G \to Y$.*

Proof. (1) $\Rightarrow$ (2) is Theorem 1. The proof of (2) $\Rightarrow$ (1) is very simple and does not require fine approximation; it is enough to know that the composition of a C^1 smooth equivalent norm of

X (which always exists under these assumptions) with a suitable real function can be uniformly approximated by a C^p smooth function with a bounded derivative. We leave the details to the reader.

REMARK 3 We do not know whether a Banach space X with no C^1-smooth bump function (for instance, $X = \ell_1$) might have the property that every C^1-smooth function $f : X \to \mathbb{R}$ can be C^1-finely approximated by C^p-smooth functions, with $p \geqslant 2$. Some results on approximation in Banach spaces with no C^1-smooth bump functions can be found in [**4**].

2.1. *Proof of Theorem* 1

We will need to use the following result, which is implicitly proved in [**2**, Proposition II.5.1]; see also [**9**].

PROPOSITION 4 *Let Z be a Banach space. The following assertions are equivalent.*

(1) *Z admits a C^p-smooth Lipschitz bump function.*

(2) *There exist numbers $a, b > 0$ and a Lipschitz function $\psi : Z \to [0, \infty)$ which is C^p-smooth on $Z \setminus \{0\}$, homogeneous (that is $\psi(tx) = |t|\psi(x)$ for all $t \in \mathbb{R}, x \in Z$), and such that $a\|\cdot\| \leqslant \psi \leqslant b\|\cdot\|$.*

For such a function ψ, the set $A = \{z \in Z : \psi(z) \leqslant 1\}$ is what we call a C^p-smooth Lipschitz *starlike body*, and the Minkowski functional of this body, $\mu_A(z) = \inf\{t > 0 : (1/t)z \in A\}$, is precisely the function ψ (see [**1**] and the references therein for further information on starlike bodies and their Minkowski functionals).

We will denote the open (resp. closed) ball of centre x and radius r, with respect to the norm $\|\cdot\|$ of X, by $B(x, r)$ (resp. $\overline{B}(x, r)$). If A is a bounded starlike body of X, we define the *open A-pseudoball* of centre x and radius r as

$$B_A(x, r) := B(x, r; \mu_A) := \{y \in X : \mu_A(y - x) < r\},$$

and we define $\overline{B}_A(x, r)$ to be the closure of $B_A(x, r)$.

According to Proposition 4 and the preceding remarks, because X has a C^p-smooth Lipschitz bump function, there is a bounded starlike body $A \subset X$ whose Minkowski functional $\mu_A = \psi$ is Lipschitz and C^p-smooth on $X \setminus \{0\}$, and there is a number $M \geqslant 1$ such that $(1/M)\|x\| \leqslant \mu_A(x) \leqslant M\|x\|$ for all $x \in X$, and $\|\mu'_A(x)\| \leqslant M$ for all $x \in X \setminus \{0\}$. Notice that in this case we have that

$$B\left(x, \frac{r}{M}\right) \subseteq B_A(x, r) \subseteq B(x, Mr)$$

for every $x \in X, r > 0$.

We next introduce some other notation used throughout the proof. Let $\{e_j, e_j^*\}$ be an unconditional Schauder basis on X, and $P_n : X \to X$ the canonical projections given by $P_n(x) = P_n\left(\sum_{j=1}^{\infty} x_j e_j\right) = \sum_{j=1}^{n} x_j e_j$. Let the unconditional basis constant be $C_1 \geqslant 1$.

Following Moulis [**10**], we put $E_n = P_n(X)$, and $E^\infty = \cup_n E_n$, noting that $\dim E_n = n$ and $\overline{E^\infty} = X$.

In the sequel the symbol $\|\cdot\|$ stands for any of the different norms of the spaces X, X^*, and $\mathcal{L}(X, Y)$.

The following lemma gives us the key to proving Theorem 1.

LEMMA 5 *Let X, Y, G be as in the statement of Theorem 1. There exists a constant $C > 0$, depending only on the space X and the basis constant, such that, for every open ball $B_0 = B(z_0, r_0)$ with $B(z_0, 3r_0) \subseteq G$, and for every C^1 function $f : G \to Y$ and numbers $\varepsilon, \eta > 0$ with $\sup_{x\in B(z_0,2r_0)} \|f'(x)\| < \eta$, there exists a C^p-smooth map $g : G \to Y$ such that*

$$\sup_{x\in B_0} \|f(x) - g(x)\| < C\varepsilon \qquad \text{and} \qquad \sup_{x\in B_0} \|g'(x)\| < C\eta.$$

Proof. We may assume that $z_0 = 0$ and $2r_0 < 1$. Choose $r > 0$ with $r < \min\{\varepsilon/C_1 M\eta,\ r_0/C_1 M\}$. Let $\varphi : \mathbb{R} \to [0, 1]$ be a C^∞-smooth function such that $\varphi(t) = 1$ if $|t| < \frac{1}{2}$, $\varphi(t) = 0$ if $|t| > 1$, $\varphi'(\mathbb{R}) \subseteq [-3, 0]$. We now construct C^1-fine smooth approximations to f on the finite-dimensional subspaces E_n. This classical integral-convolution method already appears in Whitney [**11**], but we follow Moulis [**10**] for consistency.

Consider the map $\hat{f}_n : G \to Y$, defined by

$$\hat{f}_n(x) = \frac{(a_n)^n}{c_n} \int_{E_n} f(x - y)\varphi(a_n \mu_A(y))dy,$$

where we understand that $f(x - y) = 0$ if $x - y \notin G$, $c_n = \int_{E_n} \varphi(\mu_A(y))\, dy$, and we have chosen the constants $a_n > 0$ large enough so that $\hat{f}_n$ is C^1-smooth on $B(z_0, 2r_0)$, and

$$\sup_{x\in B(z_0,2r_0)\cap E_n} \left\| \hat{f}_n(x) - f(x) \right\| < \frac{\varepsilon}{2^n}, \qquad \sup_{x\in B(z_0,2r_0)\cap E_n} \left\| \hat{f}_n'(x) - f'(x) \right\| < \frac{\eta}{2^n}.$$

With these choices one can also check that restricting $\hat{f}_n$ to E_n gives rise to a C^p-smooth map.

We next define a sequence of functions $\overline{f}_n : X \to Y$ as follows. Put $\bar{f}_0 = f(0)$, and supposing that $\overline{f}_0, \ldots, \overline{f}_{n-1}$ have been defined, we set

$$\bar{f}_n(x) = \hat{f}_n(x) + \bar{f}_{n-1}(P_{n-1}(x)) - \hat{f}_n(P_{n-1}(x)).$$

One can verify by induction that

(i) the restriction of $\bar{f}_n$ to E_n is C^p-smooth and $\bar{f}_n$ is an extension of $\bar{f}_{n-1}$;

(ii) $\sup_{x\in E_n\cap B(z_0,2r_0)} \left\| \bar{f}_n(x) - f(x) \right\| < 2\varepsilon(1 - 1/2^n)$;

(iii) $\sup_{x\in E_n\cap B(z_0,2r_0)} \left\| \bar{f}_n'(x) - f'(x) \right\| < 2\eta(1 - 1/2^n)$.

We now define the map $\bar{f} : E^\infty \to Y$ by

$$\bar{f}(x) = \lim_{n\to\infty} \bar{f}_n(x).$$

FACT 6 *The function $\bar{f}$ has the following properties:*

(i) *the restriction of $\bar{f}$ to every subspace E_n is C^p-smooth;*

(ii) $\sup_{x\in E^\infty\cap B(z_0,2r_0)} \left\| \bar{f}(x) - f(x) \right\| < 2\varepsilon$;

(iii) $\sup_{x\in E^\infty\cap B(z_0,2r_0)} \left\| \bar{f}'(x) - f'(x) \right\| < 2\eta$.

This is easily checked by using properties (i)–(iii) above.

Next, let us write $x = \sum_n x_n e_n \in X$, and define the map

$$\chi_n(x) = 1 - \varphi\left[\frac{\mu_A(x - P_{n-1}(x))}{r}\right]$$

(here we use the convention that $P_0 = 0$), and now set

$$\Psi(x) = \sum_n \chi_n(x)\, x_n e_n.$$

FACT 7 *The mapping $\Psi : X \to E^\infty$ is well defined, C^p-smooth on X, and has the following properties:*

(1) $\|\Psi'(x)\| \leqslant 4M^2 C_1(1 + C_1)$ *for all* $x \in X$*;*

(2) $\|x - \Psi(x)\| \leqslant C_1 M r$ *for all* $x \in X$*;*

(3) $\Psi(B_0) \subseteq B(z_0, 2r_0)$.

Proof. For any x_0, because $P_n(x_0) \to x_0$ and the $\|P_n\|$ are uniformly bounded, there exist a neighbourhood N_0 of x_0 and an n_0 such that $\chi_n(x) = 0$ for all $x \in N_0$ and $n \geqslant n_0$, and so $\Psi(N_0) \subset E_{n_0}$. Thus, $\Psi : X \to E^\infty$ is a well-defined C^p-smooth map. We next estimate its derivative.

We have that

$$(\chi_n(x)\, x_n)' = \chi_n'(x)\, x_n + \chi_n(x)\, e_n^*.$$

Now, since $|\varphi'(t)| \leqslant 3$, $\|\mu_A'(x)\| \leqslant M$ and $\|(I - P_{n-1})'(x)\| \leqslant 1 + C_1$ for all x, t, we get that, for any n,

$$\begin{aligned}\left\|\chi_n'(x)\right\| &\leqslant \left|\varphi'\left(\frac{\mu_A(x - P_{n-1}(x))}{r}\right)\right| \cdot r^{-1} \left\|\mu_A'(x - P_{n-1}(x))\right\| \cdot \left\|(I - P_{n-1})'(x)\right\| \\ &\leqslant 3M(1 + C_1) r^{-1}.\end{aligned}$$

Consider now the derivative of the map Ψ. We have

$$\Psi'(x)(\cdot) = \sum_n \chi_n'(x)(\cdot) x_n e_n + \sum_n \chi_n(x)\, e_n^*(\cdot).$$

For a fixed x, define $n_0 = n_0(x)$ to be the smallest integer with $\mu_A\big(x - P_{n_0-1}(x)\big) \leqslant r$. Then for all $m < n_0$, $\chi_m(x) = 1$, $\chi_m'(x) = 0$, and so, using [**3**, Lemma 6.33] since $\{e_n\}$ is unconditional with basis constant C_1, and our estimate above, we have that for every $h \in B_X$,

$$\begin{aligned}\left\|\Psi'(x)(h)\right\| &\leqslant \left\|\sum_n \chi_n'(x)(h)\, x_n e_n\right\| + \left\|\sum_n \chi_n(x)\, h_n e_n\right\| \\ &= \left\|\sum_{n \geqslant n_0} \chi_n'(x)(h)\, x_n e_n\right\| + \left\|\sum_n \chi_n(x)\, h_n e_n\right\| \\ &\leqslant C_1 \sup_n \left|\chi_n'(x)(h)\right| \left\|\sum_{n \geqslant n_0} x_n e_n\right\| + C_1 \sup_n |\chi_n(x)| \left\|\sum_n h_n e_n\right\|\end{aligned}$$

$$\leqslant 3C_1M(1+C_1)r^{-1}\|h\|\left\|\sum_{n\geqslant n_0}x_ne_n\right\|+C_1\|h\|$$
$$=3C_1M(1+C_1)r^{-1}\|h\|\left\|x-P_{n_0-1}(x)\right\|+C_1\|h\|$$
$$\leqslant 3C_1M(1+C_1)r^{-1}\|h\|\,M\mu_A\left(x-P_{n_0-1}(x)\right)+C_1\|h\|$$
$$\leqslant 3C_1M(1+C_1)r^{-1}\|h\|\,Mr+C_1\|h\|\leqslant 4M^2C_1(1+C_1)\|h\|,$$

which yields (1).

We next estimate $\|x-\Psi(x)\|$. We have, again using [**3**, Lemma 6.33] since $\{e_n\}$ is unconditional with basis constant C_1, and with $n_0=n_0(x)$ as above,

$$\|x-\Psi(x)\|=\left\|\sum_{n\geqslant n_0}x_n(1-\chi_n(x))\,e_n\right\|\leqslant C_1\sup_n|1-\chi_n(x)|\left\|\sum_{n\geqslant n_0}x_ne_n\right\|$$
$$\leqslant C_1\left\|\sum_{n\geqslant n_0}x_ne_n\right\|\leqslant C_1M\mu_A\left(x-P_{n_0-1}(x)\right)\leqslant C_1Mr,$$

which proves (2). Lastly, property (3) is immediate from (2) and the choice of r.

To end the proof of the lemma, we define

$$g(x)=\bar{f}(\Psi(x)).$$

Note that g is C^p-smooth on G, being the composition of C^p-smooth maps. Also we have that, for every $x\in B_0$, according to Facts 6, 7 and the choice of r,

$$\|f(x)-g(x)\|\leqslant\|f(x)-f(\Psi(x))\|+\left\|\bar{f}(\Psi(x))-f(\Psi(x))\right\|$$
$$\leqslant\eta\|x-\Psi(x)\|+\left\|\bar{f}(\Psi(x))-f(\Psi(x))\right\|$$
$$\leqslant\eta C_1Mr+2\varepsilon<3\varepsilon.$$

Lastly, we have, again using Facts 6 and 7, that, for $x\in B_0$,

$$\left\|g'(x)\right\|\leqslant\left\|\bar{f}'(\Psi(x))\right\|\left\|\Psi'(x)\right\|$$
$$\leqslant\left(\left\|f'(\Psi(x))\right\|+2\eta\right)4M^2C_1(1+C_1)\leqslant 12M^2C_1(1+C_1)\eta.$$

This establishes the lemma with $C=12M^2C_1(1+C_1)$.

Now we finish the proof of Theorem 1. Using separability and openness of G, as well as continuity of the functions f' and ε, we let $\left\{B(x_j,r_j/M)\right\}_{j=1}^{\infty}$ be a covering of G by open balls with centres x_j and radii r_j/M, with $B(x_j,3Mr_j)\subset G$, and such that $\|T_j'(x)-f'(x)\|<\varepsilon_j/8C$ and $\varepsilon(x)\geqslant\varepsilon_j/2$ for all $x\in B(x_j,2Mr_j)$, where T_j is the first-order Taylor polynomial to f at x_j and $\varepsilon_j=\varepsilon(x_j)$ (note in particular that $T_j'(x)$ is simply $f'(x_j)$).

Since $B(x,r/M)\subseteq B_A(x,r)\subseteq B(x,Mr)$ for every x,r, we have that

$$G=\bigcup_{j=1}^{\infty}B_A(x_j,r_j),\ \text{and}\ \left\|T_j'(x)-f'(x)\right\|<\frac{\varepsilon_j}{8C}\ \text{on}\ B_A(x_j,2r_j).$$

Next, let $\varphi_j \in C^p(X,[0,1])$ with bounded derivative so that $\varphi_j = 1$ on $B_A(x_j, r_j)$ and $\varphi_j = 0$ outside $B_A(x_j, 2r_j)$ (such a function can easily be defined as $\varphi_j(x) = \theta_j(\mu_A(x - x_j))$, where θ_j is a suitable smooth real function). Now, via Lemma 5, we may choose C^p-smooth maps $\delta_j : G \to Y$ such that on each ball $B(x_j, 2Mr_j)$ we have both $\|T_j(x) - f(x) - \delta_j(x)\| < 2^{-j-2}\varepsilon_j M_j^{-1}$, and $\left\|\delta_j'(x)\right\| < \varepsilon_j/8$, where $M_j = \sum_{k=1}^{j} \widetilde{M}_k$ and $\widetilde{M}_k = \sup_{x\in B(x_k,2Mr_k)} \left\|\varphi_k'(x)\right\|$. Then we also have

$$\left\|T_j'(x) - f'(x) - \delta_j'(x)\right\| \leqslant \left\|T_j'(x) - f'(x)\right\| + \left\|\delta_j'(x)\right\| < \varepsilon_j/8C + \varepsilon_j/8 \leqslant \varepsilon_j/4.$$

Next, we define

$$h_j = \varphi_j \prod_{k<j}(1-\varphi_k) \quad \text{and} \quad g(x) = \sum_j h_j(x)\left(T_j(x) - \delta_j(x)\right).$$

Note that for each x, if $n := n(x) := \min\{m : x \in B_A(x_m, r_m)\}$ then, because $1 - \varphi_n(x) = 0$ and $B_A(x_n, r_n)$ is open, it follows from the definition of the h_j that there is a neighbourhood N of x such that for $y \in N$, $g(y) = \sum_{j\leqslant n} h_j(y)\left(T_j(y) - \delta_j(y)\right)$, and $\sum_j h_j(y) = \sum_{j\leqslant n} h_j(y)$. Also, by a straightforward calculation, again using the fact that $\varphi_n = 1$ on $B_A(x_n, r_n)$, we have that $\sum_j h_j(y) = 1$ for $y \in B_A(x_n, r_n)$ (hence for every $y \in G$).

Now, fix any $x_0 \in G$, and let $n_0 = n(x_0)$ and a neighbourhood N_0 of x_0 be as above. Then for any $x \in N_0$, since $\operatorname{supp}(h_j) \subseteq B_A(x_j, 2r_j) \subseteq B(x_j, 2Mr_j)$,

$$\begin{aligned}
\|g(x) - f(x)\| &= \left\| \sum_{j\leqslant n_0} h_j(x)\left(T_j(x) - \delta_j(x)\right) - f(x) \right\| \\
&= \left\| \sum_{j\leqslant n_0} h_j(x)\left(T_j(x) - \delta_j(x)\right) - \sum_{j\leqslant n_0} h_j(x)\, f(x) \right\| \\
&\leqslant \sum_{j\leqslant n_0} h_j(x) \left\|\left(T_j(x) - f(x) - \delta_j(x)\right)\right\| \\
&< \sum_{j\leqslant n_0} h_j(x)\,\frac{\varepsilon_j}{4} \leqslant \varepsilon(x).
\end{aligned}$$

A straightforward calculation shows that $\left\|h_j'(x)\right\| \leqslant M_j$, and so we have

$$\begin{aligned}
&\left\|g'(x) - f'(x)\right\| \\
&= \left\| \sum_{j\leqslant n_0} h_j'(x)\left(T_j(x) - f(x) - \delta_j(x)\right) + h_j(x)\left(T_j(x) - f(x) - \delta_j(x)\right)' \right\| \\
&\leqslant \sum_{j\leqslant n_0} \left\|h_j'(x)\right\| \left\|T_j(x) - f(x) - \delta_j(x)\right\| + \sum_{j\leqslant n_0} h_j(x) \left\|\left(T_j(x) - f(x) - \delta_j(x)\right)'\right\| \\
&< \sum_{j\leqslant n_0,\ x\in B_A(x_j,2r_j)} M_j\left(2^{-j-2}\varepsilon_j M_j^{-1}\right) + \sum_{j\leqslant n_0} h_j(x)\,\frac{\varepsilon_j}{4} \\
&\leqslant \sum_{j\leqslant n_0,\ x\in B(x_j,2Mr_j)} 2^{-j}\,\frac{\varepsilon_j}{4} + \frac{\varepsilon(x)}{2} \leqslant \varepsilon(x).
\end{aligned}$$

REMARK 8 By using Moulis's ideas [**10**] and some refinements of the techniques deployed above, one can also show the following result: if X is a Banach space with an unconditional basis and a C^∞ smooth bump function with bounded derivatives, then every C^{2k-1}-smooth function can be C^k-finely approximated by C^∞-smooth functions. We do not feel that this statement justifies the inclusion of its (necessarily technically involved) proof in this note. It is also worth recalling that the assumption that the second derivative of a bump function is bounded is very strong and implies super-reflexivity of the space.

Of course, the natural problem as to whether C^k functions can be C^k-finely approximated by C^∞ functions on such spaces X remains open, even in the case where $X = \ell_2$.

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