UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS
Departamento de Álgebra

TESIS DOCTORAL

The Riemann-Roch theorem and Gysin morphism in arithmetic geometry

El teorema de Riemann-Roch y el morfismo de Gysin en geometría aritmética

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Alberto Navarro Garmendia

Directores

Fréderic Deglisé
José Ignacio Burgos Gil

Madrid, 2016

UNIVERSIDAD COMPLUTENSE DE MADRID

Facultad de Ciencias Matemáticas

Departamento de Álgebra

The Riemann-Roch theorem and Gysin morphism in arithmetic geometry

---

El teorema de Riemann-Roch y el morfismo de Gysin en geometría aritmética

Memoria de tesis doctoral presentada para optar al grado de

Doctor en Ciencias Matemáticas

Alberto Navarro Garmendia

Bajo la dirección de:

Dr. Frédéric Déglise y Dr. José Ignacio Burgos Gil

Madrid, 2015

Contents

Introduction

The original Grothendieck's Riemann-Roch theorem states that for any proper morphism $f\colon Y \to X$, between nonsingular quasiprojective irreducible varieties over a field, and any element $a \in K_0(Y)$ of the Grothendieck group of vector bundles the relation

$$\mathrm{ch}(f_!(a)) = f_*\big(\mathrm{Td}(T_f) \cdot \mathrm{ch}(a)\big)$$

holds (*cf.* [BS58]). Recall that ch denotes the Chern character, $\mathrm{Td}(T_f)$ the Todd class of the relative tangent bundle and f_* and $f_!$ the direct image in the Chow ring and K_0 respectively. Later Baum, Fulton and MacPherson proved in [BFM75] the Riemann-Roch theorem for locally complete intersection morphisms between singular projective algebraic schemes (i.e., locally of finite type separated schemes over a field). In [FG83] Fulton and Gillet proved the theorem without projective assumptions on the schemes.

The remarkable extension to higher K-theory and schemes over a regular base was proved by Gillet in [Gil81]. The Riemann-Roch theorem proved there is for projective morphisms between smooth quasiprojective schemes. However, note that in the case over a field Gillet's theorem does not recover the result of [BFM75]. The furthest generalization of the Riemann-Roch theorem I know is [Dég14] and [HS15] where Déglise and Holmstrom-Scholbach independently obtained the Riemann-Roch theorem for higher K-theory and projective lci morphisms between regular schemes over a finite dimensional noetherian base.

After the work of Cisinski in [Cis13] on Weibel's homotopy invariant K-theory one may apply Voevodsky's motivic homotopy theory to it. With Cisinski's result, we present a Riemann-Roch theorem for homotopy invariant K-theory and projective lci morphisms without smoothness assumptions on the schemes. More concretely, the theorem we prove for motivic cohomology and schemes over a finite dimensional noetherian base S is the following:

Theorem: *Let $f\colon Y \to X$ be a projective lci morphism of S-schemes and denote $T_f \in K^0(Y)$ the virtual tangent bundle and* Td *the multiplicative extension of the series given by $\frac{t}{1-e^{-t}}$. Then the diagram*

$$\begin{tikzcd} KH(Y)_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}(T_f)\mathrm{ch}"'] & KH(X)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ H_{\mathcal{M}}(Y, \mathbb{Q}) \arrow[r, "f_*"] & H_{\mathcal{M}}(X, \mathbb{Q}) \end{tikzcd}$$

commutes. In other words, for $a \in KH(Y)_{\mathbb{Q}}$ we have

$$\mathrm{ch}(f_*(a)) = f_*\big(\mathrm{Td}(T_f) \cdot \mathrm{ch}(a)\big).$$

From here, we deduce Riemann-Roch theorems for many cohomologies. In particular, for real absolute Hodge and Deligne Beilinson cohomology, rigid syntomic cohomology and mixed Weil cohomologies such as algebraic de Rham and geometric étale cohomology.

In order to prove this result, the addition we make to the theory is the construction of the Gysin morphism for regular immersions and every cohomology given by spectra. Since its beginnings, the standard construction of the Gysin morphism in motivic homotopy theory relies on the Thom space and the purity isomorphism. However, purity requires smoothness assumptions (*cf.* [MV99]). Our approach is a different one: We lift the work of Gabber for étale cohomology to the motivic homotopy setting and, thus, obtain Gysin morphisms for regular immersions without smoothness assumptions on the schemes. This leads to the construction of new Gysin morphisms for many theories like homotopy invariant K-theory, motivic cohomology, real absolute Hodge and Deligne-Beilinson cohomology, rigid syntomic cohomology, and any cohomology coming from a mixed Weil theory.

Modules over a cohomology theory are notable geometric and arithmetic invariants. Recall from [HS15] that arithmetic K-theory and arithmetic motivic cohomology are modules over K-theory and motivic cohomology respectively. In addition, for any cohomology the relative cohomology of a morphism is also module. Note that the cohomology with proper support, the cohomology with support on a closed subscheme, and the reduced cohomology are the relative cohomology of a closed immersion, an open immersion and the projection over a base point respectively.

We deduce from our results new Gysin morphisms and a Riemann-Roch theorem for abstract modules. In particular, we prove a new Riemann-Roch theorem for relative cohomology.

Theorem: *Let $f\colon Y \to X$ be a morphism of schemes, $g\colon T \to X$ be a projective lci morphism. Denote $f_T\colon Y \times_X T \to T$, $T_g \in K_0(Y)$ the virtual tangent bundle of g and $KH(f)$ and $H_{\mathcal{M}}(f, \mathbb{Q})$ the relative homotopy invariant K-theory and motivic cohomology of f respectively. Assume in addition either f is proper or g is smooth, then the diagram*

$$\begin{tikzcd} KH(f_T)_{\mathbb{Q}} \arrow[r, "g_*"] \arrow[d, "\mathrm{Td}(T_g)\mathrm{ch}"'] & KH(f)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ H_{\mathcal{M}}(f_T, \mathbb{Q}) \arrow[r, "g_*"] & H_{\mathcal{M}}(f, \mathbb{Q}) \end{tikzcd}$$

commutes. In other words, for $m \in KH(f_T)_{\mathbb{Q}}$ we have

$$\mathrm{ch}(g_*(m)) = g_*\big(\mathrm{Td}(T_g) \cdot \mathrm{ch}(m)\big).$$

We also obtain the arithmetic Riemann-Roch theorem of [HS15] and the residual Riemann-Roch theorem of [Dég14].

Finally, we improve a classic formula on this subject: the Riemann-Roch without denominators. Grothendieck conjectured it itself in [SGA6], it was first proved by Jouanolou in [Jou70] for smooth quasiprojective schemes and later generalized into the singular context ([Ful98]). The furthest generalization I know is Gillet's statement in [Gil81] for higher K-theory and smooth quasiprojective schemes over a regular base. Our theorem is proved once again in the singular context and without projective assumptions on the schemes.

Theorem: *Let $i\colon Z \to X$ be a regular immersion of codimension d. Denote $\mathfrak{q}_i\colon KH(Z) \to KH_Z(X)$ and $\mathfrak{p}_i\colon H_{\mathcal{M}}(Z, \mathbb{Z}) \to H_{\mathcal{M},Z}(X, \mathbb{Z})$ the refined Gysin morphisms, $c^Z_{q,r}\colon KH_{r,Z}(X) \to H_{\mathcal{M},Z}(X, \mathbb{Z})$ the r-th Chern class with support on Z (cf. 4.4.1 and Definition 4.2.8 respectively) and $P^d_q(r, x_1, \cdots ; y_1, \cdots)$ the polynomials with integers coefficients defined in [Jou70]. Then for any $a \in KH_r(Z)$ we have*

$$c^Z_{q,r}(\mathfrak{q}_i(a)) = \mathfrak{p}_i\big(P^d_q(\mathrm{rk}(a), c_{1,r}(a), \ldots, c_{q-d,r}(a); c_1(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X}))\big).$$

We describe the organization of this dissertation.

1. Grothendieck-Riemann-Roch

Using Panin's framework from [Pan04] we prove that Grothendieck's K-theory is the universal cohomology theory on smooth varieties over a field with Chern classes following the multiplicative law $x + y - xy$. We deduce the Grothendieck-Riemann-Roch theorem for the Grothendieck group of vector bundles and for the graduated of the K-theory by the support codimension filtration. Although no new formula is proven in this chapter, I believe it contains the main ideas of this dissertation.

2 and 3. Preliminaries and Motivic homotopy theory

We give an introduction to motivic homotopy theory for algebraic geometers. We put emphasis into the basic functoriality of the big Nisnevich site and the classification of torsors. We do not pretend to be neither complete nor self contained and most proofs are omitted. The only result not found in the literature is the classification of pseudo divisors in Proposition 3.1.43. Regarding the classification of torsors, we include complete proofs.

4. Riemann-Roch theorems and Gysin morphism

In section 4.1 we recall the notion of absolute oriented ring spectra and module over an absolute ring spectrum. We introduce the relative cohomology in the context of motivic stable homotopy theory and construct an absolute spectrum which represents relative cohomology under some conditions. In section 4.2 we construct Gysin morphisms using Gabber's ideas for the case of regular immersions and prove a unicity criteria. We prove the motivic Riemann-Roch theorem in section 4.3 (*cf.* Theorem 4.3.7) and obtain the Riemann-Roch theorem for homotopy invariant K-theory as a result. We deduce a Riemann-Roch theorem for modules and therefore obtain an arithmetic Riemann-Roch theorem and the Riemann-Roch theorem for relative cohomology. Finally, in section 4.4 we deal with the Riemann-Roch without denominators.

5. Appendix

The Appendix is devoted to the explicit construction of the real absolute Hodge spectrum using Burgos' complex (*cf.* [Bur98]) and to check that this spectrum, as well as the Deligne-Beilinson spectrum of [HS15], represent their cohomologies also in the singular context.

Introduction

Le théorème de Riemann-Roch originale affirme que pour tout morphisme propre $f\colon Y \to X$ entre variétés quasi-projectifs lisses sur un corps, et tout élément $a \in K_0(Y)$ du groupe de Grothendieck des fibrés vectoriels on a

$$\mathrm{ch}(f_!(a)) = f_*\big(\mathrm{Td}(T_f) \cdot \mathrm{ch}(a)\big)$$

(*cf.* [BS58]). Ici ch est le caractère de Chern, $\mathrm{Td}(T_f)$ est la classe de Todd du fibré tangent relative et f_* et $f_!$ sont les images directes de l'anneau de Chow et K_0 respectivement. Après, Baum, Fulton et MacPherson ont démontré en [BFM75] le théorème de Riemann-Roch pour des morphismes localement intersection complète entre des schémas algébriques (schémas séparés et localement de type fini sur un corps) projectifs et singulières. En [FG83] Fulton et Gillet ont démontré le théorème sans hypothèses projectifs.

L'extension à la théorie K supérieure pour des schémas régulières sur une base fut démontré par Gillet en [Gil81]. Le théorème de Riemann-Roch qu'il prouve est pour des morphismes projectifs entre des schémas lisses et quasi-projectifs. Donc, dans le cas des schémas sur un corps, le résultat de Gillet n'inclus pas le théorème de [BFM75]. La plus grande généralisation du théorème de Riemann-Roch que je connais est [Dég14] et [HS15], où Déglise et Holmstrom-Scholbach obtiennent indépendamment le théorème de Riemann-Roch pour la K-théorie supérieure et les morphismes projectifs lic entre schémas régulières sur une base noetherienne de dimension finie.

Après les travaux de Cisinski en [Cis13] sur la K-théorie homotopiquement invariant de Weibel on peut appliquer la théorie de l'homotopie des schémas. À l'aide des résultats de Cisinski, nous présentons un théorème de Riemann-Roch pour la K-théorie homotopiquement invariant et les morphismes projectifs lic sans hypothèses projectifs sur les schémas. Le théorème que nous prouvons pour la cohomologie motivique et les schémas de dimension finie sur une base S est

Théorème: *Soit $f\colon Y \to X$ un morphisme projectif lic entre des S-schémas et soit $T_f \in K^0(Y)$ le fibré tangent virtuel et* Td *l'extension multiplicatif de la série $\frac{t}{1-e^{-t}}$. Alors le diagramme*

$$\begin{tikzcd} KH(Y)_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}(T_f)\mathrm{ch}"'] & KH(X)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ H_{\mathcal{M}}(Y,\mathbb{Q}) \arrow[r, "f_*"] & H_{\mathcal{M}}(X,\mathbb{Q}) \end{tikzcd}$$

est commutative. C'est à dire, pour tout $a \in KH(Y)_{\mathbb{Q}}$ on a

$$\mathrm{ch}(f_*(a)) = f_*\big(\mathrm{Td}(T_f)\cdot \mathrm{ch}(a)\big).$$

On en déduit des théorèmes de Riemann-Roch pour beaucoup de cohomologies. En particulier, pour la cohomologie de Deligne-Beilinson et la cohomologie absolu de Hodge réelle, pour la cohomologie rigide syntomique et pour les cohomologies de Weil mixtes, comme la cohomologie de deRham et la cohomologie étale géométrique.

Pour démontrer ce résultat, notre contribution à la théorie est la construction du morphisme de Gysin pour des immersions régulières et tout cohomologie définie par des spectres. La construction usuelle du morphisme de Gysin dans la théorie homotopique des schémas utilise l'espace de Thom et l'isomorphisme de pureté. Mais la pureté exige des hypothèses de lissitude (*cf.* [MV99]). Notre approche est différente: on étend les idées de Gabber pour la cohomologie étale à l'homotopie des schémas. Cela nous permet de construire le morphisme de Gysin pour des immersions régulières sans des hypothèses de lissitude sur les schémas. On obtient des nouveaux morphismes de Gysin pour beaucoup des théories: la K-théorie homotopiquement invariant, la cohomologie motivique, la cohomologie absolu de Hodge et de Deligne-Beilinson réelle, la cohomologie rigide syntomique, et tout cohomologie de Weil mixte.

Les modules sur un théorie cohomologique sont des invariants arithmétiques et géométriques remarquables. La K-théorie arithmétique et la cohomologie motivique arithmétique sont des modules sur la K-théorie et la cohomologie motivique respectivement (*cf.* [HS15]). Aussi, la cohomologie relative d'un morphisme est toujours un module. Il faut remarquer que la cohomologie à supportes propres, la cohomologie à supportes dans un fermé, et la cohomologie réduite, sont la cohomologie relative d'une immersion fermée, d'une immersion ouverte, et de la projection sur un point respectivement.

On déduit de nos résultats des morphismes de Gysin et un théorème de Riemann-Roch pour des modules en général. En particulier, on prouve un nouveau théorème de Riemann-Roch pour la cohomologie relative.

Théorème: *Soit $f\colon Y \to X$ un morphisme de schémas et $g\colon T \to X$ un morphisme projectif lic. Soit $f_T\colon Y \times_X T \to T$, $T_g \in K_0(Y)$ le fibré tangent virtuel de g et $KH(f)$ et $H_{\mathcal{M}}(f, \mathbb{Q})$ la K-théorie homotopiquement invariant et la cohomologie motivique relative de f respectivement. Si f est propre ou g est lisse, alors le diagramme*

$$\begin{tikzcd}
KH(f_T)_{\mathbb{Q}} \arrow[r, "g_*"] \arrow[d, "\mathrm{Td}(T_g)\mathrm{ch}"'] & KH(f)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\
H_{\mathcal{M}}(f_T, \mathbb{Q}) \arrow[r, "g_*"] & H_{\mathcal{M}}(f, \mathbb{Q})
\end{tikzcd}$$

est commutative. C'est à dire, pour tout $m \in KH(f_T)_{\mathbb{Q}}$ on a

$$\mathrm{ch}(g_*(m)) = g_*\big(\mathrm{Td}(T_g) \cdot \mathrm{ch}(m)\big).$$

On obtient aussi le théorème de Riemann-Roch arithmétique de [HS15] et le théorème de Riemann-Roch résiduel de [Dég14].

Finalement, on améliore une formule classique: le théorème de Riemann-Roch sans dénominateurs. Conjecturé par Grothendieck en [SGA6], il était démontré par Jouanolou en [Jou70] pour des schémas projectifs et lisses. Après, il fut généralisé aux cas singulier ([Ful98]). La plus grande généralisation que je connais est celle de Gillet en [Gil81] pour la K-théorie supérieure et des schémas lisses et quasi-projectifs sur une base régulière. Notre théorème est démontré dans un contexte singulier et sans hypothèses de projectivité sur les schémas.

Théorème: *Soit $i\colon Z \to X$ une immersion régulière de codimension d. On denote $\mathfrak{q}_i\colon KH(Z) \to KH_Z(X)$ et $\mathfrak{p}_i\colon H_{\mathcal{M}}(Z, \mathbb{Z}) \to H_{\mathcal{M},Z}(X, \mathbb{Z})$ les morphismes de Gysin raffinés, $c^Z_{q,r}\colon KH_{r,Z}(X) \to H_{\mathcal{M},Z}(X, \mathbb{Z})$ la r-ème classe de Chern à support dans Z (cf. 4.4.1 et la définition 4.2.8 respectivement), et soient $P^d_q(r, x_1, \cdots; y_1, \cdots)$ les polynômes à coefficients entières définis en [Jou70]. Alors pour tout $a \in KH_r(Z)$ on a*

$$c^Z_{q,r}(\mathfrak{q}_i(a)) = \mathfrak{p}_i\big(P^d_q(\mathrm{rk}(a), c_{1,r}(a), \ldots, c_{q-d,r}(a); c_1(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X}))\big).$$

Voilà l'organisation de cette mémoire:

1. Grothendieck-Riemann-Roch

On utilise les axiomes de Panin [Pan04] et on prouve que la K-théorie de Grothendieck est la cohomologie universelle pour les variétés lisses sur un corps, lorsque les classes de Chern suivent la loi multiplicatif $x + y - xy$. On en déduit le théorème de Grothendieck-Riemann-Roch pour le K-groupe de

Grothendieck des fibrés vectoriels et pour le gradué de la K-théorie par la filtration qui donne la codimension du support. On ne donne aucun résultat nouveau dans ce chapitre, mais je crois qu'il contient les plus importantes idées de cette mémoire.

2 et 3. Préliminaires et théorie homotopique des schémas

On présente une introduction à la théorie de l'homotopie des schémas pour de géomètres algébriques. On remarque la functorialité basique du grand site de Nisnevich et la classification des torseurs. On ne prétend pas être complet ni "self contained", donc la plupart des preuves sont omises. Le seul résultat qu'on ne peut pas trouver dans la littérature est la proposition 3.1.43. Pour la classification des torseurs, on donne des preuves complètes.

4. Théorèmes de Riemann-Roch et morphismes de Gysin

Dans la section 4.1 on rappelle la notion de spectre en anneaux orienté absolu et de module sur un spectre en anneaux absolu. On introduit la cohomologie relative dans le cadre de l'homotopie des schémas et on construit le spectre absolu qui, sous des conditions convenables, représente la cohomologie relative. Dans la section 4.2 on construit le morphisme de Gysin à l'aide des idées de Gabber pour des immersions régulières et on prouve un résultat d'unicité. On démontre le théorème de Riemann-Roch motivique dans la section 4.3 (*cf.* Théorème 4.3.7) et on obtient le théorème de Riemann-Roch pour la K-théorie homotopiquement invariant comme un corollaire. On en déduit le théorème de Riemann-Roch pour les modules et on obtient un théorème arithmétique de Riemann-Roch et un théorème de Riemann-Roch pour la cohomologie relative. Finalement, dans la section 4.4 on étude le théorème de Riemann-Roch sans dénominateurs.

5. Appendice

L'appendice est dédié à la construction explicite du spectre de Hodge absolu réel à l'aide du complexe de Burgos (*cf.* [Bur98]) et on prouve que cet spectre, aussi bien que le spectre de Deligne-Beilinson de [HS15], représentent ces cohomologies dans le cadre singulier.

Introducción

El teorema de Riemann-Roch original de Grothendieck afirma que para todo morfismo propio $f\colon Y \to X$, entre variedades irreducibles quasiproyectivas lisas sobre un cuerpo, y todo elemento $a \in K_0(Y)$ del grupo de Grothendieck de fibrados vectoriales se satisface la relación

$$\mathrm{ch}(f_!(a)) = f_*\big(\mathrm{Td}(T_f) \cdot \mathrm{ch}(a)\big)$$

(*cf.* [BS58]). Recuérdese que ch denota el carácter de Chern, $\mathrm{Td}(T_f)$ la clase de Todd del fibrado tangente relativo y f_* y $f_!$ las imágenes directas en el anillo de Chow y K_0 respectivamente. Más tarde Baum, Fulton MacPherson probaron en [BFM75] el teorema de Riemann-Roch para morfismos localmente intersección completa entre esquemas algebraicos (es decir, esquemas separados localmente de tipo finito sobre cuerpo) proyectivos singulares. En [FG83] Fulton y Gillet probaron el teorema sin hipótesis proyectivas.

La notable extensión a la teoría K superior para esquemas regulares sobre una base fue probada por Gillet en [Gil81]. El teorema de Riemann-Roch allí probado es para morfismos proyectivos entre esquemas lisos quasiproyectivos. Sin embargo, obsérvese que en el caso de esquemas sobre cuerpo el resultado de Gillet no recupera el teorema de [BFM75]. La mayor generalización del teorema de Riemann-Roch que yo conozco es [Dég14] y [HS15] donde Déglise y Holmstrom-Scholbach obtuvieron independientemente el teorema de Riemann-Roch para teoría K superior y morfismos proyectivos lic entre esquemas regulares sobre una base noetheriana finito dimensional.

Tras los trabajos de Cisinski en [Cis13] sobre la teoría K homotópicamente invariante de Weibel podemos aplicar la teoría homotópica de esquemas a ella. Apoyados en los resultados de Cisinski, presentamos un teorema de Riemann-Roch para la teoría K homotópicamente invariante y morfismos proyectivos lic sin hipótesis proyectivas en los esquemas. En concreto, el teorema que probamos para la cohomología motívica y esquemas finito dimensionales sobre una base S es el siguiente

Teorema: *Sea* $f\colon Y \to X$ *un morfismo proyectivo lic entre* S*-schemes y notemos* $T_f \in K^0(Y)$ *el fibrado tangente virtual y* Td *la extensión multiplicativa de la serie* $\frac{t}{1-e^{-t}}$*. Entonces el diagrama*

$$\begin{tikzcd} KH(Y)_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}(T_f)\mathrm{ch}"'] & KH(X)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ H_{\mathcal{M}}(Y,\mathbb{Q}) \arrow[r, "f_*"] & H_{\mathcal{M}}(X,\mathbb{Q}) \end{tikzcd}$$

conmuta. Es decir, para toda $a \in KH(Y)_{\mathbb{Q}}$ *tenemos*

$$\mathrm{ch}(f_*(a)) = f_*\big(\mathrm{Td}(T_f)\cdot \mathrm{ch}(a)\big).$$

De este resultado deducimos teoremas de Riemann-Roch para muchas cohomologías. En particular, para la cohomología de Deligne-Beilinson y de Hodge absoluta reales, para la cohomología rígida sintómica y para cualquier cohomología de Weil mixta como la cohomología de deRham y étale geométrica.

Para probar este resultado, nuestra contribución a la teoría es la construcción del morfismo de Gysin para inmersiones regulares y cualquier cohomología dada por espectros. Desde sus comienzos, la construcción estándar del morfismo de Gysin en teoría de homotopía de esquemas se apoya en el espacio de Thom y el isomorfismo de pureza. Sin embargo, la pureza requiere hipótesis de lisitud (*cf.* [MV99]). Nuestro estrategia es distinta: desarrollamos las ideas de Gabber para la cohomología étale en el contexto de la homotopía de esquemas y obtenemos morfismos de Gysin para inmersiones regulares sin hipótesis de lisitud en los esquemas. Esto nos lleva a la construcción de nuevos morfismos de Gysin para muchas teorías como la teoría K homotópicamente invariante, la cohomología motívica, la cohomología de Hodge absoluta y de Deligne-Beilinson real, la cohomología rígida sintómica, y cualquier cohomología dada por una teoría de Weil mixta.

Los módulos sobre una teoría cohomológica son invariantes aritméticos y geométricos notables. La teoría K aritmética y la cohomología motívica aritmética son módulos sobre la teoría K y la cohomología motívica respectivamente (*cf.* [HS15]). Además, en cualquier cohomología la cohomología relativa a un morfismo también es un módulo. Obsérvese que la cohomología con soportes propios, la cohomología con soporte en un cerrado, y la cohomología reducida son la cohomología relativa a una inmersión cerrada, a una inmersión abierta y a la proyección sobre el punto base respectivamente.

Deducimos de nuestros resultados nuevos morfismos de Gysin y un teorema de Riemann-Roch para módulos en general. En particular, probamos un nuevo teorema de Riemann-Roch para la cohomología relativa.

Teorema: *Sea $f\colon Y \to X$ un morfismo de esquemas y $g\colon T \to X$ un morfismo proyectivo lic. Denotamos $f_T\colon Y \times_X T \to T$, $T_g \in K_0(Y)$ el fibrado tangente virtual de g y $KH(f)$ y $H_{\mathcal{M}}(f, \mathbb{Q})$ la teoría K homotópicamente invariante y cohomología motívica relativa de f respectivamente. Si además f es propio o g es liso, entonces el diagrama*

$$\begin{tikzcd} KH(f_T)_{\mathbb{Q}} \arrow[r, "g_*"] \arrow[d, "\mathrm{Td}(T_g)\mathrm{ch}"'] & KH(f)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ H_{\mathcal{M}}(f_T, \mathbb{Q}) \arrow[r, "g_*"] & H_{\mathcal{M}}(f, \mathbb{Q}) \end{tikzcd}$$

conmuta. Es decir, para todo $m \in KH(f_T)_{\mathbb{Q}}$ tenemos

$$\mathrm{ch}(g_*(m)) = g_*\big(\mathrm{Td}(T_g) \cdot \mathrm{ch}(m)\big).$$

También obtenemos el teorema de Riemann-Roch aritmético de [HS15] el teorema de Riemann-Roch residual de [Dég14].

Finalmente mejoramos una fórmula clásica en esta materia: el Riemann-Roch sin denominadores. Conjeturado por el propio Grothendieck en [SGA6], fue probado por primera vez por Jouanolou en [Jou70] para esquemas quasiproyectivos lisos y después generalizado a contexto singular ([Ful98]). La mayor generalización que yo conozco es el enunciado de Gillet en [Gil81] para la teoría K superior y esquemas lisos quasiproyectivos sobre una base regular. Nuestro teorema es probado otra vez en contexto singular y sin hipótesis proyectivas en los esquemas.

Teorema: *Sea $i\colon Z \to X$ una inmersión regular de codimensión d y denotemos $\mathfrak{q}_i\colon KH(Z) \to KH_Z(X)$ y $\mathfrak{p}_i\colon H_{\mathcal{M}}(Z, \mathbb{Z}) \to H_{\mathcal{M},Z}(X, \mathbb{Z})$ los morfismos de Gysin refinados, $c^Z_{q,r}\colon KH_{r,Z}(X) \to H_{\mathcal{M},Z}(X, \mathbb{Z})$ la r-ésima clase de Chern con soporte en Z (cf. 4.4.1 y Definición 4.2.8 respectivamente), y $P^d_q(r, x_1, \cdots ; y_1, \cdots)$ los polinomios con coeficientes enteros definidos en [Jou70]. Entonces para todo $a \in KH_r(Z)$ tenemos*

$$c^Z_{q,r}(\mathfrak{q}_i(a)) = \mathfrak{p}_i\big(P^d_q(\mathrm{rk}(a), c_{1,r}(a), \ldots, c_{q-d,r}(a); c_1(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X}))\big).$$

A continuación describimos la organización de esta memoria.

1. Grothendieck-Riemann-Roch

Usando la axiomática de Panin en [Pan04] probamos que la teoría K de Grothendieck es la cohomología universal para variedades lisas sobre un cuerpo con clases de Chern siguiendo la ley multiplicativa $x + y - xy$. Deducimos el teorema de Grothendieck-Riemann-Roch para el grupo K de Grothendieck de fibrados vectoriales y para el graduado de la teoría K por la filtración de la codimensión del soporte. Aunque no probamos ninguna fórmula nueva en este capítulo, creo que contiene las ideas principales de esta memoria.

2 y 3. Preliminares y teoría homotópica de esquemas

Damos una introducción a la teoría homotópica de esquemas para geómetras algebraicos. Ponemos énfasis en la funtorialidad básica en el lugar grande de Nisnevich y a la clasificación de torsores. No pretendemos ser ni completo y autocontenido por lo que la mayoría de las pruebas se omiten. El único resultado que no se encuentra en la literatura es la clasificación de pseudo divisores de la Proposición 3.1.43. En cuanto a la clasificación de torsores, incluimos pruebas completas.

4. Teoremas de Riemann-Roch y morfismos de Gysin

En la sección 4.1 recordamos la noción de espectro en anillos orientado absoluto y módulo sobre un espectro absoluto en anillos. Introducimos la cohomología relativa en el contexto de la teoría de homotopía de esquemas y construimos el espectro absoluto que bajo ciertas condiciones representa la cohomología relativa. En la sección 4.2 construimos el morfismo de Gysin usando las ideas de Gabber para el caso de inmersiones regulares y probamos un criterio de unicidad. Probamos el teorema motívico de Riemann-Roch en la sección 4.3 (*cf.* Teorema 4.3.7) y obtenemos el teorema de Riemann-Roch para la teoría K homotópicamente invariante como corolario. Deducimos un teorema de Riemann-Roch para módulos y por tanto obtenemos un teorema aritmético de Riemann-Roch y un teorema de Riemann-Roch para la cohomología relativa. Finalmente, en la sección 4.4. abordamos el teorema de Riemann-Roch sin denominadores.

5. Apéndice

El Apéndice está dedicado a la construcción explícita del espectro de Hodge absoluto real usando el complejo de Burgos (*cf.* [Bur98]) y para comprobar que este espectro, así como el espectro de Deligne-Beilinson de [HS15], representan estas cohomologías también en contexto singular.

Agradecimientos

Estoy profundamente agradecido a mis directores, Frédéric y José Ignacio, que tan generosa y desinteresadamente me han dedicado su tiempo y su saber. Sin su apoyo constante, tanto matemático como humano, esta tesis no habría sido posible.

Quiero agradecer también a mis profesores y compañeros del Departamento de Matemáticas de la UEX, especialmente a Adrián, Jaime, Juan Bautista, Pedro, Ricardo, mi padre y mi hermano José, que siempre me ha apoyado e inspirado. Las fantásticas Matemáticas y atmósfera que allí disfruté han alumbrado mi vida matemática.

Agradezco especialmente a mi familia, a mi amorosa madre, a mi padre y a mis hermanos Juan y José, y también a Clara, todo su apoyo. También ellos han sido imprescindibles y el tiempo que he dedicado a esta tesis era en justicia suyo.

Por último, quiero dedicar esta memoria a mi padre. No puedo expresar con palabras la deuda que esta memoria le tiene y cuánto he recibido, también en Matemáticas, de él. Me alegra saber que mi deuda impagable es signo veraz de que ha atesorado un tesoro *donde no hay polilla ni carcoma que lo roa.*

Madrid, 9 de noviembre de 2015
Nuestra Señora de la Almudena

Chapter 1

Grothendieck-Riemann-Roch

1.1 Universal property of K-theory

Let k be a field of characteristic zero. Denote $\mathbf{Var}_k$ the category of smooth algebraic varieties over k and **Rings** the category of commutative rings.

Definition 1.1.1 We call a **cohomology theory** to a contravariant functor $A\colon \mathbf{Var}_k \longrightarrow \mathbf{Rings}$ such that

1. We have $i_1^* + i_2^*\colon A(X_1 \sqcup X_2) \xrightarrow{\sim} A(X_1) \oplus A(X_2)$ for any two schemes X_1 and X_2;

2. (Homotopy invariance) We have $\pi^*\colon A(X) \xrightarrow{\sim} A(P)$ for any affine bundle $\pi\colon P \to X$;

For every projective morphism $f\colon Y \to X$ we have a morphism called **direct image** $f_*\colon A(Y) \to A(X)$ satisfying that

3. (Functoriality) $\mathrm{Id}_* = \mathrm{Id}$ and for $g\colon Z \to Y$ another projective morphism $(fg)_* = f_* g_*$;

4. (Projection formula) The map f_* is a morphism of $A(X)$-modules, *i.e.*, $a \cdot f_*(b) = f_*(f^*(a) \cdot b)$, $\forall\, a \in A(X)$, $b \in A(Y)$;

5. If f transversal to a smooth closed subvariety $i\colon Z \to X$ (in other words, $f^* N_{Z/X} \xrightarrow{\sim} N_{Z'/Y}$ for $Z' = Z \times_X Y$ and $i'\colon Z' \to Y$) then the square

$$\begin{array}{ccc} A(Z') & \xrightarrow{i'_*} & A(Y) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ A(Z) & \xrightarrow{i_*} & A(X) \end{array}$$

commutes;

6. (Localization) For any smooth closed subscheme $i\colon Z \to X$ of open complement $j\colon U \to X$ there is an exact sequence
$$A(Z) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(U);$$
7. For any projective bundle $\pi\colon \mathbb{P}(E) \to X$ and any morphism $g\colon Y \to X$ the square
$$\begin{tikzcd} A(\mathbb{P}(E)) \arrow[r, "g^*"] \arrow[d, "\pi_*"'] & A(\mathbb{P}(g^*E) \arrow[d, "\pi_*"] \\ A(X) \arrow[r, "g^*"] & A(Y) \end{tikzcd}$$
commutes;
8. (Projection bundle theorem) For any vector bundle $E \to X$ of rank r there is a natural isomorphism
$$A(\mathbb{P}(E)) = A(X) \oplus A(X)y \oplus \ldots \oplus A(X)y^{r-1}$$
where $y = s_0^* s_{0*}(1)$ the s_0 zero section of $\mathcal{O}_{\mathbb{P}^n(E)}(1)$.

A **morphism** of cohomology theories $\varphi\colon A \to \bar{A}$ is a natural transformation which preserves direct images. In other words,
$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad , \quad \varphi(ab) = \varphi(a) \cdot \varphi(b) \quad , \quad \varphi(1) = 1\,,$$
$$\varphi(f^*(a)) = f^*(\varphi(a)) \quad \text{and} \quad \varphi(f_*(a)) = f_*(\varphi(a)).$$
Let A be a cohomology theory. Let $L \to X$ be a line bundle and $s\colon X \to L$ be a section, we call the **first Chern class** of L to $c_1(L) = s^* s_{0*}(1)$ (note it does not depend on the section from axiom 2 and that in axiom 8 we have $y = c_1(\mathcal{O}_{\mathbb{P}^n(E)}(1))$). Let $Z \to X$ be a smooth closed subvariety, we call the **fundamental class** of Z in X to $\eta_Z^X = i_*(1)$.

Remark 1.1.2 These axioms are taken from [Pan04].

Remark 1.1.3 Let A be a cohomology theory, and $i\colon Z \to X$ be a smooth hypersurface. Denote L_Z the line bundle defined by the dual of the sheaf of ideals $\mathcal{I}_Z$. The line bundle L_Z always admit a section transversal to the zero section which vanishes on Z. Therefore
$$c_1(L_Z) = i_*(1) = \eta_Z^X.$$
In particular, in axiom 8 for the projective space we have $y = \eta_H^{\mathbb{P}^n} = c_1(L_H) = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ for a hyperplane $H \to \mathbb{P}^n$. Let $f\colon Y \to X$ be a morphism, note that $f^*c_1(L) = c_1(f^*L)$ since the zero section s_0 is transversal to f. The following result shows that, as expected, it is equivalent to state axiom 8 for $c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))$.

Proposition 1.1.4 *Let A be a cohomology theory, $E \to X$ be a vector bundle of rank r and denote $x = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))$. Then*

$$A(\mathbb{P}(E)) = A(X) \oplus A(X)x \oplus \ldots \oplus A(X)x^{r-1}.$$

Proof: Consider $\mathbb{P}(E) \times_X \mathbb{P}(E^*)$. The natural incidence relation defines a closed subvariety whose open complement we denote U. Note that $\pi_1 \colon U \to \mathbb{P}(E)$ and $\pi_2 \colon U \to \mathbb{P}(E^*)$ are affine bundles. Since the natural pairing

$$\pi_1^* \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}(E^*)}(-1) \to \pi_1^* E \otimes \pi_2^* E^* \to \mathcal{O}_{\mathbb{P}(E) \times_X \mathbb{P}(E^*)}$$

is an isomorphism on U then $\pi_1^* \mathcal{O}_{\mathbb{P}(E)}(-1)$ is isomorphic to $\pi_2^* \mathcal{O}_{\mathbb{P}(E^*)}(1)$ on U. By axiom 2 we have $A(\mathbb{P}(E)) \simeq A(U) \simeq A(\mathbb{P}(E^*))$ and we conclude.

□

Proposition 1.1.5 *Let A be a cohomology theory and $f = f_1 \sqcup f_2 \colon Y_1 \sqcup Y_2 \to X$ be projective morphism, we have $f_* = f_{1*} \oplus f_{2*} \colon A(Y_1) \oplus A(Y_2) \to A(X)$.*

Proof: Consider the notation of axiom 1, note that the inverse of $i_1^* + i_2^* \colon A(X_1 \sqcup X_2) \to A(X_1) \oplus A(X_2)$ is $i_{1*} \oplus i_{2*}$ since both i_1 and i_2 are transversal to i_1 and i_2.

□

Notation 1.1.6 Let A be a cohomology theory. We denote L_x a line bundle whose first Chern class is $c_1(L_x) = x$. We say that a cohomology theory *follows the law* $x + y$ or the *additive group law* if $c_1(L_x \otimes L_y) = x + y$. We say that a cohomology theory *follows the law* $x + y - xy$ or the *multiplicative group law* if $c_1(L_x \otimes L_y) = x + y - xy$.[1]

Example 1.1.7 Every classic cohomology is a cohomology theory. Let us review the main examples:

- Grothendieck's K-theory of vector bundles $K_0(X)$ is a cohomology theory (*cf.* [Nav12, §12] for example). In this case we denote the direct image of a morphism f by $f_!$. The fundamental class of a subvariety Z is $\mathcal{O}_Z$ and the first Chern class of line bundle L is $c_1(L) = 1 - L^*$. In addition, first Chern classes are nilpotent ($c_1(L)^n = 0$ for $n > \dim X$). Note that $(1 - L_1^* \otimes L_2^*) = (1 - L_1^*) + (1 - L_2^*) - (1 - L_1^*)(1 - L_2^*) \in K(X)$ so we have

  $$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) - c_1(L_1)c_1(L_2).$$

  Therefore K-theory follows the law $x + y - xy$.

[1] They are the laws of the additive and multiplicative group respectively in one variable with origin in the neutral element. Indeed, for the multiplicative law $(1 - x)(1 - y) = 1 - (x + y - xy)$.

- The graduated of the K-theory of vector bundles by the support codimension filtration $GK(X)$ is a cohomology theory (*cf.* [Nav12, §12] for example). The fundamental class of a subvariety Z of X is $[\mathcal{O}_Z]$ and they generate $GK(X)$. The first Chern class of a line bundle is $c_1(L) = [1 - L^*]$. The preceding computation concludes that in this case

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

  so that GK follows the law $x + y$.

- Denote $CH(X)$ the Chow ring of X. It is a cohomology theory (*cf.* [Ful98] for example). The fundamental class of a subvariety Z is given by the class $[Z]$ and in this case $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ so that CH follows the law $x + y$.

- Let X be a smooth algebraic variety over a field k of characteristic zero endowed with an embedding $k \to \mathbb{C}$ and denote $\bar{X} = X \times \mathrm{Spec}(\mathbb{C})$. Denote $H^{2\bullet}_{\mathrm{Bet}}(X, \mathbb{Z}) = \bigoplus_p H^{2p}_{\mathrm{Bet}}(\bar{X}, \mathbb{Z})$ where $H^p_{\mathrm{Bet}}(\bar{X}, \mathbb{Z})$ denotes the Betti cohomology of the classical topological space associated to the closed points of $\bar{X}$. Then $H^{2\bullet}_{\mathrm{Bet}}(X, \mathbb{Z})$ is a cohomology theory. The fundamental class of a subvariety Z is given by duality by the homology class defined by $\bar{Z}$. Once again we have $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ so that $H^{2\bullet}_{\mathrm{Bet}}$ follows the law $x + y$.

We review the theory of Chern classes in this context.

Definition 1.1.8 Let A be a cohomology theory, $E \to X$ be a vector bundle of rank r and denote $x = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))$. We define the **Chern classes** of E as the unique classes $c^A_1(E), \ldots, c^A_r(E) \in A(X)$ such that

$$x^r - c^A_1(E)x^{r-1} + \cdots + (-1)^r c^A_r(E) = 0.$$

We denote them simply $c_i(E)$ if no confusion is possible.

We denote the characteristic polynomial of the endomorphism $\cdot x$ by $c(E) = t^r - c_1(E)t^{r-1} + \cdots + (-1)^r c_r(E) \in A(X)[t]$.

Remark 1.1.9 Let $L \to X$ be a line bundle, then $\mathbb{P}(L) = X$ and $\mathcal{O}_{\mathbb{P}(L)}(-1) = L$ so that the above definition of first Chern class agrees with that of Definition 1.1.1.

Theorem 1.1.10 *Let A be a cohomology theory and $E \to X$ be a vector bundle. The Chern classes $c_i(E)$ are functorial. In other words, let $f\colon Y \to X$ be a morphism, then $f^*c_i(E) = c_i(f^*E)$ for all i.*

Proof: Since $f^*c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) = c_1(f^*\mathcal{O}_{\mathbb{P}(E)}(-1)) = c_1(\mathcal{O}_{\mathbb{P}(f^*E)}(-1))$ then the result follows. □

Theorem 1.1.11 *Let A be a cohomology theory. The Chern classes are additive. In other words, let $0 \to E_1 \to E \to E_2 \to 0$ be a short exact sequence of vector bundles, then*

$$c(E) = c(E_1) \cdot c(E_2) \ ,$$
$$c_k(E) = \sum_{i+j=k} c_i(E_1)c_j(E_2) \qquad i, j, k \in \mathbb{N}.$$

Proof: After the base change $\mathbb{P}(E_1) \to X$ and by induction on the rank of E_1 we can assume that E_1 is a line bundle. Therefore $i\colon X = \mathbb{P}(E_1) \to \mathbb{P}(E)$ is a section of $\mathbb{P}(E) \to X$, so that i_* is injective. Denote $j\colon U \to \mathbb{P}(E)$ the complement of $\mathbb{P}(E_1)$. The natural projection $U \to \mathbb{P}(E_2)$ is an affine bundle (of associated vector bundle $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(E_2)}(-1), E_1)$) so that the morphism $j^*\colon A(\mathbb{P}(E)) \to A(U) = A(\mathbb{P}(E_2))$ is surjective, because $j^*(x_E^n) = x_{E_2}^n$ for $x_{E_i} = c_1(\mathcal{O}_{\mathbb{P}(E_i)}(-1))$.

Since $x_{E_1} = i^*x_E$ and due to the projection formula the diagram

$$\begin{tikzcd}
0 \arrow[r] & A(\mathbb{P}(E_1)) \arrow[r, "i_*"] \arrow[d, "\cdot x_{E_1}"] & A(\mathbb{P}(E)) \arrow[r, "j^*"] \arrow[d, "\cdot x_E"] & A(\mathbb{P}(E_2)) \arrow[r] \arrow[d, "\cdot x_{E_2}"] & 0 \\
0 \arrow[r] & A(\mathbb{P}(E_1)) \arrow[r, "i_*"] & A(\mathbb{P}(E)) \arrow[r, "j^*"] & A(\mathbb{P}(E_2)) \arrow[r] & 0
\end{tikzcd}$$

is commutative. Since it is made of exact rows we conclude recalling that characteristic polynomials are additive.

□

Corollary 1.1.12 *Let A be a cohomology theory, denote $x = c_1(\mathcal{O}_{\mathbb{P}^d}(-1))$, $y = c_1(\mathcal{O}_{\mathbb{P}^d}(1))$ and $S = \mathrm{Spec}(k)$. We have that*

$$A(\mathbb{P}^d) = A(S)[x]/(x^{d+1}) = A(S)[y]/(y^{d+1}).$$

Proof: Since Chern classes are additive by the previous theorem it follows that a trivial vector bundle has null Chern classes. □

Example 1.1.13 Let $E \to X$ be a vector bundle of rank r, let us recall the splitting principle. Consider the projective bundle $\pi\colon \mathbb{P}(E) \to X$. We have the canonical exact sequence $0 \to \mathcal{O}(-1) \to \pi^*E \to Q \to 0$ where $Q = (\pi^*E)/\mathcal{O}(-1)$. Iterating this construction with Q we obtain a base change $p\colon X' \to X$ such that $p^*\colon A(X) \to A(X')$ is injective and such that p^*E is sum of line bundles L_i in $K(X')$. Since $c_1(L_i) = 1 - L_i^*$, we have in $K(X)$ that

$$c_1(E) = \mathrm{rank}\, E - E^* \ . \tag{1.1}$$

Theorem 1.1.14 (Universal property of K-theory) *Let A be a cohomology theory with multiplicative law $x + y - xy$. Then there exist a unique morphism of cohomology theories*

$$\varphi\colon K \to A.$$

Proof: Let $E \to X$ be a vector bundle. Due to equation (1.1) we have $E = \operatorname{rank} E - c_1^K(E^*)$ in $K(X)$. Therefore the unique possible morphism of cohomology theories $\varphi\colon K \to A$ is

$$\varphi(E) := \operatorname{rank} E - c_1^A(E^*) \ . \tag{1.2}$$

Let us see it is well defined. Since rank and c_1^A are additive the map φ is an additive map on vector bundles. Therefore φ defines a group morphism $\varphi\colon K(X) \to A(X)$. This morphism φ commutes with inverse images since rank and c_1^A commute with inverse images.

The map φ preserves products of line bundles since Chern classes follows the same law in A and in the K-theory:

$$\begin{aligned}\varphi(L_1 \cdot L_2) &= 1 - c_1^A(L_1^* \otimes L_2^*) = 1 - c_1^A(L_1^*) - c_1^A(L_2^*) + c_1^A(L_1^*)c_1^A(L_2^*) \\ &= \big(1 - c_1^A(L_1^*)\big)\big(1 - c_1^A(L_2^*)\big) = \varphi(L_1) \cdot \varphi(L_2).\end{aligned}$$

Let $E_1 \to X$, $E_2 \to X$ be two vector bundles. By the splitting principle we may assume they are sums of line bundles in $K(X)$. Therefore φ preserves products.

It is only left to prove that φ preserves direct images. The map φ preserves Chern classes of line bundles:

$$\varphi\big(c_1^K(L)\big) = \varphi(1 - L^*) = 1 - \varphi(L^*) = 1 - \big(1 - c_1^A(L)\big) = c_1^A(L).$$

Therefore φ preserves fundamental classes of smooth hypersurfaces. We conclude due to the following lemma. □

Panin's lemma: *Let A and $\bar{A}$ be two cohomology theories and $\varphi\colon A \to \bar{A}$ be a natural transformation which preserves the fundamental class of hypersurfaces. Then φ is a morphism of cohomology theories:*

$$\varphi(f_*(a)) = f_*(\varphi(a)) \tag{1.3}$$

for $f\colon Y \to X$ projective and $a \in A(Y)$.

Proof: Note that if the lemma is true for two morphism then it is true for the compostion, therefore it is enough to prove the lemma for closed immersions $i\colon Z \to X$ and canonical projections $\pi_X\colon \mathbb{P}^n_X \to X$.

1. *Let $i\colon Z \to X$ be a closed immersion. If equation (1.3) holds for the zero section $s\colon Z \to \bar{N} = \mathbb{P}(1 \oplus N_{Z/X})$ of the projective closure of the normal bundle then it also holds for i.*

   Proof: Consider the deformation to the projective closure of the normal bundle. That is to say, consider the commutative diagram

$$\begin{tikzcd}
\overline{N} \arrow[r, "i_0"] & X' & X \arrow[l, "i_1"'] \\
Z \arrow[u, "s_0"] \arrow[r] & \mathbb{A}^1_Z \arrow[u, "\iota"] & Z \arrow[u, "i"'] \arrow[l]
\end{tikzcd}$$

   where $X' = B_{Y\times\{0\}}\mathbb{A}^1_X$, the blow-up of $\mathbb{A}^1_X$ on $Y \times \{0\}$. Denote $U = X' - (\mathbb{A}^1_Z)$, by axiom 6 we have a commutative diagram

$$\begin{tikzcd}
 & \bar{A}(U) \\
\bar{A}(\bar{N}) & \bar{A}(X') \arrow[l, "i_0^*"'] \arrow[u, "j^*"'] \\
\bar{A}(Z) \arrow[u, "s_*"] & \bar{A}(\mathbb{A}^1_Z). \arrow[l, "i_0^*"', "\sim"] \arrow[u, "i_*"']
\end{tikzcd}$$

   Note that $(\operatorname{Ker} i_0^*) \cap (\operatorname{Ker} j^*) = 0$. Indeed the column is exact by axiom 3 and s_* is injective since $p_* s_* = \operatorname{Id}$ for the natural projection $p\colon \bar{N} \to Z$.

   Consider the commutative diagram

$$\begin{tikzcd}
 & \bar{A}(U) & \\
\bar{A}(\overline{N}) & \bar{A}(X') \arrow[l] \arrow[r] \arrow[u, "j^*"'] & \bar{A}(X) \\
A(Z) \arrow[u, "\Psi_1"] & A(\mathbb{A}^1_Z) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\Psi_2"] & A(Z) \arrow[u, "\Psi_3"']
\end{tikzcd}$$

   where the vertical arrows are the difference of the morphism that theorem states that coincide ($\Psi_i = f_*\varphi - \varphi f_*$). The map Ψ_1 is zero by hypothesis, the Ψ_2 is zero since $(\operatorname{Ker} i_0^*) \cap (\operatorname{Ker} j^*) = 0$ and we conclude that Ψ_3 is zero.

2. *Equation (1.3) holds for the zero section $s\colon Z \to \bar{E} = \mathbb{P}(1 \oplus E)$ of the projective closure of a vector bundle $E \to Z$.*

Proof: Let $E = L \to Z$ be a line bundle. Note that $s^*\colon A(\bar{L}) \to A(Z)$ is surjective, and that $\varphi(s_*(1)) = s_*(1)$ since Z is a hypersurface of $\bar{L}$. Set $a = s^*b \in A(Y)$, then

$$\varphi(s_*s^*b) = \varphi(b \cdot s_*(1)) = \varphi(b) \cdot s_*(1) = s_*(s^*\varphi(b)) = s_*(\varphi(s^*b))$$

and equation (1.3) holds.

Assume E admits a filtration $\{E_i\}$ such that the quotients E_i/E_{i-1} are line bundles. The equation holds for the zero section $Z \to \bar{E}_1$ and the morphisms $\bar{E}_1 \to \bar{E}_2 \to \ldots \to \bar{E}_r = \bar{E}$. Therefore it holds for the composition $s\colon Z \to \bar{E}$.

In the general case, due to the splitting principle there is a morphism $\pi\colon Z' \to Z$ such that π^* is injective and such that $E' = \pi^*E$ admits a filtration as before. The equation (1.3) holds for the zero section $s'\colon Z' \to \bar{E}'$, and we conclude by axiom 5 applied to $\pi\colon \bar{E}' \to \bar{E}$ and $s\colon Y \to \bar{E}$,

$$\pi^*s_*(\varphi(a)) = s'_*\pi^*\varphi(a) = s'_*\varphi(\pi^*a) = \varphi(s'_*\pi^*a) = \varphi(\pi^*s_*a) = \pi^*\varphi(s_*a).$$

3. *If equation 1.3 holds for the canonical projection $\pi\colon \mathbb{P}^n \to p$ then it also holds for the canonical projection $\pi_X\colon \mathbb{P}^n_X \to X$.*

   Proof: It follows from axiom axiom 7.

4. *The equation (1.3) holds for the canonical projection $\pi\colon \mathbb{P}^n \to p$.*

   Proof: Consider the closed immersion $i\colon \mathbb{P}^{n-1} \to \mathbb{P}^n$ and set

$$\begin{aligned} A = A(p),\ x_n = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = i_*(1) \in A(\mathbb{P}^n) \quad \text{and} \\ \bar{A} = \bar{A}(p),\ \bar{x}_n = \bar{c}_1(\mathcal{O}_{\mathbb{P}^n}(1)) = \bar{i}_*(1) \in \bar{A}(\mathbb{P}^n). \end{aligned}$$

   By hypothesis $\varphi(x_n) = \bar{x}_n$, and therefore $\varphi(x_n^r) = \bar{x}_n^r$ so that the morphism $\varphi\colon A(\mathbb{P}^n) \to \bar{A}(\mathbb{P}^n)$ induces an isomorphism of $\bar{A}$-algebras $A(\mathbb{P}^n) \otimes_A \bar{A} = \bar{A}(\mathbb{P}^n)$. We have to check that the 1-form $\bar{p}_*\colon \bar{A}(\mathbb{P}^n) \to \bar{A}$ is obtained by change of scalars from the 1-form $p_*\colon A(\mathbb{P}^n) \to A$.

   Consider the fundamental class of the diagonal $\Delta\colon \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ and denote it $\Delta_n = \Delta_*(1) \in A(\mathbb{P}^n \times \mathbb{P}^n) = A(\mathbb{P}^n) \otimes_A A(\mathbb{P}^n)$. We have

$$(p_* \otimes 1)(\Delta_n) = \pi_*\Delta_*(1) = \mathrm{Id}_*(1) = 1,$$

   where $\pi\colon \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ is the second projection. In other words, the polarity defined by the diagonal $\omega \mapsto (\omega \otimes 1)(\Delta_n)$ maps p_* to the unit.

   According to the next Proposition 1.1.15, the 1-form p_* is determined by the previous condition. Note that the fundamental class of the diagonal is stable by change of scalars (since equation (1.3) hold for closed immersions) and we obtain that p_* is stable by change of scalars.

$\square$

Proposition 1.1.15 *The polarity defined by the diagonal $A(\mathbb{P}^n)^* \to A(\mathbb{P}^n)$, $\omega \mapsto (\omega \otimes 1)(\Delta_n)$, is an isomorphism.*

Proof: In fact, by induction on n we prove that

$$\Delta_n = \sum_{r,s=0}^{n} a_{rs} x_n^r \otimes x_n^s = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & \bullet \\ 0 & \iddots & \iddots & \vdots \\ 1 & \bullet & \cdots & \bullet \end{pmatrix}$$

where $a_{rs} = 0$ when $r + s < n$, and $a_{rs} = 1$ when $r + s = n$. Indeed,

$$i_*(x_{n-1}^r) = i_* i^*(x_n^r) = x_n^r \cdot i_*(1) = x_n^{r+1},$$

and by axiom 5 we have that $(i^* \otimes 1)(\Delta_n)$ is the fundamental class for the diagonal of $\mathbb{P}^{n-1}$ in $\mathbb{P}^{n-1} \times \mathbb{P}^n$. Note tat

$$(i^* \otimes 1)(\Delta_n) = \textstyle\sum_{r,s} a_{rs} x_{n-1}^r \otimes x_n^s \quad \text{and}$$
$$(1 \otimes i_*)(\Delta_{n-1}) = \textstyle\sum_{r,s} a'_{rs} x_{n-1}^r \otimes x_n^{s+1}$$

where $\Delta_{n-1} = \sum_{rs} a'_{rs} x_{n-1}^r \otimes x_{n-1}^s$. By induction on n we obtain the result for a_{rs}, $r < n$. By symmetry we also obtain it for a_{rs}, $s < n$, and we conclude.

$\square$

1.2 Grothendieck-Riemann-Roch theorem

Definition 1.2.1 We define a **graded $\mathbb{Q}$-cohomology theory** $A^\bullet$ to be a cohomology theory with values in the category positive graded commutative $\mathbb{Q}$-algebras (i.e., $A^\bullet(X) = \oplus_{n \geq 0} A^n(X)$), such that

9. For every projective morphism $f\colon Y \to X$ between connected varieties the direct image changes the degree $f_*\colon A^n(Y) \to A^{n+d}(X_i)$ where $d = \dim X - \dim Y$;

10. It follows the additive law $c_1(L_x \otimes L_y) = x + y$.

A **morphism** of graded $\mathbb{Q}$-cohomology theories is a $\mathbb{Q}$-linear morphism of cohomology theories which preserves the degree.

Example 1.2.2 • The graduated of the K theory and the Chow ring define graded $\mathbb{Q}$-cohomology theories $GK(X)_{\mathbb{Q}} = GK(X) \otimes \mathbb{Q}$ and also $CH(X)_{\mathbb{Q}} = CH(X) \otimes \mathbb{Q}$. Betti cohomology with rational coefficients also defines a graded $\mathbb{Q}$-cohomology $H^{2\bullet}_{\mathrm{Bet}}(\bar{X}, \mathbb{Q}) = \oplus_i H^{2i}_{\mathrm{Bet}}(\bar{X}, \mathbb{Q})$.

- Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory, then $\widehat{A}^\bullet(X) := \prod_n A^n(X)$ is cohomology theory.
- Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory, the fundamental class of a smooth subvariety $Z \to X$ of codimension d belongs to $A^d(X)$. In particular, first Chern classes of line bundles belong to $A^1(X)$ and, in general, for any vector bundle $E \to X$ we have $c_i(E) \in A^i(X)$.

Let $E \to X$ be a vector bundle and $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory. By the splitting principle, after a base change $\pi\colon X' \to X$ such that $\pi^*\colon A^\bullet(X) \to A^\bullet(X')$ is injective we have $\pi^* E = L_{\alpha_1} + \cdots L_{\alpha_r}$ in $K(X')$ for $L_{\alpha_1}, \ldots, L_{\alpha_r}$ line bundles. Therefore the Chern classes of E are the elementary symmetric functions of $\alpha_1, \ldots, \alpha_r$.

Definition 1.2.3 Let $E \to X$ be a vector bundle, $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory and $F(t) = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ be a formal series with rational coefficients. We denote

$$F_+(E) = F(\alpha_1) + \ldots + F(\alpha_r) \in \widehat{A}^\bullet(X) = \prod_n A^n(X)$$

where $F_+(E)$ is a symmetric function on $\alpha_1, \ldots, \alpha_r$ and therefore a function on the Chern classes of E. Note that F_+ defines an additive function on vector bundles. Therefore it also defines a functorial morphism of groups

$$F_+\colon K(X) \longrightarrow \widehat{A}^\bullet(X)$$

which we call the **additive extension** of F.

Let $F = 1 + a_1 t + \ldots \in \mathbb{Q}[[t]]$ be a formal series where $a_0 = 1$, we denote

$$F_\times(E) = F(\alpha_1) \cdot \ldots \cdot F(\alpha_r) \in \widehat{A}^\bullet(X)^*$$

where $F_\times(E)$ is a function on the Chern classes of E. The assignation $F_\times$ also defines an additive function on vector bundles, therefore it defines a functorial morphism of groups

$$F_\times\colon K(X) \to \widehat{A}^\bullet(X)^*$$

which we call the **multiplicative extension** of F.

1.2.4 (Change of direct image) Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory. Given a series $F(t) = 1 + \ldots \in \mathbb{Q}[[t]]$ with $a_0 = 1$ we can change direct images in $\widehat{A}^\bullet$ so that so that the first Chern class of a line bundle L_x is $c_1^{\mathrm{new}}(L_x) = xF(x) = x + \ldots$, turning $\widehat{A}^\bullet$ into a (non graded) cohomology theory. Let

$f\colon Y \to X$ be a projective morphism and consider the virtual tangent bundle $T_f := T_Y - f^! T_X \in K(Y)$. Denote

$$f_*^{\mathrm{new}}(a) := f_*\big(F_\times(-T_f)\, a\big) = f_*\big(F_\times(T_f)^{-1} a\big) = F_\times(T_X) f_*\big(F_\times(T_Y)^{-1} a\big),$$

so that for $i\colon Z \to X$ a hypersurface we have

$$i_*^{\mathrm{new}}(1) = i_*\big(F_\times(N_{Z/X})\big) = i_*\big(F_\times(i^* L_Z)\big) = F_\times(L_Z) i_*(1) = F(Z)Z.$$

Proposition 1.2.5 *Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory and $F(t) = 1 + \cdots \in \mathbb{Q}[[t]]$ be a series such that $a_0 = 1$. Then $\widehat{A}^\bullet$ with the new direct images f_*^{new} given by F is a cohomology theory.*

Proof: All axioms are direct except for the projective bundle theorem. Note that $\widehat{A}^\bullet(X)$ has a natural filtration whose graded ring is $A^\bullet(X)$. Let $E \to X$ be a vector bundle and denote $x = x_E = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \in A^1(\mathbb{P}(E))$, $y = x_E^{\mathrm{new}} = xF(x) = x + \ldots \in \widehat{A}^\bullet(\mathbb{P}(E))$, so that $y^n = x^n + \ldots$. The morphism

$$\widehat{A}^\bullet(X) \oplus \widehat{A}^\bullet(X)[-1] \oplus \ldots \oplus \widehat{A}^\bullet(X)[-r] \longrightarrow \widehat{A}^\bullet(\mathbb{P}(E))$$

defined by $1, y, \ldots, y^r$ induce on its graduated the isomorphism

$$A^\bullet(X) \oplus A^\bullet(X)[-1] \oplus \ldots \oplus A^\bullet(X)[-r] \longrightarrow A^\bullet(\mathbb{P}(E))$$

defined by $1, x, \ldots, x^r$. Therefore the original map is an isomorphism and $1, y, \ldots, y^r$ define a base of the free $\widehat{A}^\bullet(X)$-module $\widehat{A}^\bullet(\mathbb{P}(E))$.

$\square$

Recall that $A^\bullet$ follows the law $x + y$. We can change direct images by the exponential series so that it follows the law $x+y-xy$. Indeed, $e^{-x} = 1-(1-e^{-x})$ and $1 - e^{-x} = x + \cdots$ so that we set

$$c^{\mathrm{new}}(L_x) = 1 - e^{-x} = x + \ldots = xF(x)$$

where $F(t) = \frac{1-e^{-t}}{t} = 1 + \ldots$. By the universal property of K-theory 1.1.14 there exists a unique morphism of cohomology theories

$$\mathrm{ch}\colon K \to \widehat{A}^\bullet.$$

Proposition 1.2.6 *Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory. The Chern classes are nilpotent.*

Proof: Consider the above new direct images on $\widehat{A}^\bullet$ so that it follows the law $x+y-xy$. Consider the morphism of cohomology theories $\mathrm{ch}\colon K \to \widehat{A}^\bullet$. Recall that in K-theory Chern classes are nilpotent. Therefore $c_1^{\mathrm{new}}(L_x) = x \cdot F(x)$ is nilpotent and we conclude that x is also nilpotent.

$\square$

Remark 1.2.7 Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory. Since Chern classes are nilpotent we can replace $\widehat{A}^\bullet$ by $A^\bullet$ in Definition 1.2.3 so that both the multiplicative and additive extension take values in $A^\bullet$. We have used $\widehat{A}^\bullet$ just to define the multiplicative and additive extension and then prove that Chern classes are nilpotent. One can prove this by using Jouanolou's trick (*cf.* [Wei89, 4.4]). Using Jouanolou's trick it is also possible to remove axiom 10.

Corollary 1.2.8 *Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory and $F(t) \in \mathbb{Q}[[t]]$ be a series such that $a_0 = 1$. Then $A^\bullet$ with the new direct images is a cohomology theory.*

$\square$

In particular, for $F(t) = \frac{1-e^{-t}}{t}$ then $A^\bullet$ follows the law $x + y - xy$. By the universal property of K-theory there is a unique morphism of cohomology theories

$$\mathrm{ch}\colon K \to A^\bullet$$

that we call the **Chern character**. Moreover, ch is the multiplicative extension of e^t because

$$\mathrm{ch}(L_x) = 1 - c_1^{\mathrm{new}}(L_x^*) = 1 - (1 - e^x) = e^x.$$

Let $f\colon Y \to X$ be a projective morphism, since ch commutes with direct images we have that

$$\mathrm{ch}(f_!(y)) = f_*^{\mathrm{new}}(\mathrm{ch}(y)) = f_*\big[F(f^*T_X - T_Y)\cdot \mathrm{ch}(y)\big] = F(T_X) f_*\big[F(T_Y)^{-1}\mathrm{ch}(y)\big].$$

We define the **Todd class** to be the multiplicative extension Td of the series

$$F(t)^{-1} = \frac{t}{1-e^{-t}} = \left(1 - \frac{t}{2!} + \frac{t^2}{3!} - \frac{t^3}{4!} + \frac{t^4}{5!} - \dots\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \dots.$$

Making the substitution in the above formula we obtain the following.

Grothendieck-Riemann-Roch theorem: *Let $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory defined on smooth varieties over a field k. For every projective morphism $f\colon Y \to X$ between smooth varieties over k the square*

$$\begin{tikzcd} K(Y) \arrow[r, "f_!"] \arrow[d, "\mathrm{Td}(T_Y)\cdot\mathrm{ch}"'] & K(X) \arrow[d, "\mathrm{Td}(T_X)\cdot\mathrm{ch}"] \\ A^\bullet(Y) \arrow[r, "f_*"] & A^\bullet(X) \end{tikzcd}$$

commutes. In other words, for $a \in K(Y)$

$$f_*\big(\mathrm{Td}(T_Y)\cdot \mathrm{ch}(a)\big) = \mathrm{Td}(T_X)\cdot \mathrm{ch}(f_!(a)).$$

$\square$

From here we deduce the following universal property.

Theorem 1.2.9 (Universal property of $GK_{\mathbb{Q}}$) *Let k be an field of characteristic zero or algebraically closed and $A^\bullet$ be a graded $\mathbb{Q}$-cohomology theory defined on smooth varieties over k. There exist a unique morphism of graded $\mathbb{Q}$-cohomology theories*

$$GK_{\mathbb{Q}} \longrightarrow A^\bullet.$$

Proof: First we construct a morphism from $GK(X)$ to $A(X)$ for all X and compatible with inverse images. Let Z be a closed subvariety of X of codimension d. If Z is smooth then by the Grothendieck-Riemann-Roch theorem applied to $i\colon Z \to X$, we have that

$$\mathrm{ch}(\mathcal{O}_Z) = \eta_Z^X + \ldots \in \textstyle\bigoplus_{p\geq d} A^p(X).$$

In the general case, since k is algebraically closed or characteristic zero, Z is smooth on the open complement of a closed subvariety Z_{sing} of codimension $> d$. Denote $j\colon U = X - Z_{\mathrm{sing}} \to X$. We have injective maps

$$j^*\colon A^p(X) \longrightarrow A^p(U),\ p \leq d.$$

Since $j^*(\mathrm{ch}(\mathcal{O}_Z)) = \mathrm{ch}(j^!\mathcal{O}_Z) = p_U(Z\cap U) + ...$ it follows that the Chern character $\mathrm{ch}\colon K(X) \to A^\bullet(X)$ preserves the filtrations so that it induces a morphism of rings $p_X\colon GK(X) \to A^\bullet(X)$ compatible with inverse images.

Due to Panin's lemma, if p preserves the first Chern class of line bundles then p is a morphism of cohomology theories. Denote $\alpha = c_1(L) \in A^1(X)$, then

$$p_X[1-L^*] = p_X[L-1] = [\mathrm{ch}(L-1)] = [e^\alpha - 1] = [\alpha + \ldots] = \alpha.$$

so p is morphism of cohomology theories.

Since the group $GK(X)$ is generated by the class of closed subvarieties, the morphism of cohomology theories p is unique by construction. $\square$

Chapter 2

Preliminaries

2.1 Simplicial sets

We recall in this section some concepts and notations regarding simplicial sets that we will use afterwards. There are many references on the subject, the reader may check [GJ99] for a detailed exposition including the classification of torsors.

Definition 2.1.1 We denote by Δ the category consisting of the finite ordered sets

$$[n] = \{0 < 1 < 2 \cdots < n\}$$

for $n \in \mathbb{N}$ and order-preserving maps. A **simplicial set** is a functor

$$\Delta^{\mathrm{op}} \to \mathbf{Sets}$$

where **Sets** is the category of sets and a **map of simplicial sets** is a natural transformation. We denote by **sSets** the category of simplicial sets with maps of simplicial sets.

Remark 2.1.2 The category Δ has distinguished maps. Consider the map $d^i \colon [n-1] \to [n]$ which is injective and "skips i ". In other words,

$$d^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i. \end{cases}$$

Consider $s^i \colon [n+1] \to [n]$ the map which is surjective and "hits i twice". More concretely,

$$s^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i. \end{cases}$$

It is easy to check that every map in Δ has a factorization as a composition of these maps. Moreover, these maps satisfy the so called *cosimplicial identities*. Let us recall them:

$$
\begin{aligned}
d^j d^i =& d^i d^{j-1} \text{ for } i < j \\
s^j s^i =& s^i s^{j+1} \text{ for } i > j \\
s^j d^i =& \begin{cases} d^i s^{j-1} & \text{for } i < j \\ \mathrm{id} & \text{for } i = j \ , \ j+1 \\ d^{i-1} s^j & \text{for } i > j+1. \end{cases}
\end{aligned}
$$

As a consequence, a simplicial set X is defined by the sets X_n image of $[n]$ and the maps d_i, s_i image of d^i and s^i respectively. In other words, by a diagram

$$
\begin{tikzcd}
X_0 \arrow[r, shift left] \arrow[r, leftarrow, shift right] & X_1 \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & X_2 \ \cdots .
\end{tikzcd}
$$

The maps d_i and s_i are called *face* and *degeneracy maps* respectively and satisfy at least the transpose of the cosimplicial identities. They can satisfy more relations.

Example 2.1.3

- Consider $n \in \mathbb{N}$. The functor

$$[q] \mapsto \mathrm{Hom}_\Delta([q], [n]) = \{\text{order preserving maps } [q] \to [n]\}$$

is a simplicial set that we call the **standard simplicial n-simplex** Δ^n. Note that the face and degeneracy maps satisfy no more than the transpose of the cosimplicial identities.

- Every set X defines a simplicial set by setting $X_i = X$ and setting all face and degeneracy maps to be the identity. We still denote this simplicial set as X so, in particular, $*$ denotes a point seen as a simplicial set.

- Let $S_0 = *$ and $S_1 = \{r, s_0(*)\}$ have only one nondegenerate element with the only possible face a degeneracy maps between them. Let S_m have only degenerate elements for $m > 1$ with the natural face and degeneracy maps. We call this simplicial set the *simplicial circle* S^1. We analogously define the simplicial n-sphere S^n who has only two non degenerate elements, one in $S^n_0 = *$ and the other in S^n_n.

- We denote $\Lambda^n[i]$ the union of all faces except the i-th one of the standard simplicial set Δ^n and we call it the **i-th horn** of Δ^n.

Let us recall two notions that will be use later.

Definition 2.1.4 We say that a simplicial set has **simplicial dimension zero** if every element on X_i for all $i > 0$ is a degeneration of an element in X_{i-1}.

We denote by $\mathbf{sSets}_\bullet$ the category of **pointed simplicial sets** whose objects are morphisms of simplicial sets $* \to X$, which we may simply denote X, and morphisms are maps of simplicial sets which maps the distinguished point onto the distinguished point.

Example 2.1.5

- Ever simplicial set X defines a pointed simplicial set $X_+ = X \sqcup *$.
- The simplicial n-sphere is naturally pointed since $S_0^n = *$.

Definition 2.1.6 Denote **Top** the category of topological spaces. We define the **standard topological n-simplex** to be

$$\Delta[n] = \Big\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \ : \ \sum_{i=0}^{n} t = 1 \text{ and } t_j \in [0,1] \ \forall\ 0 \le j \le n \ \Big\} \subset \mathbb{R}^{n+1}.$$

It is easy to check that the collection of standard topological n-simplices defines a functor $\Delta \to \mathbf{Top}$ which sends the distinguished maps d^i and s^i to:

$$\begin{aligned} d^i(t_0, \ldots, t_{n-1}) &= (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \\ s^i(t_0, \ldots, t_{n+1}) &= (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}) \end{aligned}$$

If X is a simplicial set, we define the **realization** of X to be the quotient topological space

$$|X| = (\coprod_n X_n \times \Delta[n])/\sim$$

where the equivalence relation is defined as follows: if $(x, u) \in X_m \times \Delta^n$ and $\varphi : [n] \to [m] \in \Delta$ then $(\varphi^* x, u) \sim (x, \varphi_* u)$. It is easy to check that the realization defines a functor

$$|\_| : \mathbf{sSets} \to \mathbf{Top}.$$

Example 2.1.7 As expected, the realization of the standard simplicial n-simplex Δ^n is the standard topological n-simplex $\Delta[n]$ and the realization of the simplicial sphere S^n is the classic sphere. Note that the degeneracy maps

Definition 2.1.8 Let A be a topological space, we define the **singular complex** of A to be the simplicial set $\operatorname{sing} A$ defined as

$$(\operatorname{sing} A)_n = \operatorname{Hom}_{\mathbf{Top}}(\Delta[n], Y)$$

with the natural face a degeneracy maps given by those of $\Delta[n]$. The singular complex defines a functor

$$\mathrm{sing}\colon \mathbf{Top} \to \mathbf{sSets}.$$

For the sake of completeness, let us remark the following direct result.

Proposition 2.1.9 *The realization functor is left adjoint of the singular functor. In other words, for every simplicial set X and every topological space A we have*

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, A) = \mathrm{Hom}_{\mathbf{sSets}}(X, \mathrm{sing}\, A).$$

□

Notation 2.1.10 Let $f\colon \mathbf{A} \to \mathbf{B}$ and $g\colon \mathbf{B} \to \mathbf{A}$ be two functors. On the following we will say that the pair of functors

$$f\colon \mathbf{A} \leftrightarrows \mathbf{B} : g$$

are adjoint to mean that the functor f is left adjoint to g or, equivalently, that g is right adjoint to f. We also write that the pair of functors (f, g) are adjoint

The realization of a pointed simplicial set defines a pointed topological space. Therefore, we may consider the following definitions.

Definition 2.1.11 Let X be a pointed simplicial set. We define the **n-th homotopy group** of X to be $\pi_n(X) = \pi_n(|X|)$.

Let X and Y be two simplicial sets, a map of simplicial sets $f : X \to Y$ is a **weak equivalence** if the induced map $|f|\colon |X| \to |Y|$ is a weak equivalence for any choice of base point.

Definition 2.1.12 Let X and Y be pointed sets. We define the **wedge** $X \vee Y$ to be the set

$$X \sqcup Y/{*_X} \sim {*_Y}.$$

Note that this set is the coproduct of X and Y in the category of pointed sets. We define the **smash product** of X and Y to be

$$X \wedge Y = X \times Y/X \vee Y,$$

which is the product of X and Y in the category of pointed sets. If X and Y are pointed simplicial sets we define analogously the wedge and smash product.

Let X be a pointed simplicial set, we define the **suspension** of X to be $\Sigma X = S^1 \wedge X$ where S^1 is the simplicial circle.

Example 2.1.13 It is easy to check that $S^n \wedge S^1 = S^{n+1}$.

The category of **sSets** and **sSets$_\bullet$** have *internal* Hom *objects*. More concretely, if X and Y are simplicial sets denote $\underline{\mathrm{Hom}}(X,Y)$ the simplicial set defined by

$$\underline{\mathrm{Hom}}(X,Y)_q = \mathrm{Hom}_{\mathbf{sSets}}(X \times \Delta^q, Y)$$

with natural face and degeneracy maps. If X and Y are pointed simplicial sets denote $\underline{\mathrm{Hom}}(X,Y)$ as well the pointed simplicial set defined by

$$\underline{\mathrm{Hom}}(X,Y)_q = \mathrm{Hom}_{\mathbf{sSets}_\bullet}(X \wedge \Delta^q, Y).$$

Remark 2.1.14 Let X and Y be pointed topological spaces, by abuse of notation we will still denote $\underline{\mathrm{Hom}}(X,Y)$ to the simplicial set given by

$$\underline{\mathrm{Hom}}(X,Y)_q = \mathrm{Hom}_{\mathbf{Top}}(X \wedge \Delta[q], Y).$$

The following result is immediate.

Proposition 2.1.15 *Let X be a pointed simplicial set, the pair of functors*

$$X \wedge \ \_ : \mathbf{sSets}_\bullet \leftrightarrows \mathbf{sSets}_\bullet : \underline{\mathrm{Hom}}(X, \ \_ \ )$$

are adjoint.

□

Definition 2.1.16 Let X be a pointed simplicial set, we define the **loop space** of X to be the simplicial set

$$\underline{\mathrm{Hom}}(S^1, \mathrm{sing}\,|X|).$$

We are ready for the definition of spectra.

Definition 2.1.17 Let T be a pointed simplicial set. A T-**spectrum** is a sequence of pointed simplicial sets $E = \{E^0, E^1, E^2, \ldots\}$ together with maps $T \wedge E^k \to E^{k+1}$ for $k \geq 0$. If no confusion is possible we call it simply a spectrum. A **map of spectra** $f\colon E \to F$ is a sequence of maps $f\colon E^k \to F^k$ compatible with the structural maps, i.e., such that the diagrams

$$\begin{tikzcd} T \wedge E^k \arrow[r] \arrow[d, "1_T \wedge f^k"'] & E^{k+1} \arrow[d, "f^{k+1}"] \\ T \wedge F^k \arrow[r] & F^{k+1} \end{tikzcd}$$

commute. We denote by **Spt** the category of spectra.

Finally, let us recall a concept we will use later.

Definition 2.1.18 Let f_1, $f_2\colon Y \to X$ be two maps in a category $\mathbf{C}$. We say that $g\colon Z \to Y$ is the **equalizer** of f_1 and f_2 if it is universal among morphisms $h\colon W \to Y$ such that $f_1 \circ h = f_2 \circ h$. We say that the sequence

$$Z \xrightarrow{g} Y \overset{f_1}{\underset{f_2}{\rightrightarrows}} X$$

is **exact**.

Note that if f_1 and f_2 are maps of simplicial sets then $g\colon Z \to Y$ is the equalizer of f_1 and f_2 if and only if g_n is the equalizer of $f_{1,n}$ and $f_{2,n}$. The sequence

$$Z \xrightarrow{g} Y \overset{f_1}{\underset{f_2}{\rightrightarrows}} X$$

is exact if and only if g_n, $f_{1,n}$ and $f_{2,n}$ give exact sequences of sets for every n.

2.1.1 Classification of torsors

Every group in this section is considered to be an abelian group.

Definition 2.1.19 Let (G, μ, e) be a group where μ and e denote the operation and the neutral element respectively. An **action** of G on a simplicial set X is morphism $a\colon G \times X \to X$ such that the diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu\times 1_X"] \arrow[d, "1_G\times a"'] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"] & X \end{tikzcd} \quad , \quad \begin{tikzcd} X \arrow[r, "e\times 1_X"] \arrow[dr, equal] & G \times X \arrow[d, "a"] \\ & X \end{tikzcd}$$

commute. We also say that X is a **G-set**. We say that the action is **free** if the map $G \times X \to X \times X$, which maps (g, x) to $(a(g, x), x)$, is a monomorphism. Let X and Y be G-sets, we say that a map of simplicial sets $f\colon X \to Y$ is a **morphism of G-sets** if for every $x \in X_n$ we have $f(g(x)) = g(f(x))$.

A **G-torsor** over X is a morphism $T \to X$ of simplicial sets (or sets) with a free action of G on T over X such that the canonical morphism $T/G \to X$ is an isomorphism. Let $p\colon T \to X$ and $p'\colon T' \to X$ be G-torsors over X, we say that a morphisms of G-sets $f\colon T \to T'$ is a **morphism of G-torsors** if $p = p' \circ f$. We denote by $P(X, G)$ the set of isomorphism classes of G-torsors over X.

Example 2.1.20
- Consider the second projection $\pi\colon G \times X \to X$, then $G \times X$ is the trivial G-torsor. Note that therefore the set $P(X, G)$ is non empty and we chose the trivial torsor as a base point.

- Let $f\colon Y \to X$ be a morphism of simplicial sheaves and $T \to X$ be a G torsor. Then $f^*T = T \times_X Y$ is a G-torsor over Y.

Definition 2.1.21 Let $T \to X$ be a G-torsor, $x \in X_n$ be an n-simplex and $\Delta^n \to X$ the morphism defined by x. We define the **fiber at** x to be the simplicial set $F_x = T \times_X \Delta^n$.

The following lemma is direct.

Lemma 2.1.22 *Let G be a group, T be a G-torsor over X and $x \in X_n$ be an n-simplex. Then $F_x = \Delta^n \times G$ is the trivial torsor over Δ^n. Moreover, given an action of G on a simplicial set T over X, then T is a G-torsor over X if and only if T_n is a G-torsor (of sets) over X_n for all n.*

Definition 2.1.23 We say that a morphism of simplicial sets $f\colon X \to Y$ is a **Kan fibration** if for every horn $\Lambda^n[k] \to \Delta^n$ and every commutative diagram

$$\begin{tikzcd} \Lambda^n[k] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

there exists a map $\Delta^n \to X$ making the diagram

$$\begin{tikzcd} \Lambda^n[k] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

commutative. We say that a simplicial set X is **Kan fibrant** if the projection $X \to *$ is a Kan fibration.

Proposition 2.1.24 *Let T be a G-torsor over X, then the canonical projection $T \to X$ is a Kan fibration.*

Proof: We have to prove that for every n-simplex $x\colon \Delta^n \to X$ and any i-th horn there is the lifting property for the diagram

$$\begin{tikzcd} \Lambda^n[i] \arrow[r] \arrow[d] & T \arrow[d] \\ \Delta^n \arrow[r, "x"] & X. \end{tikzcd}$$

We can replace T by $F_x = G \times \Delta^n$ and the proof is direct.

□

The abstract definition of homotopy in a model category defines in **sSets** the classical notion. We recall it.

Definition 2.1.25 Two maps of simplicial sets $f, g\colon X \to Y$ are **homotopic** if there is a commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "i_0 \sqcup i_1"] \arrow[dr, "f \sqcup g"'] & X \times \Delta^1 \arrow[d, "h"] \\ & Y. \end{tikzcd}$$

The map $h\colon X \to Y$ is called a **homotopy** between f and g.

Proposition 2.1.26 *Let T be a G-torsor over Y and $f, g\colon X \to Y$ be two homotopic maps. Then*

$$f^*T \simeq g^*T.$$

Proof: By hypothesis there is a homotopy $X \times \Delta^1 \to Y$ between f and g. Therefore it is enough to prove that for any G-torsor T over $X \times \Delta^1$ the restriction $i_0^*T \to X$ is isomorphic to $i_1^*T \to X$. Note that there exists a lifting

$$\begin{tikzcd} i_0^*T \arrow[r] \arrow[d, "i_0"'] & T \arrow[d, "p"] \\ i_0^*T \times \Delta^1 \arrow[r] \arrow[ur, dashed] & X \times \Delta^1 \end{tikzcd}$$

since i_0 is a trivial cofibration and p is a Kan fibration. Since the map $i_0^*T \times \Delta^1 \to T$ is a morphism of G-torsors it is bijective. We conclude by applying the same argument to i_1.

□

Corollary 2.1.27 *Every G-torsor T over a contractible space is trivial.*

2.1.28 Let G be a group. We denote by EG the simplicial set defined in each term as

$$EG_n = G \times \overset{n+1}{\cdots} \times G$$

with face maps given by the diagonal and degeneracy maps given by projections. Note that EG is naturally a simplicial group and also has an action of G. Denote

$$BG = EG/G$$

and note that the natural projection turns EG into a G-torsor over BG. We call EG the **universal G-torsor** and BG the **classifying space of G**.

Proposition 2.1.29 *Let G be a group. The simplicial set EG is weakly contractible.*

Proof: Note that there are natural maps $e_n\colon EG_n \to EG_{n+1}$ defined as $e_n((g_0,\dots,g_n)) = (e, g_0, \dots, g_n)$. Denote $\operatorname{cone} EG$ the simplicial set defined as

$$\operatorname{cone} EG = \varinjlim_{p\colon \Delta^n \to EG} \Delta^{n+1}.$$

Clearly $\pi_0(\operatorname{cone} EG) = \pi_0(EG)$ and $\pi_i(\operatorname{cone} EG) = 0$ for $i > 0$ and a direct computation shows $\pi_0(EG) = \{*\}$.

The coface maps $d^0\colon [n] \to [n+1]$ induce inclusions $d^0\colon \Delta^n \to \Delta^{n+1}$ and therefore an inclusion $j\colon EG \to \operatorname{cone} EG$. It is enough to prove that the identity 1_{EG} factors as

$$\begin{tikzcd} EG \arrow[r, "1_{EG}"] \arrow[d, "j"'] & EG \\ \operatorname{cone} EG. \arrow[ur, "f"'] & \end{tikzcd}$$

Define f as follows. By construction every n-simplex $x\colon \Delta^n \to \operatorname{cone} EG$ is defined by an $n-1$-simplex $y = (g_0, \dots g_n)\colon \Delta^{n-1} \to EG$ such that the diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "x"] & \operatorname{cone} EG \\ \Delta^n \arrow[u, "d_0"] \arrow[r, "y"] & EG. \arrow[u, "j"'] \end{tikzcd}$$

commutes. Define $f(x) = (e, g_0, \dots, g_n)$.

$\square$

Proposition 2.1.30 *If G is a simplicial group, then G is Kan fibrant. In particular, if G is a group the simplicial set EG is fibrant.*

Proof: Let G be a simplicial group and $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ be in G_{n-1} defining a i-th horn $\Lambda^n[i] \to G$. We give the explicit n-simplex that fill the horn:

- If $i = 0$. Set $y_n = s_{n-1}(x_n) \in G_n$ and $y_k = y_{k+1} \cdot (s_{k-1}d_k(y_{k+1}))^{-1} \cdot s_{k-1}(x_k)$ for $k = n, \cdots, 1$. Then $y_1 \in G_n$ is the filler n-simplex.

- If $0 < i < n$ let $y_0 = s_0(x_0)$ and $y_k = y_{k-1} \cdot (s_k d_k(y_{k-1}))^{-1} s_k(x_k)$ for $k = 0, \dots, i-1$. Now set $y_n = y_{i-1} \cdot (s_{n-1}d_n(y_{i-1}))^{-1} \cdot s_{n-1}(x_n)$ and take $y_k = y_{k+1} \cdot (s_{k-1}d_k(w_{k+1})^{-1} \cdot s_{k-1}(x_k)$ for $k = n, \cdots, k+1$. Then y_{i+1} is the filler.

- If $i = n$. Set $y_0 = s_0(x_0)$ and $y_k = y_{k-1} \cdot (s_k d_k(y_{k-1}))^{-1} \cdot s_k(x_k)$ for $k = 0, \dots, n-1$. Then y_{n-1} is the filler

$\square$

Corollary 2.1.31 *Let G be a group, the simplicial set BG is fibrant.*

Proof: Consider a horn $\Lambda^n[i] \to BG$. Note that, since $\Lambda^n[i]$ is contractible, the pullback of EG is therefore isomorphic to $G \times \Lambda^n[i]$. Therefore we have a diagram

$$\begin{tikzcd} G \times \Lambda^n[i] \arrow[r] \arrow[d] & EG \arrow[d] \\ \Lambda^n[i] \arrow[r] \arrow[ur, dashed] & BG. \end{tikzcd}$$

Since EG is Kan fibrant the horn $\Lambda^n[i] \to EG$ can be filled, so the horn of BG can also be filled.

□

Theorem 2.1.32 *Let G be a group and X be a simplicial set. Then the natural map*

$$[X, BG] \longrightarrow P(X, G)$$

given by $f \mapsto f^*EG$ is a bijection.

Proof: Recall that since BG is fibrant and X is cofibrant we have that the set $\mathrm{Hom}_{\mathbf{Ho(sSets)}}(X, BG) = [X, BG]$ is the set $\mathrm{Hom}_{\mathbf{sSets}}(X, BG)/\sim$, where $\sim$ denotes the homotopy relation. Due to Proposition 2.1.26 the map is well defined. It is enough to construct the inverse. Let T be a G-torsor over X and consider $EG \times_G T = EG \times T/G$. The first projection induces a map $p_1 \colon EG \times_G T \to EG/G = BG$ and the second projection induces a map $p_2 \colon EG \times_G T \to T/G = X$. Observe that p_2 is a simplicial weak equivalence since the fibers are isomorphic to EG. Therefore p_2 has an inverse in the homotopy category which defines an element in $[X, BG]$. This assignation is inverse to the map of the statement since $p_1^*EG \simeq p_2^*T \simeq EG \times T$.

□

Let us introduce some notation for a result that we will need for the case of sheaves of simplicial sets.

Definition 2.1.33 Let $f \colon X \to Y$ be a map of simplicial sets. We define the **relative** π_0 to be the simplicial set $\pi_0(f)$ defined on every n by the set

$$\pi_0(\underline{\mathrm{Hom}}(\Delta^n, X) \times_{\underline{\mathrm{Hom}}(\Delta^n, Y)} Y_n)$$

with the natural face and degeneracy maps. Note that we had abused notation to write Y_n for the simplicial set defined by the set Y_n. Also note that there is a map of simplicial sets $Y_n \to \underline{\mathrm{Hom}}(\Delta^n, Y)$ which maps any $y \in (Y_n)_0 = Y_n$ to the n-simplex $\Delta^n \to Y$ it defines.

Proposition 2.1.34 *Let $p\colon T \to X$ be a G-torsor and $f\colon X \to Y$ be a trivial Kan fibration. Then $\pi_0(f \circ p)$ is a G-torsor over Y.*

Proof: Consider a n-simplex $y\colon \Delta^n \to Y$. We have cartesian squares

$$\begin{tikzcd} F'_y \arrow[r] \arrow[d] & F_y \arrow[r, "g"] \arrow[d] & \Delta^n \arrow[d] \\ T \arrow[r, "p"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

where F_y and F'_y denote the fiber of f and $p \circ f$ respectively. By Lemma 2.1.22 it is enough to prove that $\pi_0(F'_y)$ is a G-torsor over Δ^n. Note that g is a trivial Kan fibration and therefore F_y is contractible. In addition, F'_y is a G-torsor over F_y so that $F'_y = G \times F_y$. We finish recalling that G has the discrete topology.

□

2.2 Big Nisnevich site

The original reference for topos theory is [SGA4] and every result is a consequence of this section follows from them. However it is written with a generality we do not use so we follow a more concrete approach.

In this section $\mathbf{C}$ will be a small category with fiber products.

Remark 2.2.1 Since we will not compare topologies we will not introduce the whole notation of Grothendieck topologies. In particular, we will not define the concept of sieves. Notice that the definition we are using is analogue to the *basis* of a topology in the classical sense. Apart from studying when two different basis define the same topology, all concepts and definitions of topology may be given in terms of a basis.

Definition 2.2.2 Let $\mathbf{C}$ be a category. A **Grothendieck topology** τ on $\mathbf{C}$ is the data for each object X of $\mathbf{C}$ of a set $\mathrm{Cov}_\tau(X)$ of **coverings** of X, where a covering is a set of morphisms $\{f_\alpha\colon U_\alpha \to X\}$ in $\mathbf{C}$, satisfying:

1. The identity 1_X is in $\mathrm{Cov}_\tau(X)$ for any X.

2. Let $g\colon Y \to X$ be a morphism in $\mathbf{C}$ and $\{f_\alpha\colon U_\alpha \to X\}$ be a covering of X, then $\{p_2\colon U_\alpha \times_X Y \to Y\}$ is a covering of Y.

3. Let $\{f_\alpha\colon U_\alpha \to X\}$ be a covering of X and $\{g_{\alpha\beta}\colon V_{\alpha\beta} \to U_\alpha\}$ for every index α be a covering of U_α, then $\{f_\alpha \circ g_{\alpha\beta}\colon V_{\alpha\beta} \to X\}$ is a covering of X.

We call the pair $(\mathbf{C}, \tau)$ of a category with a Grothendieck topology a **site**.

Example 2.2.3
- Let T be a topological space and let $\mathbf{C} = \mathbf{Op}(T)$ where $\mathbf{Op}(T)$ is the category of open subsets of T with morphisms the inclusion of subsets. For any open subset $U \subset T$ we let $\mathrm{Cov}_T(U)$ be the family of classic coverings. They define a Grothendieck topology on $\mathbf{Op}(T)$.

- Let T be a topological space and B be a basis of its topology such that if $U \in B$ then any open set $V \subset U$ is in B. For example, consider coordinated open sets in a smooth manifold, or open sets in $\mathbb{R}^n$ with compact closure. Consider for any open subset $U \subset T$ the family $\mathrm{Cov}'_T(U)$ of classic coverings of U made of basic open subsets (and the identity). They define another Grothendieck topology on $\mathbf{Op}(T)$. Our definition is not convenient to compare them.

Now we give some examples of topologies on categories of schemes. Denote $\mathbf{Sch}_S$ the category of schemes over a base scheme S and $\mathbf{Sm}_S$ its smooth variant.

- Let X be a scheme and consider $\mathbf{C} = \mathbf{Op}(X)$ the category of Zariski open subschemes of X and morphisms the immersions of open subschemes. For every open subscheme U of X we let $\mathrm{Cov}_{\mathrm{zar}}(Y)$ be the family of coverings (in the usual sense) of U made of open subschemes. We call *Zariski topology* and *small Zariski site* the topology and site they define. We denote the site $X_{\mathbf{zar}}$. Note that this example is a particular case of the first example.

- We define the *Zariski topology* on $\mathbf{Sch}_S$ as follows. Let X be an S-scheme, set $\mathrm{Cov}_{\mathrm{Zar}}(X) = \mathrm{Cov}_{\mathrm{zar}}(X)$. We call the *big Zariski site* the site they define. We denote the site $\mathbf{Sch}_{S,\mathbf{Zar}}$ and $\mathbf{Sm}_{S,\mathbf{Zar}}$ its smooth variant.

- Let X be a scheme and consider $\mathbf{C} = \mathbf{\acute{e}t}(X)$ the category of étale morphisms $Y \to X$ of finite type and morphisms of X-schemes. For every étale morphism of finite type $U \to X$ we let $\mathrm{Cov}_{\acute{e}t}(U)$ be the set of families $\{f_\alpha \colon U_\alpha \to U\}$ where f_α is étale of finite type and such that $\coprod |U_\alpha| \to |U|$ is surjective. We call *étale topology* and *small étale site* the topology and site they define. We denote the site $X_{\mathbf{\acute{e}t}}$.

- As in the Zariski topology, we define the *étale topology* on $\mathbf{Sch}_S$ considering for every scheme X the set of coverings $\mathrm{Cov}_{\acute{E}t}(X) = \mathrm{Cov}_{\acute{e}t}(X)$. We call the *big étale site* the site they define and denote it $\mathbf{Sch}_{S,\mathbf{\acute{E}t}}$ and $\mathbf{Sm}_{S,\mathbf{\acute{E}t}}$ its smooth variant.

The main example of topology we will use is the following one, which is finer than the Zariski topology and coarser than the étale topology.

- Let X be a scheme and consider $\mathbf{C} = \mathbf{\acute{e}t}(X)$ as before. For every étale morphism of finite type $U \to X$ we define the set of **Nisnevich coverings** $\mathrm{Cov}_{\mathrm{nis}}(U)$ to be the set of étale coverings $\{f_\alpha \colon U_\alpha \to U\}$ such that for every (possibly non-closed) point $x \in U$ there exists a point $y \in U_\alpha$ for some α such that $f_\alpha(y) = x$ and the induced map of residue fields $k(x) \to k(y)$ is an isomorphism. We call **Nisnevich topology** and *small Nisnevich site* the topology and site they define. We denote the site $X_{\mathbf{nis}}$.

- We define the *Nisnevich topology* on $\mathbf{Sch}_S$ considering for every scheme X the set of *Nisnevich coverings* $\mathrm{Cov}_{\mathrm{Nis}}(X) = \mathrm{Cov}_{\mathrm{nis}}(X)$. We call the *big Nisnevich site* the site they define and denote it $\mathbf{Sch}_{S,\mathbf{Nis}}$ and $\mathbf{Sm}_{S,\mathbf{Nis}}$ its smooth variant.

For the sake of completeness let us recall other topologies that appear in motivic homotopy theory.

- Recall that a morphism of schemes $f \colon Y \to X$ is a *topological epimorphism* if the map on the underlying topological spaces $|f| \colon |Y| \to |X|$ identifies $|X|$ with a quotient of $|Y|$ (equivalently, if $|f|$ is surjective and $U \subset |X|$ is open if and only if $|f|^{-1}(U)$ is open in $|Y|$). As usual, we say that f is a *universal* topological epimorphism if it is a topological epimorphism for any base change. We define the h-**topology** on $\mathbf{Sch}_S$ (or $\mathbf{Sm}_S$) to be the topology given by the sets $\mathrm{Cov}_h(X)$ of h-**coverings** made of finite sets $\{f_i \colon U_i \to X\}$ such that each f_i is of finite type and $\coprod_i U_i \to X$ is a universal topological epimorphism.

- We define the *qfh*-**topology** on $\mathbf{Sch}_S$ (or $\mathbf{Sm}_S$) to be the topology given by the sets $\mathrm{Cov}_{qfh}(X)$ of *qfh*-**coverings** made of h-coverings such that each f_i is quasi-finite.

- We define the *cdh*-**topology** on $\mathbf{Sch}_S$ (or $\mathbf{Sm}_S$) to be the topology given by the sets $\mathrm{Cov}_{cdh}(X)$ of *cdh*-**coverings** which is generated by:

  1. Nisnevich coverings.
  2. Families of the form $\{X' \coprod F \xrightarrow{p \coprod i} X\}$ with $p \colon X' \to X$ proper, $i \colon F \to X$ a closed immersion and $p \colon p^{-1}(X \setminus i(F)) \to X \setminus i(F)$ is an isomorphism.

Apart from Nisnevich coverings, the main example of a *cdh*-covering are those given by the blowing-up $\pi\colon B_Y X \to X$ of a closed subscheme $i\colon Y \to X$. The set consisting of one element $\{B_Y X \coprod Y \xrightarrow{\pi \coprod i} X\}$ is a *cdh*-covering.

2.2.1 Sheaves and presheaves

On the following we always consider the Nisnevich topology on $\mathbf{Sm}_S$ unless otherwise stated.

Definition 2.2.4 We define a **presheaf** (of sets) on $\mathbf{Sm}_S$ to be a contravariant functor $F\colon \mathbf{Sm}_S \to \mathbf{Sets}$. We say that a presheaf F is a **Nisnevich sheaf** (of sets) on $\mathbf{Sm}_{S,\mathbf{Nis}}$ if for every smooth scheme X and every covering family $\{f_\alpha\colon U_\alpha \to X\}$ the sequence of sets

$$F(X) \xrightarrow{\prod f_\alpha^*} \prod_\alpha F(U_\alpha) \underset{p^*_{2,\alpha,\beta}}{\overset{p^*_{1,\alpha,\beta}}{\rightrightarrows}} \prod_{\alpha,\beta} F(U_\alpha \times_X U_\beta) \tag{2.1}$$

is exact.

Analogously, we define a *presheaf of simplicial sets* on $\mathbf{Sm}_S$ to be a contravariant functor $F\colon \mathbf{Sm}_S \to \mathbf{sSets}$. We say that presheaf of simplicial sets is a *Nisnevich sheaf of simplicial sets* on $\mathbf{Sm}_S$ if for every smooth scheme X and every covering family $\{f_\alpha\colon U_\alpha \to X\}$ the analogue of the sequence (2.1) is an exact sequence of simplicial sets. We denote by $\mathbf{PreShv}(S)$ and $\mathbf{Shv}(S)$ the category of presheaves and Nisnevich sheaves of sets respectively. Note that $\mathbf{Shv}(S)$ is a full subcategory of $\mathbf{PreShv}(S)$.

Remark 2.2.5 In general, one can define presheaves with values on a general category and sheaves on any Grothendieck topology with values on a general category having arbitrary small products. In particular, the properties of this section and the following hold for sheaves of simplicial sets, sheaves of T-spectra and sheaves of abelian groups with minor changes (*cf.* Remark 2.2.26).

Definition 2.2.6 We call a **distinguished square** to a cartesian diagram on $\mathbf{Sm}_S$

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd} \tag{2.2}$$

such that j is an open immersion, f is étale such that the induce map from $f^{-1}((X-U)_{\mathrm{red}})$ to $(X-U)_{\mathrm{red}}$ is an isomorphism. Note that for every distinguished square the set $\{j\colon U \to X\ ,\ f\colon V \to X\}$ is a Nisnevich covering of X.

We prove an important property of the Nisnevich topology in Theorem 2.2.9. We learn it from [MV99].

Definition 2.2.7 Let X be a smooth scheme and $\mathcal{U} = \{f_\alpha \colon U_\alpha \to X\}$ be a Nisnevich covering of X. We say that a sequence of closed subsets of X of the form

$$\emptyset = Z_n \subset Z_{n-1} \subset \cdots \subset Z_0 = X$$

is a **splitting sequence** for $\mathcal{U}$ if there are indices $\alpha_0, \ldots, \alpha_{n-1}$ such that the morphisms

$$f_{\alpha_i}|_{f_{\alpha_i}^{-1}(Z_i - Z_{i+1})} \colon f_{\alpha_i}^{-1}(Z_i - Z_{i+1}) \to Z_i - Z_{i+1}$$

split, i.e., have sections s_i.

Lemma 2.2.8 *Let X be a smooth scheme and $\mathcal{U}$ be a Nisnevich covering of X. Then there exists a splitting sequence for $\mathcal{U}$.*

Proof: Since the scheme X is Noetherian it has a finite number of irreducible components $X = C_1 \cup \cdots \cup C_r$. Let x_1 be the generic point of C_1. By definition, there exist a U_{α_1} of $\mathcal{U}$ and $y_1 \in U_{\alpha_1}$ such that f_{α_1} induces an isomorphism $k(x_1) \to k(y_1)$ on the residue fields. As a result, there exists a closed subscheme $C_1' \subset C_1$ such that $f_{\alpha_1}|_{C_1 - C_1'}$ is an isomorphism and therefore has a section. Consider $Z_1 = C_1' \cup C_2 \cup \cdots C_r$, then $X = Z_0$ and Z_1 satisfy the condition of the splitting sequence. We apply the same argument to Z_1 and obtain Z_2. Since X is Noetherian, iterating this process we obtain $Z_n = \emptyset$ for some n. Therefore $Z_n \subset \cdots \subset Z_0$ is a splitting sequence for $\mathcal{U}$.

□

Theorem 2.2.9 *A presheaf of sets F on $\mathbf{Sm}_S$ is a sheaf for the Nisnevich topology if and only if for any distinguished square as in (2.2) the following diagram of sets*

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(W) \end{tikzcd}$$

is cartesian.

Proof: Let F be a Nisnevich sheaf. Since $\{j \colon U \to X \ , \ f \colon V \to X\}$ is a Nisnevich covering we an exact sequence

$$F(X) \longrightarrow F(U) \times F(V) \rightrightarrows F(U \times_X U) \times F(U \times_X V) \times F(V \times_X U) \times F(V \times_X V).$$

Note that the morphisms onto $F(U \times_X V)$ and $F(V \times_X U)$ give the same conditions. The morphisms onto $F(U \times_X U)$ do not give any condition since $U \times_X U = U$. The morphisms onto $F(V \times_X V)$ do not give any further condition because $W \times_U W \to V \times XV$ and $\Delta\colon V \to V \times_X V$ define a Nisnevich covering of $V \times_X V$. Therefore, the previous exact sequence is equivalent to

$$F(X) \longrightarrow F(U) \times F(V) \rightrightarrows F(U \times_X V).$$

This last sequence is clearly equivalent to the cartesian diagram of the statement of the theorem.

Let F be a presheaf satisfying the hypothesis of the theorem. Let X be a smooth scheme and $\mathcal{U} = \{f_\alpha\colon U_\alpha \to U\}$ be a Nisnevich covering of X. We want to show that the sequence

$$F(X) \longrightarrow \coprod F(U_\alpha) \rightrightarrows \coprod F(U_\alpha \times_X U_\beta)$$

is exact. We proceed by induction on the minimal length of a splitting sequence of $\mathcal{U}$. If $\mathcal{U}$ has a splitting sequence of length zero then the result is direct.

Now, assume $\mathcal{U}$ has a splitting sequence of length $n+1$ and chose a section of $s_n\colon Z_n - Z_{n+1} = Z_n \to f^{-1}_{\alpha_n|Z_n}(Z_n)$. Since f_{α_n} is étale we have a decomposition $\coprod f_{\alpha_n}|^{-1}_{Z_n}(Z_n) = \mathrm{Im}(s_n) \coprod Y \subset \coprod U_\alpha$ where Y is a closed subscheme of $\coprod U_\alpha$. Now consider $U = X - Z_n$ and $V = (\coprod U_\alpha) - Y$, by construction they form a distinguished square over X and $\mathcal{U} \times_X U \to U$ is a Nisnevich covering of U with a splitting sequence of length n. By induction and the hypothesis we have that the sequences

$$F(X) \longrightarrow F(U) \times F(V) \rightrightarrows F(U \times_X V) \quad \text{and}$$

$$F(U) \longrightarrow \coprod F(U_\alpha \times_X U) \rightrightarrows \coprod F(U_\alpha \times_X U_\beta \times U)$$

are exact. To conclude consider sections $(v_\alpha) \in \coprod F(U_\alpha)$ agreeing by the two restrictions in $\coprod F(U_\alpha \times U_\beta)$. By base change they define sections $(\bar{v}_\alpha) \in \coprod F(U_\alpha \times_X U)$ whose restrictions agree on $\coprod F(U_\alpha \times_X U_\beta \times U)$. Therefore there exists a unique $u \in F(U)$ which restricts to $(\bar{v}_\alpha)$. Since V is an open subscheme of $\coprod U_\alpha$, we also have $v = \overline{(v_\alpha)} \in F(V)$. By construction, the restrictions of (u, v) agree on $F(U \times_X V)$, and therefore we conclude the proof. □

Definition 2.2.10 Let x be a point of X. A **Nisnevich neighborhood** of x is an étale map $U \to X$ together with a point $u \in U$ mapping to x such that $k(u) \simeq k(x)$. We define the **local ring** at x in the Nisnevich topology to be

$$(\mathcal{O}_X)^{\mathbf{Nis}}_x = \varinjlim \Gamma(U, \mathcal{O}_U)$$

where $\mathcal{O}_U$ denotes the structural sheaf and the limit is taken over Nisnevich neighborhoods of x. If F is a Nisnevich sheaf, then we define the **stalk** of F at x to be

$$F_x = \varinjlim F(U).$$

We say that a local ring A is **Henselian** if it satisfies the following condition:

- Let $f(x)$ be a monic polynomial in $A[x]$ and $\bar{f}(x)$ be its image on $k[t]$, for $k = A/\mathfrak{m}$. Then any factorization of $\bar{f}$ into a product of two relatively prime polynomials lifts to a factorization in $A[x]$. In other words, if $\bar{f} = p \cdot q$ for p, $q \in k[x]$ monic and relatively prime then there exist p' and $q' \in A[x]$ such that $f = p' \cdot q'$ and $\bar{p}' = p$ and $\bar{q}' = q$.

Let A be a local ring, a Henselian ring A^h together with a morphism $A \to A^h$ is called the **Henselianization** of A if it is universal among local morphisms $A \to H$ where H is Henselian.

The next result follows from the same arguments of the étale site ([Mil13, §I.4]).

Theorem 2.2.11 *Let X be a scheme and x be a point of X. Then*

$$(\mathcal{O}_X)_x^{\mathbf{Nis}} = \mathcal{O}_{X,x}^h.$$

□

As in the classical case of sheaves on a topological space, there is a general construction which associates a sheaf to any presheaf. This general construction applies to the Nisnevich site. Find it, for example, in [Nor07, Lev.II.9.12].

Theorem 2.2.12 *The natural forgetful functor $i\colon \mathbf{Shv}(S) \to \mathbf{PreShv}(S)$ admits a left adjoint*

$$a_{\mathbf{Nis}}\colon \mathbf{PreShv}(S) \to \mathbf{Shv}(S).$$

□

Example 2.2.13 Let us point out the main examples we will use:

- Every smooth scheme X naturally gives a presheaf $\mathrm{Hom}_{\mathbf{Sm}_S}(\ \_\ , X)$. It can be checked that this presheaf is actually a sheaf in $\mathbf{Sm}_{S,\mathbf{Nis}}$ (*cf.* [Nor07, §Lev.II.9]) that we still denote X. Therefore we have a covariant functor

  $$\mathbf{Sm}_S \to \mathbf{Shv}(S)$$

  which, by Yoneda's lemma, is fully faithful.

- The base scheme S defines the following sheaf: $S(X) = *$ for every smooth scheme X.

- Let $\mathbb{A}^1_S$ be the affine line. Then for any smooth S-scheme X we know explicitly the sections of the sheaf it defines: $\mathbb{A}^1_S(X) = \mathrm{Hom}_{\mathbf{Sm}_S}(X, \mathbb{A}^1_S) = \mathcal{O}_X(X)$, the structural sheaf. We may also denote it $\mathcal{O}$.

- Let C be a set, it defines a constant presheaf on $\mathbf{Sm}_S$ by mapping any smooth scheme X to C. We still denote C its associated Nisnevich sheaf.

- Let X be a smooth scheme and $i\colon U \to X$ be an open subscheme. Then i defines a morphism of the associated sheaves $U \to X$. The presheaf $Y \mapsto X(Y)/U(Y)$ has, by Theorem 2.2.12, an associated Nisnevich sheaf which we denote X/U.

Example 2.2.14 We show an example of the main difference we have seen between the small and the big site. Denote $S = \mathrm{Spec}(\mathbb{Z})$ and consider the small Zariski site $S_{\mathbf{zar}}$ and the big Zariski site $\mathbf{Sch}_{S,\mathbf{Zar}}$ (*cf.* Example 2.2.3). Note that classic sheaves are by definition sheaves on the small Zariski site and any classic sheaf on S defines, by pullback, a sheaf on the big Zariski site.

Consider the structural sheaf of rings $\widetilde{\mathbb{Z}}$ and the classic coherent sheaf defined by $\mathbb{Z}/2\mathbb{Z}$. Note that the sheaf of homomorphism $\underline{\mathrm{Hom}}(\widetilde{\mathbb{Z}/2\mathbb{Z}}, \widetilde{\mathbb{Z}})$ is zero on the small Zariski site. However, this sheaf is not zero on the big Zariski site since $\underline{\mathrm{Hom}}(\widetilde{\mathbb{Z}/2\mathbb{Z}}, \widetilde{\mathbb{Z}})(\mathrm{Spec}(\mathbb{Z}/2\mathbb{Z})) = \{0, 1\}$.

This example shows that the big an small site relate as *restriction* in geometry. A sheaf may be zero on an open subset, but that does not imply it is zero globally.

The following is a standard property of representable functors stated in our context.

Proposition 2.2.15 *Let X be a smooth S-scheme. Then*

$$\mathrm{Hom}_{\mathbf{Shv}(S)}(X, F) = F(X).$$

Proof: Let $\xi \in F(X)$, it induces maps $\mathrm{Hom}_{\mathbf{Sm}_S}(Y, X) \to F(Y)$ where $\xi(f) = F(f)(\xi)$. Since they are compatible with restrictions they define a functor still denoted $\xi\colon X \to F$. It satisfies that $\xi(1_X) = \xi$. We have constructed $F(X) \to \mathrm{Hom}_{\mathbf{Shv}(S)}(X, F)$. The inverse map is obvious.

□

We introduce some notation that will be needed later in section § 3.1.2.

Definition 2.2.16 A morphism of presheaves $f\colon \mathcal{F} \to \mathcal{F}'$ is a **local epimorphism** if for every section $s \in F'(X)$ there exists a Nisnevich covering $\{U_\alpha \to X\}$ and sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $f_\alpha(s_\alpha) = s|_{U_\alpha}$.

The next statement follows directly from the construction of $a_{\mathbf{Nis}}$.

Proposition 2.2.17 *Let $\mathcal{F}$ be a presheaf on $\mathbf{Sm}_S$. The natural morphism $F \to a_{\mathbf{Nis}}F$ is a local epimorphism. That is to say, let $s \in a_{\mathbf{Nis}}\mathcal{F}(X)$ there exists a Nisnevich covering $\{U_\alpha \to X\}$ and sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $s|_{U_\alpha} = s_\alpha$.*

□

2.2.2 Functoriality: Inverse, direct image and $p_\sharp$

We recall constructions of functoriality in sites in our specific context. Apart from one case the reader may use the concrete description of the following Example 2.2.20 instead of this general construction.

Definition 2.2.18 We define a **map** of sites $f\colon \mathbf{Sm}_{T,\mathbf{Nis}} \to \mathbf{Sm}_{S,\mathbf{Nis}}$ to be a functor $f^{-1}\colon \mathbf{Sm}_S \to \mathbf{Sm}_T$ of the underlying categories. Note that it induces a functor $f_*\colon \mathbf{PreShv}(T) \to \mathbf{PreShv}(S)$, where $f_*F(U) = F(f^{-1}(U))$.

We say that a map of sites $f\colon \mathbf{Sm}_{T,\mathbf{Nis}} \to \mathbf{Sm}_{S,\mathbf{Nis}}$ is **continuous** if f_* maps $\mathbf{Shv}(T)$ onto $\mathbf{Shv}(S)$. We still denote $f_*\colon \mathbf{Shv}(T) \to \mathbf{Shv}(S)$ the induced functor.

The following statement is a general result from topos theory relying on the classic argument of Grothendieck's representability theorem. Find a proof, for example, in [Nor07, Lev.II.8.3].

Proposition 2.2.19 *Let $f\colon \mathbf{Sm}_{T,\mathbf{Nis}} \to \mathbf{Sm}_{S,\mathbf{Nis}}$ be a continuous map of sites, there exists a functor $f^*\colon \mathbf{Shv}(S) \to \mathbf{Shv}(T)$ left adjoint to f_*.*

□

Example 2.2.20 We will only use the following examples:

- Let $f\colon T \to S$ be a morphism of schemes. Then f induces a continuous map of sites from $\mathbf{Sm}_{T,\mathbf{Nis}}$ to $\mathbf{Sm}_{S,\mathbf{Nis}}$ defined by the functor which maps any smooth S-scheme X to $f^{-1}(X) = X \times_S T$. We denote the induced pair of adjoint functors

$$f^*\colon \mathbf{Shv}(S) \leftrightarrows \mathbf{Shv}(T) : f_*.$$

  We call them the *inverse* and *direct* image of sheaves respectively.

By definition the direct image f_*F is the sheaf $X \mapsto F(X \times_S T)$. Let F be a sheaf on S, denote F' the presheaf which maps a smooth T-scheme V to the set $F'(V) = \varinjlim F(U)$, where the limit is taken among smooth S-schemes U such that

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

commutes. By construction, the presheaf F' satisfies

$$\mathrm{Hom}_{\mathbf{PreShv}(T)}(F', G) = \mathrm{Hom}_{\mathbf{PreShv}(S)}(F, f_*G).$$

Therefore the associated sheaf is the inverse image f^*F.

Let $g \colon Y \to T$ be a morphism of schemes, then we have

$$(f \circ g)^* \simeq g^* \circ f^*,$$

$$(f \circ g)_* \simeq f_* \circ g_*.$$

- Let $p \colon X \to S$ be a smooth morphism. In this case p also induces another continuous map of sites $\Phi p \colon \mathbf{Sm}_{S,\mathbf{Nis}} \to \mathbf{Sm}_{X,\mathbf{Nis}}$ defined by the functor $(\Phi p)^{-1}$ which maps any smooth morphism $Y \to X$ to $Y \to X \to S$. The functor $(\Phi p)_*$ maps any sheaf F in $\mathbf{Shv}(S)$ onto its *restriction* $F|_X$ (defined by $F|_X(Y) = F(Y)$, for Y a smooth X-scheme). We denote by

$$p_\sharp = (\Phi p)^* \colon \mathbf{Shv}(X) \to \mathbf{Shv}(S),$$

which is left adjoint to the restriction. Let $q \colon Y \to X$ be a smooth morphism, we have

$$(p \circ q)_\sharp \simeq p_\sharp \circ q_\sharp.$$

We describe the main properties of the inverse and direct image of sheaves.

Proposition 2.2.21 *Let S be a scheme, X be a smooth S-scheme and x be a point of X:*

1. *Let $j \colon U \hookrightarrow S$ be an open immersion and F in $\mathbf{Shv}(U)$. Then $(j_*F)_x = F_y$ if $y = x \times_S U \neq \emptyset$.*
2. *Let $i \colon Z \hookrightarrow S$ be a closed immersion and F in $\mathbf{Shv}(Z)$. Then*

$$(i_*F)_x = \begin{cases} F_y & y = x \times_S Z \neq \emptyset\,; \\ * & x \times_S Z = \emptyset. \end{cases}$$

Proof: The first claim and the case $x \times_S Z = \emptyset$ of the second claim follows taking a "sufficiently small" neighborhood. The case $y = x \times_S Z \neq \emptyset$ follows from standard arguments in étale topology (*cf.* . [Mil13, 8.3]).

□

Contrary to the small site, the inverse image functor f^* on the big Nisnevich site does not preserve finite limits. It does not even preserve fibre products. The following example is based on the same idea of Example 2.2.14.

Example 2.2.22 Let $i_0 \colon p = \mathrm{Spec}(k) \to \mathbb{A}^1_k$ be the origin. Consider the two diagonals Y_+ and Y_- of $\mathbb{A}^2_k$ as smooth schemes over $\mathbb{A}^1_k$ by the first projection. Their intersection $Y_+ \times_{\mathbb{A}^2_k} Y_- = p$ is the origin, and defines the empty sheaf on $\mathbf{Sm}_{\mathbb{A}^1_k}$. However, it is easy to check that $i_0^*(Y_+) = i_0^*(Y_-) = p$ and therefore $i_0^*(Y_+) \times_{\mathbb{A}^1_k} i_0^*(Y_-) = p$ is not an empty sheaf on $\mathbf{Sm}_{\mathrm{Spec}(k)}$.

Since the inverse image functor does not preserve finite limits in particular we cannot describe the stalks of the inverse image of a sheaf as in the classical case. Nevertheless we have the following properties.

Proposition 2.2.23 *Let $f \colon T \to S$ be a morphism of schemes:*

1. *Let X be an S-smooth scheme. Then $f^*(X) = X \times_S T$.*

2. *Let F and G be in $\mathbf{Shv}(S)$, then $f^*(F \times G) = f^*F \times f^*G$.*

Let $p \colon X \to S$ be a smooth morphism:

3. *The functor p^* is the restriction functor defined in Example 2.2.20.2. In other words, let Y be a smooth X-scheme and G be in $\mathbf{Shv}(S)$, then we have*
$$p^*(G)(Y) = G|_X(Y) = G(Y).$$

Proof: For the first claim both sheaves have the same morphisms. Indeed,

$$\mathrm{Hom}_{\mathbf{Shv}(T)}(f^*(X), F) = \mathrm{Hom}_{\mathbf{Shv}(S)}(X, f_*(F)) = f_*(F)(X)$$

$$= F(X \times_S T) = \mathrm{Hom}_{\mathbf{Shv}(T)}(X \times_S T, F).$$

For the second claim, let V be a smooth T-scheme. Then we have

$$(F \times G)'(V) = \varinjlim (F \times G)(U) = \varinjlim F(U) \times \varinjlim G(U) = F'(V) \times G'(V)$$

so the claim is true for presheaves and it follows for sheaves.

For the third claim note that, since adjoints are unique, it is enough to prove that the restriction functor is left adjoint to p_*. Let F be a sheaf on X

and G be a sheaf on S. For any morphism of sheaves $\varphi\colon G|_X \to F$ and any smooth S-scheme V we have a map

$$\begin{tikzcd} G(V) \arrow[r, dashed] \arrow[d] & F(V \times_S X) \\ G(V \times_S X) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We have constructed a map $\mathrm{Hom}_{\mathbf{Shv}(Y)}(G|_Y, F) \to \mathrm{Hom}_{\mathbf{Shv}(S)}(G, p_*F)$.

Let $\Phi\colon G \to p_*F$ be a morphism of sheaves and W be a smooth X-scheme. We have a map

$$\begin{tikzcd} G(W) \arrow[r, dashed] \arrow[d, "\Phi"'] & F(W) \\ F(W \times_S X) \arrow[ur] & \end{tikzcd}$$

We have constructed a map $\mathrm{Hom}_{\mathbf{Shv}(S)}(G, p_*F) \to \mathrm{Hom}_{\mathbf{Shv}(X)}(G|_X, F)$ which is inverse to the previous one.

$\square$

Proposition 2.2.24 *Let $p\colon X \to S$ be a smooth morphism:*

1. *The functor $p_\sharp$ is left adjoint to p^*.*

2. *We abuse notation and still denote X the sheaf defined by the Yoneda embedding either in $\mathbf{Shv}(X)$ and $\mathbf{Shv}(S)$. Then $p_\sharp(X) = X$.*

3. *Projection formula: For any F in $\mathbf{Shv}(X)$ and any G in $\mathbf{Shv}(S)$ we have*
$$p_\sharp(F \times p^*(G)) \simeq p_\sharp(F) \times G.$$

4. *Let $j\colon U \hookrightarrow S$ be an open embedding then $j_\sharp$ is the "extension by zero" functor (*cf. *[Mil13, p.62]). More concretely, let F be in $\mathbf{Shv}(U)$ and consider the presheaf*
$$X \mapsto \begin{cases} F(X) & \text{if } X \text{ is } U\text{-smooth} \\ \emptyset & \text{otherwise.} \end{cases}$$
for X a smooth S-scheme. The sheaf $j_\sharp F$ is the sheaf associated to the above presheaf. In addition, let x be a point of a smooth S-scheme X and denote $y = x \times_S U$, then
$$(j_\sharp F)_x = \begin{cases} F_y & \text{if } y \neq \emptyset, \\ \emptyset & \text{if } y = \emptyset. \end{cases}$$

Proof: The first claim is a corollary of the previous result. The second one is direct from definitions. For the third claim it is enough to check that both sheaves have the same morphisms. Let H be in $\mathbf{Shv}(S)$. On one hand we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Shv}(S)}(p_\sharp(F \times p^*(G)), H) &= \mathrm{Hom}_{\mathbf{Shv}(U)}(F \times p^*G, p^*H)\\ &= \mathrm{Hom}_{\mathbf{Shv}(U)}(F, \underline{\mathrm{Hom}}_U(p^*G, p^*H)),\end{aligned}$$

where $\underline{\mathrm{Hom}}_U(p^*G, p^*H)$ denotes the internal Hom in $\mathbf{Shv}(U)$. On the other hand we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Shv}(S)}(p_\sharp F \times G, H) &= \mathrm{Hom}_{\mathbf{Shv}(S)}(p_\sharp F, \underline{\mathrm{Hom}}_S(G, H))\\ &= \mathrm{Hom}_{\mathbf{Shv}(U)}(F, p^*\underline{\mathrm{Hom}}_S(G, H)).\end{aligned}$$

We conclude by observing that $p^*\underline{\mathrm{Hom}}_S(G, H) = \underline{\mathrm{Hom}}_U(p^*G, p^*H)$.

For the last claim denote $F_\flat$ the presheaf of the statement. Note that it has the required adjunction property for presheaves:

$$\mathrm{Hom}_{\mathbf{PreShv}(S)}(F_\flat, G) \simeq \mathrm{Hom}_{\mathbf{PreShv}(U)}(F, G|_U).$$

□

Example 2.2.25 Let X be a smooth S-scheme and $j\colon U \to S$ be an open immersion. We can describe the sections of sheaf $j_\sharp j^*X$. Let Y be a smooth S-scheme:

$$(j_\sharp j^*X)(Y) = \begin{cases} X(Y) & \text{if } Y \text{ is } U\text{-smooth},\\ \emptyset & \text{otherwise.}\end{cases}$$

Note that this sheaf is representable by $X_U = X \times_S U$. We also denote it as $X - X_Z$.

Remark 2.2.26 Note that in the category of sets the initial object $\emptyset$ is different from the final object $*$. As we have seen, the condition of *being zero* for a sheaf of groups translates into two different ways for sheaf of sets.

Consider the category of sheaves of pointed sets $\mathbf{Shv}_\bullet(S)$. Note that in the category of pointed sets the initial and final object is $*$. All arguments of this section apply into this context *mutatis mutandis.*

Proposition 2.2.27 *Consider a cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f'} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

where p is a smooth morphism. Then

$$p^*f_* = f'_*p'^*.$$

Proof: Let V be a smooth X-scheme, and F be in $\mathbf{Shv}(T)$. Recall that the inverse image of a smooth morphism is the restriction functor. Then

$$(p^* f_* F)(V) = (f_* F)(V) = F(V \times_S T) = (p'^* F)(V \times_S T) = f'_* p'^* F(V)$$

since $V \times_S T = V \times_X Y$.

□

2.3 Model categories

We recall some notations and examples of model categories. The original reference is [Qui67], although we have also used [GJ99], [Hov99] and [Hir03]. For brevity's sake we omit most proofs.

In this section $\mathbf{C}$ will always denote a category with small limits and colimits, in particular with initial object $\emptyset$ and with final object $*$.

Definition 2.3.1 Let $\mathbf{C}$ be a category and W be a class of morphisms in $\mathbf{C}$. The **localization** of $\mathbf{C}$ by W is a functor $Q\colon \mathbf{C} \to \mathbf{C}[\mathrm{W}^{-1}]$ such that for all w in W the image $Q(w)$ is an isomorphism and such that it is universal for this property. In other words, for any other functor $F\colon \mathbf{C} \to \mathbf{C}'$ such that $F(w)$ is an isomorphism for all w in W there exists a unique factorization

$$\begin{tikzcd} \mathbf{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathbf{C}[\mathrm{W}^{-1}] \arrow[d, "F_{\mathrm{W}}"] \\ & \mathbf{C}'. \end{tikzcd}$$

If a localization exist, we abuse notation and call $\mathbf{C}[\mathrm{W}^{-1}]$ a localization.

The following result is straightforward.

Proposition 2.3.2 *Let $\mathbf{C}$ be a category and W be a class of morphisms. If a localization $\mathbf{C} \to \mathbf{C}[\mathrm{W}^{-1}]$ exist then it is unique up to a canonical natural transformation.*

□

Example 2.3.3
- Let $\mathbf{Top}_\bullet$ be the category of pointed topological spaces and consider Weakh the class of weak homotopy equivalences. We may consider the localization $\mathbf{Top}_\bullet[\mathrm{Weakh}^{-1}]$.
- Let $\mathbf{CW}$ be the category of CW-complexes with morphisms of topological spaces. Once again we may consider the class Weak of weak equivalences.

By Whitehead's theorem we have that the classic homotopy category of CW-complexes $\mathcal{H}$ is the localization $\mathbf{CW}[\mathrm{Weak}^{-1}]$. Therefore we have a concrete description: objects are CW-complexes and morphisms are homotopy classes of morphisms.

- Let A be a ring and denote $\mathbf{Ch}^+(A)$ the category of bounded below (cochain) complexes of A-modules (note that we assume that the differential rises the degree). Denote Qua the class of quasi-isomorphisms. The localization $\mathbf{Ch}^+(A)[\mathrm{Qua}^{-1}]$ is precisely the derived category $\mathbf{D}_+(A)$ of bounded below complexes. Once again we have concrete description of the localization: objects are bounded below complexes and morphisms are fractions of homotopy classes of morphisms of complexes. Analogously, we consider $\mathbf{Ch}^-(A)$ the category of bounded above complexes of A-modules and we have $\mathbf{Ch}^-(A)[\mathrm{Qua}^{-1}] = \mathbf{D}_-(A)$. In general, denote $\mathbf{Ch}(A)$ the category of unbounded complexes, then $\mathbf{Ch}(A)[\mathrm{Qua}^{-1}] = \mathbf{D}(A)$.

Remark 2.3.4 The localization $\mathbf{C}[\mathrm{W}^{-1}]$ may not exist in general, due to set theoretic problems. In addition, as we have seen for topological spaces, the localization may not be easy to describe.

Let us recall some notation before the main definition of this section.

Definition 2.3.5 Let $f\colon Y \to X$ be a morphism in a category $\mathbf{C}$. We say that Y is a *retract of* X (through f) if there exists a morphism $g\colon X \to Y$ such that $g \circ f = 1_Y$. We also say that g is a *retraction* (of f).

Let $f'\colon Y' \to X'$ be another morphism in $\mathbf{C}$, we abuse notation and we say that f is a **retract of** f' if it is a retract of f' in the category of arrows of $\mathbf{C}$. More concretely, if there are morphisms

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[r, "r"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "j"] & Y' \arrow[r, "t"] & Y \end{tikzcd}$$

such that $r \circ i = 1_X$ and $t \circ j = 1_Y$ (Note that (r, t) is the retraction of (i, j)).

Definition 2.3.6 Let $\mathbf{C}$ be a category and $i\colon U \to V$ and $p\colon X \to Y$ be two maps. We say that p has the **right lifting property** with respect to i and that i has the **left lifting property** with respect to p if for all commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & Y \end{tikzcd}$$

there is a map $s\colon V \to X$ making the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, "s"] & Y \end{tikzcd}$$

commutative.

The following definition, due to Quillen, provides an adequate framework for localizations.

Definition 2.3.7 We say that a category $\mathbf{C}$ together with three classes of morphisms called **weak equivalences**, **fibrations** and **cofibrations** is a **model category** if the following axioms hold:

1. The category $\mathbf{C}$ has all small limits and small colimits.

2. If f and g are two composable morphism and two out of f, g and $g \circ f$ are weak equivalences then so is the third.

3. If a morphism f is a retract of g and g is either a weak equivalence, a cofibration or a fibration then so is f.

4. Any fibration has the right lifting property with respect to cofibrations which are weak equivalences (which we call **trivial cofibrations**) and any fibration which is also a weak equivalence (which we call **trivial fibrations**) has the right lifting property with respect to cofibrations.

5. Any morphism f can be functorially factorized as a composition $p \circ i$ where p is a fibration and i is a trivial cofibration and as a composition $q \circ j$ where q is a trivial fibration and j is a cofibration.

Remark 2.3.8 By "*functorially factorized*" we mean the following: there exist two pair of functors (α, β) and (γ, δ) from the category of arrows of $\mathbf{C}$ to the category of arrows of $\mathbf{C}$ such that for any morphism f we have $f = \alpha(f) \circ \beta(f) = \gamma(f) \circ \delta(f)$ and such that $\alpha(f)$ is a fibration, $\beta(f)$ is trivial cofibration, $\gamma(f)$ is trivial cofibration and $\delta(f)$ is a cofibration (*cf.* [Hov99, 1.1.1]). Let us remark that the fifth axiom states not only that *there exists* a factorization but that we have a concrete choice of factorization for every morphism. More concretely, the *fibrant and cofibrant replacements* of paragraph 2.3.12 are also part of the definition.

Denote Weak the class of weak equivalences. We want to describe $\mathbf{C}[\mathrm{Weak}^{-1}]$ and, in order to do so, we need more notation.

Definition 2.3.9 Let $\mathbf{C}$ be a model category. We say that an object X is **fibrant** if the map $X \to *$ to the final object is a fibration. We say that X is **cofibrant** if the map $\emptyset \to X$ from the initial object is a cofibration.

Definition 2.3.10 Let $\mathbf{C}$ be a model category and X be an object. A **cylinder** for X is a commutative triangle

$$\begin{tikzcd} X \sqcup X \arrow[d, "i"'] \arrow[dr, "1_X \sqcup 1_X"] & \\ \widetilde{X} \arrow[r, "h"'] & X \end{tikzcd}$$

where i is a cofibration and h is a weak equivalence.

Let $f, g\colon X \to Y$ be two maps. A (left) **homotopy** between f and g is a commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[d, "i"'] \arrow[dr, "f \sqcup g"] & \\ \widetilde{X} \arrow[r, "h"'] & Y \end{tikzcd}$$

for some cylinder $\widetilde{X}$ for X. A morphism $f\colon X \to Y$ is a **homotopy equivalence** if there exists a morphism $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to 1_X and 1_Y respectively.

Remark 2.3.11 The fifth axiom of model categories applied to $1_X \sqcup 1_X\colon X \sqcup X \to X$ defines a cylinder for X. Therefore cylinders exist for any object.

2.3.12 Let X be an object and consider the morphism $\emptyset \to X$. Applying the fifth axiom there is a factorization

$$\begin{tikzcd} \emptyset \arrow[r, "j"'] \arrow[dr] & Q(X) \arrow[d, "q_X"] \\ & X \end{tikzcd}$$

where $Q(X)$ is cofibrant and a map $q_X\colon Q(X) \to X$ which is a trivial fibration. The functoriality of the fifth axiom implies that $Q\colon \mathbf{C} \to \mathbf{C}$, $X \mapsto Q(X)$ is a functor which we call the **cofibrant replacement**.

Analogously, if we apply the fifth axiom to the map $X \to *$ there is a factorization

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr] & R(X) \arrow[d, "p"] \\ & * \end{tikzcd}$$

where i_X is a trivial cofibration and $R(X)$ is fibrant. Once again we have a functor $R\colon \mathbf{C} \to \mathbf{C}$ which we call the **fibrant replacement**.

Denote $\mathbf{C}_{\mathrm{cf}}$ the full subcategory of $\mathbf{C}$ made of fibrant cofibrant objects. We denote

$$RQ\colon \mathbf{C} \to \mathbf{C}_{\mathrm{cf}}$$

the **fibrant cofibrant replacement** functor.

Let us restate the analogous of Whitehead's theorem in the context of model categories. Find a proof in [GJ99, 1.10].

Theorem 2.3.13 (Whitehead) *Let $\mathbf{C}$ be a model category and $f\colon X \to Y$ a morphism where both X and Y are fibrant and cofibrant objects. The morphism f is a weak equivalence if and only if f is a homotopy equivalence.*

□

In order to state the main theorem of this section let us recall a technical result. Find a proof in [Hov99, 1.2.5 and 1.2.7].

Proposition 2.3.14 *Let X be cofibrant and Y be an object. Then the homotopy relation is an equivalence relation in $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ and we denote the quotient $[X, Y]$. In addition, if $q\colon Y' \to Y$ is a trivial fibration then $[X, Y'] = [X, Y]$.*

Let X and Y be two fibrant cofibrant objects. Then the homotopy relation is an equivalence relation, stable by composition, in $\mathrm{Hom}_{\mathbf{C}}(X, Y)$.

□

Now, denote $\mathbf{Ho}(\mathbf{C}_{\mathrm{cf}})$ the category with the same objects as $\mathbf{C}_{\mathrm{cf}}$ and morphisms homotopy equivalence classes of morphisms. We are ready to state the main result of this section. Find a proof in [Hov99, 1.2.10].

Theorem 2.3.15 *Let $\mathbf{C}$ be a model category and denote Weak the class of weak equivalences. The localization of $\mathbf{C}$ by Weak^{-1} is the fibrant cofibrant replacement functor $RQ\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C}_{\mathrm{cf}})$. In other words, the localizad category $\mathbf{C}[\mathrm{Weak}^{-1}]$ exists and the obvious inclusion*

$$\mathbf{Ho}(\mathbf{C}_{\mathrm{cf}}) \xrightarrow{\sim} \mathbf{C}[\mathrm{Weak}^{-1}]$$

is an equivalence of categories.

□

Now the following notation makes sense.

Definition 2.3.16 Let $\mathbf{C}$ be a model category, we define the **homotopy category** of $\mathbf{C}$ to be $\mathbf{Ho}(\mathbf{C}) \coloneqq \mathbf{C}[\text{Weak}^{-1}] \simeq \mathbf{Ho}(\mathbf{C}_{\text{cf}})$.

Corollary 2.3.17 *Let X be a cofibrant object and Y be a fibrant object. Then*

$$\text{Hom}_{\mathbf{Ho}(\mathbf{C})}(X, Y) = [X, Y].$$

□

The following result is widely used since it assures that, in a model category, weak equivalences and either fibrations or cofibrations determine the last group of morphisms. It is an easy consequence of the axioms.

Theorem 2.3.18 *Let $\mathbf{C}$ be a model category:*

- *A map $i\colon U \to V$ is a cofibration if and only if i has the left lifting property with respect to all trivial cofibrations.*
- *A map $i\colon U \to V$ is a trivial cofibration if and only if it has the left lifting property with respect to all fibrations.*

Analogously, we have:

- *A map $p\colon X \to Y$ is a fibration if and only if it has the right lifting property with respect to all trivial fibrations.*
- *A map $p\colon X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations.*

□

Corollary 2.3.19 *Let $\mathbf{C}$ be a model category:*

- *The class of cofibrations and trivial cofibrations are closed under composition and pushout. Any isomorphism is a trivial cofibration.*
- *The class of fibrations and trivial fibrations are closed under composition and pullback. Any isomorphism is a trivial fibration.*

Definition 2.3.20 We say that a category $\mathbf{A}$ is **pointed** if the initial object exists and it is also the final object. We also call it the **null** object and denote it $*$. Let $f\colon A \to B$ be a map in a pointed model category. We call the **fiber** of f to be the cartesian product of f and $* \to B$. We denote it $\text{fib}(f)$. We call the **cofiber** of f to be the coproduct of f and $A \to *$. We denote it $\text{cofib}(f)$.

Example 2.3.21

- Let $\mathbf{A}$ be a category. Consider isomorphisms to be weak equivalences and cofibrations and fibrations to be all maps. They define the trivial model structure on $\mathbf{A}$.

- *Simplicial sets:* Let $\mathbf{sSets}$ be the category of simplicial sets. Let *weak equivalences* be weak equivalences of simplicial sets and *cofibrations* be injective maps. Taking as *fibrations* the maps having the required lifting properties they define a model structure called *Quillen's model structure*. Fibrations turn out to be *Kan fibrations* ([GJ99, I.11.3]). Pointed simplicial sets with analogous pointed morphisms define a pointed model category.

- *Topological spaces:* Denote $\mathbf{Top}$ the category of topological spaces. Let weak equivalences be weak homotopy equivalences and fibrations be the so called *Serre fibrations*. Taking as cofibrations the maps having the required lifting properties they define a model structure on $\mathbf{Top}$ called *Quillen's model structure*. Pointed topological spaces with analogous pointed morphisms define a pointed model category. In this case all topological spaces are fibrant. CW-complexes are cofibrant and the so called *CW-approximation* is the cofibrant replacement. A classic result due to Quillen states that the realization and singular complex functors induce equivalence of categories
$$|\_|\colon \mathbf{Ho}(\mathbf{sSets}_\bullet) \rightleftarrows \mathbf{Ho}(\mathbf{Top}_\bullet) : \mathrm{sing}.$$

- *Cochain complexes:* Let A be a ring and $\mathbf{Ch}^+(A)$ be the category of bounded below (cochain) complexes of A-modules. We consider quasi-isomorphisms as weak equivalences, monomorphisms in each degree as cofibrations and morphisms having the right lifting property with respect to injective quasi-isomorphisms as fibrations. They define a model structure which we call the *injective model structure*. Note that $\mathbf{Ch}^+(A)$ is naturally pointed by the zero complex. One can check that fibrations turn out to be epimorphisms with injective kernel. Therefore the injective resolution $I^\bullet(K^\bullet)$ of a complex $K^\bullet$ is a fibrant replacement. We have that
$$\mathrm{Hom}_{\mathbf{D}_+(A)}(K'^\bullet, K^\bullet) = [K'^\bullet, I^\bullet(K^\bullet)].$$

  Analogously, let $\mathbf{Ch}^-(A)$ be the category of bounded above complexes of A-modules. We define weak equivalences to be quasi-isomorphisms, fibrations to be epimorphisms and cofibrations to be those morphisms having the adequate lifting property. They define a model structure which we call the *projective model structure.* One can check that cofibrations turn out to be injective morphisms with projective cokernel so that the projective resolution $P^\bullet(K^\bullet)$ of a complex $K^\bullet$ is a cofibrant replacement. Therefore we have that

$$\mathrm{Hom}_{\mathbf{D}_-(A)}(K'^\bullet, K^\bullet) = [P^\bullet(K'^\bullet), K^\bullet].$$

- The previous construction on bounded above cochain complexes $\mathbf{Ch}^+$ also holds if one replaces A-**mod** for any abelian category with enough injectives (*cf.* [Joy84, Theorem 2]). In particular, let X be a topological space and denote $\mathbf{Ch}^+(\mathbf{Shv}(X, A))$ the category of positive degree complexes of sheaf of A-modules. The class of quasi-isomorphisms as weak equivalences, monomorphisms as cofibrations and morphisms having the right lifting property with respect to injective quasi-isomorphisms define a model structure. The injective resolution $I^\bullet(\mathcal{K}^\bullet)$ of a complex $\mathcal{K}^\bullet$ is a fibrant replacement.

- *Unbounded complexes:* Let $\mathbf{Ch}(A)$ be the category of unbounded complexes. We define weak equivalences to be quasi-isomorphisms, fibrations to be degreewise epimorphisms and cofibrations to be morphism having the adequate lifting property. They define a model structure on $\mathbf{Ch}(A)$ (*cf.* [Hov99, §2.3]). Cofibrations are degreewise split injections with cofibrant cokernel. Cofibrant complexes are degreewise projective complexes, but not all of them.

The following result is also widely used. Find a proof in [Hov99, 1.1.12].

Theorem 2.3.22 *Let $f\colon \mathbf{C} \to \mathbf{C}'$ be a functor between two model categories. If f maps trivial fibrations into weak equivalences then f preserves weak equivalences.*

□

We will introduce two concepts in model categories that will be useful: homotopy pushouts (and pullbacks) and suspension. In order to do so we will assume that the model categories satisfy the properties described below. The theory can be developed with more generality but it is needless for this memoir. For brevity's sake we just review the main results. We refer to [Hir03] for more details.

Definition 2.3.23 Let $\mathbf{A}$ be a category. We say that $\mathbf{A}$ is a **simplicial** category if there is a functor

$$\mathrm{S}(\ \_\ ,\ \_\ )\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{sSets}$$

such that for any two objects A and B of $\mathbf{A}$ we have:

- $\mathrm{S}(A, B)_0 = \mathrm{Hom}_{\mathbf{A}}(A, B)$.
- The functor $\mathrm{S}(A, \_\,)\colon \mathbf{A} \to \mathbf{sSets}$ has a left adjoint

$$A \otimes \_ : \mathbf{sSets} \to \mathbf{A}$$

which is associative. In other words, for any two simplicial sets X and Y there is an isomorphism

$$A \otimes (X \times Y) \simeq (A \otimes X) \otimes Y$$

which is functorial on each term.

- The functor $\mathrm{S}(\_\,, B)\colon \mathbf{A}^{\mathrm{op}} \to \mathbf{sSets}$ has a left adjoint

$$\underline{\mathrm{hom}}(\_\,, B)\colon \mathbf{sSets} \to \mathbf{C}^{\mathrm{op}}.$$

We call the functor $\mathrm{S}(\_\,, \_\,)$ the *simplicial mapping space.*

Definition 2.3.24 Let $\mathbf{C}$ be a model category which is simplicial. We say that $\mathbf{C}$ is a **simplicial model category** if for any cofibration $j\colon X \to X'$ and any fibration $q\colon Y' \to Y$ we have that

$$\mathrm{S}(X', Y') \xrightarrow{(j^*, q_*)} \mathrm{S}(X, Y') \times_{\mathrm{S}(X,Y)} \mathrm{S}(X', Y)$$

is a fibration of simplicial sets, which is trivial if either j or q is trivial.

Notation 2.3.25 Every model category we consider will be a simplicial model category. Note in addition that on a simplicial model category there is a natural construction of cylinder objects: For any X the object $X \otimes \Delta^1$ is a cylinder object.

Now, for the review of pullbacks and pushouts in model categories let us introduce the following concept.

Definition 2.3.26 We say that a model category $\mathbf{C}$ is **proper** if every pushout of a weak equivalence along a cofibration is a weak equivalence and every pullback of a weak equivalence along a fibration is a weak equivalence.

Let $\mathbf{C}$ be a proper model category. We say that a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

is a **homotopy pullback** square if the factorizations $Y \to QY \to T$ and $Z \to QZ \to T$ into trivial cofibrations and fibrations induce a map

$$X \longrightarrow QY \times_T QZ.$$

which is a weak equivalence. We also say that X is the homotopy pullback of $Y \to Z$ and $Z \to T$. Let $f\colon X \to Y$ be a map, we define the **homotopy fiber** of f to be the homotopy pullback of f and $* \to Y$. We denote it $\mathrm{hofib}(f)$.

We say that it is a **homotopy pushout** square if the factorizations into cofibrations and trivial fibrations $X \to RY \to Y$ and $X \to RZ \to Z$ induce a map

$$RY \sqcup_X RZ \longrightarrow T$$

which is a weak equivalence. We also say that T is the homotopy pushout of $X \to Y$ and $X \to Z$. Let $f\colon X \to Y$ be a map, we define the **homotopy cofiber** to be the pushout of f and $X \to *$. We denote it $\mathrm{hocofib}(f)$.

Notation 2.3.27 Every model category considered will be proper.

We recall a standard property that we will need. We state it just for pushouts. Find a proof in [Hir03, 13.3.8].

Proposition 2.3.28 *Let* $\mathbf{C}$ *be a model category and*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pushout square. Then it is enough to replace one morphism to obtain the homotopy pushout. In other words, let $Z \to RT \to T$ *be a factorization of* i *into a cofibration and a trivial fibration. Then the map* $RT \sqcup_Z X \to Y$ *is a weak equivalence. In particular, if* i *is a cofibration then map*

$$T \sqcup_Z X \to Y$$

is a weak equivalence.

□

2.3.1 Functoriality: total derived functors

Let $\mathbf{C}$ be a model category and $F\colon \mathbf{C} \to \mathbf{A}$ be functor into a general category $\mathbf{A}$. Recall that, by definition, F factors as

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\gamma"] \arrow[d, "F"'] & \mathbf{Ho}(\mathbf{C}) \arrow[dl, dashed] \\ \mathbf{A}. & \end{tikzcd}$$

if and only if F maps weak equivalences into isomorphisms.

As in the case of the functor "taking global sections" on complexes of sheaves of abelian groups, if F does not map weak equivalences into isomorphism but satisfies certain properties we can construct a functor which best approximates this diagram. Let us make more precise this statement. We say that a functor $G\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{A}$ *approximates* F if there is a natural transformation $\epsilon_G\colon G \circ \gamma \to F$. If the class of functors approximating F is good enough there might exist a final object $\varinjlim_{G\circ\gamma\to F} G$.

Consider functors $J\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{A}$ with a natural transformation $\tau_J\colon F \to J \circ \gamma$. As in the previous case, the class of pairs (J, τ_J) may have a initial object $\varprojlim_{F\to J\circ\gamma} J$.

Definition 2.3.29 Let $\gamma\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ be a model category and $F\colon \mathbf{C} \to \mathbf{A}$ be a functor into a general category. We say that a functor $\mathbf{L}^\gamma F\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{A}$ and a natural transformation $\epsilon\colon \mathbf{L}F \circ \gamma \to F$ is the **left derived functor** of F (with respect to γ) if for any other pair $(G\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{A},\ \epsilon_G\colon G \circ \gamma \to F)$ there is a functor $\Phi\colon G \to \mathbf{L}^\gamma F$ such that

$$\begin{tikzcd} \mathbf{L}^\gamma F \circ \gamma \arrow[r, "\epsilon"] & F \\ G \circ \gamma \arrow[u, "\Phi\circ\gamma"] \arrow[ur, "\epsilon_G"'] & \end{tikzcd}$$

commutes.

Analogously, we say that a functor $\mathbf{R}^\gamma F\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{A}$ and a natural transformation $\tau\colon F \to \mathbf{R}F \circ \gamma$ is the **right derived functor** of F (with respect to γ) if for any other pair $(G,\ \tau_G)$ there is a natural transformation $\Psi\colon \mathbf{R}^\gamma F \to G$ such that

$$\begin{tikzcd} F \arrow[r, "\tau"] \arrow[dr, "\tau_G"'] & \mathbf{R}^\gamma F \circ \gamma \arrow[d, "\Psi\circ\gamma"] \\ & G \circ \gamma \end{tikzcd}$$

commutes.

Definition 2.3.30 Let $F\colon \mathbf{C} \to \mathbf{C}'$ be a functor between model categories. We call the **total left derived functor** of F to the functor $\mathbf{L}F = \mathbf{L}^{\gamma}(F \circ \gamma')\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C}')$. We call the **total right derived functor** of F to the functor $\mathbf{R}F = \mathbf{R}^{\gamma}(F \circ \gamma')\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C}')$.

Remark 2.3.31 Let $F\colon \mathbf{C} \to \mathbf{C}'$ be a functor between model categories which preserves weak equivalences between cofibrant objects. For this case one can take the following description as the definition of total derived functors:

- For any object X we have $\mathbf{L}F \circ \gamma(X) = \gamma' \circ F(QX)$, where QX is the cofibrant replacement of X.
- Recall that the cofibrant replacement is functorial. For any map $f\colon X \to Y$ there exists a lifting $g\colon QX \to QY$ such that the diagram

$$\begin{tikzcd} QX \arrow[r, "g"] \arrow[d] & QY \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

  commutes. Then $\mathbf{L}F \circ \gamma(f) = \gamma' \circ F(g)$.

The assignation is well defined on objects. We left for the following proposition the case of morphisms and that it has the right universal property.

Let $G\colon \mathbf{C} \to \mathbf{C}'$ be a functor which maps weak equivalences between fibrant objects to weak equivalences. The total right derived functor of G has the following description: Any object X maps to $\mathbf{R}G \circ \gamma(X) = \gamma' \circ G(RX)$, where RX is the fibrant replacement of X, and a morphism $f\colon X \to Y$ maps into $G(g)$ for g the induced map between RX and RY. It is easy to check that these functors are the total derived functors of Definition 2.3.30.

Proposition 2.3.32 *Let $F, G\colon \mathbf{C} \to \mathbf{C}'$ be functors as in Remark 2.3.31. Then the total left and right derived functors exist and are the ones described there.*

□

Example 2.3.33 • Let $F\colon \mathbf{C} \to \mathbf{A}$ be a functor from a model category into a general category which maps weak equivalences between cofibrant objects into isomorphisms. Consider on $\mathbf{A}$ the trivial model structure with isomorphisms as weak equivalences. Then there is a total left derived functor $\mathbf{L}F\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{A}$. An analogous statement holds for right derived functors.

- Consider the functor

$$\mathrm{Hom}_{A\text{-}\mathbf{mod}}(M, \_\,)\colon A\text{-}\mathbf{mod} \longrightarrow A\text{-}\mathbf{mod}.$$

It induces a functor $\mathbf{Ch}^+(A) \to \mathbf{Ch}^+(A)$ denoted $\underline{\mathrm{Hom}}(M, \_\,)$. Note that this functor preserves (chain) homotopies. Recall from Example 2.3.21 that the category of bounded cochain complexes of A-modules $\mathbf{Ch}^+(A)$ has a model structure with quasi-isomorphism as weak equivalences. Recall that the derived category $\mathbf{D}_+(A)$ is the homotopy category. Since every complex is cofibrant, quasi-isomorphism between fibrant objects are homotopy equivalences. Therefore $\underline{\mathrm{Hom}}(M, \_\,)$ preserves quasi-isomorphisms between fibrant objects. Let $K^\bullet$ be a complex, recall that the injective resolution $I^\bullet(K^\bullet)$ is its fibrant replacement. We have proved that there is a total right derived functor

$$\begin{array}{rcc} \mathbf{R}\underline{\mathrm{Hom}}(M, \_\,)\colon \mathbf{D}_+(A) & \longrightarrow & \mathbf{D}_+(A) \\ K^\bullet & \mapsto & \underline{\mathrm{Hom}}(M, I^\bullet(K^\bullet)). \end{array}$$

The classic $\mathrm{Ext}^{\mathrm{n}}$-functor is precisely

$$A\text{-}\mathbf{mod} \xrightarrow{i_0} \mathbf{D}_+(A) \xrightarrow{\mathbf{R}\underline{\mathrm{Hom}}(M,\ \_\,)} \mathbf{D}_+(A) \xrightarrow{H^n} A\text{-}\mathbf{mod},$$

where $i_0(M)$ is the complex which has M in degree zero and in is zero elsewhere. The Tor-functors have an analogue description considering the projective model structure on $\mathbf{Ch}^-(A)$ (*cf.* 2.3.3).

- Let X be a topological space and denote $\mathbf{Shv}(X, A)$ the category of sheaves of A-modules. The global sections functor $\mathbf{Shv}(X, A) \to A\text{-}\mathbf{mod}$, $F \mapsto F(X)$ induces a functor

$$\Gamma\colon \mathbf{Ch}^+(\mathbf{Shv}(X, A)) \to \mathbf{Ch}^+(A)$$

which preserves homotopies. Therefore its total right derived functor

$$\mathbf{R}\Gamma\colon \mathbf{D}^+(\mathbf{Shv}(X, A)) \to \mathbf{D}^+(A)$$

exists. We can describe sheaf cohomology as

$$\mathbf{Shv}(X, A) \xrightarrow{i_0} \mathbf{D}^+(\mathbf{Shv}(X, A)) \xrightarrow{\mathbf{R}\Gamma} \mathbf{D}^+(A) \xrightarrow{H^n} A\text{-}\mathbf{mod}\,.$$

Definition 2.3.34 Let $F\colon \mathbf{C} \leftrightarrows \mathbf{C}'\colon G$ be a pair of adjoint functors between model categories:

- We say that F is a **left Quillen functor** if it preserves cofibrations and trivial cofibrations.

- We say that G is a **right Quillen functor** if it preserves fibrations and trivial fibrations.
- We say that the pair (F, G) is a **Quillen adjunction** if F is a left Quillen functor.

The following result is direct by taking adjunction.

Proposition 2.3.35 *Let $F: \mathbf{C} \leftrightarrows \mathbf{C}' : G$ be a pair of adjoint functors between model categories. Then (F, G) is a Quillen adjunction if and only if G is right Quillen functor.*

□

The following statement is a consequence of Ken Brown's lemma. Find a proof in [GJ99, II.8.9].

Lemma 2.3.36 *Every left Quillen functor preserves weak equivalences between cofibrant objects and every right Quillen functor preserves weak equivalences between fibrant objects.*

□

Quillen's total derived functor quickly follows from it.

Theorem 2.3.37 *Let $\mathbf{C}$ and $\mathbf{C}'$ be two model categories and $F: \mathbf{C} \leftrightarrows \mathbf{C}' : G$ be a pair of adjoint functors where F is right Quillen or G is left Quillen. Then the total derived functors*

$$\mathbf{L}F: \mathbf{Ho}(\mathbf{C}) \leftrightarrows \mathbf{Ho}(\mathbf{C}') : \mathbf{R}G$$

exist and form a pair of adjoint functors.

□

2.3.2 Triangulated categories

2.3.38 Let $\mathbf{C}$ be a pointed model category. Let X be an object and $\tilde{X}$ be a cylinder object for X. As in the topological case we define the **suspension** of X to be the cofiber of $X \sqcup X \to \tilde{X}$ and denote it ΣX.

Recall that we are considering simplicial model categories. Therefore for any object X and any simplicial set K we have $X \otimes K \in \mathbf{C}$ (*cf.* Definition 2.3.23). Let $p \to K$ be a pointed simplicial set, we define

$$X \wedge K = X \otimes K / X \otimes p.$$

In this case a cylinder object is $\tilde{X} = X \otimes \Delta^1$ so we always assume the suspension to be

$$\Sigma X = X \wedge S^1.$$

Therefore the suspension defines a functor which has a left adjoint

$$\Omega\colon \mathbf{C} \to \mathbf{C}$$

defined as $\Omega(X) = \underline{\mathrm{Hom}}(S^1, X)$. We call it the **loop space** functor. The following result is now direct.

Proposition 2.3.39 *Let* $\mathbf{C}$ *be a pointed model category. The suspension and loop space functor define a Quillen adjunction. In particular, we have a pair of adjoint functors*

$$\Sigma\colon \mathbf{Ho}(\mathbf{C}) \leftrightarrows \mathbf{Ho}(\mathbf{C}) : \Omega.$$

□

Notation 2.3.40 As in topology, we denote the suspension of an object X as

$$X[1] = \Sigma X = S^1 \wedge X.$$

We also write $X[n] = (\Sigma)^n X = S^n \wedge X$.

Remark 2.3.41 As we saw, in a general pointed model category (not necessarily simplicial) the suspension of a space is defined but depends on the choice of a cylinder object. One may prove that all possible cylinder objects are homotopic so the suspension and loop space of an object are well defined in $\mathbf{Ho}(\mathbf{C})$ (*cf.* [Qui67, I.2.2]).

Definition 2.3.42 Let $\mathbf{C}$ be a pointed model category and $f\colon X \to Y$ be a map between cofibrant objects. We define the **cone** of f to be the coproduct of $f \sqcup *$ and $i_0 \sqcup i_1\colon X \sqcup X \to X \otimes \Delta^1$. We denote it $\mathrm{cone}(f)$. In other words, the cone fits into a cocartesian diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "f \sqcup *"] \arrow[d] & Y \sqcup * \arrow[d] \\ X \otimes \Delta^1 \arrow[r] & \mathrm{cone}(f). \end{tikzcd}$$

Remark 2.3.43 Note that the cone can also be expressed as the coproduct of f and $i_1\colon X \to X \wedge (\Delta^1, 0)$. When needed, we also use this description in order to avoid the use of the symbol $\otimes$.

Example 2.3.44 Let X be fibrant and $X \to *$ be the null morphism. Then the cone of the null morphism is $X \wedge S^1$. Let $f\colon X \to Y$ be a map between cofibrant objects. The null morphism $Y \to *$ and the canonical projection $X \wedge \Delta^1 \to X \wedge S^1$ define a morphism $\mathrm{cone}(f) \to X \wedge S^1$. Therefore, we have a sequence of morphisms

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cone}(f) \longrightarrow X[1].$$

We remark a sufficient condition for the homotopy category of a model category to be a triangulated category (*cf.* [Nee01, 1.3.13]). Find a proof for simplicial model categories in [Rio10, 3.2]. The general case follows from analogous arguments and the properties of [Qui67, I.3.5].

Theorem 2.3.45 *Let* $\mathbf{C}$ *be a pointed model category. If the suspension functor* $[1]\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$ *is an equivalence of categories then* $\mathbf{Ho}(\mathbf{C})$ *is a triangulated category with distinguished triangles sequences*

$$U \longrightarrow V \longrightarrow W \longrightarrow U[1]$$

isomorphic, in $\mathbf{Ho}(\mathbf{C})$*, to a sequence*

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cone}(f) \longrightarrow X[1]$$

for $f\colon X \to Y$ *a map in* $\mathbf{C}$ *between cofibrant objects.*

□

We recall a useful property.

Lemma 2.3.46 *Let* $i\colon X \to Y$ *be a cofibration and* X *be cofibrant. Then the natural map*

$$\mathrm{cone}(i) \to \mathrm{cofib}(i) \simeq \mathrm{hocofib}(i)$$

is a weak equivalence.

Proof: Note that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & X \wedge (\Delta^1, 0) \arrow[r] \arrow[d] & * \arrow[d] \\ Y \arrow[r] & \mathrm{cone}(i) \arrow[r] & \mathrm{cofib}(i) \end{tikzcd}$$

made of cocartesian diagrams. From Corollary 2.3.19 we have that $X \wedge (\Delta^1, 0) \to \mathrm{cone}(i)$ is a cofibration. Note that $X \wedge (\Delta^1, 0) \to *$ is a weak equivalence. Since weak equivalences are stable under pushouts of cofibrations (*cf.* Definition 2.3.26) we conclude.

□

Proposition 2.3.47 *Consider the homotopy pushout square*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "p"'] & T \arrow[d, "p'"] \\ X \arrow[r, "i'"] & Y \end{tikzcd}$$

where i is a cofibration and every object is cofibrant. Then there is a distinguished triangle

$$Z \xrightarrow{i \oplus p} T \oplus X \xrightarrow{p' \oplus i'} Y \longrightarrow Z[1] \,.$$

□

Recall a general fact from category theory. Let $F\colon \mathbf{A} \leftrightarrows \mathbf{A'} : G$ be a pair of adjoint functors. If F is an equivalence of categories then $G = F^{-1}$.

Notation 2.3.48 Let $\mathbf{C}$ be a pointed model category and such that the suspension functor $[1]\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$ is an equivalence of categories. We denote the loop space of an object X as

$$X[-1] = \Omega X = \underline{\mathrm{Hom}}(S^1, X).$$

We also write $X[-n] = \Omega^n X$.

Recall that a triangulated category is, by definition, additive (*cf.* [Mac71, p.196]). In particular, Hom-sets are (additive) abelian groups. The following result is standard.

Proposition 2.3.49 *Let $\mathbf{A}$ be an additive category, then finite products equal finite coproducts. In other words, $X \times Y = X \sqcup Y$. We also note them $X \oplus Y$.*

□

Proposition 2.3.50 *Let $\mathbf{A}$ be a triangulated category and E be an object of $\mathbf{A}$. Every distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\partial} X[1]$$

induces a cohomological long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathbf{A}}(X[1], E) \xrightarrow{\partial^*} \mathrm{Hom}_{\mathbf{A}}(Z, E) \xrightarrow{g^*} \mathrm{Hom}_{\mathbf{A}}(Y, E) \xrightarrow{f^*} \mathrm{Hom}_{\mathbf{A}}(X, E) \to \cdots \,.$$

Proof: By the definition of distinguished triangle any two composite is zero. Let $u \in \mathrm{Hom}(Y, E)$ such that $u \circ f = 0$. Consider the commutative diagram

$$\begin{tikzcd}
* \arrow[r] & E \arrow[r, "1_E"] & E \arrow[r] & \Sigma * \\
X \arrow[u] \arrow[r, "f"] & Y \arrow[u, "u"] \arrow[r, "g"] & Z \arrow[r, "\partial"] & \Sigma X
\end{tikzcd}$$

where the upper and lower rows are distinguished triangles. By the axioms of triangulated category there exists a morphism $v\colon Z \to E$ making the diagram commutative. In other words, $u = v \circ g$ and therefore $\mathrm{Ker}\, f^* = \mathrm{Im}\, g^*$.

□

2.3.3 Bousfield localization

Let $\mathbf{C}$ be a model category and denote Weak its class of weak equivalences. Recall that in general it may not be easy to describe a localized category, but for a model category we have $\mathbf{C}[\mathrm{Weak}^{-1}] = \mathbf{Ho}(\mathbf{C})$. Let A be a set of morphisms of $\mathbf{C}$ and denote $\bar{\mathrm{A}}$ the induced set on $\mathbf{Ho}(\mathbf{C})$. Under mild hypothesis on $\mathbf{C}$, there is a general method to describe $\mathbf{C}[\mathrm{Weak}^{-1}][\mathrm{A}^{-1}] = \mathbf{Ho}(\mathbf{C})[\bar{\mathrm{A}}^{-1}]$ for any A in terms of the model structure of $\mathbf{C}$. This technique is called *Bousfield localization*. In order to avoid tedious notation we will not completely review the details. All categories we will consider satisfy the hypothesis required for the existence of such localizations (left proper cofibrantly generated cellular model category, *cf.* [Hir03, 4.1.1]).

Definition 2.3.51 Let $\mathbf{C}$ be a model category and A be a set of morphisms of $\mathbf{C}$. We say that an object U is A**-local** if U is fibrant and for every object X and every map $g\colon V \to W$ of A the induced map

$$g^*\colon \mathrm{Hom}_{\mathbf{Ho(C)}}(X \times W, U) \to \mathrm{Hom}_{\mathbf{Ho(C)}}(X \times V, U)$$

is a bijection. Denote $\mathbf{Ho}_{\mathrm{loc}}(\mathbf{C})$ the full subcategory of $\mathbf{Ho}(\mathbf{C})$ made of A-local objects.

Let $f\colon X \to Y$ be a morphism in $\mathbf{C}$:

- We say that f is an A**-local equivalence** if for every A-local object U the induced map

  $$f^*\colon \mathrm{Hom}_{\mathbf{Ho(C)}}(Y, U) \to \mathrm{Hom}_{\mathbf{Ho(C)}}(X, U)$$

  is a bijection. Denote W_{A} the class of A-local equivalences.

- We say that f is an A-**local fibration** if it has the right lifting property with respect to (original) cofibrations which also are A-local equivalences.

In addition, we say that an object X is A-**fibrant** if $X \to *$ is an A-local fibration.

Remark 2.3.52 I do not know of any geometrical reason for the use of the term *local*. To my knowledge, the reason is historical due to the original work of Bousfield.

Note that usual weak equivalences are A-local equivalences. We summarize in the following results the main properties of Bousfield localizations that we will use. Find a scattered proof in [Hir03, 4.1.1, 3.3.19, 3.3.14+3.3.16].

Theorem 2.3.53 *Let* $\mathbf{C}$ *be a model category satisfying the hypothesis of [Hir03, 4.1.1] and* A *be a set of morphisms in* $\mathbf{C}$*:*

1. *The class of A-local equivalences, (original) cofibrations and* A*-local fibrations define a model structure on* $\mathbf{C}$*. Denote* $\mathbf{Ho}_{\mathrm{A}}(\mathbf{C})$ *its homotopy category.*
2. *We have* $\mathbf{Ho}_{\mathrm{A}}(\mathbf{C}) \simeq \mathbf{Ho}(\mathbf{C})[\bar{\mathrm{A}}^{-1}]$.
3. *An object* X *is* A*-fibrant if and only if it is* A*-local.*

□

We call $\mathbf{Ho}_{\mathrm{A}}(\mathbf{C})$ the *Bousfield localization* of $\mathbf{C}$ in A. For coherence with the literature (*cf.* [MV99, 2.2.19]) denote

$$L_{\mathrm{A}} \colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}_{\mathrm{loc}}(\mathbf{C})$$

the A-local fibrant replacement induced in $\mathbf{Ho}(\mathbf{C})$. Note that it is left adjoint to the inclusion. Find a proof of the following result in [Hir03, 3.2.13].

Theorem 2.3.54 *Let* $\mathbf{C}$ *be a model category satisfying the hypothesis of [Hir03, 4.1.1] and* A *be a class of morphisms in* $\mathbf{C}$*. Assume all objects are cofibrant. Then the localization functor is* L_{A}*. In other words, we have a pair of adjoint functors*

$$L_{\mathrm{A}} \colon \mathbf{Ho}(\mathbf{C}) \leftrightarrows \mathbf{Ho}_{\mathrm{loc}}(\mathbf{C}) \simeq \mathbf{Ho}_{\mathrm{A}}(\mathbf{C}) : i.$$

□

Remark 2.3.55 For the sake of completeness, let us remark that we have reviewed the notion of *left* Bousfield localization. There is analogous notion of *right* Bousfield localization (*cf.* [Hir03, §3]).

Example 2.3.56 We recall two examples from [MV99, p. 86]:

- Consider the category of simplicial sets **sSets** with the trivial model structure. Set A to be the class of morphisms $X \times \Delta^1 \to X$ for all simplicial sets X. Then the Bousfield localization is the usual homotopy category of simplicial sets.

- Consider the category of locally contractible topological spaces with the trivial model structure. Let I be the unit interval. Set A to be the class of morphisms $X \times I \to X$ for all spaces X. Then the Bousfield localization is the usual homotopy category.

Chapter 3

Motivic homotopy theory

3.1 The homotopy category $\mathbf{H}(S)$

Let S be a finite dimensional Noetherian scheme. Recall that $\mathbf{Sm}_S$ denotes the category of smooth schemes over S. We will always consider $\mathbf{Sm}_S$ endowed with the Nisnevich topology of Example 2.2.3 so we omit the reference to the topology. Denote $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$ the category of Nisnevich sheaves of simplicial sets on $\mathbf{Sm}_S$ with morphism of sheaves. Note that the Yoneda embedding defines a fully faithful functor

$$\mathbf{Sm}_S \longrightarrow \Delta^{\mathrm{op}}\mathbf{Shv}(S)$$

which maps any smooth scheme X to the sheaf o simplicial sets defined by the sheaf of sets $\mathrm{Hom}_{\mathbf{Sm}_S}(\_\,, X)$. We still denote X to this sheaf.

Definition 3.1.1 Let $f\colon F \to G$ be a morphism of simplicial sheaves:

1. We say that f is a **local weak equivalence** if for every smooth scheme X and every point x in X the morphism of simplicial sets $f_x\colon F_x \to G_x$ is a weak equivalence of simplicial sets.

2. We say that f is a **local injective fibration** if it has the right lifting property with respect to local weak equivalences which are monomorphisms.

The following theorem comes from [Jar87, 2.7].

Theorem 3.1.2 *The classes of local weak equivalences, monomorphisms and local injective fibrations define a model structure on* $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$ *which we call the* local injective model structure*.*

□

Remark 3.1.3 In motivic homotopy theory we consider different model structures on the category $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$, three in this text and up to five that I know. We will have the same circumstance in section § 3.2 for spectra. Apart from the term *-local* for Bousfield localizations (*cf.* Definition 2.3.51) there are no standard notations in the literature with most papers using their own terminology. For example, the previous model structure is the most well known and it is referred as *simplicial* in [MV99], *motivic* in [Jar00], *local injective* in [Bla01] and *top-locale* in [Ayo07]. We have chosen to follow Blander's notation since it is the most self explanatory and the only one that allows to denote other model structures coherently.

Notation 3.1.4 Recall that the homotopy category of a model category is a localization (*cf.* Definition 2.3.1). Therefore, up to a natural transformation, it only depends on weak equivalences and not on cofibrations and fibrations. When needed, we denote homotopy categories referring their weak equivalences.

Remark 3.1.5 Recall that the model structure on simplicial sets have monomorphisms as cofibrations and Kan fibrations as fibrations (*cf.* Example 2.3.21).

Let $f\colon F \to G$ be a morphisms of simplicial sheaves. We say that f is a **local projective fibration** if for every smooth scheme X and every point x in X the morphism of simplicial sets $f_x\colon F_x \to G_x$ is a Kan fibration. Note that local projective fibrations in general *have not* the right lifting property with respect to monomorphisms which are local weak equivalences. Therefore they are not the fibrations of the local injective model structure. This is a general fact on simplicial sheaves and it would also happen for *sectionwise* weak equivalences.

However, the class of local weak equivalences, local projective fibrations and the class of morphisms having the left lifting property with respect to trivial local projective fibrations define a model structure on $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$ ([Jar87, 1.13]). We denote it the **local projective model structure**.

Many authors refer as *injective* the model structure with *natural* cofibrations and *projective* the one with natural fibrations (*cf.* [Bla01]). Note that this notation is coherent with the one on chain complexes of Example 2.3.21. Since both the injective and the projective model structures have the same weak equivalences they define the same homotopy category.

Notation 3.1.6 On the following, we will refer to model structures mentioning their weak equivalences and, if needed, *injective* or *projective* wether cofibrations or fibrations are the natural ones in the context.

The following definition comes from [MV99].

Definition 3.1.7 Denote by $\mathbf{W}_s$ the class of local weak equivalences. We define the **simplicial (or local) homotopy category** of S-schemes $\mathbf{H}_s(S)$ to be

$$\mathbf{H}_s(S) = \mathbf{Ho}(\Delta^{\mathrm{op}}\mathbf{Shv}(S)) = \Delta^{\mathrm{op}}\mathbf{Shv}(S)[\mathbf{W}_s^{-1}],$$

the homotopy category of $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$ with respect to the local injective model structure.

Example 3.1.8 Consider the sheaf defined by the base scheme S. For every smooth scheme X we have $S(X) = *$, the simplicial set of a point. Consider now the sheaf defined by the affine line $\mathbb{A}^1_S$. Recall that $\mathbb{A}^1_S(X) = \mathcal{O}_X(X)$, where $\mathcal{O}_X(X)$ denotes the simplicial set defined by the set $\mathcal{O}_X(X)$. Therefore $\mathbb{A}^1_S$ is not isomorphic to S in $\mathbf{H}_s(S)$.

Recall that in any model category there is a notion of homotopy between maps (Definition 2.3.10) and it defines an equivalence relation if the source space is cofibrant. Set $\pi(F,G)$ for the quotient of $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathbf{Shv}(S)}(F,G)$ under this equivalence relation.

Find in [Jar87, p.55] the following remark:

Proposition 3.1.9 *Let G be a local projective fibrant simplicial sheaf. For any simplicial sheaf F the canonical map*

$$\varinjlim_{p\colon F'\to F} \mathrm{Hom}_{\Delta^{op}\mathbf{Shv}(S)}(F',G)/\sim \to \mathrm{Hom}_{\mathbf{H}_s(S)}(F,G),$$

where the colimit is taken along trivial local projective fibrations and $\sim$ denotes the homotopy relation of Proposition 2.3.14, is a bijection.

$\square$

As we saw in Example 3.1.8, in order to obtain the adequate homotopy category of schemes further morphisms have to be inverted. We invert the minimum required class of morphisms.

Definition 3.1.10 A sheaf of simplicial sets F is called $\mathbb{A}^1$**-local** if for any sheaf of simplicial sets G the map

$$\mathrm{Hom}_{\mathbf{H}_s(S)}(G,F) \to \mathrm{Hom}_{\mathbf{H}_s(S)}(G\times\mathbb{A}^1,F)$$

induced by the projection $G\times\mathbb{A}^1\to G$ is a bijection. Denote $\mathbf{H}_{\mathrm{loc}}(S)$ the full subcategory of $\mathbf{H}_s(S)$ of $\mathbb{A}^1$-local objects.

Let $f\colon F\to G$ be a morphism of simplicial sheaves:

- We say that f is an $\mathbb{A}^1$**-weak equivalence** if for any $\mathbb{A}^1$-local sheaf H the induced map of simplicial sets

$$\mathrm{Hom}_{\mathbf{H}_s(S)}(G, H) \to \mathrm{Hom}_{\mathbf{H}_s(S)}(F, H)$$

is a bijection.

- We say that f is an $\mathbb{A}^1$**-fibration** if it has the right lifting property with respect to monomorphisms which are $\mathbb{A}^1$-weak equivalences.

The following result is originally proved in [MV99, §2.3], but it is nowadays a particular case of the Bousfield localization (*cf.* Theorem 2.3.53).

Theorem 3.1.11 *The classes of $\mathbb{A}^1$-weak equivalences as weak equivalences, monomorphisms as cofibrations and $\mathbb{A}^1$-fibrations as fibrations define a simplicial model structure on $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$. We call it the* $\mathbb{A}^1$-model structure.

$\square$

Definition 3.1.12 Denote $\mathbf{W}_{\mathbb{A}^1}$ the class of $\mathbb{A}^1$-weak equivalences. We define the **homotopy category** of S-schemes $\mathbf{H}(S)$ to be

$$\mathbf{H}(S) = \Delta^{\mathrm{op}}\mathbf{Shv}(S)[\mathbf{W}_{\mathbb{A}^1}^{-1}],$$

the homotopy category of $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$ with respect to the $\mathbb{A}^1$-model structure. We denote morphisms in this category as $[\ \_\ ,\ \_\ ]$.

Denote $L_{\mathbb{A}^1} \colon \mathbf{H}_s(S) \to \mathbf{H}_{\mathrm{loc}}(S)$ the $\mathbb{A}^1$-local fibrant replacement induced in $\mathbf{H}_s(S)$. The general theory of Bousfield localizations allows to conclude the following result (*cf.* Theorem 2.3.53).

Corollary 3.1.13 *The localization functor of the $\mathbb{A}^1$-model structure is $L_{\mathbb{A}^1}$. In other words, we have a pair of adjoint functors*

$$L_{\mathbb{A}^1} \colon \mathbf{H}_s(S) \leftrightarrows \mathbf{H}_{\mathrm{loc}}(S) \simeq \mathbf{H}(S) : i.$$

$\square$

3.1.14 Let $\Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}_S)$ be the category of pointed Nisnevich sheaves of simplicial sets. Consider the pointed version of the previous model structures, which we call the *pointed local model structure* and the *pointed $\mathbb{A}^1$-model structure* on $\Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}_S)$. We define the *pointed simplicial (or local) homotopy category* and the *pointed homotopy category* of schemes to be the respective homotopy categories. We denote them

$$\mathbf{H}^s_\bullet(S) \quad , \quad \mathbf{H}_\bullet(S).$$

Let F be a sheaf of pointed sets. Recall that the functor $F \wedge \_$ is left adjoint to the functor $\underline{\mathrm{Hom}}(F, \_)$. We can apply the same construction for simplicial sets instead of sets.

Proposition 3.1.15 *Let F be a sheaf of pointed simplicial sets. The pair of adjoint functors*

$$F \wedge \_ : \Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}_S) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}_S) : \underline{\mathrm{Hom}}(F, \_)$$

form a Quillen adjunction both for the local and the $\mathbb{A}^1$-model structure. We abuse notation and still denote $\underline{\mathrm{Hom}}(F, \_)$ the internal Hom *object in $\mathbf{H}^s_\bullet(S)$ and $\mathbf{H}_\bullet(S)$.*

Proof: It is direct to check that $F \wedge \_$ preserves monomorphisms and local weak equivalences so that the derived functors exist in $\mathbf{H}^s_\bullet(S)$.

Let us see that $F \wedge \_$ preserves $\mathbb{A}^1$-weak equivalences. Note that we have $\mathrm{Hom}_{\mathbf{H}^s_\bullet(S)}(F \wedge G, H) = \mathrm{Hom}_{\mathbf{H}^s_\bullet(S)}(G, \underline{\mathrm{Hom}}(F, H))$ so that if H is $\mathbb{A}^1$-local then $\underline{\mathrm{Hom}}(F, H)$ is $\mathbb{A}^1$-local for all F. Therefore if $f : G \to G'$ is an $\mathbb{A}^1$-weak equivalence note that

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{H}^s_\bullet(S)}(F \wedge G, H) \\
&\quad = \mathrm{Hom}_{\mathbf{H}^s_\bullet(S)}(G, \underline{\mathrm{Hom}}(F, H)) \simeq \mathrm{Hom}_{\mathbf{H}^s_\bullet(S)}(G', \underline{\mathrm{Hom}}(F, H)) \\
&\qquad = \mathrm{Hom}_{\mathbf{H}^s_\bullet(S)}(F \wedge G', H).
\end{aligned}$$

□

We recall an important result due to Morel and Voevodsky. Note that in motivic homotopy theory there are different notions of *spheres*. On one hand we have noted S^1 the sheaf given by the simplicial set of circle. Following [MV99], we denote

$$\begin{aligned}
\mathbb{G}_m &\;-\; \text{the sheaf represented by } \mathbb{A}^1 - \{0\} \text{ pointed by } 1, \\
T &\;-\; \text{the quotient sheaf } \mathbb{A}^1/\mathbb{A}^1 - \{0\}
\end{aligned}$$

and we call the multiplicative group $\mathbb{G}_m$ the *Tate circle*. Find a proof of the next result in [MV99, 3.2.15].

Theorem 3.1.16 *With the above notations, we have in $\mathbf{H}_\bullet(S)$ a canonical isomorphism*

$$\mathbb{G}_m \wedge S^1 \simeq T \simeq \mathbb{P}^1.$$

□

3.1.1 Functoriality, localization, blow-up and homotopy purity

In [MV99, §2.3.3] Morel and Voevodsky proved that the homotopy category of schemes satisfies good functorial properties for a general continuous map of sites satisfying certain hypothesis. We recall them here in the case of morphisms of schemes.

The same constructions from Proposition 2.2.19 and the following results apply for sheaves of simplicial sets instead of sheaves of sets. It follows that any continuous map of sites $f\colon \mathbf{Sm}_S \to \mathbf{Sm}_T$ (*cf.* Definition 2.2.18) induces a pair of adjoint functors

$$f^*\colon \Delta^{\mathrm{op}}\mathbf{Shv}(T) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Shv}(S) : f_*.$$

The pair (f^*, f_*) does not form in general a Quillen adjunction. Nevertheless, under certain *reasonable* conditions (*cf.* [MV99, 2.1.55 and 2.3.16]) they still induce total derived functors for the simplicial homotopy category and the homotopy category ([MV99, 2.1.57 and 2.3.17]). In particular, the morphisms of sites of Example 2.2.20 given by a morphism of schemes are reasonable ([MV99, 3.1.20, 3.2.8 and 3.2.9]).

We quickly review the construction. Denote R_i the fibrant replacement for the local injective model structure. Let $f\colon T \to S$ be a morphism of schemes. Then f_* induces a functor $\mathbf{H}_s(T) \to \mathbf{H}_s(S)$ defined as $F \mapsto f_*(R_iF)$ on objects and $\varphi \mapsto f_*(R_i\varphi)$ on morphisms in $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$. This functor is the total right derived functor of f_* (*cf.* [MV99, p. 62]). We abuse notation and denote it

$$f_*\colon \mathbf{H}_s(T) \to \mathbf{H}_s(S).$$

Denote $\Phi_f\colon \Delta^{\mathrm{op}}\mathbf{Shv}(S) \to \Delta^{\mathrm{op}}\mathbf{Shv}(S)$ the functor defined in [MV99, p. 65], which is a concrete local projective replacement. In other words, this functor has a natural transformation $\Phi \to \mathrm{Id}$ such that $\Phi(F) \to F$ is a trivial local projective fibration for all F. The reference then proves that the functor $f^* \circ \Phi_f\colon \Delta^{\mathrm{op}}\mathbf{Shv}(S) \to \Delta^{\mathrm{op}}\mathbf{Shv}(T)$ preserves local weak equivalences and that it is the total left derived functor of the inverse image of simplicial sheaves (*cf.* [MV99, 2.1.57]). We abuse notation and denote

$$f^*\colon \mathbf{H}_s(S) \to \mathbf{H}_s(T)$$

the functor it induces. Find a proof of the following statement in [MV99, 3.1.20].

Theorem 3.1.17 *Let $f\colon T \to S$ be a morphism of schemes. Then the pair of functors*

$$f^*\colon \mathbf{H}_s(S) \leftrightarrows \mathbf{H}_s(T) : f_*$$

defined above are adjoint. Let $g\colon U \to T$ be a morphism of schemes, we have canonical natural transformations

$$(f \circ g)_* \simeq f_* \circ g_*,$$

$$(f \circ g)^* \simeq g^* \circ f^*.$$

□

Let $p\colon X \to S$ be a smooth morphism. Recall from Example 2.2.20 that p induces another morphism of sites $\Phi p\colon \mathbf{Sm}_{S,\mathbf{Nis}} \to \mathbf{Sm}_{X,\mathbf{Nis}}$. In [MV99, p. 105] Morel and Voevodsky observe that applying an analogue construction there is an induced functor

$$p_\sharp\colon \mathbf{H}_s(U) \to \mathbf{H}_s(S).$$

Similar arguments also deduce the following result.

Corollary 3.1.18 *Let $p\colon X \to S$ be a smooth morphism. The functors*

$$p_\sharp\colon \mathbf{H}_s(X) \leftrightarrows \mathbf{H}_s(S) : p^*$$

are adjoint. Let $q\colon Y \to X$ be a smooth morphism we have a canonical natural transformation

$$(p \circ q)_\sharp \simeq p_\sharp \circ q_\sharp.$$

□

Deriving for the $\mathbb{A}^1$-model structure is now easy. Let $f\colon T \to S$ be a morphism of schemes. Note that for any sheaf F we have $f^*(F \times \mathbb{A}^1_S) = f^*(F) \times \mathbb{A}^1_T$ (*cf.* Proposition 2.2.23). Therefore it follows that f^* preserves $\mathbb{A}^1$-weak equivalences and induces a functor

$$f^*\colon \mathbf{H}(S) \longrightarrow \mathbf{H}(T).$$

It follows directly that the functor f_* preserves $\mathbb{A}^1$-local objects. Indeed,

$$\mathrm{Hom}_{\mathbf{H}_s(S)}(G\times\mathbb{A}^1_S, f_*F) = \mathrm{Hom}_{\mathbf{H}_s(T)}(f^*(G)\times\mathbb{A}^1_T, F) = \mathrm{Hom}_{\mathbf{H}_s(S)}(f^*(G)\times\mathbb{A}^1_T, F)$$

$$\simeq \mathrm{Hom}_{\mathbf{H}_s(T)}(f^*G, F) = \mathrm{Hom}_{\mathbf{H}_s(S)}(G, f_*F).$$

Therefore, the direct image induces a functor

$$f_*\colon \mathbf{H}(T) \longrightarrow \mathbf{H}(S).$$

We have proved the following result.

Proposition 3.1.19 *Let $f\colon T \to S$ be a morphism of schemes. The pair of functors*

$$f^*\colon \mathbf{H}(S) \longrightarrow \mathbf{H}(T) : f_*$$

are adjoint. Let $g\colon U \to T$ be a morphism of schemes, we have canonical natural transformations

$$(f \circ g)_* \simeq f_* \circ g_*,$$

$$(f \circ g)^* \simeq g^* \circ f^*.$$

□

Let $p\colon X \to S$ be a smooth morphism. Recall that $p_\sharp$ satisfies a projection formula (*cf.* Proposition 2.2.24) so that in particular $p_\sharp(F \times \mathbb{A}^1_X) = p_\sharp(F) \times \mathbb{A}^1_S$ and $p_\sharp$ preserves $\mathbb{A}^1$-weak equivalences. Therefore it induces a functor which we denote as well

$$p_\sharp\colon \mathbf{H}(X) \longrightarrow \mathbf{H}(S).$$

We deduce the following result.

Proposition 3.1.20 *Let $p\colon X \to S$ be a smooth morphism. Then the pair of functors*

$$p_\sharp\colon \mathbf{H}(X) \leftrightarrows \mathbf{H}(S) : p^*$$

are adjoint. Let $q\colon Y \to X$ be a smooth morphism, we have a canonical natural transformation

$$(p \circ q)_\sharp \simeq p_\sharp \circ q_\sharp.$$

□

The following is one of the main results of [MV99], which is not true in $\mathbf{H}_s(S)$ nor if we consider the Zariski topology nor the category all schemes $\mathbf{Sch}_S$. Note that this theorem is stated in the big Nisnevich site, and it would be trivial for the small site. Although deriving functors in $\mathbf{H}(S)$ is simple, the proof of the following result relies on some resolution lemmas which we have not reviewed. Find the original proof in [MV99, 3.2.21].

Theorem 3.1.21 (Localization) *Let $j\colon U \to S$ be an open embedding with complement the closed embedding $i\colon Z \to S$. Then for any simplicial sheaf F the square*

$$\begin{tikzcd} j_\sharp j^* F \arrow[r] \arrow[d] & F \arrow[d] \\ U \arrow[r] & i_* i^* F \end{tikzcd}$$

is homotopy cocartesian in $\mathbf{H}(S)$.

$\square$

Remark 3.1.22 Let $j\colon U \to S$ be an open embedding. Note that the simplicial sheaf U on $\mathbf{Sm}_S$ has one element on smooth schemes over U and is empty elsewhere. Therefore there are maps $j_\sharp F' \to U$ (cf. Proposition 2.2.24) and $U \to i_* F''$ (*cf.* Proposition 2.2.21) for any simplicial sheaves F', F''. These are the maps from the statement.

Also note that this result is the homotopy analogue to saying that for a sheaf F of abelian groups the sequence

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

is exact.

Example 3.1.23 Let $j\colon U \to S$ be an open embedding with complement the closed embedding $i\colon Z \to S$:

- Consider the simplicial sheaf defined by the base scheme S. Note that for any smooth S-scheme X we have $S(X) = *$. Observe that $j_\sharp j^* S = j_\sharp S|_U = U$. Therefore $j_\sharp j^* S = U \to U$ is the identity and the localization square is cocartesian and not just homotopy cocartesian. In particular $i_* i^* S \simeq S$.
- Consider X a smooth S-scheme and the simplicial sheaf it defines. Recall from Example 2.2.25 that $(j_\sharp j^* X)(Y)$ equals $X(Y)$ if Y is a smooth U-scheme, and equals $\emptyset$ otherwise. We denoted this sheaf $X - X_Z$. Since the upper arrow of the localization theorem is a cofibration by Proposition 2.3.28 there is no need to take cofibrant replacements. We conclude that $(i_* i^* X)(Y)$ equals a point if Y is a smooth U-scheme and $X(Y)$ if otherwise.

Recall from Remark 2.2.26 that the classical condition of "being zero" for sheaves of abelian groups translate into being $*$ or $\emptyset$ into sheaves of simplicial sets. We remark the pointed version of the localization theorem.

Corollary 3.1.24 *Let $j\colon U \to S$ be an open embedding with complement the closed embedding $i\colon Z \to S$. Then for any pointed simplicial sheaf F the square*

$$\begin{tikzcd} j_\sharp j^* F \arrow[r] \arrow[d] & F \arrow[d] \\ * \arrow[r] & i_* i^* F \end{tikzcd}$$

is homotopy cocartesian in $\mathbf{H}_\bullet(S)$.

□

Example 3.1.25 Let $j\colon U \to S$ and $i\colon Z \to S$ as before. Consider the pointed simplicial sheaf S_+. In this case it sections are always two points $S_+(Y) = * \sqcup *$. We obtain that

$$S_+/U_+ \simeq i_* i^* S_+$$

is an isomorphism in $\mathbf{H}_\bullet(S)$. In general, for X a smooth S-scheme we have that $X_+/X_{U+} \simeq i_* i^* X_+$.

We recall from [MV99, §3] a classic result stated in this context. Find the proof in [MV99, 3.2.29].

Theorem 3.1.26 (Blow-up) *Let $i\colon Z \to X$ be a closed embedding of smooth schemes over S, $\pi\colon B_Z X \to X$ be the blowing-up of Z in X and $U = X - Z = B_Z X - \pi^{-1}(Z)$. Then the square*

$$\begin{tikzcd}
\pi^{-1}(Z) \arrow[r] \arrow[d] & X_Z/U \arrow[d] \\
Z \arrow[r] & X/U
\end{tikzcd}$$

is homotopy cocartesian in $\mathbf{H}(S)$.

□

Let us remark the pointed version of this theorem.

Corollary 3.1.27 *Let $i\colon Z \to X$ be a closed embedding of smooth schemes over S, $\pi\colon B_Z X \to X$ be the blowing-up of Z in X and $U = X - Z = B_Z - \pi^{-1}(Z)$. Then the square*

$$\begin{tikzcd}
\pi^{-1}(Z)_+ \arrow[r] \arrow[d] & X_{Z+} \arrow[d] \\
Z_+ \arrow[r] & X_+
\end{tikzcd}$$

is homotopy cocartesian in $\mathbf{H}_\bullet(S)$.

□

Definition 3.1.28 Let $V \to X$ be a vector bundle. We define the **Thom space** of V as

$$\mathrm{Th}(V) = V/V - 0.$$

The following result comes from [MV99, 3.2.23].

Theorem 3.1.29 (Homotopy purity) *Let $i\colon Z \to X$ be a closed embedding of smooth schemes and denote $N_{Z/X} \to Z$ the normal bundle. Then there is a canonical isomorphism in $\mathbf{H}_\bullet(S)$*

$$\mathfrak{p}_Z^X\colon X/X - Z \xrightarrow{\sim} \mathrm{Th}(N_{Z/X}).$$

□

Remark 3.1.30 The reason why the previous result is referred as homotopy purity is the following. Loosely speaking let H denote a cohomology and $i\colon Z \to X$ be a closed immersion between schemes satisfying strong hypotheses (at least smooth). Then it is expected to have an isomorphism

$$\mathfrak{p}_Z^X\colon H(Z) \xrightarrow{\sim} H_Z(X)$$

called the *purity isomorphism.* Let $V \to X$ be a vector bundle, by construction of the Thom space there should be an isomorphism called the *Thom isomorphism*

$$H(X) \xrightarrow{\sim} H(\mathrm{Th}(V))$$

given by the *Thom class* (*cf.* Example 4.2.19).

Depending on the context there are different methods to prove the purity isomorphism. One method is by means of the Thom isomorphism and a result as Theorem 3.1.29. Very loosely speaking, assume there is a category $\mathbf{D}$ analogue to the derived category of complexes satisfying the properties of Grothendieck's six operations and where H is represented. More concretely, there is an object $\mathcal{H}$ in $\mathbf{D}$ such that $H(X) = \mathrm{Hom}_{\mathbf{D}}(X, \mathcal{H})$. Then the above result Theorem 3.1.29 implies purity since

$$\begin{aligned}\mathfrak{p}_Z^X\colon H_Z(X) = \mathrm{Hom}_{\mathbf{D}}(X/X\text{-}Z, \mathcal{H}) &\xrightarrow[3.1.29]{\sim} \mathrm{Hom}_{\mathbf{D}}(\mathrm{Th}(N_{Z/X}), \mathcal{H}) \\ &= H(\mathrm{Th}(N_{Z/X})) \xrightarrow[\text{Thom iso}]{\sim} H(Z).\end{aligned}$$

In motivic homotopy theory the stable homotopy category we construct in Section 3.2 satisfies the properties required for $\mathbf{D}$.

3.1.2 Classification of torsors

As in classic topology the simplicial homotopy category $\mathbf{H}_s(S)$ is an adequate framework to classify torsors. The proof follows from the case of simplicial sets with similar arguments. However, one has to take into account that if one considers monomorphisms as cofibrations then fibrations in $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$ are not the analogue to Kan fibrations in **sSets** (*cf.* Remark 3.1.5). The original proof of Morel and Voevodsky in [MV99, §4.1.3] is only sketched and one of the results we need is not explicitly stated there (*cf.* Proposition 3.1.43). Therefore, we include complete proofs up to the result we need.

We recall the definitions of section § 2.1.1 in the context $\Delta^{\mathrm{op}}\mathbf{Shv}(S)$.

Definition 3.1.31 Let $\mathcal{G}$ be a sheaf of groups and F be a sheaf of simplicial sets. An **action** of $\mathcal{G}$ on F is morphism $a\colon \mathcal{G} \times F \to F$ satisfying analogue diagrams to that of Definition 2.1.19. We say that the action is **free** if the map $\mathcal{G} \times F \to F \times F$, which maps any pair of sections (g, x) to $(a(g, x), x)$, is a monomorphism.

A **$\mathcal{G}$-torsor** over F is a morphism $\mathcal{T} \to F$ of sheaves of simplicial sets with a free action of $\mathcal{G}$ on $\mathcal{T}$ over F such that the canonical morphism $\mathcal{T}/\mathcal{G} \to F$ is an isomorphism. We denote by $P(F, \mathcal{G})$ the set of isomorphism classes of $\mathcal{G}$-torsors.

Remark 3.1.32 The example we are interested is when $F = S$, the sheaf of simplicial sets defined by the base scheme, and $\mathcal{G} = \mathbb{G}_m$ the functor of points of the multiplicative group. Allowing general torsors over and arbitrary sheaf F is only needed for the case of pseudodivisors in Proposition 3.1.43. The reader interested in the classification of line bundles may assume that every torsor is over S.

The following result is direct.

Proposition 3.1.33 *Let $\mathcal{G}$ be a sheaf of groups and $\mathcal{T}$ a sheaf of simplicial sets with an action of $\mathcal{G}$. A map $\mathcal{T} \to F$ is a $\mathcal{G}$-torsor if and only if $\mathcal{T}_x \to F_x$ is a $\mathcal{G}_x$-torsor (of simplicial sets) for every point x of a smooth S-scheme X.*

□

Example 3.1.34

- The second projection $\pi\colon \mathcal{G} \times F \to F$ defines the trivial $\mathcal{G}$-torsor over F so that the set $P(F, \mathcal{G})$ is not empty and we chose the trivial $\mathcal{G}$-torsor as a base point.
- Let $f\colon H \to F$ be a morphism of simplicial sheaves and $\mathcal{T} \to F$ be a $\mathcal{G}$-torsor. Then $f^*\mathcal{T} = \mathcal{T} \times_F H$ is a $\mathcal{G}$-torsor over H.
- Let X be a scheme and $L \to X$ be a line bundle. The complement of the zero section defines a $\mathbb{G}_m$-torsor (in the classical sense). Therefore the complement of the zero section defines a $\mathbb{G}_m$-torsor of sheaves of simplicial sets.

The fact that the definition of $\mathcal{G}$-torsor does not require to be "locally trivial" may surprise. Let us remark the following property.

Proposition 3.1.35 *Let $\mathcal{G}$ be a sheaf of groups and $\mathcal{T} \to S$ be a $\mathcal{G}$-torsor over S, the sheaf defined by the base scheme. Then there exists a Nisnevich covering $\{U_\alpha\}$ of S such that*

$$\mathcal{T} \times_S U_\alpha = \mathcal{G} \times U_\alpha$$

for all α.

Proof: The result follows since every torsor $\mathcal{T}$ over S is trivial on stalks. Indeed, note that S is the final object of $\mathbf{Sm}_S$ so that $S(U) = \{p_U\}$, where p_U denotes the structural morphism. Therefore $S_x = *$ for every point x of S. Then $\mathcal{T}_x = \mathcal{G}_x$ is trivial and there exists a Nisnevich neighborhood U_α of x such that $\mathcal{T} \times_S U_\alpha = \mathcal{G} \times U_\alpha$ and a sections $s_\alpha\colon U_\alpha \to \mathcal{T}$. Therefore we have a section

$$\begin{tikzcd} \mathcal{T} \times_S U_\alpha \arrow[d, shift left] \\ U_\alpha \arrow[u, shift left, "s_\alpha"] \end{tikzcd}$$

so that $\mathcal{T} \times_S U_\alpha = \mathcal{G} \times U_\alpha$.

□

Let $\mathcal{G}$ be a sheaf of groups. Denote by $E\mathcal{G}$ the sheaf of simplicial sets with n-th term $\mathcal{G}\times \overset{n+1}{\cdots} \times\mathcal{G}$. Note that $E\mathcal{G}$ is naturally a sheaf of simplicial groups and has an action of $\mathcal{G}$. Denote $B\mathcal{G} = E\mathcal{G}/\mathcal{G}$. The sheaf $E\mathcal{G}$ is naturally a $\mathcal{G}$-torsor over $B\mathcal{G}$ and we call $E\mathcal{G}$ the **universal $\mathcal{G}$-torsor** and $B\mathcal{G}$ the **classifying space**. Let $\mathcal{T}$ be a $\mathcal{G}$-torsor over F. Consider the quotient presheaf which maps a smooth S-scheme U to the quotient of $(\mathcal{T} \times E\mathcal{G})(U)$ under the relationship $(fg, e) \sim (f, ge)$ for $(f, e) \in (T \times E\mathcal{G})(U)$, $g \in G(U)$ and U an S-smooth scheme. We denote $\mathcal{T} \times_{\mathcal{G}} E\mathcal{G}$ the associated sheaf.

Lemma 3.1.36 *Let $\mathcal{G}$ be a sheaf of groups and $\mathcal{T}$ a $\mathcal{G}$-torsor over F. Denote $p_1\colon \mathcal{T} \times_{\mathcal{G}} E\mathcal{G} \to \mathcal{T}/\mathcal{G} = F$ and $p_2\colon \mathcal{T} \times_{\mathcal{G}} E\mathcal{G} \to E\mathcal{G}/\mathcal{G} = B\mathcal{G}$. Then p_1 is a trivial local projective fibration and $p_1^*\mathcal{T} \simeq p_2^*E\mathcal{G}$.*

Proof: Consider a point x of a scheme X. The first projection induces a map of simplicial sets $p_{1*}\colon (\mathcal{T} \times_{\mathcal{G}} E\mathcal{G})_x \to F_x$. Moreover, we have $(\mathcal{T} \times_{\mathcal{G}} E\mathcal{G})_x = \mathcal{T}_x \times_{\mathcal{G}_x} E\mathcal{G}_x$. Since $\mathcal{T}_x$ is a $\mathcal{G}_x$-torsor over F_x of simplicial sets p_{1*} is a trivial Kan fibration so that p_1 is a trivial local projective fibration. Finally, by construction $p_1^*\mathcal{T} \simeq p_2^*E\mathcal{G} \simeq \mathcal{T} \times E\mathcal{G}$.

□

Definition 3.1.37 Let $f\colon F \to H$ be a morphism of sheaves of simplicial sets. We define the **relative** π_0 to be the sheaf of simplicial sets $\pi_0(f)$ defined by the presheaf of simplicial sets $U \to \pi_0(f_U)$ (*cf.* Definition 2.1.33).

Lemma 3.1.38 *Let $\mathcal{G}$ be a sheaf of groups. If $f\colon F \to H$ is a trivial local projective fibration then the map*

$$\begin{array}{ccc} P(H,\mathcal{G}) & \to & P(F,\mathcal{G}) \\ \mathcal{T} & \mapsto & f^*\mathcal{T} \end{array}$$

is a bijection.

Proof: It is enough to construct the inverse. Note that there is a map $\pi_0(f) \to H$. For every point x we have that $\pi_0(f)_x \to H_x$ is a $\mathcal{G}_x$-torsor due to Proposition 2.1.34. Therefore $\pi_0(f)$ is a $\mathcal{G}$-torsor over H.

□

Theorem 3.1.39 *Let $\mathcal{G}$ be a sheaf of commutative groups. The natural map*

$$\begin{array}{ccc} P(F,\mathcal{G}) & \longrightarrow & \mathrm{Hom}_{\mathbf{H}_s(S)}(F, B\mathcal{G}) \\ \mathcal{T} & \mapsto & p_2 \circ p_1^{-1}, \end{array}$$

where p_1 and p_2 are the maps of Lema 3.1.36, is a bijection.

Proof: The map is well defined due to Lemmas 3.1.36 and 3.1.38. It is a bijection due to Proposition 3.1.9.

□

Note that both $E\mathcal{G}$ and $B\mathcal{G}$ are naturally pointed. Therefore there is a pointed version of the preceding result considering pointed torsors over sheaves of pointed simplicial sets. We denote $P_\bullet(F,\mathcal{G})$ the set of isomorphism classes of pointed torsors.

Corollary 3.1.40 *For any sheaf of groups $\mathcal{G}$, and F be a sheaf of pointed simplicial sets. The natural map*

$$P_\bullet(F,\mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathbf{H}^s_\bullet(S)}(F, B\mathcal{G})$$

is a bijection.

□

Let X be a scheme. We denote by $\mathrm{Pic}(X)$ the group of classes of isomorphisms of (Zariski) line bundles over X. Recall that the complement of the zero section naturally defines a $\mathbb{G}_m$-torsor (of sheaves of simplicial sets) over X.

Corollary 3.1.41 *Let X be a scheme, the natural map*

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Hom}_{\mathbf{H}_s(X)}(X, B\mathbb{G}_m) = \mathrm{Hom}_{\mathbf{H}^s_\bullet(X)}(X_+, B\mathbb{G}_m)$$

is a bijection.

Proof: It is enough to prove that every $\mathbb{G}_m$-torsor, as sheaf of simplicial sets, is a classical torsor. Let $\mathcal{T}$ be a $\mathbb{G}_m$-torsor over X. Since both $\mathbb{G}_m$ and X have simplicial dimension zero (Definition 2.1.4) then $\mathcal{T}$ has simplicial dimension zero. Moreover, by Proposition 3.1.35 there exists a Nisnevich covering $\{U_\alpha\}$ of X where $\mathcal{T}$ is trivial. In other words

$$\mathcal{T} \times_X (\coprod_\alpha U_\alpha) = \coprod_\alpha G \times U_\alpha.$$

Therefore $\mathcal{T}$ is locally representable by a classical $\mathbb{G}_m$-torsor, so we conclude it is represented by a classical torsor.

$\square$

Definition 3.1.42 Let X be a scheme and $U = (X - Z) \to X$ be an open subscheme. We call a **pseudo divisor** (trivialized on U) a pair $(\mathcal{L}, u)$ consisting of an invertible sheaf $\mathcal{L}$ over X (in the Zariski topology) and a trivialization $u\colon \mathcal{O}|_U \xrightarrow{\sim} \mathcal{L}|_U$. We denote $\mathrm{Pic}_Z(X)$ the group of isomorphism classes of pseudo divisors.

Let X be a scheme and $i\colon U \to X$ be an open immersion. Recall that we denoted by X/U the quotient Nisnevich sheaf of simplicial sets and $\pi\colon X_+ \to X_+/U_+ = X/U$ the natural projection. Note that we have a natural morphism $p\colon * \to X/U$ so that X/U is naturally pointed. Let $\mathcal{T}$ be a pointed $\mathbb{G}_m$-torsor over X/U, then $\pi^*\mathcal{T} \to X_+$ is a $\mathbb{G}_m$-torsor over X_+ trivialized over U_+ in the sense that $\pi^*\mathcal{T}$ has a fixed isomorphism $p'\colon \pi^*\mathcal{T} \times_{X_+} U_+ \xrightarrow{\sim} \mathbb{G}_m \times U_+$ coming from the pullback of the trivialization of $\mathcal{T}$ over p.

Proposition 3.1.43 *Let $Z \to X$ be a regular immersion of codimension 1 and denote U its open complement. Then the map*

$$\begin{array}{ccc} P_\bullet(X/U, \mathbb{G}_m) & \longrightarrow & \mathrm{Pic}_Z(X) \\ \mathcal{T} & \mapsto & \pi^*\mathcal{T} \end{array}$$

is a bijection.

Proof: It is enough to construct the inverse map. Let T be a torsor over X with a trivialization on U. We have a cartesian diagram

$$\begin{tikzcd} T_+ \times_{X_+} U_+ \simeq \mathbb{G}_m \times U_+ \arrow[r] \arrow[d] & T_+ \arrow[d] \\ U_+ \arrow[r, "i"] & X_+. \end{tikzcd}$$

The quotient $T_+/\mathbb{G}_m \times U_+$ defines a $\mathbb{G}_m$-torsor over X/U.

$\square$

Definition 3.1.44 We define the **infinite projective space** to be the sheaf of simplicial sets

$$\mathbb{P}^\infty = \varinjlim \mathbb{P}^n$$

where immersions are taken through the infinite zone. Note that $\mathbb{P}^\infty$ is naturally pointed.

We recall a result from Morel and Voevodsky (*cf.* [MV99, 4.3.7]).

Theorem 3.1.45 *There is a canonical isomorphism in* $\mathbf{H}(S)$

$$B\mathbb{G}_m \simeq \mathbb{P}^\infty.$$

□

3.2 The stable homotopy category $\mathbf{SH}(S)$

In this section we recall the construction of the stable homotopy category of schemes of Voevodsky ([Voe98]). Recall that $\Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}_S)$ denotes the category of Nisnevich sheaves of pointed simplicial sets on $\mathbf{Sm}_S$. Note that the same construction holds if one considers presheaves instead of sheaves and yields the same stable homotopy category (*cf.* [Jar00, 1.2]). For coherence with the previous section we write for sheaves.

As we saw in Theorem 2.3.45, in simplicial model categories it is enough that the suspension functor $S^1 \wedge \_$ is an equivalence of categories to have a triangulated category. Formally inverting $S^1 \wedge \_$ on $\mathbf{H}_\bullet(S)$ one obtains the Spanier-Whitehead category (*cf.* [Voe98]). However Voevodsky remarked that, as in the topological case, the Spanier-Whitehead category is not the adequate category since it lacks arbitrary coproducts (direct sums). This fact is addressed with spectra.

In addition, inverting $S^1 \wedge \_$ is not sufficient to treat cohomologies with Tate twists. Voevodsky noted that inverting the smash product by two elements, $S^1 \wedge \_$ and $\mathbb{G}_m \wedge \_$ in our case, is equivalente to invert their product. Recall that $S^1 \wedge \mathbb{G}_m \simeq \mathbb{P}^1$ by Theorem 3.1.16.

Definition 3.2.1 Denote $\mathbb{P}^1$ the sheaf of pointed simplicial sets defined by the projective line pointed by the infinity. We call a **$\mathbb{P}^1$-spectrum** E, or simply spectrum, to a sequence $(E_n)_{n\in\mathbb{N}}$ of objects of $\Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}_S)$ and morphisms of sheaves of pointed simplicial sets $\sigma_n\colon \mathbb{P}^1 \wedge E_n \to E_{n+1}$ for every n. We call a **morphism of spectra** $\varphi\colon E \to F$ to a sequence of morphisms

$(\varphi_n \colon E_n \to F_n)_{n\in\mathbb{N}}$ such that for all n the diagram

$$\begin{tikzcd}
\mathbb{P}^1 \wedge E_n \arrow[r, "\sigma_n"] \arrow[d, "1_{\mathbb{P}^1}\wedge\varphi_n"'] & E_{n+1} \arrow[d, "\varphi_{n+1}"] \\
\mathbb{P}^1 \wedge F_n \arrow[r, "\bar{\sigma}_n"] & F_{n+1}
\end{tikzcd}$$

commutes. We denote by $\mathbf{Spt}(S)$ the category of spectra with morphisms of spectra.

Let F be a pointed Nisnevich sheaf of simplicial sets. We call the **infinite suspension** of F to the spectrum $\Sigma^\infty F = ((\mathbb{P}^1)^{\wedge n} \wedge F)_{n\in\mathbb{N}}$ with structural morphisms $\sigma_n = 1_{\mathbb{P}^{1\wedge n}} \wedge 1_X$. Note that the infinite suspension defines functors

$$\Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}_S) \longrightarrow \mathbf{Spt}(S) \quad , \quad \mathbf{Sm}_s \longrightarrow \mathbf{Spt}(S).$$

We omit the reference to the infinite suspension when it is clear by the context.

As in the case of $\mathbf{H}(S)$ the stable homotopy category is obtained after a two step localization of $\mathbf{Spt}(S)$. Firs we invert the following morphisms that should naturally be isomorphism.

Definition 3.2.2 Let $\varphi \colon E \to F$ be a morphism of spectra:

- We say that φ is a **level $\mathbb{A}^1$-weak equivalence** of spectra if for every n the morphism $\varphi_n \colon E_n \to F_n$ is an $\mathbb{A}^1$-weak equivalence of sheaves.
- We say that φ is a **level $\mathbb{A}^1$-injective fibration** if it has the right lifting property with respect to trivial $\mathbb{A}^1$-injective cofibrations.

Find a proof of the following result in [Jar00, 2.1].

Theorem 3.2.3 *The classes of level $\mathbb{A}^1$-weak equivalences, monomorphisms and level $\mathbb{A}^1$-injective fibrations define a model structure on* $\mathbf{Spt}(S)$ *which we call the* level injective model structure*.*

□

Definition 3.2.4 Denote $\mathbf{W}_{\mathrm{lv}}$ the class of level $\mathbb{A}^1$-weak equivalences of spectra. We define the **level stable homotopy category** of S-schemes to be

$$\mathbf{SH}_{\mathrm{lv}}(S) = \mathbf{Spt}(S)[\mathbf{W}_{\mathrm{lv}}^{-1}],$$

the homotopy category of $\mathbf{Spt}(S)$ with respect to the level injective model structure.

Remark 3.2.5 As in the case of $\mathbf{H}(S)$ there is another possible model structure. We review it for the sake of completeness. Let $\varphi\colon E \to F$ be a morphism of spectra:

- We say that φ is an *level* $\mathbb{A}^1$*-projective fibration* if for every n the map $\varphi_n\colon E_n \to F_n$ is an $\mathbb{A}^1$-fibration.
- We say that φ is an *level* $\mathbb{A}^1$*-projective cofibration* if it has the left lifting property with respect to trivial level $\mathbb{A}^1$-projective fibrations.

The classes of level $\mathbb{A}^1$-weak equivalences, level $\mathbb{A}^1$-projective fibrations and level $\mathbb{A}^1$-projective cofibrations define a model structure on $\mathbf{Spt}(S)$ which we call the *level projective model structure* (*cf.* [Jar00, 2.1]).

We denote the $\mathbb{P}^1$-suspension functor $\Sigma\colon \mathbf{Spt}(S) \to \mathbf{Spt}(S)$, where $(\Sigma E)_n = \mathbb{P}^1 \wedge E_n$. This functor has right adjoint $\Omega\colon \mathbf{Spt}(S) \to \mathbf{Spt}(S)$, called the $\mathbb{P}^1$-loop functor, where $(\Omega E)_n = \underline{\mathrm{Hom}}(\mathbb{P}^1, E_n)$. Recall from Proposition 3.1.15 that the smash product preserves $\mathbb{A}^1$-equivalences of sheaves. Therefore it is direct that it preserves level $\mathbb{A}^1$-equivalence of spectra. The following result is a direct consequence

Proposition 3.2.6 *The $\mathbb{P}^1$-suspension functor and $\mathbb{P}^1$-loop functor induce a pair of adjoint functors*

$$\Sigma\colon \mathbf{SH}_{\mathrm{lv}}(S) \leftrightarrows \mathbf{SH}_{\mathrm{lv}}(S) : \Omega.$$

□

The following example shows that $\mathbb{P}^1$-suspension is not an equivalence of category in $\mathbf{SH}_{\mathrm{lv}}(S)$.

Example 3.2.7 Let (F, x) be a pointed simplicial sheaf of sets. There are different definitions of sheaves of homotopy groups. We call the sheaf of *naive homotopy group* $\pi_0^{\mathrm{nai}}(F, x)$ to be the sheaf associated to the presheaf defined as $U \mapsto \pi_0(F(U), x)$, where U is a smooth S-scheme. We call the $\mathbb{A}^1$*-homotopy group* $\pi_0^{\mathbb{A}^1}(F, x)$ the sheaf associated to the presheaf $U \mapsto \pi_0(L_{\mathbb{A}^1}F(U), x)$, where $L_{\mathbb{A}^1}$ denoted the $\mathbb{A}^1$-local replacement of F (*cf.* Corollary 3.1.13). We say that a simplicial sheaf F is naive connected (resp. $\mathbb{A}^1$-connected) if it has the $\pi_0^{\mathrm{nai}}(F, x)$ (resp. $\pi_0^{\mathbb{A}^1}(F, x)$) of a point.

Note that for any simplicial sheaf F we have that $\mathbb{P}^1 \wedge F$ has a trivial naive homotopy group. By a theorem of Morel it also has a trivial $\mathbb{A}^1$-homotopy group (*cf.* [Mor12, 1.18]). Consider the spectrum defined by two points $F = S \sqcup S$. We conclude that $\Sigma\Omega F$ is not level $\mathbb{A}^1$-equivalent to F.

Definition 3.2.8 We say that an spectrum E is an **Ω-spectrum** if for every integer n the map $E_n \to \underline{\mathrm{Hom}}(\mathbb{P}^1, E_{n+1})$, adjoint to $\sigma_n : \mathbb{P}^1 \wedge E_n \to E_{n+1}$, is an isomorphism in $\mathbf{H}_\bullet(S)$. We denote by $\mathbf{SH}_\Omega(S)$ the full subcategory of $\mathbf{SH}_{\mathrm{lv}}(S)$ made of Ω-spectra.

It is clear that the suspension and loop functors Σ and Ω induce equivalence of categories on $\mathbf{SH}_\Omega(S)$ (see Proposition 3.2.14 below). We could define the stable homotopy category to be the category of Ω-spectra. We review a result from Riou that shows that we can obtain the stable homotopy category as Bousfield localization.

Definition 3.2.9 Let $\varphi\colon E \to F$ be a morphism of spectra:

- We say that φ is an **$\mathbb{A}^1$-stable equivalence** if for every Ω-spectrum G the induced map

$$\mathrm{Hom}_{\mathbf{SH}_{\mathrm{lv}}(S)}(F, G) \to \mathrm{Hom}_{\mathbf{SH}_{\mathrm{lv}}(S)}(E, G)$$

  is a bijection.

- We say that φ is an **$\mathbb{A}^1$-stable fibration** if it has the right lifting property with respect monomorphisms which are $\mathbb{A}^1$-stable equivalences.

Find a proof of the following result in [Rio10, 1.22].

Proposition 3.2.10 *Let X be a smooth S-scheme and denote $i_n\Sigma^\infty F$ the spectrum defined as $(i_n\Sigma^\infty X)_k = *$ if $k < n$ and $(i_n\Sigma^\infty X)_{n+1} = (\mathbb{P}^1)^i \wedge X$. Denote* B *the class of morphisms given by*

$$i_n\Sigma^\infty(\mathbb{P}^1 \wedge X_+) \to i_n\Sigma^\infty X_+$$

Let E be an $\mathbb{A}^1$-fibrant spectrum. Then E is an Ω-spectrum if and only if E is B*-local.*

□

The next result follows from Bousfield localization (*cf.* Theorem 2.3.53).

Theorem 3.2.11 *The classes of $\mathbb{A}^1$-stable equivalences, $\mathbb{A}^1$-stable fibrations and monomorphisms define a model category structure on* $\mathbf{Spt}(S)$ *which we call* $\mathbb{A}^1$-stable model structure.

□

Definition 3.2.12 Denote $\mathbf{W}_{\mathbb{A}^1 s}$ the class of $\mathbb{A}^1$-stable equivalence. We define the **stable homotopy category** of S-schemes $\mathbf{SH}(S)$ to be

$$\mathbf{SH}(S) = \mathbf{Spt}(S)[\mathbf{W}_{s\mathbb{A}^1}^{-1}],$$

the homotopy category of $\mathbf{Spt}(S)$ with respect to the $\mathbb{A}^1$-model structure.

Let E and F be spectra. Let us remark that we define the function complex as usual: the simplicial set which has in degree n the set $\mathrm{Hom}_{\mathbf{Spt}(S)}(E \times \Delta^n, F)$ with the natural face and degeneracy maps. we abuse notation and still denote it $S(E, F)$.

Recall that in paragraph 2.3.38 we defined the suspension functor for a pointed model category. Note that in the stable homotopy category there are two natural suspensions and loops: the one coming from a simplicial model category defined by S^1 and the natural ones from $\mathbb{P}^1$-spectra given by $\mathbb{P}^1$. We review the simplicial suspension to fix the notation.

Definition 3.2.13 We define the **suspension** functor to be

$$\begin{array}{cccc} [1]\colon & \mathbf{Spt}(S) & \longrightarrow & \mathbf{Spt}(S) \\ & E & \mapsto & (E[1])_n = S^1 \wedge E_n. \end{array}$$

The suspension admits a right adjoint $[-1]\colon \mathbf{Spt}(S) \to \mathbf{Spt}(S)$, $(E[-1])_n = \underline{\mathrm{Hom}}(S^1, E_n)$ which we call the **loop** functor.

We already reviewed for general pointed model categories the following result (*cf.* Proposition 2.3.39).

Proposition 3.2.14 *The loop and suspension functor form Quillen adjunction for the level injective and $\mathbb{A}^1$-stable model structures and induce a pair of adjoint functors*

$$[1]\colon \mathbf{SH}(S) \leftrightarrows \mathbf{SH}(S) : [-1].$$

$\square$

Let us recall the $\mathbb{P}^1$-suspension and $\mathbb{P}^1$-loop functor in $\mathbf{SH}$.

Proposition 3.2.15 *The $\mathbb{P}^1$-suspension functor Σ preserves $\mathbb{A}^1$-stable weak equivalences. We denote the induced functors*

$$\Sigma\colon \mathbf{SH}(S) \leftrightarrows \mathbf{SH}(S) : \Omega.$$

Proof: Recall from Definition 3.2.9 that Σ preserves a $\mathbb{A}^1$-stable equivalence $f\colon E \to F$ if for every Ω-spectrum G the map

$$\mathrm{Hom}_{\mathbf{SH}_{\mathrm{lv}}(S)}(\Sigma F, G) \to \mathrm{Hom}_{\mathbf{SH}_{\mathrm{lv}}(S)}(\Sigma E, G)$$

is a bijection. By adjunction of the $\mathbb{P}^1$-loop and suspension functors on $\mathbf{SH}_{\mathrm{lv}}$ it is enough to prove that the map $\mathrm{Hom}_{\mathbf{SH}_{\mathrm{lv}}(S)}(F, \Omega G) \to \underline{\mathrm{Hom}}_{\mathbf{SH}_{\mathrm{lv}}(S)}(E, \Omega G)$ is a bijection. Note that $(\Omega G)_n = \underline{\mathrm{Hom}}(\mathbb{P}^1, G_n) = G_{n-1}$ since G is an Ω-spectrum. Therefore ΩG is once again an Ω-spectrum and the result follows.

□

Voevodsky remarked that inverting the smash product by $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ implies inverting the smash product by S^1 and $\mathbb{G}_m$. The proof I know requires the notion of bispectra. Find complete proofs of this fact in [Rio10, §3]. In particular, find the following result in [Rio10, 3.10]:

Theorem 3.2.16 *The functors* $[1]$*,* $[-1]$*,* Σ *and* Ω *are equivalences of categories.*

□

From here Theorem 2.3.45 allows to conclude the following result:

Corollary 3.2.17 *The stable homotopy category* $\mathbf{SH}(S)$ *is a triangulated category with distinguished triangles sequences*

$$U \longrightarrow V \longrightarrow W \longrightarrow U[1]$$

isomorphic to a sequence

$$E \xrightarrow{f} F \longrightarrow \mathrm{cone}(f) \longrightarrow E[1].$$

□

3.2.1 Symmetric spectra

As in the topological case, proving that there is an adequate smash product in the stable homotopy category requires an elaborate proof. We review just the main definitions and results of [Jar00, §4].

Definition 3.2.18 Let Σ_n denote the n-th symmetric group. A **symmetric spectrum** is a spectrum E together with symmetric group actions $\Sigma_n \times X^n \to X^n$ for all n such that the structural maps

$$(\mathbb{P}^1)^{\wedge p} \wedge E_n \to E_{p+n}$$

are $\Sigma_p \times \Sigma_n$-equivariant. A **morphism** of symmetric spectra $\varphi\colon E \to F$ is a morphism of spectra such that for all n the morphism $\varphi_n\colon E_n \to F_n$ is Σ_n-equivariant. We denote $\mathbf{Spt}_\Sigma(S)$ the category of symmetric spectra with morphism of symmetric spectra and $F\colon \mathbf{Spt}_\Sigma(S) \to \mathbf{Spt}(S)$ the forgetful functor.

Example 3.2.19 The S^0 defines a symmetric spectrum which has on level zero S^0 and $(\mathbb{P}^1)^n$ on level n. We still denote it S^0.

As in the non symmetric case, the stable homotopy category of symmetric spectra is constructed in two steps.

Definition 3.2.20 Let $\varphi\colon E \to F$ be a morphism of symmetric spectra:

- We say that φ is a **symmetric level equivalence** if for every n the map $f_n\colon E_n \to F_n$ is an $\mathbb{A}^1$-weak equivalence.
- We say that φ is a **symmetric level (injective) cofibration** if for every n the map $f_n\colon E_n \to F_n$ is a monomorphism.
- We say that φ is a **symmetric level (injective) fibration** if it has the right lifting property with respect to level cofibrations which are level equivalences.

Find a proof of the following result in [Jar00, 4.2].

Theorem 3.2.21 *The classes of symmetric level equivalences, symmetric level cofibrations and symmetric level fibrations define a model category structure on* $\mathbf{Spt}_\Sigma(S)$ *which we call the* symmetric level model structure.

□

Definition 3.2.22 Let $\varphi\colon E \to F$ be a morphism of symmetric spectra:

- We say that φ is a **symmetric (projective) $\mathbb{A}^1$-stable fibration** if the underlying map of spectra $F(\varphi)\colon F(E) \to F(F)$ is an $\mathbb{A}^1$-stable fibration of spectra.
- We say that φ is a **symmetric $\mathbb{A}^1$-stable equivalence** if for every level and stable fibrant object W the induced map

$$S(F(Y), F(X)) \to S(F(X), F(W))$$

  is a weak equivalence of simplicial sets. (Note that $S(\ \_\ ,\ \_\ )$ denotes the simplicial mapping space of Definition 2.3.23).
- We say that φ is a **symmetric (projective) $\mathbb{A}^1$-stable cofibration** if φ has the left lifting property with respect to stable fibrations which are stable equivalences.

Find a proof of the following result in [Jar00, 4.15].

Theorem 3.2.23 *The classes of stable equivalences, stable cofibrations and stable fibrations define a model category structure on* $\mathbf{Spt}_\Sigma(S)$ *which we call the* symmetric $\mathbb{A}^1$-stable model structure.

$\square$

Definition 3.2.24 Denote $\mathbf{W}_\Sigma$ the class of symmetric equivalences. We define the **stable homotopy category** of symmetric spectra $\mathbf{SH}_\Sigma(S)$ to be

$$\mathbf{SH}_\Sigma(S) = \mathbf{Spt}_\Sigma(S)[\mathbf{W}_\Sigma^{-1}],$$

the homotopy category of $\mathbf{Spt}_\Sigma(S)$ with respect to the symmetric $\mathbb{A}^1$-stable model structure.

Definition 3.2.25 We define **symmetric sequence** X to be a sequence $(X_n)_{n\in\mathbb{N}}$ of pointed simplicial sheaves which have a group action $\Sigma_n \times X_n \to X_n$. We define a **morphism** of symmetric sequences $f\colon X \to Y$ to be a sequence of maps $f_n\colon X_n \to Y_n$ which are Σ_n-equivariant. We define the product of two symmetric sequences $X \otimes Y$ to be the symmetric sequence defined as

$$(X \otimes Y)_n = \bigvee_{p+q=n} \Sigma_n \otimes_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \simeq \bigvee_{\substack{p+q=n \\ \Sigma_n/(\Sigma_p \times \Sigma_q)}} X_p \wedge Y_q.$$

Remark 3.2.26 The product of symmetric sequences is commutative. There is a canonical isomorphism $X \otimes Y \xrightarrow{\sim} Y \otimes X$ defined by the maps

$$X_p \wedge Y_q \longrightarrow Y_q \wedge X_p \xrightarrow{in_e} (Y \otimes X)_n$$

where $in_e\colon Y_q \wedge X_p \to \bigvee_{\Sigma_n/(\Sigma_q \times \Sigma_p)} Y_q \wedge X_p$ is the inclusion by defined by the class of the neutral element.

Remark 3.2.27 Note that every symmetric spectrum E defines a symmetric sequence, still denoted E. Moreover, spectra are identified with symmetric sequences which are S^0-*modules*, in the sense that there is morphism of symmetric sequences

$$S^0 \otimes E \to E.$$

Definition 3.2.28 Let E and F be symmetric spectra and denote the maps as S^0-module $\mu\colon S^0 \otimes E \to E$ and $\nu\colon S^0 \otimes F \to F$ respectively. We define the **smash product** of spectra $E \wedge F$ as the symmetric sequence coequalizer

$$S^0 \otimes E \otimes F \rightrightarrows E \otimes F \to E \wedge F$$

of the maps $\mu \otimes 1_F \colon S^0 \otimes E \otimes F \to F$ and the map

$$1_E \otimes \nu \colon S^0 \otimes E \otimes F = E \otimes S^0 \otimes F \to E \otimes F.$$

Note that $E \wedge F$ is a S^0-module with the map induced by $\mu \otimes 1_F$ and therefore it is a symmetric spectrum.

Find a proof of the following theorem in [Jar00, 4.31].

Theorem 3.2.29 *The forgetful functor* $F \colon \mathbf{Spt}_\Sigma(S) \to \mathbf{Spt}(S)$ *is a left Quillen functor and induces adjoint equivalence of categories*

$$\mathbf{SH}_\Sigma(S) \leftrightarrows \mathbf{SH}(S).$$

Moreover, $\mathbf{SH}_\Sigma(S)$ *is a symmetric monoidal category in the sense of [Mac71, p.251] and therefore so is* $\mathbf{SH}(S)$.

□

3.2.2 Functoriality

Recall from section § 2.2.2 that any morphism of schemes $f \colon T \to S$ induce a pair of adjoint functors

$$f^* \colon \Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(S) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(T) : f_*.$$

Recall from Proposition 2.2.23 that we know that for any F in $\Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(S)$ we have that $f^*(\mathbb{P}^1 \wedge F) \simeq \mathbb{P}^1 \wedge f^*F$. We abuse notation and still denote $f^* \colon \mathbf{Spt}(S) \to \mathbf{Spt}(T)$ the following functor. Let E be a spectrum, then $(f^*E)_n = f^*E_n$ for $n \in \mathbb{N}$ and with $f^*\sigma_n$ as structural morphisms. Let $\varphi = \{\varphi_n \colon E_n \to F_n\}$ be a morphism of spectra, then $f^*\varphi = \{f^*\varphi_n\}$.

By adjunction we deduce from Proposition 2.2.23 an isomorphism of sheafs $\underline{\mathrm{Hom}}(\mathbb{P}^1, f_*H) \simeq f_*\underline{\mathrm{Hom}}(\mathbb{P}^1, H)$. We abuse notation and still denote the functor $f_* \colon \mathbf{Spt}(T) \to \mathbf{Spt}(S)$ which maps any spectrum F in $\mathbf{Spt}(T)$ to the spectrum defined by $(f_*F)_n = f_*F_n$ with structural morphisms the adjoint of the composite

$$f_*E_n \to \underline{\mathrm{Hom}}(\mathbb{P}^1, f_*E_{n+1}) \simeq f_*\underline{\mathrm{Hom}}(\mathbb{P}^1, E_{n+1}).$$

It is clear that the pair of functors (f^*, f_*) are adjoint. It is also direct from the definition that f_* preserves level epimorphisms and level $\mathbb{A}^1$-weak equivalences. In other words, f_* is a right Quillen functor for the level $\mathbb{A}^1$-model structure and there are total derived functors

$$f^* \colon \mathbf{SH}_{\mathrm{lv}}(S) \leftrightarrows \mathbf{SH}_{\mathrm{lv}}(T) : f_*.$$

Therefore we have proved the following result.

Proposition 3.2.30 *Let $f\colon T \to S$ be a morphism of schemes. The pair of functors*
$$f^*\colon \mathbf{SH}_{\mathrm{lv}}(S) \leftrightarrows \mathbf{SH}_{\mathrm{lv}}(T) : f_*$$
are adjoint. If $g\colon U \to S$ is a morphism of schemes then
$$(f \circ g)_* \simeq f_* \circ g_*,$$
$$(f \circ g)^* \simeq g^* \circ f^*.$$
□

Let $p\colon X \to S$ be a smooth morphism of schemes. Recall from section § 2.2.2 that p also induces a pair of adjoint functors
$$p_\sharp\colon \Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(X) \to \Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(S) : p^*.$$
It follows from the projection formula that $p_\sharp(H \times \mathbb{P}^1_X) \simeq p_\sharp(X) \times \mathbb{P}^1_S$ (*cf.* Proposition 2.2.24). We abuse notation and consider $p_\sharp\colon \mathbf{Spt}(X) \to \mathbf{Spt}(S)$ the functor which maps any spectrum F in $\mathbf{Spt}(X)$ to the spectrum defined by $(p_\sharp F)_n = p_\sharp F_n$ with structural morphisms $\{p_\sharp \sigma_n\}$. By construction $p_\sharp$ is left adjoint to p^*. It is clear from the definitions that the restriction p^* is a right Quillen functor for the level $\mathbb{A}^1$-projective model structure. We abuse notation once again and denote
$$p_\sharp\colon \mathbf{SH}_{\mathrm{lv}}(X) \to \mathbf{SH}_{\mathrm{lv}}(S).$$
the total left derived functor. We have proven the following result.

Proposition 3.2.31 *Let $p\colon X \to S$ be a smooth morphism. Then the pair of functors*
$$p_\sharp\colon \mathbf{SH}_{\mathrm{lv}}(X) \leftrightarrows \mathbf{SH}_{\mathrm{lv}}(S) : p^*$$
are adjoint. Let $q\colon Y \to X$ be a smooth morphism, we have a canonical natural transformation
$$(p \circ q)_\sharp \simeq p_\sharp \circ q_\sharp.$$
□

For the case of $\mathbf{SH}$ recall from Definition 3.2.9 that E is an Ω-spectrum if the adjoints of the structural maps $E_n \to \underline{\mathrm{Hom}}(\mathbb{P}^1, E_{n+1})$ are bijections. The fact that $\underline{\mathrm{Hom}}(\mathbb{P}^1, f_*F) \simeq f_*\underline{\mathrm{Hom}}(\mathbb{P}^1, F)$ readily implies that f_* preserves Ω-spectra. Therefore f^* preserves $\mathbb{A}^1$-stable weak equivalences. Indeed, let $\varphi\colon E \to F$ be a $\mathbb{A}^1$-stable weak equivalence and G an Ω-spectrum, then
$$\mathrm{Hom}_{\mathbf{SH}(T)_{\mathrm{lv}}}(f^*F, G) \simeq \mathrm{Hom}_{\mathbf{SH}(S)_{\mathrm{lv}}}(F, f_*G) \simeq \mathrm{Hom}_{\mathbf{SH}(S)_{\mathrm{lv}}}(E, f_*G)$$

$$\simeq \mathrm{Hom}_{\mathbf{SH}(T)_{\mathrm{lv}}}(f^*E, G).$$

We conclude that (f^*, f_*) induce a pair of adjoint functors on $\mathbf{SH}$. We have proven the following result.

Proposition 3.2.32 *Let $f\colon T \to S$ be a morphism of schemes. The pair of functors*

$$f^*\colon \mathbf{SH}(S) \leftrightarrows \mathbf{SH}(T) : f_*$$

are adjoint. If $g\colon U \to S$ is a morphism of schemes then

$$(f \circ g)_* \simeq f_* \circ g_*,$$

$$(f \circ g)^* \simeq g^* \circ f^*.$$

Let $p\colon X \to S$ be a smooth morphism. Then pair of functors

$$p_\sharp\colon \mathbf{SH}(U) \leftrightarrows \mathbf{SH}(S) : p^*$$

are adjoint. Let $q\colon Y \to X$ be a smooth morphism, we have a canonical natural transformation

$$(p \circ q)_\sharp \simeq p_\sharp \circ q_\sharp.$$

□

3.2.3 Localization, Mayer-Vietoris, blow-up and homotopy purity

We are ready to deduce cohomological type properties: the local, Mayer-Vietoris and blow-up distinguished triangles. I learnt these results from [CD12, §2 and 3] where a much more complete and general treatment on the subject can be found. We only cover few of their results. Find another complete treatment in [Ayo07].

The following result is a consequence of Proposition 2.3.47 and Corollary 3.1.24.

Theorem 3.2.33 (Localization) *Let $j\colon U \to S$ be an open embedding with complement the closed embedding $i\colon Z \to S$. Then for any spectrum E we have a distinguished triangle*

$$j_\sharp j^* E \to E \to i_* i^* E \to j_\sharp j^* E[1].$$

□

The next result follows from Theorem 2.2.9.

Theorem 3.2.34 (Mayer-Vietoris) *Let X be a smooth S-scheme, $j\colon U \to X$ be an open immersion and $p\colon V \to X$ be an étale morphism inducing and isomorphism $p^{-1}(X-U) \to X-U$. Then we have a distinguished triangle*

$$p^{-1}(U)_+ \xrightarrow{p \oplus j} U_+ \oplus V_+ \xrightarrow{j \oplus (-p)} X_+ \longrightarrow p^{-1}(U)_+[1].$$

$\square$

We deduce from Proposition 2.3.47 and Corollary 3.1.27 the following theorem.

Theorem 3.2.35 (Blow-up) *Let $i\colon Z \to X$ be a closed embedding of smooth schemes over S, $\pi\colon B_Z X \to X$ be the blowing-up of Z in X and $U = X - Z = B_Z X - \pi^{-1}(Z)$. Then there is a distinguished triangle*

$$\pi^{-1}(Z)_+ \xrightarrow{\pi \oplus i'} Z_+ \oplus B_Z X_+ \xrightarrow{i \oplus (-\pi)} X_+ \longrightarrow \pi^{-1}(Z)_+[1].$$

$\square$

We restate Theorem 3.1.29 in **SH** as follows.

Theorem 3.2.36 (Homotopy purity) *Let $i\colon Z \to X$ be a closed embedding of smooth schemes and denote $N_{Z/X} \to Z$ the normal bundle. Then there is a canonical isomorphism in $\mathbf{SH}(X)$*

$$\mathfrak{p}_Z^X\colon X/X - Z \xrightarrow{\sim} \mathrm{Th}(N_{Z/X}).$$

$\square$

3.2.4 Exceptional functors

We recall the properties of the exceptional functors on **SH**. With the exception of the purity property for a morphism this result completes Grothendieck's six operations formalism. The original reference is [Ayo07]. We state the slightly more general result of [CD09, 2.2.14].

Theorem 3.2.37 *Let $f\colon Y \to X$ be a separated morphism of finite type. There exist a pair of adjoint functors*

$$f_!\colon \mathbf{SH}(Y) \to \mathbf{SH}(X) : f^!$$

such that:

1. *Functoriality:* $\mathrm{Id}_! = 1$, $\mathrm{Id}^! = 1$ *and for* $g\colon Z \to Y$ *another separated morphism of finite type* $(f \circ g)_! = f_! \circ g_!$ *and* $(f \circ g)^! = f^! \circ g^!$.
2. *There is a natural transformation* $\alpha_f\colon f_! \to f_*$ *which is an isomorphism if* f *is proper.*
3. *For any open immersion* j *we have* $j_! = j_\sharp$ *and* $j^! = j^*$.
4. *For any cartesian square*
$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$
where f *is separated of finite type we have*
$$g^* f_! \simeq f'_! g'^* \text{ and } g'_* f'^! \simeq f^! g_*.$$
5. *For any* E *in* $\mathbf{SH}(Y)$ *and* F, F' *in* $\mathbf{SH}(X)$ *we have*
$$f_! E \wedge F \simeq f_!(E \wedge f^* F),\ \underline{\mathrm{Hom}}(f_! E, F) \simeq f_* \underline{\mathrm{Hom}}(E, f^! F) \text{ and}$$
$$f^! \underline{\mathrm{Hom}}(F, F') \simeq \underline{\mathrm{Hom}}(f^* F, f^! F').$$

□

Chapter 4

Riemann-Roch theorems and Gysin morphisms

4.1 Cohomology and its operations

Definition 4.1.1 A (commutative) **ring spectrum** is an associative commutative unitary monoid in $\mathbf{SH}(X)$. In other words, a ring spectrum is a triple $(E,\, \mu\colon E \wedge E \to E,\, \eta\colon \mathbb{1}_X \to E)$ consisting of a spectrum, the product morphism and the unit morphism satisfying that the diagrams

$$\begin{tikzcd} E \arrow[r, "1\wedge\eta"] \arrow[dr, equal] & E \wedge E \arrow[d, "\mu"] \\ & E \end{tikzcd} \quad , \quad \begin{tikzcd} E \wedge E \wedge E \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \wedge E \arrow[r, "\mu"] \arrow[d, "\gamma"'] & E \\ E \wedge E, \arrow[ur, "\mu"'] & \end{tikzcd} \quad ,$$

where γ is the permutation isomorphism, commute.

Let $(E,\, \mu,\, \eta)$ and $(F,\, \bar{\mu},\, \eta)$ be two ring spectra, a **morphism** of ring spectra $\varphi\colon E \to F$ is a morphism of spectra such that $\varphi \circ \eta = \bar{\eta}$ and such that the diagram

$$\begin{tikzcd} E \wedge E \arrow[r, "\varphi\wedge\varphi"] \arrow[d, "\mu"'] & F \wedge F \arrow[d, "\bar{\mu}"] \\ E \arrow[r, "\varphi"] & F \end{tikzcd}$$

commutes.

An **absolute spectrum** $\mathbb{E}$ is a family stable by pullback of spectra $\mathbb{E}_X \in \mathbf{SH}(X)$ for every scheme X. That is to say, for every morphism $f\colon Y \to X$ we have fixed an isomorphism $\epsilon_f\colon f^*\mathbb{E}_X \to \mathbb{E}_Y$ satisfying the usual cocycle condition[1]. A **morphism** of absolute spectra $\varphi\colon \mathbb{E} \to \mathbb{F}$ is a collection of morphism of spectra $\varphi_X\colon \mathbb{E}_X \to \mathbb{F}_X$ for every scheme X such that for every morphism $f\colon Y \to X$ the diagram

$$\begin{tikzcd} f^*\mathbb{E}_X \arrow[r, "f^*\varphi_X"] \arrow[d, "\epsilon_f"'] & f^*\mathbb{F}_X \arrow[d, "\bar{\epsilon}_f"] \\ \mathbb{E}_Y \arrow[r, "\varphi_Y"] & \mathbb{F}_Y \end{tikzcd}$$

commutes. An absolute ring spectrum is an absolute spectrum made of ring spectra and morphisms of absolute ring spectra are morphisms of absolute spectra made of morphisms of ring spectra.

Notation 4.1.2 Every absolute spectrum $\mathbb{E}$ is naturally isomorphic to the absolute spectrum obtained by pullback of the spectrum $\mathbb{E}_S \in \mathbf{SH}(S)$, where S is the base scheme. Instead of considering schemes over a fixed base S, one may work over a general category $\mathbf{S}$ without a final object. All definitions and proofs of this thesis may be carried into this context using the notion of $\mathbf{S}$-absolute spectra (*cf.* [Dég14]). However, since we will not make use of this generality we have chosen otherwise. In addition, we will abuse notation and call $\mathbb{E}_S$ also the absolute spectrum.

Let X be an S-scheme. We call a family stable by pullback of spectra $\mathbb{E}_Y$ for Y an X-scheme an absolute spectra *over* X.

Notation 4.1.3 The invertible element $\mathbb{1}_X(1) := \operatorname{coker}(\Sigma^\infty X \to \Sigma^\infty \mathbb{P}^1)[-2]$ is called the *Tate object*. For any spectrum E we denote the *Tate twist* by $E(1) := E \wedge \mathbb{1}_X(1)$ and denote $E(q)[p]$ for twisting and shifting $q, p \in \mathbb{Z}$ times respectively.

Absolute ring spectra is the adequate framework to describe cohomology.

Definition 4.1.4 Let X be a scheme and $\mathbb{E}$ be an absolute spectrum over X. We define the **$\mathbb{E}$-cohomology** of X to be

$$\mathbb{E}^{p,q}(X) = \operatorname{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathbb{E}_X(q)[p]) \quad \text{for } p, q \in \mathbb{Z}$$

[1]The cocycle condition is the following: Let $f\colon Y \to X$ and $g\colon Z \to Y$ be two morphism of schemes, then the diagram

$$\begin{tikzcd} g^*f^*\mathbb{E}_X \arrow[r, "g^*\epsilon_f"] \arrow[d, "\wr"'] & g^*\mathbb{E}_Y \arrow[d, "\epsilon_g"] \\ (g\circ f)^*\mathbb{E}_X \arrow[r, "\epsilon_{g\circ f}"] & \mathbb{E}_Z \end{tikzcd}$$

commutes.

and $\mathbb{E}(X) = \bigoplus_{p,q} \mathbb{E}^{p,q}(X)$. Let $i\colon Z \to X$ be a closed immersion, we define the **$\mathbb{E}$-cohomology with support on** Z to be

$$\mathbb{E}_Z^{p,q}(X) = \mathrm{Hom}_{\mathbf{SH}(X)}(i_*\mathbb{1}_Z, \mathbb{E}_X(q)[p]) \quad \text{for } p,q \in \mathbb{Z}.$$

For any $f\colon T \to X$ and a closed immersion $i\colon Z \to X$ we define the **inverse image of** f which maps any $a\colon i_*\mathbb{1}_Z \to \mathbb{E}_X(q)[p]$ in $\mathbb{E}_Z^{p,q}(X)$ to the composition

$$f^*(a)\colon f^*i_*\mathbb{1}_Z \simeq i'_*\mathbb{1}_{Z'} \to f^*(\mathbb{E}_X(q)[p]) \simeq \mathbb{E}_T(q)[p] \ \in \mathbb{E}_{Z'}^{p,q}(T)$$

where $i'\colon Z' = Z \times_T X \to T$. We denote it $f^*\colon \mathbb{E}_Z(X) \to \mathbb{E}_{Z'}(T)$.

Let us recall a generalization of the morphism of forgetting support.

Definition 4.1.5 Let $\mathbb{E}$ be an absolute spectrum. Consider $Z \xrightarrow{j} Y \xrightarrow{i} X$ closed immersions, we define a morphism

$$j_\flat\colon \mathbb{E}_Z(X) \to \mathbb{E}_Y(X)$$

as follows. The adjunction morphism $\mathrm{ad}\colon \mathbb{1}_Y \to j_*j^*\mathbb{1}_Y$ gives a morphism $i_*(\mathrm{ad})\colon i_*(\mathbb{1}_Y) \to i_*j_*j^*\mathbb{1}_Y = (ij)_*\mathbb{1}_Z$. Let $a\colon (ij)_*\mathbb{1}_Z \to \mathbb{E}_X$ be in $\mathbb{E}_Z(X)$, we define

$$j_\flat(a) \coloneqq i_*(\mathbb{1}_Y) \xrightarrow{i_*(\mathrm{ad})} (ij)_*\mathbb{1}_Z \xrightarrow{a} \mathbb{E}_X \ \in \mathbb{E}_Y(X).$$

In order for the $\mathbb{E}$-cohomology groups to have all the usual product properties of a cohomology the spectrum $\mathbb{E}$ has to be a ring spectrum.

Definition 4.1.6 Let $\mathbb{E}$ be an absolute ring spectrum and $Z \xrightarrow{j} Y \xrightarrow{i} X$ be closed immersions, we call **refined product** to the morphism

$$\mathbb{E}_Z^{p,q}(Y) \otimes \mathbb{E}_Y^{r,s}(X) \to \mathbb{E}_Z^{p+r,q+s}(X)$$

constructed as follows. For any element $a\colon j_*(\mathbb{1}_Z) \to \mathbb{E}_Y(q)[p]$ belonging to $\mathrm{Hom}_{\mathbf{SH}(Y)}(j_*(\mathbb{1}_Z), \mathbb{E}_Y(q)[p]) = \mathbb{E}_Z^{p,q}(Y)$ we have a morphism

$$\gamma\colon i_*j_*(\mathbb{1}_Z) \to i_*i^*\mathbb{E}_X(q)[p] \simeq \mathbb{E}_X \wedge i_*\mathbb{1}_Y(q)[p].$$

We define the product of a with $b\colon i_*\mathbb{1}_Y \to \mathbb{E}_X(s)[r]$ in $\mathbb{E}_Y^{r,s}(X)$ to be

$$a \cdot b\colon i_*j_*(\mathbb{1}_Z) \xrightarrow{\gamma} \mathbb{E}_X \wedge i_*\mathbb{1}_Y(q)[p] \xrightarrow{\mathrm{Id}\wedge\rho} \mathbb{E}_X \wedge \mathbb{E}_X(q+s)[p+r] \xrightarrow{\mu} \mathbb{E}_X(q+s)[p+r].$$

The properties one may expect from the refined product and the morphism of forgetting support are summarized in the following result, which comes from [Dég14, 1.2.9].

Proposition 4.1.7 *Let $\mathbb{E}$ be an absolute spectrum, the following properties hold:*

1. *If $j\colon Z \to Y$ and $i\colon Y \to X$ are closed immersion then $i_\flat j_\flat = (ij)_\flat$.*

2. *If i is a closed nil-immersion then $i_\flat$ is an isomorphism.*

3. *Consider the following cartesian squares:*

$$\begin{tikzcd} Z' \arrow[r, "j'"] \arrow[d] & Y' \arrow[r] \arrow[d, "g"] & X' \arrow[d, "f"] \\ Z \arrow[r, "j"] & Y \arrow[r] & X \end{tikzcd}$$

   where the horizontal arrows are closed immersions. Then, for $a \in \mathbb{E}_Z(X)$ we have $f^* j_\flat(a) = j'_\flat f^*(a)$.

Let now $\mathbb{E}$ be an absolute ring spectrum:

4. *With the preceding notations, for any pair $(a, b) \in \mathbb{E}_Z(Y) \times \mathbb{E}_Y(X)$ we have $f^*(a \cdot b) = g^*(a) \cdot f^*(b)$ and $j_\flat(a \cdot b) = j_\flat(a) \cdot b$.*

5. *Consider closed immersions $T \to Z \to Y \to X$. Then for any triple $(a,\, b,\, c) \in \mathbb{E}_T(Z) \times \mathbb{E}_Z(Y) \times \mathbb{E}_Y(X)$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.*

6. *Consider the following commutative diagram*

$$\begin{tikzcd} Z' \arrow[r, "j'"] \arrow[d, "h"'] & Y' \arrow[d, "g"] & \\ Z \arrow[r, "j"] & Y \arrow[r] & X \end{tikzcd}$$

   of closed immersions and such the square is cartesian. Then for any $(a,\, b) \in \mathbb{E}_Z(Y) \times \mathbb{E}_{Y'}(X)$ the relation $h_\flat(g^*(a) \cdot b) = a \cdot g_\flat(b)$ holds.

Proof: Point 1 is direct by definition. Since i_* is an equivalence of categories for a nil-immersion point 2 follows. Point 3 follows from the base change for proper morphisms since $f^* j_* = j'_* f^*$. Point 4 and 5 follow from the definition of ring spectra and the fact that the pullback functor is monoidal. Point 6 is direct.

□

Example 4.1.8
- We recall some basic examples of absolute spectra. The simplicial set of the n-sphere defines constant simplicial presheaves on $\mathbf{Sm}/X$ for every X. Their infinite suspension define the spectra S^n_X for every X, which form an absolute spectrum. Recall that for every morphism $f\colon X \to Y$ the pullback functor f^* is monoidal, therefore the unit for the smash product $\mathbb{1}_X \in \mathbf{SH}(X)$ form an absolute spectrum $\mathbb{1}$. From here one deduces that both the projective spaces $\mathbb{P}^n_X$ and the multiplicative group $\mathbb{G}_{m,X}$ form absolute spectra.

The following are examples of absolute ring spectra:

- Let k be a perfect field and consider $S = \operatorname{Spec}(k)$. In [CD12, 2.1.4] Cisinski and Déglise defined the notion of mixed Weil theories with coefficient in a field of characteristic zero. In [CD12], every such theory is proved to define a ring spectrum on $\mathbf{SH}(S)_{\mathbb{Q}}$ stable by pullback and, therefore, an absolute ring spectrum. Recall that there is an algebraic de Rham ring spectrum for k of characteristic zero, an analytic de Rham ring spectrum for k algebraically closed of characteristic zero, and a $\mathbb{Q}_l$ geometric étale ring spectrum for k countable perfect and l a prime different from the characteristic of k.

- Consider the category of all finite dimensional noetherian schemes with $\operatorname{Spec}(\mathbb{Z})$ as final object. The K-theory absolute ring spectrum KGL is defined in [Voe98] and [Rio10]. By constructions it is periodic, meaning that there are isomorphisms
$$\mathrm{KGL} \simeq \mathrm{KGL}(i)[2i]\,,\ \forall i.$$
It represents Weibel's homotopy invariant K-theory for every scheme (*cf.* [Cis13, 2.15]), and therefore represents Quillen's algebraic K-theory for regular schemes. We denote the cohomology groups they define as $KH_i(\ \_\ )$.

  Following [Rio10, 5.3], the $\mathbb{Q}$-localization of the K-theory spectrum admits a decomposition induced by the Adams operations, i.e.,
$$\mathrm{KGL}_{\mathbb{Q}} = \bigoplus_{i\in\mathbb{Z}} \mathrm{KGL}^{(n)}_{\mathbb{Q}},$$
where $\mathrm{KGL}^{(n)}_{\mathbb{Q}}$ denotes the eigenspaces for the Adams operations. The Beilinson's absolute motivic cohomology spectrum is defined as $\mathrm{H}_{\mathrm{B}} = \mathrm{KGL}^{(0)}_{\mathbb{Q}}$ and it is also an absolute ring spectrum.

  Finally, Voevodsky's absolute algebraic cobordism ring spectrum MGL is constructed out of the Thom spaces of the canonical bundle of Grassmannians ([Voe98]) and its cohomology is called algebraic cobordism.

- In [Spi12] Spitzweck defines the absolute motivic cohomology ring spectrum H_Λ with coefficients in Λ for schemes over a Dedekind domain S. Over a field, this spectrum coincides with the motivic Eilenberg-MacLane spectrum, so it represents motivic cohomology. Rationally Spitzweck's spectrum coincides with the Beilinson's motivic cohomology spectrum H_{B}. For coherence with notations in [BMS87] we denote the motivic cohomology groups as $H^*_{\mathcal{M}}(\_\,,\Lambda(*))$ for motivic cohomology with coefficients in Λ. Let $S = \mathrm{Spec}(k)$ for k a perfect field. Recall that Voevodsky proved that for X a smooth k-scheme we have

$$H^{2n}_{\mathcal{M}}(X, \mathbb{Z}(n)) = CH^n(X)$$

  where CH^n denotes the Chow group of n-codimensional cycles.

- Let $S = \mathrm{Spec}(k)$ for k an arithmetic field (*cf.* Appendix 5.1.2). In [HS15] Holmstrom and Scholbach defined the real Deligne-Beilinson ring spectrum $\mathbb{E}_{\mathbb{R}\mathrm{D},X} \in \mathbf{SH}(X)_{\mathbb{Q}}$ for X a smooth S-scheme. The absolute ring spectrum $\mathbb{E}_{\mathbb{R}\mathrm{D},S}$ represents the Deligne-Beilinson cohomology with real coefficients. In his thesis, Brad Drew constructed the absolute ring spectrum representing absolute Hodge cohomology with rational coefficients. His construction also holds for any subfield of the real numbers [Dre13, 2.1.8]. We include in Appendix 5.1 a construction of the real Hodge absolute spectrum $\mathbb{E}_{\mathbb{R}\mathrm{AH}}$ with the same assumptions of [HS15].

- Let K be a p-adic field, k its residue field and $S = \mathrm{Spec}(k)$. The absolute rigid syntomic ring spectrum $\mathbb{E}_{\mathrm{syn}} \in \mathbf{SH}(S)_{\mathbb{Q}}$, which represents Besser's rigid syntomic cohomology, is constructed in [DM14].

Remark 4.1.9 In the reference [HS15] the Deligne-Beilinson absolute spectrum is proved to represent the real Deligne-Beilinson cohomology on smooth schemes asking explicitly for the nonsmooth case. In the Appendix 5.1 we check that it also represent the real Deligne-Beilinson cohomology for general schemes.

Another important class of objects in **SH** is the following.

Definition 4.1.10 Let E in $\mathbf{SH}(X)$ be a ring spectrum, an E**-module** is a spectrum M in $\mathbf{SH}(X)$ together with a morphism of spectra $\upsilon\colon E \wedge M \to M$ in $\mathbf{SH}(X)$ satisfying that the diagrams

$$\begin{tikzcd} E \wedge E \wedge M \arrow[r, "1_E\wedge\upsilon"] \arrow[d, "\mu\wedge 1_M"'] & E \wedge M \arrow[d, "\upsilon"] \\ E \wedge M \arrow[r, "\upsilon"] & M \end{tikzcd}\quad , \qquad \begin{tikzcd} \mathbb{1}_X \wedge M \arrow[r, "\eta\wedge 1_E"] \arrow[dr, equal] & E \wedge M \arrow[d, "\upsilon"] \\ & M \end{tikzcd}$$

commute. Let (M, v) be an E-module and (M', v') be an F-module. Let $\varphi\colon E \to F$ be a morphism of ring spectra. A **φ-morphism** of modules $\Phi\colon M \to M'$ is a morphism of spectra in $\mathbf{SH}(X)$ such that the diagram

$$\begin{tikzcd} E \wedge M \arrow[r, "\varphi\wedge\Phi"] \arrow[d, "v"'] & F \wedge M' \arrow[d, "v'"] \\ M \arrow[r, "\Phi"] & M' \end{tikzcd}$$

commutes.

Let $\mathbb{E}$ be an absolute ring spectrum, an **absolute $\mathbb{E}$-module** is an absolute spectrum $\mathbb{M}$ such that $\mathbb{M}_X$ is a $\mathbb{E}_X$-module for every X and the isomorphisms ϵ_f are isomorphisms of modules. Let $\mathbb{M}$ be an absolute $\mathbb{E}$-module, $\mathbb{M}'$ be an absolute $\mathbb{F}$-module and $\varphi\colon \mathbb{E} \to \mathbb{F}$ be a morphism of absolute ring spectra. A morphism of absolute $\mathbb{E}$-modules $\Phi\colon \mathbb{M} \to \mathbb{M}'$ is a morphism of absolute spectra such that Φ_X is a φ_X-morphism of modules which are stable by pullback. On the following, we may omit the adjective *absolute* when it is clear by the context and notation.

4.1.11 Let $\mathbb{M}$ be an absolute $\mathbb{E}$-module and $Z \xrightarrow{j} Y \xrightarrow{i} X$ be closed immersions. Denote $\mu\colon \mathbb{E} \wedge \mathbb{M} \to \mathbb{M}$ the structure morphism. We have as well a refined product

$$\begin{array}{ccc} \mathbb{M}_Z^{p,q}(Y) \times \mathbb{E}_Y^{r,s}(X) & \to & \mathbb{M}_Z^{p+r,q+s}(X) \\ (m, a) & \mapsto & m \cdot a \end{array}$$

defined as follows: let $m\colon j_*(\mathbb{1}_Z) \to \mathbb{M}_Y(q)[p]$ and $a\colon i_*\mathbb{1}_Y \to \mathbb{E}_X(s)[r]$, then

$$m \cdot a\colon i_*j_*\mathbb{1}_Z \longrightarrow i_*\mathbb{M}_Y(q)[p] \simeq \mathbb{M}_X \wedge i_*\mathbb{1}_Y(q)[p] \xrightarrow{\mathrm{id}\wedge a} \mathbb{M}_X \wedge \mathbb{E}_X(q+s)[p+r]$$

$$\xrightarrow{\mu} \mathbb{M}_X(q+s)[p+r].$$

Note that the same construction defines a product

$$\mathbb{E}_Z^{p,q}(Y) \times \mathbb{M}_Y^{r,s}(X) \longrightarrow \mathbb{M}_Z^{p+r,q+s}(X).$$

The following result follows from the same proofs as in Proposition 4.1.7.

Proposition 4.1.12 *Let $\mathbb{E}$ be an absolute ring spectrum and $\mathbb{M}$ be an absolute $\mathbb{E}$-module:*

1. *With the preceding notations, for any pair $(a, m) \in \mathbb{E}_Z(Y) \times \mathbb{M}_Y(X)$ we have $f^*(a \cdot m) = g^*(a) \cdot f^*(m)$ and $j_\flat(a \cdot m) = j_\flat(a) \cdot m$. Analogue formulas hold classes in $\mathbb{M}_Z(Y) \times \mathbb{E}_Y(X)$.*

2. *Consider closed immersions* $T \to Z \to Y \to X$. *Then for any triple* $(a, b, m) \in \mathbb{E}_T(Z) \times \mathbb{E}_Z(Y) \times \mathbb{M}_Y(X)$ *we have* $a \cdot (b \cdot m) = (a \cdot b) \cdot m$. *Analogue formulas hold for classes in* $\mathbb{E}_T(Z) \times \mathbb{M}_Z(Y) \times \mathbb{E}_Y(X)$ *and* $\mathbb{M}_T(Z) \times \mathbb{E}_Z(Y) \times \mathbb{E}_Y(X)$.

3. *Consider the following commutative diagram*

$$\begin{tikzcd} Z' \arrow[r, "j'"] \arrow[d, "h"'] & Y' \arrow[d, "g"] & \\ Z \arrow[r, "j"] & Y \arrow[r] & X \end{tikzcd}$$

of closed immersions and such that the square is cartesian. Then for any $(a, m) \in \mathbb{E}_Z(Y) \times \mathbb{M}_{Y'}(X)$ *the relation* $h_\flat(g^*(a) \cdot m) = a \cdot g_\flat(m)$ *holds. Analogue formulas hold for classes in* $\mathbb{M}_Z(Y) \times \mathbb{E}_{Y'}(X)$.

□

Example 4.1.13
- Every absolute ring spectrum is an absolute module over itself.

- In [Rio10], Riou lifted the Adams operations as an endomorphism of $\mathrm{KGL}_\mathbb{Q}$ in $\mathbf{SH}(S)_\mathbb{Q}$ resulting in the isomorphism $\mathrm{KGL}_\mathbb{Q} \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_\mathrm{B}(i)[2i]$ (see [Rio10, 5.3.17]). It induces a morphism of spectra

$$\mathrm{ch}\colon \mathrm{KGL}_\mathbb{Q} \to \mathrm{H}_\mathrm{B}.$$

  We call this morphism the **Chern character** since for any regular scheme X it induces the classical higher Chern characters

$$\mathrm{ch}_{r,n}\colon K_r(X)_\mathbb{Q} \to H^{2n-r}_\mathcal{M}(X, \mathbb{Q}(n)).$$

  The Chern character is a morphism of absolute ring spectra. In particular H_B is a module over $\mathrm{KGL}_\mathbb{Q}$.

- Denote $\mathbb{E}$ the absolute ring spectrum defined by a mixed Weil theory as in [CD12] (for example, algebraic and analytic de Rham or $\mathbb{Q}_l$ geometric étale). Cisinski and Déglise constructed a morphism of absolute ring spectra

$$\mathrm{cl}\colon \mathrm{H}_\mathrm{B} \to \mathbb{E}.$$

  We call this morphism the **cycle class**. Therefore any mixed Weil spectrum $\mathbb{E}$ is a module over H_B. Note that there is an analogue construction of the cycle class for Besser's rigid syntomic, absolute Hodge and Deligne-Beilinson spectra (*cf.* [DM14] and [HS15]).

4.1.1 Examples of modules

In the classical case the kernel of a morphism of rings is a module. In our setting the kernel will be replaced by the homotopy fiber. We show that many constructions of cohomology are particular examples of this: the *relative* cohomology and the spectra of arithmetic cohomologies defined in [HS15].

We use the theory of monoids and modules in model categories. This theory can be found written in the context of motivic homotopy theory in [CD09, §7] and a more accesible summary for **SH** in [Dég13, §2.2].

Definition 4.1.14 A **strict ring spectrum** E is a commutative monoid in the category $\mathbf{Spt}_\Sigma(X)$. In other words, a strict ring spectrum is a triple $(E, \mu\colon E \wedge E \to E, \eta\colon \mathbb{1}_X \to E)$ such that product and the unit morphism are in $\mathbf{Spt}_\Sigma(X)$ and satisfy the same diagrams of Definition 4.1.1. A **morphism** of strict ring spectra $\varphi\colon E \to F$ is a morphism in $\mathbf{Spt}_\Sigma(X)$ satisfying the diagram of 4.1.1. We denote by $\mathbf{Mon}(X)$ the category of strict ring spectra with morphism morphisms of strict ring spectra. A **strict absolute ring spectrum** $\mathbb{E}$ a strict ring spectrum $\mathbb{E}_S$ in $\mathbf{Spt}_\Sigma(S)$ and morphism of absolute ring spectra $\varphi\colon \mathbb{E} \to \mathbb{F}$ is a morphism of strict ring spectra $\varphi_S\colon \mathbb{E}_S \to \mathbb{F}_S$.

Let E be a strict ring spectrum in $\mathbf{Spt}_\Sigma(X)$, a **strict** E**-module** is a symmetric spectrum M with a morphism $\mu\colon M \wedge E \to M$ in $\mathbf{Spt}_\Sigma(X)$ satisfying the diagrams of Definition 4.1.10. A **morphism** of strict E-modules $\varphi\colon M \to M'$ is a morphism in $\mathbf{Spt}_\Sigma(X)$ satisfying the diagram of 4.1.10. We denote E-**mod** the category of strict E-modules with morphisms of E-modules. Let $\mathbb{E}$ be a strict absolute ring spectrum, a **strict absolute** $\mathbb{E}$**-module** is a strict $\mathbb{E}_S$-module $\mathbb{M}_S$.

Example 4.1.15 Every absolute spectrum of Example 4.1.8 representing a cohomology is constructed in their respective references as a strict absolute ring spectrum. Moreover, the Chern character and the cycle class map of Example 4.1.13 are morphism of strict ring spectra so that $\mathrm{H_B}$ is a strict absolute $\mathrm{KGL}_\mathbb{Q}$-module and any strict ring spectrum $\mathbb{E}$ coming from a mixed Weil theory is a strict absolute $\mathrm{H_B}$-module.

Notation 4.1.16 We abuse notation and say that a ring spectrum E in $\mathbf{SH}(X)$ is *strict* if it can be represented by a strict ring spectrum. Analogously, we say that a module, (absolute module, absolute ring spectrum or morphism) in **SH** is strict if it can be represented by a strict module (absolute module, absolute ring spectrum or strict morphism respectively).

Remark 4.1.17 The categories $\mathbf{Mon}(X)$ and E-**mod** inherit a model structure from the $\mathbb{A}^1$-stable symmetric model structure in $\mathbf{Spt}_\Sigma(X)$. The categories

$\mathbf{Ho}(\mathbf{Mon}(\ \_\ ))$ are well behaved with respect to inverse and direct image as described in [CD09, 7.1.11]. We will use the following fact: let $f: Y \to X$ be a morphism of schemes, $E \in \mathbf{Mon}(Y)$ and $F \in \mathbf{Mon}(X)$, then f_*E in $\mathbf{SH}(X)$ and f^*E in $\mathbf{SH}(Y)$ are given by strict ring spectra.

Let $\mathbb{E}$ be a strict absolute ring spectrum. The categories $\mathbf{Ho}(\mathbb{E}_X\textbf{-mod})$, where X is an S-scheme, have good functorial properties as $\mathbf{Ho}(\mathbf{Mon})$. Moreover, they are triangulated categories and the forgetful functor

$$\mathbf{Ho}(\mathbb{E}_X\textbf{-mod}) \to \mathbf{SH}(X)$$

is triangulated.

4.1.18 Let $\varphi: \mathbb{E} \to \mathbb{F}$ be a morphism of strict absolute ring spectra and X be an S-scheme. Denote $\varphi_X: \mathbb{E}_X \to \mathbb{F}_X$ the morphism of strict ring spectra in $\mathbf{Spt}_\Sigma(X)$ and $\mathrm{hofib}(\varphi_X)$ the homotopy fiber of φ_X. The spectrum $\mathrm{hofib}(\varphi_S)$ in $\mathbf{Spt}_\Sigma(S)$ defines by pullback an absolute spectrum, which we denote $\mathrm{hofib}(\varphi)$. Recall that the homotopy fiber fits into a distinguished triangle. In other words, for X an S-scheme we have that

$$\mathrm{hofib}(\varphi_X) \longrightarrow \mathbb{E}_X \xrightarrow{\varphi_X} \mathbb{F}_X \longrightarrow \mathrm{hofib}(\varphi_X)[1]. \tag{4.1}$$

Since $\mathbf{Ho}(\mathbb{E}_X\textbf{-mod})$ is triangulated and φ is a morphism of absolute spectra the following result is direct.

Proposition 4.1.19 *Let $\varphi: \mathbb{E} \to \mathbb{F}$ be a morphism of strict absolute ring spectra. With above notations,* $\mathrm{hofib}(\varphi)$ *is a strict absolute* $\mathbb{E}$*-module and for every S-scheme X we have* $\mathrm{hofib}(\varphi)_X = \mathrm{hofib}(\varphi_X)$.

□

Remark 4.1.20 Still in above notations, after a replacement we can assume φ_S to be a fibration and $\mathbb{F}_S$ to be fibrant so that $\mathrm{hofib}(\varphi_S)$ fits into a cartesian square

$$\begin{tikzcd} \mathrm{hofib}(\varphi_S) \arrow[r] \arrow[d] & \mathbb{E}_S \arrow[d, "\varphi_S"] \\ * \arrow[r] & \mathbb{F}_S. \end{tikzcd}$$

Note that the replacement is functorial so we have a commutative diagram

$$\begin{tikzcd} \mathrm{hofib}(\varphi_S) \wedge \mathrm{hofib}(\varphi_S) \arrow[r] \arrow[d, dashed] & \mathbb{E}_S \wedge \mathbb{E}_S \arrow[r, "\varphi_S \wedge \varphi_S"] \arrow[d, "\mu"] & \mathbb{F}_S \wedge \mathbb{F}_S \arrow[d, "\bar{\mu}"] \\ \mathrm{hofib}(\varphi_S) \arrow[r] & \mathbb{E}_S \arrow[r, "\varphi"] & \mathbb{F}_S \end{tikzcd}$$

Therefore, the groups $\mathrm{Hom}_{\mathbf{SH}(\_)}(\mathbb{1}_{\_}, \mathrm{hofib}\,\varphi)$ not only are modules over $\mathbb{E}(\_)$ but also have an inner product. Note that they do not have a unit. As in the classical case with the kernel, $\mathrm{hofib}(\varphi)$ is an *ideal* of $\mathbb{E}$. We do not introduce this notation since we do not make use of it.

Let X be a scheme, the distinguished triangle (4.1) gives raise to a long exact sequence

$$\cdots \to \mathbb{F}^{*-1,*}(X) \to \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathrm{hofib}(\varphi)(*)[*]) \to \mathbb{E}^{*,*}(X) \to \mathbb{F}^{*,*}(X) \to \cdots$$

where arrows are compatible with products.

We introduce the relative cohomology in the context of motivic homotopy. Let $f\colon Y \to X$ be a morphism of schemes, then $f_*\mathbb{E}_Y$ represents in $\mathbf{SH}(X)$ the cohomology of Y. Indeed,

$$\mathbb{E}^{*,*}(Y) = \mathrm{Hom}_{\mathbf{SH}(Y)}(f^*\mathbb{1}_X, \mathbb{E}_Y(*)[*]) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, f_*\mathbb{E}_X(*)[*]).$$

Since $\mathbb{E}_Y \simeq f^*\mathbb{E}_X$ we have an adjunction morphism $\mathbb{E}_X \to f_*f^*\mathbb{E}_X$.

Proposition 4.1.21 *Let $\mathbb{E}$ be an absolute ring spectrum and $f\colon Y \to X$ be a morphism of schemes:*

1. *The spectrum $f_*\mathbb{E}_Y$ is a ring spectrum. The adjunction $\mathbb{E}_X \to f_*\mathbb{E}_Y$ is a morphism of ring spectra and it induces the inverse image on cohomology $f^*\colon \mathbb{E}(X) \to \mathbb{E}(Y)$.*

2. *If in addition $\mathbb{E}$ is strict, then $f_*\mathbb{E}_Y$ is also strict and the adjunction map $\mathbb{E}_X \to f_*\mathbb{E}_X$ is represented by a morphism of strict ring spectra.*

Proof: The unit morphism $\mathbb{1}_Y \simeq f^*\mathbb{1}_X \to \mathbb{E}_Y$ induce by adjunction a morphism $\mathbb{1}_X \to f_*\mathbb{E}_Y$. Recall that the pullback functor is monoidal, therefore we have a morphism

$$(f^*f_*\mathbb{E}_Y) \wedge (f^*f_*\mathbb{E}_Y) \longrightarrow \mathbb{E}_Y \wedge \mathbb{E}_Y \xrightarrow{\mu} \mathbb{E}_Y .$$

By adjunction, we deduce a morphism $f_*\mathbb{E}_Y \wedge f_*\mathbb{E}_Y \to f_*\mathbb{E}_Y$. It is a direct computation to check the diagrams of Definition 4.1.1 and that the adjunction $\mathbb{E}_X \to f_*\mathbb{E}_Y$ is morphism of ring spectra.

For the second point note that in Remark 4.1.17 we observed that $f_*\mathbb{E}_Y$ is strict. After a fibrant replacement, we can choose a representative of the adjunction morphism $R\mathbb{E}_X \to Rf_*\mathbb{E}_Y$ which is a morphism of strict ring spectra.

□

Definition 4.1.22 Let $\mathbb{E}$ be a strict absolute ring spectrum, $f\colon Y \to X$ be a morphism of schemes. Abuse notation and denote $\mathrm{hofib}_{\mathbb{E}}(f_X)$ (or simply $\mathrm{hofib}(f_X)$ if it is clear by the context) the strict $\mathbb{E}_X$-module defined as the homotopy fiber of the morphism of strict ring spectra $\mathbb{E}_X \to f_*\mathbb{E}_Y$. We define the **relative cohomology** of f to be

$$\mathbb{E}^{p,q}(f) := \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathrm{hofib}(f_X)(q)[p]) \quad \text{for } p, q \in \mathbb{Z}.$$

We also denote $\mathrm{hofib}_{\mathbb{E}}(f)$ (or simply $\mathrm{hofib}(f)$) the strict absolute $\mathbb{E}$-module over X that $\mathrm{hofib}(f_X)$ defines by pullback.

Remark 4.1.23 Consider above notations. We have constructed a morphism of strict ring spectra $\mathbb{E}_X \to f_*\mathbb{E}_Y$ in $\mathbf{Spt}_\Sigma(X)$ and the strict absolute morphism it defines by pullback. Therefore note that, although $\mathbb{E}$ is an absolute spectrum, the spectrum $\mathrm{hofib}(f)$ need not to have good cohomological properties. More concretely, consider a cartesian square

$$\begin{tikzcd} Y_T \arrow[r] \arrow[d, "f_T"'] & Y \arrow[d, "f"] \\ T \arrow[r, "g"] & X. \end{tikzcd}$$

Then the spectrum $g^*f_*\mathbb{E}_Y$ may not be isomorphic to $f_{T*}\mathbb{E}_{Y_T}$. In other words, the family of spectra $f_{T*}\mathbb{E}_{Y_T}$ for $T \to X$ may not define an absolute spectrum. Therefore $\mathrm{hofib}(f)_T = g^* \mathrm{hofib}(f_X)$ may not be isomorphic to $\mathrm{hofib}(f_T)$. Since he relative cohomology of f_T is represented by $\mathrm{hofib}(f_T)$ the absolute spectrum $\mathrm{hofib}(f)$ may not represent the relative cohomology of f_T for a general base change $T \to X$.

Proposition 4.1.24 *Let $\mathbb{E}$ be a strict absolute spectrum, $f\colon Y \to X$ and $g\colon T \to X$ be two morphism of schemes. If either f is proper or g is smooth then we have*

$$g^* \mathrm{hofib}(f_X) \simeq \mathrm{hofib}(f_T).$$

Proof: Denote $g'\colon Y_T \to Y$ and $f_T\colon Y_T \to T$. It is enough to prove that $g^*f_*\mathbb{E}_Y \simeq f_{T*}g'^*\mathbb{E}_Y$. The result follows from the smooth base change property of Proposition 2.2.27 and the base change for proper morphism of Theorem 3.2.37.

□

Example 4.1.25 Let $\mathbb{E}$ be a strict absolute ring spectrum. The construction of relative cohomology generalizes many concrete situations:

1. Let $S = \mathrm{Spec}(k)$ and $p\colon X \to S$ be the structural morphism. Then $\mathbb{E}(p) = \widetilde{\mathbb{E}}(X)$, the reduced cohomology of X.

2. Let $i\colon Z \to X$ be a closed immersion with open complement $j\colon U \to X$. Recall from Theorem 3.2.33 that we also have a distinguished triangle

$$j_\sharp j^*\mathbb{E}_X \to \mathbb{E}_X \to i_*i^*\mathbb{E}_X \to j_\sharp j^*\mathbb{E}_X[1].$$

In this case $\mathrm{hofib}(i) \simeq j_\sharp j^*\mathbb{E}_X$. Recall that $j_\sharp = j_!$ since j is an open immersion. Although we have not reviewed it, $j_!j^*\mathbb{E}_X$ represents, by definition, the cohomology of U with *compact support* $\mathbb{E}_c(U)$ in $\mathbf{SH}(X)$. There is as well a natural product of elements in $\mathbb{E}_c(U)$ with elements of $\mathbb{E}(X)$ which is the same as the one as $\mathbb{E}$-module.

3. Consider above notations. Recall from [CD09, 2.3.3] that by duality we also have a distinguished triangle

$$i_*i^!\mathbb{E}_X \to \mathbb{E}_X \to j_*j^*\mathbb{E}_X \to i_*i^!\mathbb{E}_X[1].$$

Therefore $\mathrm{hofib}(j) \simeq i_*i^!\mathbb{E}_X$. Note that $i_*i^!\mathbb{E}$ represents in $\mathbf{SH}(X)$ the cohomology of X with support on Z. Indeed,

$$\mathbb{E}_Z^{p,q}(X) = \mathrm{Hom}_{\mathbf{SH}(X)}(i_*i^*\mathbb{1}_X, \mathbb{E}_X) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, i_*i^!\mathbb{E}_X).$$

The product as $\mathbb{E}$-module of Paragraph 4.1.11 is the refined product of Definition 4.1.6.

4. Let $i\colon Z \to X$ be a closed immersion and consider the blow-up cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi'"'] & B_Z X \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X. \end{tikzcd}$$

It follows from upcoming Corollary 4.2.6 that $\mathbb{E}(\pi) = \mathbb{E}(P)/\mathbb{E}(Z)$. The product as $\mathbb{E}(X)$-module is the product through $(i\pi')^*\colon \mathbb{E}(X) \to \mathbb{E}(P)$.

5. Let R be a Dedekind domain and F be its field fractions. Denote $K(R) = K(\mathrm{Spec}(R))$, $K(F) = K(\mathrm{Spec}(F))$ and $\gamma\colon \mathrm{Spec}(F) \to \mathrm{Spec}(R)$ the localization morphism. Then $K_2(\gamma) = \coprod_{\mathfrak{p}} K_2(R/\mathfrak{p})$ where $\mathfrak{p}$ denote prime ideals of R, $K_1(\gamma) = \coprod_{\mathfrak{p}}(R/\mathfrak{p})^\times$ the connecting $\delta\colon K_2(F) \to K_2(\gamma)$ satisfies $\delta = \coprod \delta_{\mathfrak{p}}$ where $\delta_{\mathfrak{p}}$ are the tame symbols (*cf.* [Wei89, III.6.5]).

Theorem 4.1.26 *Let $f\colon Y \to X$ be a morphism of regular schemes and denote $K(f)$ the relative algebraic K-theory (*cf.* [**?**, IV.8.5.3]). Then*

$$K_i(f) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathrm{hofib}_{\mathrm{KGL}}(f_X)[-i]) \quad \textit{for } i \in \mathbb{Z}.$$

Proof: We use notation from [CD09]. Recall from [CD09, §3.2] that there is total derived global section functor

$$\mathrm{R}\Gamma\colon \mathbf{SH}(X) \to \mathbf{Ho}(\mathbf{Spt}_{S^1}).$$

where $\mathbf{Spt}_{S^1}$ denotes the classic category of S^1-spectra of simplicial sets. Recall from [CD09, 13.4] that $K_n(X) = \pi_n(\mathrm{R}\Gamma(X, \mathrm{KGL}_X)) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathrm{KGL}_X)$. Applying the total derived global section functor to the homotopy fiber sequence

$$\mathrm{hofib}_{\mathrm{KGL}}(f_X) \to \mathrm{KGL}_X \to f_*\mathrm{KGL}_Y$$

we obtain a (classic) homotopy fiber sequence

$$\mathrm{R}\Gamma(X, \mathrm{hofib}_{\mathrm{KGL}}(f_X)) \to \mathrm{R}\Gamma(X, \mathrm{KGL}_X) \to \mathrm{R}\Gamma(X, f_*\mathrm{KGL}_Y).$$

We conclude by recalling that the relative K-theory of f is defined in [**?**, IV.8.5.3] as the classic homotopy fiber.

□

We review the construction of the arithmetic counterparts of K-theory and motivic cohomology of [HS15], which are another example of an homotopy fiber.

Let A be an arithmetic ring (Appendix 5.1.2) and denote $S = \mathrm{Spec}(A)$ and η its generic point. Recall from Example 4.1.8 that we have the Deligne-Beilinson cohomology strict ring spectrum $\mathbb{E}_{\mathbb{R}\mathrm{D},\eta} \in \mathbf{SH}(\eta)$, which defines a strict absolute ring spectrum $\eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta} \in \mathbf{SH}(S)$. Recall from Example 4.1.13 that we have the cycle class map $\mathrm{cl}\colon \mathrm{H}_{\mathrm{B},\eta} \to \mathbb{E}_{\mathbb{R}\mathrm{D},\eta}$ which induces a map

$$\varphi\colon \mathrm{H}_{\mathrm{B},S} \to \eta_*\mathrm{H}_{\mathrm{B},\eta} \xrightarrow{\eta_*\mathrm{cl}} \eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta}$$

in $\mathbf{SH}(S)_{\mathbb{Q}}$. This map is actually strict. Recall that in $\mathbf{SH}(S)_{\mathbb{Q}}$ we have $\mathrm{KGL}_{S,\mathbb{Q}} = \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_{\mathrm{B},S}(i)[2i]$ (*cf.* [CD09, §14]). We have a map

$$\oplus(\mathrm{ch}_i \circ \varphi_i)\colon \mathrm{KGL}_{S,\mathbb{Q}} \to \eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta}(i)[2i]$$

where $\varphi_i\colon \mathrm{H}_{\mathrm{B},S}(i)[2i] \to \eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta}(i)[2i]$. This map is also strict.

Definition 4.1.27 In above notations, we define the **arithmetic motivic cohomology** strict absolute spectrum as $\widehat{\mathbf{H}}_{\mathrm{B},S} = \mathrm{hofib}(\varphi)$. Let X be a smooth S-scheme, we denote the cohomology it defines as

$$\widehat{H}^p_{\mathcal{M}}(X, q) := \mathrm{Hom}_{\mathbf{SH}(X)_{\mathbb{Q}}}(\mathbb{1}_X, \widehat{\mathbf{H}}_{\mathrm{B},X}(q)[p]) \quad \text{for } p, q \in \mathbb{Z}.$$

Analogously, we define the **arithmetic homotopy invariant K-theory** strict absolute spectrum as $\widehat{\mathrm{KGL}}_{S,\mathbb{Q}} = \mathrm{hofib}(\oplus(\mathrm{ch}_i \circ \varphi_i))$. Note that the periodicity of the K-theory makes $\widehat{\mathrm{KGL}}$ also periodic. Let X be a smooth S-scheme, we denote the cohomology it defines as

$$\widehat{KH}_i(X)_{\mathbb{Q}} := \mathrm{Hom}_{\mathbf{SH}(X)_{\mathbb{Q}}}(\mathbb{1}_X, \widehat{\mathrm{KGL}}_{\mathbb{Q}}[-i]) \quad \text{for } i \in \mathbb{Z}.$$

Remark 4.1.28 Note that our definition is written differently from [HS15], where they considered the spectra

$$\mathrm{hofib}(\mathrm{H}_{\mathrm{B},S} \xrightarrow{\mathrm{id}\wedge 1_{\mathrm{D}}} \mathrm{H}_{\mathrm{B},S} \wedge \eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta}) \text{ and } \mathrm{hofib}(\mathrm{KGL}_S \xrightarrow{\mathrm{id}\wedge 1_{\mathrm{D}}} \mathrm{KGL}_S \wedge \eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta}).$$

Recall that in $\mathbf{SH}(S)_{\mathbb{Q}}$ we have $\mathrm{H}_{\mathrm{B},S} \wedge \eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta} \simeq \eta_*\mathbb{E}_{\mathbb{R}\mathrm{D},\eta}$ (*cf.* [CD09, 14.2.8]) so that both definitions agree.

Remark 4.1.29 By construction, both $\widehat{\mathrm{H}}_{\mathrm{B}}$ and $\widehat{\mathrm{KGL}}_{\mathbb{Q}}$ are strict absolute H_{B} and $\mathrm{KGL}_{\mathbb{Q}}$ modules respectively, but not rings. In particular note that they do not have a unit morphism. Therefore both the arithmetic homotopy invariant K-theory and arithmetic motivic cohomology do not have the unit.

Nevertheless, the Chern character $\mathrm{ch}\colon \mathrm{KGL}_{\mathbb{Q}} \to \mathrm{H}_{\mathrm{B}}$ induces an *arithmetic Chern character* $\widehat{\mathrm{ch}}\colon \widehat{\mathrm{KGL}}_{\mathbb{Q}} \to \widehat{\mathrm{H}}_{\mathrm{B}}$. In addition, the square

$$\begin{tikzcd} \widehat{\mathrm{KGL}}_{\mathbb{Q}} \arrow[r] \arrow[d, "\widehat{\mathrm{ch}}"'] & \mathrm{KGL}_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ \widehat{\mathrm{H}}_{\mathrm{B}} \arrow[r] & \mathrm{H}_{\mathrm{B}} \end{tikzcd}$$

commutes.

4.1.2 Orientations

We review the theory of orientations (i.e., Chern classes) for spectra. As in the classical case, they are determined by the first Chern class of the tautological line bundle of projective spaces.

Recall the definition of the Tate object as $\mathbb{1}_S(1) = \mathrm{coker}(\Sigma^\infty S \to \Sigma^\infty \mathbb{P}^1)[-2]$. For any ring spectrum $\mathbb{E}$ with unit $\eta\colon S \to \mathbb{E}_S$ there is a canonical class in $\mathbb{E}^{2,1}(\mathbb{P}^1)$ defined as the morphism

$$\mathbb{P}^1 \to \mathbb{1}_S(1)[2] = S \wedge \mathbb{1}_S(1)[2] \xrightarrow{\eta\wedge\mathrm{Id}} \mathbb{E}_S(1)[2].$$

By abuse of notation we will still denote it $\eta \in \mathbb{E}^{2,1}(\mathbb{P}^1)$.

The definition of $\mathbb{E}$-cohomology may be extended to general spectra. In particular, recall that the infinite projective space is defined to be $\mathbb{P}^\infty_X = \varinjlim \mathbb{P}^n_X$ and we denote

$$\mathbb{E}^{p,q}(\mathbb{P}^\infty_X) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{P}^\infty_X, \mathbb{E}_X(q)[p]).$$

Definition 4.1.30 We define an **orientation** on an absolute ring spectrum $\mathbb{E}$ to be a class $c_1 \in \mathbb{E}^{2,1}(\mathbb{P}^\infty)$ such that for $i_1\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$ satisfies $i_1^*(c_1) = \eta$. We also say that $\mathbb{E}$ is **oriented**.

Let X be a scheme and $\mathcal{V}$ be a locally free $\mathcal{O}_X$-module. We call the **vector bundle** given by $\mathcal{V}$ to the scheme $V = \mathbf{Spec}_X(S^\bullet\mathcal{V}^*) \to X$ and the **projective bundle** given by $\mathcal{V}$ to the scheme $\mathbb{P}(V) = \mathbf{Proj}_X(S^\bullet\mathcal{V}^*) \to X$.

Let $B\mathbb{G}_m$ be the classifying space for $\mathbb{G}_m$-torsors ([MV99, 4.1.16]), due to Corollary 3.1.41 Theorem 3.1.45 we have a natural map

$$\mathrm{Pic}(X) \to \mathrm{Hom}_{\mathbf{H}_\bullet(X)}(X_+, B\mathbb{G}_m) \simeq \mathrm{Hom}_{\mathbf{H}_\bullet(X)}(X_+, \mathbb{P}^\infty) \to \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathbb{P}^\infty_X)$$

so that any line bundle $L \in \mathrm{Pic}(X)$ defines a morphism $f\colon X \to \mathbb{P}^\infty_X$ in $\mathbf{SH}(X)$.

Let $(\mathbb{E}, c_1)$ be an oriented spectrum absolute ring spectrum and denote $p\colon \mathbb{P}^\infty_X \to \mathbb{P}^\infty$. For any line bundle L we have

$$\begin{array}{ccc} \mathbb{E}^{2,1}(\mathbb{P}^\infty_X) & \xrightarrow{f^*} & \mathbb{E}^{2,1}(X) \\ p^*c_1 & \mapsto & c_1(L) \end{array}$$

and we say that $c_1(L) \coloneqq f^*p^*c_1$ is **the first Chern class** of L.

Example 4.1.31 Every example of Example 4.1.8 representing a cohomology is oriented. We quickly review the references: Mixed Weil theories are oriented in [CD12, 2.2.8], the algebraic K-theory KGL and Beilinson's motivic cohomology H_{B} in [CD09, 13.2.2] and [CD09, 14.1.5] respectively, algebraic cobordism MGL in [PPR08, 1.4]. In particular, the orientation of K-theory is given by $c_1^{\mathrm{KGL}}(L) = 1 - [L^*]$. Spitzweck's motivic cohomology spectrum H_Λ is oriented in [Spi12, 11.1]. In the Appendix 5.1 we give an orientation for the absolute Hodge spectrum and the Deligne-Beilinson is done in [HS15, 3.6]. Finally, every cohomology considered in [DM14] is represented by an oriented spectrum (*cf.* [DM14, 1.4.11.(1) and 2.1.2.(1)]). In particular, Besser's absolute rigid syntomic spectrum is oriented.

Remark 4.1.32 Let $\varphi\colon \mathbb{E} \to \mathbb{F}$ be a morphism of absolute ring spectra and let $c_1 \in \mathbb{E}^{2,1}(\mathbb{P}^\infty)$ be an orientation on $\mathbb{E}$. Since φ is a morphism of rings, it maps the unit $\mathbb{1}_S \to \mathbb{E}_S$ onto the unit $\mathbb{1}_S \to \mathbb{F}_S$. We conclude that the element $\varphi_{\mathbb{P}^\infty}(c_1) \in \mathbb{F}^{2,1}(\mathbb{P}^\infty)$ is an orientation on $\mathbb{F}$.

Remark 4.1.33 The immersion $\mathbb{P}^{n-1} \to \mathbb{P}^n$ is defined by a sheaf canonically isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$. Therefore if we denote $i_n\colon \mathbb{P}^n \to \mathbb{P}^\infty$ we have that $i_n^*(c_1) = c_1(\mathcal{O}_{\mathbb{P}^n}(-1))$ and we write $c_1 = c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))$.

To fix notations, we recall the theory of Chern classes in the context of spectra. Proof of the following result in the context of stable homotopy theory may be found in [Dég14, 2.1.13 and 2.1.22].

Theorem 4.1.34 (Projective bundle) *Let $V \to X$ be a vector bundle of rank $(n+1)$, $\mathbb{E}$ an oriented absolute ring spectrum and $x = c_1(\mathcal{O}_{\mathbb{P}(V)}(-1))$. There is a canonical isomorphism*

$$\begin{array}{ccc} \bigoplus_{i=0}^{n} \mathbb{E}^{*-2i,*-i}(X) & \xrightarrow{\sim} & \mathbb{E}^{*,*}(\mathbb{P}(V)) \\ (a_0, \ldots, a_n) & \mapsto & \sum_i \pi^*(a_i)x^i. \end{array}$$

□

Definition 4.1.35 Let $V \to X$ be a vector bundle of rank n. We define the **i-th Chern classes** $c_i(V) \in \mathbb{E}^{2i,i}(X)$ for $i = 1, \ldots, n$ as the unique ones satisfying

$$c_1(\mathcal{O}_{\mathbb{P}(V)}(-1))^n + \sum_{i=1}^{n} (-1)^i c_i(V) c_1(\mathcal{O}_{\mathbb{P}(V)}(-1))^{n-i} = 0.$$

4.1.36 *Formal groups laws $F(x,y)$* are certain type of series (see for example [Ada74]). In particular, they satisfy the property that any formal group law $F(x,y)$ is of the form

$$F(x,y) = x + y + f(x,y)$$

for $f(x,y)$ a symmetric series. A formal group law is called **additive** if $f(x,y) = 0$ and **abelian** if $F(x,y) = F(y,x)$. The following is a classic result. See for example [Dég08, 3.7].

Theorem 4.1.37 *Let $\mathbb{E}$ be an oriented absolute ring spectrum. There exists a formal abelian group law $F(x,y) \in \mathbb{E}^{**}(S)[[x,y]]$ such that*

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$$

for any line bundles L_1, L_2 over X.

□

Lemma 4.1.38 *There is a short exact sequence*

$$0 \to \varprojlim_n{}^1 \mathbb{E}^{i-1,*}(\mathbb{P}^n) \to \mathbb{E}^{i,*}(\mathbb{P}^\infty) \to \varprojlim_n \mathbb{E}^{i,*}(\mathbb{P}^n) \to 0$$

where $\varprojlim^1$ is the first derived functor of $\varprojlim$.

Proof: We can assume $\mathbb{E}_S$ is fibrant. The result follows from [Hov99, 7.3.2] provided that $\operatorname{hocolim} \Sigma^\infty \mathbb{P}^n \simeq \Sigma^\infty \mathbb{P}^\infty$. This is true since $\mathbb{P}^n \xrightarrow{i} \mathbb{P}^{n+1}$ is a cofibration therefore it is the homotopy colimit of cofibrations between cofibrant objects.

□

Proposition 4.1.39 *Let $(\mathbb{E}, c_1)$ be an oriented absolute ring spectrum, then*

$$\mathbb{E}(\mathbb{P}^\infty) = \mathbb{E}(S)[[c_1]]$$

Proof: The result follows from Lemma 4.1.38. Note that from Theorem 4.1.34 we have that $\mathbb{E}^{i-1,*}(\mathbb{P}^{n+1}) \to \mathbb{E}^{i-1,*}(\mathbb{P}^n)$ is constant for $n \geq i$ and therefore $\varprojlim^1 \mathbb{E}^{i-1,*}(\mathbb{P}^n) = 0$.

□

Example 4.1.40 Every cohomology from Example 4.1.8 apart from K-theory and algebraic cobordism have additive formal group laws. That is to say: motivic cohomology, cohomologies coming from mixed Weil theories, real absolute Hodge and Deligne-Beilinson cohomology and Besser's rigid syntomic cohomology have additive formal group laws.

We define the **total Chern class** of a vector bundle V of rank n to be $c(V) = 1 + c_1(V) + \cdots + c_n(V)$. Standard arguments yields the following classic formula:

Theorem 4.1.41 (Whitney sum) *The total Chern class is multiplicative. In other words, let $0 \to V' \to V \to V'' \to 0$ be a short exact sequence of vector bundles, then we have the following equivalent formulas*

$$c(V) = c(V')c(V''),$$

$$c_k(V) = \sum_{i+j=k} c_i(V')c_j(V'') \quad i, j, k \in \mathbb{N}.$$

□

Proposition 4.1.42 *Let $(\mathbb{E}, c_1)$ be an oriented absolute spectrum and c_1^{new} be another orientation. Then there exist $G(t) \in \mathbb{E}(S)[[t]]$ with leading coefficient 1 such that for any line bundle L we have*

$$c_1^{\mathrm{new}}(L) = G(c_1(L))c_1(L).$$

Proof: Since $\mathbb{P}^\infty \simeq \mathrm{B}\mathbb{G}_m$, the classifying space for line bundles, it is enough to check the formula for $x = c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))$. Recall that $\mathbb{E}(\mathbb{P}^\infty) = \mathbb{E}(S)[[x]]$ and therefore we have $c_1^{\mathrm{new}}(\mathcal{O}_{\mathbb{P}^\infty}(-1)) = ax + \ldots = G(x)x$ for $a \in \mathbb{E}(S)$ invertible. Finally, both classes satisfy $i_1^*(c_1(\mathcal{O}_{\mathbb{P}^\infty})) = i_1^*(c_1^{\mathrm{new}}(\mathcal{O}_{\mathbb{P}^\infty})) = \eta \in \mathbb{E}^{2,1}(\mathbb{P}^1)$ and we conclude that $a = 1$.

□

Let us recall the construction of the Todd class in our context. Denote as $K^0 \colon \mathbf{Sch}/S \to Ab$ the functor which maps X to $K^0(X)$, the Grothendieck group of vector bundles over X. Let $\mathbb{E}$ be an absolute ring spectrum, denote $\mathbb{E}^\times$ the functor which maps X to $\mathbb{E}^\times(X)$, the group of invertible elements of $\mathbb{E}(X)$.

Corollary 4.1.43 *Let $(\mathbb{E}, c_1)$ be an oriented absolute spectrum and c_1^{new} be another orientation on $\mathbb{E}$. With above notations, there exist a unique natural transformation*

$$\mathrm{Td}_G \colon K^0 \to \mathbb{E}^\times$$

which is multiplicative (in the sense that $\mathrm{Td}_G(V+W) = \mathrm{Td}_G(V) \cdot \mathrm{Td}_G(W)$) and such that for any line bundle L we have $\mathrm{Td}_G(L) = G^{-1}(c_1(L))$.

Proof: Let $V \to X$ be a vector bundle. By the splitting principle we may assume $V = L_1 + \cdots + L_n$. We define $\mathrm{Td}_G(V) = G^{-1}(L_1) \cdot \cdots \cdot G^{-1}(L_n) \in \mathbb{E}^\times(X)$. It is clear that this class satisfies that for any short exact sequence $0 \to V' \to V \to V'' \to 0$ we have $\mathrm{Td}_G(V) = \mathrm{Td}_G(V')\,\mathrm{Td}_G(V'')$. Therefore, it induces a map $K^0(X) \to \mathbb{E}^\times(X)$ with all desired properties.

□

Remark 4.1.44 With the preceding notations, if $V = L_1 + \cdots + L_n \in K^0(T)$ then $\mathrm{Td}(V) = G^{-1}(c_1(L_1)) \cdots G^{-1}(c_1(L_n))$. In particular, for any n-rank vector bundle V we have

$$c_n^{\mathrm{new}}(V) = \mathrm{Td}(-V) c_n(V).$$

4.1.3 Chern class with support

Let X be a scheme and $U = (X - Z) \to X$ be an open subscheme. Recall from Definition 3.1.42 that a pseudo divisor (trivialized on U) is a pair $(\mathcal{L}, u)$ consisting of an invertible sheaf $\mathcal{L}$ over X (in the Zariski topology) and a trivialization $u \colon \mathcal{O}|_U \xrightarrow{\sim} \mathcal{L}|_U$. Recall that $\mathrm{Pic}_Z(X)$ denoted the group of isomorphism classes of pseudo divisors.

For convenience of the reader we recall Proposition 3.1.43:

Proposition 4.1.45 *Let $Z \to X$ be a regular immersion of codimension 1 and U be its open complement. Then*

$$\mathrm{Pic}_Z(X) \to \mathrm{Hom}_{\mathbf{H}_\bullet^s(X)}(X/U, \mathbb{P}_X^\infty)$$

is a bijection.

□

It follows from above identification that there is a map

$$\mathrm{Pic}_Z(X) \to \mathrm{Hom}_{\mathbf{H}_\bullet(X)}(X/U, \mathbb{P}^\infty_X) \to \mathrm{Hom}_{\mathbf{SH}(X)}(\Sigma^\infty X/U, \mathbb{P}^\infty_X) \to$$

$$\stackrel{(c_1)_*}{\to} \mathrm{Hom}_{\mathbf{SH}(X)}(\Sigma^\infty X/U, \mathbb{E}_X(1)[2]) = \mathbb{E}^{2,1}_Y(X).$$

Note that in the last equality we have used that $\Sigma^\infty X/U = i_*\mathbb{1}_Z$ for $i\colon Z \to X$. Finally, for an oriented absolute spectrum $\mathbb{E}$ and any line bundle L we have

$$\begin{array}{ccc} \mathrm{Pic}_Z(X) & \xrightarrow{\varphi^*_{L,u}} & \mathbb{E}^{2,1}_Z(X) \\ (L,u) & \mapsto & c_1^Z(L,u) \end{array}$$

and we say that $c_1^Z(L,u) := \varphi^*_{L,u}(L,u)$ is the first Chern class of L with support on Z. We omit the reference to the trivialization when no confusion is possible. The next statement follows from the definition:

Proposition 4.1.46 *Let $f\colon X' \to X$ be a morphism of schemes and $(\mathcal{L}, u)$ be a pseudo divisor over the open $X - Z$, then*

$$f^* c_1^Z(L) = c_1^{f^{-1}Z}(f^* L).$$

$\square$

Proposition 4.1.47 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves over X and $u_i\colon \mathcal{O}|_U \to \mathcal{L}_i|_U$, $i = 1, 2$, trivializations. Denote $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $u = u_1 \otimes u_2$. Let $F(x,y) \in \mathbb{E}(S)[[x,y]]$ be the formal group law of c_1 given by Theorem 4.1.37. Then*

$$c_1^Z(L) = F(c_1^Z(L_1), c_1^Z(L_2)).$$

Proof: The pseudo divisors (L_1, u_1), (L_2, u_2) and (L, u) correspond to morphism f_1, f_2, $f\colon X/U \to \mathbb{P}^\infty_X$ respectively. Denote the Segre embedding $\sigma\colon \mathbb{P}^\infty_X \times \mathbb{P}^\infty_X \to \mathbb{P}^\infty_X$. By construction, the diagram

$$\begin{tikzcd} X/U \arrow[r, "f_1\times f_2"] \arrow[dr, "f"'] & \mathbb{P}^\infty_X \times \mathbb{P}^\infty_X \arrow[d, "\sigma"] \\ & \mathbb{P}^\infty_X \end{tikzcd}$$

commutes and, after applying the functors $\mathrm{Hom}_{\mathbf{SH}(X)}(\_\,, \mathbb{E}_X(*)[*])$, we have the commutative diagram

$$\begin{tikzcd} \mathbb{E}(\mathbb{P}^\infty_X) \arrow[r, "\sigma^*"] \arrow[dr, "f^*"'] & \mathbb{E}(\mathbb{P}^\infty_X \times \mathbb{P}^\infty_X) \arrow[d, "(f_1\times f_2)^*"] \\ & \mathbb{E}_Y(X). \end{tikzcd}$$

Recall that $\mathbb{E}(\mathbb{P}^\infty) = \mathbb{E}(S)[[t]]$ and $\mathbb{E}(\mathbb{P}^\infty \times \mathbb{P}^\infty) = \mathbb{E}(S)[[u,v]]$ where $t = c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))$, $u = c_1(p_1^*\mathcal{O}_{\mathbb{P}^\infty}(-1))$, $v = c_1(p_2^*\mathcal{O}_{\mathbb{P}^\infty}(-1))$ and p_1, p_2 are the canonical projections. With this notations, the Segre morphism maps $t \mapsto F(u,v)$ where F is the formal group law of the orientation of $\mathbb{E}$. We conclude by the commutativity last diagram.

□

4.2 Gysin morphism

4.2.1 Regular immersions

We construct the Gysin morphism for a regular immersion following Gabber's ideas for étale cohomology (see [Fuj02] and [Rio07a]). Gabber's method reduces the case of general codimension to that of codimension one by means of describing the cohomology of the blow-up (*cf.* Corollary 4.2.6). In order to prove the needed functoriality properties the versatile context of modified blow-up is used.

Definition 4.2.1 Let $i\colon Z \to X$ be a regular immersion of codimension 1. The sheaf $\mathcal{I}_Z = \mathcal{L}_{-Z}$ is locally principal and it has a natural trivialization on $X - Y$, and so does its dual. We define the **refined fundamental class** (of Z in X) to be

$$\bar{\eta}_Z^X := c_1^Z(\mathcal{I}_Z^*) = c_1^Z(L_Z) \in \mathbb{E}_Z^{2,1}(X)$$

and the **fundamental class** to be $\eta_Z^X := c_1(L_Z) = i_\flat(c_1^Z(L_Z)) \in \mathbb{E}^{2,1}(X)$. We define the **refined Gysin morphism** as

$$\begin{array}{cccc} \mathfrak{p}_i\colon & \mathbb{E}^{*,*}(Z) & \longrightarrow & \mathbb{E}_Z^{*+2,*+1}(X) \\ & a & \mapsto & a \cdot \bar{\eta}_Z^X \end{array}$$

and the **Gysin morphism** as $i_*\colon \mathbb{E}^{*,*}(Z) \longrightarrow \mathbb{E}^{*+2,*+1}(X)$, $a \mapsto i_\flat(a \cdot \bar{\eta}_Z^X)$.

More generally, let $(\mathcal{L}, u\colon \mathcal{O}|_U \xrightarrow{\sim} \mathcal{L}|_U)$ be a pseudo divisor where $U = X - Z$. We define the **refined Gysin morphism (given by** $(\mathcal{L}, u)$**)** as

$$\begin{array}{cccc} \mathfrak{p}_{\mathcal{L}}\colon & \mathbb{E}^{*,*}(Z) & \longrightarrow & \mathbb{E}_Z^{*+2,*+1}(X) \\ & a & \mapsto & a \cdot c_1^Z(L). \end{array}$$

Remark 4.2.2 Let $Z \xrightarrow{i} X$ be a regular immersion of codimension 1. Due to Proposition 4.1.7 it is easy to check that

$$c_1^Z(L_Z) \cdot c_1(N_{Z/X}) = c_1^Z(L_Z)^2$$

where $c_1^Z(L_Z) \in \mathbb{E}_Z^{2,1}(X)$, $N_{Z/X} = \mathrm{Spec}(S^\bullet(\mathcal{I}_Z/\mathcal{I}_Z^2)^*)$ and $c_1(N_{Z/X}) \in \mathbb{E}^{2,1}(Z)$.

We now define a refined fundamental class for any closed subscheme $Z \to X$ and any epimorphism $\mathcal{F}^* \to \mathcal{I}_Z/\mathcal{I}_Z^2$ where $\mathcal{F}$ is a locally free $\mathcal{O}_Z$-module. This more general context, due to Gabber, is the suitable one to prove the basic properties of the Gysin morphism.

Definition 4.2.3 Let $Z \to X$ be a closed immersion defined by a sheaf of ideals $\mathcal{I}_Z$. Let $\mathcal{F}^* \to \mathcal{I}_Z/\mathcal{I}_Z^2$ be an epimorphism of $\mathcal{O}_Z$-modules where $\mathcal{F}$ is locally free and consider the $\mathcal{O}_X$-graded algebra $\mathcal{A} = \bigoplus \mathcal{A}_n$ defined on each degree as the fibre product

$$\begin{array}{ccc} \mathcal{A}_n & \longrightarrow & \mathcal{I}_Z^n \\ \downarrow & & \downarrow \\ S^n\mathcal{F}^* & \longrightarrow & \mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}. \end{array}$$

We define the **modified blow-up** as the projective scheme $B_{Z,\mathcal{F}}X \coloneqq \mathbf{Proj}_X(\mathcal{A})$.

See [Rio07a, 2.2.1.3; 2.2.1.4 and 2.1.5] for a proof of the following properties of the modified blow-up:

Proposition 4.2.4 *Let $\pi\colon B_{Z,\mathcal{F}}X \to X$ be a modified blowing-up:*

1. *The epimorphism $\mathcal{A} \to \bigoplus \mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}$ defines a closed embedding into the classic blow-up $B_{Z,\mathcal{F}}X \hookrightarrow B_ZX$.*
2. *If $\mathcal{F}^* = \mathcal{I}_Z/\mathcal{I}_Z^2$ then $B_{Z,\mathcal{F}}X = B_ZX$ is the classic blow-up B_ZX.*
3. *Denote $U = X - Z$, then $B_{Z,\mathcal{F}}X|_{\pi^{-1}(U)} \simeq U$.*
4. *$\pi^{-1}(Z) = \mathbb{P}(F) = \mathbf{Proj}_Z(S^\bullet\mathcal{F})$.*
5. *For any morphism $p\colon X' \longrightarrow X$ there is a canonical morphism*

$$B_{Z',p^*\mathcal{F}}X' \longrightarrow B_{Z,\mathcal{F}}X \times_X X'$$

which is a nil-immersion.

□

Proposition 4.2.5 *Let $i\colon Z \to X$ be a closed immersion and let $\mathcal{F}^* \to \mathcal{I}_Z/\mathcal{I}_Z^2$ be an epimorphism of $\mathcal{O}_Z$-modules where $\mathcal{F}$ is locally free. Let $B = B_{Z,\mathcal{F}}X$ be the modified blowing up, $\pi\colon B \to X$ the canonical morphism and $P = \mathbb{P}(F)$ the exceptional divisor. Then for any $q \in \mathbb{Z}$ there is a long exact sequence*

$$\cdots \to \mathbb{E}_Z^{p,q}(X) \to \mathbb{E}_P^{p,q}(B) \oplus \mathbb{E}^{p,q}(Z) \to \mathbb{E}^{p,q}(P) \to \mathbb{E}_Z^{p+1,q}(X) \to \cdots$$

Proof: First recall that we call a *cdh*-distinguished square any cartesian square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

such that i is a closed immersion, π is proper and defines an isomorphism $\pi^{-1}(X - Z) \simeq X - Z$. Following [CD09, 3.3.8] any *cdh*-distinguished square gives homotopy bicartesian squares in $\mathbf{SH}(X)$. In particular, for any absolute spectrum $\mathbb{E}$ the diagram in $\mathbf{SH}(X)$

$$\begin{tikzcd} \mathbb{E}_X \arrow[r] \arrow[d] & \pi_*\mathbb{E}_{X'} \arrow[d] \\ i_*\mathbb{E}_Z \arrow[r] & i'_*\pi_*\mathbb{E}_{Z'} \end{tikzcd}$$

is homotopy bicartesian. As a result, we obtain that there is a distinguished triangle

$$\mathbb{E}_X \to \pi_*\mathbb{E}_{X'} \oplus i_*\mathbb{E}_Z \to i_*\pi'_*\mathbb{E}_{Z'} \to \mathbb{E}_X[1].$$

Applying the functor $\mathrm{Hom}_{\mathbf{SH}(X)}(i_*\mathbb{1}_Z(-q),\ \_\ )$ to this triangle in the case $X' = B = B_{Z,\mathcal{F}}X$ and $Z' = P = \mathbb{P}(F)$ we deduce a long exact sequence where we can compute each term:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{SH}(X)}(i_*\mathbb{1}_Z(-q), \mathbb{E}) &= \mathbb{E}_Z^{0,q}(X) \\
\mathrm{Hom}_{\mathbf{SH}(X)}(i_*\mathbb{1}_Z(-q), \pi_*\mathbb{E}_B) &= \mathrm{Hom}_{\mathbf{SH}(B)}(\pi^* i_*\mathbb{1}_Z, \mathbb{E}_B(q)) = \\
&= \mathrm{Hom}_{\mathbf{SH}(B)}(i'_*\pi'^*\mathbb{1}_Z, \mathbb{E}_B(q)) = \mathbb{E}_P^{0,q}(B) \\
\mathrm{Hom}_{\mathbf{SH}(X)}(i_*\mathbb{1}_Z(-q), i_*\mathbb{E}_Z) &= \mathrm{Hom}_{\mathbf{SH}(Z)}(\mathbb{1}_Z, \mathbb{E}_Z(q)) = \mathbb{E}^{0,q}(Z) \\
\mathrm{Hom}_{\mathbf{SH}(X)}(i_*\mathbb{1}_Z(-q), i_*\pi'_*\mathbb{E}_P) &= \mathrm{Hom}_{\mathbf{SH}(Z)}(i^* i_*\mathbb{1}_Z, \pi'_*\mathbb{E}_P(q)) = \\
&= \mathrm{Hom}_{\mathbf{SH}(P)}(\pi'^*\mathbb{1}_Z, \mathbb{E}_P(q)) = \mathbb{E}^{0,q}(P).
\end{aligned}$$

The first equality is the definition of cohomology with coefficients in $\mathbb{E}$, the second is deduced by adjunction followed by base change for proper morphisms and the third one follows from the fact that the functor i_* is fully faithful.

□

Corollary 4.2.6 *With the preceding notations let p, $q \in \mathbb{Z}$, we have a split short exact sequence*

$$0 \longrightarrow \mathbb{E}_Z^{p,q}(X) \longrightarrow \mathbb{E}_P^{p,q}(B) \overset{s}{\leftrightarrows} \mathbb{E}^{2n,q}(P)/\mathbb{E}^{p,q}(Z) \longrightarrow 0.$$

Proof: The preceding long exact sequence may be rewritten as

$$\cdots \to \mathbb{E}^{p,q}_Z(X) \xrightarrow{\pi^*} \mathbb{E}^{p,q}_P(B) \xrightarrow{i^*} \mathbb{E}^{p,q}(P)/\mathbb{E}^{p,q}(Z) \to \cdots .$$

Denote $x = c_1(\mathcal{O}_P(-1))$, by the projective bundle theorem we have

$$\mathbb{E}^{p,q}(P)/\mathbb{E}^{p,q}(Z) = \mathbb{E}^{p-2,q-1}(Z)x \oplus \cdots \oplus \mathbb{E}^{p-2(n-1),q-n+1}(Z)x^{n-1}.$$

Since $i^*(c_1^P(\mathcal{O}_B(-1))) = c_1(\mathcal{O}_P(-1)) = x$ we define the section s by giving its value at the generators $s(x^i) = (c_1^P(\mathcal{O}_B(-1)))^i$ and linearity.

□

4.2.7 Consider once again the notations of Proposition 4.2.5. Recall that Z a is closed subscheme of X, $B \to X$ is a modified blowing-up of X over Z and $P = \pi^{-1}(Z)$. We now construct a distinguished class in $\mathbb{E}_Z(X)$ to define the Gysin morphism.

Although P is not in general of codimension 1, the invertible sheaf $\mathcal{L}_P = \mathcal{O}_B(-1)$ has a canonical trivialization on $B - P$. Therefore we consider the refined Gysin morphism $\mathfrak{p}_{\mathcal{O}_B(-1)}$ and the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{E}^{2n,n}_Z(X) \arrow[r, "\pi^*"] & \mathbb{E}^{2n,n}_P(B) \arrow[r, "i^*"] & \mathbb{E}^{2n,n}(P)/\mathbb{E}^{2n,*}(Z) \arrow[r] & 0 \\
 & & \mathbb{E}^{2n-2,n-1}(P) \arrow[u, "\mathfrak{p}_{\mathcal{O}_B(-1)}"] & &
\end{tikzcd}$$

where $n = \operatorname{rank} \mathcal{F}$. Since $\Sigma_0^{n-1}(-1)^{n+1+i}c_i(F)x^{n-i} = c_n(F) = 0$ in $\mathbb{E}(P)/\mathbb{E}(Z)$, we define

$$Cl^X_{Z,\mathcal{F}} := \Sigma_0^{n-1}(-1)^{n+1+i}c_i(F)x^{n-i-1} \in \mathbb{E}^{2n-2,n-1}(P).$$

Note that

$$\begin{aligned} i^*\mathfrak{p}_{\mathcal{O}_B(-1)}(Cl^X_{Z,\mathcal{F}}) &= i^*(c_1^P(\mathcal{O}_B(-1))(\Sigma_0^{n-1}(-1)^{n+1+i}c_i(F)x^{n-i-1})) \\ &= \Sigma_0^{n-1}(-1)^{n+1+i}c_i(F)x^{n-i} = 0. \end{aligned}$$

Therefore, there exist a unique class $\bar{\eta}^X_{Z,\mathcal{F}} \in \mathbb{E}^{2n,n}_Z(X)$ such that $\pi^*\bar{\eta}^X_{Z,\mathcal{F}} = \mathfrak{p}_{\mathcal{O}_B(-1)}(Cl^X_{Z,\mathcal{F}})$.

Definition 4.2.8 Let $Z \to X$ be a closed subscheme and let $\mathcal{F}^* \to \mathcal{I}_Z/\mathcal{I}_Z^2$ be an epimorphism of $\mathcal{O}_Z$-modules where $\mathcal{F}$ is locally free of rank n. With the preceding notations, we define the **refined fundamental class of** Z

in X **modified by** $\mathcal{F}$ to be the unique class $\bar{\eta}^X_{Z,\mathcal{F}} \in \mathbb{E}^{2n,n}_Z(X)$ such that $\mathfrak{p}_{\mathcal{O}_B(-1)}(Cl^X_{Z,\mathcal{F}}) = \pi^*(\bar{\eta}^X_{Z,\mathcal{F}})$.

In the case $i\colon Z \to X$ is a regular immersion of codimension n and $\mathcal{F}^* = \mathcal{I}_Z/\mathcal{I}_Z^2$ we call this class the **refined fundamental class of** Z **in** X and we denote it $\bar{\eta}^X_Z \in \mathbb{E}^{2n,n}_Z(X)$. We define the **refined Gysin morphism** to be

$$\begin{array}{rccc} \mathfrak{p}_i : & \mathbb{E}^{*,*}(Z) & \to & \mathbb{E}^{*+2n,*+n}_Z(X) \\ & a & \mapsto & a \cdot \bar{\eta}^X_Z \end{array}$$

and the **Gysin morphism** to be $i_*\colon \mathbb{E}^{*,*}(Z) \longrightarrow \mathbb{E}^{*+2n,*+n}(X)$, $a \mapsto i_\flat(a \cdot \bar{\eta}^X_Z)$.

Corollary 4.2.9 *Let $i\colon Z \to X$ be a regular immersion of codimension n, then*

$$i^*\eta^X_Z = c_n(N_{Z/X}) \in \mathbb{E}^{2n,n}(Z)$$

where $N_{Z/X} = \mathrm{Spec}(S^\bullet \mathcal{I}_Z/\mathcal{I}_Z^2)$. In particular, let $V \to X$ be a rank n vector bundle and $s_0\colon X \to V$ be the zero section, then $s_0^*\eta^V_X = c_n(V) \in \mathbb{E}^{2n,n}(X)$.

Proof: Consider the commutative square

$$\begin{tikzcd} P \arrow[r, "i'"] \arrow[d, "\pi'"'] & B_Z X \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X. \end{tikzcd}$$

Note that $\pi'^*\colon \mathbb{E}(Z) \to \mathbb{E}(P)$ is injective. By construction we have

$$(i' \circ \pi)^*\eta^X_Z = i'^*\, Cl^X_{Z,\mathcal{I}_Z/\mathcal{I}_Z^2} = \pi'^* c_n(N_{Z/X})$$

□

Corollary 4.2.10 (Projection formula) *With the preceding notations, the Gysin morphism i_* is a morphism of $\mathbb{E}(X)$-modules. In other words,*

$$a \cdot i_*(b) = i_*(i^*(a) \cdot b) \ \forall\ a \in \mathbb{E}(X)\ ,\ b \in \mathbb{E}(Z).$$

Proof: We have

$$i_*(i^*(a) \cdot b) = i_\flat(i^*(a) \cdot b \cdot \bar{\eta}^X_Z) = a \cdot i_\flat(b \cdot \bar{\eta}^X_Z) = a \cdot i_*(b)$$

where we have used point 6 of Proposition 4.1.7.

□

Corollary 4.2.11 *With the preceding notations, let $r\colon X \to Z$ be a retraction of i. Then i_* is a morphism of $\mathbb{E}(Z)$-modules with respect to r^*. In other words,*

$$i_*(a \cdot b) = r^*(a) \cdot i_*(b) \quad \forall\ a \in \mathbb{E}(Z).$$

In particular, $i_*(a) = r^*(a) \cdot \eta_Z^X$.

Proof: We have

$$r^*(a) \cdot i_*(b) = r^*(a) \cdot i_\flat(b \cdot \bar{\eta}_Z^X) = i_\flat(r^*(a) \cdot b \cdot \bar{\eta}_Z^X)$$

$$= i_\flat(a \cdot b \cdot \bar{\eta}_Z^X) = i_*(a \cdot b)$$

where we have used point 4 and 6 of Proposition 4.1.7. □

Proposition 4.2.12 *The refined fundamental class is stable under base change. In other words, let $p\colon X' \longrightarrow X$ be a morphism of schemes, $Z \to X$ a closed subscheme and $\mathcal{F}^* \to \mathcal{I}_Z/\mathcal{I}_Z^2$ an epimorphism of $\mathcal{O}_Z$-modules where $\mathcal{F}$ is locally free, then*

$$p^*(\bar{\eta}_{Z,\mathcal{F}}^X) = \bar{\eta}_{p^{-1}(Z),p^*(\mathcal{F})}^{X'}.$$

Proof: It is enough to check

$$p^*(Cl_{Z,\mathcal{F}}^X) = Cl_{p^{-1}(Z),p^*\mathcal{F}}^{X'} \in \mathbb{E}^{2r-2,r-1}(\mathbb{P}(p^*\mathcal{F}))$$

where $r = \operatorname{rank} \mathcal{F}$. This follows from the fact that Chern classes are functorial and the induced morphism $\bar{p}\colon B_{p^{-1}Z,p^*\mathcal{F}}X' \to B_{Z,\mathcal{F}}X$ satisfies

$$p^*\mathcal{O}_{B_{p^{-1}Z,p^*\mathcal{F}}X'}(-1) = \mathcal{O}_{B_{Z,\mathcal{F}}X}(-1).$$

□

Proposition 4.2.13 *With the preceding notations, let $\mathcal{F}'^* \longrightarrow \mathcal{F}^*$ be an epimorphism of locally free $\mathcal{O}_Z$-modules of constant rank r', r respectively. Denote $\mathcal{K}^*$ be the kernel and F, F', and K the vector bundles they define. We have the relation*

$$\bar{\eta}_{Z,\mathcal{F}'}^X = c_{r'-r}(K)\bar{\eta}_{Z,\mathcal{F}}^X.$$

Proof: Due to the splitting principle we are reduced to prove the case where $r' = r + 1$. In order to do so it is enough to check that

$$j^* Cl_{Z,\mathcal{F}'}^X = c_1(K) Cl_{Z,\mathcal{F}}^X.$$

By construction there is short exact sequence $0 \to F \to F' \to K \to 0$ of vector bundles. Recall from Theorem 4.1.41 that $c_i(F') = c_i(F) + c_1(K)c_{i-1}(F)$ and therefore

$$\begin{aligned} j^* Cl^X_{Z,\mathcal{F}'} &= j^*((-1)^{r'+1}\Sigma_0^{r'-1}(-1)^i c_i(F')x^{r'-1-i}) \\ &= (-1)^r \Sigma_0^r (-1)^i (c_i(F) + c_1(K)c_{i-1}(F))x^{r-i} \\ &= (-1)^r \Sigma_0^r (-1)^i c_i(F) x^{r-i} \ + (-1)^{r+1} c_1(K)\Sigma_0^{r-1}(-1)^j c_j(F)x^{r-1-j} \\ &= c_1(K)\, Cl^X_{Z,\mathcal{F}}. \end{aligned}$$

□

In Gabber's versatile context of modified blow-up the so called *key formula* (*cf.* [Ful98, 6.7]) and the more general *excess intersection formula* are a direct consequence of the definition by Propositions 4.2.12 and 4.2.13.

Corollary 4.2.14 (Excess intersection formula) *Consider the cartesian square*

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where both i and j are regular immersions of codimension n and m respectively. If $K = \pi'^* N_{Z/X}/N_{P/X'}$ is the excess vector bundle then

$$\pi^* i_*(a) = j_*(c_{n-m}(K)\pi'^*(a)).$$

Moreover, we have the refined version

$$\pi^* \mathfrak{p}_i(a) = \mathfrak{p}_j(c_{n-m}(K)\pi'^*(a)).$$

□

Remark 4.2.15 The preceding properties characterize the Gysin morphism and the refined Gysin morphism for regular immersions due to Corollary 4.2.6. More concretely, if $(\mathbb{E}, c_1)$ is an oriented absolute ring spectrum there exist a unique family of group morphisms $\mathfrak{p}_i \colon \mathbb{E}^{*,*}(Z) \to \mathbb{E}_Z^{*+2d,*+d}(X)$ indexed by regular closed immersions $i \colon Z \to X$ of codimension d such that:

1. If $d = 1$ then $\mathfrak{p}_i(a) = ac_1^Z(L_Z)$.
2. For any blow-up they satisfy the *key formula*, in Corollary 4.2.14 notations
$$\pi^* \mathfrak{p}_i(a) = \mathfrak{p}_j(c^Z_{n-1}(K)\pi'^*(a)).$$

The general Gysin morphism is characterized by analogous conditions (see Theorem 4.2.39 for the complete statement).

4.2.2 Functoriality

In order for the definition of the Gysin morphism to be of any use it has to be functorial. In other words, if $Z \xrightarrow{j} Y \xrightarrow{i} X$ are regular immersions then the morphism $(ij)_*$ should be equal to $i_* j_*$. It is clear that if the classes $\bar{\eta}_Z^X$ and $\bar{\eta}_Y^X \bar{\eta}_Z^Y \in \mathbb{E}_Z(X)$ coincide this readily implies the functoriality.

Theorem 4.2.16 *If $Z \xrightarrow{j} Y \xrightarrow{i} X$ are two regular immersions then*

$$\bar{\eta}_Z^X = \bar{\eta}_Z^Y \cdot \bar{\eta}_Y^X \in \mathbb{E}_Z(X). \tag{4.2}$$

Proof: Let n be the codimension of j and m that of i, that we may assume constant. We split the proof into two parts:

Lemma 4.2.17 *With the preceding notations, if equation (4.2) holds for $m =$ 1 then it holds for any m.*

Proof: Consider $B = B_Y X$ the blow-up of Y in X and denote $P = \mathbb{P}(N_{Y/X})$, $P' = \mathbb{P}(N_{Y/X}|_Z)$. We have the diagram

$$\begin{tikzcd}
P' \arrow[r, "j'"] \arrow[d, "p'"'] & P \arrow[r] \arrow[d, "p"'] & B \arrow[d, "\pi"] \\
Z \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

where both squares are cartesian, $P \to B$ is a regular closed immersion of codimension 1 and j' is of codimension n. Since the morphism

$$\mathbb{E}_Y^{2(n+m),n+m}(X) \xrightarrow{\pi^*} \mathbb{E}_{P'}^{2(n+m),n+m}(B)$$

is injective (*cf.* Corollary 4.2.6) it is enough to check on $\mathbb{E}_{P'}^{2(n+m),n+m}(B)$ the relation. Denote $\mathcal{K}^*$ the kernel of the epimorphism $p'^* \mathcal{I}_Z/\mathcal{I}_Z^2 \to \mathcal{I}_{P'}/\mathcal{I}_{P'}^2$ and K its associated vector bundle. Using that the refined fundamental class are stable under base change (Proposition 4.2.12), the formula from Proposition 4.2.13 and the equation (4.2) for $m = 1$ we get

$$\pi^* \bar{\eta}_Z^X = \bar{\eta}^B_{P', \pi^* N_{Z/X}} = c_{n-1}(K) \bar{\eta}^B_{P'} = c_{n-1}(K) \bar{\eta}^P_{P'} \bar{\eta}^B_P.$$

Now, consider the commutative diagram of vector bundles on P'

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & K & \longrightarrow & j'^*K' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p'^*N_{Y/X} & \longrightarrow & p'^*N_{Z/X} & \longrightarrow & j'^*p^*N_{Y/X} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N_{P'/P} & \longrightarrow & N_{P'/B} & \longrightarrow & j'^*N_{P/B} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where K'^* is the vector bundle associated to the kernel of $p^*\mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{I}_P/\mathcal{I}_P^2$. Taking into account that $j'^*K' \xrightarrow{\sim} K$ and using Proposition 4.2.12 and Proposition 4.2.13 once again we conclude

$$\begin{aligned}
\pi^*\bar{\eta}_Z^X &= c_{n-1}(K')\bar{\eta}_{P'}^P\bar{\eta}_P^B = (c_{n-1}(K)\bar{\eta}_P^B)\bar{\eta}_{P'}^P = \bar{\eta}^{X_Y}_{P,p^*\mathcal{I}_Y/\mathcal{I}_Y^2}\bar{\eta}_{P'}^P \\
&= \pi^*(\bar{\eta}_Y^X)\bar{\eta}_{P'}^P = \pi^*\bar{\eta}_Y^X\pi^*\bar{\eta}_Z^Y.
\end{aligned}$$

□

Lemma 4.2.18 *With the preceding notations, the equation (4.2) is true for $m = 1$.*

Proof: Denote $P = \mathbb{P}(N_{Z/X})$, $P' = \mathbb{P}(N_{Z/Y})$, $n = \operatorname{codim}_Y Z$ and Y_Z, X_Z for the blow-up of Z in Y and X respectively. Consider the commutative diagram

$$\begin{tikzcd}
P' \arrow[r, "h"] \arrow[d, "v"'] & Y_Z \arrow[d, "w"] & \\
P \arrow[r, "g"] \arrow[d] & \pi^{-1}(Y) \arrow[r] \arrow[d] & X_Z \arrow[d, "\pi"] \\
Z \arrow[r, "j"] & Y \arrow[r] & X
\end{tikzcd}$$

where every square is cartesian. Since $\pi^*\colon \mathbb{E}_Z^{2n+2,n+1}(X) \to \mathbb{E}_P^{2n+2,n+1}(X_Z)$ is injective (*cf.* Corollary 4.2.6) it is enough to prove

$$\pi^*\bar{\eta}_Z^X = \pi^*\bar{\eta}_Z^Y\pi^*\bar{\eta}_Y^X \tag{4.3}$$

where $\pi^*\bar{\eta}_Z^X \in \mathbb{E}_P^{2n+2,n+1}(X_Z)$, $\pi^*\bar{\eta}_Z^Y \in \mathbb{E}_P^{2n,n}(\pi^{-1}(Y))$ and $\pi^*\bar{\eta}_Y^X \in \mathbb{E}^{2,1}_{\pi^{-1}(Y)}(X_Z)$.

We make the explicit computations of these two terms. Let $\mathcal{I}_Y^X$ be the sheaf of ideals of Y in X and L_Y its associated line bundle. To begin with,

$$\begin{aligned}\pi^*\bar{\eta}_Y^X &= c_1^{\pi^{-1}Y}(\pi^*L_Y) = c_1^{\pi^{-1}Y}(L_P \otimes L_{Y_Z}) = c_1^{\pi^{-1}Y}(\mathcal{O}_{X_Z}(-1) \otimes L_{Y_Z}) \\ &= c_1^{\pi^{-1}Y}(\mathcal{O}_{X_Z}(-1)) + c_1^{\pi^{-1}Y}(L_{Y_Z}) + c_1^{\pi^{-1}Y}(\mathcal{O}_{X_Z}(-1))c_1^{\pi^{-1}Y}(L_{Y_Z})f\end{aligned}$$

where $f \in \mathbb{E}^{**}(S)[[x,y]]$ is the series given by the formal group law (*cf.* 4.1.36). Therefore, the right hand side of equation (4.3) is the sum of the preceding three terms multiplied by $\pi^*\bar{\eta}_Z^Y$.

We compute each one of those three terms. From now on, we use the notation $u = c_1^P(\mathcal{O}_{X_Z}(-1)) \in \mathbb{E}_P^{2,1}(X_Z)$. For the first term

$$\pi^*\bar{\eta}_Z^Y \cdot c_1^{\pi^{-1}Y}(\mathcal{O}_{X_Z}(-1)) = g^*\pi^*\bar{\eta}_Z^Y \cdot u = c_n(\pi^*N_{Z/Y})u = \mathrm{I}_1$$

where the first equality is due to 6 of Proposition 4.1.7 applied to the map $g_\flat\colon \mathbb{E}_P(X_Z) \to \mathbb{E}_{\pi^{-1}Y}(X_Z)$ and the second one to $g^*\pi^* = \pi^*j^*$ together with Corollary 4.2.9. For the second term

$$\begin{aligned}&\pi^*\bar{\eta}_Z^Y \cdot c_1^{\pi^{-1}Y}(L_{Y_Z}) = \pi^*\bar{\eta}_Z^Y \cdot w_\flat c_1^{Y_Z}(L_{Y_Z}) = v_\flat(w^*\pi^*\bar{\eta}_Z^Y \cdot c_1^{Y_Z}(L_{Y_Z})) \\ &= v_\flat\big((-1)^{n+1}[\sum_{i=0}^{n-1}(-1)^i c_i(N_{Z/Y})c_1(\mathcal{O}_{P'}(-1))^{n-1-i}]c_1^{P'}(\mathcal{O}_{Y_Z}(-1))c_1^{Y_Z}(L_{Y_Z})\big) \\ &\qquad = v_\flat\big((-1)^{n+1}[\sum_{i=0}^{n-1}(-1)^i c_i(N_{Z/Y})c_1^{P'}(\mathcal{O}_{Y_Z}(-1))^{n-i}]c_1^{Y_Z}(L_{Y_Z})) \\ &\qquad\qquad = (-1)^{n+1}[\sum_{i=0}^{n-1}(-1)^i c_i(N_{Z/Y})u^{n-i}c_1(L_{Y_Z})] = \mathrm{I}_2\end{aligned}$$

is due to Proposition 4.1.7 for $w_\flat\colon \mathbb{E}_{Y_Z}(X_Z) \to \mathbb{E}_{\pi^{-1}Y}(X_Z)$, Proposition 4.1.46 and Corollary 4.2.2. For the third and last term

$$\begin{aligned}&\pi^*\bar{\eta}_Z^Y c_1^{\pi^{-1}Y}(L_{Y_Z})c_1^{\pi^{-1}Y}(\mathcal{O}_{X_Z}(-1))f \\ &\qquad = (-1)^{n+1}[\sum_{i=0}^{n-1}(-1)^i c_i(N_{Z/Y})u^{n-i}]c_1(L_{Y_Z})c_1^{\pi^{-1}Y}(\mathcal{O}_{X_Z}(-1))f \\ &\qquad\qquad = (-1)^{n+1}[\sum_{i=0}^{n-1}(-1)^i c_i(N_{Z/Y})u^{n-i+1}c_1(L_{Y_Z})f] = \mathrm{I}_3\end{aligned}$$

where we use the preceding computation.

Consider the short exact sequence

$$0 \to \mathcal{K}^* \to \mathcal{I}_Z^X/\mathcal{I}_Z^{X2} \to \mathcal{I}_Z^Y/\mathcal{I}_Z^{Y2} \to 0$$

With it, we compute the other side of the equation (4.3):

$$\begin{aligned}\pi^*\bar{\eta}_Z^X &= (-1)^n[\sum_{j=0}^{n}(-1)^j c_j(N_{Z/X})c_1(\mathcal{O}_P(-1))^{n-j})]u \\ &= (-1)^n[\sum_{j=0}^{n}(-1)^j(c_j(N_{Z/Y}) + c_{j-1}(N_{Z/Y})c_1(K))u^{n+1-j}]\end{aligned}$$

Note that $\mathcal{I}_{P'}^P = \mathcal{K} \otimes \mathcal{O}_P(-1)$ and therefore $\mathcal{K} = \mathcal{I}_{Y_Z}^{X_Z} \otimes \mathcal{O}_P(1)$ so that $c_1(K) = c_1(L_{Y_Z} \otimes \mathcal{O}_P(-1))$.

$$= \sum_{j=0}^{n}(-1)^{n+j}[c_j(N_{Z/Y}) + c_{j-1}(N_{Z/Y})(c_1(L_{Y_Z}) + c_1(\mathcal{O}_P(-1)) + c_1(L_{Y_Z})c_1(\mathcal{O}_P(-1))f)]u^{n+1-j}$$

Therefore, $\pi^*\bar{\eta}_Z^X$ is the sum of three terms:

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{n+j}[c_j(N_{Z/Y}) + c_{j-1}(N_{Z/Y})c_1(\mathcal{O}_P(-1))]u^{n+1-j} = \\ =&\sum_{j=0}^{n}(-1)^{n+j}c_j(N_{Z/Y})u^{n+1-j} + \sum_{i=0}^{n-1}(-1)^{n+i+1}c_i(N_{Z/Y})u^{n+1-i} = \\ =&c_n(N_{Z/Y})u = \mathrm{I}_1\end{aligned}$$

which is given by Corollary 4.2.9 and the definition of the Chern class,

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{n+j}c_{j-1}(N_{Z/Y})(c_1(L_{Y_Z})))u^{n+1-j} = \\ =&(-1)^{n+1}[\sum_{i=0}^{n-1}(-1)^i c_i(N_{Z/Y})u^{n-i}c_1(L_{Y_Z})] = \mathrm{I}_2\end{aligned}$$

and finally

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{n+j}[c_{j-1}(N_{Z/Y})c_1(L_{Y_Z})c_1(\mathcal{O}_{X_Z}(-1))f]u^{n+1-j} = \\ =&(-1)^{n+1}[\sum_{i=0}^{n-1}(-1)^i c_i(N_{Z/Y})u^{n+1-i}c_1(L_{Y_Z})f] = \mathrm{I}_3.\end{aligned}$$

□

Example 4.2.19 Let $V \to X$ be a vector bundle of rank n and $(\mathbb{E}, c_1)$ be an oriented absolute ring spectrum. The **Thom class** of V is defined to be

$$\mathbf{t}(V) \coloneqq \sum_{i=0}^{n} (-1)^i c_i(V) x^i \in \mathbb{E}^{2n,n}(\bar{V})$$

where $x = c_1(\mathcal{O}_{\bar{V}}(-1))$ and $\bar{V} = \mathbb{P}(V \oplus 1)$. It has being standard in motivic homotopy theory since its beginning to define fundamental classes out of Thom classes. More concretely, denote $s_0 \colon X \to \bar{V}$ the zero section. Its fundamental class was, by definition, $\mathbf{t}(V)$. Therefore, the unicity of Gysin morphisms in the context of regular schemes (*cf.* [Dég14]) proves that for a regular scheme X then

$$\eta_X^{\bar{V}} = \mathbf{t}(V).$$

4.2.20 Let us check that $\mathbf{t}(V)$ coincides with $\eta_X^{\bar{V}}$ for arbitrary schemes. In order to do so we recall some facts of the theory of Thom classes.

For convenience of the reader we recall the definitions. We define the **Thom space** of V as

$$\mathrm{Th}(V) = V/V - 0 \simeq \bar{V}/\mathbb{P}(V).$$

Its cohomology fits into a long exact sequence

$$\ldots \to \mathbb{E}^{**}(\mathrm{Th}(V)) \xrightarrow{\pi^*} \mathbb{E}^{**}(\bar{V}) \to \mathbb{E}^{**}(V) \to \ldots$$

where, from Theorem 4.1.34, the third arrow is always a split epimorphism. Since $\mathbf{t}(V)$ is zero in $\mathbb{E}(\mathbb{P}(V))$, we call the **refined Thom class** to the unique element

$$\bar{\mathbf{t}}(V) \in \mathbb{E}(\mathrm{Th}(V)) \simeq \mathbb{E}_X(\bar{V}) = \mathbb{E}_X(V)$$

such that $\pi^*(\bar{\mathbf{t}}(V)) = \mathbf{t}(V)$. Clearly, proving that $\bar{\mathbf{t}}(V)$ coincides with $\bar{\eta}_X^{\bar{V}}$ is equivalent to proving that $\bar{\mathbf{t}}(V)$ coincides with $\bar{\eta}_X^{\bar{V}}$.

One last technical recall (*cf.* [Dég14] for example): if $0 \to V' \to V \to V'' \to 0$ is exact, the refined Thom classes satisfy

$$\bar{\mathbf{t}}(V) = \bar{\mathbf{t}}(V')\bar{\mathbf{t}}(V_{V'}) \in \mathbb{E}_X(V).$$

Here $\mathbf{t}(V_{V'})$ denotes V considered as a bundle over V' [2] and the product is that of Definition 4.1.6.

Proposition 4.2.21 *Let $V \to X$ be a vector bundle and denote $\bar{V}$ its projective completion. Then*

$$\eta_X^{\bar{V}} = \mathbf{t}(V).$$

[2]The scheme which parametrises the sections of $V \to V''$ is a torsor (of group $\mathrm{Hom}(V'', V')$)and the pullback of the short exact sequence is naturally split there.

Proof: It is clear that this formula is equivalent to its refined counterpart, $\bar{\eta}_X^{\bar{V}} = \bar{\mathbf{t}}(V)$. Due to Theorem 4.2.16 and the previous remark on refined Thom classes it is enough to prove it for the case of a line bundle $V = L$.

In this case $\mathbf{t}(V) = c_1(L) - c_1(\mathcal{O}_{\bar{L}}(-1))$ and $\eta_X^{\bar{L}} = c_1(\mathcal{I}^*)$, where $\mathcal{I}$ stands for the sheaf of ideals of the zero section in $\bar{L}$. This sheaf may be computed explicitly: the composition $\mathcal{O}_{\bar{L}}(-1) \to L \oplus \mathcal{O} \to L$ of the canonical morphism and the projection is an isomorphism out of the zero section, which induces $L^* \otimes \mathcal{O}_{\bar{L}}(-1) \simeq \mathcal{I} \to \mathcal{O}$.

We consider the canonical short exact sequence $0 \to \mathcal{O}_{\bar{L}}(-1) \to L \oplus \mathcal{O} \to Q \to 0$, where Q is the canonical quotient bundle. Taking second exterior product it induces $L = \bigwedge^2(L \oplus \mathcal{O}) = Q \otimes \mathcal{O}_{\bar{L}}(-1)$ so that $Q = L \otimes \mathcal{O}_{\bar{L}}(1) = \mathcal{I}^*$. Therefore we conclude

$$\eta_X^{\bar{L}} = c_1(Q) = c_1(L) - c_1(\mathcal{O}_{\bar{L}}(-1)) = \mathbf{t}(L).$$

□

Let $\mathbb{E}$ be an absolute ring spectrum and $\mathbb{M}$ be an absolute $\mathbb{E}$-module. Note that $\mathbb{M}$ does not have a unit and therefore there are no orientations. As a consequence there are no fundamental nor Chern classes in the $\mathbb{M}$-cohomology. However, $\mathbb{M}(X)$ is an $\mathbb{E}(X)$-module and therefore we can still multiply classes in the $\mathbb{M}$-cohomology by fundamental classes and Chern classes of the $\mathbb{E}$-cohomology. This suffices to define the Gysin morphism.

Definition 4.2.22 Let $i\colon Z \to X$ be a regular immersion. We define in the $\mathbb{M}$-cohomology the **Gysin morphism** i_* and the **refined Gysin morphism** $\mathfrak{p}_i$ to be

$$\begin{array}{cccc} i_*\colon & \mathbb{M}^{p,q}(Z) & \longrightarrow & \mathbb{M}^{p+2n,q+n}(X) \\ & m & \mapsto & m \cdot \eta_Z^X \end{array} \quad , \quad \begin{array}{cccc} \mathfrak{p}_i\colon & \mathbb{M}^{p,q}(Z) & \longrightarrow & \mathbb{M}_Z^{p+2n,q+n}(X) \\ & m & \mapsto & m \cdot \bar{\eta}_Z^X . \end{array}$$

Note that the product is that of paragraph 4.1.11.

Corollaries 4.2.9, 4.2.10, 4.2.14 and 4.2.11 readily allow to conclude the following properties for modules:

Theorem 4.2.23 *Let $\mathbb{E}$ be an oriented absolute ring spectrum, $\mathbb{M}$ be an absolute $\mathbb{E}$-module and $i\colon Z \to X$ be a regular immersion.*

- *Functoriality: Let $j\colon Y \to Z$ be a regular immersion, then $(ij)_* = i_* j_*$.*
- *Projection formula: The Gysin morphism is $\mathbb{E}(X)$-linear. In other words,*

$$a \cdot i_*(m) = i_*(i^*(a) \cdot m) \qquad \forall\ a \in \mathbb{E}(X)\ ,\ m \in \mathbb{M}(Z).$$

Note that an analogue formula also holds for $n \in \mathbb{M}(X)$ and $b \in \mathbb{E}(Z)$.

- *Denote $n = \operatorname{codim}_X Z$, we have*
$$i^* i_*(m) = c_n(N_{Z/X}) \cdot m \quad \forall\, m \in \mathbb{M}^{2n,n}(Z).$$
- *Let $r\colon X \to Z$ be a retraction of i, then the Gysin morphism i_* is $\mathbb{M}(Z)$-linear (with r^*). In other words,*
$$i_*(m) = r^*(m) \cdot \eta_Z^X \quad \forall\, m \in \mathbb{M}(Z).$$
- *Excess intersection formula: Consider a cartesian square*
$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$
where both i and j are regular immersions of codimension n and m respectively. Denote $K = \pi'^* N_{Z/X}/N_{P/X'}$ the excess vector bundle, then
$$\begin{aligned} \pi^* i_*(m) &= j_*(c_{n-m}(K) \cdot \pi'^*(m)) && \text{and} \\ \pi^* \mathfrak{p}_i(m) &= \mathfrak{p}_j(c_{n-m}(K) \cdot \pi'^*(m)) && \forall\, m \in \mathbb{M}(Z). \end{aligned}$$

□

4.2.3 The projective lci case

We construct the Gysin morphism for the projection $p_X\colon \mathbb{P}^n_X \to X$ of a projective space onto its base. We prove afterwards that the Gysin morphism for projective lci morphism, without smoothness assumptions, have all usual properties.

The main reference we have used is [Dég08, §5] where Déglise thoroughly studied the Gysin morphism for projective lci morphisms. However, the reference works on the smooth case and on a general category of premotives satisfying certain axioms, which do not hold for **SH**. Nevertheless, we will see the arguments still hold *mutatis mutandis* in our context.

4.2.24 Denote $\mathbb{E}(\mathbb{P}^n_X)^\vee = \operatorname{Hom}_{\mathbb{E}(X)-\mathbf{mod}}(\mathbb{E}(\mathbb{P}^n_X), \mathbb{E}(X))$ the dual of $\mathbb{E}(\mathbb{P}^n_X)$ in the category of $\mathbb{E}(X)$-modules. Recall that from the projective bundle theorem we have
$$\mathbb{E}(\mathbb{P}^n_X \times_X \mathbb{P}^n_X) \simeq \mathbb{E}(\mathbb{P}^n_X) \otimes_{\mathbb{E}(X)} \mathbb{E}(\mathbb{P}^n_X).$$
Consider the diagonal embedding $\Delta_n\colon \mathbb{P}^n_X \to \mathbb{P}^n_X \times \mathbb{P}^n_X$. The fundamental class of the diagonal $\eta_{\Delta_n} = \eta_{\Delta_n}^{\mathbb{P}^n_X \times \mathbb{P}^n_X} \in \mathbb{E}(\mathbb{P}^n_X) \otimes \mathbb{E}(\mathbb{P}^n_X)$ defines a $\mathbb{E}(X)$-bilinear pairing $g^\vee$ on $\mathbb{E}(\mathbb{P}^n_X)^\vee$, and therefore a polarity
$$\Phi\colon \mathbb{E}(\mathbb{P}^n_X)^\vee \to \mathbb{E}(\mathbb{P}^n_X)\ ,\ \omega \mapsto\ (\omega \otimes 1)(\eta_{\Delta_n}).$$

Proposition 4.2.25 *The polarity Φ defined above is an isomorphism.*

Proof: We proceed by induction on n, the dimension of the projective space. We prove that

$$\eta_{\Delta_n} = \sum_{r,s=0}^{n} a_{r,s} x_n^r \otimes x_n^s \quad \text{where} \quad (a_{r,s}) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & \bullet \\ 0 & \iddots & \iddots & \vdots \\ 1 & \bullet & \cdots & \bullet \end{pmatrix} \tag{4.4}$$

where $x_n = c_1(\mathcal{O}_{\mathbb{P}^n_X}(-1))$. That is to say, $a_{rs} = 0$ if $r + s < n$ and $a_{rs} = 1$ if $r + s = n$.

Denote $i\colon \mathbb{P}^{n-1}_X \to \mathbb{P}^n_X$, we have

$$i_* x^r_{n-1} = i_* i^* x^r_n = x^r_n \cdot i_*(1) = x^{r+1}_n.$$

Consider the cartesian diagram

$$\begin{tikzcd} \mathbb{P}^{n-1}_X \arrow[r] \arrow[d, "i"'] & \mathbb{P}^{n-1}_X \times \mathbb{P}^n_X \arrow[d, "i\times 1"] \\ \mathbb{P}^n_X \arrow[r, "\Delta_n"] & \mathbb{P}^n_X \times \mathbb{P}^n_X. \end{tikzcd}$$

Since $i \times 1$ is transversal to Δ_n the excess intersection formula for the regular immersion gives that the class $(i^* \otimes 1)(\eta_{\Delta_n})$ is the fundamental class of the diagonal of $\mathbb{P}^{n-1}_X$ seen in $\mathbb{P}^{n-1}_X \times \mathbb{P}^n_X$. Since

$$(i^* \otimes 1)(\eta_{\Delta_n}) = \sum_{r=0}^{n-1} \sum_{s=0}^{n} a_{rs} x^r_{n-1} \otimes x^s_n$$
$$(1 \otimes i_*)(\eta_{\Delta_{n-1}}) = \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} a'_{rs} x^r_{n-1} \otimes x^{s+1}_n$$

we conclude $a_{rs} = a'_{r(s-1)}$ for $r = 0, \ldots, n-1$ and $s = 0, \ldots, n-1$ and $a_{r0} = 0$ for $r = 0, \ldots n-1$. Therefore, provided that $r < n$, we conclude by the induction hypothesis that $a_{rs} = 0$ if $r + s < n$ and $a_{rs} = 1$ if $r + s = n$. By symmetry we have the same result if $s < n$ which concludes the proof.

□

Definition 4.2.26 Let $\mathbb{E}$ be an oriented absolute ring spectrum and denote $p_X\colon \mathbb{P}^n_X \to X$ the natural projection, we define the **direct image** of p_X to be

$$p_{X*}\colon \mathbb{E}(\mathbb{P}^n_X) \overset{\Phi^{-1}}{\simeq} \mathbb{E}(\mathbb{P}^n_X)^\vee \xrightarrow{(p^*)^\vee} \mathbb{E}(X)^\vee = \mathbb{E}(X).$$

Remark 4.2.27 Denote $p\colon \mathbb{P}^n \to S$, since the square

$$\begin{tikzcd}
\mathbb{P}^n_X \arrow[r] \arrow[d] & \mathbb{P}^n_X \times \mathbb{P}^n_X \arrow[d, "\pi"] \\
\mathbb{P}^n \arrow[r] & \mathbb{P}^n \times \mathbb{P}^n
\end{tikzcd}$$

is transversal, the excess intersection formula gives $\eta_{\Delta_{nX}} = \pi^*\eta_{\Delta_n}$ and we have

$$p_{X*} = p_* \otimes 1_X.$$

Remark 4.2.28 Denote M_n the matrix from (4.4), the matrix of the bilinear pairing $g^\vee$ in the dual of the standard basis $\{1, x = c_1(\mathcal{O}^n_{\mathbb{P}}(-1)), \ \dots\ , x^n\}$. Through the polarity Φ, it defines as well a bilinear pairing g on $\mathbb{E}(\mathbb{P}^n)$ which, in the standard basis, has the matrix

$$M_n^{-1} = \begin{pmatrix} \bullet & \cdots & \bullet & 1 \\ \vdots & \iddots & \iddots & 0 \\ \bullet & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Recall that the identification $\mathbb{E}(S)^\vee \to \mathbb{E}(S)$ is given by $\omega \mapsto \omega(1)$. Therefore we conclude that for all $a \in \mathbb{E}(\mathbb{P}^n)$ we have that

$$p_*(a) = < a, 1 >_g$$

where $< \_ , \_ >_g$ denotes the product with g. If we have $\pi\colon X \to S$, then $p_{X*}(b) = (b, 1)_{\pi^* g}$.

Lemma 4.2.29 *Let $\Delta_n\colon \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n = \mathbb{P}^n_{\mathbb{P}^n}$ be the diagonal embedding. Then $p_{\mathbb{P}^n *}\Delta_* = 1_{\mathbb{P}_n}$.*

Proof: Denote $P = \mathbb{P}^n$ and $\pi\colon P \to S$ the structural morphism. Recall from the previous remark that

$$p_{\mathbb{P}^n *}(a) = < a, 1 >_{\pi^* g}$$

for $a \in \mathbb{E}(\mathbb{P}^n_P)$. In particular, if we denote η_Δ the fundamental class of the diagonal $\mathbb{P}^n_P \to \mathbb{P}^n_P \times \mathbb{P}^n_P$ we directly obtain

$$p_{\mathbb{P}^n *}(\Delta_*(1)) = < \eta_\Delta, 1 >_{\pi^* M_n^{-1}} = < \pi^*\eta_{\Delta_n}, 1 >_{\pi^* g} = 1$$

since $\pi^* g$ is represented by $\pi^* M_n$. Denote $x = c_1(\mathcal{O}^n_{\mathbb{P}}(-1))$. We conclude by observing that

$$p_{\mathbb{P}^n *}(\Delta_*(x^j)) = p_{\mathbb{P}^n *}(\Delta_*(\Delta^*(1 \otimes x^j)) = p_{\mathbb{P}^n *}(1 \otimes x^j) \cdot p_{\mathbb{P}^n *}(\Delta_*(1)) = 1 \otimes x^j$$

since $p_{\mathbb{P}^n *} = p_* \otimes 1_{\mathbb{P}^n}$.

□

Corollary 4.2.30 *Let $s\colon X \to \mathbb{P}^n_X$ be a section of $p_X\colon \mathbb{P}^n_X \to X$. Then $p_{X*}s_* = 1_X$.*

Proof: Note that preceding lemma also holds for a projective space over a general base $\mathbb{P}^n_X$. Consider the cartesian squares

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "s"'] & \mathbb{P}^n \times X \arrow[r, "p_X"] \arrow[d, "1_{\mathbb{P}^n}\times s"] & X \arrow[d, "s"] \\
\mathbb{P}^n_X \arrow[r, "\Delta"] & \mathbb{P}^n \times \mathbb{P}^n \times X \arrow[r, "p_{\mathbb{P}^n_X}"] & \mathbb{P}^n_X.
\end{tikzcd}$$

We may apply Corollary 4.2.14 to the left square so that $s^*s_* = \Delta_*(1_{\mathbb{P}^n_X} \times s)^*$. From Definition 4.2.26 we also have that $p_{X*}(1_{\mathbb{P}^n_X} \times s)^* = p_{\mathbb{P}^n_X *}$. Together with the previous lemma we deduce $s^* = p_*s_*s^*$. Since s^* is surjective we conclude that $p_*s_* = 1_X$.

□

Remark 4.2.31 We have proved the commutativity of the following triangle:

$$\begin{tikzcd}
\mathbb{E}(\mathbb{P}^n) \arrow[r, "\Delta_*"] \arrow[dr, "1_{\mathbb{P}^n}"'] & \mathbb{E}(\mathbb{P}^n) \otimes \mathbb{E}(\mathbb{P}^n) \arrow[d, "p_{\mathbb{P}^n *}=p_*\otimes 1_S"] \\
 & \mathbb{E}(\mathbb{P}^n).
\end{tikzcd}$$

Note that $p_* \in \mathbb{E}(\mathbb{P}^n)^\vee$. Since the polarity Φ of 4.2.24 is an isomorphism, p_* is totally determined by the fact that its image through Φ is

$$\Phi(p_*) = (p_* \otimes 1_{\mathbb{P}^n})(\eta_\Delta) = 1.$$

Lemma 4.2.32 *Let $i\colon Z \to X$ be a regular immersion and consider the cartesian diagram*

$$\begin{tikzcd}
\mathbb{P}^n_Z \arrow[r, "k"] \arrow[d, "p_Z"'] & \mathbb{P}^n_X \arrow[d, "p_X"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

Then $k_* = (1_{\mathbb{P}^n} \times i)_* = 1_{\mathbb{P}^n} \otimes i_*$ and $p_{X*}k_* = i_*p_{Z*}$.

Proof: For the first claim, applying the excess intersection formula and the projective bundle theorem (Corollary 4.2.14, Theorem 4.1.34) we get that the diagram

$$\begin{tikzcd}
\mathbb{E}(\mathbb{P}^n) \otimes_{\mathbb{E}(S)} \mathbb{E}(Z) \arrow[r, "k_*"] & \mathbb{E}(\mathbb{P}^n) \otimes_{\mathbb{E}(S)} \mathbb{E}(X) \\
\mathbb{E}(Z) \arrow[r, "i_*"] \arrow[u, "p_Z^*"] & \mathbb{E}(X) \arrow[u, "p_X^*"']
\end{tikzcd}$$

commutes so that $k_*(1 \otimes a) = 1 \otimes i_*(a)$ for $a \in \mathbb{E}(Z)$. Applying the projection formula from Corollary 4.2.10 we get that $k_*(b \otimes 1) = (b \otimes 1) \cdot k_*(1 \otimes 1) = b \otimes \eta_Z^X$ for $b \in \mathbb{E}(\mathbb{P}^n)$ so we conclude $k_* = (1_{\mathbb{P}^n} \times i)_* = 1_{\mathbb{P}^n} \otimes i_*$. From here and the previous definition the formula $p_{X*}k_* = i_* p_{Z*}$ follows.

□

Theorem 4.2.33 *Consider a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "i"'] & \mathbb{P}^n_X \arrow[d, "p"] \\ \mathbb{P}^m_X \arrow[r, "q"] & X \end{tikzcd}$$

where i and k are regular immersions of codimension r and s respectively and p and q are the natural projections. Then, $p_* k_* = q_* i_*$.

Proof: Consider the following commutative diagram:

$$\begin{tikzcd} & & \mathbb{P}^n_X \arrow[dr, "p"] & \\ Y \arrow[urr, bend left, "k"] \arrow[r, "v"] \arrow[drr, bend right, "i"'] & \mathbb{P}^n_X \times_X \mathbb{P}^m_X \arrow[ur, "q'"] \arrow[dr, "p'"'] & & X. \\ & & \mathbb{P}^m_X \arrow[ur, "q"'] & \end{tikzcd}$$

Since it is clear that $p_* q'_* = q_* p'_*$ it is enough to prove that $p'_* v_* = i_*$. For that case, denote $T = \mathbb{P}^m_X$, $v = \rho \times i$ where $\rho \colon Y \to \mathbb{P}^n$ and consider

$$\begin{tikzcd} Y \arrow[dr, "s"] \arrow[drr, bend left, "v"] \arrow[ddr, equal] & & \\ & \mathbb{P}^n_Y \arrow[r, "l"'] \arrow[d, "\pi"] & \mathbb{P}^n_T \arrow[d, "p'"] \\ & Y \arrow[r, "i"] & T \end{tikzcd}$$

where i, $s = \rho \times 1_Y$, l and v are regular immersions. By the functoriality of Theorem 4.2.16 we have $v_* = l_* s_*$ and by the previous Lemma 4.2.32 we also have $\pi_* s_* = 1_Y$ and $p'_* l_* = i_* \pi_*$. Considering all together we conclude

$$p'_* v_* = p'_* l_* s_* = i_* \pi_* s_* = i_*.$$

□

Definition 4.2.34 We define an X-scheme $Y \to X$ to be a **local complete intersection** (**lci**) if it locally admits a factorization by a regular immersion into $\mathbb{A}^n_X$ ([SGA6, VIII 1.1]). In [SGA6, VIII 1.2] it is proved that a projective lci morphism $f\colon Y \to X$ admits a factorization of the form $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ where i is a regular closed immersion and p is the canonical projection.

Let $f\colon Y \to X$ be a projective lci morphism and $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} Y$ be a factorization, we define the **direct image** of f as $f_* \coloneqq p_* i_*$ (by Theorem 4.2.33 it does not depend on the choice of factorization).

Finally, let us prove the main properties of the Gysin morphism (Theorem 4.2.16, Corollaries 4.2.10 and 4.2.14) once again in this context. They are a direct consequence of the definition and the case of regular immersions.

Theorem 4.2.35 (Functoriality) *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be two projective lci morphism, then*

$$(f \circ g)_* = f_* \circ g_*.$$

Proof: Consider factorizations $Z \xrightarrow{j} \mathbb{P}^m_X \xrightarrow{q} X$ of $f \circ g$ and $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ of f. We can compute explicitly the base change

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "\pi'"'] & \mathbb{P}^m_X \times \mathbb{P}^n_X \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & \mathbb{P}^n_X \end{tikzcd}$$

as $Y' = (\mathbb{P}^m_X \times_X \mathbb{P}^n_X) \times_{\mathbb{P}^n_X} Y = \mathbb{P}^m_Y$. If we denote $j = v \times (fg)\colon Z \to \mathbb{P}^n \times X$ and consider $k = v \times g\colon Z \to \mathbb{P}^n \times Y$ it fits into a commutative diagram

$$\begin{tikzcd} & & & \mathbb{P}^m_X \arrow[dddrrr, bend left, "q"] & & & \\ & & & \mathbb{P}^m_X \times \mathbb{P}^n_X \arrow[u, "p'"'] \arrow[drr, "\pi"] & & & \\ & \mathbb{P}^m_Y \arrow[urr, "i'"] \arrow[drr, "\pi'"] & & & & \mathbb{P}^n_X \arrow[dr, "p"] & \\ Z \arrow[uuurrr, bend left, "j"] \arrow[ur, "k"] \arrow[rrr, "g"] & & & Y \arrow[urr, "i"] \arrow[rrr, "f"] & & & X. \end{tikzcd}$$

The preceding lemmas allow to conclude

$$f_* g_* = p_* i_* \pi'_* k_* = p_* \pi_* i'_* k_* = q_* p'_* i'_* k_* = q_* j_* = (fg)_*.$$

□

Proposition 4.2.36 (Excess intersection formula) *Consider a cartesian square*

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where f and g are projective lci morphisms of codimension n and m respectively. Choose a factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ of f and let $K = q^* N_{Y/\mathbb{P}^n_X}/N_{Y'/\mathbb{P}^n_{X'}}$. Then

$$p^* f_*(a) = g_*(c_{n-m}(K) \cdot q^*(a)) \quad \forall\ a\ \in\ \mathbb{E}(Y).$$

Proof: Recall the factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$. Changing base on the regular immersion we get a cartesian diagram

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "q"'] & \mathbb{P}^n_{X'} \arrow[d, "p'"] \\ Y \arrow[r, "i"] & \mathbb{P}^n_X \end{tikzcd}$$

which has the same excess bundle K and where j and i are regular immersions. Therefore we can apply Corollary 4.2.14 to obtain that for any $a \in \mathbb{E}(Y)$ the relation

$$\pi^* i_*(a) = j_*(c_{n-m}(K) \cdot q^*(a))$$

holds. We conclude by remarking that if we consider the diagram

$$\begin{tikzcd} \mathbb{P}^n_{X'} \arrow[r, "\pi'"] \arrow[d, "p'"'] & X' \arrow[d, "p"] \\ \mathbb{P}^n_X \arrow[r, "\pi"] & X \end{tikzcd}$$

then $p^* \pi_* = \pi'_* p'^*$.

□

Remark 4.2.37 Recall that the definition of K in the previous proposition does not depend on the choice of factorization (*cf.* [Ful98, 6.6]).

Proposition 4.2.38 (Projection formula) *Let $f\colon Y \to X$ be a projective lci morphism, then f_* is a morphism of $\mathbb{E}(Y)$-modules. In other words,*

$$f_*(f^*(a) \cdot b) = a \cdot f_*(b) \quad \forall\ a \in \mathbb{E}(X)\ ,\ b \in \mathbb{E}(Y).$$

Proof: Consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\gamma_f"'] & X \arrow[d, "\Delta"] \\ Y \times X \arrow[r, "f \times 1_X"] & X \times X \end{tikzcd}$$

where γ_f denotes the graphic of f and Δ denotes the diagonal. Since Δ is transversal to $f \times 1_X$ we may apply the excess intersection formula: for any $a \in \mathbb{E}(X)$ and $b \in \mathbb{E}(Y)$ we have that

$$\Delta^*(f \times 1_X)_*(b \times a) = \Delta^*((f_* b) \times a) = f_*(b) \cdot a$$

equals

$$f_* \gamma_f^*(b \times a) = f_*(b \cdot f^*(a)).$$

□

As a result of the construction we can characterize direct images. The result is a reword of [Pan09, 4.1.4] in the context of stable homotopy theory, following [Dég14, 3.3.1].

Theorem 4.2.39 *Let $(\mathbb{E}, c_1)$ be an oriented absolute spectrum, there exist a unique way of assign for any projective lci morphisms $f\colon Y \to X$ a group morphism $f_*\colon \mathbb{E}(Y) \to \mathbb{E}(X)$ satisfying the following properties:*

1. *Functoriality: $(fg)_* = f_* g_*$.*
2. *Normalization: For regular immersions $i\colon Y \to X$ of codimension one they satisfy $i_*(a) = i_\flat(a \cdot c_1^Y(L_Y))$.*
3. *Key formula: If $i\colon Y \to X$ is a regular immersion of codimension n, $\pi^*\colon B_Y X \to X$ is the blowing-up of Y in X with exceptional divisor $j : \mathbb{P}(N_{Y/X}) \to B_Y X$, we have $\pi^* i_*(a) = j_*(c_{n-1}(K) \cdot \pi'^*(a))$.*
4. *Projection formula: They are $\mathbb{E}(X)$-linear, i.e., $f_*(f^*(a) \cdot b) = a \cdot f_*(b)$ for $a \in \mathbb{E}(X)$ and $b \in \mathbb{E}(Y)$.*

When considering regular immersions, properties 2 and 3 characterize them.

Proof: The functoriality property reduces the proof to the case of regular immersions and the projection of a projective space onto its base. The case of closed immersions follows directly from properties 2, 3 and the long exact sequence of the blow-up (*cf.* Corollary 4.2.6).

For the projection of a projective space we apply the excess formula to the commutative square

$$\begin{tikzcd} \mathbb{P}^n_X \arrow[r, "p_X"] \arrow[d, "\pi_1"'] & X \arrow[d] \\ \mathbb{P}^n \arrow[r, "p"] & S. \end{tikzcd}$$

Together with the projective bundle theorem 4.1.34 and the projection formula we obtain that $p_{X*} = p_* \otimes 1_X$. The functoriality property implies the commutativity of the triangle

$$\begin{tikzcd}
\mathbb{E}(\mathbb{P}^n) \arrow[r, "\Delta_*"] \arrow[dr, "1_{\mathbb{P}^n_X}"'] & \mathbb{E}(\mathbb{P}^n) \otimes \mathbb{E}(\mathbb{P}^n) \arrow[d, "p_*"] \\
& \mathbb{E}(\mathbb{P}^n).
\end{tikzcd}$$

The argument from Remark 4.2.31 shows that $p_* \in \mathbb{E}(\mathbb{P}^n)^\vee$ is uniquely determined by its image through the polarity, which is $\Phi(p_*) = (p_* \otimes 1)(\eta_\Delta) = 1$. □

As in the case of regular immersion, the direct image for absolute ring spectra induces a natural direct image for absolute modules.

Definition 4.2.40 Let $\mathbb{M}$ be an absolute $\mathbb{E}$-module and $p_X \colon \mathbb{P}^n_X \to X$ be the natural projection. We define the **direct image** p_{X*} in the $\mathbb{M}$-cohomology as the morphism

$$\begin{tikzcd}
\mathbb{M}(\mathbb{P}^n_X) \arrow[r, dashed, "p_{X*}"] \arrow[d, equal] & \mathbb{M}(X) \arrow[d, equal] \\
\mathbb{E}(\mathbb{P}^n) \otimes_{\mathbb{E}(S)} \mathbb{M}(X) \arrow[r, "p_* \otimes 1"] & \mathbb{E}(S) \otimes_{\mathbb{E}(S)} \mathbb{M}(X).
\end{tikzcd}$$

Let $f \colon Y \to X$ be a projective lci morphism and $f = pi$ be a factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ where i is a regular immersion and p is the natural projection. We define the direct image of f as $f_* := p_* i_*$.

We summarize the main properties of the direct image for the case of modules which follow directly from Theorem 4.2.35 and Propositions 4.2.36 and 4.2.38.

Theorem 4.2.41 *Let $f \colon Y \to X$ be a projective lci morphism, $\mathbb{E}$ be an absolute oriented ring spectrum and $\mathbb{M}$ be an absolute $\mathbb{E}$-module:*

- *Functoriality: If $g \colon Z \to Y$ is another projective lci morphism then*

$$(fg)_* = f_* g_* \colon \mathbb{M}(Z) \longrightarrow \mathbb{M}(X).$$

- *Projection formula: f_* is a morphism of $\mathbb{E}(X)$-modules. That is to say,*

$$f_*(f^*(a) \cdot m) = a \cdot f_*(m) \quad \forall\ a \in \mathbb{E}(X)\ ,\ m \in \mathbb{M}(Y).$$

- *Excess intersection formula: Consider a cartesian square*

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where f and g are projective lci morphisms of codimension r and s respectively. Choose a factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ of f and denote $K = q^* N_{Y/\mathbb{P}^n_X} / N_{Y'/\mathbb{P}^n_{X'}}$. Then

$$p^* f_*(m) = g_*(c_{r-s}(K) \cdot q^*(m)) \quad \forall\ m \ \in \ \mathbb{M}(Y).$$

□

4.3 Motivic Riemann-Roch

We devote this section to prove the motivic Riemann-Roch theorem in the context of the algebraic stable homotopy category as in [Dég14]. We follow the same ideas of Chapter 1. The morphism of cohomology theories is replaced in the general setting by a morphism of oriented ring spectra $\varphi\colon (\mathbb{E}, c_1) \to (\mathbb{F}, \bar{c}_1)$. For clarity in the exposition, overlined morphisms and elements will refer to the $\mathbb{F}$-cohomology. For example, we denote the orientation $\bar{c}_1 \in \mathbb{F}^{2,1}(\mathbb{P}^\infty)$.

The following result is the analogue of Panin's lemma.

Theorem 4.3.1 *Let $\varphi\colon (\mathbb{E}, c_1) \to (\mathbb{F}, \bar{c}_1)$ be a morphism of oriented absolute spectra such that $\varphi_{\mathbb{P}^\infty}(c_1) = \bar{c}_1$ and let $f\colon Y \to X$ be a projective lci, then the diagram*

$$\begin{tikzcd} \mathbb{E}(Y) \arrow[r, "f_*"] \arrow[d, "\varphi_Y"'] & \mathbb{E}(X) \arrow[d, "\varphi_X"] \\ \mathbb{F}(Y) \arrow[r, "\bar{f}_*"] & \mathbb{F}(X) \end{tikzcd}$$

commutes. In other words, for $a \in \mathbb{E}(Y)$

$$\varphi_X(f_*(a)) = \bar{f}_*(\varphi_Y(a)).$$

Proof: Following the standard approach, it is enough to check the theorem for a regular immersion $i\colon X \to \mathbb{P}^n_Y$ and the projection $p\colon \mathbb{P}^n_Y \to Y$.

Lemma 4.3.2 (Regular immersions) *Theorem 4.3.1 holds for a regular immersion $i\colon Z \to X$.*

Proof: We split the proof in two parts:

Lemma 4.3.3 *Let $i\colon Z \to X$ be a regular immersion, if Theorem 4.3.1 holds for the zero section $s_0\colon Z \to \mathbb{P}(1 \oplus N_{Z/X}) = \overline{N}$ of the projective closure of the normal bundle then it also holds for i.*

Proof: Consider the deformation to the projective closure of the normal bundle. That is to say, consider the commutative diagram

$$\begin{tikzcd} \overline{N} \arrow[r, "i_0"] & X' & X \arrow[l, "i_1"'] \\ Z \arrow[u, "s_0"] \arrow[r] & \mathbb{A}^1_Z \arrow[u, "\iota"] & Z \arrow[l] \arrow[u, "i"'] \end{tikzcd} \tag{4.5}$$

where $X' = B_{Y\times\{0\}}\mathbb{A}^1_X$. For $U = X' - \mathbb{A}^1_Z$, taking $\mathbb{E}$-cohomology gives

$$\begin{tikzcd} & \mathbb{E}(U) \\ \mathbb{E}(\overline{N}) \arrow[d, shift left, "p_*"] & \mathbb{E}(X') \arrow[l, "i_0^*"'] \arrow[u, "j^*"'] \\ \mathbb{E}(Z) \arrow[u, shift left, "s_*"] & \mathbb{E}(\mathbb{A}^1_Z) \arrow[l, "\sim"'] \arrow[u] \end{tikzcd}$$

where the middle column is exact and s_0 is injective (since p is a retract). Chasing the diagram we have that if $a \in \mathbb{E}(X')$ has $j^*(a) = 0$ and $i_0^*(a) = 0$ then $a = 0$. Now consider the commutative diagram

$$\begin{tikzcd} & \mathbb{F}(U) & \\ \mathbb{F}(\overline{N}) & \mathbb{F}(X') \arrow[l] \arrow[r] \arrow[u, "j^*"'] & \mathbb{F}(X) \\ \mathbb{E}(Z) \arrow[u, "\Psi_1"] & \mathbb{E}(\mathbb{A}^1_Z) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\Psi_2"] & \mathbb{E}(Z) \arrow[u, "\Psi_3"'] \end{tikzcd}$$

where the vertical arrows are the difference of the morphism that Theorem 4.3.1 states that coincide: $\Psi_1 = \bar{f}_*\varphi_Z - \varphi_{\bar{N}} f_*$, $\Psi_2 = \bar{f}_*\varphi_{\mathbb{A}^1_Z} - \varphi_{X'} f_*$ and $\Psi_3 = \bar{f}_*\varphi_Z - \varphi_X f_*$. The morphism Ψ_1 is zero by hypothesis. Taking into account that $\operatorname{Ker} j^* \cap \operatorname{Ker} i_0^* = 0$ we have that Ψ_2 is also zero since $\bar{j}^*\Psi_2 = 0$. We conclude that Ψ_3 is also zero.

□

Lemma 4.3.4 *Theorem 4.3.1 holds for zero section of the projective closure* $s\colon X \to \mathbb{P}(1 \oplus E) = \overline{E}$ *of any vector bundle* E.

Proof: Let $E = L$ be a line bundle. Note that $s^*\colon \mathbb{E}(L) \to \mathbb{E}(X)$ is surjective and that $\varphi_L(s_*(1)) = \varphi_L(c_1(L)) = \bar{c}_1(L)$. Let $a \in \mathbb{E}(X)$ where for $a = s^*(b)$, then

$$\varphi_L(s_*(a)) = \varphi_L(s_*(s^*(b))) = \varphi_L(b \cdot s_*(1)) = \varphi_L(b) \cdot \bar{s}_*(1) = \bar{s}_*(\varphi_L(a))$$

so the lemma holds for line bundles.

In the general case, due to the splitting principle, we may assume that there exist a flag $E_1 \subset E_2 \subset \cdots \subset E_n = E$ of vector bundles such that E_i/E_{i-1} are line bundles for all i. Therefore the theorem holds for $X \to \bar{E}_1$ and $\bar{E}_i \to \bar{E}_{i+1}$ for all i. Then it also holds for $s\colon X \to \bar{E}$.

□

Lemma 4.3.5 (Projection) *Theorem 4.3.1 holds for the canonical projection* $p\colon \mathbb{P}^n_X \to X$.

Proof: Applying Theorem 4.3.1 to the diagonal embedding $\Delta_n\colon \mathbb{P}^n_X \to \mathbb{P}^n_X \times \mathbb{P}^n_X$ we obtain that $\varphi_{\mathbb{P}^n_X \times \mathbb{P}^n_X}$ preserves the fundamental class of the diagonal. Recall from Definition 4.2.26 that $p_* = \Phi^{-1}(p^*)^\vee$ where $\Phi\colon \mathbb{E}(\mathbb{P}^n_X) \xrightarrow{\sim} \mathbb{E}(\mathbb{P}^n_X)^\vee$ is the polarity defined by the fundamental class of the diagonal (*cf.* Paragraph 4.2.24). Since φ commute with inverse images the diagram

$$\begin{tikzcd}
\mathbb{E}(\mathbb{P}^n_X) \arrow[r, "\Phi^{-1}"] \arrow[d, "\varphi_{\mathbb{P}^n_X}"'] & \mathbb{E}(\mathbb{P}^n_X)^\vee \arrow[r, "(p^*)^\vee"] \arrow[d, "\varphi^\vee_{\mathbb{P}^n_X}"] & \mathbb{E}(X)^\vee \simeq \mathbb{E}(X) \arrow[d, "\varphi_X"] \\
\mathbb{F}(\mathbb{P}^n_X) \arrow[r, "\bar{\Phi}^{-1}"] & \mathbb{F}(\mathbb{P}^n_X)^\vee \arrow[r, "(\bar{p}^*)^\vee"] & \mathbb{F}(X)^\vee \simeq \mathbb{F}(X)
\end{tikzcd}$$

is made of commutative squares.

□

Lemma 4.3.6 (Change of direct image) *Let* $(\mathbb{E}, c_1)$ *be an oriented absolute ring spectrum. Let* $c_1^{\mathrm{new}} = G(c_1) \cdot c_1$ *be a new orientation (cf. Proposition 4.1.42) and denote* Td_φ *the multiplicative extension of* $G^{-1} \in \mathbb{E}(S)[[t]]$ *(cf. Corollary 4.1.43). Let* $f\colon Y \to X$ *be a projective lci morphism and denote* $T_f = i^*T_{\mathbb{P}^n_Y} - N_i \in K^0(Y)$ *the virtual tangent bundle of* $f = p \circ i$*, then*

$$f_*^{\mathrm{new}}(a) = f_*\big(\mathrm{Td}_\varphi(T_f) \cdot a\big) \quad \forall\ a\ \in \mathbb{E}(Y).$$

Proof: Since the case of the canonical projection $p\colon \mathbb{P}^n_X \to X$ is immediate, it is enough to prove the formula for a regular immersion $i\colon Y \to X$. The we have to check that the family of morphism $i_*^{\mathrm{new}}\colon \mathbb{E}^{*,*}(Y) \to \mathbb{E}^{2d+*,d+*}(X)$, $a \mapsto i_*\big(\mathrm{Td}(-N_{Y/X})\cdot a\big)$ satisfy the properties 2 and 3 of Theorem 4.2.39. The first one follows from directly from the definition of Todd class. For the key formula consider Corollary **??** notations and recall that we denote the canonical quotient bundle by $K = \pi'^* N_{Y/X}/\mathcal{O}_P(-1)$. We then have

$$\begin{aligned}\pi' i_*^{\mathrm{new}}(a) = \pi^* i_*\big(\mathrm{Td}(-N_{Y/X})\cdot a\big) &= j_*\big(c_{d-1}(K)\cdot \mathrm{Td}(-K-\mathcal{O}_P(.-1))\cdot \pi^*(a)\big)\\ &= j_*^{\mathrm{new}}\big(c_{d-1}^{\mathrm{new}}(K)\cdot \pi^*(a)\big)\end{aligned}$$

where the last equality comes from Remark 4.1.44.

□

Theorem 4.3.7 (Motivic Riemann-Roch) *Let $\varphi\colon (\mathbb{E}, c_1) \to (\mathbb{F}, \bar{c}_1)$ be a morphism of oriented absolute spectra. Denote $G \in \mathbb{F}[S][[t]]$ the series such that $\varphi(c_1) = G(\bar{c}_1)\cdot \bar{c}_1 \in \mathbb{F}(\mathbb{P}^\infty_S)$ and Td_φ be the multiplicative extension of G^{-1}. Let $f\colon Y \to X$ be a projective lci where $f = p\circ i$ and denote $T_f = i^*T_p - N_i \in K^0(Y)$ the virtual tangent bundle. Then the diagram*

$$\begin{tikzcd} \mathbb{E}(Y) \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}_\varphi(T_f)\varphi_Y"'] & \mathbb{E}(X) \arrow[d, "\varphi_X"] \\ \mathbb{F}(Y) \arrow[r, "\bar{f}_*"] & \mathbb{F}(X) \end{tikzcd}$$

commutes. In other words, for $a \in \mathbb{E}(Y)$ we have

$$\varphi_X(f_*(a)) = \bar{f}_*\big(\mathrm{Td}_\varphi(T_f)\cdot \varphi_Y(a)\big).$$

Proof: We define $\bar{c}_1^{\mathrm{new}} = \varphi_{\mathbb{P}^\infty}(c_1) = G(\bar{c}_1)\cdot \bar{c}_1$ that gives a direct image $\bar{f}_*^{\mathrm{new}}$ satisfying $\varphi_X(f_*(c_1)) = \bar{f}_*^{\mathrm{new}}(\varphi_Y(\bar{c}_1))$ due to Theorem 4.3.1. We conclude recalling Lemma 4.3.6.

□

Example 4.3.8 Consider the identity $\mathrm{Id}\colon (\mathbb{E}, c_1) \to (\mathbb{E}, \bar{c}_1)$ between a ring spectrum with two different orientations c_1 and $\bar{c}_1$. The explicit computations of Td_φ is a classic subject on formal group laws (*cf.* [Dég14, §5.2] for a review in the context of the Riemann-Roch theorem).

The simplest example of a change of orientations is

$$\bar{c}_1(L) = -c_1(L^*).$$

Recall from the proof of Proposition 4.2.21 that the canonical short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to Q \to 0$ satisfies that $Q = \mathcal{O}_{\mathbb{P}^1}(1)$

so we have $c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = -c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$ for any orientation. Therefore the class $\bar{c}_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))$ as defined above is always an orientation. If F is the formal group law of c_1 then the series G of Proposition 4.1.42 in this case is the *formal inverse* μ of F, i.e., the series satisfying $F(x, \mu(x)) = 0$. However, it is much easier to compute Chern classes explicitly by the splitting principle and the projective bundle theorem obtaining

$$\bar{c}_i(E) = (-1)^i c_i(E^*) \qquad \text{and} \qquad \sum_{i=0}^{n} (-1)^i c_i(E^*) y^{n-i} = 0 \in \mathbb{E}(\mathbb{P}(E))$$

where $y = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{E})}(1))$.

The Riemann-Roch theorem for modules is a direct consequence of the case of rings. We state the results in two steps as before.

Theorem 4.3.9 *Let $\varphi\colon (\mathbb{E}, c_1) \to (\mathbb{F}, \bar{c}_1)$ be a morphism of oriented absolute spectra such that $\varphi_{\mathbb{P}^\infty}(c_1) = \bar{c}_1$. Let $\Phi\colon \mathbb{M} \to \mathbb{M}'$ be a φ-morphism of absolute modules and $f\colon Y \to X$ be a projective lci, then the diagram*

$$\begin{array}{ccc} \mathbb{M}(Y) & \xrightarrow{f_*} & \mathbb{M}(X) \\ {\scriptstyle \Phi_Y}\downarrow & & \downarrow{\scriptstyle \Phi_X} \\ \mathbb{M}'(Y) & \xrightarrow{\bar{f}_*} & \mathbb{M}'(X) \end{array}$$

commutes. In other words, for $m \in \mathbb{M}(Y)$

$$\Phi_X(f_*(m)) = \bar{f}_*(\Phi_Y(m)).$$

Proof: Consider the case of a regular immersion $i\colon Z \to X$ and the case of a projection $p_X\colon \mathbb{P}^n_X \to X$ of a projective space onto its base. Both cases follow from Theorem 4.3.1.

Let $i\colon Z \to X$ be a regular immersion. Since $\varphi_X(i_*(1)) = \bar{i}_*(1)$ then φ preserves the fundamental class of Z in X. Therefore $\bar{i}_*\Phi_Y = \Phi_X i_*$. Denote $p\colon \mathbb{P}^n \to S$ the natural projection and recall that $p_{X*} = p_* \otimes 1_{\mathbb{M}(X)}\colon \mathbb{M}(\mathbb{P}^n(X)) \to \mathbb{M}(X)$ where $p_*\colon \mathbb{E}(\mathbb{P}^n) \to \mathbb{E}(S)$. Since $\varphi_S p_* = \bar{p}_* \varphi_{\mathbb{P}^n}$ we have that $\Phi_X p_{X*} = \bar{p}_{X*}\Phi_{\mathbb{P}^n_X}$.

□

Theorem 4.3.10 *Let $\varphi\colon (\mathbb{E}, c_1) \to (\mathbb{F}, \bar{c}_1)$ be a morphism of oriented absolute spectra. Denote $G \in \mathbb{F}[S][[t]]$ the series such that $\varphi(c_1) = G(\bar{c}_1) \cdot \bar{c}_1 \in \mathbb{F}(\mathbb{P}^\infty_S)$ and Td_φ be the multiplicative extension of G^{-1}. Let $\Phi\colon \mathbb{M} \to \mathbb{M}'$ be a φ-morphism of absolute modules and $f\colon Y \to X$ be a projective lci where $f = p \circ i$*

and denote $T_f = i^*T_p - N_i \in K^0(Y)$ *the virtual tangent bundle. Then the diagram*

$$\begin{tikzcd} \mathbb{M}(Y) \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}_\varphi(T_f)\Phi_Y"'] & \mathbb{M}(X) \arrow[d, "\Phi_X"] \\ \mathbb{M}'(Y) \arrow[r, "\bar{f}_*"] & \mathbb{M}'(X) \end{tikzcd}$$

commutes. In other words, for $m \in \mathbb{M}(Y)$ *we have*

$$\Phi_X(f_*(m)) = \bar{f}_*\big(\mathrm{Td}_\varphi(T_f) \cdot \Phi_Y(m)\big).$$

Proof: Consider the case of a regular immersion $i\colon Z \to X$ and the projection $p_X\colon \mathbb{P}^n_X \to X$. From Theorem 4.3.7 we have that $\varphi_X(i_*(1)) = \bar{i}_*(\mathrm{Td}_\varphi(-N_{Z/X}))$ from which the formula

$$\Phi_X i_*(m) = \bar{i}_*\big(\mathrm{Td}_\varphi(-N_{Z/X}) \cdot \Phi_Z(m)\big)$$

follows. From Theorem 4.3.7 we have that $\varphi_X p_{X*}(\alpha) = \bar{p}_{X*}\big(\mathrm{Td}_\varphi(T_p) \cdot \varphi_{\mathbb{P}^n_X}(\alpha)\big)$ from which the formula

$$\Phi_X p_{X*}(m) = \bar{p}_{X*}(\mathrm{Td}_\varphi(T_p) \cdot \Phi_{\mathbb{P}^n_X}(m))$$

follows.

□

Example 4.3.11 We review some concrete examples of this formula. Let $\varphi\colon \mathbb{E} \to \mathbb{F}$ a morphism of strict absolute ring spectra:

- Let $i\colon Z \to X$ be a closed immersion and consider the cartesian square

$$\begin{tikzcd} P \arrow[r, "i'"] \arrow[d, "\pi'"'] & B_Z X \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

  where $B_Z X$ denotes the blow-up of Z in X. Recall from Example 4.1.25 that $\mathbb{E}(\pi) \simeq \mathbb{E}(\pi') = \mathbb{E}(P)/\mathbb{E}(Z)$. We deduce from the Riemann-Roch theorem for modules that the square

$$\begin{tikzcd} \mathbb{E}(P)/\mathbb{E}(Z) \arrow[r, "i_*"] \arrow[d, "\mathrm{Td}_\varphi(-N_{Z/X})\varphi"'] & \mathbb{E}(P)/\mathbb{E}(Z) \arrow[d, "\varphi"] \\ \mathbb{F}(P)/\mathbb{F}(Z) \arrow[r, "\bar{i}_*"] & \mathbb{F}(P)/\mathbb{F}(Z) \end{tikzcd}$$

  commutes. Note that $i_*(m) = m \cdot \big((i \circ \pi')^*\eta^X_Z\big) = m \cdot c_n(N_{Z/X})$. Recall from Remark 4.1.44 that $c_n(N_{Z/X}) = \mathrm{Td}_\varphi(-N_{Z/X}) \cdot \bar{c}_n(N_{Z/X})$ so the formula also follows from the Riemann-Roch theorem for rings.

- Let $S = \operatorname{Spec}(k)$ be a point. Denote $f\colon X \to S$ be k-scheme and $g\colon Y \to S$ be a smooth projective k-scheme. We have that $\mathbb{E}(f) = \widetilde{\mathbb{E}}(X) = \mathbb{E}(X)/\mathbb{E}(S)$ and that $\mathbb{E}(f_T) = \mathbb{E}(X \times Y)/\mathbb{E}(Y)$. Then the square
$$\begin{tikzcd} \mathbb{E}(X \times Y)/\mathbb{E}(Y) \arrow[r, "g_*"] \arrow[d, "\mathrm{Td}_\varphi(T_g)\varphi"'] & \widetilde{\mathbb{E}}(X) \arrow[d, "\varphi"] \\ \mathbb{F}(X \times Y)/\mathbb{F}(Y) \arrow[r, "\bar{g}_*"] & \widetilde{\mathbb{F}}(X) \end{tikzcd}$$
commutes. Assume both $\mathbb{E}$ and $\mathbb{F}$ satisfy the Künneth formula. Then $\mathbb{E}(X \times Y)/\mathbb{E}(Y) = \widetilde{\mathbb{E}}(X) \otimes \mathbb{E}(Y)$ and
$$g_* = 1 \otimes g_*\colon \widetilde{\mathbb{E}}(X) \otimes \mathbb{E}(Y) \to \widetilde{\mathbb{E}}(X) \otimes \mathbb{E}(S) = \widetilde{\mathbb{E}}(X)$$
and $\bar{g}_* = 1 \otimes \bar{g}_*$. In this case the formula follows directly from the Riemann-Roch theorem for ring spectra.

- *Residual Riemann-Roch:* Let $i\colon Z \to X$ be a closed immersion of open complement $j\colon U \to X$. Recall that $\mathbb{E}(j) = \mathbb{E}_Z(X)$. Note that the morphism of modules $\varphi\colon \operatorname{hofib}_{\mathbb{E}}(j) \to \operatorname{hofib}_{\mathbb{F}}(j)$ fits into a morphism of distinguished triangles. We deduce that the square
$$\begin{tikzcd} \mathbb{E}(U) \arrow[r, "\delta"] \arrow[d, "\varphi"'] & \mathbb{E}_Z(X) \arrow[d, "\varphi"] \\ \mathbb{F}(U) \arrow[r, "\bar{\delta}"] & \mathbb{F}_Z(X), \end{tikzcd}$$
where δ denotes the connecting, commutes. If both Z and X are smooth then we have the purity isomorphism $\mathbb{E}(Z) \simeq \mathbb{E}_Z(X)$ and we deduce Déglise's residual Riemann-Roch theorem (*cf.* [Dég14, 4.2.3]): The square
$$\begin{tikzcd} \mathbb{E}(U) \arrow[r, "\delta"] \arrow[d, "\varphi"'] & \mathbb{E}(Z) \arrow[d, "\mathrm{Td}_\varphi(-N_{Z/X})\varphi"] \\ \mathbb{F}(U) \arrow[r, "\bar{\delta}"] & \mathbb{F}(Z) \end{tikzcd}$$
commutes.

- Let $i\colon Z \to X$ be a closed immersion of open complement $j\colon U \to X$. We have that $\mathbb{E}(i) = \mathbb{E}_c(U)$. Let $g\colon T \to X$ be a projective lci morphism, the square
$$\begin{tikzcd} \mathbb{E}_c(U \times_X T) \arrow[r, "g_*"] \arrow[d, "\varphi"'] & \mathbb{E}_c(U) \arrow[d, "\varphi"] \\ \mathbb{F}_c(U \times_X T) \arrow[r, "\bar{g}_*"] & \mathbb{F}_c(U), \end{tikzcd}$$
commutes.

4.3.1 Applications

The main application we are interested in is the Grothendieck-Riemann-Roch theorem for higher K-theory. We afterwards review some other Riemann-Roch type formulas as well and the arithmetic Riemann-Roch theorem. Recall from Example 4.1.13 that the Chern character is a morphism of strict absolute ring spectra $\mathrm{ch}\colon \mathrm{KGL}_{\mathbb{Q}} \to \mathrm{H}_{\mathrm{B}}$. Denote Td the multiplicative extension of the Todd series $\frac{t}{1-e^{-t}}$ (*cf.* Corollary 4.1.43).

Theorem 4.3.12 (Riemann-Roch) *Let $f\colon Y \to X$ be a projective lci where $f = p \circ i$ and denote $T_f := i^*T_p - N_i \in K_0(Y)$ the virtual tangent bundle. Then the diagram*

$$\begin{tikzcd} KH(Y)_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}(T_f)\mathrm{ch}"'] & KH(X)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ H_{\mathcal{M}}(Y, \mathbb{Q}) \arrow[r, "f_*"] & H_{\mathcal{M}}(X, \mathbb{Q}) \end{tikzcd}$$

commutes. In other words,

$$\mathrm{ch}(f_*(a)) = f_*\big(\mathrm{Td}(T_f) \cdot \mathrm{ch}(a)\big).$$

Proof: The result follows from Theorem 4.3.7 applied to ch. Recall that $\mathrm{ch}(L) = e^{c_1^{\mathrm{H}_{\mathrm{B}}}(L)}$, that $c_1^{\mathrm{KGL}}(L) = 1 - L^*$ and that $c_1^{\mathrm{H}_{\mathrm{B}}}$ is additive so, in particular, $c_1^{\mathrm{H}_{\mathrm{B}}}(\mathcal{O}_{\mathbb{P}^n}(1)) = -c_1^{\mathrm{H}_{\mathrm{B}}}(\mathcal{O}_{\mathbb{P}^n}(-1))$. Denote $x = c_1^{\mathrm{H}_{\mathrm{B}}} = c_1^{\mathrm{H}_{\mathrm{B}}}(\mathcal{O}_{\mathbb{P}^\infty}(-1))$ and $y = c_1^{\mathrm{KGL}} = c_1^{\mathrm{KGL}}(\mathcal{O}_{\mathbb{P}^\infty}(-1))$, we have

$$\mathrm{ch}(y) = 1 - e^{-x} = x \cdot \frac{1 - e^{-x}}{x}.$$

Therefore $G = \frac{1-e^{-t}}{t}$ and $\mathrm{Td}_{\mathrm{ch}} = \mathrm{Td}$ the multiplicative extension of $\frac{t}{1-e^{-t}}$. □

A general Riemann-Roch statement as in [Gil81] follows from the fact that Beilinson motivic cohomology spectrum is universal for spectra with additive orientations (*cf.* [CD09, 14.2.16] and [Dég14, 5.3.9]).

Proposition 4.3.13 *Let $(\mathbb{E}, c_1)$ be an oriented absolute ring spectrum in $\mathbf{SH}(S)_{\mathbb{Q}}$ with c_1 additive. Then there exist a unique morphism of absolute spectra*

$$\varphi\colon \mathrm{H}_{\mathrm{B}} \to \mathbb{E}.$$

Moreover, the morphism satisfies that $\varphi_{\mathbb{P}^\infty}(c_1^{\mathrm{H}_{\mathrm{B}}}) = c_1 \in \mathbb{E}^{2,1}(\mathbb{P}^\infty)$.

Recall that some examples of oriented absolute ring spectra with additive orientations are those coming from real absolute Hodge and Deligne-Beilinson cohomology, rigid syntomic cohomology, and mixed Weil theories. Let now $S = \mathrm{Spec}(k)$ for k a perfect field for mixed Weil theories, a field of characteristic zero for real absolute Hodge and Deligne-Beilinson cohomology, or a residue field of a p-adic field for rigid syntomic cohomology. The next result follows from the Riemann-Roch theorem and Proposition 4.3.13:

Corollary 4.3.14 *Let* H *denote either real absolute Hodge cohomology, real Deligne-Beilinson cohomology, rigid syntomic cohomology or any cohomology coming from a mixed Weil theory. Let* S *be as above so that* H *is defined and let* $f\colon Y \to X$ *be a projective lci morphism of* S*-schemes. Then, with previous notations, the diagram*

$$\begin{tikzcd} KH(Y)_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}(T_f)\mathrm{ch}"'] & KH(X)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ \mathrm{H}(Y) \arrow[r, "f_*"] & \mathrm{H}(X) \end{tikzcd}$$

commutes. In other words, for $a \in KH(Y)_{\mathbb{Q}}$ *we have*

$$\mathrm{ch}(f_*(a)) = f_*\big(\mathrm{Td}(T_f) \cdot \mathrm{ch}(a)\big).$$

□

Another general type of morphism of oriented absolute ring spectra to which the motivic Riemann-Roch theorem applies are those coming from algebraic cobordism MGL. Recall that MGL is the universal oriented absolute ring spectrum (see [Vez01]).

Proposition 4.3.15 *Let* $(\mathbb{E}, c_1)$ *be an oriented absolute ring spectrum. Then there exist a unique morphism of absolute ring spectra*

$$\varphi\colon \mathrm{MGL} \to \mathbb{E}$$

such that $\varphi(c_1^{\mathrm{MGL}}) = (c_1) \in \mathbb{E}^{2,1}(\mathbb{P}^\infty)$.

□

Since this morphism preserves the orientation then Theorem 4.3.1 applies to them.

Corollary 4.3.16 *Let* $(\mathbb{E}, c_1)$ *be an oriented absolute ring spectrum and* $f\colon Y \to X$ *be a projective lci morphism. Then, with previous notations, for* $a \in \mathrm{MGL}(Y)$ *we have*

$$\varphi_X(f_*(a)) = f_*(\varphi_Y(a)).$$

□

We apply the Riemann Roch theorem for modules 4.3.10 to the examples we described in Section 4.1.1.

Theorem 4.3.17 *Let $f\colon Y \to X$ be a morphism of schemes, $g\colon T \to X$ be a projective lci morphism and denote $f_T\colon Y \times_X T \to T$ and $T_g \in K_0(Y)$ the virtual tangent bundle of g. Assume in addition either f is proper or g is smooth, then the diagram*

$$\begin{tikzcd} KH(f_T)_{\mathbb{Q}} \arrow[r, "g_*"] \arrow[d, "\mathrm{Td}(T_g)\mathrm{ch}"'] & KH(f)_{\mathbb{Q}} \arrow[d, "\mathrm{ch}"] \\ H_{\mathcal{M}}(f_T, \mathbb{Q}) \arrow[r, "g_*"] & H_{\mathcal{M}}(f, \mathbb{Q}) \end{tikzcd}$$

commutes. In other words, for $m \in KH(f_T)_{\mathbb{Q}}$ we have

$$\mathrm{ch}(g_*(m)) = g_*\big(\mathrm{Td}(T_g) \cdot \mathrm{ch}(m)\big).$$

□

We also obtain an arithmetic Riemann-Roch theorem as a consequence of the Riemann-Roch theorem for modules.

Theorem 4.3.18 (Arithmetic Riemann-Roch) *Let $f\colon Y \to X$ be a projective morphism between smooth schemes over an arithmetic ring and $T_f \in K_0(Y)$ the virtual tangent bundle. Then the diagram*

$$\begin{tikzcd} \widehat{KH}(Y)_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\mathrm{Td}(T_f)\widehat{\mathrm{ch}}"'] & \widehat{KH}(X)_{\mathbb{Q}} \arrow[d, "\widehat{\mathrm{ch}}"] \\ \widehat{H}_{\mathcal{M}}(Y, \mathbb{Q}) \arrow[r, "f_*"] & \widehat{H}_{\mathcal{M}}(f, \mathbb{Q}) \end{tikzcd}$$

commutes. In other words, for $m \in \widehat{KH}(Y)_{\mathbb{Q}}$ we have

$$\widehat{\mathrm{ch}}(f_*(m)) = f_*\big(\mathrm{Td}(T_f) \cdot \widehat{\mathrm{ch}}(m)\big).$$

□

4.4 Riemann-Roch without denominators

In this section we prove the Riemann-Roch theorem without denominators. To this date, every proof of this result relies at some point on already standard arguments and computations for universal polynomials involving higher Chern classes. Let us briefly recall these notions.

4.4.1 There is an obvious natural transformation

$$c_i \colon K_0(\_\,) \to \mathbb{E}^{2i,i}(\_\,)$$

of presheaves of sets on $\mathbf{Sm}/S$ which maps every locally free module to its i-th Chern class. After Riou's results, one has an isomorphism (cf. [Rio10, 1.1.6])

$$\mathrm{Hom}(K_0(\_\,), \mathbb{E}^{2i,i}(\_\,)) \simeq \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{Z} \times Gr, \Omega^\infty \mathbb{E}(i)[2i])$$

where Gr denotes the infinite Grassmannian. This allows to define higher Chern classes with support in a closed subscheme. Let $Z \to X$ be a closed immersion and denote $T = S^r \wedge X/X - Z$, then

$$c^Z_{i,r} \colon KH_{Z,r}(X) \simeq [T, \mathbb{Z} \times Gr] \to [T, \Omega^\infty \mathbb{E}(i)[2i]] \simeq \mathbb{E}^{2i-r,i}_Z(X).$$

The following property is a direct consequence of the definition.

Proposition 4.4.2 *Higher Chern classes with support are functorial. In other words, let $Z \to X$ be a closed subscheme and let $f \colon X' \to X$ be a morphism of schemes. Then for any $a \in KH_r(X)$ and any i we have*

$$f^*(c^Z_{i,r}(a)) = c^{f^{-1}(Z)}_{i,r}(f^* a).$$

□

Recall that from the construction of KGL in [Cis13] it follows that for every X the map

$$K(X) \to KH(X)$$

from Thomason-Trobaugh's K-theory to Weibel's homotopy invariant K-theory is a morphism of rings.

Denote $P^d_q(\xi, c_1, \dots c_{q-d}; c'_1, \dots, c'_{q-d})$ the universal polynomial with integer coefficients defined in [Jou70, §1].

Theorem 4.4.3 (Riemann-Roch without denominators) *Let $i \colon Z \to X$ be a regular immersion of codimension d and denote $\mathfrak{q}_i \colon KH(Z) \to KH_Z(X)$ and $\mathfrak{p}_i \colon H_{\mathcal{M}}(Z, \mathbb{Z}) \to H_{\mathcal{M},Z}(X, \mathbb{Z})$ the respective refined Gysin morphisms. Then for any $q > 0$ and any $a \in KH_r(Z)$ we have*

$$c^Z_{q,r}(\mathfrak{q}_i(a)) = \mathfrak{p}_i(P^d_q(\mathrm{rk}(a), c_{1,r}(a), \dots, c_{q-d,r}(a); c_1(N_{Z/X}), \dots, c_{q-d}(N_{Z/X}))). \tag{4.6}$$

Proof: Denote $P^d_q(a, E) = P^d_q(\mathrm{rk}(a), c_{1,r}(a), \dots, c_{q-d,r}(a); c_1(E), \dots, c_{q-d}(E))$ for any vector bundle E on Z. We consider once again the deformation to the projective completion of the normal bundle of Lemma 4.3.3 and its notations. Taking motivic cohomology in diagram (4.5) we get

$$\begin{tikzcd}
 & H_{\mathcal{M}}(U,\mathbb{Z}) & \\
H_{\mathcal{M},Z}(\bar{N},\mathbb{Z}) \arrow[d, shift left] & H_{\mathcal{M},\mathbb{A}^1_Z}(X',\mathbb{Z}) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] \arrow[u, "h"] & H_{\mathcal{M},Z}(X,\mathbb{Z}) \\
H_{\mathcal{M}}(Z,\mathbb{Z}) \arrow[u, shift left, "\mathfrak{p}_s"] & H_{\mathcal{M}}(\mathbb{A}^1_Z,\mathbb{Z}) \arrow[l, "\sim"'] \arrow[r, "v^*"] \arrow[u, "\mathfrak{p}_\iota"] & H_{\mathcal{M}}(Z,\mathbb{Z}) \arrow[u, "\mathfrak{p}_i"']
\end{tikzcd}$$

where $h = j^*\iota_\flat$ and $\mathfrak{p}_s$ is injective.

We now prove that if formula (4.6) holds for $\iota\colon \mathbb{A}^1_Z \to X'$ then it also holds for $i\colon Z \to X$. Since $v^*N_{\mathbb{A}^1_Z/X'} = N_{Z/X}$ the refined versions of the excess intersection formula applied to the right square and the functoriality of higher Chern classes gives

$$c^Z_{q,r}\mathfrak{q}_i(v^*(a)) = c^Z_{q,r}i_1^*(\mathfrak{q}_i(a)) = i_1^*c^{\mathbb{A}^1_Z}_{q,r}\mathfrak{q}_\iota(a)$$

and

$$i_1^*(\mathfrak{p}_\iota(P^d_q(a, N_{\mathbb{A}^1_Z/X'}))) = \mathfrak{p}_{i_1}((P^d_q(v^*a, N_{Z/X}))).$$

The last and the first term respectively are elements that theorem state that coincide.

We deduce from the same arguments of Lemma 3.3 that if $a \in H_{\mathcal{M},Z}(X',\mathbb{Z})$ satisfies $h(a)$ and $i_0^*(a) = 0$ then $a = 0$. Since $h(c^{\mathbb{A}^1_Z}_{q,r}\mathfrak{q}_\iota(a)) = c_{q,r}j^*\mathfrak{q}_\iota(a) = 0$ and $h\mathfrak{p}_\iota = j^*\iota_* = 0$ the last case left to prove is formula (4.6) for the zero section $s\colon Z \to \bar{N}$ of the projective completion of the normal bundle. This is the case treated in the literature when Z is smooth.

First recall that in 4.2.20 we observed that $H_{\mathcal{M}}(\mathrm{Th}(N),\mathbb{Z}) \simeq H_{\mathcal{M},Z}(\bar{N},\mathbb{Z}) \to H_{\mathcal{M}}(\bar{N},\mathbb{Z})$ is injective, so it is enough to prove formula (4.6) in $H_{\mathcal{M}}(\bar{N},\mathbb{Z})$.

We summarize in two lemmas computations which only involve standard arguments in $K_0(\bar{N})$ that do not need the smoothness assumption (*cf.* [KY14, 4.3 and 4.4]).

Lemma 4.4.4 *With the previous notations, denote Q the canonical quotient bundle of $\bar{N}$ and $\mathbf{t}^{KH}(N)$ and $\mathbf{t}(N)$ the Thom class in KH and $H_{\mathcal{M}}$ respectively. Then for any $b \in KH_r(\bar{N})$ we have*

$$c_{q,r}(b \cdot \mathbf{t}^{KH}(N)) = P^d_q(b, Q) \cdot \mathbf{t}(N).$$

□

Lemma 4.4.5 *Let $p\colon \bar{N} \to Z$ be the projection. For any $a \in KH_r(Z)$ we have*

$$p^*(P_q^d(a, N)) \cdot \mathbf{t}(N) = P_q^d(p^*a, Q) \cdot \mathbf{t}(N).$$

□

To conclude the proof of the theorem recall that the fundamental class of the zero section coincide with the Thom class (*cf.* Proposition 4.2.21). From here, the projection formula gives $s_*(a) = p^*(a) \cdot \mathbf{t}^{KH}(N)$ and the analogous formula for motivic cohomology. With them, we conclude

$$c_{q,r}(i_*(a)) = c_{q,r}(p^*(a) \cdot \mathbf{t}(N)) = i_*(P_q^d(a, N)).$$

□

The original Riemann-Roch theorem without denominators, as conjectured by Grothendieck and proved by Jouanolou, was stated without supports. Let us make this precise statement for the sake of completeness.

Corollary 4.4.6 *Let $i\colon Z \to X$ be a regular immersion of codimension d, then for any $q > 0$ and any $a \in KH_r(Z)$ we have in $H_{\mathcal{M}}(X, \mathbb{Z})$ that*

$$c_{q,r}(i_*(a)) = i_*(P_q^d(\mathrm{rk}(a), c_{1,r}(a), \ldots, c_{q-d,r}(a); c_1(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X}))).$$

Chapter 5

Appendix

5.1 Absolute Hodge cohomology

In this Appendix we apply a theorem of Déglise and Mazzari to give direct construction of the (real) absolute Hodge spectrum representing absolute Hodge cohomology with real coefficients with no smoothness assumption. Since in [HS15] the authors asked explicitly if the Deligne-Beilinson spectrum represented the Deligne-Beilinson cohomology on singular schemes we also prove it for the Deligne-Beilinson spectrum.

We refer to Brad Drew's thesis [Dre13] for the original construction of the (rational) absolute Hodge spectrum and a more complete treatment on the subject. We refer to [Bei83], [Jan88], [Bur98] and [Bei85], [EV88], [Bur94] for more details of the following constructions for absolute Hodge and Deligne-Beilinson cohomology respectively.

5.1.1 Let X be a smooth complex variety. We can find a proper complex variety $\bar{X}$ and an open embedding $j\colon X \to \bar{X}$ such that $D = \bar{X} - X$ is a normal crossing divisor. We denote by $A^*_{\bar{X}}(\log D)$ the complex of smooth differential forms with logarithmic singularities along D (*cf.* [Bur97]). Taking limit over all suitable compactifications we define

$$A^*_{\log}(X) = \lim A^*_{\bar{X}}(\log D)$$

the complex of smooth differential forms with logarithmic singularities along infinity. The complex $A^*_{\log}(X)$ has a natural filtration W which assigns weight zero to the sections of $A^*(X)$ and weight one to the sections $\mathrm{d}z_i/z_i$ and $\mathrm{d}\bar{z}_i/z_i$. The complex $A^*_{\log}(X)$ also has a natural Hodge filtration F, as well as a subcomplex $A^*_{\log,\mathbb{R}}(X)$ of differential forms invariant under complex conjugation. Therefore it defines an $\mathbb{R}$-Hodge complex. The absolute Hodge cohomology of

X with real coefficients is defined as

$$H^p_{\mathrm{AH}}(X, \mathbb{R}(q)) = H^p(\tilde{\Gamma}(A_{\log}(X), q))$$

where

$$\tilde{\Gamma}(A_{\log}(X), q) = \operatorname{cone}((2\pi i)^q \hat{W}_{2q} A_{\log,\mathbb{R}}(X) \oplus \hat{W}_{2q} \cap F^q A_{\log}(X) \to \hat{W}_{2q} A_{\log}(X))[-1]$$

and $\hat{W}$ denotes the decalé filtration of W (*cf.* [Del71, 1.1.2]).

The *real Deligne-Beilinson cohomology* is obtained by ignoring the weight filtration. That is to say, we define it as

$$H^p_{\mathrm{D}}(X, \mathbb{R}(q)) = H^p(\Gamma(A_{\log}(X), q))$$

where

$$\Gamma(A_{\log}(X), q) = \operatorname{cone}((2\pi i)^q A_{\log,\mathbb{R}}(X) \oplus F^q A_{\log}(X) \to A_{\log}(X))[-1].$$

Both the real absolute Hodge and the Deligne-Beilinson cohomology can also be computed by means of the *Thom-Whitney simple* introduced in [Nav87]. Following [Bur98], the Thom-Whitney simple has a concrete description in these cases. Denote L_1^* the differential graded commutative $\mathbb{R}$-algebra of algebraic forms over $\mathbb{A}^1_{\mathbb{R}}$, then the Thom-Whitney simple $\tilde{\Gamma}_{\mathrm{TW}}(A_{\log}(X), q)$ for the real absolute Hodge cohomology is the subcomplex of

$$(2\pi i)^q \hat{W}_{2q} A_{\log,\mathbb{R}}(X) \oplus \hat{W}_{2q} \cap F^q A_{\log}(X) \oplus (L_1^* \otimes \hat{W}_{2q} A_{\log}(X))$$

made by elements (a, b, ω) such that $\omega(0) = a$ and $\omega(1) = b$. The differential is the natural one on each summand. These complex satisfy that

$$H^p_{\mathrm{AH}}(X, \mathbb{R}(q)) = H^p(\tilde{\Gamma}_{\mathrm{TW}}(A_{\log}(X), q)).$$

Definition 5.1.2 An **arithmetic field** is a triple (k, Σ, Fr) where k is a field, $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ is a set of embeddings $k \to \mathbb{C}$ and $\mathrm{Fr}\colon \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ is a $\mathbb{C}$-antilinear involution such that the image of k in $\mathbb{C}^\Sigma$ by $\sigma_1 \times \cdots \times \sigma_n$ is invariant under Fr. We call *Frobenius* to the map Fr. If X is a k-scheme we write $X_\Sigma = \coprod \big(X \times_{\sigma_i} \operatorname{Spec} k\big)$, which is naturally a complex variety.

Let X be a smooth scheme over an arithmetic field. The Frobenius Fr induces a $\mathbb{C}$-linear action on $A_{\log}(X_\Sigma)$ by taking the action it induces on a compactification $\bar{X}_\Sigma$ together with the complex conjugation. That is to say, $\mathrm{Fr} f(x) = \bar{f}(\mathrm{Fr}(x))$. This action is compatible with the weight and Hodge filtration. Therefore we consider

$$\tilde{\Gamma}(A_{\log}(X_{\mathbb{R}}), q) = \tilde{\Gamma}(A_{\log}(X_\Sigma), q)^{\mathrm{Fr}},$$

$$\Gamma(A_{\log}(X_{\mathbb{R}}), q) = \Gamma(A_{\log}(X_{\Sigma}), q)^{\mathrm{Fr}}.$$

We denote the cohomology they define as

$$H^p_{\mathrm{D}}(X_{\mathbb{R}}, \mathbb{R}(q)) = H^p(\tilde{\Gamma}(A_{\log}(X_{\mathbb{R}}), q)),$$

$$H^p_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(q)) = H^p(\Gamma(A_{\log}(X_{\mathbb{R}}), q)).$$

As before, their respective Thom-Whitney simple also compute the cohomology. In the case of the complex for absolute Hodge cohomology we denote it $\tilde{\Gamma}_{\mathrm{TW}}(A_{\log}(X_{\mathbb{R}}), q)$.

In [HS15] Holmstrom and Scholbach proved that there exist an absolute spectrum $\mathbb{E}_{\mathrm{D}} \in \mathbf{SH}(S)_{\mathbb{Q}}$ that represents the Deligne-Beilinson cohomology for smooth schemes. In other words, there exist an absolute spectrum $\mathbb{E}_{\mathrm{D}}$ such that for every X smooth and every integers p, q we have

$$\mathbb{E}^{p,q}_{\mathrm{D}}(X) = H^p_D(X, \mathbb{R}(q)).$$

This argument has been vastly generalized by Déglise and Mazzari in [DM14, 1.4.10] by giving sufficient conditions for a family of presheaves $X \mapsto \mathcal{F}_i(X)$ for $i \in \mathbb{N}$ so that the cohomology they define is represented by an absolute ring spectrum. That is to say, they give sufficient conditions on $(\mathcal{F}_i)_{i\in\mathbb{N}}$ so that there exist an absolute ring spectrum $\mathbb{E}_{\mathcal{F}}$ satisfying

$$H^p(\mathcal{F}_q(X)) = \mathbb{E}^{p,q}_{\mathcal{F}}(X).$$

5.1.3 In order to check that the family $X \mapsto \tilde{\Gamma}_{TW}(A^*_{\log}(X_{\mathbb{R}}), i)$ satisfy the hypothesis of *loc. cit.* let us introduce some notation. Consider $\mathbb{C}$ as an arithmetic field with $\Sigma = \{\mathrm{Id}, \sigma\}$ where σ denotes the complex conjugation. Denote $c\colon \mathbb{R} \to \tilde{\Gamma}_{TW}(A^*_{\log}(\mathbb{G}_{m\,\mathbb{R}}), 1)[1]$ the section given by

$$\Big(\frac{\mathrm{d}z}{z} + \frac{\mathrm{d}\bar{z}}{\bar{z}},\, (\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}\bar{z}}{\bar{z}})i,\, (1-x)(\frac{\mathrm{d}z}{z} + \frac{\mathrm{d}\bar{z}}{\bar{z}}) + x(\frac{\mathrm{d}z}{z} - \frac{d\bar{z}}{\bar{z}})i + (\ln z\bar{z} + i\ln\frac{z}{\bar{z}})\mathrm{d}x\Big)$$

where the first term is in $(2\pi i)\hat{W}_2 A_{\log 0,\infty,\mathbb{R}}((\mathbb{P}^1)_{\mathbb{R}})$, the second belongs to $\hat{W}_2 \cap F^1 A_{\log 0,\infty}((\mathbb{P}^1)_{\mathbb{R}})$ and the third belongs to $L^*_1 \otimes \hat{W}_2 A_{\log 0,\infty}((\mathbb{P}^1)_{\mathbb{R}})$. For a general arithmetic field we still denote c the section defined by taking c on each component of $\mathbb{G}_{m\,\Sigma}$.

Also recall that a *distinguished square* is a commutative cartesian diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

in $\mathbf{Sm}/S$ such that $Y \to X$ is an open immersion and $X' \to X$ is étale and induces an isomorphism $(X' \setminus Y')_{\mathrm{red}} \to (X \setminus Y)_{\mathrm{red}}$. We say that a complex of presheaves of R-modules $\mathcal{F}$ on $\mathbf{Sm}/S$ has the *Brown-Gersten property with respect to the Nisnevich topology* if for every distinguished square the diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(X') \arrow[d] \\ \mathcal{F}(Y) \arrow[r] & \mathcal{F}(Y') \end{tikzcd}$$

is a homotopy pullback square in the category of complexes of R-modules.

Proposition 5.1.4 *Let k be an arithmetic field and denote $S = \mathrm{Spec}(k)$. Consider the family of presheaves $X \mapsto \tilde{\Gamma}_{TW}(A^*_{\log}(X_{\mathbb{R}}), i)$ on $\mathbf{Sm}/S$ together with the section $c\colon \mathbb{R} \to \tilde{\Gamma}_{TW}(A^*_{\log}(\mathbb{G}_{m\,\mathbb{R}}), 1)[1]$ defined above:*

1. *They form an $\mathbb{N}$-commutative graded monoid (*cf.* [DM14, 1.4.9]) in the category of complexes of $\mathbb{R}$-linear presheaves on the category of affine and smooth S-schemes.*

2. *They have the Brown-Gersten property with respect to the Nisnevich topology.*

3. *They are homotopy invariant, i.e., $H^p_{\mathrm{AH}}(\mathbb{A}^1_X, \mathbb{R}(q)) = H^p_{\mathrm{AH}}(X, \mathbb{R}(q))$.*

4. *If $\bar{c} \in H^1_{\mathrm{AH}}(\mathbb{G}_{m\,\mathbb{R}}, \mathbb{R}(1))$ denotes the class of c, then for any smooth scheme X and any integers p, q the map*

$$\begin{array}{ccc} H^p_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(q)) & \longrightarrow & H^{p+1}_{\mathrm{AH}}((X \times \mathbb{G}_m)_{\mathbb{R}}, \mathbb{R}(q+1))/H^p_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(q)) \\ x & \mapsto & [x \times \bar{c}] \end{array}$$

   where [_] denotes the class defined in the quotient, is an isomorphism.

5. *If $u\colon \mathbb{G}_m \to \mathbb{G}_m$ is the inverse map of the group scheme $\mathbb{G}_m$ and $\bar{c}'$ is the image of c in $H^1_{\mathrm{AH}}(\mathbb{G}_{m\,\mathbb{R}}, \mathbb{R}(1))$ then $u^*(\bar{c}') = -\bar{c}'$.*

Proof: Although these properties are well known for experts let us review them for the sake of completeness. The Thom-Whitney simple has a well defined associative and commutative product (*cf.* [Bur97, §6]), from which point 1 follows.

For point 2, first notice that the Brown-Gersten property is stable by direct sums and cones of maps. Therefore it is enough to prove it for $\hat{W}_q A_{\log,\mathbb{R}}$, $\hat{W}_{2q} \cap F^q A_{\log}$ and $\hat{W}_q A_{\log}$. The étale descent for $A_{\log,\mathbb{R}}$, $F^q A_{\log}$ and $A_{\log}$ may be found in [HS15, 2.9], from which the Brown-Gersten property follows. For the weight filtration, consider a distinguished square as in 5.1.3. We have a short exact sequence

$$0 \to A_{\log}(X) \to A_{\log}(X') \bigoplus A_{\log}(Y) \to A_{\log}(Y') \to 0$$

as well as for $A_{\log,\mathbb{R}}$ and $F^q A_{\log}$. These morphisms are strict both for the Hodge and the decalé Weight filtration, so the Brown-Gersten property readily follows for $\hat{W}_q A_{\log,\mathbb{R}}$, $\hat{W}_{2q} \cap F^q A_{\log}$ and $\hat{W}_q A_{\log}$. We conclude by taking invariants on the action induced by the Frobenius.

For point 4 note that the absolute Hodge cohomology of $\mathbb{G}_m$ is null apart from the groups $H^0(\mathbb{G}_m, \mathbb{R}(0)) = \mathbb{R}$ and $H^1(\mathbb{G}_m, \mathbb{R}(1)) = \mathbb{R}$. It is easy to see from the definition of absolute Hodge cohomology that group $H^{p+1}_{\mathrm{AH}}((X \times \mathbb{G}_m)_{\mathbb{R}}, \mathbb{R}(q+1))$ equals

$$\big(H^p_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(q)) \otimes H^1_{\mathrm{AH}}(\mathbb{G}_{m\,\mathbb{R}}, \mathbb{R}(1))\big) \oplus \big(H^{p+1}_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(q+1)) \otimes H^0_{\mathrm{AH}}(\mathbb{G}_{m\,\mathbb{R}}, \mathbb{R}(0))\big)$$

from which point 4 follows. Finally, a direct computation concludes point 5. □

Corollary 5.1.5 *Denote $\mathbb{E}_{\mathrm{AH}}$ the oriented absolute ring spectrum constructed in [DM14, 1.4.10] out of the presheaves $(\tilde{\Gamma}_{TW}(A^*_{\log}(\_\ {}_{\mathbb{R}}), i))_{i \in \mathbb{N}}$, which we call the absolute Hodge spectrum. Then $\mathbb{E}_{\mathrm{AH}}$ represents real absolute Hodge cohomology on smooth schemes. In other words, for any smooth S-scheme X and any $p, q \geq 0$*

$$\mathbb{E}^{p,q}_{\mathrm{AH}} = H^p_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(q)).$$

□

Let us now prove that the absolute Hodge spectrum represents absolute Hodge cohomology for general schemes. The same method will apply also for the Deligne-Beilinson cohomology.

5.1.6 Let Z be complex variety, following [Del74, 8.3] we can find a diagram $\bar{X}_\bullet \hookleftarrow X_\bullet \xrightarrow{p} Z$ so that $X_\bullet$ and $\bar{X}_\bullet$ are simplicial complex varieties, p satisfies cohomological descent (in particular, it is a hypercovering for the h-topology), $\bar{X}_\bullet$ is proper smooth and $\bar{X}_\bullet - X_\bullet$ is a normal crossing divisor. If $\bar{X}'_\bullet \hookleftarrow X'_\bullet \to Z$ is a second diagram we have the isomorphisms $H_{\mathrm{AH}}(X_\bullet, \mathbb{R}) \simeq H_{\mathrm{AH}}(X'_\bullet, \mathbb{R})$ and $H_{\mathrm{D}}(X_\bullet, \mathbb{R}) \simeq H_{\mathrm{D}}(X'_\bullet, \mathbb{R})$ for both the absolute Hodge and Deligne-Beilinson cohomology of those simplicial varieties. Therefore, we call the absolute Hodge and the Deligne-Beilinson cohomology of Z to

$$H_{\mathrm{AH}}(Z, \mathbb{R}) := H_{\mathrm{AH}}(X_\bullet, \mathbb{R}) \qquad H_{\mathrm{D}}(Z, \mathbb{R}) := H_{\mathrm{D}}(X_\bullet, \mathbb{R})$$

This construction is compatible with the Frobenius action so we define the groups $H_{\mathrm{AH}}(Z_{\mathbb{R}}, \mathbb{R}) := H_{\mathrm{AH}}(X_{\bullet\,\mathbb{R}}, \mathbb{R})$ and $H_{\mathrm{D}}(Z_{\mathbb{R}}, \mathbb{R}) := H_{\mathrm{D}}(X_{\bullet\,\mathbb{R}}, \mathbb{R})$.

Recall that any rational oriented absolute ring spectrum is also a Beilinson motive (*cf.* [CD09, 14.2.16]). The category of Beilinson motives $\mathbf{DM}_{\mathrm{B}}(S)$ satisfies the h-descent (*cf.* [CD09, 3.1]). This implies that if X is a scheme and $X_\bullet \to X$ is a h-hypercover then for any oriented absolute ring spectrum we have

$$\mathbb{E}(X_\bullet) = \mathbb{E}(X),$$

where $\mathbb{E}(X_\bullet)$ denotes the cohomology of the simplicial scheme (*cf.* [CD09, §3.1] or [DM14, §2.2.1] for $\mathbf{DM}_{\mathrm{B}}$).

Corollary 5.1.7 *Let $\mathbb{E}_{\mathrm{AH}}$ and $\mathbb{E}_D$ be the real absolute Hodge and Deligne-Beilinson spectra, then for any scheme Z over an arithmetic field and every p, $q \geq 0$ we have*

$$\mathbb{E}^{p,q}_{\mathrm{AH}}(Z) = H^p_{\mathrm{AH}}(Z_{\mathbb{R}}, \mathbb{R}(q)) \quad \textit{and} \quad \mathbb{E}^{p,q}_{\mathrm{D}}(Z) = H^p_{\mathrm{D}}(Z_{\mathbb{R}}, \mathbb{R}(q)).$$

Bibliography

[Ada74] J.F. Adams: *Stable homotopy and generalised homology.* University of Chicago Press (1974)

[Ayo07] J. Ayoub: *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique I-II.* Asterisque, no. 314-315 (2007)

[Bei83] A. Beilinson: *Notes on absolute Hodge cohomology.* Applications of algebraic K-theory to Algebraic Geometry and Number Theory (S. Bloch, ed.), Contemporary Mathematics, vol. 55, AMS, 35-68 (1983)

[Bei85] A. Beilinson: *Higher regulators and values of L-functions.* J. Soviet Math. 30, 2036-2070 (1985)

[BMS87] A. Beilinson, R. MacPherson and V. Schechtman: *Notes on motivic cohomology.* Duke Math Journal, vol. 54, 2, 679-710 (1987)

[BFM75] P. Baum, W. Fulton and R. MacPherson: *Riemann-Roch for singular varieties* Publ. Math. IHES, no. 45, 101-145 (1975)

[BFM79] P. Baum, W. Fulton and R. MacPherson: *Riemann-Roch and topological K-theory for singular varieties.* Acta Math; 143, (1979)

[Bla01] B. Blander: *Local projective model structures on simplicial presheaves.* K-theory, 24, 283-301 (2001)

[BS58] A. Borel and J.P. Serre: *Le théorème de Riemann-Roch.* Bull. Soc. Math. France, 86, 97-136 (1958)

[Bur94] J.I. Burgos: *A C^∞-logarithmic Dolbeaut complex.* Compositio Math. 92, 61-86 (1994)

[Bur97] J.I. Burgos: *Arithmetic Chow rings and the Deligne-Beilinson cohomology* J. Algebraic Geom; 6 (2), 335-377 (1997)

[Bur98] J.I. Burgos, S. Wang: *Higher Bott-Chern forms and Beilinson's regulator.* Invent. Math. 132, 261-305 (1998)

[Cis13] D.C. Cisinski: *Descente par éclatements en K-théorie invariente par homotopie.* Ann. of Math., vol. 117 n 2, 425-448 (2013)

[CD09] D.C. Cisinski and F. Déglise: *Triangulated Categories of Mixed Motives.* arxiv:0912.2110 (2009)

[CD12] D.C. Cisinski and F. Déglise: *Mixed Weil cohomologies.* Adv. Math., no. 1, 55-130 (2012)

[Dég08] F. Déglise: *Around the Gysin triangle II.* Doc. Math (2008)

[Dég13] F. Déglise: *Orientable homotopy modules.* Amer. J. of Math., 135(2), 519-560 (2013)

[Dég14] F. Déglise: *Orientation theory in arithmetic geometry.* arXiv:1111.4203 (2014)

[DM14] F. Déglise and N. Mazzari: *The rigid sintomic ring spectrum.* J. Inst. Math. Jussieu, 1-47 (2014)

[Del71] P. Déligne: *Théorie de Hodge: II* Publ. Math. IHES, tome. 40, 5-57 (1971)

[Del74] P. Déligne: *Théorie de Hodge: III* Publ. Math. IHES, tome. 44, 5-77 (1974)

[Del80] P. Déligne: *La conjecture de Weil: II* Publ. Math. IHES, tome. 52, 137-252 (1980)

[Dre13] B. Drew: *Réalisations tannakienes des motifs mixtes triangulés.* Thèse de Doctorat, U. Paris 13 (2013)

[EV88] H. Esnault and E. Viehweg: *Deligne-Beilinson cohomology.* Beilinson's conjectures on special values of L-functions. Perspectives in Mathematics 4, Academic Press, Inc; 43-92 (1988)

[Fuj02] K. Fujiwara: *A proof of the absolute purity conjecture (after Gabber).* Algebraic Geometry 2000, Azumino (Hotaka), Adv. Stud. in Pure Math., vol 36, Math. Soc. Japan, Tokyo 144-173 (2002)

[Ful98] W. Fulton: *Intersection theory.* Second Edition. Springer-Verlag.(1998)

[FG83] W. Fulton and H. Gillet: *Riemann-Roch for general algebraic varieties.* Bull. Soc. Math. France 111, 287300. (1983)

[Gil81] H. Gillet: *Riemann-Roch theorems for higher algebraic K-theory.* Adv. in Math. 40, no. 3, 203289. (1981)

[GJ99] P.G. Goerss and J.F. Jardine: *Simplicial homotopy theory.* Progr. in Math. 140. (1999)

[HS15] A. Holmstrom and J. Scholbach: *Arakelov motivic cohomology I.* J. of Algebraic Geom. (2015)

[Hir03] P. Hirschhorn: *Model cateogries and their localizations.* Math Surveys and Monographs, vol. 99 (2003)

[Hov99] M. Hovey: *Model categories.* Mathe Surveys and Monographs, vol 63 (1999)

[Jan88] U. Jannsen: *Deligne homology, Hodge D-conjectures, and motives.* Beilinson's conjectures on special values of L-functions. Perspectives in Mathematics 4, Academic Press, Inc; 43-92 (1988)

[Jar00] J.F. Jardine: *Motivic symmetric spectra.* Doc. Math; 5, 445-552 (2000)

[Jar15] J.F. Jardine: *Local homotopy theory.* Springer Monographs in Mathematics (2015)

[Jar87] J.F. Jardine: *Simplicial presheaves.* J. Pure Appl. Algebra; 47, 35-87 (1987)

[Joy84] A. Joyal: *Letter to A. Grothendieck.* (1984)

[Jou70] J.P. Jouanolou: *Riemann-Roch sans dénominateurs.* Inventiones Math; 11, 15-26 (1970)

[KY14] S. Kondo and S. Yasuda: *The Riemann-Roch theorem without denominators in motivic homotopy theory* J. Pure Appl. Algebra, no. 8, 1478-1495 (2014)

[Mac71] S. ac Lane: *Categories for the working mathematician,* Graduate Texts in Math; vol. 5, Springer (1971)

[Mil13] J. Milne: *Lectures on Étale cohomology.* `http://www.jmilne.org/math/CourseNotes/LEC.pdf` (2013)

[Mor12] F. Morel: *$\mathbb{A}^1$-algebraic topology over a field.* Lect. Notes in Math; 2052 (2012)

[MV99] F. Morel and V. Voevodsky: *$\mathbb{A}^1$-homotopy theory of schemes* Publ. Math. IHES, tome 90, 45-143 (1999)

[Nav87] V. Navarro Aznar: *Sur la théorie de Hodge-Deligne.* Invent. Math. 90, 11-76 (1987)

[Nav12] J.A. Navarro González: *Notes for a degree in mathematics.* `http://matematicas.unex.es/navarro/degree.pdf` (2012)

[Nee01] A. Neeman: *Triangulated categories.* Ann. of Math. Studies, 148. Princ. Univ. Press (2001)

[Nor07] B. I. Dundas, M. Levine, P.A. Ostvær, O. Röndigs, V. Voevodsky: *Motivic Homotopy theory. Lecture at a Summer school at Nordfjordeid, August 2002.* Springer-Verlag. (2007)

[Pan04] I. Panin: *Riemann-Roch theorems for oriented cohomologies.* Axiomatic, enriched and motivc homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordecht, 261-333 (2004)

[Pan09] I. Panin: *Oriented cohomology theories of algebraic varieties II.* Homology, Homotopy Appl. 11, no. 1, 349-405 (2009)

[PPR08] I. Panin, K. Pimenov and O. Röndigs: *A universality theorem for Voevodsky's algebraic cobordism spectrum.* Homology, Homotopy Appl. 10, no. 2, 211-226 (2008)

[Qui67] D. Quillen: *Homotopical algebra.* Lect. Notes in Math; vol. 43, Springer-Verlag (1967)

[Rio07a] J. Riou: *Exposé XVI Travaux de Gabber sur luniformisation locale et la cohomologie étale des schémas quasi-excellents.* arXiv:1207.3648 (2007)

[Rio07b] J. Riou: *Catégorie homotopique stable d'un site suspendu avec intervalle.* Bull. Soc. Math. France, 495-547 (2007)

[Rio10] J. Riou: *Algebraic K-theory, $\mathbb{A}^1$-homotopy and Riemann-Roch theorems.* J. Topol. 3, no. 2, 229-264 (2010)

[SGA4] M. Artin, A. Grothendieck and J-L. Verdier: *SGA 4 - Theorie des topos et cohomologie etale des schemas.* Lect. Notes in Math, vol. 269, 270, 305. (1972-73)

[SGA6] P. Berthelot, A. Grothendieck and L. Illusie: *SGA 6 - Théorie des intersections et théorème de Riemann-Roch*. Lect. Notes in Math, vol. 225 (1971)

[Spi12] M. Spitzweck: *A commutative* $\mathbb{P}^1$*-spectrum representing motivic cohomology over Dedekind domains.* arXiv:1207.4078 (2012)

[SV96] A. Suslin and V. Voevodsky: *Singular homology of abstract algebraic varieties*. Invent. Math. 123, n 1, 61-94 (1996)

[Vez01] G. Vezzosi: *Brown-Peterson spectra in the stable* $\mathbb{A}^1$*-homotopy theory.* Rend. Sem. Mat. Univ. Padova 106, 47-64 (2001)

[Voe98] V. Voevodsky: $\mathbb{A}^1$*-homotopy theory*. Doc. Math. ICM 1998, vol. I, 579-604 (1998)

[Wei89] C. Weibel: *Homotopy invariant* K*-theory*. Contemporary Mathematics, vol. 83, 461-488 (1989)

A.M.D.G.