Evolution of density perturbations in $f(R)$ theories of gravity

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In the context of $f(R)$ theories of gravity, we study the evolution of scalar cosmological perturbations in the metric formalism. Using a completely general procedure, we find the exact fourth-order differential equation for the matter density perturbations in the longitudinal gauge. In the case of sub-Hubble modes, the expression reduces to a second-order equation which is compared with the standard (quasistatic) equation used in the literature. We show that for general $f(R)$ functions the quasistatic approximation is not justified. However, for those functions adequately describing the present phase of accelerated expansion and satisfying local gravity tests, it provides a correct description for the evolution of perturbations.

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I. INTRODUCTION

The present phase of accelerated expansion of the universe [1] poses one of the most important problems of modern cosmology. It is well-known that ordinary Einstein's equations in either a matter or radiation dominated universe give rise to decelerated periods of expansion. In order to have acceleration, the total energy-momentum tensor appearing on the right-hand side of the equations should be dominated at late times by a hypothetical negative pressure fluid usually called dark energy (see [2] and references therein).

However, there are other possibilities to generate a period of acceleration in which no new sources are included on the right-hand side of the equations, but instead Einstein's gravity itself is modified [3]. In one of such possibilities, new functions of the curvature scalar [$f(R)$ terms] are included in the gravitational action, which amounts to modifiying the left-hand side of the equations of motion. Although such theories are able to describe the accelerated expansion on cosmological scales correctly, they typically give rise to strong effects on smaller scales. In any case viable models can be constructed to be compatible with local gravity tests and other cosmological constraints [4].

The important question that arises is therefore how to discriminate dark energy models from modified gravities using present or future observations. It is known that by choosing particular $f(R)$ functions, one can mimic any background evolution (expansion history), and, in particular, that of ΛCDM. Accordingly, the exclusive use of observations such as high-redshift Hubble diagrams from supernovae type Ia [1], baryon acoustic oscillations [5] or CMB shift factor [6], based on different distance measurements which are sensitive only to the expansion history, cannot settle the question of the nature of dark energy [7].

However, there exist observations of a different type which are sensitive, not only to the expansion history, but also to the evolution of matter density perturbations. The fact that the evolution of perturbations depends on the specific gravity model, i.e., it differs in general from that of Einstein's gravity even though the background evolution is the same, means that observations of this kind will help distinguish between different models for acceleration.

In this work we study the problem of determining the exact equation for the evolution of matter density perturbations for arbitrary $f(R)$ theories. Such a problem had been previously considered in the literature ([8–13]) and approximated equations have been widely used. They are typically based on the so-called quasistatic approximation in which all the time derivative terms for the gravitational potentials are discarded, and only those including density perturbations are kept [14]. From our exact result, we will be able to determine under which conditions such an approximation can be justified.

The paper is organized as follows: In Sec. II, we briefly review the perturbations equations for the standard ΛCDM model. In Sec. III we obtain the perturbed equations for general $f(R)$ theories. In Sec. IV we describe the procedure to obtain the general equation for the density perturbation. In Sec. V we summarize the main viability condition for $f(R)$ theories. Section VI is devoted to the study of the validity of the quasistatic approximation. In Sec. VII we apply our results to some particular models, and finally in Sec. VIII we include the main conclusions. In Appendixes A and B we have also included complete expressions for the relevant coefficients of the perturbation equation.

II. DENSITY PERTURBATIONS IN ΛCDM

Let us start by considering the simplest model for dark energy described by a cosmological constant Λ. The cor-

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responding Einstein's equations read

$$G^{\mu}{}_{\nu} = -8\pi G T^{\mu}{}_{\nu} - \Lambda \delta^{\mu}{}_{\nu}, \tag{1}$$

where $G^{\mu}{}_{\nu}$ is the Einstein's tensor and $T^{\mu}{}_{\nu}$ is the energy-momentum tensor for matter.

In the metric formalism for the ΛCDM model it is possible to obtain a second-order differential equation for the growth of matter density perturbation $\delta \equiv \delta\rho/\rho_0$. Let us consider the scalar perturbations of a flat Friedmann-Robertson-Walker metric in the longitudinal gauge and in conformal time:

$$ds^2 = a^2(\eta)[(1+2\Phi)d\eta^2 - (1-2\Psi)(dr^2 + r^2 d\Omega_2^2)], \tag{2}$$

where $\Phi \equiv \Phi(\eta, \vec{x})$ and $\Psi \equiv \Psi(\eta, \vec{x})$ are the scalar perturbations. From this metric, we obtain the first-order perturbed Einstein's equation:

$$\delta G^{\mu}{}_{\nu} = -8\pi G \delta T^{\mu}{}_{\nu}, \tag{3}$$

where the perturbed energy-momentum tensor reads

$$\delta T^0{}_0 = \delta\rho = \rho_0 \delta, \qquad \delta T^i{}_j = -\delta P \delta^i{}_j = -c_s^2 \delta^i{}_j \rho_0 \delta$$
$$\delta T^0{}_i = -\delta T^i{}_0 = -(1+c_s^2)\rho_0 \partial_i v, \tag{4}$$

with ρ_0 the unperturbed energy density and v the potential for velocity perturbations. We assume that the perturbed and unperturbed matter have the same equation of state, i.e., $\delta P/\delta\rho \equiv c_s^2 \equiv P_0/\rho_0$, where $c_s = 0$ for matter perturbations. The resulting differential equation for δ in Fourier space is written as

$$\delta'' + \mathcal{H}\frac{k^4 - 6\tilde{\rho}k^2 - 18\tilde{\rho}^2}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)}\delta' - \tilde{\rho}\frac{k^4 + 9\tilde{\rho}(2\tilde{\rho} - 3\mathcal{H}^2) - k^2(9\tilde{\rho} - 3\mathcal{H}^2)}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)}\delta = 0, \tag{5}$$

where $\tilde{\rho} \equiv 4\pi G \rho_0 a^2 = -\mathcal{H}' + \mathcal{H}^2$ and $\mathcal{H} \equiv a'/a$ with prime denoting derivative with respect to time η. We point out that it is not necessary to explicitly calculate potentials Φ and Ψ to obtain Eq. (5), but algebraic manipulations in the field equations are enough to get this result. In the extreme sub-Hubble limit, i.e., $k\eta \gg 1$ or equivalently $k \gg \mathcal{H}$, (5) is reduced to the well-known expression

$$\delta'' + \mathcal{H}\delta' - 4\pi G\rho_0 a^2 \delta = 0. \tag{6}$$

In this regime and at early times, the matter energy density dominates over the cosmological constant, and it is easy to show that δ solutions for (6) grow as $a(\eta)$. At late times (near today) the cosmological constant contribution is not negligible and power-law solutions for (6) no longer exist. It is necessary in this case to assume an ansatz for δ. One which works very well is the one proposed in [7,15]:

$$\frac{\delta(a)}{a} = e^{\int_{a_i}^{a}[\Omega_m(a)^{\gamma} - 1]\mathrm{d}\ln a}. \tag{7}$$

This expression fits with high precision the numerical solution for δ with a constant parameter $\gamma = 6/11$.

III. PERTURBATIONS IN $f(R)$ THEORIES

Let us consider the modified gravitational action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R + f(R)), \tag{8}$$

where R is the scalar curvature.[1] The corresponding equations of motion read

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}f(R) + R_{\mu\nu}f_R(R) - g_{\mu\nu}\Box f_R(R) + f_R(R)_{;\mu\nu} = -8\pi G T_{\mu\nu}, \tag{9}$$

where $f_R(R) = df(R)/dR$. For the background flat Robertson-Walker metric they read

$$\frac{3\mathcal{H}'}{a^2}(1+f_R) - \frac{1}{2}(R_0 + f_0) - \frac{3\mathcal{H}}{a^2}f_R' = -8\pi G\rho_0 \tag{10}$$

and

$$\frac{1}{a^2}(\mathcal{H}' + 2\mathcal{H}^2)(1+f_R) - \frac{1}{2}(R_0 + f_0) - \frac{1}{a^2}(\mathcal{H}f_R' + f_R'') = 8\pi G c_s^2 \rho_0, \tag{11}$$

where R_0 denotes the scalar curvature corresponding to the unperturbed metric, $f_0 \equiv f(R_0)$, $f_R \equiv \mathrm{d}f(R_0)/\mathrm{d}R_0$, and prime means derivative with respect to time η. A very useful equation to use in the following calculations is the (11) and (10) combination

$$2(1+f_R)(-\mathcal{H}' + \mathcal{H}^2) + 2\mathcal{H}f_R' - f_R'' = 8\pi G\rho_0(1+c_s^2)a^2. \tag{12}$$

Finally we have the conservation equation

$$\rho_0' + 3(1+c_s^2)\mathcal{H}\rho_0 = 0. \tag{13}$$

Using the perturbed metric (2) and the perturbed energy-momentum tensor (4), the first-order perturbed equations, assuming that the background equations hold, may be written as

$$\begin{aligned}(1+f_R)\delta G^{\mu}_{\nu} &+ (R_{0\nu}{}^{\mu} + \nabla^{\mu}\nabla_{\nu} - \delta^{\mu}_{\nu}\Box)f_{RR}\delta R \\ &+ [(\delta g^{\mu\alpha})\nabla_{\nu}\nabla_{\alpha} - \delta^{\mu}_{\nu}(\delta g^{\alpha\beta})\nabla_{\alpha}\nabla_{\beta}]f_R \\ &- [g_0^{\alpha\mu}(\delta\Gamma^{\gamma}_{\alpha\nu}) - \delta^{\mu}_{\nu}g_0^{\alpha\beta}(\delta\Gamma^{\gamma}_{\beta\alpha})]\partial_{\gamma}f_R \\ &= -8\pi G\delta T^{\mu}_{\nu},\end{aligned} \tag{14}$$

[1]The Riemann tensor definition is $R^{\mu}{}_{\nu\alpha\beta} = \partial_{\beta}\Gamma^{\mu}_{\nu\alpha} - \partial_{\alpha}\Gamma^{\mu}_{\nu\beta} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta}$, which has an opposite sign to the one proposed in [16].

where $f_{RR} = \mathrm{d}^2 f(R_0)/\mathrm{d}R_0^2$, $\Box \equiv \nabla_\alpha \nabla^\alpha$, and ∇ is the usual covariant derivative with respect to the unperturbed Friedmann-Robertson-Walker metric (see [16] for perturbed metric, connection symbols, and other useful perturbed quantities). Notice that unlike the ordinary Einstein-Hilbert (EH) case, with second-order equations, this is a set of fourth-order differential equations. By computing the covariant derivative with respect to the perturbed metric $\tilde{\nabla}$ of the perturbed energy-momentum tensor $\tilde{T}^\mu_\nu$, we find the conservation equations

$$\tilde{\nabla}_\mu \tilde{T}^\mu_\nu = 0 \tag{15}$$

which do not depend on $f(R)$.

For the linearized Einstein's equations, the components (00), (ii), $(0i) \equiv (i0)$, and (ij), where $i, j = 1, 2, 3$, $i \neq j$, in Fourier space, read, respectively,

$$\begin{aligned}(1 + f_R)[&-k^2(\Phi + \Psi) - 3\mathcal{H}(\Phi' + \Psi') \\ &+ (3\mathcal{H}' - 6\mathcal{H}^2)\Phi - 3\mathcal{H}'\Psi] + f_R'(-9\mathcal{H}\Phi \\ &+ 3\mathcal{H}\Psi - 3\Psi') \\ &= 2\tilde{\rho}\delta,\end{aligned} \tag{16}$$

$$\begin{aligned}(1 + f_R)[&\Phi'' + \Psi'' + 3\mathcal{H}(\Phi' + \Psi') + 3\mathcal{H}'\Phi \\ &+ (\mathcal{H}' + 2\mathcal{H}^2)\Psi] + f_R'(3\mathcal{H}\Phi - \mathcal{H}\Psi + 3\Phi') \\ &+ f_R''(3\Phi - \Psi) \\ &= 2c_s^2\tilde{\rho}\delta,\end{aligned} \tag{17}$$

$$\begin{aligned}(1 + f_R)[\Phi' + \Psi' + \mathcal{H}(\Phi + \Psi)] + f_R'(2\Phi - \Psi) \\ = -2\tilde{\rho}(1 + c_S^2)v,\end{aligned} \tag{18}$$

$$\Phi - \Psi = -\frac{f_{RR}}{1 + f_R}\delta R, \tag{19}$$

where δR is given by

$$\begin{aligned}\delta R = -\frac{2}{a^2}[&3\Psi'' + 6(\mathcal{H}' + \mathcal{H}^2)\Phi + 3\mathcal{H}(\Phi' + 3\Psi') \\ &- k^2(\Phi - 2\Psi)].\end{aligned} \tag{20}$$

Finally, from the energy-momentum tensor conservation (15), we get to first order:

$$3\Psi'(1 + c_s^2) - \delta' + k^2(1 + c_s^2)v = 0 \tag{21}$$

and

$$\Phi + \frac{c_s^2}{1 + c_s^2}\delta + v' + \mathcal{H}v(1 - 3c_s^2) = 0 \tag{22}$$

for the temporal and spatial components, respectively.

In a dust matter dominated universe, i.e., $c_s^2 = 0$, (21) and (22) can be combined to give

$$\delta'' + \mathcal{H}\delta' + k^2\Phi - 3\Psi'' - 3\mathcal{H}\Psi' = 0, \tag{23}$$

which will be very useful in future calculations.

IV. EVOLUTION OF DENSITY PERTURBATIONS

Our porpose is to derive a fourth-order differential equation for matter density perturbation δ alone. This can be performed by means of the following process.

Let us consider Eqs. (16) and (18) for a matter dominated universe i.e., $c_s^2 = 0$, and combine them to express the potentials Φ and Ψ in terms of $\{\Phi', \Psi', \delta, \delta'\}$ by means of algebraic manipulations. The resulting expressions are the following:

$$\begin{aligned}\Phi = \frac{1}{\mathcal{D}(\mathcal{H}, k)}\Big\{&[3(1 + f_R)\mathcal{H}(\Psi' + \Phi') + f_R'\Psi' + 2\tilde{\rho}\delta] \\ &\times (1 + f_R)(\mathcal{H} - f_R') + \Big[(1 + f_R)(\Phi' + \Psi') + \frac{2\tilde{\rho}}{k^2} \\ &\times (\delta' - 3\Psi')\Big][(1 + f_R)(-k^2 - 3\mathcal{H}') + 3f_R'\mathcal{H}]\Big\}\end{aligned} \tag{24}$$

and

$$\begin{aligned}\Psi = \frac{1}{\mathcal{D}(\mathcal{H}, k)}\Big\{&[-3(1 + f_R)\mathcal{H}(\Psi' + \Phi') - 3f_R'\Psi' \\ &- 2\tilde{\rho}\delta][(1 + f_R)\mathcal{H} + 2f_R'] - \Big[(1 + f_R)(\Phi' \\ &+ \Psi') + \frac{2\tilde{\rho}}{k^2}(\delta' - 3\Psi')\Big][(1 + f_R)(-k^2 + 3\mathcal{H}' \\ &- 6\mathcal{H}^2) - 9\mathcal{H}f_R']\Big\},\end{aligned} \tag{25}$$

where

$$\begin{aligned}\mathcal{D}(\mathcal{H}, k) \equiv -6(1 + f_R)^2\mathcal{H}^3 + 3\mathcal{H}[&f_R'^2 \\ + 2(1 + f_R)^2\mathcal{H}'] + 3(1 + f_R)&f_R'(-2\mathcal{H}^2 \\ + k^2 + \mathcal{H}').&\end{aligned} \tag{26}$$

The second step will be to derive Eqs. (24) and (25) with respect to η and obtain Φ' and Ψ' algebraically in terms of $\{\Phi'', \Psi''; \delta, \delta', \delta''\}$. These last results can be substituted in Eqs. (16) and (18) to obtain potentials Φ and Ψ just in terms of $\{\Phi'', \Psi'', \delta, \delta', \delta''\}$. So at this stage we are able to express, but we do not do here explicitly, the following:

$$\begin{aligned}\Phi &= \Phi(\Phi'', \Psi''; \delta, \delta', \delta''), \\ \Psi &= \Psi(\Phi'', \Psi''; \delta, \delta', \delta''), \\ \Phi' &= \Phi'(\Phi'', \Psi''; \delta, \delta', \delta''), \\ \Psi' &= \Psi'(\Phi'', \Psi''; \delta, \delta', \delta''),\end{aligned} \tag{27}$$

where we mean that the functions on the left-hand side are algebraically dependent on the functions inside the parenthesis on the right-hand side.

The natural reasoning at this point would be to try to obtain the potentials' second derivatives $\{\Phi'', \Psi''\}$ in terms of $\{\delta, \delta', \delta''\}$ by an algebraic process. The chosen equations to do so will be (19) and (23), the first derivative with respect to η. In (23) it is necessary to substitute Φ and Ψ' by the expressions obtained in (27), whereas in (19) the first derivative may be sketched as follows:

$$\Phi' - \Psi' = -\frac{f_{RR}}{1+f_R}\delta R' + \left[\frac{f_{RR}f_R' - f_{RR}'(1+f_R)}{(1+f_R)^2}\right]\delta R. \tag{28}$$

Before deriving, we are going to substitute the Ψ'' that appears in (19) by lower derivatives' potentials $\{\Phi, \Psi, \Phi', \Psi'\}$, δ, and its derivatives. To do so we consider in (16) and (18) the first derivatives with respect to η where the quantity v has been previously substituted by its expression in (21). Following this process we may express Ψ'' as follows:

$$\Psi'' = \Psi''(\Phi, \Psi, \Phi', \Psi'; \delta, \delta', \delta''), \tag{29}$$

and now substituting in (19) we can derive that equation with respect to η. Solving a two algebraic equations system with Eqs. (23) and (28) and introducing (27) we are able to express $\{\Phi'', \Psi''\}$ in terms of $\{\delta, \delta', \delta'', \delta'''\}$:

$$\Phi'' = \Phi''(\delta, \delta', \delta'', \delta'''); \qquad \Psi'' = \Psi''(\delta, \delta', \delta'', \delta'''). \tag{30}$$

We substitute the results obtained in (30) straightforwardly in (27) in order to express $\{\Phi, \Psi, \Phi', \Psi'\}$ in terms of $\{\delta, \delta', \delta'', \delta'''\}$. With the two potentials and their first derivatives as algebraic functions of $\{\delta, \delta', \delta'', \delta'''\}$, we perform the last step: we consider $\Phi(\delta, \delta, \delta'', \delta''')$ and derive it with respect to η. The result should be equal to $\Phi'(\delta, \delta, \delta'', \delta''')$, so we only need to express together these two results obtaining a fourth-order differential equation for δ. Note that this procedure is completely general to first order for scalar perturbations in the metric formalism for $f(R)$ gravities.

Once this fourth-order differential equation has been solved, we may go backward, and by using the results for δ we obtain $\{\Phi'', \Psi''\}$ from (30) as functions of time. Analogously from (27) the behavior of the potentials $\{\Phi, \Psi\}$ and their first derivatives could be determined.

The resulting equation for δ can be written as follows:

$$\beta_{4,f}\delta^{iv} + \beta_{3,f}\delta''' + (\alpha_{2,\mathrm{EH}} + \beta_{2,f})\delta'' + (\alpha_{1,\mathrm{EH}} + \beta_{1,f})\delta' + (\alpha_{0,\mathrm{EH}} + \beta_{0,f})\delta = 0, \tag{31}$$

where the coefficients $\beta_{i,f}$ $(i = 1, \ldots, 4)$ involve terms with f_R' and f_R'', i.e., terms disappearing if we take f_R constant. Equivalently, $\alpha_{i,\mathrm{EH}}$ $(i = 0, 1, 2)$ contain terms coming from the linear part of f_0 in R_0.

It is very useful to define the parameter $\epsilon \equiv \mathcal{H}/k$ since it will allow us to perform a perturbative expansion of the previous α and β coefficients in the sub-Hubble limit. Other dimensionless parameters which will be used are the following: $\kappa_i \equiv \mathcal{H}^{\prime(i)}/\mathcal{H}^{i+1}$ $(i = 1, 2, 3)$ and $f_i \equiv f_R^{\prime(j)}/(\mathcal{H}^j f_R)$ $(j = 1, 2)$.

Expressing the α and β coefficients with those dimensionless quantities we may write

$$\alpha_{i,\mathrm{EH}} = \sum_{j=1}^{3}\alpha_{i,\mathrm{EH}}^{(j)} \qquad i = 0, 1, 2,$$
$$\beta_{i,f} = \sum_{j=1}^{7}\beta_{i,f}^{(j)} \qquad i = 3, 4, \tag{32}$$
$$\beta_{i,f} = \sum_{j=1}^{8}\beta_{i,f}^{(j)} \qquad i = 0, 1, 2,$$

where two consecutive terms in each series differ in the ϵ^2 factor. The expressions for the coefficients are too long to be written explicitly. Instead, in the following sections we will show different approximated formulas useful in certain limits.

V. VIABLE $f(R)$ THEORIES

Results obtained so far are valid for any $f(R)$ theory. However, as mentioned in the introduction, this kind of model is severely constrained in order to provide consistent theories of gravity. In this section we review the main conditions [9]:

(1) $f_{RR} > 0$ for high curvatures [17]. This is the requirement for a classically stable high-curvature regime and the existence of a matter dominated phase in the cosmological evolution.

(2) $1 + f_R > 0$ for all R_0. This condition ensures the effective Newton's constant to be positive at all times and the graviton energy to be positive.

(3) $f_R < 0$ ensures ordinary general relativity behavior is recovered at early times. Together with the condition $f_{RR} > 0$, it implies that f_R should be a negative and monotonically growing function of R_0 in the range $-1 < f_R < 0$.

(4) $|f_R| \ll 1$ at recent epochs. This is imposed by local gravity tests [17], although it is still not clear what is the actual limit on this parameter. This condition also implies that the cosmological evolution at late times resembles that of ΛCDM. In any case, this constraint is not required if we are only interested in building models for cosmic acceleration.

VI. EVOLUTION OF SUB-HUBBLE MODES AND THE QUASISTATIC APPROXIMATION

We are interested in the possible effects on the growth of density perturbations once they enter the Hubble radius in the matter dominated era. In the sub-Hubble limit $\epsilon \ll 1$, it can be seen that the $\beta_{4,f}$ and $\beta_{3,f}$ coefficients are suppressed by ϵ^2 with respect to $\beta_{2,f}$, $\beta_{1,f}$, and $\beta_{0,f}$, i.e., in this limit the equation for perturbations reduces to the following second-order expression:

$$\delta'' + \mathcal{H}\delta' + \frac{(1+f_R)^5\mathcal{H}^2(-1+\kappa_1)(2\kappa_1-\kappa_2) - \frac{16}{a^8}f_{RR}^4(\kappa_2-2)k^8 8\pi G\rho_0 a^2}{(1+f_R)^5(-1+\kappa_1) + \frac{24}{a^8}f_{RR}^4(1+f_R)(\kappa_2-2)k^8}\delta = 0, \tag{33}$$

where we have taken only the leading terms in the ϵ expansion for the α and β coefficients.

This expression can be compared with that usually considered in literature, obtained after performing strong simplifications in the perturbed equations—(16)–(19), (21), and (22)—by neglecting time derivatives of Φ and Ψ potentials (see [14]). Thus in [10,18] the authors obtain

$$\delta'' + \mathcal{H}\delta' - \frac{1 + 4\frac{k^2}{a^2}\frac{f_{RR}}{1+f_R}}{1 + 3\frac{k^2}{a^2}\frac{f_{RR}}{1+f_R}}\frac{\tilde{\rho}\delta}{1+f_R} = 0. \tag{34}$$

This approximation has been considered as too aggressive in [11] since neglecting time derivatives can remove important information about the evolution.

Note also that there exists a difference in a power k^8 between those terms coming from the f part and those coming from the EH part in (33). This result differs from that in the quasistatic approximation where difference is in a power k^2 according to (34).

In order to compare the evolution for both equations, we have considered a specific function $f_{\text{test}}(R) = -4R^{0.63}$, where H_0^2 units have been used, which gives rise to a matter era followed by a late-time accelerated phase with the correct deceleration parameter today. Initial conditions in the matter era were given at redshift $z = 485$ where the EH part was dominant. Results for $k = 600H_0$ are presented in Fig. 1. We see that, as expected, both expressions give rise to the same evolutions at early times (large redshifts) where they also agree with the standard ΛCDM evolution. However, at late times the quasistatic approximation fails to correctly describe the evolution of perturbations.

Notice that the model example satisfies all the viability conditions described in the previous section except for the local gravity tests. As we will show in the following, it is precisely this last condition $|f_R| \ll 1$ that will ensure the validity of the quasistatic approximation.

We will now restrict ourselves to models satisfying all the viability conditions, including $|f_R| \ll 1$.

In Appendix A we have reproduced all the α and the first four β coefficients for each δ term in (31). These are the dominant ones for sub-Hubble modes (i.e., $\epsilon \ll 1$) once the condition $|f_R| \ll 1$ has been imposed. Thus, keeping only $\sum_{j=1}^4 \beta_{i=0,\ldots,4,f}^{(j)}$ and $\alpha_{i=0,1,2,\text{EH}}^{(1)}$ as the relevant contributions for the general coefficients, the full differential equation (31) can be simplified as

$$c_4\delta^{iv} + c_3\delta''' + c_2\delta'' + c_1\delta' + c_0\delta = 0, \tag{35}$$

where the c coefficients are written in Appendix B.

We see that indeed in the sub-Hubble limit the c_4 and c_3 coefficients are negligible and the equation can be reduced to a second-order expression.

As a consistency check, we find that both in a matter dominated universe and in ΛCDM all β coefficients vanish identically since f_1, $f_2 \equiv 0$. For these cases, Eq. (31) becomes Eq. (6) as expected. For instance, in the pure matter dominated case, the κ coefficients are constant, and they take the following values: $\kappa_1 = -1/2$, $\kappa_2 = 1/2$, $\kappa_3 = -3/4$, and $\kappa_4 = 3/2$.

Another important feature from our results is that, in general, without imposing $|f_R| \ll 1$, the quotient $(\alpha_{1,\text{EH}} + \beta_{1,f})/(\alpha_{2,\text{EH}} + \beta_{2,f})$ is not always equal to $\mathcal{H}$. In fact only the quotients $\alpha_{1,\text{EH}}^{(1)}/\alpha_{2,\text{EH}}^{(1)}$ and $\beta_{1,f}^{(1)}/\beta_{2,f}^{(1)}$ are identically equal to $\mathcal{H}$, which is in agreement with the δ' coefficient in (6). However, for our approximated expressions it is true that $c_1/c_2 \equiv \mathcal{H}$.

From expressions in Appendix B, the second-order equation for δ becomes

$$\delta'' + \mathcal{H}\delta' - \frac{4}{3}\frac{\left[\frac{6f_{RR}k^2}{a^2} + \frac{9}{4}\left(1 - \sqrt{1 - \frac{8}{9}\frac{2\kappa_1-\kappa_2}{-2+\kappa_2}}\right)\right]\left[\frac{6f_{RR}k^2}{a^2} + \frac{9}{4}\left(1 + \sqrt{1 - \frac{8}{9}\frac{2\kappa_1-\kappa_2}{-2+\kappa_2}}\right)\right]}{\left[\frac{6f_{RR}k^2}{a^2} + \frac{5}{2}\left(1 - \sqrt{1 - \frac{24}{25}\frac{-1+\kappa_1}{-2+\kappa_2}}\right)\right]\left[\frac{6f_{RR}k^2}{a^2} + \frac{5}{2}\left(1 + \sqrt{1 - \frac{24}{25}\frac{-1+\kappa_1}{-2+\kappa_2}}\right)\right]}(1-\kappa_1)\mathcal{H}^2\delta = 0, \tag{36}$$

which can also be written as

$$\delta'' + \mathcal{H}\delta' - \frac{4}{3}\frac{\left(\frac{6f_{RR}k^2}{a^2} + \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{9}{2}\frac{2\kappa_1-\kappa_2}{-2+\kappa_2}}{\left(\frac{6f_{RR}k^2}{a^2} + \frac{5}{2}\right)^2 - \frac{25}{4} + 6\frac{-1+\kappa_1}{-2+\kappa_2}}(1-\kappa_1)\mathcal{H}^2\delta = 0. \tag{37}$$

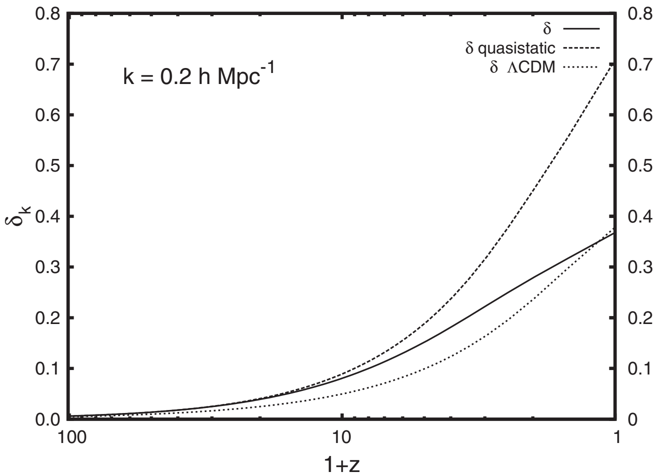

FIG. 1. δ_k with $k = 0.2\ \mathrm{h\,Mpc^{-1}}$ for $f_{\mathrm{test}}(R)$ model and ΛCDM. Both standard quasistatic evolution and Eq. (33) have been plotted in the redshift range from 100 to 0.

Note that the quasistatic expression (34) is only recovered in the matter era (i.e., for $\mathcal{H} = 2/\eta$) or for a pure ΛCDM evolution for the background dynamics. Nevertheless in the considered limit $|f_R| \ll 1$ it can be proven using the background equations of motion that

$$1 + \kappa_1 - \kappa_2 \approx 0 \tag{38}$$

and therefore $2\kappa_1 - \kappa_2 \approx -2 + \kappa_2 \approx -1 + \kappa_1$, which allows simplifying expression (37) to approximately become (34). This is nothing but the fact that for viable models the background evolution resembles that of ΛCDM [9].

In other words, although for general $f(R)$ functions the quasistatic approximation is not justified, for those viable functions describing the present phase of accelerated expansion and satisfying local gravity tests, it gives a correct description for the evolution of perturbations.

VII. SOME PROPOSED MODELS

In order to check the results obtained in the previous section, we propose two particular $f(R)$ theories which allow us to determine—at least numerically—all the quantities involved in the calculations and therefore to obtain solutions for (31). As commented before, for viable models the background evolution resembles that of ΛCDM at low redshifts and that of a matter dominated universe at high redshifts, i.e., the quantity $(R + f(R))/R$ tends to one in the high-curvature regime. Nevertheless the $f(R)$ contribution gives the dominant contribution to the gravitational action for small curvatures, and therefore it may explain the cosmological acceleration. For the sake of concreteness we will fix the model parameters imposing a deceleration parameter today $q_0 \approx -0.6$.

Thus, our first model (A) will be $f(R) = c_1 R^p$. According to the results presented in [12,19] viable models of this type include both a matter dominated and late-time

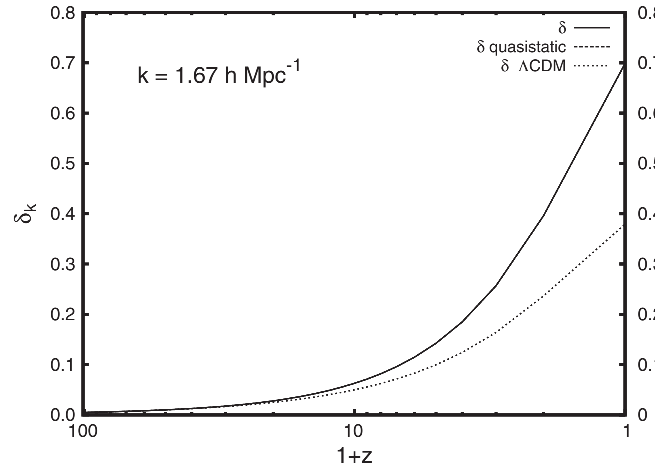

FIG. 2. δ_k with $k = 1.67\ \mathrm{h\,Mpc^{-1}}$ for $f(R)$ model A evolving according to (36), ΛCDM and quasistatic approximation given by Eq. (34) in the redshift range from 1000 to 0. The quasistatic evolution is indistinguishable from that coming from (36), but diverges from ΛCDM behavior as z decreases.

accelerated universe provided the parameters satisfy $c_1 < 0$ and $0 < p < 1$. We have chosen $c_1 = -4.3$ and $p = 0.01$ in H_0^2 units. This choice does verify all the viability conditions, including $|f_R| \ll 1$ today. For the second model (B), $f(R) = \frac{1}{c_1 R^{e_1} + c_2}$, we have chosen $c_1 = 2.5 \times 10^{-4}$, $e_1 = 0.3$, and $c_2 = -0.22$ also in the same units.

For each model, we compare our result (36) with the standard ΛCDM and the quasistatic approximation (34) (see Figs. 2 and 3). In both cases, the initial conditions are given at redshift $z = 1000$ where δ is assumed to behave as in a matter dominated universe, i.e., $\delta_k(\eta) \propto a(\eta)$ with no k dependence. We see that for both models the quasistatic

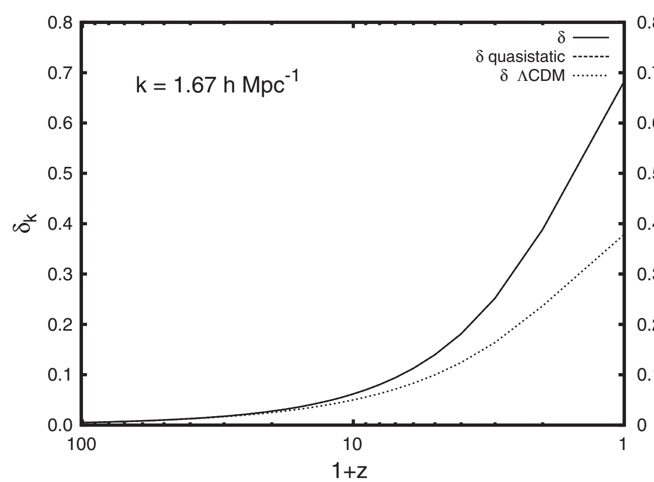

FIG. 3. δ_k with $k = 1.67\ \mathrm{h\,Mpc^{-1}}$ for $f(R)$ model B evolving according to (36), ΛCDM and quasistatic evolution given by Eq. (34) in the redshift range from 1000 to 0. The quasistatic evolution is indistinguishable from that coming from (36), but diverges from ΛCDM behavior as z decreases.

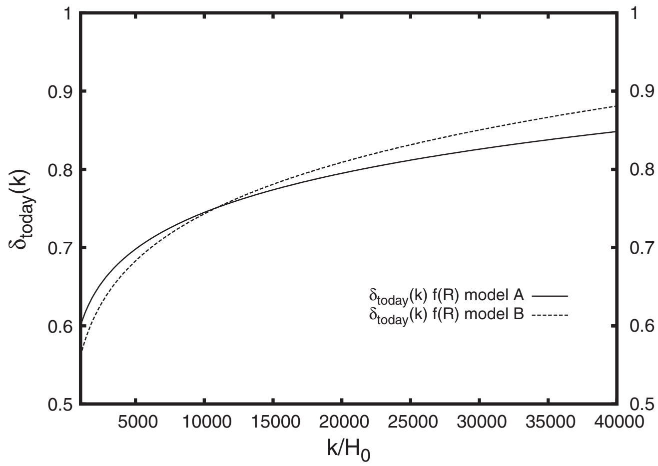

FIG. 4. Scale dependence of δ_k evaluated today ($z = 0$) for k/H_0 in the range from 1000 to 40 000.

approximation gives a correct description for the evolution which clearly deviates from the ΛCDM case.

In Fig. 4 the density contrast evaluated today was plotted as a function of k for both models. The growing dependence of δ with respect to k is verified. This modified k dependence with respect to the standard matter dominated universe could give rise to observable consequences in the matter power spectrum, as shown in [13], and could be used to constrain or even discard $f(R)$ theories for cosmic acceleration.

VIII. CONCLUSIONS

In this work we have studied the evolution of matter density perturbations in $f(R)$ theories of gravity. We have presented a completely general procedure to obtain the exact fourth-order differential equation for the evolution of perturbations. We have shown that for sub-Hubble modes the expression reduces to a second-order equation. We have compared this result with that obtained within the quasistatic approximation used in the literature and found that for arbitrary $f(R)$ functions such an approximation is not justified.

However, if we limit ourselves to theories for which $|f_R| \ll 1$ today, then the perturbative calculation for sub-Hubble modes requires taking into account, not only the first terms, but also higher-order terms in $\epsilon = \mathcal{H}/k$. In that case, the resummation of such terms modifies the equation which can be seen to be equivalent to the quasistatic case but only if the universe expands as in a matter dominated phase or in a ΛCDM model. Finally, the fact that for models with $|f_R| \ll 1$ the background behaves today precisely as that of ΛCDM makes the quasistatic approximation correct in those cases.

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APPENDIX A: α AND β COEFFICIENTS

Coefficients for the δ^{iv} term:

$$\begin{aligned}
\beta_{4,f}^{(1)} &\simeq 8 f_R^4 (1+f_R)^6 f_1^4 \epsilon^2, \\
\beta_{4,f}^{(2)} &\simeq 72 f_R^3 f_1^3 \epsilon^4 (-2+\kappa_2), \\
\beta_{4,f}^{(3)} &\simeq 216 f_R^2 f_1^2 \epsilon^6 (-2+\kappa_2)^2, \\
\beta_{4,f}^{(4)} &\simeq 216 f_R f_1 \epsilon^8 (-2+\kappa_2)^3.
\end{aligned} \tag{A1}$$

Coefficients for the δ''' term:

$$\begin{aligned}
&\beta_{3,f}^{(1)} \simeq 8 f_R^4 (1+f_R)^5 f_1^4 \mathcal{H} \epsilon^2 [3 + f_R(3+f_1)], \qquad \beta_{3,f}^{(2)} \simeq 6 f_R^3 f_1^2 \mathcal{H} \epsilon^4 \{8 f_2(-2+\kappa_2) + 4 f_1[12\kappa_1 + 9\kappa_2 - 2(9+\kappa_3)]\}, \\
&\beta_{3,f}^{(3)} \simeq -72 f_R^2 f_1 \mathcal{H} \epsilon^6 (-2+\kappa_2)[-4 f_2(-2+\kappa_2) + f_1(19 - 23\kappa_1 - 10\kappa_2 + 4\kappa_3)], \\
&\beta_{3,f}^{(4)} \simeq -216 f_R \mathcal{H} \epsilon^8 (-2+\kappa_2)^2 [-2 f_2(-2+\kappa_2) + f_1(7 - 11\kappa_1 - 4\kappa_2 + 2\kappa_3)].
\end{aligned} \tag{A2}$$

Coefficients for the δ'' term:

$$\begin{aligned}
&\alpha_{2,\mathrm{EH}}^{(1)} = 432(1+f_R)^{10} \mathcal{H}^2 \epsilon^8 (-1+\kappa_1)(-2+\kappa_2)^3, \qquad \alpha_{2,\mathrm{EH}}^{(2)} = 1296(1+f_R)^{10} \mathcal{H}^2 \epsilon^{10} (-1+\kappa_1)^2(-2+\kappa_2)^3, \\
&\alpha_{2,\mathrm{EH}}^{(3)} = 3888(1+f_R)^{10} \mathcal{H}^2 \epsilon^{12} (-1+\kappa_1)^2(-2+\kappa_2)^3, \qquad \beta_{2,f}^{(1)} \simeq 8 f_R^4 (1+f_R)^6 f_1^4 \mathcal{H}^2, \\
&\beta_{2,f}^{(2)} \simeq 88 f_R^3 f_1^3 \mathcal{H}^2 \epsilon^2 (-2+\kappa_2), \qquad \beta_{2,f}^{(3)} \simeq 24 f_R^2 f_1^2 \mathcal{H}^2 \epsilon^4 (-2+\kappa_2)(-28 + 2\kappa_1 + 13\kappa_2), \\
&\beta_{2,f}^{(4)} \simeq 72 f_R f_1 \mathcal{H}^2 \epsilon^6 (-2+\kappa_2)^2(-14 + 4\kappa_1 + 5\kappa_2).
\end{aligned} \tag{A3}$$

Coefficients for the δ' term:

$$\alpha^{(1)}_{1,\mathrm{EH}} = 432(1+f_R)^{10}\mathcal{H}^3\epsilon^8(-1+\kappa_1)(-2+\kappa_2)^3, \qquad \alpha^{(2)}_{1,\mathrm{EH}} = 2592(1+f_R)^{10}\mathcal{H}^3\epsilon^{10}(-1+\kappa_1)^2(-2+\kappa_2)^3,$$
$$\alpha^{(3)}_{1,\mathrm{EH}} = -7776(1+f_R)^{10}\mathcal{H}^3\epsilon^{12}(-1+\kappa_1)^3(-2+\kappa_2)^3, \qquad \beta^{(1)}_{1,f} \simeq 8f_R^4(1+f_R)^6 f_1^4\mathcal{H}^3,$$
$$\beta^{(2)}_{1,f} \simeq 88f_R^3 f_1^3\mathcal{H}^3\epsilon^2(-2+\kappa_2), \qquad \beta^{(3)}_{1,f} \simeq 24f_R^2 f_1^2\mathcal{H}^3\epsilon^4(-2+\kappa_2)(-28+2\kappa_1+13\kappa_2),$$
$$\beta^{(4)}_{1,f} \simeq 72f_R f_1\mathcal{H}^3\epsilon^6(-2+\kappa_2)^2(-14+4\kappa_1+5\kappa_2). \tag{A4}$$

Coefficients for the δ term:

$$\alpha^{(1)}_{0,\mathrm{EH}} = 432(1+f_R)^{10}\mathcal{H}^4\epsilon^8(-1+\kappa_1)(2\kappa_1-\kappa_2)(-2+\kappa_2)^3,$$
$$\alpha^{(2)}_{0,\mathrm{EH}} = 1296(1+f_R)^{10}\mathcal{H}^4\epsilon^{10}(-1+\kappa_1)^2(-1+4\kappa_1-\kappa_2)(-2+\kappa_2)^3,$$
$$\alpha^{(3)}_{0,\mathrm{EH}} = 3888(1+f_R)^{10}\mathcal{H}^4\epsilon^{12}(-1+\kappa_1)^2(2\kappa_1^2-\kappa_2)(-2+\kappa_2)^3,$$
$$\beta^{(1)}_{0,f} \simeq -\frac{16}{3}f_R^4(1+f_R)^5 f_1^4\mathcal{H}^4[2+f_R(2+2f_1-f_2-2\kappa_1)-2\kappa_1],$$
$$\beta^{(2)}_{0,f} \simeq 112f_R^3 f_1^3\mathcal{H}^4\epsilon^2(-1+\kappa_1)(-2+\kappa_2),$$
$$\beta^{(3)}_{0,f} \simeq 48f_R^2 f_1^2\mathcal{H}^4\epsilon^4(-1+\kappa_1)(-2+\kappa_2)(-16+2\kappa_1+7\kappa_2),$$
$$\beta^{(4)}_{0,f} \simeq 144f_R f_1\mathcal{H}^4\epsilon^6(-1+\kappa_1)(-2+\kappa_2)^2(-6+4\kappa_1+\kappa_2). \tag{A5}$$

APPENDIX B: c COEFFICIENTS

$$c_4 = -f_R f_1[-f_R f_1 k^2 - 3\mathcal{H}^2(-2+\kappa_2)]^3,$$
$$c_3 = -3f_R\mathcal{H}[-f_R f_1 k^2 - 3\mathcal{H}^2(-2+\kappa_2)]\{f_R^2 f_1^3 k^4 + 6f_2\mathcal{H}^4(-2+\kappa_2)^2 + f_1\mathcal{H}^2(-2+\kappa_2)[2f_R f_2 k^2$$
$$+ 3\mathcal{H}^2(-7+11\kappa_1+4\kappa_2-2\kappa_3)] + 2f_R f_1\mathcal{H}^2k^2(-6+6\kappa_1+3\kappa_2-\kappa_3)\},$$
$$c_2 = [-f_R f_1 k^2 - 3\mathcal{H}^2(-2+\kappa_2)]^2[f_R^2 f_1^2 k^4 + 5f_R f_1\mathcal{H}^2k^2(-2+\kappa_2) + 6\mathcal{H}^4(-1+\kappa_1)(-2+\kappa_2)], \qquad c_1 = \mathcal{H}c_2,$$
$$c_0 = \tfrac{2}{3}\mathcal{H}^2(-1+\kappa_1)[-f_R f_1 k^2 - 3\mathcal{H}^2(-2+\kappa_2)]^2[2f_R^2 f_1^2 k^4 + 9f_R f_1\mathcal{H}^2k^2(-2+\kappa_2) + 9\mathcal{H}^4(2\kappa_1-\kappa_2)(-2+\kappa_2)]. \tag{B1}$$

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