CUADERNOS DE TRABAJO
ESCUELA UNIVERSITARIA DE ESTADÍSTICA

Inaccurate parameters in Gaussian Bayesian networks

Miguel A. Góomez-Villegas
Paloma Main
Rosario Susi

Cuaderno de Trabajo número 02/2008

UCM
UNIVERSIDAD COMPLUTENSE MADRID

Los Cuadernos de Trabajo de la Escuela Universitaria de Estadística constituyen una apuesta por la publicación de los trabajos en curso y de los informes técnicos desarrollados desde la Escuela para servir de apoyo tanto a la docencia como a la investigación.

Los Cuadernos de Trabajo se pueden descargar de la página de la Biblioteca de la Escuela www.ucm.es/BUCM/est/ y en la sección de investigación de la página del centro www.ucm.es/centros/webs/eest/

CONTACTO: Biblioteca de la E. U. de Estadística
Universidad Complutense de Madrid
Av. Puerta de Hierro, S/N
28040 Madrid
Tlf. 913944035
buc_est@buc.ucm.es

ISSN: en trámite

Inaccurate parameters in Gaussian Bayesian networks

Miguel A. Gómez-Villegas and Paloma Main
Departamento de Estadística e Investigación Operativa I
Universidad Complutense de Madrid
28040 Madrid, Spain

Rosario Susi
Departamento de Estadística e Investigación Operativa III
Universidad Complutense de Madrid
28040 Madrid, Spain

Abstract

To determine the effect of a set of inaccurate parameters in Gaussian Bayesian networks, it is necessary to study the sensitivity of the model. With this aim we propose a sensitivity analysis based on comparing two different models: the original model with the initial parameters assigned to the Gaussian Bayesian network and the perturbed model obtained after perturbing a set of inaccurate parameters with specific characteristics.

The network's outputs obtained for both models, after the evidence propagation, are going to be compared with the Kullback-Leibler divergence. This measure is useful to discriminate between two probability distributions, comparing the whole behavior of the considered probability distributions.

Depending on the set of parameters that are going to be perturbed, different expressions for the Kullback-Leibler are obtained. It is possible to determine the set of parameters that mostly disturb the network's output, detecting the variables that must be accurately described in the model.

The methodology developed in this work is for a Gaussian Bayesian network with a set of variables of interest and a set of evidential variables.

One example is introduced to show the sensitivity analysis proposed.

1 Introduction

In Bayesian networks some sensitivity analysis had been proposed to study the effect of inaccurate parameters over the network's output. Most of them, like the analyses and methodologies proposed by Laskey (1995), Coupé, van der Gaag and Habbema (2000), Kjærulff and van der Gaag (2000), Bednarski, Cholewa and Frid (2004) or Chan and Darwiche (2005), to name a few, are developed to study the sensitivity in discrete Bayesian networks.

In Gaussian Bayesian networks Castillo and Kjærulff (2003) performed a methodology based on studying small changes in the parameters, with one variable of interest in the model, and Gómez-Villegas, Main and Susi (2007) developed a sensitivity analysis to study any kind of perturbations, small or large changes in the parameters, when there exists one variable of interest in the Gaussian Bayesian network.
In the present work, we study a generalization of the sensitivity analysis proposed by Gómez-Villegas, Main and Susi (2007), because now we consider a Gaussian Bayesian network with a set of variables of interest and a set of evidential variables.

This paper is organized as follows. In Section

2 a brief introduction is presented, defining first a Bayesian network and a Gaussian Bayesian network and reviewing the evidence propagation for these models. Moreover, we introduce the working example. In Section 3, we present the methodology developed to study the sensitivity of a Gaussian Bayesian network with a set of variables of interest and in Section 4, we perform the sensitivity analysis proposed with the working example. Finally, the paper ends with some conclusions.

2 Gaussian Bayesian networks

A Bayesian network is a probabilistic graphical model useful to study a set of random variables with a specified dependence structure.
Bayesian networks have been studied by authors like Pearl (1988), Lauritzen (1996) or Jensen (2001), among others.

Definition 1 (Bayesian network)**.** A Bayesian network is a couple (G,P) where G is a directed acyclic graph (DAG) whose nodes are random variables $\mathbf{X} = \{X_1, ..., X_n\}$ and edges represent probabilistic dependencies, P=$\{p(x_1|pa(x_1)), ..., p(x_n|pa(x_n))\}$ being a set of conditional probability distributions (one for each variable), $pa(x_i)$ the set of parents of node X_i in G and $pa(x_i) \subseteq \{X_1, ..., X_{i-1}\}$.
The set P defines the joint probability distribution as

$$p(\mathbf{x}) = \prod_{i=1}^{n} p(x_i|pa(x_i)). \quad (1)$$

Because of this modular structure, Bayesian networks are useful to study real life problems in complex domains.

Depending on the kind of variables of the problem, it is possible to describe discrete, Gaussian and mixed Bayesian networks. The results presented in this paper are developed for Gaussian Bayesian networks defined next

Definition 2 (Gaussian Bayesian network)**.** A Gaussian Bayesian network is a Bayesian network where the joint probability distribution of $\mathbf{X} = \{X_1, ..., X_n\}$ is a multivariate normal distribution $N(\mu, \Sigma)$, then the joint density

$$f(\mathbf{x}) = (2\pi)^{-n/2}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\right\} \quad (2)$$

where μ is the n-dimensional mean vector and Σ the $n{\times}n$ positive definite covariance matrix.

Moreover, the conditional probability distribution of X_i, that verifies the expression (1), is a univariate normal distribution with density

$$f(x_i|pa(x_i)) \sim N(x_i|\mu_i + \sum_{j=1}^{i-1} \beta_{ij}(x_j - \mu_j), \nu_i)$$

where μ_i is the mean of the variables X_i, β_{ij} are the regression coefficients of X_i on its parents, and $\nu_i = \Sigma_{ii} - \Sigma_{iPa(x_i)}\Sigma^{-1}_{Pa(xi)}\Sigma^{'}_{iPa(x_i)}$ is the conditional variance of X_i given its parents at the DAG. It can be pointed that $pa(x_i) \subseteq \{X_1, ..., X_{i-1}\}$.

It is possible to work with a Bayesian network introducing evidence at a variable of the network and computing the probability distribution of the rest of the variables given the evidence. This process in known as *evidence propagation*. Therefore, when there exists evidence about one variable of the problem, knowing its value, the evidence propagation updates the probability distributions of the rest of the variables of the network given the evidence.
Different algorithms had been developed to propagate the evidence in Bayesian networks. In Gaussian Bayesian networks most of the algorithms proposed are based on computing the conditional probability distribution for a multivariate normal distribution given a set of evidential variables.
Thereby, to perform the evidence propagation in a Gaussian Bayesian network we consider a partition of the set of variables, where $\mathbf{X} = (\mathbf{E}, \mathbf{Y})'$, being $\mathbf{E} = \mathbf{e}$ the set of evidential variables, $\mathbf{e}$ the evidence about the variables in $\mathbf{E}$, and $\mathbf{Y}$ the rest of variables of the problem that will be considered as the set of variables of interest. After performing the evidence propagation, the conditional

probability distribution of the variables of interest $\mathbf{Y}$ given the evidence $\mathbf{E} = \mathbf{e}$ is a multivariate normal distribution, being $\mathbf{Y}|\mathbf{E} = \mathbf{e} \sim N(\mathbf{y}|\mu^{\mathbf{Y}|\mathbf{E}=\mathbf{e}}, \Sigma^{\mathbf{Y}|\mathbf{E}=\mathbf{e}})$ where

$$\mu^{\mathbf{Y}|\mathbf{E}=\mathbf{e}} = \mu_{\mathbf{Y}} + \Sigma_{\mathbf{YE}}\Sigma_{\mathbf{EE}}^{-1}(\mathbf{e} - \mu_{\mathbf{E}}) \qquad (3)$$

and

$$\Sigma^{\mathbf{Y}|\mathbf{E}=\mathbf{e}} = \Sigma_{\mathbf{YY}} - \Sigma_{\mathbf{YE}}\Sigma_{\mathbf{EE}}^{-1}\Sigma_{\mathbf{EY}} \qquad (4)$$

Next, the working example of a Gaussian Bayesian network is introduced.

Example 1. The interest of the problem is about the duration of time that a machine works for. The machine is made up of 7 elements, connected as shown the DAG in Figure 1.

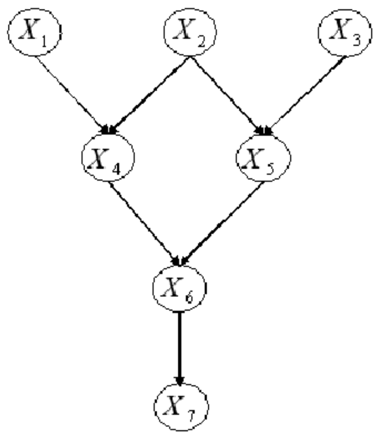

Figure 1: DAG of the Gaussian Bayesian network in Example 1

It is known that the time that each element is working is a normal distribution, being the joint probability distribution of $\mathbf{X} = \{X_1, X_2, ..., X_7\}$ a multivariate normal distribution, where $\mathbf{X} \sim N(\mu, \Sigma)$ with parameters

$$\mu = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 4 \\ 5 \\ 8 \end{pmatrix}; \mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 & 8 & 8 \\ 0 & 0 & 2 & 0 & 2 & 4 & 4 \\ 1 & 2 & 0 & 6 & 4 & 20 & 20 \\ 0 & 2 & 2 & 4 & 10 & 28 & 28 \\ 2 & 8 & 4 & 20 & 28 & 97 & 97 \\ 2 & 8 & 4 & 20 & 28 & 97 & 99 \end{pmatrix}$$

The Gaussian Bayesian network that represents the problem is given by the joint probability distribution of $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma)$ and by the DAG in Figure 1, showing the dependence structure between the variables of the example.

Experts know that the evidence is given by

$$\mathbf{E} = \{X_1 = 2, X_2 = 2, X_3 = 1\}$$

Then, performing the evidence propagation over the initial model that describes the Gaussian Bayesian network, the probability distribution of the rest of the variables given the evidence, $\mathbf{Y}|\mathbf{E} = \mathbf{e}$, is a multivariate normal distribution $\mathbf{Y}|\mathbf{E} \sim N(\mathbf{y}|\mu^{\mathbf{Y}|\mathbf{E}}, \Sigma^{\mathbf{Y}|\mathbf{E}})$ with parameters

$$\mu^{\mathbf{Y}|\mathbf{E}} = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 0 \end{pmatrix}; \mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}} = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 4 & 8 & 8 \\ 2 & 8 & 21 & 21 \\ 2 & 8 & 21 & 23 \end{pmatrix}$$

The effect of introducing the evidence updates the parameters of the marginal distribution of the variables $\mathbf{Y}$ given by

$$\mu^{\mathbf{Y}} = \begin{pmatrix} 1 \\ 4 \\ 5 \\ 8 \end{pmatrix}; \mathbf{\Sigma}^{\mathbf{Y}} = \begin{pmatrix} 6 & 4 & 20 & 20 \\ 4 & 10 & 28 & 28 \\ 20 & 28 & 97 & 97 \\ 20 & 28 & 97 & 99 \end{pmatrix}$$

and the independence relations because X_4 and X_5 become dependent.

3 Sensitivity Analysis

The aim of this work is to generalize the one way sensitivity analysis developed by

Gómez-Villegas, Main and Susi (2007), useful to study the effect of inaccuracy over the parameters of a Gaussian Bayesian network for one variable of interest after the evidence propagation.

In this paper we want to study the effect of inaccuracy over a set of parameters of a Gaussian Bayesian network considering a set of variables of interest.

The proposed methodology consists in comparing two different network's outputs: the first one, given by the network's output after the evidence propagation at the *original model*, and the other one, given by the network's output after the evidence propagation with a *perturbed model*. The perturbed model is obtained after adding a set of perturbations to the inaccurate parameters, as will be shown in Subsection 3.2. In this case, both network's outputs are the conditional probability distributions of the set of variables of interest, given the evidence.

3.1 Kullback-Leibler divergence

To compare the network's outputs we work with the n-dimensional Kullback-Leibler divergence (Kullback-Leibler, 1951). This measure takes into account the whole behavior of the distributions to be considered, therefore, this measure is a good way to compare the network's outputs being multivariate normal distributions of the variables of interest, given the evidence. Furthermore, the Kullback-Leibler (KL) divergence has been used in statistical inference in the past by authors like Jeffreys, Fisher and Lindley.

Definition 3 (Kullback-Leibler divergence)**.** Let $f(w)$ and $f'(w)$ be two probability densities defined over the same domain. The Kullback-Leibler divergence is given by

$$KL(f(w), f'(w)) = \int_{-\infty}^{\infty} f(w) \ln \frac{f(w)}{f'(w)} dw \quad (5)$$

When the probability densities to be compared with the KL divergence are multivariate normal distributions expression (5) can be written as

$$KL(f, f') =$$

$$= \frac{1}{2}\left[\ln \frac{|\mathbf{\Sigma}'|}{|\mathbf{\Sigma}|} + tr\left(\mathbf{\Sigma}\mathbf{\Sigma}'^{-1}\right) - \dim(\mathbf{X})\right] +$$

$$+\frac{1}{2}\left[(\mu' - \mu)^T \mathbf{\Sigma}'^{-1} (\mu' - \mu)\right] \quad (6)$$

where f is the joint probability density of $\mathbf{X} \sim N(\mathbf{x}|\mu, \mathbf{\Sigma})$ and f' is the joint probability density of $\mathbf{X} \sim N(\mathbf{x}|\mu', \mathbf{\Sigma}')$.

For small KL divergences, next to zero, it can be concluded that the distributions to be compared are similar.

3.2 Sensitivity Analysis: methodology

The sensitivity analysis consists in comparing, with the KL divergence, two different network's output, obtained for two different models. These models are the original and the perturbed model.

The original model is the initial description of the parameters of the network, given by $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma)$. The perturbed model quantifies the uncertainty about the inaccurate parameters of the original model, as a set of additive perturbations. These perturbations are given by the *perturbed mean vector* δ and the *perturbed covariance matrix* Δ, where

$$\delta = \begin{pmatrix} \delta_{\mathbf{E}} \\ \delta_{\mathbf{Y}} \end{pmatrix}; \mathbf{\Delta} = \begin{pmatrix} \Delta_{\mathbf{EE}} & \Delta_{\mathbf{EY}} \\ \Delta_{\mathbf{YE}} & \Delta_{\mathbf{YY}} \end{pmatrix}$$

Depending on the inaccurate parameters it is possible to consider five different perturbed models obtained when the uncertainty is about the evidential means, the means of interest, the variances-covariances between evidential variables, the variances-covariances between variables of interest and about the covariances between evidential variables and variables of interest. Therefore, each perturbed model considers a set of perturbations, having next perturbed models:

- $\mathbf{X} \sim N(\mathbf{x}|\mu^{\delta_{\mathbf{E}}}, \Sigma)$ where

$$\mu^{\delta_{\mathbf{E}}} = \begin{pmatrix} \mu_{\mathbf{E}} + \delta_{\mathbf{E}} \\ \mu_{\mathbf{Y}} \end{pmatrix}$$

- $\mathbf{X} \sim N(\mathbf{x}|\mu^{\delta_{\mathbf{Y}}}, \Sigma)$ being

$$\mu^{\delta_{\mathbf{Y}}} = \begin{pmatrix} \mu_{\mathbf{E}} \\ \mu_{\mathbf{Y}} + \delta_{\mathbf{Y}} \end{pmatrix}$$

- $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma^{\Delta_{\mathbf{EE}}})$ with

$$\mathbf{\Sigma}^{\mathbf{\Delta}_{\mathbf{EE}}} = \begin{pmatrix} \Sigma_{\mathbf{EE}} + \Delta_{\mathbf{EE}} & \Sigma_{\mathbf{EY}} \\ \Sigma_{\mathbf{YE}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$$

- $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma^{\Delta_{\mathbf{YY}}})$ where

$$\mathbf{\Sigma}^{\mathbf{\Delta}_{\mathbf{YY}}} = \begin{pmatrix} \Sigma_{\mathbf{EE}} & \Sigma_{\mathbf{EY}} \\ \Sigma_{\mathbf{YE}} & \Sigma_{\mathbf{YY}} + \Delta_{\mathbf{YY}} \end{pmatrix}$$

- $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma^{\Delta_{\mathbf{YE}}})$ where

$$\mathbf{\Sigma}^{\mathbf{\Delta}_{\mathbf{YE}}} = \begin{pmatrix} \Sigma_{\mathbf{EE}} & \Sigma_{\mathbf{EY}} + \Delta_{\mathbf{EY}} \\ \Sigma_{\mathbf{YE}} + \Delta_{\mathbf{YE}} & \Sigma_{\mathbf{YY}} \end{pmatrix}$$

In this way, at the proposed sensitivity analysis the network's outputs of all the perturbed models are going to be compared with the network's output of the original model given by the conditional probability distribution obtained after the evidence propagation for the model $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma)$. Thereby, five different KL divergences are obtained, one for each perturbed model.
When the KL divergence is large for a specific perturbed model we can conclude that the set of parameters perturbed must be reviewed to describe the network more accurately. However, when the KL divergence is small, close to zero, it can be concluded that the network is not sensitive to the proposed perturbations.

3.3 Main results

After computing the KL divergence for each perturbed model, the results are in Propositions 1 and 2.

Proposition 1 (Uncertainty about the mean vector)**.** *Let (G, P) be a Gaussian Bayesian network with $\mathbf{X} = \{\mathbf{E}, \mathbf{Y}\}$ and $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma)$ where the mean vector μ is uncertain. Giving values to the perturbed mean vector $\delta = (\delta_E, \delta_Y)^T$, the following results are obtained*

1. *When the perturbation $\delta_{\mathbf{E}}$ is added to the mean vector of the evidential variables, the perturbed model after the evidence propagation is $\mathbf{Y}|\mathbf{E}, \delta_{\mathbf{E}} \sim N(\mathbf{y}|\mu^{\mathbf{Y}|\mathbf{E},\delta_{\mathbf{E}}}, \mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}})$ with $\mu^{\mathbf{Y}|\mathbf{E},\delta_{\mathbf{E}}} = \mu^{\mathbf{Y}|\mathbf{E}} - \mathbf{\Sigma}_{\mathbf{YE}}\mathbf{\Sigma}_{\mathbf{EE}}^{-1}\delta_{\mathbf{E}}$. The KL divergence is*

$$KL^{\mu_{\mathbf{E}}} = \frac{1}{2}\left[\delta_{\mathbf{E}}^T M_1^T \left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right)^{-1} M_1 \delta_{\mathbf{E}}\right]$$

with $M_1 = \mathbf{\Sigma}_{\mathbf{YE}}\mathbf{\Sigma}_{\mathbf{EE}}^{-1}$

2. *When the perturbation $\delta_{\mathbf{Y}}$ is added to the mean vector of the variables of interest, after the evidence propagation the perturbed model is $\mathbf{Y}|\mathbf{E}, \delta_{\mathbf{Y}} \sim N(\mathbf{y}|\mu^{\mathbf{Y}|\mathbf{E},\delta_{\mathbf{Y}}}, \Sigma^{\mathbf{Y}|\mathbf{E}})$ where $\mu^{\mathbf{Y}|\mathbf{E},\delta_{\mathbf{Y}}} = \mu^{\mathbf{Y}|\mathbf{E}} + \delta_Y$ and the KL divergence is*

$$KL^{\mu_{\mathbf{Y}}} = \frac{1}{2}\left[\delta_{\mathbf{Y}}^T \left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right)^{-1} \delta_{\mathbf{Y}}\right]$$

Proof. When there is uncertainty about the mean vector, we work with two perturbed models, depending on the set of inaccurate parameters. Parameters of the perturbed models are obtained after performing the evidence propagation.

In both perturbed models the covariance matrix $\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}$ is the same for the original model and for the perturbed model, then $tr\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right)^{-1}\right) = \dim(\mathbf{Y})$. Then, working with expression (6) and dealing with the perturbed models, the KL divergences follow directly.

□

The KL divergence obtained when there exists uncertainty about the mean vector of the evidential variables coincides with the KL divergence computed for a perturbation in the evidence vector **e**. This gives us a tool to evaluate evidence influence on the network's outputs, as can be seen in Susi (2007).

Proposition 2 (Uncertainty about the covariance matrix)**.** *Let (G, P) be a Gaussian Bayesian network with $\mathbf{X} = \{\mathbf{E}, \mathbf{Y}\}$ and $\mathbf{X} \sim N(\mathbf{x}|\mu, \Sigma)$ where the covariance matrix Σ is uncertainty. Giving values to the perturbed covariance matrix $\mathbf{\Delta} = \begin{pmatrix} \Delta_{\mathbf{EE}} & \Delta_{\mathbf{EY}} \\ \Delta_{\mathbf{YE}} & \Delta_{\mathbf{YY}} \end{pmatrix}$, the following results are obtained*

1. *When the perturbation $\mathbf{\Delta_{EE}}$ is added to the variances-covariances of the evidential variables, after the evidence propagation, the perturbed model is*

$$\mathbf{Y}|\mathbf{E}, \mathbf{\Delta_{EE}} \sim N(\mathbf{y}|\mu^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}}, \mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}})$$

   with $\mu^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}} = \mu_{\mathbf{Y}} + \mathbf{\Sigma_{YE}}\left(\mathbf{\Sigma_{EE}} + \mathbf{\Delta_{EE}}\right)^{-1}(\mathbf{e} - \mu_{\mathbf{E}})$ *and* $\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}} = \mathbf{\Sigma_{YY}} - \mathbf{\Sigma_{YE}}\left(\mathbf{\Sigma_{EE}} + \mathbf{\Delta_{EE}}\right)^{-1}\mathbf{\Sigma_{EY}}$

   The KL divergence is
   $KL^{\mathbf{\Sigma_{EE}}} =$

$$= \frac{1}{2}\left[\ln \frac{\left|\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}}\right|}{\left|\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right|} - \dim(\mathbf{Y})\right] +$$

$$+\frac{1}{2}\left[tr\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}}\right)^{-1}\right)\right] +$$

$$+\frac{1}{2}\left[M_2^T\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}}\right)^{-1}M_2\right]$$

   where $M_2 = \mu^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{EE}}} - \mu^{\mathbf{Y}|\mathbf{E}}$.

2. *When the perturbation $\mathbf{\Delta_{YY}}$ is added to the variances-covariances between the variables of interest, after the evidence propagation the perturbed model is*

$$\mathbf{Y}|\mathbf{E}, \mathbf{\Delta_{YY}} \sim N(\mathbf{y}|\mu^{\mathbf{Y}|\mathbf{E}}, \mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{YY}}})$$

   with $\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{YY}}} = \mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}} + \mathbf{\Delta_{YY}}$.

   The obtained KL divergence is
   $KL^{\mathbf{\Sigma_{YY}}} =$

$$= \frac{1}{2}\left[\ln \frac{\left|\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}} + \mathbf{\Delta_{YY}}\right|}{\left|\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right|} - \dim(\mathbf{Y})\right] +$$

$$+\frac{1}{2}\left[tr\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}} + \mathbf{\Delta_{YY}}\right)^{-1}\right)\right]$$

3. *If the perturbation $\mathbf{\Delta_{YE}}$ is added to the covariances between $\mathbf{Y}$ and $\mathbf{E}$, the perturbed model after the evidence propagation is*

$$\mathbf{Y}|\mathbf{E}, \mathbf{\Delta_{YE}} \sim N(\mathbf{y}|\mu^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{YE}}}, \mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{YE}}})$$

   with $\mu^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{YE}}} = \mu_{\mathbf{Y}} + \left(\mathbf{\Sigma_{YE}} + \mathbf{\Delta_{YE}}\right)\mathbf{\Sigma_{EE}^{-1}}(\mathbf{e} - \mu_{\mathbf{E}})$ *and* $\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{\Delta_{YE}}} = \mathbf{\Sigma_{YY}} - \left(\mathbf{\Sigma_{YE}} + \mathbf{\Delta_{YE}}\right)\mathbf{\Sigma_{EE}^{-1}}\left(\mathbf{\Sigma_{EY}} + \mathbf{\Delta_{EY}}\right)$

   Then, the KL divergence is
   $KL^{\mathbf{\Sigma_{YE}}} =$

$$= \frac{1}{2}\left[\ln \frac{\left|\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}} - M(\mathbf{\Delta_{YE}})\right|}{\left|\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right|} - \dim(\mathbf{Y})\right] +$$

$$+\frac{1}{2}\left[tr\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}} - M(\mathbf{\Delta_{YE}})\right)^{-1}\right)\right] +$$

$$+\frac{1}{2}\left[(\mathbf{e} - \mu_{\mathbf{E}})^T\left(\mathbf{\Sigma_{EE}^{-1}}\right)^T M_3\mathbf{\Sigma_{EE}^{-1}}(\mathbf{e} - \mu_{\mathbf{E}})\right]$$

   where
   $M_3 = \mathbf{\Delta_{YE}^{T}}\left(\mathbf{\Sigma}^{\mathbf{Y}|\mathbf{E}} - \mathbf{\Delta_{YE}}\mathbf{\Sigma_{EE}^{-1}}\mathbf{\Sigma_{YE}^{T}} - \mathbf{\Sigma_{YE}}\mathbf{\Sigma_{EE}^{-1}}\mathbf{\Delta_{EY}} - \mathbf{\Delta_{YE}}\mathbf{\Sigma_{EE}^{-1}}\mathbf{\Delta_{EY}}\right)^{-1}\mathbf{\Delta_{YE}}$

Proof. We work with three perturbed models defined for different sets of inaccurate parameters. The parameters of the perturbed models are obtained after performing the evidence propagation. Then, computing the KL divergence with (6) to compare the network's output of the original model with the network's outputs obtained for the perturbed models, the obtained expressions follow directly. □

4 Experimental results

Next, we will run the sensitivity analysis proposed in Section 3 for the Example 1.

Example 2. There are different opinions between experts about the parameters of the

Gaussian Bayesian network shown in Example 1. Quantifying this uncertainty we obtain the perturbed mean vector δ and the perturbed covariance matrix Δ as next partitions

$$\delta_{\mathbf{E}} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}; \delta_{\mathbf{Y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{\Sigma_{EE}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{\Sigma_{YY}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

$$\mathbf{\Sigma_{EY}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Taking into account the evidence $\mathbf{E} = \{X_1 = 2, X_2 = 2, X_3 = 1\}$ and the variables of interest $\mathbf{Y} = \{X_4, X_5, X_6, X_7\}$, it is possible to perform the sensitivity analysis proposed.

Then, for the KL divergence with the expressions presented in Propositions 1 and 2, next values are obtained:

$$\begin{aligned} &KL^{\mu_{\mathbf{E}}} = 2.125 \\ &KL^{\mu_{\mathbf{Y}}} = 2.375 \\ &KL^{\mathbf{\Sigma_{EE}}} = 0.596 \\ &KL^{\mathbf{\Sigma_{YY}}}(f, f^{\mathbf{\Sigma_{YY}}}) = 1.629 \\ &KL^{\mathbf{\Sigma_{YE}}}(f, f^{\mathbf{\Sigma_{YE}}}) = 0.265 \end{aligned}$$

With the obtained results it is possible to conclude that some parameters must be reviewed to describe the network more accurately.

The parameter that must be reviewed is the mean vector, because the possible perturbations make the KL divergence larger than 1 and, moreover, it is necessary to review the parameters that describe the variances-covariances between the variables of interest because the network is sensitive to uncertainty about these parameters.

Uncertainty about the variances-covariances between evidential variables and about the covariances between variables of interest and evidential variables do not change the network's output so much, therefore the network is not sensitive to these inaccurate parameters.

In fact, experts must review the information about the variables of interest and about the mean of the evidential variables, to describe more accurately $\mu_{\mathbf{Y}}$, $\Sigma_{\mathbf{YY}}$ and $\mu_{\mathbf{E}}$ respectively.

5 Conclusions

In this paper we propose a sensitivity analysis for Gaussian Bayesian networks useful to determine the set or sets of inaccurate parameters that must be reviewed to be introduced in the network more accurately, or if the network is not sensitive to the perturbations proposed.

The analysis performed is a generalization of the one way sensitivity analysis developed by Gómez-Villegas, Main and Susi (2007). Now we work with a set of variables of interest and a set of evidential variables.

In a Gaussian Bayesian network, some inaccuracies about the parameters that describes the network, involve a sensitivity analysis of the model.

The sensitivity analysis we propose in which five different sets of parameters are considered depending on the type of variables and if they describe the mean or the covariance of the model. After computing the expressions of the KL divergence obtained in Propositions 1 and 2, it is possible to conclude the set or sets of parameters that must be reviewed to describe the network more accurately. In this way when a KL divergence is small, close to zero, we can conclude that the network is not sensitive to the proposed perturbations, otherwise it is necessary to review the uncertainty parameters.

The sensitivity analysis proposed is easy to perform with any Gaussian Bayesian network, being able to evaluate any kind of inaccurate parameters, that is, large and small perturbations associated to uncertainty parameters.

Acknowledgments

This research was supported by the Ministerio de Educación y Ciencia from Spain Grant MTM2005–05462 and Comunidad de Madrid–Universidad Complutense Grant 910395 CCG06–UCM / ESP–1165.

References

Bednarski, M., Cholewa, W. and Frid, W. 2004. Identification of sensitivities in Bayesian networks. *Engineering Applications of Artificial Intelligence*, 17:327–335.

Castillo, E. and Kjærulff, U. 2003. Sensitivity analysis in Gaussian Bayesian networks using a symbolic-numerical technique.*Reliability Engineering and System Safety*, 79:139–148.

Chan, H. and Darwiche, A. 2005. A distance Measure for Bounding Probabilistic Belief Change.*International Journal of Approximate Reasoning*, 38(2):149–174.

Coupé, V.M.H., van der Gaag, L.C. and Habbema, J.D.F. 2000. Sensitivity analysis: an aid for belief-network quantification.*The Knowledge Engineering Review*, 15(3):215–232.

Gómez-Villegas, M.A., Main, P. and Susi, R. 2007. Sensitivity Analysis in Gaussian Bayesian Networks Using a Divergence Measure, *Communications in Statistics: Theory and Methods*, 36(3):523–539.

Jensen. F. 2001. *Bayesian Networks and Decision Graphs*. New York, Springer Verlag.

Kjærulff, U. and van der Gaag, L.C. 2000. Making Sensitivity Analysis Computationally Efficient. In *Proceedings of the 16^{th} Conference on Uncertainty in Artificial Intelligence*, San Francisco, CA, USA, pages 315–325. Morgan Kaufmann.

Kullback, S. y Leibler, R.A. 1951. On Information and Sufficiency. *Annals of Mathematical Statistics*, 22:79–86.

Laskey, KB. 1995. Sensitivity Analysis for Probability Assessments in Bayesian Networks. *IEEE Transactions on Systems, Man and Cybernetics*, 25:901–909.

Lauritzen, S.L. 1996. *Graphical Models*. Oxford, Clarendon Press.

Pearl, J. 1988. *Probabilistic Reasoning in Intelligent Systems. Networks of Plausible Inference*. Morgan Kaufmann, Palo Alto.

Susi R. 2007. *Análisis de Sensibilidad en Redes Bayesianas Gaussianas*. Ph.D. Thesis, Departamento de Estadística e Investigación Operativa, Universidad Complutense de Madrid, Spain.