

# Vectorial Ribaucour Transformations for the Lamé Equations

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## Abstract

The vectorial extension of the Ribaucour transformation for the Lamé equations of orthogonal conjugates nets in multidimensions is given. We show that the composition of two vectorial Ribaucour transformations with appropriate transformation data is again a vectorial Ribaucour transformation, from which it follows the permutability of the vectorial Ribaucour transformations. Finally, as an example we apply the vectorial Ribaucour transformation to the Cartesian background.

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1. The connection between Soliton Theory and Differential Geometry of Surfaces in Euclidean Space is well established. Many systems considered in Geometry have been analyzed independently in Soliton Theory, as examples we cite the Liouville and sine-Gordon equations which characterize minimal and pseudo-spherical surfaces, respectively. An important case is given by the the Darboux equations for conjugate systems of coordinates that were solved 12 years ago in its matrix generalization, using the  $\bar{\partial}$ -dressing, by Zakharov and Manakov [19], and further the Lamé equations for orthogonal conjugate nets were solved only very recently [18] by Zakharov imposing appropriate constraints in the Marchenko integral equation associated with the Darboux equations.

In this note we present a vectorial extension of a transformation that preserves the Lamé equations which is known as Ribaucour transformation [15]. This vectorial extension can be thought as the result of the iteration of the standard Ribaucour transformation; i. e. sequences of Ribaucour transformations. The expressions that we found are expressed in terms of multi-Grammian type determinants, as in the fundamental transformation case.

The layout of this latter is as follows. In §2 we recall the reader the Darboux system for conjugate nets and its vectorial fundamental transformations, then in §3 we present the Lamé equations for orthogonal conjugate nets and show how the vectorial fundamental transformation reduces to the vectorial Ribaucour transformation. Here we also prove that, given an orthonormal basis of tangent vectors to the orthogonal conjugate coordinate lines, the vectorial Ribaucour transformation preserves this character. Next, in §4 we prove the permutability for the vectorial Ribaucour transformation basing the discussion in a similar existing result for the vectorial fundamental transformation. Finally, in §5 we present an example: we dress the zero background, specifically the Cartesian coordinates.

## 2. The Darboux equations

$$\frac{\partial \beta_{ij}}{\partial u_k} = \beta_{ik} \beta_{kj}, \quad i, j, k = 1, \dots, N, \text{ with } i, j, k \text{ different}, \quad (1)$$

for the  $N(N-1)$  functions  $\{\beta_{ij}\}_{\substack{i,j=1,\dots,N \\ i \neq j}}$  of  $\mathbf{u} := (u_1, \dots, u_N)$ , characterize  $N$ -dimensional submanifolds of  $\mathbb{R}^D$ ,  $N \leq D$ , parametrized by conjugate coordinate systems [3, 7], and are the compatibility conditions of the following

linear system

$$\frac{\partial \mathbf{X}_j}{\partial u_i} = \beta_{ji} \mathbf{X}_i, \quad i, j = 1, \dots, N, \quad i \neq j, \quad (2)$$

involving suitable  $D$ -dimensional vectors  $\mathbf{X}_i$ , tangent to the coordinate lines. The so called Lamé coefficients satisfy

$$\frac{\partial H_j}{\partial u_i} = \beta_{ij} H_i, \quad i, j = 1, \dots, N, \quad i \neq j, \quad (3)$$

and the points of the surface  $\mathbf{x}$  can be found by means of

$$\frac{\partial \mathbf{x}}{\partial u_i} = \mathbf{X}_i H_i, \quad i = 1, \dots, N, \quad (4)$$

which is equivalent to the more standard Laplace equation

$$\frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} = \frac{\partial \ln H_i}{\partial u_j} \frac{\partial \mathbf{x}}{\partial u_i} + \frac{\partial \ln H_j}{\partial u_i} \frac{\partial \mathbf{x}}{\partial u_j}, \quad i, j = 1, \dots, N, \quad i \neq j.$$

The fundamental transformation for the Darboux system was introduced by [11, 8], see also [9, 12], and its vectorial extension was given, in a discrete framework in [14, 6]. It requires the introduction of a potential in the following manner: given vector solutions  $\boldsymbol{\xi}_i \in V$  and  $\boldsymbol{\zeta}_i^* \in W^*$  of (2) and (3),  $i = 1, \dots, N$ , respectively, where  $V, W$  are linear spaces and  $W^*$  is the dual space of  $W$ , one can define a potential matrix  $\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) : W \rightarrow V$  through the equations

$$\frac{\partial \Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*)}{\partial u_i} = \boldsymbol{\xi}_i \otimes \boldsymbol{\zeta}_i^*. \quad (5)$$

We give here the continuous version of the vectorial fundamental transformation for quadrilateral lattices [14, 6].

**Vectorial Fundamental Transformation.** *Given solutions  $\boldsymbol{\xi}_i \in V$  and  $\boldsymbol{\xi}_i^* \in V^*$  of (2) and (3),  $i = 1, \dots, N$ , respectively, new rotation coefficients  $\hat{\beta}_{ij}$ , tangent vectors  $\hat{\mathbf{X}}_i$ , Lamé coefficients  $\hat{H}$  and points of the surface  $\hat{\mathbf{x}}$  are given by*

$$\begin{aligned} \hat{\beta}_{ij} &= \beta_{ij} - \langle \boldsymbol{\xi}_j^*, \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1} \boldsymbol{\xi}_i \rangle, \\ \hat{\mathbf{X}}_i &= \mathbf{X}_i - \Omega(\mathbf{X}, \boldsymbol{\xi}^*) \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1} \boldsymbol{\xi}_i, \\ \hat{H}_i &= H_i - \boldsymbol{\xi}_i^* \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1} \Omega(\boldsymbol{\xi}, H), \\ \hat{\mathbf{x}} &= \mathbf{x} - \Omega(\mathbf{X}, \boldsymbol{\xi}^*) \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1} \Omega(\boldsymbol{\xi}, H). \end{aligned} \quad (6)$$

Here we are assuming that  $\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$  is invertible. We shall refer to this transformation as vectorial fundamental transformation with transformation data  $(V, \boldsymbol{\xi}_i, \boldsymbol{\xi}_i^*)$ .

**3.** The Lamé equations describe  $N$ -dimensional conjugate orthogonal systems of coordinates [13, 4, 17]:

$$\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik} \beta_{kj} = 0, \quad i, j, k = 1, \dots, N, \text{ with } i, j, k \text{ different}, \quad (7)$$

$$\frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_{\substack{k=1, \dots, N \\ k \neq i, j}} \beta_{ki} \beta_{kj} = 0, \quad i, j = 1, \dots, N, \quad i \neq j. \quad (8)$$

An important observation, that in the scalar case appears in [16], is:

**Lemma 1.** *Given a solution  $\boldsymbol{\xi}_i \in V$  of (2) then*

$$\boldsymbol{\xi}_i^* := \left( \frac{\partial \boldsymbol{\xi}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \boldsymbol{\xi}_k \beta_{ki} \right)^t, \quad (9)$$

where  $^t$  means transpose, is a  $V^*$ -valued solution of (3) if and only if (8) holds.

*Proof.* Just notice that from (1) and (2) it follows that

$$\frac{\partial \boldsymbol{\xi}_j^*}{\partial u_i} = \beta_{ij} \boldsymbol{\xi}_i^* + \left( \frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_{\substack{k=1, \dots, N \\ k \neq i, j}} \beta_{ki} \beta_{kj} \right) \boldsymbol{\xi}_i^t.$$

□

A second observation is that,

**Lemma 2.** *Given  $\beta$ 's solving the Lamé equations (7) and (8),  $\boldsymbol{\xi}_i \in V$  and  $\boldsymbol{\zeta}_i \in W$  solutions of (2) and  $\boldsymbol{\xi}_i^*$  and  $\boldsymbol{\zeta}_i^*$  as prescribed in (9),  $i = 1, \dots, N$ , then:*

$$\frac{\partial}{\partial u_i} \left( \Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) + \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)^t - \sum_{k=1, \dots, N} \boldsymbol{\xi}_k \otimes \boldsymbol{\zeta}_k^t \right) = 0, \quad i = 1, \dots, N,$$

*Proof.* Using (5) and the definition (9) we have

$$\begin{aligned} \frac{\partial}{\partial u_i}(\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) + \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)^\dagger) &= \boldsymbol{\xi}_i \otimes \left( \frac{\partial \boldsymbol{\zeta}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \boldsymbol{\zeta}_k \beta_{ki} \right)^\dagger \\ &\quad + \left( \frac{\partial \boldsymbol{\xi}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \boldsymbol{\xi}_k \beta_{ki} \right) \otimes \boldsymbol{\zeta}_i^\dagger, \end{aligned}$$

and recalling that  $\boldsymbol{\xi}_i \beta_{ki} = (\partial/\partial u_i)\boldsymbol{\xi}_k$  and  $\boldsymbol{\zeta}_i \beta_{ki} = (\partial/\partial u_i)\boldsymbol{\zeta}_k$  we get the statement.  $\square$

Therefore, as  $\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*)$  and  $\Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)$  are defined by (5) up to additive constant matrices, the previous lemma is telling us that we can take those constants such that

$$\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) + \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)^\dagger = \sum_{k=1, \dots, N} \boldsymbol{\xi}_k \otimes \boldsymbol{\zeta}_k^\dagger.$$

We now can prove the following lemma:

**Lemma 3.** *Suppose given a solution  $\beta_{ij}$  of the Lamé equations (7) and (8),  $\boldsymbol{\xi}_i \in V$  and  $\boldsymbol{\zeta}_i \in W$  solving (2) and  $\boldsymbol{\xi}_i^*$  and  $\boldsymbol{\zeta}_i^*$  as prescribed in (9). Then, if*

$$\begin{aligned} \Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) + \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)^\dagger &= \sum_{k=1, \dots, N} \boldsymbol{\xi}_k \otimes \boldsymbol{\zeta}_k^\dagger, \\ \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*) + \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^\dagger &= \sum_{k=1, \dots, N} \boldsymbol{\xi}_k \otimes \boldsymbol{\xi}_k^\dagger \end{aligned} \tag{10}$$

the vectorial fundamental transformation (6):

$$\begin{aligned} \hat{\beta}_{ij} &= \beta_{ij} - \langle \boldsymbol{\xi}_j^*, \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1} \boldsymbol{\xi}_i \rangle, \\ \hat{\boldsymbol{\zeta}}_i &= \boldsymbol{\zeta}_i - \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*) \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1} \boldsymbol{\xi}_i, \\ \hat{\boldsymbol{\zeta}}_i^* &= \boldsymbol{\zeta}_i^* - \boldsymbol{\xi}_i^* \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1} \Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*), \end{aligned}$$

is such that

$$\hat{\boldsymbol{\zeta}}_i^* := \left( \frac{\partial \hat{\boldsymbol{\zeta}}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \hat{\boldsymbol{\zeta}}_k \hat{\beta}_{ki} \right)^\dagger.$$

*Proof.* Using (5), (6) and (9) we find that

$$\begin{aligned} \frac{\partial \hat{\zeta}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \hat{\zeta}_k \hat{\beta}_{ki} &= (\zeta_i^*)^t - \Omega(\zeta, \xi^*) \Omega(\xi, \xi^*)^{-1} (\xi_i^*)^t \\ &\quad - \sum_{k=1, \dots, N} \langle \xi_i^*, \Omega(\xi, \xi^*)^{-1} \xi_k \rangle (\zeta_k - \Omega(\zeta, \xi^*) \Omega(\xi, \xi^*)^{-1} \xi_k), \end{aligned}$$

that together with the identity

$$\langle \xi_i^*, \Omega(\xi, \xi^*)^{-1} \xi_k \rangle = \xi_k^t (\Omega(\xi, \xi^*)^{-1})^t (\xi_i^*)^t,$$

implies

$$\begin{aligned} \frac{\partial \hat{\zeta}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \hat{\zeta}_k \hat{\beta}_{ki} &= (\zeta_i^*)^t - \left[ \Omega(\zeta, \xi^*) \Omega(\xi, \xi^*)^{-1} \right. \\ &\quad \left. + \sum_{k=1, \dots, N} \left( \zeta_k - \Omega(\zeta, \xi^*) \Omega(\xi, \xi^*)^{-1} \xi_k \right) \otimes \xi_k^t (\Omega(\xi, \xi^*)^{-1})^t \right] (\xi_i^*)^t. \end{aligned}$$

Now, the constraints (10) applied to the above expression gives

$$\frac{\partial \hat{\zeta}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \hat{\zeta}_k \hat{\beta}_{ki} = (\zeta_i^*)^t - \Omega(\xi, \zeta^*)^t (\Omega(\xi, \xi^*)^{-1})^t (\xi_i^*)^t,$$

that when transposed gives the desired equality.  $\square$

With these lemmas available we are able to state the main theorem of this letter:

**Theorem.** *The vectorial fundamental transformation (6) when applied to a solution of the Lamé equation preserves the orthogonal character of the conjugate net whenever the transformation data  $(V, \xi_i, \xi_i^*)$  satisfy*

$$\begin{aligned} (\xi_i^*)^t &= \frac{\partial \xi_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \xi_k \beta_{ki}, \\ \Omega(\xi, \xi^*) + \Omega(\xi, \xi^*)^t &= \sum_{k=1, \dots, N} \xi_k \otimes \xi_k^t. \end{aligned}$$

*Proof.* Lemma 3 together with Lemma 1 imply that the new  $\hat{\beta}$  are a solution of the Lamé equations (7) and (8).  $\square$

A vectorial fundamental transformation with data  $(V, \boldsymbol{\xi}_i, \boldsymbol{\xi}_i^*)$  as in the Theorem will be referred as a vectorial Ribaucour transformation with data  $(V, \boldsymbol{\xi}_i)$ . In the scalar case the vectorial Ribaucour transformation reduces to the Ribaucour transformation [15, 4, 9, 17].

The Lamé equations (7) and (8) are the compatibility conditions for

$$\begin{aligned} \frac{\partial \mathbf{X}_j}{\partial u_i} &= \beta_{ji} \mathbf{X}_i, \quad i, j = 1, \dots, N \quad i \neq j, \\ \frac{\partial \mathbf{X}_i}{\partial u_i} &= - \sum_{\substack{k=1, \dots, N \\ k \neq i}} \mathbf{X}_k \beta_{ki}. \end{aligned} \quad (11)$$

These conditions are equivalent to the fact that the independent tangent vectors  $\{\mathbf{X}_i(\mathbf{u})\}_{i=1, \dots, N}$  form an orthonormal basis for all  $\mathbf{u}$  if they do for a particular value  $\mathbf{u}_0$ ; i. e.  $\mathbf{X}_i^t \mathbf{X}_j = \delta_{ij}$ . We now show that the vectorial Ribaucour transformation preserves this orthonormal character for the transformed basis. Indeed, (11) together with (9) implies  $\mathbf{X}_i^* = 0$  and the vectorial fundamental transformation gives  $\hat{\mathbf{X}}_i^* = 0$  if  $\Omega(\boldsymbol{\xi}, 0)$ , which is an arbitrary constant matrix, is taken as zero. Hence, Lemma 3 implies that

$$\frac{\partial \hat{\mathbf{X}}_i}{\partial u_i} = - \sum_{\substack{k=1, \dots, N \\ k \neq i}} \hat{\mathbf{X}}_k \hat{\beta}_{ki},$$

and recalling that

$$\frac{\partial \hat{\mathbf{X}}_j}{\partial u_i} = \hat{\beta}_{ji} \hat{\mathbf{X}}_i, \quad i, j = 1, \dots, N \quad i \neq j,$$

is satisfied, we find out that the new tangent vectors  $\{\hat{\mathbf{X}}_i(\mathbf{u})\}_{i=1, \dots, N}$  form an orthonormal basis for all  $\mathbf{u}$  if they do for some value of  $\mathbf{u} = \mathbf{u}_0$ . Indeed, by choosing  $\Omega(\mathbf{X}, \boldsymbol{\xi}^*) = 0$  one gets  $\mathbf{X}_i(\mathbf{u}_0) = \hat{\mathbf{X}}_i(\mathbf{u}_0)$ ; i. e. the initial basis and the transformed one coincide at that point.

Notice that the above results constitute an alternative proof of the Theorem.

4. In [6] it was proven a permutability theorem for the vectorial fundamental transformation of quadrilateral lattices, here we give its continuous limit to conjugate nets:

**Permutability of Vectorial Fundamental Transformations.** *The vectorial fundamental transformation with transformation data*

$$\left( V_1 \oplus V_2, \left( \begin{array}{c} \boldsymbol{\xi}_{i,(1)} \\ \boldsymbol{\xi}_{i,(2)} \end{array} \right), (\boldsymbol{\xi}_{i,(1)}^*, \boldsymbol{\xi}_{i,(2)}^*) \right)$$

*coincides with the following composition of vectorial fundamental transformations:*

1. *First transform with data*

$$(V_2, \boldsymbol{\xi}_{i,(2)}, \boldsymbol{\xi}_{i,(2)}),$$

*and denote the transformation by '.*

2. *On the result of this transformation apply a second one with data*

$$(V_1, \boldsymbol{\xi}'_{i,(1)}, \boldsymbol{\xi}'_{i,(1)}).$$

Therefore, the composition of two vectorial fundamental transformations yields, independently of the order, a new vectorial fundamental transformation; hence the permutability character of these transformations. Moreover, from this result it also follows that the vectorial fundamental transformation is just a superposition of a number of fundamental transformations.

One can easily conclude that this result can be extended to the vectorial Ribaucour transformation for orthogonal conjugate nets.

**Proposition.** *The vectorial Ribaucour transformation with transformation data*

$$\left( V_1 \oplus V_2, \left( \begin{array}{c} \boldsymbol{\xi}_{i,(1)} \\ \boldsymbol{\xi}_{i,(2)} \end{array} \right) \right),$$

*as prescribed in our Theorem, coincides with the following composition of vectorial Ribaucour transformations:*

1. *First transform with data*

$$(V_2, \boldsymbol{\xi}_{i,(2)}),$$

*and denote the transformation by '.*

2. On the result of this transformation apply a second one with data

$$(V_1, \boldsymbol{\xi}'_{i,(1)}).$$

*Proof.* Because the transformation data follows the prescription of our Theorem they must satisfy

$$\begin{aligned} (\boldsymbol{\xi}^*_{i,(s)})^t &= \frac{\partial \boldsymbol{\xi}_{i,(s)}}{\partial u_i} + \sum_{\substack{k=1,\dots,N \\ k \neq i}} \boldsymbol{\xi}_{k,(s)} \beta_{ki}, \quad s = 1, 2 \\ \Omega(\boldsymbol{\xi}_{(s)}, \boldsymbol{\xi}^*_{(s)}) + \Omega(\boldsymbol{\xi}_{(s)}, \boldsymbol{\xi}^*_{(s)})^t &= \sum_{k=1,\dots,N} \boldsymbol{\xi}_{k,(s)} \otimes \boldsymbol{\xi}^t_{k,(s)}, \quad s = 1, 2 \\ \Omega(\boldsymbol{\xi}_{(1)}, \boldsymbol{\xi}^*_{(2)}) + \Omega(\boldsymbol{\xi}_{(2)}, \boldsymbol{\xi}^*_{(1)})^t &= \sum_{k=1,\dots,N} \boldsymbol{\xi}_{k,(1)} \otimes \boldsymbol{\xi}^t_{k,(2)}. \end{aligned}$$

Thus, we see that the first vectorial fundamental transformation is a vectorial Ribaucour transformation with data  $(V, \boldsymbol{\xi}_{i,(2)})$ . Now, applying Lemma 3, we see that the vectorial fundamental transformation of point 2. is also a vectorial Ribaucour transformation.  $\square$

5. For the zero background  $\beta_{ij} = 0$  we have that the solutions of (2) are any set of functions  $\{\boldsymbol{\xi}_i\}_{i=1,\dots,N}$  of the form

$$\boldsymbol{\xi}_i = \boldsymbol{\xi}_i(u_i) \in \mathbb{R}^M$$

and for the adjoint we have

$$\boldsymbol{\xi}_i^* = \frac{d \boldsymbol{\xi}_i^t}{d u_i}.$$

We also have

$$\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)(\mathbf{u}) = \sum_{k=1,\dots,N} \Omega_k(u_k)$$

with

$$\begin{aligned} \Omega_i(u_i) &= \int_{u_{i,0}}^{u_i} d u_i \boldsymbol{\xi}_i \otimes \frac{d \boldsymbol{\xi}_i^t}{d u_i} + \Omega_{i,0}, \\ \Omega_{i,0} + \Omega_{i,0}^t &= (\boldsymbol{\xi}_i \otimes \boldsymbol{\xi}_i^t)|_{u_{i,0}}. \end{aligned}$$

In particular, the Cartesian background has  $\mathbf{X}_i = \mathbf{e}_i$ ,  $\{\mathbf{e}_i\}_{i=1,\dots,N}$  an canonical basis of  $\mathbb{R}^N$ ,  $H_i = 1$  and the coordinates are  $\mathbf{x}(\mathbf{u}) = \mathbf{u}$ . This implies that

$$\begin{aligned}\Omega(\mathbf{X}, \boldsymbol{\xi}^*)(\mathbf{u}) &= A + \sum_{k=1,\dots,N} \mathbf{e}_i \otimes \boldsymbol{\xi}_i^t(u_i), \\ \Omega(\boldsymbol{\xi}, H)(\mathbf{u}) &= \mathbf{c} + \sum_{k=1,\dots,N} \int_{u_{i,0}}^{u_i} d u_i \boldsymbol{\xi}_i(u_i),\end{aligned}$$

where  $A$  is a constant  $N \times M$  matrix and  $\mathbf{c} \in \mathbb{R}^N$  is a constant vector, and the orthogonal conjugate net is given by

$$\begin{aligned}\mathbf{x}(\mathbf{u}) = \mathbf{u} - \left[ A + \sum_{k=1,\dots,N} \mathbf{e}_i \otimes \boldsymbol{\xi}_i^t(u_i) \right] \left[ \sum_{k=1,\dots,N} \Omega_i(u_i) \right]^{-1} \\ \times \left[ \mathbf{c} + \sum_{k=1,\dots,N} \int_{u_{i,0}}^{u_i} d u_i \boldsymbol{\xi}_i(u_i) \right].\end{aligned}$$

**6.** In contrast with the well known Laplace and Levy transformations there is no literature on sequences of Ribaucour transformations, however in [5] a permutability theorem was proven for the 2-dimensional case iterating the Ribaucour transformation twice. Later in [1], see also [2], it was done in three dimensional case and in [9] one can find the extension to any dimension. Recently, in [10] three Ribaucour transformations were iterated in 3-dimensional space to get some results related with permutability. The permutability theorem for the scalar fundamental was established in [11].

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