UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA

TESIS DOCTORAL

MOTIVOS DE ESPACIOS DE MÓDULI DE PARES Y APLICACIONES
MOTIVES OF MODULI SPACES OF PAIRS AND APPLICATIONS

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Jonatan Sánchez Hernández

Director

Vicente Muñoz Velázquez

Madrid, 2014

Motivos de espacios de módulo de pares y aplicaciones

Motives of moduli spaces of pairs and applications

UNIVERSIDAD COMPLUTENSE MADRID

Memoria presentada para optar al grado de
Doctor en Ciencias Matemáticas por

Jonatan Sánchez Hernández

Dirigida por

Dr. Vicente Muñoz Velázquez

Departamento de Geometría y Topología
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid

Agradecimientos

En un proyecto a largo plazo como puede ser una tesis doctoral, el número de personas a las que hay que agradecer su apoyo puede ser inmenso. No puedo prentender ser exhaustivo nombrando a todos ellos: para ello, más abajo he dejado la oportunidad de enmendar mi olvido para que aquellos que realmente creen que deberían estar.

A nivel profesional, hay unos cuantos nombres a los que quería aquí nombrar. Entre ellos está mi director Vicente Muñoz, e inmediatamente después está el de André Gama Oliveira e Indranil Biswas con los que colaboré profesionalmente y que tan amables fueron conmigo. Sería imperdonable no mencionar a Tomás Luis Gómez de Quiroga, a Giovanni Bazzoni y, muy en especial por la estima y el aprecio que me une a ellos, a Álvaro Antón Sancho, a Jesús Pascual Moreno Damas y a Raquel Díaz Sánchez.

Más a nivel personal, a mis amigos, en especial a los más cercanos: a Inmaculada Soriano, con la que he pasado tantas horas, tanto alegres como amargas; a Sonia Bouzas, por su carácter de luchadora y su magnética personalidad; y a Ángel Carrascosa al que agradezco que compartiese conmigo su vasto conocimiento discográfico.

A nivel familiar, mi hermano Juan Pablo Sánchez Hernández, el único de mi entorno que pocos realmente pueden llegar a comprender lo que es hacer una tesis.

Finalmente, a nivel espiritual, tengo que agradecer el refugio y el placer que me ha proporcionado la Música, en especial la de Bach, Beethoven, Wagner y Schubert (¡entre otros muchos!). Asímismo, mencionar a Spinoza y a Nietzsche por sus *useful conversations.* También al sistema nacional de becas, cuya ayuda ha sido inestimable a lo largo de mi etapa educativa y de doctorado, y que ahora más que nunca sigue siendo una ayuda esencial para muchos.

Mi último agradecimiento lo reservo para ti, querido lector, que has hecho el esfuerzo de acercarte a estas páginas.

El espacio consagrado a ello no es suficiente como para evitar las omisiones. Espero que nadie sienta ningún desagravio si no le menciono aquí pero, si realmente crees que me he olvidado de ti, recibe mis disculpas y a continuación coge un bolígrafo o abre el editor de ficheros *pdf* y, por favor, corrige mi error por mí:

Agradezco personalmente a ____________________ por ______________________________

____

______________________________.

Index

Introduction . . . 3
1. Triples and pairs . . . 3
2. Hodge Theorem and Hodge conjecture . . . 4
3. Toric varieties . . . 5

Introducción (español) . . . 7
1. Triples y pares . . . 7
2. El Teorema y la Conjetura de Hodge . . . 8
3. Variedades tóricas . . . 10

1. Motives and Grothendieck groups . . . 13
1. The Grothendieck group . . . 13
2. Motives . . . 18
3. The Jacobian Variety . . . 20
4. The Brauer group . . . 22

2. Triples and pairs . . . 27
1. Definition of holomorphic triple . . . 27
2. The moduli space of holomorphic triples . . . 28
3. Extension of triples . . . 31
4. Cohomological definition of Ext group . . . 32
5. Filtrations . . . 34
6. Stratification . . . 34

3. Rank 2 . . . 39
1. The class . . . 39
2. Computation of Hodge-Deligne polynomial . . . 41
3. The class of moduli of bundles of rank 2 with odd degree . . . 42
4. The class of moduli of bundles of rank 2 with even degree . . . 44

4. Rank 3 . . . 47
1. Computation of the critical value . . . 47
2. Set of strata . . . 47
3. The sum . . . 52
4. The class of the moduli space of stable bundles of rank 3 . . . 55
5. Appendix: A Mathematica sheet . . . 57

5. The Moduli of stable bundle of rank 3 . . . 65
1. Introduction . . . 65
2. The situation . . . 65
3. Strata of length 1 . . . 69
4. Strata of length 2 . . . 72
5. Strata of length 3 . . . 80
6. Another stratificaton for the flip loci . . . 82
7. The Mathematica sheet . . . 96

6. Rank 4 . . . 101
1. Critical values . . . 101
2. The first set of strata . . . 103
3. The second set of strata . . . 104

4. The third set of strata . . . 105
5. The Mathematica show . . . 108

7. The Hodge conjecture for pairs . . . 115
1. The Generalized Hodge Conjecture . . . 115
2. The Standard Hodge Conjecture . . . 116
3. The Arapura-Kang functor . . . 117
4. Some family of strata . . . 119
5. Some cases with non-coprime rank and degree . . . 133
6. Proof of the main results . . . 135

8. Toric Variety and Equivariant Bundles . . . 137
1. Toric Variety . . . 137
2. What is a connection? The Atiyah Sequence . . . 137
3. The log-connection on G-pairs . . . 139
4. A counterexample . . . 141
5. Cones and Fans on Toric Varieties . . . 142
6. Chern clases of equivariant bundles . . . 144
7. Semistability and restriction to invariant curves . . . 146
8. Equivariant Krull-Schmidt decomposition . . . 147

Bibliography . . . 149

Introduction

This PhD Thesis deals with the following three topics:

1) The class in the Grothendieck ring of varieties of the moduli spaces of bundles and of pairs,
2) The Hodge conjecture for such moduli spaces,
3) Equivariant bundles on toric varieties and G-pairs.

In the following pages, we will briefly introduce the aforementioned topics, one by one, and at the end of each section of this introduction we will give further details about the techniques applied and our results.

1 Triples and pairs

Sometimes, the mathematical world is like a river: there are two sides, two ways to understand the same object. In our situation, these two sides are Algebraic Geometry and Differential (complex) Geometry Each side has its own and particular techniques. The renowned GAGA's Theorem is on of the bridges joining the two sides of the river. Parametrizing geometric objects, the moduli problem, has produced a lot of tools in this interplay. The current thesis lies at the heart of one of the most studied moduli problems, that of bundles over algebraic curves.

Let C be an algebraic curve. In 1965, Narashiman and Seshadri [NS65] gave a GIT construction (an algebro-geometric construction) of the moduli space of bundles $M_C^s(n,d)$ on the curve C (here n is the rank and d the degree of the bundles). This algebraic technique introduces the notion of stability on bundles. In fact, the stability notion has had several versions, although the most used one involves the slope (quotient of the degree by the rank of the bundle) [Mum63]. On the other hand we have the differential-geometric notion known as Hermitian-Einstein condition, introduced by Kobayashi. Vector bundles satisfying this condition are called Hermitian-Einstein bundles. In [Kob82], Kobayashi proved that an irreducible Hermitian-Einstein bundle over a compact Kähler manifold is in fact stable. After Donaldson gave the equivalence of both notions in a particular case, Hitchin and Kobayashi conjetured independently the correspondence Hitchin-Kobayashi, which stablishes the equivalence between stable bundles and Hermitian-Einstein bundles. This correspondence was proved by Donaldson in [Don85] for algebraic surfaces and in [Don87] for manifolds.

A holomorphic pair (E,ϕ) is a pair consisting of a bundle and a holomorphic section $\phi \in H^0(C,E)$. Nowadays we have at our disposal several constructions of the moduli space parametrizing these objects. The first ones came from an analytic point of view: by dimensional reduction [GP94] or via the vortex equations [BGP97]. In this occasion the stability notion depends on a real parameter (see [Bra91]) and hence the moduli space of pairs $\mathfrak{M}_\tau(n,d)$ of rank n and degree d depends on the real parameter τ. Some properties are studied in [BD91]. From the algebraic point of view, the construction was given by Schmitt [Sch08], but before him Thaddeus [Tha94] made an algebraic construction in the case of rank 2 to prove the Verlinde formula.

A more general object than pairs is that of holomorphic triples and, in a certain sense, they enjoy nicer properties. An holomorphic triple $T=(E_1,E_2,\phi)$ is a 3-upla consisting of two bundles and a map $\phi: E_2 \to E_1$. Note that if E_2 is a line bundle, the triple is equivalent to a pair $(E_1 \otimes E_2^\vee, \phi)$, where $E^\vee$ denotes the dual bundle. The reader will find constructions and properties of moduli space of triples in [BGP96] and [BGPG04] (and also later in this thesis). Also, a good reference for this topic is the survey [Bra95]. As in the case of pairs, the stability notion depends on a real parameter σ giving for each parameter a moduli space of semistable triples $\mathcal{N}_\sigma = \mathcal{N}_\sigma(n_1,n_2,d_1,d_2)$ where $(\operatorname{rk} E_i, \deg E_i) = (n_i, d_i)$ for $i=1,2$. Furthermore, the existence of this moduli space depends on such a parameter: the moduli space exists (i.e. it is non empty) if and only if σ belongs to a closed interval $I=[\sigma_m,\sigma_M] \subset \mathbb{R}$. Usually, σ_m and σ_M are called minimal and maximal critical values respectively. When it exists, the moduli space is a complex projective variety. In such interval, there are some special values called *critical values*: whenever the parameter σ crosses a critical value σ_c, the underlying set of $\mathcal{N}_\sigma$ changes. This is due to some $(\sigma_c-\epsilon)$-unstable ($(\sigma_c-\epsilon)$-stable) triples becoming $(\sigma_c+\epsilon)$-stable (resp., $(\sigma_c+\epsilon)$-

unstable) and viceversa. The set of such triples defines the flip locus $\mathcal{S}_{\sigma_c^-}$ (resp., $\mathcal{S}_{\sigma_c^+}$). In [Mn10], Muñoz introduces a stratification for these flip loci in case $n_1 = 1$. Such a stratification generalizes the method used in [VMnVG07] and [Mn08] to compute the Hodge-Deligne polynomials of moduli space of pairs and stable bundles in the cases of rank 2 and 3. Chapter 2 is devoted to this and provides full references for this topic.

The following chapters (Chapters 3–6) use this stratification to compute the class in the Grothendieck ring of the category of varieties $K_0(\mathfrak{Var}_{\mathbb{C}})$ (some properties of this ring are advanced in the Chapter 1, accompanied by an introduction to Motives, as we use them later). We compute the class of $\mathcal{S}_{\sigma_c^+}$ and $\mathcal{S}_{\sigma_c^-}$ in $K_0(\mathfrak{Var}_{\mathbb{C}})$ for all critical values σ_c, for ranks $n = 3$ and $n = 4$, and then sum (total contribution) through all the critical values. Analogous computations were done in [VMnVG07] and [Mn08] where the authors obtained the Hodge-Deligne polynomials of $\mathcal{N}_\sigma(n, 1, d_1, d_2)$ for ranks $n = 2$ and $n = 3$.

It is also possible to compute the class of $M^s(n, d)$. The technique (used mainly in [Mn08]) to compute such classes lies in the possibility to obtain two different expressions for the class $\mathcal{N}_{\sigma_m}(n, 1, d_1, d_2)$ where σ_m denotes the minimal critical value: the first one obtained via the sum of the classes of all flip loci for all critical values $\sigma_c > \sigma_m$; only the information about the classes of $M^s(n', d')$ for $n' < n$ are necessary. The second one is to compute directly this space by the equality $\mathcal{N}_{\sigma_m}(n, 1, d_1, d_2) = \mathcal{S}_{\sigma_m^+}$. The class of $M^s(n, d_1)$ appears in the last space $\mathcal{S}_{\sigma_m^+}$. As we have pointed out, this trick is used in [Mn08] to compute the Hodge-Deligne polynomial of $M^s(3, 1)$. We perform analogous computations to get the class of $M^s(2, 1)$, $M^s(2, 0)$ (whose Hodge-Deligne polynomial was computed in [VMnVG09]) and $M^s(3, 1)$. The larger Chapter 5 is devoted to compute the class of $M^s(3, 0)$ by this technique.

Finaly, in Section 5.6 we offer an alternative computation of $\mathcal{S}_{\sigma_m^+}$ for the case $M^s(3, 0)$. To do this, we introduce a new method to stratify this flip locus by looking at the short exact sequences produced by the Harder-Narashimhan filtration of the semistable bundle E of a triple $T = (E, L, \phi)$. The construction of any stratum is done in two steps: First, we parametrize bundles E with some conditions of their short exact sequences; second, we parametrize the bundle L and the map $\phi : L \to E$ which defines a σ_m^+-stable triple $T = (E, L, \phi) \in \mathcal{S}_{\sigma_m^+}$.

At the end of some of the chapters, we conclude with some instructions for computing and checking the results by a Mathematica worksheet.

2 Hodge Theorem and Hodge conjecture

In the beginning of the 20th century, W. Hodge proves two capital theorems in complex geometry. The first one claims that any cohomology class in $H^k(X, \mathbb{C})$ admits only one harmonic representative. The second claims that the cohomology $H^k(X, \mathbb{C})$ admits a decomposition according to the type of the harmonic representative of each class.

Theorem 0.1. [Hodge's Decomposition Theorem] *Let X be a compact Kähler manifold. Let $H^{p,q}(X)$ be the space of cohomology classes whose harmonic representative is of type (p, q). Then there is a direct sum decomposition*

$$H^k_{\text{de Rham}}(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X).$$

Moreover, $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

In a certain way, such a decomposition encodes the complex structure of a compact complex manifold. This leads us to the abstract notion of Hodge structure: fix a ring R, a Hodge structure is an R-module V endowed with a decreasing filtration F on $V \otimes \mathbb{C}$ satisfying $F^p V \cap \overline{F^q V} = 0$; or, equivalently, a decomposition of $V = \bigoplus_{p+q=n} V^{p,q}$ where $\overline{V^{q,p}} = V^{p,q}$. The structure of R-module provides an extra information which corresponds with the map $H^k(X, R) \to H^k(X, \mathbb{C})$ in cohomology.

Subvarieties of any algebraic variety X are related to cohomology classes by the cycle class map $\mathrm{cl} : Z_k(X) \to H^{2(n-k)}(X)$ defined by the equality

$$\int_X \alpha \wedge \mathrm{cl}(Z) = \int_Z i^* \alpha$$

The set of *algebraic classes* is the image of this map. The cycle class map induces a correspondence between cycles in $Z_k(X)$ and classes in $H^{n-k}(X)$. This is a purely *differential geometry* property. In complex geometry, this correspondence leads naturally to the *Hodge conjecture* which remains open up to now. We need additional remarks for a correct formulation.

The image of cl map lives in a special subset of $H^{2(n-k)}(X)$. First, it is not difficult to show that $\mathrm{cl}(Z) \in H^{n-k,n-k}(X)$ for any cycle $Z \in Z_k(X) \otimes \mathbb{Q}$. Moreover, as Z is a rational class, $\mathrm{cl}(Z)$ then lives in the image of the map $H^{2(n-k)}(X, \mathbb{Q}) \to H^{2(n-k)}(X, \mathbb{C})$. The intersection of both two sets are the Hodge class

$$\mathrm{Hdg}^k(X) = H^{k,k}(X) \cap \mathrm{Im}\left(H^{2(n-k)}(X, \mathbb{Q}) \to H^{2(n-k)}(X, \mathbb{C})\right)$$

The Hodge conjecture states an analogous correspondence for projective varieties:

Hodge Conjecture 0.2. *Let X be a smooth projective variety. Every (k, k)-class with rational coefficients is algebraic. In other words, every class in* $\mathrm{Hdg}^k(X)$ *is a rational combination of fundamental cohomology classes of subvarieties of X.*

There is some evidence indicating that the Hodge conjecture may be true. For example, the $(1, 1)$-Leftschetz theorem proves the Hodge conjecture for cycles of codimension 1. If we combine this theorem with the Hard Lefschetz theorem, we can also see that the Hodge conjecture is valid for cycles of dimension 1. Furthermore, the Hodge conjecture also holds for some families of varieties, such as the Prym varieties.

One of the most popular versions of the Hodge Conjecture, which turns out to be equivalent to it, is the Generalized Hodge Conjecture, which is written in terms of filtrations.

Generalized Hodge Conjecture 0.3. *Let X be a smooth projective variety. The largest rational Hodge substructure of $F^pH^k(X, \mathbb{C}) \cap H^k(X, \mathbb{Q})$ is the union of all the rational Hodge substructures supported on codimension $\geq p$ subvarieties of X.*

Although the Hodge Conjecture still remains unanswered, the different approaches to this conjecture have produced a number of useful objects in Algebraic Geometry, like the work of P. Deligne on mixed Hodge structures ([Del71a], [Del71b] and [Del74]), who proved that the cohomology groups of an algebraic variety (quasi-projective or singular) over a closed field admits a natural mixed hodge structure.

The present thesis gives a proof of the Generalized Hodge Conjecture for the moduli spaces of bundles and of pairs with rank less or equal than 4 (for a generic curve). Our approach to this problem is based on techniques developed by D. Arapura and S. Kang in [AK06]. This work makes possible to check the Generalized Hodge Conjecture of a algebraic complex variety X by applying a functor $\Theta : K_0(\mathfrak{Var}_{\mathbb{C}}) \to K_0(\mathfrak{fhs})$, where $\mathfrak{Var}_{\mathbb{C}}$ is the category of complex varieties and $\mathfrak{fhs}$ the category of filtered Hodge structures. If $\Theta([X])$ is zero, then X satifies the Hodge Conjecture. As we have said in the precedent section of this introduction, the class in $K_0(\mathfrak{Var})$ of the moduli spaces of bundles and of pairs is computed via the stratification of the flip loci $\mathcal{S}_{\sigma_m^\pm}$. To make easier the proof, we consider the subring $\mathcal{R}_C \subset \hat{K}_0(\mathfrak{Var}_{\mathbb{C}})$ defined as the completion by $\mathbb{L}^{-1}$ of the minimal subring which contains the class of the curve $[C]$, the Jacobian variety of such a curve, the Leftschetz class $\mathbb{L} = [\mathbb{P}^1] - 1$ and is closed by the symmetric power operator λ^k, for any k. Usually, we say that X is motivated by the curve if its class $[X]$ belongs to $\mathcal{R}_C$. We prove that any variety motivated by a generic curve C satisfies the Hodge conjecture by showing $\Theta(\mathcal{R}_C) = 0$. Finally, we prove that the strata of the flip loci $\mathcal{S}_{\sigma_m^\pm}$ are motivated by the curve C. In general, this was done in [Mn10] for cases where the group of symmetries of the stratum is trivial. In the general case, some difficulties arise from proving this due to the fact that some bundles are not Zariski locally trivial bundles. Some techniques related to the Brauer group are employed to get around these problems.

3 Toric varieties

Chapter 8 of this PhD Thesis is devoted to a different topic.

A toric variety is an algebraic complex variety X cointaining an algebraic torus as an open dense subset such that the action of the torus on itself extends to the whole variety. They can be considered as toy examples in several fields, mainly for its easy construction and description. Nevertheless, it is possible to find in this category some counterexamples for some statements, e. g., an algebraic complex variety which is non-projective. Other interesting applications are, as well, the resolution of singularities: compare the solution suggested by Fujiki in [Fuj74] and the toric way to remove the cyclic quotient singularities. The latter method can be found in the first chapters of [Oda12].

This chapter deals with equivariant bundles. Let X be an algebraic variety and consider a group G which acts on it. A bundle $\pi : E \to X$ is said to be equivariant if the G-action lifts to E. A characterization of equivariant bundles in terms of flat connections can be found in these pages. This issue has been studied by Klyachko (see [Kly90]): he obtains an equivalence between equivariant bundles and a subset of cones satisfying some properties. This work should be consider as part of the traditional language of toric varieties as it presents any geometric objects in terms of cones, points of lattices, etc. Our approach uses the formal definitions of bundles keeping away from the traditional approach. This makes possible to extend our results to a larger category: log-parallelizable G-pairs. They keep some characteristics of toric varieties and can be viewed as a natural extension of this category: let X be a smooth projective variety and $D \subset X$ a divisor, a pair (X, D) is a log-parallelizable G-pair if it satisfies the following three properties:

- G acts on X leaving invariant D;
- D is a normal crossing divisor of X;
- the natural map $op : \mathcal{O}_X \otimes \mathfrak{g} \to TX$ is an isomorphism.

In the last pages of this chapter we compute the Chern classes of an equivariant bundle from this point of view. The result obtained is in agreement with Klyachko's.

Introducción

Esta tesis trata con los siguientes tres temas:

1) La clase del anillo de Grothendieck de variedades del espacio de móduli de fibrados y de pares,
2) la conjetura de Hodge para tales espacios de móduli,
3) Los fibrados equivariantes sobre variedades tóricas y G-pares.

En las siguientes páginas introduciremos uno por uno y brevemente los temas mencionados arriba y, al final de cada sección de esta introducción, daremos detalles sobre las técnicas aplicadas y nuestros resultados.

1 Triples y pares

A veces, el mundo matemático es como un río: hay dos lados, dos modos de entender el mismo objeto. En nuestra situación, estos dos lados son la Geometría Algebraica y la Geometría (compleja) Diferencial. Cada lado del río tiene sus propias y particulares técnicas. El conocido Teorema GAGA es uno de los puentes que une los dos lados del río. La parametrización de objetos, el problema de móduli, ha producido numerosas herramientas en esta conexión. La actual tesis está en el corazón de uno de los más estudiados problemas de móduli, el de fibrados sobre curvas algebraicas.

Sea C una curva algebraica. En 1965, Narashiman y Seshadri [NS65] dieron una construcción GIT (una construcción algebro-geométrica) de los espacios de móduli de fibrados $M_C^s(n,d)$ sobre una curva C (aquí n es el rango y d el grado de los fibrados). Esta técnica algebraica introduce la noción de estabilidad de fibrados. De hecho, esta noción de estabilidad ha tenido varias versiones, aunque la más usada es la que involucra la pendiente (el cociente de el grado por el rango del fibrado) [Mum63]. Por otro lado, tenemos la noción geométrico-diferencial conocido como la condición Hermitian-Einstein, introducido por Kobayashi. Los fibrados vectoriales que satisfacen esta condición son llamados fibrados de Hermitian-Einstein. En [Kob82], Kobayashi demostró que un fibrado de Hermitian-Einstein irreducible sobre una variedad compacta Kähler es de hecho estable. Poco después de que Donaldson diese la equivalencia entre ambas nociones en un caso particular, Hitchin y Kobayashi conjeturó de manera independiente la correspondencia Hitchin-Kobayashi, la cual establece la equivalencia entre fibrados estables y fibrados de Hermitian-Einstein. Esta correspondencia fue probada por Donaldson en [Don85] para superficies algebraicas y en [Don87] para variedades.

Un par holomorfo (E,ϕ) es un par que consiste en un fibrados E y en una sección holomorfa $\phi \in H^0(C,E)$. Actualmente tenemos a nuestra disposición varias construcciones del espacio de moduli parametrizando estos objetos. La primera construcción proviene del punto de vista analítico: por reducción dimensional [GP94] o *via* ecuaciones vorticiales [BGP97]. En esta ocasión la noción de estabilidad depende de un parámetro real (ver [Bra91]) y por lo tanto el espacio de móduli de pares $\mathfrak{M}_\tau(n,d)$ de rango n y grado d depende del parámetro real τ. Algunas propiedades están estudiadas en [BD91]. Desde el punto de vista algebraico, la construcción fue dada por Schmitt [Sch08], pero antes que él Thaddeus [Tha94] dio una construcción algebraica para el caso de rango 2 para probar la fórmula de Verlinde.

Un objeto más general que los pares es el de triples holomorfos y, en cierto sentido, disfrutan de mejores propiedades. Un triple holomorfo $T=(E_1,E_2,\phi)$ es una 3-upla que consiste en dos fibrados y una aplicación $\phi : E_2 \to E_1$. Observa que para el caso en que E_2 sea un fibrado de línea, el triple es equivalente al par $(E_1 \otimes E_2^\vee, \phi)$, donde $E^\vee$ denota el fibrado dual. El lector encontrará construcciones y propiedades de los espacios de móduli de triples en [BGP96] y en [BGPG04] (y también después en esta tesis). También, una buena referencia para este tema es el *survey* [Bra95]. Como en el caso de los pares, la noción de estabilidad depende de un parámetro real σ dando para cada valor del parámetro un espacio de móduli de triples semiestables $\mathcal{N}_\sigma = \mathcal{N}_\sigma(n_1,n_2,d_1,d_2)$ donde $(\operatorname{rk} E_i, \deg E_i) = (n_i, d_i)$ para $i=1,2$. Además, la existencia de este espacio de móduli depende de este mismo parámetro: el espacio de móduli existe (esto es, es no vacío) si y sólo si σ pertenece a un intervalo cerrado $I=[\sigma_m,\sigma_M] \subset \mathbb{R}$. Es usual llamar a σ_m y σ_M valores críticos minimal y maximal respectivamente. Cuando $\mathcal{N}_\sigma$ existe, el espacio de móduli es una variedad proyectiva compleja. En este intervalo, hay algunos valores que son especiales, llamados *valores críticos*: siempre y cuando el parámetro σ cruza un valor crítico σ_c, el conjunto subyaciente de $\mathcal{N}_\sigma$ cambia. Esto es debido a que algunos triples $(\sigma_c - \epsilon)$-inestables ($(\sigma_c - \epsilon)$-estables) se hacen $(\sigma_c + \epsilon)$-estables (resp., $(\sigma_c + \epsilon)$-inestables) y viceversa. El conjunto de tales triples define el *flip locus* $\mathcal{S}_{\sigma_c^-}$ (resp., $\mathcal{S}_{\sigma_c^+}$). En [Mn10], Muñoz presenta una estratificación para esos *flip loci* en el caso $n_1 = 1$. Tal estratificación generaliza los métodos usados en [VMnVG07] y en [Mn08] para calcular los polinomios de Hodge-Deligne de los espacios de móduli de pares y fibrados estabes para los caso de rango 2 y 3. El Capítulo 2 está dedicado a esto y proporcina completas referencias para este tema.

Los siguientes capítulos (Capítulos del 3 al 6) usan esta estratificación para calcular la clase en el anillo de Grothendieck de la categoría de variedades $K_0(\mathfrak{Var}_\mathbb{C})$ (algunas propiedades de este anillo se proporcionan en el capítulo 1 así como una introducción a Motivos que usaremos más adelante). Calculamos las clses $\mathcal{S}_{\sigma_c^+}$ y $\mathcal{S}_{\sigma_c^-}$ en $K_0(\mathfrak{Var}_\mathbb{C})$ para todos los valores críticos σ_c para rangos $n = 3$ y $n = 4$, y entonces sumamos todas esas contribuciones a lo largo de todos los valores críticos. Cálculos análogos han sido hechos en [VMnVG07] y en [Mn08] donde los autores obtienen los polinomios de Hodge-Deligne de $\mathcal{N}_\sigma(n, 1, d_1, d_2)$ para los rangos $n = 2$ y $n = 3$.

También es posible calcular las clases de $M^s(n, d)$. Esta técnica (usada principalmente en [Mn08]) se basa en la posibilidad de calcular la clase $\mathcal{N}_{\sigma_m}(n, 1, d_1, d_2)$ de dos maneras diferentes donde σ_m denota el valor crítico minimal: la primera de ellas se obtiene sumando todas las clases de todos los *flip loci* para todos los valores críticos $\sigma_c > \sigma_m$; sólo la información sobre las clases de $M^s(n', d')$ para $n' < n$ son necesarios. La segunda forma se realiza calculando directamente este espacio por medio de la igualdad $\mathcal{N}_{\sigma_m}(n, 1, d_1, d_2) = \mathcal{S}_{\sigma_m^+}$. La clase de $M^s(n, d_1)$ aparece en este último espacio $\mathcal{S}_{\sigma_m^+}$. Como se ha señalado antes, este truco se ha usado en [Mn08] para calcular el polinomio de Hodge-Deligne de $M^s(3, 1)$. Aquí realizamos cálculos análogos para obtener las clases de $M^s(2, 1)$, $M^s(2, 0)$ (cuyo polinomio de Hodge-Deligne fue calculado en [VMnVG09]) y $M^s(3, 1)$. El largo Capítulo 5 está dedicado a calcular la clase de $M^s(3, 0)$ con esta técnica.

Finalmete, en la Sección 5.6 ofrecemos un cálculo alternativo para $\mathcal{S}_{\sigma_m^+}$ para el caso $M^s(3, 0)$. Para hacer esto, presentamos un nuevo método para estratificar este *flip locus* mediante las sucesiones exactas cortas que producen la filtración Harder-Narashiman de un fibrado semiestable E del triple $T = (E, L, \phi)$. La construcción de cualquiera de estos estratos se realiza en dos pasos: primero parametrizamos los fibrados E con algunas condiciones sobre sus sucesiones exactas cortas; y segundo parametrizamos el fibrado L junto con la aplicación holomorpha $\phi : L \to E$ que definen un triple σ_m^+-estable $T = (E, L, \phi) \in \mathcal{S}_{\sigma_m^+}$.

Al final de algunos de estos capítulos, concluiremos con algunas instrucciones para calcular y comprobar los resultados usando una hoja de cálculo del Mathematica.

2 El Teorema y la Conjetura de Hodge

En los albores del siglo XX, W. Hodge demostró dos teoremas capitales en geometría compleja. La primera de ellas afirma que toda clase de cohomología en $H^k(X, \mathbb{C})$ admite un único representante armónico. La segunda establece que la cohomología $H^k(X, \mathbb{C})$ admite una descomposición de manera natural de acuerdo con el tipo del representante armónico de cada clase.

Teorema 0.4. [Teorema de descomposición de Hodge] *Sea X una variedad compacta Kähler. Sea $H^{p,q}(X)$ el espacio de cohomología de clases cuyos representantes armónicos son del tipo (p, q). Entonces existe una descomposición en suma directa*

$$H^k_{\text{de Rham}}(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X).$$

Además, $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

En cierto modo, tal descomposición codifica la estructura compleja de una variedad compleja compacta. Esto nos lleva a la noción abstracta de estructura de Hodge: fijamos un anillo R, una estructura de Hodge es un R-módulo V dotado de una filtración decreciente F en $V \otimes \mathbb{C}$ satisfaciendo que $F^pV \cap \overline{F^qV} = 0$; o equivalentemente, una descomposición de $V = \bigoplus_{p+q=n} V^{p,q}$ donde $\overline{V^{q,p}} = V^{p,q}$.

La estructura de R-modulo proporciona una información extra que corresponde con la aplicación en cohomología $H^k(X,R) \to H^k(X,\mathbb{C})$.

Las subvariedades de cualquier variedad algebraica X están relacionadas con las clases de cohomología mediante la aplicación de clases de ciclo (en inglés, *cycle class map*) $\mathrm{cl} : Z_k(X) \to H^{2(n-k)}(X)$ definidas por la igualdad

$$\int_X \alpha \wedge \mathrm{cl}(Z) = \int_Z i^* \alpha.$$

El conjunto de *clases algebraicas* es la imagen de $Z_k(X)$ por esta aplicación. La aplicación de clases de ciclo induce una correspondencia entre los ciclos en $Z_k(X)$ y las clases en $H^{n-k}(X)$. Esto es una propiedad geométrica puramente diferencial. En geometría compleja, esta correspondencia lleva de manera natural a la *conjetura de Hodge* que hasta hoy permanece abierta. Para dar una formulación correcta necesitamos realizar ciertas observaciones.

La imagen de cualquier clase por cl vive en un subconjunto muy especial de $H^{2(n-k)}(X)$. Primero (y esto no es difícil de comprobar) $\mathrm{cl}(Z) \in H^{n-k,n-k}(X)$ para cualquier ciclo $Z \in Z_k(X) \otimes \mathbb{Q}$. Además, al ser Z una clase racional, $\mathrm{cl}(Z)$ tentonces vive en la imagen por la aplicación $H^{2(n-k)}(X,\mathbb{Q}) \to H^{2(n-k)}(X,\mathbb{C})$. La intersección de ambos conjuntos son lo que se conoce como las *clases de Hodge*:

$$\mathrm{Hdg}^k(X) = H^{k,k}(X) \cap \mathrm{Im}\left(H^{2(n-k)}(X,\mathbb{Q}) \to H^{2(n-k)}(X,\mathbb{C})\right).$$

La conjetura de Hodge establece una correspondencia análoga para variedades proyectivas

Conjetura de Hodge 0.5. *Sea X una variedad proyectiva no singular. Toda clase de tipo (k,k) con coeficientes racionalees es algebraico. En otras palabras, toda clase en $\mathrm{Hdg}^k(X)$ es una combinación racional de clases de cohomología fundamentales de subvariedades de X.*

Hay ciertas evidencias que inclinan a pensar que la conjetura de Hodge puede ser cierta. Por ejemplo, el Teorema $(1,1)$ de Leftchetz demuestra que la conjetura de Hodge es cierta para ciclos de codimensión 1. Si combinamos este teorema con el Propiedad fuerte de Leftschetz, podemos ver que la conjetura de Hodge sigue siendo válido para ciclos de dimensión 1. Además, la conjetura de Hodge ha sido probada para algunas familias de variedades como las variedades de Prym.

Una de las más populares versiones de la conjetura de Hodge, que a la postre es equivalente a la original, es la *Conjetura de Hodge Generalizada*, escrita en términos de filtraciones.

Conjetura de Hodge Generalizada 0.6. *Sea X una variedad proyectiva no singular. La mayor subestructura de Hodge racional de $F^pH^k(X,\mathbb{C}) \cap H^k(X,\mathbb{Q})$ es la unión de todas las subestructuras racionales de hodge soportadas por subvariedades de X de codimensión mayor o igual que p.*

A pesar de que la conjetura de Hodge sigue sin una respuesta, las diferentes aproximaciones a esta conjetura han producido un buen numero de objetos y resultados interesantes en Geometría Algebraica, como el trabajo de P. Deligne sobre las estructuras mixtas de Hodge ([Del71a], [Del71b] and [Del74]), quien demuestra que los grupos de cohomología de una variedad algebraica (quasi-proyectiva o singular) sobre un cuerpo algebraicamente cerrado admite de manera natural una estructura de Hodge mixta.

La tesis presente da una demostración de la conjetura de Hodge generalizada para los espacios de móduli de fibrados y pares con rango menor o igual que 4 (para una curva genérica). Nuestra aproximación a este problema está basado en las técnicas desarrolladas por D. Arapura y S. Kang en [AK06]. Este trabajo hace posible comprobar la conjetura de Hodge generalizada de una variedad compleja algebraica X aplicando un funtor $\Theta : K_0(\mathfrak{Var}_{\mathbb{C}}) \to K_0(\mathfrak{fhs})$, donde $\mathfrak{Var}_{\mathbb{C}}$ denota la categoría de variedades complejas algebraicas y $\mathfrak{fhs}$ la de estructuras de Hodge con una filtración. Si $\Theta([X])$ es cero, entonces X satisface la conjetura de Hodge. Como se ha indicado en la sección precedente de esta introducción, la clase en $K_0(\mathfrak{Var}_{\mathbb{C}})$ del espacio de móduli de fibrados y pares se calcula *via* la estratificación de los *flip loci* $\mathcal{S}_{\sigma_m^\pm}$. Para hacer más fácil la prueba, consideraremos un subanillo $\mathcal{R}_C \subset \hat{K}_0(\mathfrak{Var}_{\mathbb{C}})$ definido como la compleción por $\mathbb{L}^{-1}$ del subanillo más pequeño que contiene a la clase de la curva, $[C]$, a su variedad de Jacobi $\mathrm{Jac}\, C$ y a la clase de Leftschetz $\mathbb{L} = [\mathbb{P}^1] - 1$, siendo cerrado para el operador simétrico λ^k para cualquier k. Usualmente diremos que X está motivado por la curva si su clase $[X]$ pertenece a $\mathcal{R}_C$. Demostraremos que cualquier variedad motivada por una curva genérica C satisface la conjetura de Hodge al probar que $\Theta(\mathcal{R}_C) = 0$. Finalmente, demostraremos que todos los estratos de los *flip loci* $\mathcal{S}_{\sigma_c^{\pm}}$ están motivadas por la curva C. En general, esto ya ha sido hecho en [Mn10] para los casos en los que el grupo de simetrías del estrato es trivial. En el caso general, algunas dificultades aparecen en la demostrción debido al hecho de que algunos fibrados no son localmente Zariski triviales. Algunas técnicas relacionadas con el grupo de Brauer han sido empleadas para solventar estos problemas.

3 Variedades tóricas

El capítulo 8 de esta tesis se dedica a un tema completamente diferente.

Una variedad tóricas es una variedad algebraica X que contiene un toro algebraico (el grupo multiplicativo $(\mathbb{C} \setminus \{0\})^n$ para algún n) como un abierto denso tal que la propia acción del toro se extiende a toda la variedad. Por su aparente simplicidad y su facilidad de construcción suelen considerarse como ejemplos sencillos en varios campos de la matemática. Sin embargo, es posible encontrar en ellas algunos contraejemplos para algunas afirmaciones, e. g., una variedad compleja que no es proyectiva. Otras interesantes aplicaciones son, además, la resolución de singularidades: compare la solución que da Fujiki en [Fuj74] y la que las variedades tóricas aportan para resolver singularidades cíclicas cociente. El último método puede encontrarse en los primeros capítulos de [Oda12].

Este capítulo trata con fibrados equivariantes. Sea X una variedad algebraica tórica, y consideremos el grupo G que actúa sobre él. Un fibrado $\pi : E \to X$ se dice que es equivariante si la acción de G en X induce una acción en E. Una caracterización de fibrados equivariantes en términos de conexiones planas puede encontrarse en estas páginas. Este tema ha sido estudiado por Klyachko (ver [Kly90]): él obtiene una equivalencia entre fibrados equivariantes y un subconjunto de conos satisfaciendo ciertas propiedades. Este trabajo está enmarcado en el lenguaje tradicional de variedades tóricas, puesto que presenta cualquier objeto geométrico en términos de conos, puntos de *lattices*, etc. Nuestra aproximación usa la definición formal de fibrados dejando aparte el tratamiento tradicional. Esto hace posible extender nuestros resultados a una categoría más grande: los G-pares log-paralelizables. Estos conservan algunas características de las variedades tóricas y pueden verse como una extensión natural de esta categoría: sea X una variedad proyectiva no singular y sea $D \subset X$ un divisor, un par (X, D) es un G-par log-paralelizable si satisface las siguientes tres propiedades:

- G actúa en X dejando invariante al divisor D;
- D es un divisor con intersecciones normales (más conocido como *normal crossing divisor*) de X;
- la aplicación natural $op : \mathcal{O}_X \otimes \mathfrak{g} \to TX$ es un isomorfismo.

En las últimas páginas de este capítulo calculamos las clases de Chern de un fibrado equivariante desde este punto de vista. El resultado obtenido coincide con el de Klyachko.

Chapter 1
Motives and Grothendieck groups

1 The Grothendieck group

There are a lot of useful functors on the category of quasi-projective varieties over $\mathbb{C}$, $\mathfrak{Var}_{\mathbb{C}}$, which share some standard properties like good behaviour under disjoint unions (excision theorems) or fibers products. We have examples like the Euler characteristic $\chi(X) = \sum_n (-1)^n h^n(X, \mathbb{C})$ or the Hodge-Deligne polynomial $h(X) = \sum_{p,q} h^{p,q}(X) u^p v^q$. This induces to think that these properties has some geometric meaning: this defines the spirit of the Grothendieck group.

Definition 1.1. *The Grothendieck ring $K_0(\mathfrak{Var}_{\mathbb{C}})$ is the quotient of the free abelian group generated by isomorphism classes of k-varieties by the relation $[X \backslash Y] = [X] - [Y]$, where Y is a closed subscheme of X. The fibre product over k induces a ring structure defined by $[X] \cdot [X'] = [(X \times X')_{\text{red}}]$.*

The unit of the sum this group is $0 = [\varnothing]$ and the unit of the product is $\mathbf{1} = [\operatorname{Spec} k]$ (easily, $\operatorname{Spec} k \times_k X \cong X$). We define the Lefschetz class $\mathbb{L} = [\mathbb{A}^1_k]$. Then

$$[\mathbb{P}^n{}_k] = \mathbf{1} + \mathbb{L} + \cdots + \mathbb{L}^n$$

In 2004, Franziska Bittner [Bit04] shows that the Grothendieck group has an alternative presentation.

Theorem 1.2. [Bit04, Theorem 5.1] *The Grothendieck group of k-varieties has the following alternative presentation: it is the abelian group generated by the isomorphism classes of smooth complete k-varieties subject to the relation $[\varnothing] = 0$ and $[\operatorname{Bl}_Y X] = [X] - [Y] + [E]$, where X is smooth and complete, $Y \subset X$ is a closed smooth subvariety, $\operatorname{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

Definition 1.3. *Let R and S be rings. An additive invariant λ from the categroy $\mathfrak{Var}_R$ of algebraic varieties over R with values in S, assigns for any X in $\mathfrak{Var}_R$ an element $\lambda(X)$ of S such that $\lambda(X) = \lambda(X')$ for $X \cong X'$, $\lambda(X) = \lambda(X') + \lambda(X \setminus X')$ for X' closed in X, and $\lambda(X \times X') = \lambda(X) \cdot \lambda(X')$ for every X and X'.*

The following objects are classical examples of this object:

- The Euler characteristic map

$$\chi(X) = \sum_n (-1)^n \operatorname{rk} H^n(X_{\mathbb{C}}, \mathbb{C}),$$

- the Hodge-Deligne polynomial

$$e(X)(u,v) = \sum_{k,p,q} (-1)^{p+q} h^{k,p,q} u^p v^q$$

- and the map of virtual motives —see below for a definition of motives—

$$\chi_{\mathbb{C}} : \mathfrak{Var}_k \to K_0(\mathfrak{Mot}_k).$$

Indeed, this is a theorem:

Theorem 1.4. [Ser91 and Jan94] *Let k be a field of characteristic zero. There exists an unique morphism of rings*

$$\chi_c : K_0(\mathfrak{Var}_k) \to K_0(\mathfrak{Mot}_k)$$

where $\mathfrak{Mot}_k$ denotes the category of Chow motives, such that $\chi_c([X]) = [h(X)]$ for X projective and smooth.

Another useful operator in $K_0(\mathfrak{Var}_\mathbb{C})$ is the induced by the symmetric operator in $\mathfrak{Var}_\mathbb{C}$. Let us define the symmetric power of a variety X to be

$$\operatorname{Sym}^n X = X^n/\mathfrak{S}_n$$

where n is a positive integer number. When $n = 0$, we fix $\operatorname{Sym}^0 X = \operatorname{Spec}\mathbb{C}$. Then, this operator induces a map $\lambda^n : K_0(\mathfrak{Var}_\mathbb{C}) \to K_0(\mathfrak{Var}_\mathbb{C})$ to be $\lambda([X]) = [\operatorname{Sym}^k X]$. There are some well-known properties:

$$\lambda^n(X+Y) = \sum_{i+j=n} \lambda^i(X) \cdot \lambda^j(Y), \tag{1.1}$$

where $X, Y \in K_0(\mathfrak{Var})$. Also

$$\lambda^n(\mathbb{L}^k) = \mathbb{L}^{k\cdot n}. \tag{1.2}$$

From the Totaro's lemma [Got01, Lemma 4.4]

$$[\operatorname{Sym}^k(Y \times \mathbb{A}^l)] = [\operatorname{Sym}^k(Y)] \times [\mathbb{A}^{kl}],$$

we have

$$\lambda^k([Y]\mathbb{L}^l) = \lambda^k([Y])\mathbb{L}^{kl}.$$

Related to this operator we define the additive operator δ^n for $n \in \mathbb{N}$. We compute this operator only for cases $n = 2, 3$, but it can defined for any integer number n.

$$\begin{aligned} \delta^2(X) &= 2\lambda^2(X) - X^2, \\ \delta^3(X) &= \frac{1}{2}\left(6\lambda^3(X) - 3\delta^2(X)X - X^3\right) \\ &= 3\lambda^3(X) - 3\lambda^2(X)X + 2X^2. \end{aligned} \tag{1.3}$$

So defined we have the properties $\delta^n(X+Y) = \delta^n(X) + \delta^n(Y)$ for any class $X, Y \in K_0(\mathfrak{Var}_\mathbb{C})$ and $n = 2, 3$. Also, $\delta^n(\mathbb{L}^k X) = \mathbb{L}^{nk}\delta^n(X)$. We check this for $n = 3$; the reader can be check this for $n = 2$.

Lemma 1.5. *The δ^3 operator is additive.*

Proof. This is a constructive proof: we define an additive operator by forcing it to satisfy the additivity condition. Let us begin with $3! \cdot \lambda^3(X) - X^3$ and we compute the "excess":

$$6\lambda^3(X+Y) - (X+Y)^3 - \left[6(\lambda^3(X) + \lambda^3(Y)) - X^3 - Y^3\right] = 3\left(2\lambda^2(X) - X^2\right)Y + 3\left(2\lambda^2(Y) - Y^2\right)X,$$

so we add $3\delta^2(X)X + 3\delta^2(Y)Y$ to obtain the operator

$$X \mapsto 6\lambda^3(X) - 3\delta^2(X)X - X^3$$

which preserves the sum in $K_0(\mathfrak{Var}_\mathbb{C})$ by construction. □

Remark 1.6. *By the equalities in Hodge-Deligne polynomials for symmetric powers*

$$\begin{aligned} e(\operatorname{Sym}^2 X)(u,v) &= \frac{1}{2}\left(e(X)(u,v)^2 + e(X)(-u^2,-v^2)\right) \\ e(\operatorname{Sym}^3 X)(u,v) &= \frac{1}{6}\left(e(X)(u,v)^3 + 3e(X)(-u^2,-v^2)e(x)(u,v) + 2e(X)(u^3,v^3)\right), \end{aligned} \tag{1.4}$$

we match (1.3) with its respective pieces and conclude that

$$
\begin{aligned}
e(\delta^2(X))(u,v) &= e(X)(-u^2,-v^2),\\
e(\delta^3(X))(u,v) &= e(X)(u^3,v^3).
\end{aligned}
\tag{1.5}
$$

An interesting question is to prove if the product on $K_0(\mathfrak{Var}_{\mathbb{C}})$ is preserved by δ^n, that is, $\delta^n(X\cdot Y)=\delta^n(X)\cdot\delta^n(Y)$ for $X,Y\in K_0(\mathfrak{Var}_{\mathbb{C}})$ and n an integer number. Notice that its corresponding Hodge-Deligne polynomials (e. g. $e(X)(u^3,v^3)$) does it.

The presentation of the Grothendieck group by blow-ups gives an useful map in $K_0(\mathfrak{Var}_{\mathbb{C}})$ induced by functors: Arapura and Kang [AK06] shows that there is a homomorphism of groups $AK: K_0(\mathfrak{Var}_{\mathbb{C}})\to K_0(\mathfrak{fhs})$ so that $AK([X])=0$ if and only if X satisfies the Generalized Hodge Conjecture. Here, $\mathfrak{fhs}$ is the category of filtered Hodge structure. Nevertheless, althought AK preserves the sum, the product does not holds.

This really huge group $K_0(\mathfrak{Var}_{\mathbb{C}})$ is still unknown. Recently, we have learned that it is not a domain.

Theorem 1.7. [Poo02] *Let k be an algebraic closed field of characteristic zero. The Grothendieck ring of varieties $K_0(\mathfrak{Var}_k)$ is not a domain.*

It is possible to define a *power structure* in $K_0(\mathfrak{Var}_k)$ as [GZLMH04b] and [GZLMH04a].

Definition 1.8. *A power structure over a ring R with a unit is a map*

$$
\begin{aligned}
(1+t\cdot R[[t]])\times R &\longrightarrow 1+t\cdot R[[t]]\\
(A(t),m) &\longmapsto A(t)^m
\end{aligned}
$$

which posseses the following properties:

1) $A(t)^0=1$,
2) $A(t)^1=A(t)$,
3) $(A(t)\cdot B(t))^m=A(t)^m\cdot B(t)^m$,
4) $A(t)^{m+n}=A(t)^m\cdot A(t)^n$,
5) $A(t)^{m\cdot n}=(A(t)^m)^n$,
6) $(1+t)^m=1+mt+$ *terms of higher degree,*
7) $A(t^k)^m=A(t)^m|_{t\mapsto t^k}$.

To define a power structure, we introduce the following concept:

Definition 1.9. *A pre-λ structure on a ring R is given by a series $\lambda_a(t)\in 1+t\cdot R[[t]]$ defined for each $a\in R$ so that*

1) $\lambda_a(t)=1+at \pmod{t^2}$
2) $\lambda_{a+b}(t)=\lambda_a(t)\lambda_b(t)$ *for* $a,b\in R$.

The following theorem allows us to define a power structure on any ring R by having a pre-λ structure:

Proposition 1.10. [GZLMH04a, Prop. 2] *To define a finitely determined power structure over a ring R it is sufficient to define the series $A_0(t)^m$ for any fixed series $A_0(t)$ of the form $1+t+\cdots$ (terms of higher degree), and for each $m\in R$, so that*

1) $A_0(t)^m=1+mt+$ *terms of higher degree,*
2) $A(t)^{m+n}=A(t)^m\cdot A(t)^n$,

Thus, we introduce the candidate of pre-λ structure.

Definition 1.11. *Define the Kapranov zeta function of a quasi-projective variety X as the series*

$$
\zeta_X(t)=1+[X]t+[\mathrm{Sym}^2 X]t^2+[\mathrm{Sym}^3 X]t^3+\cdots\in K_0(\mathfrak{Var}_{\mathbb{C}})[[t]].
$$

For example, by (1.2) we have
$$\zeta_{\mathbb{L}^n}(t) = \frac{1}{1-\mathbb{L}^n t}.$$

The equation (1.1) gives the equality
$$\zeta_{X+Y}(t) = \zeta_X(t)\cdot\zeta_Y(t). \tag{1.6}$$

This means that $\zeta_\bullet(t)$ defines a pre-λ structure on the Grothendieck ring $K_0(\mathfrak{Var})$. Thus, we write
$$\left(\sum_{n=0}^{\infty} t^n\right)^{[X]} = \left(\frac{1}{1-t}\right)^{[X]} = \sum_{n=0}^{\infty} \lambda^n([X])t^n.$$

Usually, we work in a larger group. Recall $\mathbb{L}$ the Lefschetz object, we shall consider the localization $K(\mathfrak{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$, and its completion
$$\hat{K}(\mathfrak{Var}_{\mathbb{C}}) = \left\{ \sum_{r\geq 0} [Y_r]\,\mathbb{L}^{-r}\,;\, \dim Y_r - r \to -\infty \right\}.$$

The λ^n-operations can be extended to $\hat{K}(\mathfrak{Var}_{\mathbb{C}})$.

■ **The class of a curve** Let C be a smooth projective algebraic curve over $\mathbb{C}$ of genus g. Define the class $\bar{C} = [C] - \mathbf{1} - \mathbb{L}$ in $K_0(\mathfrak{Var}_{\mathbb{C}})$. It is well-known that
$$\begin{aligned} &\lambda^n(\bar{C}) = \mathbb{L}^{g-n}\lambda^{2g-n}(\bar{C}),\ \text{for } n = 0,\dots,2g,\\ &\lambda^n(\bar{C}) = 0,\ \text{for } n > 2g. \end{aligned} \tag{1.7}$$

This implies that $\zeta_{\bar{C}}(t)$ is a polynomial of degree $2g$. Furthermore, the class $\lambda^n([C])$ is
$$\lambda^n([C]) = \operatorname*{coeff}_{x^n} \zeta_{\mathbf{1}+\mathbb{L}+\bar{C}}(x) = \operatorname*{coeff}_{x^n} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)}$$

by applying the properties of function ζ. Using the equalities (1.7) it is possible to write the a explicit class for $\zeta_{\bar{C}}(\mathbb{L}^k)$.

Lemma 1.12. *We have the following equality*
$$\zeta_{\bar{C}}(\mathbb{L}^k) = \lambda^g(\bar{C}\mathbb{L}^k+1) + \mathbb{L}^{kg}\lambda^g(\tilde{C}+\mathbb{L}^{k+1}) - \lambda^g(\tilde{C}\mathbb{L}^k), \tag{1.8}$$

for k an integer number.

Proof. The first equality of (1.7) yields that $\zeta_{\bar{C}}(t) = \sum_{n=0}^{\infty}\lambda^n(\bar{C})t^n$ is a polynomial of degree $2g$. We use the second equality to get a shorter expression for $\zeta_{\bar{C}}(t)$:
$$\begin{aligned} \zeta_{\bar{C}}(t) &= \sum_{k=0}^{g-1}\lambda^k(\bar{C})t^k + \lambda^g(\bar{C})t^g + \sum_{k=g+1}^{2g} \mathbb{L}^{k-g}\lambda^{2g-k}(\bar{C})t^k \\ &= \sum_{k=0}^{g}\lambda^k(\bar{C})t^k - \lambda^g(\bar{C})t^g + \sum_{k=0}^{g}\mathbb{L}^k\lambda^{g-k}(\bar{C})t^{k+g} \\ &= \sum_{k=0}^{g}\lambda^k(\bar{C})t^k - \lambda^g(\bar{C})t^g + t^g\sum_{k=0}^{g}\lambda^{g-k}(\bar{C})(\mathbb{L}t)^k. \end{aligned}$$

Now, we substitute t by $\mathbb{L}^k$ and we use $\lambda^k(\mathbb{L}^s) = \mathbb{L}^{ks}$. We have the desired equality. □

Lemma 1.13. *The following equality holds:*

$$\zeta_{\bar{X}}(\mathbb{L}^{-a}) = \mathbb{L}^{(1-2a)g}\zeta_{\bar{X}}(\mathbb{L}^{a-1}). \tag{1.9}$$

Proof. Consider the equality (1.8) and multiply by $\mathbb{L}^{ag}$ to obtain

$$\lambda^g(\bar{C}\mathbb{L}^{a+1} + \mathbb{L}^a) + \lambda^g(\bar{C}\mathbb{L}^{a+1} + \mathbb{L}^{2(a+1)}) - \lambda^g(\bar{C}\mathbb{L}^{a+1}).$$

We relate this class to $\mathbb{L}^{2ga}\zeta_{\bar{C}}(\mathbb{L}^{-a})$. Such class is written as follows

$$\mathbb{L}^{2(a+1)g}\lambda^g(\bar{C}\mathbb{L}^{-a-1} + \mathbb{L}^{-2(a+1)+1}) + \mathbb{L}^{2(a+1)g}\lambda^g(\bar{C}\mathbb{L}^{-a-1} + 1) + \mathbb{L}^{2(a+1)g}\lambda^g(\bar{C}\mathbb{L}^{-a-1})$$

which is clearly $\mathbb{L}^{2g(a+1)}\zeta_{\bar{C}}(\mathbb{L}^{-a-1})$. So rephrasing the equality $\mathbb{L}^g\zeta_{\bar{C}}(\mathbb{L}^a) = \mathbb{L}^{2(a+1)g}\zeta_{\bar{C}}(\mathbb{L}^{-1-a})$ we have the desired result. □

It will be useful to develop some technique to compute classes in a easy way. The following geometric results goes in such direction.

Theorem 1.14. [Existence of Residue map] *Let k be a algebraically closed field and let C be a complete non-singular curve over k. For each closed point $p \in C$, there is an unique k-linear map* $\mathrm{Res}_p : \Omega_C \to k$ *with the following properties*

1) $\mathrm{Res}_p\,\tau = 0$ *for all* $\tau \in \Omega^1_C$*;*
2) $\mathrm{Res}_p(f^n \mathrm{d}f) = 0$ *for all* $f \in K^*$*, and* $n \neq -1$*;*
3) $\mathrm{Res}_p(f^{-1}\mathrm{d}f) = v_P(f) \cdot 1$*, where* v_P *is the valuation associated to* P*.*

Theorem 1.15. [Residue theorem] *For any* $\tau \in \Omega_C$*, we have* $\sum_{p\in C} \mathrm{Res}_p\,\tau = 0$*.*

Let us consider $\mathbb{C}(\mathbb{L})$ the subring of $K_0(\mathfrak{Var}_{\mathbb{C}})$ generated by the Leftschetz class. Unlike $\mathcal{R}_C$, it is obvious that $\mathbb{C}(\mathbb{L})$ is a domain, so we are allowed to apply the Residue Theorem on the curve $\mathbb{P}^1_{\mathcal{K}}$ where $\mathcal{K}$ denotes the algebraic clousure of $\mathbb{C}(\mathbb{L})$.

Usually, in this work we compute residues of rational polynomials like

$$\frac{\zeta_{\bar{C}}(x)}{\prod_{i=1}^b(1-\mathbb{L}^{a_i}x)x^n}.$$

where a_i are integer numbers and all distinct each other. Since $\zeta_{\bar{C}}(x)$ is a polynomial of degree $2g$, we have $2g$ monomials

$$\sum_{k=0}^{2g}\frac{x^k}{\prod_{i=1}^b(1-\mathbb{L}^{a_i}x)x^n}\lambda^k(\bar{C}).$$

Then, we compute for each monomial its coefficient treating them as a rational function on $\mathbb{P}^1_{\mathcal{K}}$. In other words, frecuently we will write

$$\operatorname*{coeff}_{x^0}\frac{\zeta_{\bar{C}}(x)}{\prod_{i=1}^b(1-\mathbb{L}^{a_i}x)x^n} = \mathrm{Res}_{x=0}\frac{\zeta_{\bar{C}}(x)}{\prod_{i=1}^b(1-\mathbb{L}^{a_i}x)x^{n+1}}\mathrm{d}x,$$

where by the last notation we mean

$$\sum_{k=0}^{2g}\left(\mathrm{Res}_{x=0}\frac{x^k}{\prod_{i=1}^b(1-\mathbb{L}^{a_i}x)x^{n+1}}\mathrm{d}x\right)\lambda^k(\bar{C}).$$

Formally, the result is like computing the residue as if the ring $\mathcal{R}_C$ is a domain and apply the Residue Theorem in $\mathbb{P}^1_{\bar{\mathcal{R}}_C}$, where $\bar{\mathcal{R}}_C$ is the algebraic closure of $\mathcal{R}_C$.

2 Motives

■ **Motives** Let us review the definition of Chow motives. A standard reference for the basic theory of classical motives, including the material presented here, is [Sch94]. Given a smooth projective variety X, let $CH^d(X)$ denote the abelian group of $\mathbb{Q}$-cycles on X, of codimension d, modulo rational equivalence. Let $\mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}$ be the category of quasi-projective varieties. If $X, Y \in \mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}$, suppose that X is connected and $\dim(X) = d$. The group of correspondences (of degree 0) from X to Y is $\mathrm{Corr}(X,Y) = CH^d(X \times Y)$.

For varieties $X, Y, Z \in \mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}$, the composition of correspondences

$$\mathrm{Corr}(X,Y) \otimes \mathrm{Corr}(Y,Z) \to \mathrm{Corr}(X,Z)$$

is defined as

$$g \circ f = p_{XZ*}(p_{XY}^*(f) \cdot p_{YZ}^*(g)),$$

where $p_{XZ} : X \times Y \times Z \to X \times Z$ is the projection, and similarly for p_{XY} and p_{YZ}.

Definition 1.16. *The category of (effective Chow) motives is the category $\mathfrak{Mot}$ such that:*

- *its objects are pairs (X,p) where $X \in \mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}$, and $p \in \mathrm{Corr}(X,X)$ is an idempotent ($p = p \circ p$);*
- *if $(X,p), (Y,q)$ are effective motives, then the morphisms are $\mathrm{Hom}((X,p),(Y,q)) = q \circ \mathrm{Corr}(X,Y) \circ p$.*

There is a natural functor

$$h : \mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}{}^{\mathrm{opp}} \to \mathfrak{Mot} \tag{1.10}$$

such that, for a smooth projective variety X,

$$h(X) = (X, \Delta_X),$$

where $\Delta_X \in \mathrm{Corr}(X,X)$ is the graph of the identity $id_X : X \to X$. We say that $h(X)$ is the *motive of* X.

The category $\mathfrak{Mot}$ is pseudo-abelian, where direct sums and tensor products are defined as follows:

$$(X,p) \oplus (Y,q) = (X \sqcup Y, p+q),$$
$$(X,p) \otimes (Y,q) = (X \times Y, p_{X\times X}^*(p) \cdot p_{Y\times Y}^*(q)).$$

In particular

$$h(X \sqcup Y) = h(X) \oplus h(Y),$$
$$h(X \times Y) = h(X) \otimes h(Y).$$

This allows us to define $K(\mathfrak{Mot})$ as the abelian group generated by elements $[M]$, for $M \in \mathfrak{Mot}$, subject to the relations $[M] = [M_1] + [M_2]$, when $M = M_1 \oplus M_2$. This is a ring with the product $[M_1] \cdot [M_2] = [M_1 \otimes M_2]$.

In $\mathfrak{Mot}$, we have that $\mathbf{1} = h(pt)$ is the identity of the tensor product, so it is called the *unit motive*. It is easily seen that there is an isomorphism $\mathbf{1} = (\mathbb{P}^1, \mathbb{P}^1 \times pt)$. Set $\mathbb{L} = (\mathbb{P}^1, pt \times \mathbb{P}^1)$, which is called the *Lefschetz motive* (the reason for using the same notation as for the Lefschetz object will be clear in a moment). Therefore $h(\mathbb{P}^1) = \mathbf{1} \oplus \mathbb{L}$, and more generally,

$$h(\mathbb{P}^n) = \mathbf{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n.$$

Denote also by $\mathbb{L} \in K(\mathfrak{Mot})$ the class of the Lefschetz motive $\mathbb{L} \in \mathfrak{Mot}$. We formally invert $\mathbb{L} \in K(\mathfrak{Mot})$, and then consider the completion of $K(\mathfrak{Mot})[\mathbb{L}^{-1}]$, namely

$$\hat{K}(\mathfrak{Mot}) = \left\{ \sum_{r \geq 0} [M_r]\, \mathbb{L}^{-r} \,;\, \dim M_r - r \to -\infty \right\}.$$

In [Man68] it was shown that the motive of the blow-up of a smooth projective variety X along a codimension r smooth subvariety Y is $h(\mathrm{Bl}_Y(X)) = h(X) \oplus \Big(\bigoplus_{i=1}^{r-1} (h(Y) \otimes \mathbb{L}^i)\Big)$, being thus compatible with the relation defining $K^{bl}(\mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}})$. So the map h in (1.10) descends to $K^{bl}(\mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}) \to K(\mathfrak{M}ot)$, hence defining a ring homomorphism

$$\chi : \hat{K}(\mathfrak{V}\mathrm{ar}_{\mathbb{C}}) \to \hat{K}(\mathfrak{M}ot). \tag{1.11}$$

When X is smooth and projective, we have

$$\chi([X]) = [h(X)],$$

so we can think of the map χ as the natural extension of the notion of motives to all quasi-projective varieties. Notice that $\chi(\mathbb{L}) = \mathbb{L}$, which justifies the use of the same notation for the Lefschetz object and the Lefschetz motive.

Let X be a smooth projective variety, and F a finite group acting on X. Then, from Proposition 1.2 of [dBRNA98], we have the equation

$$h(X/F) = \Big(X, \frac{1}{|F|} \sum_{g \in F} \Gamma_g\Big),$$

where Γ_g is the graph of $g \in F$. In particular, $h(X/F)$ is an effective sub-motive of $h(X)$, that is

$$h(X) = h(X/F) \oplus N \tag{1.12}$$

for an effective motive N.

From [dB01], there are operations $\lambda^i(M)$ in $K(\mathfrak{M}ot)$ such that for X smooth projective, we have $\lambda^i([h(X)]) = [h(\mathrm{Sym}^i X)]$. These operations satisfy the relation $\lambda^n(a+b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b)$. Note that $\lambda^k(\mathbb{L}^r) = \mathbb{L}^{rk}$. The map χ in (1.11) therefore commutes with the corresponding λ-operations on $K(\mathfrak{V}\mathrm{ar}_{\mathbb{C}})$ and on $K(\mathfrak{M}ot)$.

Let C be a smooth projective curve. The motive of C decomposes as

$$h(C) = \mathbf{1} \oplus h^1(C) \oplus \mathbb{L}.$$

It can be seen that

$$h(\mathrm{Jac}\, C) = \sum_{i=0}^{2g} h^i(\mathrm{Jac}\, C),$$

where $[h^i(\mathrm{Jac}\, C)] = \lambda^i([h^1(C)])$, $0 \le i \le 2g$.

The following will be used later. Let C be a smooth projective curve over $\mathbb{C}$. We define the subring $\tilde{\mathcal{R}}_C \subset \hat{K}(\mathfrak{V}\mathrm{ar}_{\mathbb{C}})$ as the smallest set such that:

1) $[C] \in \tilde{\mathcal{R}}_C$;
2) if $a, b \in \tilde{\mathcal{R}}_C$, then $a \cdot b \in \tilde{\mathcal{R}}_C$;
3) if $a \in \tilde{\mathcal{R}}_C$, then $\mathbb{L}^k \cdot a \in \tilde{\mathcal{R}}_C$ for all $k \in \mathbb{Z}$;
4) $\tilde{\mathcal{R}}_C$ is complete: if $a_k \in \tilde{\mathcal{R}}_C$, $\dim a_k \to -\infty$, then $\sum_{k \ge 0} a_k \in \tilde{\mathcal{R}}_C$;
5) if $a \in \tilde{\mathcal{R}}_C$, then $\lambda^n(a) \in \tilde{\mathcal{R}}_C$ for all $n \ge 0$.

Note that

$$[\mathrm{Jac}\, C] \in \tilde{\mathcal{R}}_C. \tag{1.13}$$

This holds because there is a Zariski locally trivial fibration $\mathbb{P}^{g-1} \to \mathrm{Sym}^{2g-1} C \to \mathrm{Jac}\, C$, hence $[\mathrm{Jac}\, C] = \lambda^{2g-1}([C])[\mathbb{P}^{g-1}]^{-1}$.

Finally, define

$$\mathcal{R}_C = \chi(\tilde{\mathcal{R}}_C) \subset \hat{K}(\mathfrak{M}ot). \tag{1.14}$$

Definition 1.17. *We say that $X \in \mathfrak{V}\mathrm{ar}_{\mathbb{C}}$ is motivated by C if $\chi([X]) \in \mathcal{R}_C$.*

Henceforth we shall use the notation $[X] \in \hat{K}(\mathfrak{M}ot)$ for $\chi([X])$.

3 The Jacobian Variety

■ Definition: an algebraic approach

Let C be a curve over a field k (for us, $k = \mathbb{C}$). We have the *Picard group* $\operatorname{Pic}(C)$ which is the group of isomorphism classes of invertible sheaves (bundles of rank 1) with the tensor product $L \otimes L'$ as the law group and dualization $L \mapsto L^\vee$ as the inverse.

For any scheme T, let $\operatorname{Pic}^\circ(C \times T) \subseteq \operatorname{Pic}(C \times T)$ be the set of invertible sheaves $\mathcal{L}$ on $C \times T$ such that the restriction $\mathcal{L}|_{C_t}$ on each fibre for $t \in T$ has degree 0. The elements of $\operatorname{Pic}^\circ(C \times T)$ are families of invertible sheaves of degree 0 on C, parametrised by T. Nevertheless, we have "repeated" families. Let $p : C \times T \to T$ the second projection, we have the subgroup $p^* \operatorname{Pic}(T)$ of $\operatorname{Pic}^\circ(C \times T)$: for any $\mathcal{N} \in \operatorname{Pic}(T)$, the pull-back $p^*\mathcal{N}$ is a trivial invertible sheaf on each fibre C_t. These elements are *trivial families* of invertible sheaves of degree 0 over C. We quotient $\operatorname{Pic}^\circ(C \times T)$ by this subgroup and we define

$$\operatorname{Pic}^\circ(C/T) = \operatorname{Pic}^\circ(C \times T)/p^* \operatorname{Pic}(T).$$

This quotient group is, indeed, the set of families of invertible sheaves where two families $\mathcal{L}$ and $\mathcal{M}$ are equivalent if and only if $\mathcal{L} \otimes \mathcal{M}^{-1} \in p^* \operatorname{Pic}(T)$, or equivalently, $\mathcal{L}|_{C_t} \cong \mathcal{M}|_{C_t}$ for all $t \in T$.

This gives the functor $\mathfrak{J} : \mathfrak{Var} \to \mathfrak{Ab}$ defined by $\mathfrak{J}(T) = \operatorname{Pic}^\circ(C/T)$ (here, $\mathfrak{Ab}$ is the category of abelian groups).

Theorem 1.18. *For any algebraic curve C, there exists a variety $\operatorname{Jac} C$ which represents the functor $\mathfrak{J}$.*

Definition 1.19. *Let C be a curve of genus g over k. The* Jacobian variety *of C is a pair $(\operatorname{Jac} C, \mathcal{U})$ where $\operatorname{Jac} C$ is a scheme over k and $\mathcal{U} \in \operatorname{Pic}^\circ(C/\operatorname{Jac} C)$ is a family of invertible sheaves of degree 0 over C having the following universal property. For any other pair $(T, \mathcal{L})$, there exists a map $f : T \to \operatorname{Jac} C$ such that $(1_C \times f)^*\mathcal{U} \cong \mathcal{L}$.*

Trivially, this definition translates to any degree d, and we get the Jacobian variety $(\operatorname{Jac}^d C, \mathcal{U}_d)$. Tensoring by an invertible sheaf $\mathcal{L}$ of degree d over C, we have $\mathcal{U} \otimes p_1^*\mathcal{L} \cong \mathcal{U}_d$ up to $p_1^* \operatorname{Pic}(\operatorname{Jac} X)$, where $p_1 : X \times T \to T$ natural projection. Indeed, $\operatorname{Jac}^d C \cong \operatorname{Jac} C$ for any d.

The closed points of the variety $\operatorname{Jac} C$ (which is also called Jacobian variety) is in correspondence of elements of the group $\operatorname{Pic}^\circ(C)$:

$$\begin{aligned} \operatorname{Jac} C &\longrightarrow \operatorname{Pic}^\circ C \\ t &\longmapsto \mathcal{U}|_{C_t} \end{aligned}$$

Then, this gives a group structure on $\operatorname{Jac} C$.

Other important property is the $\operatorname{Jac} C$ is a smooth proper variety over k of dimension g: in analytic language, $\operatorname{Jac} C$ is a complex compact variety (or manifold).

There is a natural map $\ell_n : \operatorname{Sym}^n X \to \operatorname{Jac} X$: fix a closed point $x_0 \in X$, for any set of n points $\bar{x} = \{x_1, x_2, \dots, x_n\} \in \operatorname{Sym}^n X$, we have the divisor $D_{\bar{x}} = x_1 + x_2 + \cdots + x_n - n \cdot x_0$ of degree 0 and, hence, an invertible sheaf of degree 0, $\mathcal{L}_{\bar{x}} = \mathcal{O}_X(x_1 + x_2 + \cdots + x_n - n \cdot x_0)$. This defines the map $\ell_n : \operatorname{Sym}^n X \to \operatorname{Jac} X$ to be $\ell_n(\bar{x}) = \mathcal{L}_{\bar{x}}$.

There is a variation of this map $\ell_n : \operatorname{Sym}^n X \to \operatorname{Jac}^n X$ to be

$$\bar{x} = \{x_1, \dots, x_n\} \mapsto \mathcal{O}_X(x_1 + \cdots + x_n).$$

It has an interesting property, since for $n > 2g-2$ we have $\mathcal{O}_X(x_1 + \cdots + x_n)$ is a non-special invertible sheaf and there is an effective divisor D_{eff} for any divisor D since $|D| \cong \mathbb{P}\Gamma(X, \mathcal{O}_X(x_1 + \cdots + x_n))$ and $\dim |D| = n + 1 - g \geq g$. Hence, the map

$$\operatorname{Sym}^n X \to \operatorname{Jac} X \tag{1.15}$$

is surjective. Moreover,

Theorem 1.20. *The map (1.15) is a fibration whose fibre at $L \in \operatorname{Jac} X$ is isomorphic to $\mathbb{P}H^0(X, \mathcal{O}_X(D)) \cong \mathbb{P}^{n+1-g}$.*

■ **The class of the Jacobian variety** Let X be a smooth projective curve. We come back to the Jacobian variety $\operatorname{Jac} X$. Recall that $\operatorname{Jac} X$ is an abelian variety. The motive of an abelian variety admits a decomposition. Let us present the following theorem:

Theorem 1.21. [Sch94] *Let A be an abelian variety, then we have*

1) *There is a unique decomposition in $\mathcal{M}_k$*

$$h(A) = \bigoplus_{k=0}^{2g} h^k(A)^{\mathrm{can}}$$

which is stable under the product $[\times n]^*$ where $[\times n]^*|_{h^i(A)}$ is the product by the scalar n^i, for every $n \in \mathbb{Z}$.

2) *The iterated product maps*

$$h(A) \otimes \cdots \otimes h(A) = h(A \times \cdots \times A) \to h(A)$$

induce for every $i \geq 0$ isomorphisms

$$\bigwedge^{2g} h^k(A)^{\mathrm{can}} \to h^k(A)^{\mathrm{can}}.$$

3) *Let $\xi \in A^1(A)$ be the class of an ample symmetric line bundle on A. Then there is a commutative diagram*

$$\begin{tikzcd} h^k(A) \arrow[r, hook] \arrow[d, "\sim"'] & h(A) \arrow[d, "\xi^{g-k}"] \\ h^{2g-k}(A) \otimes \mathbb{L}^{k-g} \arrow[r, hook] & h(A) \otimes \mathbb{L}^{k-g} \end{tikzcd}$$

in which the horizontal arrows are the obvious inclusions.

Here, $h^k(A)^{\mathrm{can}} = (A, p_k^{\mathrm{can}})$ where $\{p_k^{\mathrm{can}}\}$ is a set of canonical projections related to the cohomology decomposition $H^*(A) = \bigoplus_k H^k(A)$.

This implies the following decomposition for the Jacobian variety

$$h(\operatorname{Jac} C) = \bigoplus_{k=0}^{2g} h^k(\operatorname{Jac} C)$$

Furthermore, $h^1(\operatorname{Jac} C) = h^1(C)$ where $h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C)$, that is, the same decomposition $[C] = 1 + \bar{C} + \mathbb{L}$. Hence combining the previous theorem with this equality we have the decomposition

$$h(\operatorname{Jac} C) = \bigoplus_{k=0}^{2g} h^k(\bar{C})$$

Furthermore, by the third point of Theorem 1.21, we have the equality on motives

$$h^{2g-k}(\bar{C}) = \mathbb{L}^{g-k} h^k(\bar{C}) \text{ for } k = 0, \ldots, g.$$

Now, we have the following equalities in $K_0(\mathfrak{Mot})$:

$$\begin{aligned} \mathbb{J} = \sum_{k=0}^{2g} \lambda(\bar{C}) &= \sum_{k=0}^{g-1} \lambda^k(\bar{C}) + \lambda^g(\bar{C}) + \sum_{k=g+1}^{2g} \lambda^k(\bar{C}) \\ &= \sum_{k=0}^{g} \lambda^k(\bar{C}) + \sum_{k=g}^{2g} \lambda^k(\bar{C}) - \lambda^g(\bar{C}) \\ &= \lambda^g(\bar{C}+1) + \sum_{k=0}^{g} \mathbb{L}^{g-k} \lambda^k(\bar{C}) - \lambda^g(\bar{C}) \\ &= \lambda^g(\bar{C}+1) + \lambda^g(\bar{C}+\mathbb{L}) - \lambda^g(\bar{C}). \end{aligned}$$

Then,

Proposition 1.22. *The following equality holds in $K_0(\mathfrak{Mot})$:*

$$h(\operatorname{Jac} C) = \zeta_{\bar{C}}(1)\,.$$

Remark 1.23. *The last equality holds in $K_0(\mathfrak{V}\mathrm{ar})$. This follows from Theorem 1.20.*

4 The Brauer group

■ **Definition of the Brauer group** Let $(X, \mathcal{O}_X)$ be an algebraic smooth scheme. We are interested on the following two spaces:

- The space $\mathfrak{V}ect_r(X) = \{E \to X \,:\, E \text{ is a vector bundle over } X \text{ of rank } r\}$. We denote $\mathfrak{V}ect(X) = \bigsqcup_r \mathfrak{V}ect_r(X)$. Recall the operations on bundles (direct sum, torsion product and duality) induce the corresponding ones in the category $\mathfrak{V}ect_r(X)$.
- The space $\mathrm{Proj}_r(X) = \{P \to X \,:\, P \text{ is a projective bundle over } X \text{ of rank } r\}$, that is, $P \in \mathrm{Proj}_r(X)$ means the fibres of $P \to X$ are isomorphic to $\mathbb{P}^r$. Also, we denote $\mathrm{Proj}(X) = \bigsqcup_r \mathrm{Proj}_r(X)$.

Definition 1.24. *A projective bundle $P \in \mathrm{Proj}_{r-1}(X)$ is said to be* insignificant *if there exists $E \in \mathfrak{V}ect_r(X)$ such that $P = \mathbb{P}(E)$. Otherwise, we say that P is* significant.

The composition law $\otimes$ on $\mathfrak{V}ect_r(X)$ induces a composition law on $\mathrm{Proj}_{r-1}(X)$ and the involution $E \mapsto E^\vee$ extends to $\mathrm{Proj}_{r-1}(X)$. This extension is done as follows: given a projective bundle $P \in \mathrm{Proj}_{r-1}(X)$, there exists a stratification $X = \sqcup_{i=1}^d X_i$ of X such that $P|_{X_i}$ is insignificant; on each pieces, we are allowed to define the involution and the composition map; finally, the union of these new pieces gives another P.

Using this extension, we define an equivalence relation $\sim$ on $\mathrm{Proj}(X)$: for $P, Q \in \mathrm{Proj}(X)$ projective bundle, we say

$$P \sim Q \iff P \otimes \mathbb{P}(E) \sim Q \otimes \mathbb{P}(F) \text{ for some } E, F \in \mathfrak{V}ect(X) \tag{1.16}$$

Definition 1.25. *The* Brauer group *(in the sense of Grothendieck) of the complex space X is the quotient set*

$$\mathrm{Br}(X) = \mathrm{Proj}(X)/\sim\,.$$

The composition law $\otimes$ induces a group law on $\mathrm{Br}(X)$ whose inverse is the involution $P \mapsto P^\vee$.

The main idea of the Brauer group is that the set of significant projective bundles makes a group. This link between both notions is explained below.

■ **Why Brauer group?** The standard topology on algebraic geometry is the Zariski topology. Nevertheless, it is very coarse for some purposes. For example, this topology fails on the cohomology of the constant sheaves: it does not correspond with the cohomology on the usual complex topology. But for our current case the problem is to define projective bundles in the same way as vector bundles, that is, by triviality on open subsets: there exists projective bundles locally trivial on the complex topology but not on the Zariski topology. To avoid this misfunction of the Zariski topology, Grothendieck defined the étale topology whose open subsets are, roughly speaking, unramified coverings of the Zariski open subsets. But this is not our story... the important key here is that there is some projective bundles which are not locally Zariski trivial. As we have seen, we can check when a projective bundle P on an algebraic variety X is locally Zariski trivial (or not) by computing its Brauer class $[P] \in \mathrm{Br}(X)$: if $[P] = 0$, then P is Zariski locally trivial, since it is the projectivization of some vector bundle naturally Zariski locally trivial. Recall the locally triviality on Zariski topologies gives the factorization of $K_0(\mathfrak{V}\mathrm{ar})$ namely $[P] = [\mathbb{P}^n][X]$.

■ Brauer group and cohomology

We have a fundamental exact sequence of sheaves on X:

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathrm{GL}(r, \mathcal{O}_X) \longrightarrow \mathrm{PGL}(r, \mathcal{O}_X) \longrightarrow 1$$

which gives a long exact sequence in cohomology

$$\cdot \longrightarrow H^1(X, \mathrm{GL}(r, \mathcal{O}_X)) \xrightarrow{\mathbb{P}} H^1(X, \mathrm{PGL}(r, \mathcal{O})) \xrightarrow{\delta_r} H^2(X, \mathcal{O}_X^*) \tag{1.17}$$

This long exact sequence translates to the equivalence:

$$P \in \mathrm{Proj}_{r-1}(X) \text{ is insignificant } \iff \delta_r(P) = 1 \in H^2(X, \mathcal{O}^*),$$

so δ_r is the obstruction for P to come from a vector bundle. When a projective bundle $P \in \mathrm{Proj}_{r-1}(X)$ is insignificant its class $[P] \in \mathrm{Br}(X)$ is the neutral element, so we have the following:

Proposition 1.26. *The obstruction maps δ_r induces a morphism of groups*

$$\delta : \mathrm{Br}(X) \to H^2(X, \mathcal{O}_X^*)$$

making the folowing diagram commutative

$$\begin{tikzcd} & \mathrm{Proj}(X) \arrow[dl, "quotient\ \sim"'] \arrow[dr, "\sqcup_r \delta_r"] & \\ \mathrm{Br}(X) \arrow[rr, "\delta"'] & & H^2(X, \mathcal{O}_X^*) \end{tikzcd}$$

From this, we have two consequences:

Corollary 1.27. *If P is a projective bundle with $[P] = 1$ in $\mathrm{Br}(X)$, then P is insignificant.*

Notice that from the definition of the Brauer group it does not follow trivially such a corollary.

Corollary 1.28. *The map $\delta : \mathrm{Br}(X) \to H^2(X, \mathcal{O}_X^*)$ is injective.*

Now, we focus on the image of δ.

Consider the analogous short exact sequence (1.17) for SL groups:

$$1 \longrightarrow \mu_r \longrightarrow \mathrm{SL}(r, \mathcal{O}_X) \longrightarrow \mathrm{PGL}(r, \mathcal{O}_X) \longrightarrow 1$$

where $\mu_r = \{\xi \in \mathbb{C} : \xi^r = 1\}$. This gives the long exact sequence in cohomology

$$H^1(X, \mathrm{SL}(r, \mathcal{O})) \xrightarrow{\mathbb{P}} \mathrm{Proj}_{r-1}(X) \xrightarrow{\varepsilon_r} H^2(X, \mathcal{O}_X^*).$$

The inclusion $\mu_r \hookrightarrow \mathcal{O}_X^*$ gives the commutative diagram

$$\begin{tikzcd} & \mathrm{Proj}_{r-1}(X) \arrow[dl, "\varepsilon_r"'] \arrow[dr, "\delta_r"] & \\ H^2(X, \mu_r) \arrow[rr, "j_*"'] & & H^2(X, \mathcal{O}_X^*) \end{tikzcd}$$

Observe that P could be an insignificant projective bundle with $\varepsilon_r(P) \neq 0$.

On the other hand, from the Kummer exact sequence

$$1 \longrightarrow \mu_r \xrightarrow{j} \mathcal{O}_X^* \xrightarrow{(\cdot)^r} \mathcal{O}_X^* \longrightarrow 1$$

yields the long exact sequence in cohomology

$$\mathrm{Pic}(X) \xrightarrow{(\cdot)^{\otimes r}} \mathrm{Pic}(X) \longrightarrow H^2(X,\mu_r) \xrightarrow{j_*} H^2(X,\mathcal{O}_X^*) \xrightarrow{(\cdot)^r} H^2(X,\mathcal{O}_X^*)$$

using the well-known identification $\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}_X^*)$. This exact sequence gives

$$1 \longrightarrow \frac{\mathrm{Pic}(X)}{(\mathrm{Pic}(X))^{\otimes r}} \longrightarrow H^2(X,\mu_r) \xrightarrow{j_*} H^2(X,\mathcal{O}_X^*)^{r-\mathrm{torsion}} \longrightarrow 1$$

from which we have

Proposition 1.29. *The obstruction element $\delta_r(P)$ of any $P \in \mathrm{Proj}_{r-1}(X)$ is of r-torsion in $H^2(X,\mathcal{O}_X^*)$.*

Indeed, a refined version for compact smooth projective curves is

Proposition 1.30. *The equality*

$$\mathrm{Im}\,\delta = H^2(X,\mathcal{O}_C^*)_{\mathrm{torsion}}$$

holds for C smooth projective curve.

■ The algebraic version

As it is usual in Algebraic Geometry, any definition of algebro-geometric nature has its counterpart in Algebra. Indeed, the algebraic version of the Brauer groups is the most widely spread in mathematics.

Firstly, we define the objects. Let k be a field.

Definition 1.31. *A k-*algebra *A is a ring containing k in its centre and finite dimensional as a k-vector space.*

On k-algebras we have two important operations:

- the tensor product $A \otimes A'$, whose elements are $a \otimes a'$.
- the opposite k-algebra A^{opp}, where the product of the ring A^{opp} is defined to be $a \cdot b = ba$.

We are interested on some k-algebras.

Definition 1.32. *A k-algebra is said to be* simple *if it is nonzero and contains no k-algebras apart from the obvious A and k.*

Definition 1.33. *A k-algebra is said to be* central *if its center is just the field k.*

The most important example of a central simple k-algebra is the matrix algebra $\mathfrak{M}_n(k)$. A very relevant property is:

Fact 1.34. *Central simple k-algebras are closed by the two operations on k-algebras, torsion product and opposite algebra.*

Then, given a field k, we can consider the set of all central simple k-algebras $\mathrm{CSA}(k)$ up to isomorphism and these operators $\otimes$ and $^{\mathrm{opp}}$ run inside this set. Now, we define the following equivalence relation

$$A \sim B \iff A \otimes_k \mathfrak{M}_n(k) \cong B \otimes_k \mathfrak{M}_m(k) \text{ for some } n, m \text{ integers.} \tag{1.18}$$

Observe the obvious parallelism between the equivalence relation in (1.16) on $\mathrm{Proj}(X)$ (indeed, (1.18) is the local version for (1.16)). Then,

Definition/Theorem 1.35. *The* Brauer group *of k is defined to be*

$$\mathrm{Br}(k) = \mathrm{CSA}(X)/\sim .$$

The torsion product $\otimes_k$ induces a well-defined operation on $\mathrm{Br}(k)$, that is,

$$[A]\cdot[B] = [A \otimes_k B].$$

which gives a group structure to $\mathrm{Br}(k)$*: the identity element is* $[\mathfrak{M}_n(k)]$ *(no matter with* n *since* $\mathfrak{M}_n(k) \sim \mathfrak{M}_m(k)$*) and the inverse of* $[A]$ *is* $[A^{\mathrm{opp}}]$.

It is convenient to spare some lines on *Azumaya algebras.* Most books on Brauer groups introduce this topic using these algebras. Indeed, Grothendieck's formulation follows this direction.

Definition 1.36. *An* Azumaya algebra A *over* k *is a finite dimension* k*-algebra such that there exists a finite separable field extension* k'/k *such that* $A \otimes_k k' = \mathfrak{M}_n(k')$.

Such algebras are closely related to central simple algebras:

Theorem 1.37. [Milne, p. 94] *Any simple* k*-algebra is isomorphic to* $\mathfrak{M}_n(D)$ *for some* n *and some division* k*-algebra* D.

Recall a division k-algebra A is a k-algebra where every non-zero element $a \in A$ has an inverse, that is, the ring structure of A satisfies all the axioms to be a field except commutativity.

Now, both algebraic structures are tied by this theorem: A central simple algebra A is $\mathfrak{M}_n(D) \cong \mathfrak{M}_n(k) \otimes D$ for some D division k-algebra. This shows the equivalence with Azumaya algebras. Notice that D is a finite extension since A is a finite dimensional k-vector space.

A variety X over a field k is Brauer-Severy if $X_{\bar{k}} = X \times_k \bar{k}$ is isomorphic to a projective space where $\bar{k}$ is the closure field of k.

Theorem 1.38. [Gabber] *Let* $\pi : X \to S$ *be a Brauer-Severi scheme. Then*

$$\pi^*_{\mathrm{torsion}} : H^2(S, \mathcal{O}^*_S)_{\mathrm{torsion}} \to H^2(X, \mathcal{O}^*_X)_{\mathrm{torsion}}$$

is surjective and $\ker(\pi^*_{\mathrm{torsion}}) = \delta(\mathrm{cl}(X)) \cdot H^0(S, \mathbb{Z})$.

Chapter 2

Triples and pairs

Let C be a smooth projective curve of genus $g \geq 2$. In this chapter we define the category of objects which shall be the main topic of this Thesis. They are called 'triples' since they consist of three objects: two bundles a map. These objects and their morphisms form an abelian category. Furthermore, fixing degrees and ranks of the bundles, it is possible to construct the moduli space of triples $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$. This space depends on a real parameter $\sigma \in \mathbb{R}$ where the complex structure changes when σ changes. Furthermore, there are some special values of this parameter where the underlying set of the moduli space changes: these values are called critical values. On these values, some triples become unstable, others becomes stables — the set of these triples forms a subvariety called flip locus. Obviously, there are two flip loci: the corresponding to the 'right' side and the 'left' side. The important key to understand this is the filtration defined on such triples which leads us to define a stratification.

1 Definition of holomorphic triple

Let C be an smooth projective curve of genus $g \geq 2$. The following paragraphs will describe the main objects.

Definition 2.1. *A holomorphic triple $T = (E_1, E_2, \varphi)$ consists in two holomorphic vector bundle E_1 and E_2 over C and in a holomorphic map $\varphi : E_2 \to E_1$.*

A triple T is called *of type* (n_1, n_2, d_1, d_2) if $\operatorname{rk} E_i = n_i$ and $\deg E_i = d_i$.

Let $T = (E_1, E_2, \varphi)$ and $T' = (E_1', E_2', \varphi')$ be two triples. A *holomorphic map* $\Psi : T \to T'$ is a pair (ψ_1, ψ_2) of two holomorphic maps between vector bundles which makes the following diagram

$$\begin{tikzcd} E_1 \arrow[r, "\psi_1"] \arrow[d, "\varphi"'] & E_1' \arrow[d, "\varphi'"] \\ E_2 \arrow[r, "\psi_2"'] & E_2' \end{tikzcd}$$

commutative. There is an obvious definition of composition of maps: if $\Psi = (\psi_1, \psi_2) : T \to T'$ and $\Psi' = (\psi_1', \psi_2') : T' \to T''$, the composition is defined to be

$$\Psi' \circ \Psi = (\psi_1' \circ \psi_1, \psi_2' \circ \psi_2) : T \to T''.$$

These objects with their morphisms constitue the category of triples on an algebraic curve C denoted by $\mathfrak{T}_C$.

The usual definitions of isomorphism and inverse coincide with the categorical definition of such notions. Analogously, the notions of subtriple and quotient triple follow the usual definition: A triple $T' = (E_1', E_2', \varphi')$ is a subtriple of $T = (E_1, E_2, \varphi)$ if $E_i' \subset E_i$ are subbundles for $i = 1, 2$, $\varphi(E_1') \subset E_2'$ and $\varphi|_{E_1'} = \varphi'$; A subtriple $T' \subset T$ is *proper* if $T' \neq 0$ and $T' \neq T$. A quotient triple $T'' = T/T'$ is given by $E_i'' = E_i/E_i'$ for $i = 1, 2$ and $\varphi'' : E_2'' \to E_1''$ being the map induced by φ.

With these definitions, $\mathfrak{T}_X$ is an abelian category.

We shall now define some natural operations on triples.

■ Operations between triples

We define some basic operations on triples analogous to the corresponding ones on bundles. Given two triples $T = (E_1, E_2, \varphi)$ and $T' = (E_1', E_2', \varphi')$, we have:

- **Direct sum:** We define the direct sum to be

$$T \oplus T' = (E_1 \oplus E_1', E_2 \oplus E_2', \varphi \oplus \varphi')$$

- **Tensor product** We define the tensor product of two triples as follows

$$T \otimes T' = (E_1 \otimes E_1', E_2 \otimes E_2', \varphi \otimes \varphi')$$

- **Dual triple** For any triple $T = (E_1, E_2, \varphi)$ we have the dual triple $T^* = (E_2^*, E_1^*, \varphi^*)$.
- **Pull-back**: Let $f : E_2 \to E_2$ be a holomorphic map, then $f^*T = (E_2, f^*E_1, f^*\phi)$ which makes the diagram

$$\begin{tikzcd} f^*E_1 \arrow[r, "f^*\phi"] \arrow[d] & E_2 \arrow[d, "f"] \\ E_1 \arrow[r, "\phi"'] & E_2 \end{tikzcd}$$

commutative. Here, f^*E denotes the pull-back of the bundle E by f.
- **Push forward**: Let $g : E_1 \to E_1'$ be a holomorphic map, define $g_*T = (g_*E_2, E_1', g_*\phi)$ which makes the following square

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "g"'] & E_2 \arrow[d] \\ E_1' \arrow[r, "g_*\phi"'] & g_*E_2 \end{tikzcd}$$

where g_*E is the push-forward of the bundle E by g.

2 The moduli space of holomorphic triples

■ **Moduli of triples** The works of O. García Prada and Bradlow [BGPG04] show the existence of a moduli space of stable holomorphic triples.

Analogous to the case of moduli space of holomorphic bundles, we need to define the notions of slope and stability before.

Definition 2.2. *For any $\sigma \in \mathbb{R}$ the σ-slope of a triple $T = (E_1, E_2, \varphi)$ of type (n_1, n_2, d_1, d_2) is defined by*

$$\mu_\sigma(T) = \frac{d_1 + d_2}{n_1 + n_2} + \sigma \frac{n_2}{n_1 + n_2}$$

It is costumary to shorten the notation by defining the μ-slope as $\mu(T) = \mu(E_1 \oplus E_2)$ and the λ-slope as $\lambda(T) = \frac{n_2}{n_1+n_2}$. So $\mu_\sigma(T) = \mu + \sigma \cdot \lambda$.

Observe that the slope depends on a real parameter σ unlike the slope of a holomorphic vector bundle.

Definition 2.3. *A triple $T = (E_1, E_2, \varphi)$ is called σ-stable (resp. σ-semistable) if the inequality*

$$\mu_\sigma(T') < \mu_\sigma(T) \quad (\textit{resp. } \mu_\sigma(T') < \mu_\sigma(T))$$

holds for any proper subtriple $T' \subset T$. A triple is σ-polystable if it is a direct sum of σ-stable triples of the same σ-slope. A triple is σ-unstable if it is not σ-semistable. Finally, we say that a triple T is properly σ-semistable if it is σ-semistable but not σ-stable.

When a triple T is non σ-stable, by definition there exists a subtriple $T' \subset T$ such that $\mu_\sigma(T') \geq \mu_\sigma(T)$. This kind of subtriples are called *σ-destabilizing subtriples.*

We denote by $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ the moduli space of σ-polystable triples T of type (n_1, n_2, d_1, d_2). To shorten notation, we drop the type from this notation when it is clear from the context. The open subset of σ-stable triples is denoted by $\mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$. The construction of this moduli space (and hence its existence) is done in [BGPG04] by using dimensional reduction. A direct construction is given by Schmitt [Sch08] using geometric invariant theory.

The existence of σ-stable triples depends on the real parameter σ. For a triple $T = (E_1, E_2, \varphi)$ of type (n_1, n_2, d_1, d_2) denote by $\mu_i = \mu(E_i)$ the slope of the bundles of T, for $i = 1, 2$. We set the values

$$\begin{aligned} \sigma_m &= \mu_1 - \mu_2, \\ \sigma_M &= \begin{cases} \left(1 + \frac{n_1+n_2}{|n_1-n_2|}\right)(\mu_1 - \mu_2), & \text{if } n_1 \neq n_2, \\ \infty & \text{if } n_1 = n_2. \end{cases} \end{aligned} \tag{2.1}$$

we hay that σ_m (σ_M, resp.) is the minimal (maximal, resp.) critical value.

Proposition 2.4. [BGPG04] *The moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ is a complex projective variety. Let I denote the interval $I = [\sigma_m, \sigma_M]$ if $n_1 \neq n_2$, or $I = [\sigma_m, \infty)$ if $n_1 = n_2$. A necessary condition for $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ to be non-empty is that $\sigma \in I$.*

For the smallest value for the real parameter σ we have a nice description for the moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$. Let $\sigma_m^+ = \sigma_m + \varepsilon$ be where ε is sufficiently small, then

Proposition 2.5. [VMnVG07, Prop. 4.10] *There is a map*

$$\pi : \mathcal{N}_{\sigma_m^+}(n_1, n_2, d_1, d_2) \to M(n_1, d_1) \times M(n_2, d_2)$$

which sends $T = (E_1, E_2, \varphi)$ to (E_1, E_2). If $\gcd(n_1, d_1) = \gcd(n_2, d_2) = 1$ and $\mu_1 - \mu_2 > 2g - 2$, then $\mathcal{N}_{\sigma_m^+}$ is a projective bundle over $M(n_1, d_1) \times M(n_2, d_2)$, whose fibers are projective spaces of dimension $n_2 d_1 - n_1 d_2 - n_1 n_2 (g - 1) - 1$.

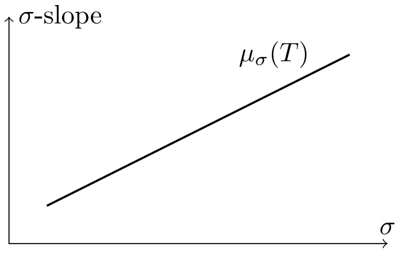
Later, we recover this proposition as a result of the stratification.

Critical Values

The moduli space $\mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$ depends on the real parameter σ. It is useful to study this dependence in order to understand and get a nice description of the moduli space. Look at the formula for the critical value Definition 2.2

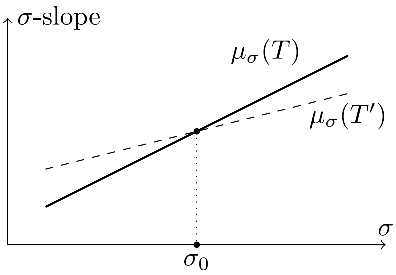
$$\mu_\sigma(T) = \mu(T) + \sigma\lambda(T)$$

We understand this formula as a affine line on the plane as shown in the picture.

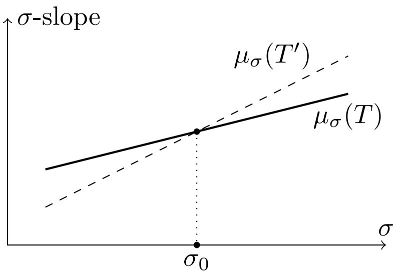


In this case, $\lambda(T)$ is the slope of the line.

Now, consider a subtriple T'. Associated to this subtriple we have another line in the plane. It could be above or below from the line drawn by $\mu_\sigma(T)$ or —this is more interesting— it could cross this line as it is shown in the following picture:



Case 1



Case 2

In both cases, these pictures represent the following situation: at σ_0 we have the equality

$$\mu_{\sigma_0}(T) = \mu_{\sigma_0}(T'), \tag{2.2}$$

and $\lambda(T) \neq \lambda(T')$ (which is equivalent to $n_1'n_2 \neq n_1n_2'$). In case 1, the triple T is σ-unstable for $\sigma > \sigma_0$; in the other case, is σ-unstable for $\sigma < \sigma_0$.

We have the case $\lambda(T) = \lambda(T')$. Clearly, from (2.2) this implies that $\mu_\sigma(T) = \mu_\sigma(T')$ for all σ. If a triple T has a subtriple T' satisfying this equality, and there is no more σ-desestabilizing triple with different slope λ, it is easy to see that T is strictly σ-semistable for σ close to σ_0. Summing up all, we have:

Proposition 2.6. [VMnVG07] *Let $\sigma_0 \in I$ and let $T = (E_1, E_2, \varphi) \in \mathcal{N}_{\sigma_0}(n_1, n_2, d_1, d_2)$ be a strictly σ_0-semistable triple. Then one of the following conditins holds:*

1) *For all σ_0-destabilizing subtriple $T' = (E_1', E_2', \varphi)$, we have $\lambda' = \frac{n_2'}{n_1'+n_2'} = \lambda = \frac{n_1}{n_1+n_2}$. Then T is strictly σ-semistable for $\sigma \in (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$, for some small $\varepsilon > 0$.*
2) *There exists a σ_0-destabilizing subtriple $T' = (E_1', E_2', \varphi)$ with $\lambda' = \frac{n_2'}{n_1'+n_2'} \neq \lambda = \frac{n_1}{n_1+n_2}$. Then either:*
   a) *$\lambda' > \lambda$. Then for any $\sigma > \sigma_0$, T is σ-unstable.*
   b) *$\lambda' < \lambda$. Then for any $\sigma < \sigma_0$, T is σ-unstable.*

Conversely, if T is a triple such that it is σ-semistable for $\sigma < \sigma_0$ and σ-unstable for $\sigma > \sigma_0$ (respectively, σ-semistable for $\sigma > \sigma_0$ and σ-unstable for $\sigma < \sigma_0$) then T is strictly σ_0-semistable and case a (respectively, case b) holds.

The above arguments show the following

Corollary 2.7. *Where $n_1 \neq n_2$. There are finitely many critical values on the interval $I = [\sigma_m, \sigma_M]$. Moreover, they are among the numbers*

$$\sigma_c = \frac{(n_1+n_2)(d_1'+d_2') - (n_1'+n_2')(d_1+d_2)}{n_1'n_2 - n_1n_2'},$$

where $0 \leq n_i' \leq n_i$ and $n_1'n_2 \neq n_1n_2'$.

Proof. It follows directly from the equality

$$\mu_{\sigma_c}(T') = \frac{d_1'+d_2'}{n_1'+n_2'} + \sigma_c\frac{n_2'}{n_1'+n_2'} = \frac{d_1+d_2}{n_1+n_2} + \sigma_c\frac{n_2}{n_1+n_2} = \mu_{\sigma_c}(T)$$

where $T' \subset T$ is a subtriple of type (n_1', n_2', d_1', d_2') of a triple T of type (n_1, n_2, d_1, d_2). $\square$

Definition 2.8. *The values of (2.7) are called virtual critical values. The rest of the values $\sigma \in I$ are called generic.*

Remark 2.9. *The adjetive "virtual" is due to the possibility that not all the virtual critical values are critical values.*

Lemma 2.10. [VMnVG07] *Given a triple T, the values $\sigma \in I$ where T is σ-semistable is a closed interval of $\mathbb{R}$. The set of values where T is stable is either empty or it is the interior of the set*

$$\{\sigma \in \mathbb{R} : T \text{ is } \sigma\text{-semistable}\}.$$

Finally, the following proposition collects the main results

Proposition 2.11. [BGPG04, Prop. 3.6] *Fix the type (n_1, n_2, d_1, d_2). Then*

1) *The critical values are a finite number of values $\sigma_c \in I$.*
2) *The stability and semistability criteria for two values of σ lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.*
3) *If σ is not a critical value and $\gcd(n_1, n_2, d_1 + d_2) = 1$, then σ-semistability is equivalent to σ-stability, that is, $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$.*

We have seen that when we pass through a critical value σ_c the moduli space $\mathcal{N}^s_\sigma(n_1, n_2, d_1, d_2)$ changes, not only its complex structure but also its underlying set. We write

$$\sigma_c^+ = \sigma_c + \varepsilon, \quad \sigma_c^- = \sigma_c - \varepsilon,$$

where $\varepsilon > 0$ is small enough so that σ_c is the only critical value in the interval (σ_c^-, σ_c^+).

Definition 2.12. *We define the flip loci as*

$$\mathcal{S}_{\sigma_c^+} = \{T \in \mathcal{N}_{\sigma_c^+}(n_1, n_2, d_1, d_2) \,:\, T \text{ is } \sigma_c^-\text{-unstable}\} \subset \mathcal{N}_{\sigma_c^+},$$
$$\mathcal{S}_{\sigma_c^-} = \{T \in \mathcal{N}_{\sigma_c^-}(n_1, n_2, d_1, d_2) \,:\, T \text{ is } \sigma_c^+\text{-unstable}\} \subset \mathcal{N}_{\sigma_c^-},$$

and $(\mathcal{S}^\pm_{\sigma_c})^s = \mathcal{S}^\pm_{\sigma_c} \cap (\mathcal{N}^\pm_{\sigma_c})^s$ *for the stable part of the flip loci.*

The flip loci consists of the triples which drop when we pass to the other side of the critical value σ_c. Furthermore, we have from the definition the following relation

Lemma 2.13. [VMnVG07] *Let σ_c be a critical value. Then we have*

1) $\mathcal{N}_{\sigma_c^+} \setminus \mathcal{S}^+_{\sigma_c} = \mathcal{N}_{\sigma_c^-} \setminus \mathcal{S}^-_{\sigma_c}$,
2) $\mathcal{N}^s_{\sigma_c^+} \setminus (\mathcal{S}^+_{\sigma_c})^s = \mathcal{N}^s_{\sigma_c^-} \setminus (\mathcal{S}^-_{\sigma_c})^s = \mathcal{N}^s_{\sigma_c}$,

Remark 2.14. *Clearly, in case that $\sigma = \sigma_m$, the flip locus $\mathcal{S}^-_{\sigma_m}$ is the empty set. So $\mathcal{N}_{\sigma_m^+} = \mathcal{S}^+_{\sigma_m}$ and $\mathcal{N}_{\sigma_m^-} = \varnothing$.*

In the following section we shall focus on these spaces. The points of these spaces, that is, the σ_c-destabilizing triples have nice properties.

Theorem 2.15. [BGPG04] *Let C be a compact complex curve of genus g; fix $\sigma \in \mathbb{R}$ a real parameter, fix the ranks r_1 and r_2 and the degrees d_1 and d_2. There exists a moduli space of σ-stable triples $\mathcal{N}^s_\sigma(n_1, n_2, d_1, d_2)$ of type (n_1, n_2, d_1, d_2). This space is a complex smooth quasiprojective variety. The dimension of the smooth locus is*

$$1 + n_2 d_1 - n_1 d_2 + (n_1^2 + n_2^2 - n_1 n_2)(g - 1).$$

The moduli space $\mathcal{N}^s_\sigma(n_1, n_2, d_1, d_2)$ is non vacuous if and only if the real parameter σ belongs to the interval (σ_m, σ_M) defined in (2.1). $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ is projective (maybe singular).

Remark 2.16. *The case $\mathcal{N}(1, 1, d_2, d_2)$ does not depend on the parameter σ, that is, there is no critical value and*

$$\mathcal{N}(1, 1, d_1, d_2) \cong \operatorname{Sym}^{d_1 - d_2} X \times \operatorname{Jac}^{d_2} X$$

by the isomorphism $(L_1, L_2, \varphi) = (L_1^* \otimes L_2, s)$ where s is a section of $L_1^* \otimes L_2$ induced by the map φ.

3 Extension of triples

We come back now to the category of triples. In this section we study the theory of extension of triples. We have two reasons:

1) The homological algebra of triples, in particular the derivative functor $R^k \operatorname{Hom}$, has an important rôle in the study of the moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$. In particular, the smoothness and the dimension.
2) We shall define the flip loci of the moduli space. Roughly speaking, the flip locus is the subspace which appears or disappears from the moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ when the real parameter σ "cross" a critical value when we run along a real interval $I \subseteq \mathbb{R}$.

The category of triples $\mathfrak{T}_X$ is an abelian category, we have short exact sequence of triples. A short exact sequence of triples

$$0 \longrightarrow T'' \xrightarrow{p} T \xrightarrow{q} T' \longrightarrow 0 \tag{2.3}$$

is a commutative diagram where rows are exact

$$\begin{tikzcd} 0 \arrow[r] & E_1' \arrow[r, "p_1"] & E_1 \arrow[r, "q_1"] & E_1'' \arrow[r] & 0 \\ 0 \arrow[r] & E_2' \arrow[r, "p_2"'] \arrow[u, "\varphi'"'] & E_2 \arrow[r, "q_2"'] \arrow[u, "\varphi"'] & E_2'' \arrow[r] \arrow[u, "\varphi''"'] & 0 \end{tikzcd}$$

In this situation, we say T is an *extension of* T' *by* T''.

Now, we define the group (although in this case is a vector space) $\mathrm{Ext}^1(T'', T')$. There are two ways to define this set: as a set of triples $[p, T, q]$ under a certain relation, or a cohomological definition as the first derivative functor $R^1 \operatorname{Hom}$.

We shall make an outline of the definition of the group $\mathrm{Ext}^1(T'', T')$ as a set. All the details are in the very interesting book [HS97]. Firstly, we define the following set

$$\widetilde{\mathrm{Ext}}^1(T'', T') = \{(p, T, q) \ : \ T \in \mathfrak{T}_X \text{ and the short exact sequence (2.3) holds}\},$$

On this set we define the following equivalence relation. We say two elements (p_1, T_1, q_1) and (p_2, T_2, q_2) of $\widetilde{\mathrm{Ext}}^1(T'', T')$ are equivalent if there exists an isomorphism of triples $\psi : T_1 \to T_2$ such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & T' \arrow[r, "p_1"] \arrow[d, equal] & T_1 \arrow[r, "q_1"] \arrow[d, "\psi"'] & T'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T' \arrow[r, "p_2"] & T_2 \arrow[r, "q_2"] & T'' \arrow[r] & 0 \end{tikzcd}$$

If this occurs then we denote $(p_1, T_1, q_1) \sim (p_2, T_2, q_2)$ or $T_1 \sim T_2$ where we drop the maps when it is clear from the context. It is easy to prove that this relation satisfies the properties of an equivalence relation. We denote $\mathrm{Ext}^1(T'', T')$ to be the quotient of $\widetilde{\mathrm{Ext}}^1(T'', T')$ by the relation $\sim$, and we denote $[p, T, q]$ the classes in this set: these classes are called *extensions*.

The set of extensions $\mathrm{Ext}^1(T'', T')$ can be given a structure of vector space. The sum is the well-known Bauer sum from the literature in homological algebra. The product by scalars is defined as follows:

$$\lambda \cdot [p, T, q] = [\lambda \cdot p, T, q] \sim [p, T, \lambda^{-1} q].$$

This makes $\mathrm{Ext}^1(T'', T')$ a vector space.

4 Cohomological definition of Ext group

From the vector space $\operatorname{Hom}(\_, \_)$ we define two functors. The contravariant functor

$$\operatorname{Hom}(\_, T'') : \mathfrak{T} \to \mathfrak{Vect}_{\mathbb{C}},$$

and the covariant functor

$$\operatorname{Hom}(T', \_) : \mathfrak{T} \to \mathfrak{Vect}_{\mathbb{C}}.$$

The k-derivate of these functors gives the well-known Ext^k-groups. Concretely, we define $\mathrm{Ext}^k(T', T'') = R^k \operatorname{Hom}(T', T'')$, that is, the functor $R^k \operatorname{Hom}(\_, T'')$ evaluated on T' (or $R^k \operatorname{Hom}(T', \_)$ evaluated on T'').

The most important result in this section is the possibility of writing these groups as the hypercohomology of a complex of sheaves. For two triples $T' = (E_1', E_2', \varphi')$ and $T'' = (E_1'', E_2'', \varphi'')$ we define a complex of sheaves

$$C^\bullet(T'', T') : \ ((E_1'')^* \otimes E_1') \oplus ((E_2'')^* \oplus E_2') \xrightarrow{c} (E_2'')^* \otimes E_1',$$

where the map c is defined by

$$c(s_1, s_2) = \varphi' \circ s_2 - s_1 \circ \varphi''.$$

We introduce the following notation:

$$\begin{aligned}
\mathbb{H}^k(T'', T') &= \mathbb{H}^k(C^\bullet(T'', T')),\\
h^i(T'', T') &= \dim \mathbb{H}^i(T'', T'),\\
\chi(T'', T') &= h^0(T'', T') - h^1(T'', T') + h^2(T'', T').
\end{aligned}$$

The following proposition links the usual Ext-groups with the hypercohomology of this complex

Proposition 2.17. [BGPG04, Prop. 3.1] *There are natural isomorphisms*

$$\begin{aligned}
\mathrm{Hom}(T'', T') &\cong \mathbb{H}^0(T'', T'),\\
\mathrm{Ext}^1(T'', T') &\cong \mathbb{H}^1(T'', T'),
\end{aligned}$$

and a long exact sequence associated to the complex $C^\cdot(T'', T')$:

$$\begin{aligned}
0 &\longrightarrow \mathbb{H}^0(T'', T') \longrightarrow H^0\left(((E_1'')^* \otimes E_1') \oplus ((E_2'')^* \oplus E_2')\right) \longrightarrow H^0\left((E_2'')^* \otimes E_1'\right)\\
&\longrightarrow \mathbb{H}^1(T'', T') \longrightarrow H^1\left(((E_1'')^* \otimes E_1') \oplus ((E_2'')^* \oplus E_2')\right) \longrightarrow H^1\left((E_2'')^* \otimes E_1'\right)\\
&\longrightarrow \mathbb{H}^2(T'', T') \longrightarrow 0
\end{aligned}$$

Applying the functor χ to this long exact sequence we have

Proposition 2.18. [BGPG04, Prop. 3.2] *For any holomorphic triples T' and T'' we have*

$$\begin{aligned}
\chi(T'', T') &= \chi((E_1'')^* \otimes E_1') + \chi((E_2'')^* \otimes E_2') - \chi((E_2'')^* \otimes E_1')\\
&= (1-g)\left(n_1''n_1' + n_2''n_2' - n_2''n_1'\right) + n_1''d_2' - n_1'd_1'' + n_2''d_2' - n_2'd_2'' -\\
&\quad - n_2''d_1' + n_1'd_2'',
\end{aligned}$$

where $\chi(E) = \dim H^0(E) - \dim H^1(E)$ is the Euler characteristic of E.

Some dimensions of $\mathbb{H}^i$ are easily computed. This lemma computes the $\mathbb{H}^0(T'', T')$ in some cases.

Lemma 2.19. [BGPG04, Prop. 3.5] *Suppose that T' and T'' are σ-semistable, for some value σ. Then*

1) If $\mu_\sigma(T') < \mu_\sigma(T'')$ then $\mathbb{H}^0(T'', T') = 0$.
2) If $\mu_\sigma(T') = \mu_\sigma(T'')$ and T', T'' are σ-stable, then

$$\mathbb{H}^0(T'', T') = \begin{cases} \mathbb{C} & \text{if } T' \cong T'',\\ 0 & \text{if } T' \ncong T''. \end{cases}$$

And the following gives a sufficient condition for the nullity of $\mathbb{H}^2(T'', T')$.

Lemma 2.20. [Mn08, Prop. 3.10] *If $T'' = (E_1'', E_2'', \varphi'')$ is an injective triple, that is $\varphi'' : E_2'' \to E_1''$ is injective, then $\mathbb{H}^2(T'', T') = 0$.*

The study of this complex gives infinitesimal information of the moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$.

Theorem 2.21. [BR94, Thm 2.3] *The space of infinitesimal deformations of a triple T is canonically isomorphic to $\mathbb{H}^1(T, T)$.*

The identification $\mathbb{H}^1(T, T)$ and $T_T\mathcal{N}_\sigma^s$ gives the following theorem

Theorem 2.22. [BGPG04, Thm 3.8] *Let $T = (E_1, E_2, \varphi)$ be a σ-stable triple of type (n_1, n_2, d_1, d_2), then*

1) *If $\mathbb{H}^2(T, T) = 0$, then the moduli space of σ-stable triples is smooth in a neighbourhood of the point defined by T.*
2) *At a smooth point $T \in \mathcal{N}^s_\sigma(n_1, n_2, d_1, d_2)$ the dimension of the moduli space of σ-stable triples is*

$$\begin{aligned}\dim \mathcal{N}^s_\sigma(n_1, n_2, d_2, d_2) =&(1-g)\left(n_1''n_1' + n_2''n_2' - n_2''n_1'\right) + \\ &+ n_1''d_2' - n_1'd_1'' + n_2''d_2' - n_2'd_2'' - n_2''d_1' + n_1'd_2'' + 1.\end{aligned}$$

3) *Let $T = (E_1, E_2, \varphi)$ be a σ-stable triple. If T is an injective triple, then the moduli space is smooth at T.*

5 Filtrations

When a triple belongs to a flip locus $\mathcal{S}_{\sigma_c^\pm}$, there is a filtration which allows us to stratify nicely $\mathcal{S}^\pm_{\sigma_c}$.
This filtration is similar to the Harder-Narashiman filtration.

Lemma 2.23. [Mn10, Lemma 4.5] *Let σ_c be a critical value. Let $T \in \mathcal{S}^\pm_{\sigma_c}$ be a triple. Then there exists an unique filtration*

$$0 = T_0 \subsetneq T_1 \subsetneq T_2 \subsetneq \ldots \subsetneq T_r = T \tag{2.4}$$

such that T_i/T_{i-1} is the maximal σ_c-polystable subtriple of T/T_{i-1} of the same σ_c-slope.

Sketch of proof. The uniqueness follows from the fact the sum of σ_c-polystable triples is σ_c-polystable: if exists S and S' two maximal σ_c-polystable subtriples of T/T_i, then $S + S' \subset T/T_i$ is a σ_c-polystable subtriple contradicting the maximality of S and S'. ☐

Definition 2.24. *We call standard filtration to the filtration (2.4). We denote by $\bar{T}_i = T_{i+1}/T_i$ and $\tilde{T}_i = T/T_i$ so that $\bar{T}_i \subset \tilde{T}_i$ is the maximal σ_c-polystable subtriple.*

The standard filtration for triples of type $(n, 1, d_1, d_2)$ satisfies nice properties. For the case $T \in \mathcal{S}_{\sigma_c^+}$, the σ_c^+-stability of T means the quotient $\bar{T}_{r-1} = T/T_{r-1}$ is of rank $(n', 1)$, hence T_{r-1} is of rank $(n - n', 0)$, and its subtriples T_i are of rank $(n_i, 0)$:

Lemma 2.25. [Mn10, Lemma 4.8] *Let σ_c be a critical value for $\mathcal{N}^s_\sigma(n_1, 1, d_1, d_2)$. Let $T \in \mathcal{S}^\pm_{\sigma_c}$ be a triple. Consider its standard filtration (2.4) and let $\bar{T}_i = T_i/T_{i-1}$ for $i = 1, \ldots, r$. Then:*

- *If $T \in \mathcal{S}^+_{\sigma_c}$, then $\bar{T}_r$ is of type $(n', 1, d_1', d_2)$ and $\bar{T}_i$ are triples isomorphic to $(F_i, 0, 0)$ for $i < r$ where F_i are polystable bundles, all of them of the same slope. Furthermore, $n' > 0$ if $\sigma_c > \sigma_m$ and $n' = 0$ if $\sigma_c = \sigma_m$.*
- *If $T \in \mathcal{S}^-_{\sigma_c}$, then $\bar{T}_1$ is a triple of type $(n', 1, d_1', d_2)$ and $\bar{T}_i$ are isomorphic to $(F_i, 0, 0)$ for $i > 1$ where F_i are polystable bundles, all of them of the same slope. Furthermore, $n' > 0$ for all critical values σ_c.*

6 Stratification

Consider the moduli space $\mathcal{N}^s_\sigma(n_1, 1, d_1, d_2)$. Fix a critical value $\sigma_c \in I$. We show how the standard filtration induces a stratification on the flip locus $\mathcal{S}^\pm_{\sigma_c}$. First, we focus on $\mathcal{S}^+_{\sigma_c}$.
The standard filtration of a triple $T \in \mathcal{S}^+_{\sigma_c}$ carries two main informations:

- The length of the filtration which we denote by $\ell(T)$.
- By Lemma 2.25, the subquotient triples $\bar{T}_k$ for $k = 1, \cdots, \ell(T)$ are of rank $(n_i, 0)$ and also are σ_c-polystable triples. This means that they are triples $(E_k, 0, 0)$ where E_k decomposes into a sum of stable bundles

$$E_k = \bigoplus_{i=1}^{s_k} F_i^{a_k^i}.$$

Then, the data of each stratum consists on

- the length of the standard filtration $\ell(T)$ (which it shall controlled by the constant r),
- the stable bundles F_i (without repetitions) appearing on all E_k (it shall control by the integer vector $\mathbf{n} = (n_1, \ldots, n_s)$, $n_i = \operatorname{rk} F_i$),
- and the number of occurrences a_k^i such stable bundles F_i in each E_k (it shall control by the matrix $\mathbf{A} = (a_k^i)_{i,k}$).

Now, let $\mathbf{n} = (n_1, \ldots, n_s)$ be a vector of s integers such that $n_i > 0$. Also, consider a matrix $s \times r$

$$\mathbf{A} = \begin{pmatrix} a_r^1 & \cdots & a_r^b \\ \vdots & \ddots & \vdots \\ a_1^1 & \cdots & a_1^b \end{pmatrix},$$

where $a_i^k \geq 0$ and there is no zero row or zero column. Fixing these, define the following subspace

$$X(\mathbf{A}, \mathbf{n}) = \left\{ T \in \mathcal{S}_{\sigma_c}^+ : \begin{array}{l} E_k \text{ has a decomposition } \bigoplus_{i=1}^s F_i^{a_k^i} \text{ simultaneously} \\ \text{for all } k = 1, \ldots, r-1, \text{ where } F_i \not\cong F_j \text{ for } i \neq j \text{ and} \\ F_i \in M^s(n_i, d_i) \text{ for } i, j = 1, \ldots, b, \text{ and } \ell(T) = r. \end{array} \right\},$$

Clearly, $\mathcal{S}_{\sigma_c}^+ = \sqcup X(\mathbf{A}, \mathbf{n})$ where the disjoint union runs along all the possibilities for $\mathbf{A}$ and $\mathbf{n}$. These possibilities satisfy the equation

$$\sum_{k=1}^{r} \sum_{i=1}^{b} a_k^i n_i + n' = n$$

where $(n', 1)$ is the rank of $\bar{T}_r$.

It is possible to give a description of the stratum. This description is based on the short exact sequence satisfied by T:

$$0 \to T_k \to T \to \tilde{T}_k \to 0, \text{ for } k = 1, \ldots, r$$

where $\tilde{T}_k = T/T_k$ denotes the quotient triple. These triples have a common subtriple T_{k-1}. Modulo T_{k-1} we obtain another useful short exact sequence

$$0 \longrightarrow \bar{T}_k \xrightarrow{p_k} \tilde{T}_{k-1} \xrightarrow{q_k} \tilde{T}_k \longrightarrow 0. \tag{2.5}$$

where $\bar{T}_k = T_k/T_{k-1}$. Hence $\tilde{T}_{k-1} \in \operatorname{Ext}^1(\tilde{T}_k, \bar{T}_k)$. The main idea is that $T \in X(\mathbf{A}, \mathbf{n})$ can be obtained by inductive extensions by k, that is, the stratum $X(\mathbf{A}, \mathbf{n})$ is a subvariety of a big variety obtained recursively by considering, on every k, the vector bundle over the space which parametrizes the triple $\tilde{T}_k$ and $\bar{T}_k$ with fibres isomorphic to $\operatorname{Ext}^1(\tilde{T}_k, \bar{T}_k)$.

First, not every extension $\tilde{T}_k \in \operatorname{Ext}^1(\tilde{T}_{k-1}, \bar{T}_k)$ satisfies the maximality condition of Lemma 2.23: we have to assure that $\bar{T}_k$ is the maximal σ_c-polystable subtriple of $\tilde{T}_{k-1}$. This fact led us to consider two groups:

1) Let

$$0 \longrightarrow \bar{T}'_k \xrightarrow{\tau} \bar{T}_k \xrightarrow{\rho} \bar{T}''_k \longrightarrow 0$$

and suppose that the map $\rho_* \tilde{T}_{k-1} = 0$. This implies that there exists $\tilde{T}'_{k-1} \in \operatorname{Ext}^1(\tilde{T}_k, \bar{T}'_k)$ which is a subtriple of $\tilde{T}_{k-1}$ such that $\tau_* \tilde{T}'_{k-1} = \tilde{T}_{k-1}$. Consider the commutative diagram

$$\begin{tikzcd} \operatorname{Ext}^1(\tilde{T}_k, \bar{T}'_k) \arrow[r, "\tau_*"] \arrow[d, "p_{k+1}^*"'] & \operatorname{Ext}^1(\tilde{T}_k, \bar{T}_k) \arrow[r, "\rho_*"] \arrow[d, "p_{k+1}^*"'] & \operatorname{Ext}^1(\tilde{T}_k, \bar{T}''_k) \arrow[d, "p_{k+1}^*"'] \\ \operatorname{Ext}^1(\bar{T}_{k+1}, \bar{T}'_k) \arrow[r, "\tau_*"'] & \operatorname{Ext}^1(\bar{T}_{k+1}, \bar{T}_k) \arrow[r, "\rho_*"'] & \operatorname{Ext}^1(\bar{T}_{k+1}, \bar{T}''_k) \end{tikzcd} \tag{2.6}$$

The map $p_{k+1}^*\tilde{T}'_{k-1}$ (first column) sends to an extension of two σ_c-polystable triples. This is still a subtriple. Since map ρ_* sends $\tilde{T}_{k-1}$ to the zero extension, and the commutativity of (2.6), the subtriple $p_{k+1}^*\tilde{T}'_{k-1}$ admits a σ_c-polystable subtriple. This contradicts the maximality of $\bar{T}_k$. Therefore, the extension corresponding to the $\tilde{T}_{k-1}$ lives in

$$\prod_{i=1}^{s} \mathcal{V}(a_k^j, \operatorname{Ext}^1(\tilde{T}_k, S_j))$$

where $\mathcal{V}(r, V)$ denotes the Stiefel variety.

2) Any subbundle $\tau : \tilde{T}'_k \hookrightarrow \tilde{T}_k$ gives the pull-back diagram

$$\begin{tikzcd} \bar{T}_k \arrow[r] \arrow[d, equal] & \tilde{T}'_{k-1} \arrow[r] \arrow[d, hook] & \tilde{T}'_k \arrow[d, hook] \\ \bar{T}_k \arrow[r, "p_k"] & \tilde{T}_{k-1} \arrow[r, "q_k"] & \tilde{T}_k \end{tikzcd} \tag{2.7}$$

which is nothing but the map $\tau^* : \operatorname{Ext}^1(\tilde{T}_k, \bar{T}_k) \to \operatorname{Ext}^1(\tilde{T}'_k, \bar{T}_k)$. By assumptions, the subbundles of $\tilde{T}_k$ are subbundles of the σ_c-polystable bundle $\bar{T}_{k+1}$. If there exists a map τ such that $\tau^*\tilde{T}_{k-1} = 0$, then $\tilde{T}_{k-1}$ admits the σ_c-polystable subtriple $\tilde{T}'_k \oplus \bar{T}_k$. This contradicts the σ_c-maximality of $\bar{T}_k$. Therefore, $p_{k+1}^*\tilde{T}_{k-1}$ must live in

$$\prod_{j=1}^{s} \mathcal{V}\left(a_{k+1}^j, \operatorname{Ext}^1(S_j, \bar{T}_k)\right).$$

Both conditions assure that the subtriple $\bar{T}_k \subset \tilde{T}_{k-1}$ is the maximal σ_c-polystable subtriple, and T lives in $X^+(\mathbf{A}, \mathbf{n})$. Details can be found in [Mn10].

By this argument we have that $\tilde{T}_{k-1}$ lives in a subset of $\operatorname{Ext}^1(\tilde{T}_k, \bar{T}_k)$. Nevertheless, recall that an extension is a triple $\xi = [p, \tilde{T}_{k-1}, q]$. This means that there are several extensions which produce the same triple $\tilde{T}_{k-1}$. To avoid this repetition, we identify extensions by the action of the group: for any extension $[p, T, q] \in \operatorname{Ext}^1(T', T'')$ we have two collections of triples $[p \circ \varphi, T, q]$ for any $\varphi \in \operatorname{Aut}(T'')$ and $[p, T, \psi \circ q]$ for any automorphism $\psi \in \operatorname{Aut}(T')$, both non-equivalent to the former extension giving the same triple T. We conclude that the group $\operatorname{Aut}(\tilde{T}_k) \otimes_{\mathbb{C}^\times} \operatorname{Aut}(\bar{T}_k)$ acts on $\operatorname{Ext}^1(\tilde{T}_k, \bar{T}_k)$. By Lemma 2.19 we know $\mathbb{H}^0(\tilde{T}_k, \tilde{T}_k) \cong \mathbb{C}$, thus $\operatorname{Aut}(\tilde{T}_k) \cong \mathbb{C}^\times$, and $\operatorname{Aut}(\bar{T}_k) \cong \prod_{i=1}^s \mathrm{GL}(a_k^i, \mathbb{C})$. Then, we have to quotient the subspace of all extensions ξ which gives a triple $\tilde{T}_{k-1}$ satisfying the condition of Lemma 2.23.

These summarize as follows. Define

$$M(\mathbf{n}) = \left[\left(\prod_{i=1}^{b} M^s(n_i, d_i)\right) \setminus \Delta\right] \times \mathcal{N}(n', 1, d'_1, d_2)$$

where $n' < n$ and

$$\frac{d'_1 + d_2 + \sigma_c}{n' + 1} = \frac{d_1 + d_2 + \sigma_c}{n + 1} \tag{2.8}$$

and the degree of the stable bundles satisfies the equality

$$\frac{d_i}{n_i} = \frac{d_1 + d_2 + \sigma_c}{n + 1} \tag{2.9}$$

Then,

Proposition 2.26. [Mn10, Prop. 5.1] *Let $(E_1, E_2, \ldots, E_b, T') \in M(\mathbf{n})$. Define triples $\tilde{T}_k$ by downward recursion as follows $\tilde{T}_{r+1} = \bar{T}_{r+1}$ and for $1 \leq k \leq r$ define $\tilde{T}_k$ as an extension*

$$0 \to S(\mathbf{a}_{k+1}) \to \tilde{T}_k \to \tilde{T}_{k+1} \to 0 \tag{2.10}$$

(equivalent to (2.5)). Let $\xi_k \in \operatorname{Ext}^1(\tilde{T}_{k+1}, S(\mathbf{a}_{k+1}))$ *be the extension class corresponding to (2.10). Denote* $T = \tilde{T}_1$*. Then* $T \in X(\mathbf{A}, \mathbf{n})$ *if and only if the following conditions are satisfied:*

1) *The extension class* $\xi_k \in \operatorname{Ext}^1(\tilde{T}_{k+1}, S(\mathbf{a}_{k+1})) = \prod_{i=1}^{b} \operatorname{Ext}^1(\tilde{T}_{k+1}, S_i)^{a_k^i}$ *lives in* $\prod_{i=1}^{b} \mathcal{V}(a_k^i, \operatorname{Ext}^1(\tilde{T}_{k+1}, S_i))$*, where* $\mathcal{V}(a, V)$ *denotes the Stiefel variety of the vector space* V*.*
2) *Consider the map* $S(\mathbf{a}_{k+1}) \to \tilde{T}_{k+1}$ *and the element* ξ_k' *which is the image of* ξ_k *under* $\operatorname{Ext}^1(\tilde{T}_{k+1}, S(\mathbf{a}_{k+1})) \to \operatorname{Ext}^1(S(\mathbf{a}_{k+2}), S(\mathbf{a}_{k+1})) = \prod_{i=1}^{b} \operatorname{Ext}^1(S_i, S(\mathbf{a}_{k+1}))^{a_{k+1}^i}$ *lives in* $\prod_{i=1}^{b} \mathcal{V}(a_{k+1}^i, \operatorname{Ext}^1(S_i, S(\mathbf{a}_{k+1})))$*.*

Two extensions ξ_k *give rise to isomorphic* $\tilde{T}_k$ *if and only if the triples* $\tilde{T}_{k+1}$ *are isomorphic and the extension classes are the same up to action of the group* $\mathrm{GL}(\mathbf{a}_{k+1}) = \mathrm{GL}(a_k^1) \times \cdots \times \mathrm{GL}(a_k^b)$*.*

We define in a inductively way the bundle X_k over the $(k-1)$-th step X_{k-1} with fibres isomorphic to

$$F_k \cong \left\{ \left(\prod_{i=1}^{s} \mathcal{V}(a_k^i, \operatorname{Ext}^1(\tilde{T}_k, S_i)) \right) \cap \left((q_k^*)^{-1} \left[\prod_{i=1}^{s} \mathcal{V}(a_{k+1}^i, \operatorname{Ext}^1(S_i, \bar{T}_k)) \right] \right) \right\} / \prod_{i=1}^{s} \mathrm{GL}(a_k^i, \mathbb{C}).$$

Here we define $X_0 = M(\mathbf{n})$. We denote by $\tilde{X}(\mathbf{A}, \mathbf{n}) = X_r$. Now, define the subgroup

$$G = \left\{ \sigma \in \mathfrak{S}_s \;:\; a_k^{\sigma(j)} = a_k^j \text{ and } n_{\sigma(j)} = n_j \text{ for all } k = 1, \ldots, r \text{ and for all } j = 1, ..., b \right\}.$$

This group acts on $\tilde{X}(\mathbf{A}, \mathbf{n})$ by swapping the triples S_i and then identifies isomorphic triples $T \in \tilde{X}(\mathbf{A}, \mathbf{n})$. Then, the stratum is

$$X(\mathbf{A}, \mathbf{n}) = \tilde{X}(\mathbf{A}, \mathbf{n})/G.$$

An analogous statement holds for the clase $\mathcal{S}_{\sigma_c}^-$:

For strata in $\mathcal{S}_{\sigma_c}^-$ we consider the short exact sequence

$$0 \to T_{k-1} \to T_k \to \bar{T}_k \to 0$$

and we have

$$0 \longrightarrow \tilde{T}_{k-1} \xrightarrow{p_{k-1}} \tilde{T}_{k-1} \xrightarrow{q_{k-1}} \bar{T}_k \longrightarrow 0 \tag{2.11}$$

(we denote with the same letters the maps in such short exact sequence).

Proposition 2.27. [Mn10, Prop. 5.3] *Let* $(E_1, \ldots, E_b, T') \in M(\mathbf{n})$*. Define triples* $\tilde{T}_k$ *by recursion as follows:* $\tilde{T}_1 = T_1$*, and for* $2 \le k \le r+1$ *define* $\tilde{T}_k$ *as extension*

$$0 \to \tilde{T}_{k-1} \to \tilde{T}_k \to S(\mathbf{a}_k) \to 0.$$

Let $\xi_k \in \operatorname{Ext}^1(S(\mathbf{a}_k), \tilde{T}_{k-1})$ *be the corresponding extension class. Denote* $T = \tilde{T}_{r+1}$*. Then* $T \in X^-(\mathbf{A}, \mathbf{n})$ *if and only if the following conditions are satisfies:*

1) *The extension class* $\xi_k \in \operatorname{Ext}^1(S(\mathbf{a}_k), \tilde{T}_{k-1}) = \prod_{i=1}^{b} \operatorname{Ext}^1(S_i, \tilde{T}_{k-1})^{a_k^i}$ *lives in* $\mathcal{V}(a_k^i, \operatorname{Ext}^1(S_i, \tilde{T}_{k-1}))$*.*
2) *Consider the map* $\tilde{T}_{k-1} \to S(\mathbf{a}_{k-1})$ *and the element* ξ_k' *which is the image of* ξ_k *under* $\operatorname{Ext}^1(S(\mathbf{a}_k), \tilde{T}_{k-1}) \to \operatorname{Ext}^1(S(\mathbf{a}_k), S(\mathbf{a}_{k-1}))$*. Then the element* $\xi_k' \in \operatorname{Ext}^1(S(\mathbf{a}_k), S(\mathbf{a}_{k-1})) = \prod_{i=1}^{b} \operatorname{Ext}^1(S(\mathbf{a}_k), S_i)^{a_{k-1}^i}$ *lives in* $\prod_{i=1}^{b} \mathcal{V}(a_{k-1}^i, \operatorname{Ext}^1(S(\mathbf{a}_k), S_i))$*.*

Two extensions ξ_k *give rise to isomorphic* $\tilde{T}_k$ *if and only if the triples* $\tilde{T}_{k-1}$ *are isomorphic and the extension classes are the same up to action of the group* $\mathrm{GL}(\mathbf{a}_k) = \mathrm{GL}(a_k^1) \times \cdots \times \mathrm{GL}(a_k^b)$ *and the finite group* G*.*

Combining Proposition 2.26 and Lemma 2.25 we have the following proposition:

Proposition 2.28. [VMnVG07, Prop. 4.10] *There is a map*

$$\pi : \mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(n_1, n_2, d_1, d_2) \to M(n_1, d_1) \times M(n_2, d_2)$$

which sends $T = (E_1, E_2, \phi)$ *to* (E_1, E_2)*. If* $\gcd(n_1, d_1) = 1$*,* $gcd(n_2, d_2) = 1$ *and* $\mu_1 - \mu_2 > 2g - 2$*, then* $\mathcal{N}_{\sigma_m^+}$ *is a projective bundle over* $M(n_1, d_1) \times M(n_2, d_2)$*, whose fibers are projective spaces of dimension* $n_2 d_1 - n_1 d_2 - n_1 n_2 (g-1) - 1$*.*

Chapter 3

Rank 2

In this chapter we are going to compute the case of rank 2. The main reference is [VMnVG07] where the authors compute the Hodge polynomials for flip-loci and hence those of $\mathcal{N}_\sigma^s(2,1,d_1,d_2)$. We use similar arguments to compute the class of $\mathcal{N}_\sigma^s(2,1,d_1,d_2)$. in $K_0(\mathfrak{Var})$ and $K_0(\mathfrak{Mot})$.

1 The class

■ Computation of the critical value

Lemma 3.1. [Mn08, Lemma 5.3] *Let $\sigma_c = 3d_M - d_1 - d_2$ be a critical value. Then*

$$\mu_1 \leq d_M \leq d_1 - d_2,$$

and $\sigma_c = \sigma_m$ if and only if $d_M = \mu_1$.

Since $n' > 0$ by Lemma 2.25, then there is only one possibility: the standard filtration has length 1 (so $r = 1$) and $n_1 = 1$ and $a_1^1 = 1$.

■ Construction of the stratum We begin with the case $\mathcal{S}_{\sigma_c}^+$. Let $T' \in \mathcal{N}_{\sigma_c}^s(1,1,d_1',d_2)$ the initial triple and let L denote a line bundle on C. By (2.8) we have $d_1' = d_1 - d_M$. Let S denote the triple $(L,0,0)$. The triple $T \in \mathcal{S}_{\sigma_c}^+$ as an extension, lives in $\operatorname{Ext}^1(T', 0 \to S)$. We remove the trivial extension since it does not satisfies properties to given an standard filtration. The group $\mathbb{C}^\times$ identifies extensions which give the same triple and the quotient is the projective space $\mathbb{P}\operatorname{Ext}^1(T',S)$. Let $\tilde{X}^+$ denote the bundle over $\mathcal{N}_{\sigma_c}^s(1,1,d_1',d_2) \times \operatorname{Jac} C$ whose fibre is $\mathbb{P}\operatorname{Ext}^1(T',S)$. Since G is the trivial group, then $X^+(\mathbf{A},\mathbf{n}) = \tilde{X}^+$.

The case $\mathcal{S}_{\sigma_c}^-$ is similar: the triple $T \in X(\mathbf{A},\mathbf{n})$ lives in $\operatorname{Ext}^1(S,T') \backslash \{0\}$ as an extension; the action of $\mathbb{C}^\times$ identifies extensions giving same triple. Let $\tilde{X}^-$ be the bundle over $\mathcal{N}_{\sigma_c}^s(1,1,d_1',d_2) \times \operatorname{Jac} C$ with fibres $\mathbb{P}\operatorname{Ext}^1(S,T')$. Since G is trivial, then $X^-(\mathbf{A},\mathbf{n}) = \tilde{X}^-$.

Since $\tilde{X}^+$ and $\tilde{X}^-$ are locally Zariski trivial, then its class in $K(\mathfrak{Var})$ are

$$\begin{aligned}
[X^+(\mathbf{A},\mathbf{n})] &= [\mathbb{P}\operatorname{Ext}^1(T',S)][\mathcal{N}_{\sigma_c}^s(1,1,d_1-d_M,d_2)][\operatorname{Jac} C],\\
[X^-(\mathbf{A},\mathbf{n})] &= [\mathbb{P}\operatorname{Ext}^1(S,T')][\mathcal{N}_{\sigma_c}^s(1,1,d_1-d_M,d_2)][\operatorname{Jac} C].
\end{aligned} \tag{3.1}$$

■ Computation the class Let $\mathbb{L} = [\mathbb{P}^1] - 1$ be the Lefschetz symbol in $K_0(\mathfrak{Var})$. Clearly $[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \ldots + \mathbb{L} + 1$ and

$$[\mathbb{P}^n] = \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1}$$

in $\hat{K}_0(\mathfrak{Var})$. On the other hand, the dimensions of $\operatorname{Ext}^1(T',S)$ are computed in [VMnVG07]. For a critical value $\sigma_c = 3d_M - d_1 - d_2$,

$$\begin{aligned}
\dim \operatorname{Ext}^1(T',S) &= d_1 - d_M - d_2,\\
\dim \operatorname{Ext}^1(S,T') &= 2d_M - d_1 + g - 1.
\end{aligned}$$

From (3.1) we have

$$\begin{aligned}
[X^+(\mathbf{A},\mathbf{n})] &= [\mathbb{P}^{d_1-d_M-d_2-1}][\mathcal{N}_{\sigma_c}^s(1,1,d_1-d_M,d_2)][\operatorname{Jac} C] =\\
&= \frac{\mathbb{L}^{d_1-d_M-d_2} - 1}{\mathbb{L} - 1}[\operatorname{Jac} C]^2 \lambda^{d_1-d_M-d_2}([C]);\\
[X^-(\mathbf{A},\mathbf{n})] &= [\mathbb{P}^{2d_M-d_1+g-2}][\mathcal{N}_{\sigma_c}^s(1,1,d_1-d_M,d_2)][\operatorname{Jac} C] =\\
&= \frac{\mathbb{L}^{2d_M-d_1+g-1} - 1}{\mathbb{L} - 1}[\operatorname{Jac} C]^2 \lambda^{d_1-d_M-d_2}([C]).
\end{aligned}$$

Recall from Remark 2.16 that

$$[\mathcal{N}_\sigma(1,1,d_1,d_2)] = \lambda^{d_1-d_2}([C])[\mathrm{Jac}\, C].$$

Fix $\sigma = 3d - d_1 - d_2 \in [\sigma_m, \sigma_M]$ a non critical value. Then $[\mathcal{N}_\sigma^s(2,1,d_1,d_2)]$ is the sum of all flip loci for all critical values $\sigma_c = d_M - d_1 - d_2$ such that

$$d_M \geq d_0 = \left\lceil \frac{1}{3}(\sigma + d_1 + d_2) \right\rceil.$$

Hence

$$\begin{aligned}
[\mathcal{N}_\sigma^s(2,1,d_1,d_2)] &= \sum_{d_M=d_0}^{d_1-d_2} \big([X^-(\mathbf{A},\mathbf{n})] - [X^+(\mathbf{A},\mathbf{n})]\big) \\
&= \sum_{d_M=d_0}^{d_1-d_2} \left(\frac{\mathbb{L}^{2d_M-d_1+g-1}-1}{\mathbb{L}-1} [\mathrm{Jac}\, C]^2 \lambda^{d_1-d_M-d_2}([C]) \right) - \\
&\quad - \sum_{d_M=d_0}^{d_1-d_2} \left(\frac{\mathbb{L}^{d_1-d_M-d_2}-1}{\mathbb{L}-1} [\mathrm{Jac}\, C]^2 \lambda^{d_1-d_M-d_2}([C]) \right).
\end{aligned}$$

First, we focus on the sum

$$\sum_{d_M=d_0}^{d_1-d_2} \mathbb{L}^{2d_M-d_1+g-1} \lambda^{d_1-d-d_2}([C]).$$

Take $d = d_1 - d_M - d_2$, that is, $d_M = d_1 - d - d_2$. Thus the sum becomes

$$\begin{aligned}
\sum_{d=0}^{d_1-d_0-d_2} \mathbb{L}^{2(d_1-d-d_2)-d_1+g-1} \lambda^d([C]) &= \mathbb{L}^{2d_0-d_1+g-1} \left(\sum_{d=0}^{d_1-d-d_2} \mathbb{L}^{2((d_1-d_0-d_2)-d)} \lambda^d([C]) \right) \\
&= \mathbb{L}^{2d_0-d_1+g-1} \lambda^{d_1-d_0-d_2}([C] + \mathbb{L}^2),
\end{aligned}$$

where in the last equality we use the formula

$$\lambda^n(C + \mathbb{L}^k) = \sum_{i=0}^{n} \mathbb{L}^{k(n-i)} \lambda^i(C).$$

We do the same change of variable on the sum

$$\sum_{d_M=d_0}^{d_1-d_2} \mathbb{L}^{d_1-d-d_2} \lambda^{d_1-d-d_2}([C]).$$

and the sum becomes

$$\sum_{d=0}^{d_1-d_0-d_2} \mathbb{L}^d \lambda^d([C]) = \lambda^{d_1-d_0-d_2}([C]\mathbb{L} + 1),$$

where we use the equality

$$\lambda^n(C\mathbb{L}^k + 1) = \sum_{i=0}^{n} \mathbb{L}^{ik} \lambda^i(C).$$

Summing up both equalities we have

Theorem 3.2. *The class of* $\mathcal{N}_\sigma(2,1,d_1,d_2)$ *is*

$$[\mathcal{N}_\sigma^s(2,1,d_1,d_2)] = \frac{[\mathrm{Jac}\, C]^2}{\mathbb{L}-1} \left(\mathbb{L}^{2d_0-d_1+g-1} \lambda^{d_1-d_0-d_2}([C] + \mathbb{L}^2) - \lambda^{d_1-d_0-d_2}([C]\mathbb{L} + 1) \right) \tag{3.2}$$

where $d_0 = \left\lceil \frac{1}{3}(\sigma - d_1 - d_2) \right\rceil$*, where* $\lceil a \rceil$ *is the lowest integer number greater than* a*.*

2 Computation of Hodge-Deligne polynomial

The functor $e : \mathfrak{Var}_{\mathbb{C}} \to \mathbb{C}[x, y]$ which gives the Hodge-Deligne polynomial from a variety descends to $K_0(\mathfrak{Var}_{\mathbb{C}})$, we call $\bar{e} : K_0(\mathfrak{Var}_{\mathbb{C}}) \to \mathbb{C}[u, v]$.

Knowing the formula

$$e(\mathrm{Sym}^n C) = \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^n}$$

and letting $\lambda^n([C]) = 0 \in \hat{K}_0(\mathfrak{Var})$ for $n < 0$, we have

$$\lambda^n([C] + \mathbb{L}^k) = \sum_{m=0}^{\infty} \mathbb{L}^{mk} \lambda^{n-m}([C])$$

and we have that

$$\begin{aligned} \lambda^n([C] + \mathbb{L}^k) &\overset{\bar{e}}{\longmapsto} \operatorname*{coeff}_{x^0} \left\{ \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)(1-(uv)^k x)x^n} \right\}, \\ \lambda^n([C]\mathbb{L}^k + 1) &\overset{\bar{e}}{\longmapsto} \operatorname*{coeff}_{x^0} \left\{ \frac{(1+ux)^g(1+vx)^g(uv)^{kn}}{(1-x)(1-uvx)(1-\frac{x}{(uv)^k})x^n} \right\}. \end{aligned} \tag{3.3}$$

We apply the following lemma

Lemma 3.3. *The equality*

$$\prod_{i=1}^{n} \frac{1}{1-a_i x} = \sum_{i=1}^{n} \frac{A_i}{1-a_i x}$$

holds if

$$A_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_i}{a_i - a_j}.$$

Proof. Denote

$$P(x) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_i(1-a_j x)}{a_i - a_j}.$$

Clearly,

$$\sum_{i=1}^{n} \frac{A_i}{1-a_i x} = \frac{P(x)}{\prod_{i=1}^{n}(1-a_i x)},$$

We require that $P(x)$ is the polynomial 1. We substitute on the values $\frac{1}{a_i}$ for $i = 1, \ldots, n$. Since $\deg P = n-1$, these values determine completely P. Now,

$$P\left(\frac{1}{a_k}\right) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{a_i(1-\frac{a_j}{a_k})}{a_i - a_j}.$$

Notice $1 - \frac{a_j}{a_k} = 0$ if $j = k$, so the ith-summand

$$\prod_{j \neq i} \frac{a_i(1-\frac{a_j}{a_k})}{a_i - a_j} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Then, $P\left(\frac{1}{a_k}\right) = 1$ for all $k = 1, \ldots, n$. $\square$

Now, we write the formulæ (3.3) in separate factors

$$\begin{aligned}\bar{e}\left(\lambda^n(C+\mathbb{L}^2)\right) =& \operatorname*{coeff}_{x^0}\Big\{(1+ux)^g(1+vx)^g x^{-n}\times \\ &\times\Big(\frac{1}{(1-uv)(1-(uv)^2)(1-x)}-\frac{(uv)^2}{(1-uv)^2(1-uvx)}+ \\ &+\frac{(uv)^3}{(1-uv)(1-(uv)^2)(1-(uv)^2x)}\Big)\Big\}, \\ \bar{e}\left(\lambda^n(C\mathbb{L}+1)\right) =& \operatorname*{coeff}_{x^0}\Big\{(1+ux)^g(1+vx)^g(uv)^n x^{-n}\times \\ &\times\Big(-\frac{uv}{(1-t)^2(1-x)}+\frac{(uv)^3}{(1-uv)(1-(uv)^2)(1-uvx)}+ \\ &+\frac{1}{(1-uv)(1-(uv)^2)(1-\frac{x}{uv})}\Big)\Big\}.\end{aligned}$$

For $n \geq 2g-2$, the function $f_n(x) = (1+ux)^g(1+vx)^g x^{-n}$ is holomorphic on $\mathbb{C}\setminus\{0\}$ and has no pole at infinity. Then

$$\operatorname*{coeff}_{x^0}\left\{\frac{f_n(x)}{1-a_ix}\right\} = \operatorname{Res}\left\{\frac{f_{n+1}(x)}{1-a_ix};\frac{1}{a_i}\right\} = -\frac{1}{a_i}f_{n+1}\left(\frac{1}{a_i}\right).$$

Therefore,

$$\begin{aligned}\bar{e}\left(\lambda^n([C]+\mathbb{L}^2)\right) =& \frac{(1+u)^g(1+v)^g}{(1-uv)(1-(uv)^2)}\left((uv)^{n+1-g}(uv+1)-1\right)- \\ &-\frac{(uv)^{2n-3g+3}(1+uv^2)^g(1+u^2v)^g}{(1-uv)(1-(uv)^2)},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\bar{e}\left(\lambda^n(C\mathbb{L}+1)\right) =& \frac{(1+u)^g(1+v)^g}{(1-uv)(1-(uv)^2)}\left((uv)^{n+1}(uv+1)-(uv)^{2n+3-g}\right)- \\ &-\frac{(1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)}\end{aligned} \tag{3.5}$$

These formulæ give the following expression for the class $[M^s(2,d_1)]$ in $\hat{K}_0(\mathfrak{Var})$ for odd degree. The difference

$$\mathbb{L}^g\lambda^{\lfloor\sigma_m\rfloor}([C]+\mathbb{L}^2)-\lambda^{\lfloor\sigma_m\rfloor}([C]\mathbb{L}+1)$$

gives the Hodge-Deligne polynomial

$$\frac{(1+u)^g(1+v)^g(uv)^g}{(1-uv)(1-(uv)^2)}\left((uv)^{2\lfloor\mu\rfloor+3-2g}-1\right)-\frac{(1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)}\left((uv)^{2\lfloor\mu\rfloor+3-2g}-1\right).$$

Recall $2\lfloor\mu\rfloor = 2\mu-1$, then we have

$$\bar{e}(\mathbb{M}_2^{\mathrm{odd}}) = \frac{(1+u)^{2g}(1+v)^{2g}(uv)^g-(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)} \tag{3.6}$$

which coincides with Muñoz's results (see. [VMnVG07]) and it does not depend on the value of μ.

3 The class of moduli of bundles of rank 2 with odd degree

Let us compute the class of $\mathbb{M}_2^{\mathrm{odd}}$. To do this, we compute the class $[\mathcal{N}_{\sigma_m^+}(2,1,d_1,d_2)]$ in two different ways. On the one hand, we have the class of $[\mathcal{N}_\sigma(2,1,d_1,d_2)]$ written in (3.2). In this case the parameter σ is $\sigma_m = \frac{d_1}{2}-d_2$, that we shall denote by μ, (see (2.1)) and $d_M = \lceil\frac{d_1}{2}\rceil$, so $d_1-d_M-d_2 = \lfloor\mu\rfloor$, and hence we have

$$[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)] = \frac{\mathbb{J}^2}{\mathbb{L}-1}\left(\mathbb{L}^g\lambda^{\lfloor\mu\rfloor}([C]+\mathbb{L}^2)-\lambda^{\lfloor\mu\rfloor}([C]\mathbb{L}+1)\right), \tag{3.7}$$

where $\mathbb{J} = [\operatorname{Jac} C]$ On the other hand, Proposition 2.28 shows that

$$[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)] = \frac{\mathbb{L}^{d_1-2d_2-2(g-1)}-1}{\mathbb{L}-1}\mathbb{J}\cdot\mathbb{M}_2^{\mathrm{odd}} = \frac{\mathbb{L}^{2(\mu+g-1)}-1}{\mathbb{L}-1}\mathbb{J}\cdot\mathbb{M}_2^{\mathrm{odd}}. \tag{3.8}$$

Alternatively, there is a unique stratum where $n'=0$, $n_1=2$ and $a_1^1=1$, and we obtain the same class as (3.8)

Combining both classes (3.7) and (3.8) we have that the class of $\mathbb{M}_2^{\mathrm{odd}}$ is

$$\mathbb{M}_2^{\mathrm{odd}} = \frac{\mathbb{J}}{\mathbb{L}^{2(\mu+1-g)}-1}\Big(\mathbb{L}^g\lambda^{\lfloor\mu\rfloor}([C]+\mathbb{L}^2) - \lambda^{\lfloor\mu\rfloor}([C]\mathbb{L}+1)\Big).$$

This class can be computed using residues as before. Fixing $n=\lfloor\mu\rfloor$, we have the formula

$$\mathbb{M}_2^{\mathrm{odd}} = \frac{\mathbb{J}}{\mathbb{L}^{2(n-g)+3}-1}\left(\mathbb{L}^g\lambda^n([C]+\mathbb{L}^2) - \lambda^n([C]\mathbb{L}+1)\right).$$

On one hand,

$$\lambda^n([C]+\mathbb{L}^2) = \operatorname*{coeff}_{x^0}\frac{\zeta_{[C]+\mathbb{L}^2}(x)}{x^n} = \operatorname*{coeff}_{x^0}\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^2x)x^n}$$

denoting $\omega = \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^2x)x^{n+1}}\mathrm{d}x$ a 1-form of $\mathbb{P}^1_{\bar{\mathcal{K}}}$. Recall the definition of $\bar{\mathcal{K}}$ in Section 1.1.

$$\lambda^n([C]+\mathbb{L}^2) = \operatorname{Res}_{x=0}\omega = -\operatorname{Res}_{x=1}\omega - \operatorname{Res}_{x=\mathbb{L}^{-1}}\omega - \operatorname{Res}_{x=\mathbb{L}^{-2}}\omega$$

by the Residue theorem. Then

$$\begin{aligned}\lambda^n([C]+\mathbb{L}^2) = &-\frac{\zeta_{\bar{C}}(1)}{(1-\mathbb{L})(1-\mathbb{L}^2)}\cdot(-1)\cdot\operatorname{Res}_{x=0}\frac{\mathrm{d}(-x)}{1-x} - \frac{\mathbb{L}^{n+1}\zeta_{\bar{C}}\big(\mathbb{L}^{-1}\big)}{(1-\mathbb{L}^{-1})(1-\mathbb{L})}\frac{-1}{\mathbb{L}}\operatorname{Res}_{x=\mathbb{L}^{-1}}\frac{d(-\mathbb{L}x)}{1-\mathbb{L}x}\\ &-\frac{\mathbb{L}^{2(n+1)}\zeta_{\bar{C}}\big(\mathbb{L}^{-2}\big)}{(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-1})}\frac{-1}{\mathbb{L}^2}\operatorname{Res}_{x=\mathbb{L}^{-2}}\frac{\mathrm{d}(-\mathbb{L}^2x)}{1-\mathbb{L}^2x}\end{aligned}$$

and by the equality $\zeta_{\bar{C}}(\mathbb{L}^{-a}) = \mathbb{L}^{(1-2a)g}\zeta_{\bar{C}}\big(\mathbb{L}^{a-1}\big)$,

$$= \frac{\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)} - \frac{\mathbb{L}^{n+1-g}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)^2} + \frac{\mathbb{L}^{2n+3-3g}\zeta_{\bar{C}}(\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)}.$$

A similar argument works for $\lambda^n([C]\mathbb{L}+1)$: let $\omega = \frac{\zeta_{\bar{C}}(\mathbb{L}x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^2x)x^{n+1}}\mathrm{d}x$ be a 1-form of $\mathbb{P}^1_{\bar{\mathcal{K}}}$. Then,

$$\lambda^n([C]\mathbb{L}+1) = \operatorname{Res}_{x=0}\omega = -\operatorname{Res}_{x=1}\omega - \operatorname{Res}_{x=\mathbb{L}^{-1}}\omega - \operatorname{Res}_{x=\mathbb{L}^{-2}}\omega$$

by the Residue Theorem, and

$$\begin{aligned}\lambda^n([C]\mathbb{L}+1) = &-\frac{\zeta_{\bar{C}}(\mathbb{L})}{(1-\mathbb{L})(1-\mathbb{L}^2)}\cdot(-1)\cdot\operatorname{Res}_{x=1}\frac{\mathrm{d}(-x)}{1-x} - \frac{\mathbb{L}^{n+1}\zeta_{\bar{C}}(1)}{(1-\mathbb{L}^{-1})(1-\mathbb{L})}\frac{-1}{\mathbb{L}}\operatorname{Res}_{x=\mathbb{L}^{-1}}\frac{\mathrm{d}(-\mathbb{L}x)}{1-\mathbb{L}x}\\ &-\frac{\mathbb{L}^{2(n+1)}\zeta_{\bar{C}}\big(\mathbb{L}^{-1}\big)}{(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-1})}\frac{-1}{\mathbb{L}^2}\operatorname{Res}_{x=\mathbb{L}^{-2}}\frac{\mathrm{d}(-\mathbb{L}^2x)}{1-\mathbb{L}^2x}\\ = &\frac{\zeta_{\bar{C}}(\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)} - \frac{\mathbb{L}^{n+1}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)^2} + \frac{\mathbb{L}^{2n+3-g}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)}.\end{aligned}$$

We compute the difference $\mathbb{L}^g\lambda^n([C]+\mathbb{L}^2) - \lambda^n([C]\mathbb{L}+1)$: The summands divided by $\frac{1}{(1-\mathbb{L})^2}$ are cancelled, so we have

$$\mathbb{L}^g\lambda^n([C]+\mathbb{L}^2) - \lambda^n([C]\mathbb{L}+1) = \frac{\mathbb{L}^{2(n-g)+3}-1}{(1-\mathbb{L})(1-\mathbb{L}^2)}\zeta_{\bar{C}}(\mathbb{L}) - \mathbb{L}^g\frac{\mathbb{L}^{2(n-g)+3}-1}{(1-\mathbb{L})(1-\mathbb{L}^2)}\zeta_{\bar{C}}(1)$$

and hence

$$\mathbb{M}_2^{\mathrm{odd}} = \frac{\mathbb{J}}{(1-\mathbb{L})(1-\mathbb{L}^2)}\left(\zeta_{\bar{C}}(\mathbb{L}) - \mathbb{L}^g\zeta_{\bar{C}}(1)\right). \tag{3.9}$$

4 The class of moduli of bundles of rank 2 with even degree

Let us define a modification of the Grothendieck group: let

$$\bar{K}_0(\mathfrak{Var}_{\mathbb{C}}) = \frac{\hat{K}_0(\mathfrak{Var}_{\mathbb{C}})}{\langle [P] - [\mathbb{P}^r][B] \,:\, \text{if } \mathbb{P}^r \to P \to B \text{ is a projective fibration} \rangle}.$$

We have to add this relation: some analytic projective bundles should be non locally Zariski trivial (see Section 1.4).

Let $\mathbb{M}_2^{\text{even}}$ be the class in $K_0(\mathfrak{Var}_{\mathbb{C}})$ or $K_0(\mathfrak{Mot})$ of the moduli space of bundles of rank 2 with even degree. To do this, we write in another way the class $[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)]$ in terms of the class of the curve $[C]$ and the $\mathbb{L}$ class, where $\sigma_m^+ = \sigma_m + \epsilon$ with no critical values in the interval (σ_m, σ_m^+).

By Remark 2.14, $[\mathcal{N}_{\sigma_m^+}(2,1,d_1,d_2)] = [\mathcal{S}^+_{\sigma_m}]$. Furthermore, Lemma 2.25 claims that the initial triple must be of type $(0,1,0,d_2)$. Consider d_1 an even number, we have five strata corresponding to all posibilities for $\mathbf{A}$ and $\mathbf{n}$ (observe that $d_1/2$ is an integer number, so the strata X_1, X_2, X_3 and X_5 below are well-defined in this case).

Firstly, we list the class of the flip loci:

- For the case
$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$
the stratum has the class
$$[X_1^+] = \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu}-\mathbb{L}^{\mu+1-g})}{(\mathbb{L}-1)^2}(\mathbb{J}^3-\mathbb{J}^2).$$
where $\mathbb{J} = h(\operatorname{Jac} C)$.
- For the case
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$
we have the following class of the stratum
$$[X_2^+] = \frac{\mathbb{L}^{2(\mu+1-g)}-1}{(\mathbb{L}^2-1)}\lambda^2(\mathbb{J})\cdot\mathbb{J} + \mathbb{L}\frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^3 - \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+2-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2. \quad (3.10)$$
Using the δ^2 operator, this can be rewritten as follows by the equality $\lambda^2([C]) = \frac{1}{2}\left(\delta^2([C]) + [C]^2\right)$:
$$[X_2^+] = \frac{1}{2}\frac{\mathbb{L}^{2(\mu+1-g)}-1}{(\mathbb{L}^2-1)}\delta^2(\mathbb{J}) + \frac{1}{2}\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^2\mathbb{J}^3 - \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+2-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2. \quad (3.11)$$
- The third case is $\mathbf{A} = \begin{pmatrix} 2 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 1 \end{pmatrix}$ and its class is
$$[X_3^+] = \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu-g}-1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\mathbb{J}^2. \quad (3.12)$$
- The following is the most interesting case: the constants are given by the matrix $\mathbf{A} = \begin{pmatrix} 1 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 2 \end{pmatrix}$ and its class is
$$[X_4^+] = \frac{\mathbb{L}^{2(\mu+1-g)}-1}{\mathbb{L}-1}\mathbb{M}_2^{\text{even}}\mathbb{J} \quad (3.13)$$
where $\mathbb{M}_2^{\text{even}} = h(M^s(2,0))$.
- The fifth case is given by the matrix
$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 1 \end{pmatrix}$$
and its class is
$$[X_5^+] = \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu}-\mathbb{L}^{\mu-g})}{(\mathbb{L}-1)^2}\mathbb{J}^2 \quad (3.14)$$

The sum of these classes gives the class $[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)]$.

On the other hand, we have already computed the class $[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)]$. For parameter σ, the class is

$$[\mathcal{N}^s_\sigma(2,1,d_1,d_2)] = \frac{\mathbb{J}^2}{\mathbb{L}-1}\left(\mathbb{L}^{2d_0-d_1+g-1}\lambda^{d_1-d_0-d_2}([C]+\mathbb{L}^2) - \lambda^{d_1-d_0-d_2}([C]\mathbb{L}+1)\right),$$

where d_0 is the lowest integer such that in the interval $(\frac{\sigma+d_1+d_2}{3},\infty)\subset\mathbb{R}$. For the case $\sigma=\sigma_m^+$ we have

$$[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)] = \frac{\mathbb{J}^2}{\mathbb{L}-1}\left(\mathbb{L}^{g+1}\lambda^{\mu-1}([C]+\mathbb{L}^2) - \lambda^{\mu-1}([C]\mathbb{L}+1)\right),$$

since $\sigma_m^+ = \frac{d_1}{2}-d_2+\epsilon = \mu+\epsilon$ (for $\epsilon>0$ sufficiently small), hence $d_0=\frac{d_1}{2}+1$ (notice d_1 is even).

Now, we know the expansion of $[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)]$ in terms of $\bar{C}$ (this is a simple computation using the same residues computed above — changing n by μ):

$$\begin{aligned}[\mathcal{N}^s_{\sigma_m^+}(2,1,d_1,d_2)] &= \left\{\frac{\mathbb{L}^{2\mu-2g}-1}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\zeta_{\bar{C}}(\mathbb{L}) + \left[\frac{\mathbb{L}^\mu-\mathbb{L}^{\mu-1}}{(\mathbb{L}-1)^2} - \frac{\mathbb{L}^{2\mu-g+1}-\mathbb{L}^{g-1}}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\right]\zeta_{\bar{C}}(1)\right\}\frac{\mathbb{J}^2}{\mathbb{L}-1}\\ &= \frac{\mathbb{L}^{2\mu-2g}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(\mathbb{L})\zeta_{\bar{C}}(1)^2 - \frac{\mathbb{L}^{g+1}(\mathbb{L}^{\mu-g-1}+1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(1)^3,\end{aligned} \tag{3.15}$$

where we use the equality $\mathbb{J}=\zeta_{\bar{C}}(1)$.

Now, we deal with the other side of the equality The right hand side is formed by

$$[X_1^+]+[X_2^+]+\cdots+[X_5^+].$$

- the terms multiplied bby $\mathbb{J}^2$ add up to... zero!

$$\begin{aligned}&-\frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^\mu-\mathbb{L}^{\mu+1-g})}{(\mathbb{L}-1)^2} - \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+2-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)} +\\ &\frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)} + \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^\mu-\mathbb{L}^{\mu-g})}{(\mathbb{L}-1)^2} =\\ &= \frac{\mathbb{L}^{\mu+1-g}-1}{(\mathbb{L}-1)^2}\left\{\mathbb{L}^{\mu+1-g}-\mathbb{L}^{\mu-g}+\frac{\mathbb{L}^{\mu-g}-\mathbb{L}^{\mu+2-g}}{\mathbb{L}+1}\right\} =\\ &= \frac{\mathbb{L}^{\mu+1-g}-1}{(\mathbb{L}-1)^2}\left\{\mathbb{L}^{\mu-g}(\mathbb{L}-1)+\mathbb{L}^{\mu-g}\frac{1-\mathbb{L}^2}{\mathbb{L}+1}\right\} = 0.\end{aligned}$$

- the terms multiplied by $\mathbb{J}^2$ are:

$$\begin{aligned}&\mathbb{L}\frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)} + \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^\mu-\mathbb{L}^{\mu+1-g})}{(\mathbb{L}-1)^2}\\ &= \frac{\mathbb{L}^{\mu+1-g}-1}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\left(\mathbb{L}(\mathbb{L}^{\mu-g}-1)+(\mathbb{L}+1)(\mathbb{L}^\mu-\mathbb{L}^{\mu+1-g})\right)\\ &= \frac{\mathbb{L}^{\mu+1-g}-1}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\left(-\mathbb{L}^{\mu+2-g}+\mathbb{L}^{\mu+1}+\mathbb{L}^\mu-\mathbb{L}\right),\end{aligned}$$

although, if we use δ^2 operator instead of λ operator (see the second stratum), this coefficient is

$$\frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^\mu-\mathbb{L}^{\mu+1-g})}{(\mathbb{L}-1)^2} + \frac{1}{2}\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^2 = \frac{(\mathbb{L}^{\mu+1-g}-1)(2\mathbb{L}^\mu-\mathbb{L}^{\mu+1-g}-1)}{2(\mathbb{L}-1)^2}.$$

Hence, we have the following equality

$$\begin{aligned}
&\frac{\mathbb{L}^{2\mu-2g+2}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(\mathbb{L})\zeta_{\bar{C}}(1)^2-\frac{(\mathbb{L}^{\mu-g+1}-1)(\mathbb{L}^{\mu+2}+\mathbb{L}^{g-1})}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(1)^3= \\
&=-\frac{(\mathbb{L}^{\mu+1-g}-1)(2\mathbb{L}^{\mu}-\mathbb{L}^{\mu+1-g}-1)}{2(\mathbb{L}-1)^2}\zeta_{\bar{C}}(1)^3+ \\
&+\frac{\mathbb{L}^{2(\mu-g+1)}-1}{\mathbb{L}-1}\mathbb{M}_2^{\text{even}}\zeta_{\bar{C}}(1)+\frac{\mathbb{L}^{2(\mu-g+1)}-1}{\mathbb{L}^2-1}\delta^2(\zeta_{\bar{C}}(1))\zeta_{\bar{C}}(1)
\end{aligned}$$

from where we get the class of $\mathbb{M}_2^{\text{even}}$:

$$\begin{aligned}
\frac{\mathbb{L}^{2(\mu-g+1)}-1}{\mathbb{L}-1}\mathbb{M}_2^{\text{even}}\zeta_{\bar{C}}(1)&=\frac{\mathbb{L}^{2\mu-2g+2}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(\mathbb{L})\zeta_{\bar{C}}(1)^2-\frac{\mathbb{L}^{2(\mu-g+1)}-1}{\mathbb{L}^2-1}\delta^2(\zeta_{\bar{C}}(1))\zeta_{\bar{C}}(1)- \\
&\quad-\frac{\mathbb{L}^{\mu+1-g}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\big(2\mathbb{L}^{g+1}(\mathbb{L}^{\mu-g-1}-1)- \\
&\quad-(\mathbb{L}^2-1)(2\mathbb{L}^{\mu}-\mathbb{L}^{\mu+1-g}-1)\big)\zeta_{\bar{C}}(1)^3= \\
&=\frac{\mathbb{L}^{2\mu-2g+2}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(\mathbb{L})\zeta_{\bar{C}}(1)^2-\frac{\mathbb{L}^{2(\mu-g+1)}-1}{\mathbb{L}^2-1}\delta^2(\zeta_{\bar{C}}(1))\zeta_{\bar{C}}(1)- \\
&\quad-\frac{(\mathbb{L}^{2(\mu-g+1)}-1)(2\mathbb{L}^{g+1}-\mathbb{L}^2+1)}{2(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(1)^3
\end{aligned}$$

This yields the following equality

$$\mathbb{M}_2^{\text{even}}=\frac{\zeta_{\bar{C}}(\mathbb{L})-\zeta_{\bar{C}}(1)(\mathbb{L}^{g-1}-\mathbb{L}^2+\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\zeta_{\bar{C}}(1)-\frac{\lambda^2(\zeta_{\bar{C}}(1))}{\mathbb{L}+1} \tag{3.16}$$

If we wish to write this in terms of the δ^2 operator, we apply the equality $\lambda^2(Y)=\frac{1}{2}\left(Y^2+\delta^2(Y)\right)$ for any $Y\in K_0(\mathfrak{Var}_{\mathbb{C}})$ and we have

$$\mathbb{M}_2^{\text{even}}=\frac{2\zeta_{\bar{C}}(\mathbb{L})-\zeta_{\bar{C}}(1)(2\mathbb{L}^{g+1}-\mathbb{L}^2+1)}{2(\mathbb{L}-1)(\mathbb{L}^2-1)}\zeta_{\bar{C}}(1)-\frac{1}{2(\mathbb{L}+1)}\delta^2(\zeta_{\bar{C}}(1)). \tag{3.17}$$

Chapter 4

Rank 3

In this chapter we repeat the same arguments developed in the previous chapter to obtain the class of the moduli of pairs of rank 3, $\mathcal{N}_\sigma(3,1,d_1,d_2)$. A paralell computation can be found in [Mn08] where Muñoz computes the Hodge-Deligne polynomial of this variety.

1 Computation of the critical value

By Proposition 2.4 the critical values belong to the interval

$$[\sigma_m, \sigma_M] = \left[\frac{d_3 - 3d_2}{3}, d_1 - 3d_2\right] \tag{4.1}$$

These critical values are indexed by an integer n as shown by the following proposition.

Proposition 4.1. [Mn08, Prop. 6.2] *The critical values σ_c for triples of type$(3,1,d_1,d_2)$ such that $\sigma_c > \sigma_m$ are the numbers*

$$\sigma_n = 2n - d_1 - d_2, \qquad \frac{2}{3}d_1 < n \le d_1 - d_2, \quad n \in \mathbb{Z}$$

Remark 4.2. *For $n = \frac{2}{3}d_1$ (possibly non integer), $\sigma_n = \sigma_m$.*

On every critical value not all possible strata appear.

2 Set of strata

The list of the possible strata are the same as the ones obtained in the computation of the class of $M^s(2,0)$ in the precedent chapter. These are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix},$$

whose respective $\mathbf{n}$ are

$$\mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix}.$$

Since the construction of these strata are similar to the previous case, we do not develop the arguments. We list the class in $\hat{K}_0(\mathfrak{Var}_\mathbb{C})$ of these strata.

- **Stratum No. 1** It only appears for $d_1 \equiv 0 \pmod 3$, that is, n even. Recall that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Then, this class is

$$[\mathcal{N}(1,1,d_1-n,d_2)]([\mathrm{Jac}^{\frac{d_1}{3}} C]^2 - [\mathrm{Jac}^{\frac{d_1}{3}} C])[\mathbb{P}\,\mathrm{Ext}^1(T', S_1)]([\mathbb{P}\,\mathrm{Ext}^1(\tilde{T}_1, S_2)] - [\mathbb{P}\,\mathrm{Ext}^1(T', S_2)]). \tag{4.2}$$

The dimension of each projective space is computed as follows. We compute the degrees of the bundles of T' by the equation (2.9)

$$\mu(T') = \mu(T) = \frac{d_1 + d_2 + \sigma_n}{4} = \frac{n}{2}.$$

Combining with the equality (4.1), we have that T' is of type $(1,1,d_1-n,d_2)$. Then, by Proposition 2.18 we have

$$\chi(T',S_1)=n+d_2-d_1. \tag{4.3}$$

Since $\operatorname{Hom}(T',S_1)=0$ by Lemma 2.19, then

$$\dim\operatorname{Ext}^1(T',S_1)=d_1-d_2-n=\delta-n,$$

where we denote by δ the difference d_1-d_2.

On the other hand, the dimension of $\operatorname{Ext}^1(\tilde{T}_1,S_2)$ is computed in a similar way. By the equality $\mu(\tilde{T}_1)=\mu(T)=\frac{n}{2}$ we obtain the type of $\tilde{T}_1$, which is $(2,1,d_1-\frac{n}{2},d_2)$. Again, from Proposition 2.18 we have

$$\chi(\tilde{T}_1,S_2)=(1-g)+n-\delta. \tag{4.4}$$

Since $\operatorname{Hom}(\tilde{T}_1,S_2)=0$, we have $\dim\operatorname{Ext}^1(\tilde{T}_1,S_2)=\delta-n+g-1$. Then,

$$\begin{aligned}[X(\mathbf{A},\mathbf{n})^+] &= \lambda^{\delta-n}([C])[\operatorname{Jac}^{d_2}C]([\operatorname{Jac}^{n/2}C]^2-[\operatorname{Jac}^{n/2}C])\frac{(\mathbb{L}^{\delta-n}-1)(\mathbb{L}^{\delta-n+(g-1)}-\mathbb{L}^{\delta-n})}{(\mathbb{L}-1)^2}\\ &= \lambda^{\delta-n}([C])\mathbb{J}^2(\mathbb{J}-1)\frac{(\mathbb{L}^{\delta-n}-1)(\mathbb{L}^{\delta-n+(g-1)}-\mathbb{L}^{\delta-n})}{(\mathbb{L}-1)^2}.\end{aligned} \tag{4.5}$$

The corresponding stratum in the other flip locus $X(\mathbf{A},\mathbf{n})^-$ is obtained similarly. The structure of the stratum is the same, only changing the dimensions:

$$\begin{aligned}\chi(S_1,T') &= 1-g+d_1-\frac{3}{2}n,\\ \chi(S_2,\tilde{T}_1) &= 2(1-g)+d_1-\frac{3}{2}n.\end{aligned} \tag{4.6}$$

Notice that $\operatorname{Hom}(S_1,T')$, where $\tilde{T}_1'=(F,L_0,\varphi)$, is isomorphic to $\operatorname{Hom}(L_1,L)$. But this space is zero since $\deg(L_1^*\otimes L)=d_1-\frac{3}{2}n<0$, since $n>\frac{2}{3}d_1$. On the other hand, we claim $\operatorname{Hom}(L_2,F)=0$. The short exact sequence

$$0\to L_0\to F\to L_1\to 0$$

gives the long exact sequence in cohomology

$$0\to\operatorname{Hom}(L_2,L_0)\to\operatorname{Hom}(L_2,F)\to\operatorname{Hom}(L_2,L_1)\to\cdots$$

Since $\operatorname{Hom}(L_2,L_1)=0$, we have $\operatorname{Hom}(L_2,F)$ is zero. So that, $\operatorname{Hom}(S_2,\tilde{T}_1)$, which is isomorphic to $\operatorname{Hom}(L_2,F)$, is zero.

Then, we have

$$[X(\mathbf{A},\mathbf{n})^-]=\lambda^{\delta-n}([C])\mathbb{J}^2(\mathbb{J}-1)\frac{(\mathbb{L}^{g-1+\frac{3}{2}n-d_1}-1)(\mathbb{L}^{2(g-1)+\frac{3}{2}n-d_1}-\mathbb{L}^{g-1+\frac{3}{2}n-d_1})}{(\mathbb{L}-1)^2} \tag{4.7}$$

This stratum needs even n, so this stratum appears when $n\in 2\mathbb{Z}$.

■ **Stratum No. 2** This stratum is determined by

$$\mathbf{A}=\begin{pmatrix}1\\1\end{pmatrix}\text{ and }\mathbf{n}=(\,1\,).$$

Observe that this is possible only for values $d_1\equiv 0 \pmod 3$, that is, for n even number.

The construction of this stratum follows exactly the same argument as the previous one. There is only one difference: on the last step, the cohomology group $\operatorname{Hom}(L_2,L_1)$ does not vanish, we have $\operatorname{Hom}(L_2,L_1)\cong\mathbb{C}$. This fact modifies the class (4.2): the class of this stratum is now

$$[\mathcal{N}(1,1,d_1-n,d_2)][\operatorname{Jac}^{\frac{d_1}{3}}C][\mathbb{P}\operatorname{Ext}^1(T',S_1)]([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1,S_2)]-[\mathbb{P}\ker p_1^*]).$$

where $\ker p_1^*$ is not isomorphic to $\mathrm{Ext}^1(T', S_2)$ but to $\mathrm{Ext}^1(T', S_2)/\operatorname{Hom}(S_1, S_2)$ by the long exact sequence

$$0 \longrightarrow \operatorname{Hom}(S_1, S_2) \longrightarrow \mathrm{Ext}^1(T', S_2) \xrightarrow{q_1^*} \mathrm{Ext}^1(\tilde{T}_1, S_2) \xrightarrow{p_1^*} \mathrm{Ext}^1(S_1, S_2) \longrightarrow 0$$

The computation on (4.6) holds for this case, hence we have

$$[X(\mathbf{A}, \mathbf{n})^+] = \lambda^{\delta-n}([C])\mathbb{J}^2 \frac{(\mathbb{L}^{\delta-n} - 1)(\mathbb{L}^{(g-1)+\delta-n} - \mathbb{L}^{\delta-n-1})}{(\mathbb{L}-1)^2}. \tag{4.8}$$

Similar reasoning gives the stratum at the other side:

$$[X(\mathbf{A}, \mathbf{n})^-] = \lambda^{\delta-n}([C])\mathbb{J}^2 \frac{(\mathbb{L}^{g-1+\frac{3}{2}n-d_1} - 1)(\mathbb{L}^{2(g-1)+\frac{3}{2}n-d_1} - \mathbb{L}^{g-2+\frac{3}{2}n-d_1})}{(\mathbb{L}-1)^2}. \tag{4.9}$$

■ Stratum No. 3

This stratum is determined by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \end{pmatrix} \text{ and } \mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

This only appears for $d_1 \equiv 0 \pmod 3$, that is, for even n

As before, the details of the computation of this stratum can be found in the equivalent stratum in the precedent chapter, written in (3.10). We have to adjust the dimension of the cohomology groups and make a slight modification:

$$\begin{aligned}
[X(\mathbf{A}, \mathbf{n})^+] &= \Big(\lambda^2(\mathbb{P}^{\delta-n-1}[\operatorname{Jac}^{n/2} C]) - \lambda^2(\mathbb{P}^{\delta-n-1})[\operatorname{Jac}^{n/2} C]\Big)\,[\mathcal{N}(1,1,d_1-n,d_2)] \\
&= \lambda^{\delta-n}([C])\mathbb{J}\cdot\left(\frac{\mathbb{L}^{2(\delta-n)}-1}{\mathbb{L}^2-1}\lambda^2(\mathbb{J}) + \mathbb{L}\frac{(\mathbb{L}^{\delta-n-1}-1)(\mathbb{L}^{\delta-n}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2\right. \\
&\qquad \left. - \frac{(\mathbb{L}^{\delta-n}-1)(\mathbb{L}^{\delta-n+1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}\right).
\end{aligned} \tag{4.10}$$

To get this, recall the formula (4.6).

Recall the δ^2 operator in $K_0(\mathfrak{Var}_{\mathbb{C}})$ (see Section 1.1). Writing the class $[X(\mathbf{A}, \mathbf{n})^+]$ using $\delta^2(\mathbb{J})$ instead of $\lambda^2(\mathbb{J})$ we have

$$\begin{aligned}
[X(\mathbf{A}, \mathbf{n})^+] = \lambda^{\delta-n}([C])\mathbb{J}\cdot&\left(\frac{1}{2}\frac{\mathbb{L}^{2(\delta-n)}-1}{\mathbb{L}^2-1}\delta^2(\mathbb{J}) + \frac{1}{2}\left(\frac{\mathbb{L}^{\delta-n}-1}{\mathbb{L}-1}\right)^2\mathbb{J}^2 - \right. \\
&\left. - \frac{(\mathbb{L}^{\delta-n}-1)(\mathbb{L}^{\delta-n+1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}\right).
\end{aligned}$$

The other flip locus is written in a similar way:

$$\begin{aligned}
[X(\mathbf{A}, \mathbf{n})^-] = &\left(\frac{\mathbb{L}^{2(g-1)+3n-2d_1}-1}{\mathbb{L}^2-1}\lambda^2(\mathbb{J}) + \mathbb{L}\frac{(\mathbb{L}^{g-2+\frac{3}{2}n-d_1}-1)(\mathbb{L}^{g-1+\frac{3}{2}n-d_1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2 - \right. \\
&\left. \frac{(\mathbb{L}^{g-1+\frac{3}{2}n-d_1}-1)(\mathbb{L}^{g+\frac{3}{2}n-d_1}-1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\mathbb{J}\right)\mathbb{J}\lambda^{\delta-n}([C]).
\end{aligned}$$

or else

$$\begin{aligned}
[X(\mathbf{A}, \mathbf{n})^-] = \lambda^{\delta-n}([C])\mathbb{J}\cdot&\left(\frac{1}{2}\frac{\mathbb{L}^{2(g-1+\frac{3}{2}n-d_1)}-1}{\mathbb{L}^2-1}\delta^2(\mathbb{J}) + \frac{1}{2}\left(\frac{\mathbb{L}^{g-1+\frac{3}{2}n-d_1}-1}{\mathbb{L}-1}\right)^2\mathbb{J}^2 - \right. \\
&\left. - \frac{(\mathbb{L}^{g-1+\frac{3}{2}n-d_1}-1)(\mathbb{L}^{g-\frac{3}{2}n-d_1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}\right).
\end{aligned}$$

■ **Stratum No. 4** This stratum is given by

$$\mathbf{A} = (\,2\,) \text{ and } \mathbf{n} = (\,1\,)$$

Evidently, the line bundle has degree $d_1/3$, thus this appear only for n even.

Furthermore, this is analogous to (3.12). As before, we use our previous computations of the type of T' and the dimension (see (4.6)) to adapt to this case:

$$\begin{aligned}[X(\mathbf{A},\mathbf{n})^+] &= [\mathrm{Gr}(2,\mathrm{Ext}^1(T',S_1))][\mathrm{Jac}^{n/2}\,C][\mathcal{N}(1,1,d_1-n,d_2)] = \\ &= \frac{(\mathbb{L}^{\delta-n}-1)(\mathbb{L}^{\delta-n-1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2\lambda^{\delta-n}([C]).\end{aligned} \tag{4.11}$$

The other flip locus is

$$[X(\mathbf{A},\mathbf{n})^-] = \frac{(\mathbb{L}^{g-1+\frac{3}{2}n-d_1}-1)(\mathbb{L}^{g+\frac{3}{2}n-d_1-1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2\lambda^{\delta-n}([C]). \tag{4.12}$$

■ **Stratum No. 5** This stratum is given by

$$\mathbf{A} = (\,1\,) \text{ and } \mathbf{n} = (\,2\,).$$

Unlike the other strata, this appears for any n. Nevertheless, it contains the of $M^s(2,d)$ and depends on n: if n is odd, the class is in (3.9), else the class is (3.17) living in $\bar{K}_0(\mathfrak{Var})$. To simplify our notation, we denote $\mathbb{M}_2^d = [M^s(2,d)] \in \bar{K}_0(\mathfrak{Var})$.

Recall the equality $\mu(S_1) = \mu(T) = \frac{n}{2}$, from which we have $\deg(F) = n$ where $S_1 = (F,0,0)$. On the other hand, the triple T' is of type $(1,1,d_1-n,d_2)$. Then, the class in $\hat{K}_0(\mathfrak{Var})$ is

$$[X(\mathbf{A},\mathbf{n})^+] = \mathbb{M}_2^n\mathbb{J}\lambda^{\delta-n}([C])\mathbb{P}\,\mathrm{Ext}^1(T',S_1)$$

We compute $\dim \mathrm{Ext}^1(T',S_1) = -\chi(T',S_1)$ since $\mathrm{Hom}(T',S_1) = 0$:

$$\chi(T',S_1) = -2(\delta-n).$$

Then, we have

$$[X(\mathbf{A},\mathbf{n})^+] = \mathbb{M}_2^n\mathbb{J}\lambda^{\delta-n}([C])\frac{\mathbb{L}^{2(\delta-n)}-1}{\mathbb{L}-1} \tag{4.13}$$

Substituing $\mathbb{M}_2^n$ for its value we have

$$[X(\mathbf{A},\mathbf{n})^+] = \begin{cases} \mathbb{J}\dfrac{\mathbb{L}^{2(\delta-n)}-1}{\mathbb{L}-1}\left(\dfrac{2\zeta_{\bar{C}}(\mathbb{L})\,\mathbb{J}-\mathbb{J}^2(2\mathbb{L}^{g-1}-\mathbb{L}^2+1)}{2(\mathbb{L}-1)(\mathbb{L}^2-1)} - \dfrac{1}{2}\dfrac{\delta^2(\mathbb{J})}{\mathbb{L}+1}\right)\lambda^{\delta-n}([C]), & \text{for } n \text{ even,} \\ \mathbb{J}^2\dfrac{\mathbb{L}^{2(\delta-n)}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\left(\mathbb{L}^g\mathbb{J}-\zeta_{\bar{C}}(\mathbb{L})\right)\lambda^{\delta-n}([C]), & \text{for } n \text{ odd.}\end{cases}$$

The other stratum is very similar

$$[X(\mathbf{A},\mathbf{n})^-] = \mathbb{M}_2^n\mathbb{J}\lambda^{\delta-n}([C])\frac{\mathbb{L}^{2(g-1)+3n-2d_1}-1}{\mathbb{L}-1} \tag{4.14}$$

since

$$\chi(S_1,T') = 2(1-g)+2d_1-3n.$$

Then,

$$[X(\mathbf{A},\mathbf{n})^-] = \begin{cases} \mathbb{J}\dfrac{\mathbb{L}^{2(g-1)+3n-2d_1}-1}{\mathbb{L}-1}\left(\dfrac{2\zeta_{\bar{C}}(\mathbb{L})-\mathbb{J}(2\mathbb{L}^{g-1}-\mathbb{L}^2+1)}{2(\mathbb{L}-1)(\mathbb{L}^2-1)}\mathbb{J}-\dfrac{\delta^2(\mathbb{J})}{2(\mathbb{L}+1)}\right)\lambda^{\delta-n}([C]), & \text{for } n \text{ even,} \\ \mathbb{J}^2\dfrac{\mathbb{L}^{2(g-1)+3n-2d_1}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\left(\mathbb{L}^g\mathbb{J}-\zeta_{\bar{C}}(\mathbb{L})\right)\lambda^{\delta-n}([C]), & \text{for } n \text{ odd.} \end{cases}$$

Remark 4.3. *This is the unique stratum which appears when n is odd.*

■ **Stratum No. 6** This last stratum is determined by the constants

$$\mathbf{A} = (\,1\,) \text{ and } \mathbf{n} = (\,1\,),$$

and it appears only for even n. This is

$$[X(\mathbf{A},\mathbf{n})^+] = [\mathcal{N}^s_{\sigma_n}(2,1,d_1-\frac{n}{2},d_2)][\mathrm{Jac}^{n/2}\,C][\mathbb{P}\,\mathrm{Ext}^1(T',S_1)],$$
$$[X(\mathbf{A},\mathbf{n})^-] = [\mathcal{N}^s_{\sigma_n}(2,1,d_1-\frac{n}{2},d_2)][\mathrm{Jac}^{n/2}\,C][\mathbb{P}\,\mathrm{Ext}^1(S_1,T')].$$

From Proposition 2.18 we have

$$\chi(T',S_1) = 1-g+n-\delta, \qquad \chi(S_1,T') = 2(1-g)+d_1-\frac{3}{2}n.$$

It is easy to check that σ_n is a critical value for pairs of rank 2. Critical values of $\mathcal{N}_{\sigma_n}(2,1,d_1-\frac{n}{2},d_2)$ are $\sigma_k = 3k-(d_1-\frac{n}{2}-d_2)$ by $\sigma_k=\sigma_n=2n-d_1-d_2$, we have

$$3k-(d_1-\frac{n}{2})-d_2 = 2n-d_1-d_2 \iff 3k=\frac{3}{2}n \iff k=\frac{n}{2}.$$

Because n is even, k is an integer number and σ_n is a critical value for $\mathcal{N}_{\sigma_n}(2,1,d_1,d_2)$.

Lemma 4.4. *The class of $\mathcal{N}^s_{\sigma_c}(2,1,d_1,d_2)$ in $K_0(\mathfrak{Var})$ is*

$$[\mathcal{N}^s_{\sigma_c}(2,1,d_1,d_2)] = \frac{\mathbb{J}^2}{\mathbb{L}-1}\left(\mathbb{L}^{2m-d_1+g+1}\lambda^{\delta-m-1}([C]+\mathbb{L}^2)-\lambda^{\delta-m}([C]\mathbb{L}+1)+\lambda^{\delta-m}([C])\right). \quad (4.15)$$

where $m=\frac{1}{3}(\sigma_c-d_1-d_2)$ for σ_c a critical value.

Proof. Let σ_c be a critical value for $\mathcal{N}^s_{\sigma_c}(2,1,d_1,d_2)$. This critical value should be $\sigma_c=\sigma_m=3m-d_1-d_2$ for m an integer number. By Lemma 2.13 we have the equality

$$[\mathcal{N}^s_{\sigma_c}(2,1,d_1,d_2)] = [\mathcal{N}_{\sigma_c-\epsilon}(2,1,d_1,d_2)]-[\mathcal{S}^-_{\sigma_c}],$$

for a sufficiently small $\epsilon>0$. Their clases are computed in Theorem 3.2 (equation (3.2))

$$[\mathcal{N}_{\sigma_c-\epsilon}(2,1,d_1,d_2)] = \frac{\mathbb{J}^2}{\mathbb{L}-1}\left(\mathbb{L}^{2m-d_1+g-1}\lambda^{\delta-m}([C]+\mathbb{L}^2)-\lambda^{\delta-m}([C]\mathbb{L}+1)\right)$$

and (see (3.1) and below)

$$[\mathcal{S}^-_{\sigma_c}] = \mathbb{J}^2\frac{\mathbb{L}^{2m-d_1+g-1}-1}{\mathbb{L}-1}\lambda^{\delta-m}([C]).$$

Here, $m = \left\lceil \frac{1}{3}(\sigma_c + d_1 + d_2) \right\rceil + 1$. Their difference is

$$[\mathcal{N}^s_{\sigma_c}(2,1,d_1,d_2)] = \frac{\mathbb{J}^2}{\mathbb{L}-1}\big(\mathbb{L}^{2m-d_1+g-1}\left(\lambda^{\delta-m}([C]+\mathbb{L}^2) - \lambda^{\delta-m}([C])\right) - \\ - \left(\lambda^{\delta-m}([C]\mathbb{L}+1) - \lambda^{\delta-m}([C])\right)\big) = \\ = \frac{\mathbb{J}^2}{\mathbb{L}-1}\big(\mathbb{L}^{2m-d_1+g+1}\lambda^{\delta-m-1}([C]+\mathbb{L}^2) - \lambda^{\delta-m}([C]\mathbb{L}+1) + \lambda^{\delta-m}([C])\big).$$

And we have the desired equality. □

Remark 4.5. *In terms of $\bar{C}$, we have*

$$[\mathcal{N}^s_{\sigma_c}(2,1,d_1,d_2)] = \frac{\mathbb{J}^2}{\mathbb{L}-1}\operatorname*{coeff}_{x^0}\left\{\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)x^{\delta-m}}\left(\frac{\mathbb{L}^{2m-d_1+g+1}x}{1-\mathbb{L}^2x} - \frac{\mathbb{L}^{\delta-m}}{1-\frac{x}{\mathbb{L}}} + 1\right)\right\}.$$

Compare this formula (4.15) with the formula obtained in Mn08, Theorem 5.3.

Substituting $[\mathcal{N}^s_{\sigma_n}(2,1,d_1-\frac{n}{2},d_2)]$ by its class in $\hat{K}_0(\mathfrak{V}\mathrm{ar})$ (here, $m = \left\lceil \frac{1}{3}(\sigma_n + (d_1 - \frac{n}{2}) + d_2) \right\rceil = \frac{n}{2}$ where $\sigma_n = 2n - d_1 - d_2$ and $n \in \mathbb{Z}$) we have that the class of the stratum 6 is

$$[X(\mathbf{A},\mathbf{n})^+] = \mathbb{J}^3\frac{\mathbb{L}^{g-1+\delta-n}-1}{(\mathbb{L}-1)^2}\left(\mathbb{L}^{g+1+\frac{3}{2}n-d_1}\lambda^{\delta-\frac{n}{2}-1}([C]+\mathbb{L}^2) - \lambda^{\delta-\frac{n}{2}}([C]\mathbb{L}+1) + \lambda^{\delta-\frac{n}{2}}([C])\right),$$
$$[X(\mathbf{A},\mathbf{n})^-] = \mathbb{J}^3\frac{\mathbb{L}^{2(g-1)+\frac{3}{2}n-d_1}-1}{(\mathbb{L}-1)^2}\left(\mathbb{L}^{g+1+\frac{3}{2}n-d_1}\lambda^{\delta-\frac{n}{2}-1}([C]+\mathbb{L}^2) - \lambda^{\delta-\frac{n}{2}}([C]\mathbb{L}+1) + \lambda^{\delta-\frac{n}{2}}([C])\right). \tag{4.16}$$

3 The sum

Now we aim to compute the class of the moduli space of pairs of rank 3. Fix a critical value σ and the integer number

$$n_0 = \left\lceil \frac{\sigma + d_1 + d_2}{2} \right\rceil$$

so that the smallest critical value greater than σ is $\sigma_{n_0} = 2n_0 - d_1 - d_2$. By Lemma 2.13 we write

$$[\mathcal{N}_\sigma(3,1,d_1,d_2)] = \sum_{n=n_0}^{d_1-d_2}\left([\mathcal{S}^-_{\sigma_n}] - [\mathcal{S}^+_{\sigma_n}]\right) \tag{4.17}$$

where for $n = d_1 - d_2$ we have the maximal critical value σ_M.

First, we compute $[\mathcal{S}^{\pm}_{\sigma_c}]$. Recall that these class depends on n: for even integer number n,

$$[\mathcal{S}^{\pm}_{\sigma_n}] = [X^{\pm}_1] + \cdots + [X^{\pm}_6]$$

while for odd integer number n,

$$[\mathcal{S}^{\pm}_{\sigma_n}] = [X^{\pm}_5]$$

We focus on the even case where the computations are more problematic. To do this, it should be convenient to rewrite the stratum 5: we replace $\mathbb{M}^2_{\mathrm{even}}$ by the class $\mathbb{M}^2_{\mathrm{even}} - \mathbb{M}^2_{\mathrm{odd}}$. Hence the sum (4.17) becomes

$$\sum_{\substack{n=0\\ n\mathrm{even}}}^{d_1-d_2}\left([\mathcal{S}^-_{0,\sigma_n}] - [\mathcal{S}^+_{0,\sigma_n}]\right) + \sum_{n=n_0}^{d_1-d_2}\left([\mathcal{S}^-_{1,\sigma_n}] - [\mathcal{S}^+_{1,\sigma_n}]\right).$$

where $\mathcal{S}^{\pm}_{0,\sigma_n}$ denotes the modified class of $\mathcal{S}^{\pm}_{\sigma_n}$ by the mentioned replacement for the even case, and $\mathcal{S}^{\pm}_{1,\sigma_n}$ the case odd, that is, the class $[X^{\pm}_5]$ with $\mathbb{M}^2_{\mathrm{odd}}$ class.

Now, we compute the class of $[\mathcal{S}^+_{0,\sigma_n}]$ by computing the terms accompanying $\mathbb{J}^2$, $\mathbb{J}^3$ and $\delta^2(\mathbb{J})$. The Appendix contains the source code of Mathematica for computing these classes. In order to simplify the exponents, we define

$$a = d_1 - d_2 - n,$$
$$b = g - 1 + \frac{3}{2}n - d_1,$$

Furthermore, we use the δ^2 operator instead of λ^2.

- Terms with $\mathbb{J}^2$:

$$\begin{aligned} -\frac{(\mathbb{L}^a-1)(\mathbb{L}^{a+g-1}-\mathbb{L}^a)}{(\mathbb{L}-1)^2} + \frac{(\mathbb{L}^a-1)(\mathbb{L}^{a+g-1}-\mathbb{L}^{a-1})}{(\mathbb{L}-1)^2} - \\ -\frac{(\mathbb{L}^a-1)(\mathbb{L}^{a+1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)} + \frac{(\mathbb{L}^a-1)(\mathbb{L}^{a-1}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)} = \\ = \frac{\mathbb{L}^a-1}{(\mathbb{L}-1)^2}\left(\mathbb{L}^a - \mathbb{L}^{a-1} - \frac{\mathbb{L}^{a+1}-\mathbb{L}^{a-1}}{\mathbb{L}+1}\right) = 0. \end{aligned}$$

- Terms with $\mathbb{J}^3$:

$$\begin{aligned} &\left(\mathbb{L}^a\frac{(\mathbb{L}^a-1)(\mathbb{L}^{g-1}-1)}{(\mathbb{L}-1)^2} + \frac{1}{2}\left(\frac{\mathbb{L}^a-1}{\mathbb{L}-1}\right)^2 - \frac{1}{2}\frac{(\mathbb{L}^{2a}-1)(2\mathbb{L}^{g-1}-\mathbb{L}^2+1)}{2(\mathbb{L}-1)^2(\mathbb{L}^2-1)} - \mathbb{L}^{g-1}\frac{\mathbb{L}^{2a}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\right)\lambda^a([C]) + \\ &+ \frac{\mathbb{L}^{a+g-1}-1}{(\mathbb{L}-1)^2}\left(\mathbb{L}^b\lambda^a([C]+\mathbb{L}^2) - \lambda^a([C]\mathbb{L}+1) + \lambda^a([C])\right) = \\ &= \mathbb{L}^{g-1}\frac{(\mathbb{L}^a-1)(\mathbb{L}^a-\mathbb{L})}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2\lambda^a([C]) + \frac{\mathbb{L}^{a+g-1}-1}{(\mathbb{L}-1)^2}\left(\mathbb{L}^b\lambda^a([C]+\mathbb{L}^2) - \lambda^a([C]\mathbb{L}+1) + \lambda^a([C])\right) \end{aligned}$$

- Coefficient of $\delta^2(\mathbb{J})$ is zero: the coefficient of $\delta^2(\mathbb{J})$ of X_2^+ are cancelled by the coefficient of X_5^+. The same result we obtain for the coefficient of $\zeta(\mathbb{L})\mathbb{J}^2$.

Analogously, we have the class of $[\mathcal{S}^-_{0,\sigma_n}]$: the classes of $\mathbb{J}^2$ and $\delta^2(\mathbb{J})$ do not appear as in case $[\mathcal{S}^+_{0,\sigma_c}]$, but the coefficient of the class $\mathbb{J}^3$ is

$$\mathbb{L}^{g-1}\frac{(\mathbb{L}^b-1)(\mathbb{L}^b-\mathbb{L})}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\mathbb{J}^2\lambda^a([C]) + \frac{\mathbb{L}^{b+g-1}-1}{(\mathbb{L}-1)^2}\left(\mathbb{L}^b\lambda^a([C]+\mathbb{L}^2) - \lambda^a([C]\mathbb{L}+1) + \lambda^a([C])\right)$$

The difference is

$$\begin{aligned} \Delta_n = \mathbb{L}^{g-1}\frac{(\mathbb{L}^a-\mathbb{L}^b)(\mathbb{L}^b+\mathbb{L}^a-(\mathbb{L}+1))}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\mathbb{J}^3\lambda^a([C]) + \\ + \mathbb{L}^{g-1}\frac{\mathbb{L}^b-\mathbb{L}^a}{(\mathbb{L}-1)^2}\left(\mathbb{L}^b\lambda^a([C]+\mathbb{L}^2) - \lambda^a([C]\mathbb{L}+1) + \lambda^a([C])\right) \end{aligned}$$

We use the formula

$$\lambda^a([C]) = \operatorname*{coeff}_{x^0}\frac{\zeta_{[C]}(x)}{x^a}$$

and the "additivity" of the ζ function to obtain, after some algebraic computation, a simplified expression for Δ_n:

$$\Delta_n = \operatorname*{coeff}_{x^0}\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)x^a}\left(\mathbb{L}^{g-1}\frac{\mathbb{L}^a-\mathbb{L}^b}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\left(\mathbb{L}^a\frac{\mathbb{L}^2+x}{\mathbb{L}-x} + \mathbb{L}^b\frac{1+\mathbb{L}^3x}{\mathbb{L}^2x-1}\right)\mathbb{J}^3\right).$$

This cover the even case.

Now, we compute the odd case, that is, Δ_n for n an odd integer number. Since $[\mathcal{S}^{\pm}_{\sigma_c}] = X_5^{\pm}$, then

$$\begin{aligned}
\Delta_n &= X_5^- - X_5^+ \\
&= \mathbb{J}^2 \frac{\mathbb{L}^{2b} - \mathbb{L}^{2a}}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} \left(\mathbb{L}^g \mathbb{J} - \zeta_{\bar{C}}(\mathbb{L})\right) \lambda^{\delta-n}([C]) \\
&= \frac{\mathbb{J}^2(\mathbb{L}^g \mathbb{J} - \zeta_{\bar{C}}(\mathbb{L}))}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} \cdot \operatorname*{coeff}_{x^0} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)x^{\delta-n}} (\mathbb{L}^{2b} - \mathbb{L}^{2a}).
\end{aligned}$$

Then, for a non-critical value σ, let $n = \lceil \frac{n-d_1-d_2}{2} \rceil$ and then

$$[\mathcal{N}_\sigma(3,1,d_1,d_2)] = \sum_{k=n_0}^{\delta} \Delta_n$$

Notice $\zeta_{\bar{C}}(x)$ is a polynomial of degree $2g$, and $\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)}$ is a series whose coefficient of x^n for $n<0$ is zero. This implies

$$\sum_{n=n_0}^{\delta} \Delta_n = \sum_{n=n_0}^{\infty} \Delta_n$$

Consider the following sums:

$$\begin{aligned}
\sum_{n=n_0}^{\infty} \frac{\mathbb{L}^{2a}}{x^{\delta-n}} &= \frac{\mathbb{L}^{2(\delta-n_0)}}{x^{\delta-n_0}} \frac{1}{1-\mathbb{L}^{-2}x}, \\
\sum_{n=n_0}^{\infty} \frac{\mathbb{L}^{2b}}{x^{\delta-n}} &= \frac{\mathbb{L}^{2(g-1)+3n_0-d_1}}{x^{\delta-n_0}} \frac{1}{1-\mathbb{L}^3 x}, \\
\sum_{n=n_0,\text{even}}^{\infty} \frac{\mathbb{L}^{2a}}{x^{\delta-n}} &= \frac{\mathbb{L}^{2(\delta-n_0)}}{x^{\delta-n_0}} \frac{1}{1-(\mathbb{L}^{-2}x)^2}, \\
\sum_{n=n_0,\text{even}}^{\infty} \frac{\mathbb{L}^{2b}}{x^{\delta-n}} &= \frac{\mathbb{L}^{2(g-1)+3n_0-d_2}}{x^{\delta-n_0}} \frac{1}{1-(\mathbb{L}^3 x)^2}, \\
\sum_{n=n_0,\text{even}}^{\infty} \frac{\mathbb{L}^{a+b}}{x^{\delta-n}} &= \frac{\mathbb{L}^{g-1+\frac{n_0}{2}-d_2}}{x^{\delta-n_0}} \frac{1}{1-\mathbb{L}x^2},
\end{aligned}$$

Plugging these equalities we have

Theorem 4.6. *Let $\mathcal{N}^s_\sigma(3,1,d_1,d_2)$ be the moduli space of pairs of fixed degrees d_1 and d_2 of rank 3. Let σ be a non-critical value, then*

$$\begin{aligned}
[\mathcal{N}^s_\sigma(3,1,d_1,d_2)] = \mathbb{J}^2 \operatorname*{coeff}_{x^0} & \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)x^{\delta-n_0}} \times \\
\times \Bigg\{ & \left(\frac{\mathbb{L}^{2(g-1)+3n_0-2d_1}}{1-\mathbb{L}^3x} - \frac{\mathbb{L}^{2(\delta-n_0)}}{1-\mathbb{L}^{-2}x} \right) \frac{\zeta_{\bar{C}}(\mathbb{L}) - \mathbb{L}^g\mathbb{J}}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} + \\
& + \frac{\mathbb{L}^{g-1}}{(\mathbb{L}-1)(\mathbb{L}^2-1)} \left(\frac{\mathbb{L}^{2(g-1)+3\bar{n}_0-2d_1}}{(1-\mathbb{L}^2x)(1-\mathbb{L}^3x)} + \frac{\mathbb{L}^{2(\delta-\bar{n}_0)+1}}{(1-\mathbb{L}^{-1}x)(1-\mathbb{L}^{-2}x)} \right) - \\
& - \frac{\mathbb{L}^{g-1}}{(\mathbb{L}-1)^2} \frac{\mathbb{L}^{g-1+\bar{n}_0/2-d_2}}{(1-\mathbb{L}^2x)(1-\mathbb{L}^{-1}x)} \Bigg\}.
\end{aligned} \tag{4.18}$$

where n_0 is the greatest integer lower than $\frac{\sigma-d_1-d_2}{2}$, that is, $n_0 = \lceil \frac{\sigma-d_1-d_2}{2} \rceil$, and $\bar{n}_0$ is the even integer such than $n_0 \le \bar{n}_0 \le n_0+1$, that is, $\bar{n}_0 = 2\lceil \frac{n_0}{2} \rceil$.

4 The class of the moduli space of stable bundles of rank 3

We compute the class of $M^s(3,1)$. We follow similar arguments as in the computation of the class of $M^s(2,1)$. First, we compute the class of $[\mathcal{N}^s_{\sigma_m}(3,1,d_1,d_2)]$. To do this, we choose d_1 and d_2 so that $\delta - n_0$ is bigger than $2g$ in the formula (4.18) and apply our residues formula to obtain the class. On the other hand, it is possible to compute such a class by the stratification of the flip loci $\mathcal{S}^+_{\sigma_m}$; in that case there is an unique stratum. Equivalently, we can apply Proposition 2.28.

Let us begin computing some residues to apply formula (4.18). To get the class $[\mathcal{N}^+_{\sigma_m}(3,1,d_1,d_2)]$ for $\delta - n_0 > 2g$ and $d_1 \not\equiv 0 \pmod 3$, we take $n_0 = \lceil \frac{2}{3} d_1 \rceil = \frac{2d_1+1}{3}$ and $\bar{n}_0$ the even number such that $n_0 \leq \bar{n}_0 \leq n_0 + 1$. We have to compute the coefficient of some rational functions on $\mathbb{P}^1_{\bar{\mathcal{K}}}$. This shall be done by computing residues of some 1-forms. Let

$$\omega_1 = \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^3 x)x^{\delta-n_0+1}}\mathrm{d}x$$

1-form on $\mathbb{P}^1_{\bar{\mathcal{K}}}$. By Residue Theorem (see Theorem 1.15),

$$\begin{aligned}
\mathrm{Res}_{x=0}\,\omega_1 &= -\mathrm{Res}_{x=1}\,\omega_1 - \mathrm{Res}_{x=\mathbb{L}^{-1}}\,\omega_1 - \mathrm{Res}_{x=\mathbb{L}^{-3}}\,\omega_1,\\
&= -\frac{\zeta_{\bar{C}}(1)}{(1-\mathbb{L})(1-\mathbb{L}^3)}(-1)\,\mathrm{Res}_{x=1}\frac{\mathrm{d}(-x)}{(1-x)} - \frac{\mathbb{L}^{\delta-n_0+1}\zeta_{\bar{C}}\big(\mathbb{L}^{-1}\big)}{(1-\mathbb{L}^{-1})(1-\mathbb{L}^2)}\frac{-1}{\mathbb{L}}\,\mathrm{Res}_{x=\mathbb{L}^{-1}}\frac{\mathrm{d}(-\mathbb{L}x)}{1-\mathbb{L}x} -\\
&\quad - \frac{\mathbb{L}^{3(\delta-n_0+1)}\zeta_{\bar{C}}\big(\mathbb{L}^{-3}\big)}{(1-\mathbb{L}^{-3})(1-\mathbb{L}^{-2})}\frac{-1}{\mathbb{L}^3}\,\mathrm{Res}_{x=\mathbb{L}^{-3}}\frac{\mathrm{d}(-\mathbb{L}^3x)}{1-\mathbb{L}^3x}\\
&= \frac{\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^3-1)} - \frac{\mathbb{L}^{\delta-n_0-g+1}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)} + \frac{\mathbb{L}^{3(\delta-n_0)+5-5g}\zeta_{\bar{C}}\big(\mathbb{L}^2\big)}{(\mathbb{L}^2-1)(\mathbb{L}^3-1)}.
\end{aligned}$$

For the 1-form

$$\omega_2 = \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^{-2}x)x^{\delta-n_0+1}}\mathrm{d}x$$

in $\mathbb{P}^1_{\bar{\mathcal{K}}}$, its residue at $x=0$ is exactly the coefficient of x^0 of the funcion $\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^{-2}x)x^{\delta-n_0}}$ that we are interested on. The Residue theorem gives the equality

$$\begin{aligned}
\mathrm{Res}_{x=0}\,\omega_2 &= -\mathrm{Res}_{x=1}\,\omega_2 - \mathrm{Res}_{x=\mathbb{L}^{-1}}\,\omega_2 - \mathrm{Res}_{x=\mathbb{L}^{2}}\,\omega_2\\
&= -\frac{\zeta_{\bar{C}}(1)}{(1-\mathbb{L})(1-\mathbb{L}^{-2})}(-1)\,\mathrm{Res}_{x=1}\frac{\mathrm{d}(-x)}{1-x} - \frac{\mathbb{L}^{\delta-n_0+1}\zeta_{\bar{C}}\big(\mathbb{L}^{-1}\big)}{(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-3})}\frac{-1}{\mathbb{L}}\,\mathrm{Res}_{x=\mathbb{L}^{-1}}\frac{\mathrm{d}(-\mathbb{L}x)}{1-\mathbb{L}x} -\\
&\quad - \frac{\zeta_{\bar{C}}\big(\mathbb{L}^2\big)}{\mathbb{L}^{2(\delta-n_0+1)}(1-\mathbb{L}^2)(1-\mathbb{L}^3)}\frac{-1}{\mathbb{L}^{-2}}\,\mathrm{Res}_{x=\mathbb{L}^2}\frac{\mathrm{d}(-\mathbb{L}^{-2}x)}{1-\mathbb{L}^{-2}x}\\
&= -\frac{\mathbb{L}^2\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)} + \frac{\mathbb{L}^{\delta-n_0+4-g}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^3-1)} + \frac{\zeta_{\bar{C}}\big(\mathbb{L}^2\big)}{\mathbb{L}^{2(\delta-n_0)}(\mathbb{L}^2-1)(\mathbb{L}^3-1)}.
\end{aligned}$$

Let us consider the 1-form

$$\omega_3 = \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^2x)(1-\mathbb{L}^3x)x^{\delta-n_0+1}}\mathrm{d}x$$

where $\mathrm{Res}_{x=0}\,\omega_3 = \mathrm{coeff}_{x^0} f(x)$ where $f(x) = \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^2x)(1-\mathbb{L}^3x)x^{\delta-n_0}}$. The Residue theorem says that

$$\begin{aligned}
\mathrm{Res}_{x=0}\,\omega_3 &= -\mathrm{Res}_{x=1}\,\omega_3 - \mathrm{Res}_{x=\mathbb{L}^{-1}}\,\omega_3 - \mathrm{Res}_{x=\mathbb{L}^{-2}}\,\omega_3 - \mathrm{Res}_{x=\mathbb{L}^{-3}}\,\omega_3\\
&= -\frac{\zeta_{\bar{C}}(1)}{(1-\mathbb{L})(1-\mathbb{L}^2)(1-\mathbb{L}^3)}(-1)\,\mathrm{Res}_{x=1}\frac{\mathrm{d}(-x)}{1-x} -\\
&\quad - \frac{\mathbb{L}^{\delta-n_0+1}\zeta_{\bar{C}}\big(\mathbb{L}^{-1}\big)}{(1-\mathbb{L}^{-1})(1-\mathbb{L})(1-\mathbb{L}^2)}\frac{-1}{\mathbb{L}}\,\mathrm{Res}_{x=\mathbb{L}^{-1}}\frac{\mathrm{d}(-\mathbb{L}x)}{1-\mathbb{L}x} -
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\mathbb{L}^{2(\delta-n_0+1)}\zeta_{\bar{C}}\big(\mathbb{L}^{-2}\big)}{(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-1})(1-\mathbb{L})}\frac{-1}{\mathbb{L}^2}\operatorname{Res}_{x=\mathbb{L}^{-2}}\frac{\mathrm{d}(-\mathbb{L}^2x)}{1-\mathbb{L}^2x}-\\
&-\frac{\mathbb{L}^{3(\delta-n_0+1)}\zeta_{\bar{C}}\big(\mathbb{L}^{-3}\big)}{(1-\mathbb{L}^{-3})(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-1})}\frac{-1}{\mathbb{L}^3}\operatorname{Res}_{x=\mathbb{L}^{-3}}\frac{\mathrm{d}(-\mathbb{L}^3x)}{1-\mathbb{L}^3x}\\
=&-\frac{\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}+\frac{\mathbb{L}^{\delta-n_0+1-g}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}-\frac{\mathbb{L}^{2(\delta-n_0)+3(1-g)}\zeta_{\bar{C}}(\mathbb{L})}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}+\\
&+\frac{\mathbb{L}^{3(\delta-n_0+2)-5g}\zeta_{\bar{C}}\big(\mathbb{L}^2\big)}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}.
\end{aligned}
$$

Now we use the 1-form

$$
\omega_4=\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^{-1}x)(1-\mathbb{L}^{-2}x)x^{\delta-n_0+1}}\mathrm{d}x
$$

in $\mathbb{P}^1_{\bar{\mathcal{K}}}$, and its residue at $x=0$ is the coefficient of x^0 of the function

$$
f(x)=\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^{-1}x)(1-\mathbb{L}^{-2}x)x^{\delta-n_0}}.
$$

As before, the Residue theorem allows us to compute the residue at $x=0$ as follows:

$$
\begin{aligned}
\operatorname{Res}_{x=0}\omega_4=&-\operatorname{Res}_{x=1}\omega_4-\operatorname{Res}_{x=\mathbb{L}^{-1}}\omega_4-\operatorname{Res}_{x=\mathbb{L}}\omega_4-\operatorname{Res}_{x=\mathbb{L}^2}\omega_4\\
=&-\frac{\zeta_{\bar{C}}(1)}{(1-\mathbb{L})(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-2})}(-1)\operatorname{Res}_{x=1}\frac{\mathrm{d}(-x)}{1-x}-\\
&-\frac{\mathbb{L}^{\delta-n_0+1}\zeta_{\bar{C}}\big(\mathbb{L}^{-1}\big)}{(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-3})}\frac{-1}{\mathbb{L}}\operatorname{Res}_{x=\mathbb{L}^{-1}}\frac{\mathrm{d}(-\mathbb{L}x)}{1-\mathbb{L}x}-\\
&-\frac{\zeta_{\bar{C}}(\mathbb{L})}{\mathbb{L}^{\delta-n_0+1}(1-\mathbb{L})(1-\mathbb{L}^2)(1-\mathbb{L}^{-1})}\frac{-1}{\mathbb{L}^{-1}}\operatorname{Res}_{x=\mathbb{L}}\frac{\mathrm{d}(-\mathbb{L}^{-1}x)}{1-\mathbb{L}^{-1}x}-\\
&-\frac{\zeta_{\bar{C}}\big(\mathbb{L}^2\big)}{\mathbb{L}^{2(\delta-n_0+1)}(1-\mathbb{L}^2)(1-\mathbb{L}^3)(1-\mathbb{L})}\frac{-1}{\mathbb{L}^{-2}}\operatorname{Res}_{x=\mathbb{L}^2}\frac{\mathrm{d}(-\mathbb{L}^{-2})}{1-\mathbb{L}^{-2}x}\\
=&-\frac{\mathbb{L}^3\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}+\frac{\mathbb{L}^{\delta-n_0+6-g}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}+\frac{\zeta_{\bar{C}}(\mathbb{L})}{\mathbb{L}^{\delta-n_0-1}(\mathbb{L}-1)^2(\mathbb{L}^2-1)}-\\
&-\frac{\zeta_{\bar{C}}\big(\mathbb{L}^2\big)}{\mathbb{L}^{2(\delta-n_0)}(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}.
\end{aligned}
$$

And finally, the last computation with residues. In this case, the 1-form is

$$
\omega_5=\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^2x)(1-\mathbb{L}^{-1}x)x^{\delta-n_0+1}}\mathrm{d}x
$$

and clearly

$$
\operatorname{Res}_{x=0}\omega_5=\operatorname*{coeff}_{x^0}\frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^2x)(1-\mathbb{L}^{-1}x)x^{\delta-n_0}}.
$$

The Residue theorem yields the following equalities:

$$
\begin{aligned}
\operatorname{Res}_{x=0}\omega_5=&-\operatorname{Res}_{x=1}\omega_5-\operatorname{Res}_{x=\mathbb{L}^{-1}}\omega_5-\operatorname{Res}_{x=\mathbb{L}^{-2}}\omega_5-\operatorname{Res}_{x=\mathbb{L}}\omega_5\\
=&-\frac{\zeta_{\bar{C}}(1)}{(1-\mathbb{L})(1-\mathbb{L}^2)(1-\mathbb{L}^{-1})}(-1)\operatorname{Res}_{x=1}\frac{\mathrm{d}(-x)}{1-x}-\\
&-\frac{\mathbb{L}^{\delta-n_0+1}\zeta_{\bar{C}}\big(\mathbb{L}^{-1}\big)}{(1-\mathbb{L}^{-1})(1-\mathbb{L})(1-\mathbb{L}^{-2})}\frac{-1}{\mathbb{L}}\operatorname{Res}_{x=\mathbb{L}^{-1}}\frac{\mathrm{d}(-\mathbb{L}x)}{1-\mathbb{L}x}-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\mathbb{L}^{2(\delta-n_0+1)}\zeta_{\bar{C}}(\mathbb{L}^{-2})}{(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-3})}\frac{-1}{\mathbb{L}^2}\operatorname{Res}_{x=\mathbb{L}^{-2}}\frac{\mathrm{d}(-\mathbb{L}^2x)}{1-\mathbb{L}^2x}-\\
&-\frac{\zeta_{\bar{C}}(\mathbb{L})}{\mathbb{L}^{\delta-n_0+1}(1-\mathbb{L})(1-\mathbb{L}^2)(1-\mathbb{L}^3)}\frac{-1}{\mathbb{L}^{-1}}\operatorname{Res}_{x=\mathbb{L}}\frac{\mathrm{d}(-\mathbb{L}^{-1}x)}{1-\mathbb{L}^{-1}x}\\
&=\frac{\mathbb{L}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}-\frac{\mathbb{L}^{\delta-n_0+3-g}\zeta_{\bar{C}}(1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}+\frac{\mathbb{L}^{2(\delta-n_0+3)-3g}\zeta_{\bar{C}}(\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}-\\
&-\frac{\zeta_{\bar{C}}(\mathbb{L})}{\mathbb{L}^{\delta-n_0}(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\\
&=\frac{1}{\mathbb{L}^{\delta-n_0}}\frac{\mathbb{L}^{3(\delta-n_0+2-g)}-1}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\zeta_{\bar{C}}(\mathbb{L})-\mathbb{L}\frac{\mathbb{L}^{\delta-n_0+2-g}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\zeta_{\bar{C}}(1)\,.
\end{aligned}
$$

We plug these equalities in the formula of the class $[\mathcal{N}^s_{\sigma_m^+}(3,1,d_1,d_2)]$ (see Theorem 4.6) for $\sigma=\sigma_m=\frac{d_1}{3}-d_2$ and we get

$$
\begin{aligned}
[\mathcal{N}_{\sigma_m^+}(3,1,6g-5,0)] = \zeta_{\bar{C}}(1)^2\,\frac{\mathbb{L}^{3(1+\sigma_m-g)}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\Big(&\mathbb{L}^{3g-1}(1+\mathbb{L}+\mathbb{L}^2)\zeta_{\bar{C}}(1)^2-\\
&-\mathbb{L}^{2g-1}(1+\mathbb{L})^2\zeta_{\bar{C}}(1)\,\zeta_{\bar{C}}(\mathbb{L})+\zeta_{\bar{C}}(\mathbb{L}^2)\Big)
\end{aligned}
$$

On the other hand, it is possible to compute the same class $[\mathcal{N}^s_{\sigma}(3,1,d_1,d_2)]$ for these values of d_1 and d_2. For values of d_1 not a multiple of 3 we can apply Proposition 2.5. This gives the class

$$
[\mathcal{N}_{\sigma_m^+}(3,1,d_1,d_2)] = \mathbb{M}_3^{\epsilon}\mathbb{J}\frac{\mathbb{L}^{d_1-3d_2+3(g-1)}-1}{\mathbb{L}-1},
$$

Joining these two expressions for $[\mathcal{N}^s_{\sigma_m^+}(3,1,d_1,d_2)]$ we have

Theorem 4.7. *The class of the moduli space of bundles of rank* 3 *and degree not multiple of* 3 *is*

$$
\begin{aligned}
\mathbb{M}_3^{\epsilon} = \zeta_{\bar{C}}(1)\,\frac{1}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\Big(&\mathbb{L}^{3g-1}(1+\mathbb{L}+\mathbb{L}^2)\zeta_{\bar{C}}(1)^2-\\
&-\mathbb{L}^{2g-1}(1+\mathbb{L})^2\zeta_{\bar{C}}(1)\,\zeta_{\bar{C}}(\mathbb{L})+\zeta_{\bar{C}}(\mathbb{L})\,\zeta_{\bar{C}}(\mathbb{L}^2)\Big)
\end{aligned}
$$

5 Appendix: A Mathematica sheet

To compute these class one can use some mathematical software. We choose Mathematica but a very similar tool is Wxmaxima, which is an open source solution.

■ The class of moduli of pairs of rank 3 Firstly, we first put some basic definition which makes easier writing the class of the strata. The label or name of these commands or shorthands are clear and it is related with this definition.

```
SetAttributes[g, Constant];
Element[g, Integers && g > 2];
Element[m, Integers];
Element[d1, Integers];
Element[d2, Integers];
```

The function `J[k,t]` is defined by $(1+tu^k)^g(1+tv^k)^g$. In this way, we have

$$
\begin{aligned}
J[0,t] &= \mathbb{J} \text{ or } \zeta_{\bar{C}}(1)\,,\\
J[-1,2] &= \delta^2(\zeta_{\bar{C}}(1)),\\
J[1,t^s] &= \zeta_{\bar{C}}(\mathbb{L}^s)\,.
\end{aligned}
$$

```
P[g_] := (t^(g)-1)/(t-1)

(*Variedad de Stiefel*)
V[k_, N_] := Product[t^N-t^i,{i, 0, k-1}]

(*Grassmaniano*)
Gr[k_, N_] := Product[(1-t^(N-i))/(1-t^(i+1)), {i, 0, k-1}]

(*Moduli stable bundles rk=2 degree even*)
M2[0] = 1/(2*(1-t)*(1-t^2))*(2*J[1, 1]*J[1, t] -
     J[1, 1]^2*(1+2*t^(g+1)-t^2)-J[2,-1]*(1-t)^2);

(*Moduli stable bundles rk=2 degree odd*)
M2[1] = (J[1, 1]*J[1, t]-t^g*J[1, 1]^2)/((1-t)*(1-t^2)) // Simplify;

(* Moduli of pairs of rank 2 *)
N2 = (N1*(1-t^\[Alpha]/(1-x/t)+(t^(\[Beta]+2)x)/(1-t^2 x))J[1, 1])/(-1 + t);

Sym2[f_] :=
 Simplify[1/2*(f^2 + (f /. {t -> t^2, J[k_, t_] -> J[2*k, -(t)^2]})),
  g \[Element] Integers]

(* \[Chi](T'',T') *)
\[Chi] = (1 - g)*(nn[1]*n[1] + nn[2]*n[2] - nn[2]*n[1]) + nn[1]*d[1] -
    n[1]*dd[1] + nn[2]*d[2] - n[2]*dd[2] - nn[2]*d[1] + n[1]*dd[2];

(* We have written dd[1]=d1-(3-N-k)*m/2 *)
a[N_, k_] := -\[Chi] /. {nn[1] -> N - k, nn[2] -> 1,
   dd[1] -> d1 - (3 - N + k)*m/2, dd[2] -> d2, n[1] -> k,
   d[1] -> k*m/2, d[2] -> 0, n[2] -> 0}
b[N_, k_] := -\[Chi] /. {n[1] -> N - k, n[2] -> 1,
   d[1] -> d1 - (3 - N + k)*m/2, d[2] -> d2, nn[1] -> k,
   dd[1] -> k*m/2, dd[2] -> 0, nn[2] -> 0}
```

These last two definitions are related with the dimension of the cohomology groups. The formula can be found in Proposition 2.18. The formula can be reduced as follows:

```
a[N_, k_] := k*\[Alpha] + k*(N - k - 1)*(g - 1);
b[N_, k_] := k*\[Beta] + k*(N - k - 1)*(g - 1);
```

where

$$\alpha = \texttt{a[2,1]} = d_1 - d_2 - m,$$
$$\beta = \texttt{b[2,1]} = g - 1 - d_1 + \frac{m}{2}.$$

Using this notation we define the class of $\mathcal{N}^s_{\sigma_n}(2,1,d_1,d_2)$:

```
N2 = (N1*(1-t^\[Alpha]/(1-x/t) + (t^\[Beta]x)/(1-t^2 x))*J[1, 1])/(t-1)
```

and then, we write the classes:

```
S[3, 1] = P[a[3, 1]]*J[1, 1]*N2;
SS[3, 1] = P[b[3, 1]]*J[1, 1]*N2;

(* 0 1 / 1 0 *)
S[2, 1] = (P[a[3, 1]]-P[a[2, 1]])*P[a[2, 1]]*(J[1, 1]^2-J[1, 1])*N1;
SS[2, 1] = (P[b[3, 1]]-P[b[2, 1]])*P[b[2, 1]]*(J[1, 1]^2-J[1, 1])*N1;

(* 1 / 1 *)
S[2, 2] = (P[a[3, 1]]-P[a[2, 1]-1])*P[a[2, 1]]*J[1, 1]*N1;
SS[2, 2] = (P[b[3, 1]]-P[b[2, 1]-1])*P[b[2, 1]]*J[1, 1]*N1;
```

```
(* 1 1 *)
S[2, 3] =
  Refine[(Sym2[P[a[2, 1]]*J[1, 1]] - Sym2[P[a[2, 1]]]*J[1, 1])*N1,
    t > 0] // Simplify;
SS[2, 3] =
  Refine[(Sym2[P[b[2, 1]]*J[1, 1]] - Sym2[P[b[2, 1]]]*J[1, 1])*N1,
    t > 0] // Simplify;
(* 2 *)
S[2, 4] = Gr[2, a[2, 1]] *J[1, 1]*N1;
SS[2, 4] = Gr[2, b[2, 1]]*J[1, 1]*N1;
```

Here, we modify the class of the fifth stratum. This stratum is different in even case or odd case. But we write the "purely" even part and the part that appears for any n integer number.

```
(* 1_ 2  n even *)
S[2, 5] = P[a[3, 2]]*(M2[0] - M2[1])*N1;
SS[2, 5] = P[b[3, 2]]*(M2[0] - M2[1])*N1;
(* 1_2 any n *)
Sb[2, 5] = P[a[3, 2]]*M2[1]*N1;
SSb[2, 5] = P[b[3, 2]]*M2[1]*N1;
```

The next step is sum all together as follows:

```
TheSum[m_] := (Sum[SS[2, k]-S[2, k], {k, 1, 5}] + SS[3, 1]-S[3, 1])/ x^\[Alpha]
               /. {\[Alpha] -> d1-d2-m, \[Beta] -> g-1-d1+3*m/2, N1 -> NN1};
TheSumb[m_] := (SSb[2, 5] - Sb[2, 5])/ x^\[Alpha]
               /. {\[Alpha] -> d1-d2-m, \[Beta] -> g-1-d1+3*m/2, N1 -> NN1};
```

The class `TheSum[m]` is the class of $\mathcal{S}_{\sigma_n}$ in case n even *except* the part of `SSb[2,5]-Sb[2,5]`; `TheSumb[m]` is the class of $\mathcal{S}_{\sigma_n}$ for n odd, with the difference that we sum this along any n integer number.

We are ready to sum this class from $m = n_0$ (or $m = \bar{n}_0$) to infinity

```
XX = Sum[TheSum[m], {m, m0, Infinity, 2}] // FullSimplify
XXb = Sum[TheSumb[m], {m, n0, Infinity}] // FullSimplify
```

Here, we obtain without any great effort the class of $\mathcal{N}_{\sigma_n^-}(3, 1, d_1, d_2)$. We just have to work a bit to put this formula in a nice way. On one hand we have the raw output of `XXb`:

$$\frac{t^{-2(1+d_1+d_2+n_0)}x^{-d_1+d_2+n_0}\left(t^{2(d_2+g)+3n_0}\left(t^2-x\right)+t^{4+4d_1}(-1+tx)\right)\mathbb{J}\left(t^g\mathbb{J}-\zeta_{\bar{C}}(\mathbb{L})\right)}{(-1+t)^3(1+t)\left(t^2-x\right)(-1+tx)}[\mathcal{N}(1,1,d_1-m/2,d_2)]$$

which corresponds with the first part of the obtained formula. On the other hand, we have a quite large output of `XX`, but using the `Apart` command, we chop it in three pieces, and we have

$$\frac{t^{-2d_1+3(-1+g+\bar{n}_0)}x^{-d_1+d_2+\bar{n}_0}}{(-1+t)^2(1+t)\left(-1+t^2x\right)\left(-1+t^3x\right)},\quad \frac{t^{-1-d_2+2g+\frac{\bar{n}_0}{2}}x^{-d_1+d_2+\bar{n}_0}}{(-1+t)^2(t-x)\left(-1+t^2x\right)},\quad \frac{t^{3+2d_1-2d_2+g-2\bar{n}_0}x^{-d_1+d_2+\bar{n}_0}}{(-1+t)^2(1+t)(t-x)\left(t^2-x\right)}$$

which corresponds with the previous computation.

■ **The stable part of moduli of pairs on a critical value** It is possible to continue this worksheet to compute the stable subset of the moduli of pairs of rank 3. Fix a critical point σ_n. According to the equality (see Lemma 2.13)

$$[\mathcal{N}^s_{\sigma_n}] = [\mathcal{N}_{\sigma_n^-}(2,1,d_1,d_2)] - [\mathcal{S}^-_{\sigma_n}]$$

that is, we have to remove to the class of $[\mathcal{N}_{\sigma_n^-}(2,1,d_1,d_2)]$ the last class of $[\mathcal{S}^-_{\sigma_n}]$ in the sum before done. Observe that the class of $[\mathcal{S}^-_{\sigma_n}]$ depends on whether n is even or not.

The big task has been done in the previous worksheet, we only have to add a few lines. Write the following:

```
Sp[m_] = (Sum[S[2, k],{k, 1, 5}]+S[3, 1])/x^\[Alpha]
         /.{\[Alpha] -> d1-d2-m,\[Beta] -> g-1-d1+3*m/2,N1 -> NN1}//FullSimplify
```

```
Sm[m_] = (Sum[SS[2, k], {k, 1, 5}] + SS[3, 1])/x^\[Alpha]
          /.{\[Alpha] -> d1-d2-m,\[Beta] -> g-1-d1+3*m/2,N1 -> NN1}//FullSimplify
```

`Sp[m]` is the class of $[\mathcal{S}^-_{\sigma_n}]$ when n is even. In the case n odd we define:

```
Spb[m_] := Sb[2, 5]/x^\[Alpha]
           /.{\[Alpha] -> d1-d2-m,\[Beta] -> g-1-d1+3*m/2,N1 -> NN1}//FullSimplify
Smb[m_] := SSb[2, 5]/ x^\[Alpha]
           /.{\[Alpha] -> d1-d2-m,\[Beta] -> g-1-d1+3*m/2,N1 -> NN1}//FullSimplify
```

For the even case, we compute the difference as follows:

```
N3sEven = XX - Sp[m0]/(NN1*J[1, 1]^2) // FullSimplify
```

Here, we prefer to do the sum in two steps. On one hand, the one corresponding to the even part (in our case, `XX` class): the class `Sp[m]` corresponds to the even case. After, we deal with the `Spb[b]` variable, which corresponds to `XXb` class.

The raw output deserves a bit work. You can use the `Apart` command as follows

```
aux = t^(-3-2*d1-2*d2-2*m0)*x^(-d1 + d2 + m0)/((-1+t)^2*(t-x)*(-1+t^2*x));
N3sEven/aux // Apart
```

and we get a nice expression for the class of $\mathcal{N}^s_{\sigma_n}(3,1,d_1,d_2)$ which needs a small amount of effort to get a reasonable expression. We have

$$\begin{aligned}\texttt{N3sEven} = \frac{\mathbb{L}^{g-1}[\mathcal{N}(1,1,d_1-\bar{n}_0,d_2)]}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\left(\frac{\mathbb{L}^{2a+1}}{(1-\frac{x}{\mathbb{L}})(1-\frac{x}{\mathbb{L}^2})}+\frac{\mathbb{L}^{2b}x}{(1-\mathbb{L}^2x)(1-\mathbb{L}^3x)}-\mathbb{L}\right)-\\ -\frac{[\mathcal{N}(1,1,d_1-\bar{n}_0,d_2)]}{(\mathbb{L}-1)^2}\left(\frac{\mathbb{L}^{a+b+g-1}x}{(1-\frac{x}{\mathbb{L}})(1-\mathbb{L}^2x)}+\frac{\mathbb{L}^bx}{1-\mathbb{L}^2x}+\frac{\mathbb{L}^a}{1-\frac{x}{\mathbb{L}}}-1\right)\end{aligned}$$

Analogous tricks works for the “odd” part. We define the difference:

```
N3sodd = XXb - Spb[n0] // FullSimplify
```

and we work a bit the raw output to get a nice expression. This is

$$\operatorname*{coeff}_{x^0}\frac{[\mathcal{N}^s_{\sigma_c}(1,1,d_1-n_0,d_2)]}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\left\{\frac{\mathbb{L}^{2a}}{1-\frac{x}{\mathbb{L}^2}}-\frac{\mathbb{L}^{2b+1}x}{1-\mathbb{L}^3x}-1\right\}.$$

Then,

Proposition 4.8. *The class of $\mathcal{N}^s_{\sigma_n}(3,1,d_1,d_2)$ where σ_n is a critical value and n is an odd integer number is*

$$\begin{aligned}[\mathcal{N}^s_{\sigma_c}(3,1,d_1,d_2)] = \operatorname*{coeff}_{x^0}\frac{[\mathcal{N}^s_{\sigma_c}(1,1,d_1-n_0,d_2)]}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\left\{\frac{\mathbb{L}^{2a}}{1-\frac{x}{\mathbb{L}^2}}-\frac{\mathbb{L}^{2b+1}x}{1-\mathbb{L}^3x}-1\right\}+\\ +\operatorname*{coeff}_{x^0}\frac{[\mathcal{N}^s_{\sigma_c}(1,1,d_1-\bar{n}_0,d_2)]}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\left\{\frac{\mathbb{L}^{2a+1}}{(1-\frac{x}{\mathbb{L}^2})(1-\frac{x}{\mathbb{L}})}+\frac{\mathbb{L}^{2b}}{(1-\mathbb{L}^3x)(1-\mathbb{L}^2x)}-\frac{(1+\mathbb{L})\mathbb{L}^{a+b}}{(1-\mathbb{L}^2x)(1-\frac{x}{\mathbb{L}})}\right\}.\end{aligned}$$

If n is an even integer number, the class of $\mathcal{N}^s_{\sigma_n}(3,1,d_1,d_2)$ is

$$\begin{aligned}[\mathcal{N}^s_{\sigma_c}(3,1,d_1,d_2)] = \operatorname*{coeff}_{x^0}\frac{[\mathcal{N}^s_{\sigma_c}(1,1,d_1-n_0,d_2)]}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\left\{\frac{\mathbb{L}^{2a}}{1-\frac{x}{\mathbb{L}^2}}-\frac{\mathbb{L}^{2b+1}x}{1-\mathbb{L}^3x}-1\right\}+\\ +\frac{\mathbb{L}^{g-1}[\mathcal{N}(1,1,d_1-\bar{n}_0,d_2)]}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\left(\frac{\mathbb{L}^{2a+1}}{(1-\frac{x}{\mathbb{L}})(1-\frac{x}{\mathbb{L}^2})}+\frac{\mathbb{L}^{2b}x}{(1-\mathbb{L}^2x)(1-\mathbb{L}^3x)}-\mathbb{L}\right)-\\ -\frac{[\mathcal{N}(1,1,d_1-\bar{n}_0,d_2)]}{(\mathbb{L}-1)^2}\left(\frac{\mathbb{L}^{a+b+g-1}x}{(1-\frac{x}{\mathbb{L}})(1-\mathbb{L}^2x)}+\frac{\mathbb{L}^bx}{1-\mathbb{L}^2x}+\frac{\mathbb{L}^a}{1-\frac{x}{\mathbb{L}}}-1\right)\end{aligned}$$

Here, $a = d_1-d_2-n_0$ and $b = g-1-d_1+\frac{3}{2}n_0$; the integer $n_0 = \frac{\sigma_n+d_1+d_2}{2} = n$ and $\bar{n}_0$ is the even integer number such that $n_0 \le \bar{n}_0 \le n_0+1$.

■ The class of stable bundles of rank 3 We can use the same worksheet to compute the class of $M^s(3,1)$. To do this, we have to rewrite a bit the class of `XX` and `XXb` to our convenience. Define the functions `F3[a]` and `F4[a,b]` as follows:

$$\texttt{F3[a]} = \frac{[\mathcal{N}(1,1,d_1-n_0,d_2)]}{1-\mathbb{L}^a x},$$
$$\texttt{F4[a,b]} = \frac{[\mathcal{N}(1,1,d_1-n_0,d_2)]}{(1-\mathbb{L}^a x)(1-\mathbb{L}^b x)}.$$

Indeed, `XX` should be

```
N3a = J[1, 1]*(t^g*J[1, 1]-J[1, t])/((-1+t)^3*(1+t))*
      (t^(2*\[Alpha])*F3[-2]-t^(2*\[Beta])*F3[3]);
```

and `XXb` is

```
N3b = t^(g-1)*J[1, 1]^2/((t-1)*(t^2-1))*
      (t^(2*\[Beta])*F4[2, 3]+t^(2*\[Alpha]+1)*F4[-1,-2]
      -(1+t)*t^(\[Alpha]+\[Beta])*F4[-1, 2])
```

Recall that the class of $\mathcal{N}(1,1,d_1-n,d_2)$ is

$$\lambda^{d_1-d_2-n_0}(C)\mathbb{J},$$

so using the function $\zeta_{[C]}(x)$ we can rewrite this as

$$[\mathcal{N}(1,1,d_1-n,d_2)] = \mathbb{J} \operatorname*{coeff}_{x^{d_1-d_2-n_0}} \zeta_{[C]}(x) = \mathbb{J}\operatorname*{coeff}_{x^0} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)x^{d_1-d_2-n_0}}.$$

Thus, functions `F3` and `F4` should be

$$\texttt{F3[a]} = \mathbb{J}\operatorname*{coeff}_{x^0} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^a x)x^{d_1-d_2-n_0}},$$
$$\texttt{F4[a,b]} = \mathbb{J}\operatorname*{coeff}_{x^0} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^a x)(1-\mathbb{L}^b x)x^{d_1-d_2-n_0}},$$

Now, we work to rewrite this in the `Mathematica` language.

First, we have to compute the coefficient of x^0 in these expressions. We do not trust in `Coefficient` or `CoefficientList` commands built in `Mathematica`; we construct our tools to compute such classes. We use the Residue theorem to compute this as follows.

To compute the coefficient of x^0 of the function

$$f(x) = \frac{\zeta_{\bar{C}}(x)}{\prod_{i=1}^n (1-\mathbb{L}^{k_i}x)x^N}$$

we define the 1-form in $\mathbb{P}^1_{\mathcal{K}}$

$$\omega = \frac{f(x)}{x}\mathrm{d}x = \frac{\zeta_{\bar{C}}(x)}{\prod_{i=1}^n (1-\mathbb{L}^{k_i}x)x^{N+1}}\mathrm{d}x.$$

Thus,

$$\operatorname*{coeff}_{x^0} f(x) = \mathrm{Res}_{x=0}\,\omega = -\sum_{i=1}^n \mathrm{Res}_{x=\mathbb{L}^{-k_i}}\,\omega$$

where the last equality is due to the Residue Theorem. Nevertheless, we have to impose some assumptions to avoid a pole at infinity: by the change $x \mapsto \frac{1}{x}$ we have

$$\omega \mapsto \frac{\zeta_{\bar{C}}\left(\frac{1}{x}\right) x^{N+1}}{\prod_{i=1}^n (1-\frac{\mathbb{L}^{k_i}}{x})}\mathrm{d}\left(\frac{1}{x}\right) = \frac{\zeta_{\bar{C}}(x)\, x^{N+1-n}}{\prod_{i=1}^n (x-\mathbb{L}^{k_i})}\frac{1}{x^2}\mathrm{d}x.$$

Thus we have a pole at infinity if and only if $N-1-n-2g<0$; in other words, we always assume that $N \geq 2g+1-n$ to avoid poles at infinity of the 1-form ω.

Now, the poles at $x=\mathbb{L}^{-k_i}$ are simple poles (we assume that $k_i \neq k_j$ for $i \neq j$, $i,j=1,\dots,n$), so

$$\mathrm{Res}_{x=\mathbb{L}^{-k_i}}\,\omega = \lim_{x\to\mathbb{L}^{-k_i}} (1-\mathbb{L}^{k_i}x)\cdot f(x)$$

(where, we understand this limit as an evaluation in $x=\mathbb{L}^{-k_i}$) Then,

$$\underset{x^0}{\mathrm{coeff}}\, f(x) = -\sum_{i=1}^{n} f(\mathbb{L}^{-k_i}) = \sum_{i=1}^{n} \frac{\zeta_{\bar{C}}(\mathbb{L}^{-k_i})}{\prod_{j\neq i}(1-\mathbb{L}^{k_j-k_i})\mathbb{L}^{k_i N}}$$

Recall the equality $\zeta_{\bar{C}}(\mathbb{L}^{-k_i}) = \mathbb{L}^{2gk_i}\zeta_{\bar{C}}(\mathbb{L}^{k_i-1})$ to turn a negative exponent into a positive exponent. At this point, we can write this procedure as a function in `Mathematica`. Recall our notation that we use in this mathematical software: `t` means the class $\mathbb{L}$, the family of the Jacobian class $\zeta_{\bar{C}}(\mathbb{L}^k)$ are denoted by `J[1,t^k]`.

The following function computes the coefficient of x^0 of $f(x)$:

```
Resf[i_, k_, NN_] := If[k[[i]] <= 0,-1/t^k[[i]]*J[1, t^(-k[[i]])]*
t^(Sum[k[[i]]-k[[j]], {j,1,i-1}])/(t^(-k[[i]]*(NN+1))*
Product[t^(k[[i]]-k[[j]])-1,{j, 1, i-1}]*
Product[1-t^(k[[j]]-k[[i]]),{j, i+1, Length[k]}]),
1/t^k[[i]]*J[1,t^(k[[i]]-1)]*t^(k[[i]]*(NN+1))*
t^(Sum[k[[i]]-k[[j]], {j, 1, i-1}])/(Product[t^(k[[i]]-k[[j]])-1,{j, 1, i-1}]*
Product[1-t^(k[[j]]-k[[i]]),{j, i+1, Length[k]}]*t^(-(2*(-k[[i]])+1)*g))]
```

Observe the `If` statement is used to switch the negative exponent of $\zeta_{\bar{C}}(\mathbb{L}^k)$ into a positive one. Now, the functions `F3` and `F4` are defined by

```
F3[a_] := Sum[-Resf[i, Sort[{a, 0, 1}], N], {i, 1, 3}]
F4[a_, b_] := Sum[-Resf[i, Sort[{a, b, 0, 1}], N], {i , 1, 4}]
```

Here, `N` is $d_1-d_2-n_0$. Observe that by the definition of `Resf`, we have to use the `Sort` command to order the list `{a, 0, 1}` or `{a, b, 0, 1}`.

Our next task is to set some constants. The most important constant is n_0 and $\bar{n}_0$ and they are given by the critical value. In this case, the critical value is $\sigma_m^+ = \mu = \frac{d_1}{3} - d_2$. Recall that

$$n_0 = \left[\frac{\sigma_m^+ + d_1 + d_2}{2}\right] + 1 = \left[\frac{2}{3}d_1\right] + 1,$$

where brackets represent the integer part of the rational number inside them, that is, the floor function. We work for degrees d_1 not multiple of 3. So we have two cases, $d_1 \equiv 1 \pmod 3$ or $d_2 \equiv 2 \pmod 3$.

For $d \equiv 1 \pmod 3$, we have $n_0 = \frac{2}{3}d_1 + \frac{1}{3}$, and $\bar{n}_0 = \frac{2}{3}(d_1+2)$ (the lowest even integer number greater than n_0); so

$$\begin{cases} a = d_1 - d_2 - n_0 = \mu - \dfrac{1}{3}, \\ b = g - 1 - d_1 + \dfrac{3}{2}n_0 = g - \dfrac{1}{2} \end{cases} \qquad \begin{cases} \bar{a} = d_1 - d_2 - \bar{n}_0 = \mu - \dfrac{4}{3}, \\ \bar{b} = g - 1 - d_2 + \dfrac{2}{3}\bar{n}_0 = g + 1 \end{cases}$$

In this case, we redefine the functions

```
F3[a_] := Sum[-Resf[i, Sort[{a, 0, 1}], m-1/3], {i, 1, 3}] // Simplify
F4[a_, b_] := Sum[-Resf[i, Sort[{a, b, 0, 1}], m-4/3], {i , 1, 4}] // Simplify
```

(here `m` denotes the constant μ) and we define

```
N3a = J[1, 1]*(t^g*J[1, 1]-J[1, t])/((-1+t)^3*(1+t))*
      (t^(2*\[Alpha])*F3[-2]-t^(2*\[Beta])*F3[3])
      /. {\[Alpha] -> m-1/3, \[Beta] -> g-1/2}
N3b = t^(g-1)*J[1, 1]^2/((t-1)*(t^2-1))*(t^(2*\[Beta])*F4[2, 3] +
      t^(2*\[Alpha]+1)*F4[-1, -2]-(1+t)*t^(\[Alpha]+\[Beta])*F4[-1, 2])
```

```
        /. {\[Alpha] -> m-4/3, \[Beta] -> g+1}
```

and then Mathematica magically computes the class of $\mathcal{N}_{\sigma_m^+}(3,1,d_1,d_2)$ for $d_1 \equiv 1 \pmod 3$:

```
N3 = N3a + N3b;
CoefficientList[N3, J[1, t^2]] // FullSimplify
CoefficientList[%[[1]], J[1, t]] // FullSimplify
CoefficientList[%[[1]], J[1, 1]] // FullSimplify
```

The output is

$$-\frac{t^{-3g}\left(t^{3g}-t^{3+3m}\right)J[1,1]J[1,t]}{(-1+t)^5(1+t)^2\left(1+t+t^2\right)},\quad \frac{t^{-1-g}\left(t^{3g}-t^{3+3m}\right)J[1,1]^2}{(-1+t)^5\left(1+t+t^2\right)},\quad \frac{-t^{3g}+t^{3+3m}}{(-1+t)^5t(1+t)^2},$$

which are the coefficients of $\zeta_{\bar{C}}(\mathbb{L}^2)$, $\zeta_{\bar{C}}(\mathbb{L})$ and $\mathbb{J}^3$ respectively.

The same computations can be done with the case $d_1 \equiv 2 \pmod 3$. In this case, $n_0 = \frac{2}{3}(d_1+1)$, and $\bar{n}_0 = n_0$ because of n_0 clearly is even. In this case

$$\begin{cases} a = d_1 - d_2 - n_0 = \mu - \dfrac{2}{3}, \\ b = g - 1 - d_1 + \dfrac{3}{2}n_0 = g. \end{cases}$$

Since $n_0 = \bar{n}_0$, there is not need to distinguish between a and $\bar{a}$ and so on. The code is the following.

```
F3[a_] := Sum[-Resf[i, Sort[{a, 0, 1}], m-2/3], {i, 1, 3}] // Simplify
F4[a_, b_] := Sum[-Resf[i, Sort[{a, b, 0, 1}], m-2/3], {i , 1, 4}] // Simplify
N3 = J[1, 1]*(t^g*J[1, 1]-J[1, t])/((-1+t)^3*(1+t))*(t^(2*\[Alpha])*F3[-2]
     -t^(2*\[Beta])*F3[3])+t^(g-1)*J[1, 1]^2/((t-1)*(t^2-1))*
     (t^(2*\[Beta])*F4[2, 3]+t^(2*\[Alpha]+1)*F4[-1, -2]-
     (1+t)*t^(\[Alpha]+\[Beta])*F4[-1, 2])
     /. {\[Alpha] -> m-2/3, \[Beta] -> g} // Simplify
```

If we run the code

```
N3 = N3a + N3b;
CoefficientList[N3, J[1, t^2]] // FullSimplify
CoefficientList[%[[1]], J[1, t]] // FullSimplify
CoefficientList[%[[1]], J[1, 1]] // FullSimplify
```

we will obtain the same class for $\mathcal{N}_{\sigma_m^+}(3,1,d_1,d_2)$.

This gives almost automatically the class of $M^s(3,1)$.

$$[M^s(3,1)] = \frac{\mathbb{J}}{\mathbb{L}-1}\left(\frac{\mathbb{L}^{3g-1}\mathbb{J}^2}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2} + \frac{\mathbb{L}^{2g-1}\mathbb{J}\zeta_{\bar{C}}(\mathbb{L})}{(\mathbb{L}-1)^2(\mathbb{L}^3-1)} - \frac{\zeta_{\bar{C}}(\mathbb{L})\,\zeta_{\bar{C}}(\mathbb{L}^2)}{(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\right)$$

Remark 4.9. *There is a mistake in Mn08, Theorem 7.1: there is an extra* $e(\mathbb{J}) = (1+u)^g(1+v)^g$.

Chapter 5

The Moduli of stable bundle of rank 3

In the previous chapter the class $M^s(3,d)$ for degree d and 3 coprime was computed. Indeed, we used the Mathematica software to avoid some boring operations. Here, we compute the class of $M^s(3,0)$. This work requires a greater effort because of the number of the strata of $\mathcal{S}^+_{\sigma_m}$ increases a lot. Furthermore, some strata are locally trivial projective bundles but not locally Zariski trivial: this occurs when $M^s(2,0)$ was computed. To do this, we recover the definition of $\bar{K}(\mathfrak{V}\mathrm{ar}_{\mathbb{C}})$.

Taking into account these differences, the computation of $M^s(3,0)$ uses analogous arguments as those developed in previous chapters.

1 Introduction

As we have advanced in the abstract, the computation of $M^s(3,0)$ involves a great effort which makes that this chapter occupies a lot of pages. We present the procedure. This should be taken as an index.

1) We define a new Grothendieck group $K_0'(\mathfrak{V}\mathrm{ar})$.
2) We list all possible strata which compound the flip locus $\mathcal{S}^+_{\sigma_m}$.
3) We compute the dimension of cohomology groups. After that, we do not mention it anymore: the reader should be able to find the correct formula that we have applied in each case.
4) We compute the class in $K_0'(\mathfrak{V}\mathrm{ar}_{\mathbb{C}})$ (or $K_0'(\mathfrak{M}ot)$) of every stratum.

   Here, we take a break. There is a way to compute the class of the flip locus $\mathcal{S}^+_{\sigma_m}$ from the scratch. We present some results and we give a new criterium to stratify the flip locus. In principle, this should not be related with the previous one, but it does! This also assures that the classes obtained are well computed.
5) Finally, we sum these strata. Mathematica helps us to give the correct conclusion of this chapter.

2 The situation

■ **A weaker Grothendieck group** It is well known that not all locally trivial complex bundle is a Zariski locally trivial bundle. For example, the universal bundle of $M^s(2,0)$ is not locally Zariski trivial but it is trivial locally in the usual complex topology. To solve this, we add the new following relation to the Grothendieck group

$$[P] \sim [\mathbb{P}^r][X] \text{ if and only if } P \to X \text{ is a locally trivial projective bundle of rank } r.$$

This new equivalence relation defines the group $\bar{K}_0(\mathfrak{V}\mathrm{ar}_{\mathbb{C}}) = K_0(\mathfrak{V}\mathrm{ar}_{\mathbb{C}})/\sim$. Analogously, we define $K_0'(\mathfrak{M}ot)$.

■ **Strata: all possible configurations** First, recall the formula

$$n' + \sum_{i=1}^{r} \sum_{j=1}^{b} a_i^j n_j = n, \tag{5.1}$$

where

- n' is the rank of the *initial triple* T' (in our case, $n'=0$);
- r is the length of the filtration, that is, how many steps we have to do to complete the stratum;
- b is the number of non-isomorphic bundles involved in the stratum;

- n_j is the rank of the j-th bundle;
- a_i^j is the number of times that the triple $S_j = (E_j, 0, 0)$ appears in the i-th step, that is, the bundle of the i-th step is $\bar{\bar{T}}_i = S_1^{a_i^1} \oplus \cdots \oplus S_b^{a_i^b}$;
- n is the rank of the triple T (in our case $n = 3$).

Any solution of this equation gives a possible stratum.

To compute all posible strata, we proceed by setting the length of the tower r, and then the number of non-isomorphic stable bundle b.

▪ $r = 1$ In this case, the equation (5.1) produces three subcases: the equality

$$\sum_{j=1}^{b} a_1^j n_j = 3$$

gives few possibilities. First, $b \leq 3$ and thus

▪▪ $b = 3$ there is only one possibility: $n_i = 1$ and $a_1^i = 1$, for $i = 1, 2, 3$. This is written in a matricial fashion as

$$X_1 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\ \mathbf{n} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{cases}$$

▪▪ $b = 2$ In this case, we have the equation

$$a_1^1 n_1 + a_1^2 n_2 = 3.$$

So one of these $a_1^i n_i$ must value 2: namely $a_1^1 n_1$. So we have two possibilities: if $a_1^2 = 2$, so we have the stratum

$$X_2 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 2 & 1 \end{pmatrix} \\ \mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix} \end{cases}$$

or $n_1 = 2$, so

$$X_3 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 1 \end{pmatrix} \\ \mathbf{n} = \begin{pmatrix} 2 & 1 \end{pmatrix} \end{cases}$$

▪▪ $b = 1$ We have the equation $a_1^1 n_1 = 3$, so we have two possibilities: The first one is the case $a_1^1 = 3$, so we have the stratum given by the matrix

$$X_4 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 3 \end{pmatrix} \\ \mathbf{n} = \begin{pmatrix} 1 \end{pmatrix} \end{cases}$$

the other case, $n_1 = 3$, gives the most relevant stratum

$$X_5 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 \end{pmatrix} \\ \mathbf{n} = \begin{pmatrix} 3 \end{pmatrix} \end{cases}$$

In the stratum X_5 it appears the class of $M^s(3, d)$, which we are interested on.

▪ $r = 2$ For this case, the equation sets $b \leq 3$:

▪▪ $b = 3$ As in the previous case, $b = 3$ implies that $n_j = 1$ for $j = 1, 2, 3$. Furtheremore, for $j = 1, 2, 3$, we have the equation $a_1^j + a_2^j = 1$, so one of these must be 0 (and hence the other 1). This gives two *non-isomorphic* strata. The first one is

$$X_6 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ \mathbf{n} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{cases}$$

and the second one is

$$X_7 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1 \quad 1\,) \end{cases}$$

▪ ▪ $b = 2$ Equation (5.1) is

$$(a_1^1 + a_2^1)n_1 + (a_1^2 + a_2^2)n_2 = 3$$

We distinguish between two possibilities: if $n_i = 1$ for $i = 1, 2$, then $a_1^i + a_2^i = 2$ for some $i = 1, 2$; suppose equality holds for the case $i = 1$, so we have several possibilities:

- the case $(a_1^1, a_1^2) = (2, 0)$: this forces $(a_2^1, a_2^2) = (0, 1)$ and the stratum is

$$X_8 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1\,) \end{cases}$$

- the case $(a_1^1, a_1^2) = (0, 2)$, so $(a_2^1, a_2^2) = (1, 0)$ and the stratum is

$$X_9 \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1\,) \end{cases}$$

- the case $(a_1^1, a_1^1) = (1, 1)$ produces two possibilities for (a_2^1, a_2^2):

$$\begin{cases} \text{for the case } (a_2^1, a_2^2) = (1, 0), \ X_{10} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1\,) \end{cases} \\ \text{for the case } (a_2^1, a_2^2) = (0, 1), \ X_{11} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1\,) \end{cases} \end{cases}$$

Finally, it remains the possibility $n_i = 2$ for some $i = 1, 2$. We can suppose $i = 1$, and the equation (5.1) is $a_i^j = 1$ for any i and j. In this case, the strata are given by

$$X_{12} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{n} = (\,2 \quad 1\,) \end{cases} \qquad X_{13} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1_2 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 2\,) \end{cases}$$

▪ ▪ $b = 1$ In this case equation (5.1) becomes

$$(a_1^1 + a_2^1)n_1 = 3.$$

Because nor a_1^1 neither a_1^2 can be zero, this implies $n_1 \neq 3$, hence $n_1 = 1$. This gives two possibilities

$$X_{14} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \mathbf{n} = (\,1\,) \end{cases} \qquad X_{15} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \mathbf{n} = (\,1\,) \end{cases}$$

▪ $r = 3$ We are in the last case. Equation (5.1) is now

$$\sum_{j=1}^{b} (a_1^j + a_2^j + a_3^j)n_j = 3, \tag{5.2}$$

As in previous cases, we study the possibilities by values of b:

▪ ▪ $b = 3$ As before, $n_1 = n_2 = n_3 = 1$ and hence $a_1^j + a_2^j + a_3^j = 1$. The unique possibility is

$$X_{16} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1 \quad 1\,) \end{cases}$$

▪ ▪ $b = 2$ In this case, we have two possibilities: because $n_1 = n_2 = 1$, there is no case with $n_1 = 2$, since this implies that one of the three rows must be 0. Thus, $a_1^j + a_2^j + a_3^j = 2$ for some j. We can suppose $j = 1$. Since r (the length of the filtration) is 3, all $a_i^j \leq 1$ and no row (and no column) must be 0, so there are three cases:

$$X_{17} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1\,) \end{cases} \quad X_{18} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1\,) \end{cases} \quad X_{19} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{n} = (\,1 \quad 1\,) \end{cases}$$

▪ ▪ $b = 1$ Here, the unique solution is

$$X_{20} \equiv \begin{cases} \mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \mathbf{n} = (\,1\,) \end{cases}$$

This list contains all twenty strata which form the flip locus $\mathcal{S}^+_{\sigma_m}$.

▪ Cohomology groups

We compute the dimensions of the Ext^1 cohomology groups which are going to appear frequently. These are:

$$\mathrm{Ext}^1(T', S_i), \quad \mathrm{Ext}^1(\tilde{T}_1, S_i), \quad \mathrm{Ext}^1(\tilde{T}_2, S_i),$$

where $T' = (0, L_0, 0)$, $\tilde{T}_1$ is in the middle of the short exact sequence of triples

$$0 \to S_1^{a_1^1} \oplus \cdots \oplus S_b^{a_b^1} \to \tilde{T}_1 \to T' \to 0 \tag{5.3}$$

and $\tilde{T}_2$ is in a similar short exact sequence:

$$0 \to S_1^{a_1^2} \oplus \cdots \oplus S_b^{a_b^2} \to \tilde{T}_2 \to \tilde{T}_1 \to 0 \tag{5.4}$$

These equalities shall be used along this chapter without further reference.

The first cohomology group is quite easy to compute using Riemann-Roch formula: the special shape of the triple T' yields the equivalence between $\mathrm{Ext}^1(T', S_i)$ and $\mathrm{Hom}(L_0, E_i) \cong H^0(L_0^* \otimes E_i)$. Now, by Riemann-Roch, we have

$$h^0(L_0^* \otimes E_i) - h^1(L_0^* \otimes E_1) = \mathrm{rk}\, E_i\,(1 - g) + \deg(L_0^* \otimes E_i).$$

On the one hand, using the well-known formula

$$\deg(E \otimes F) = \deg E \,\mathrm{rk}\, F + \mathrm{rk}\, E \deg F,$$

we have

$$\deg(L_0^* \otimes E_i) = n_i \frac{d_1}{n} - n_i d_2 = n_i \mu.$$

where $n_i = \operatorname{rk} E_i$ (thus $\deg E_i = n_i \frac{d_1}{n}$) and $\deg L_0 = d_2$ and $\operatorname{rk} L_0 = 1$. On the other hand, we usually take $d_1 - d_2 \geq 3(2g-1)$ so that $\deg(L_0^* \otimes E_i) \geq (2g-2)n_i$ and $h^1(L_0^* \otimes E_i) = 0$. So

$$\dim \operatorname{Ext}^1(T', S_i) = n_i(1-g) + n_i\mu.$$

The dimension of the second cohomology group $\operatorname{Ext}^1(\tilde{T}_1, S_j)$ is computed as follows: by the short exact sequence (5.3) we have the long exact sequence:

$$\begin{array}{c} 0 \longrightarrow \operatorname{Hom}(\bigoplus_{i=1}^b S_i^{a_i^1}, S_j) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1(T', S_j) \longrightarrow \operatorname{Ext}^1(\tilde{T}_1, S_j) \longrightarrow \operatorname{Ext}^1(\bigoplus_{i=1}^b S_i^{a_i^1}, S_j) \longrightarrow 0 \end{array}$$

So, $\chi(\tilde{T}_1, S_j) = \chi(T', S_j) + \chi(\bigoplus_{i=1}^b S_i^{a_i^1}, S_j)$ and $\chi(\tilde{T}_1, S_j) = -\dim \operatorname{Ext}^1(\tilde{T}_1, S_j)$. The number $\chi(T', S_j) = -\dim \operatorname{Ext}(T', S_j)$ was computed as $-n_j\mu - n_j(1-g)$. The computation of $\chi(S_i, S_j)$ is a direct application of Riemann-Roch.

$$\begin{aligned} \chi(S_i, S_j) =& \operatorname{rk}(E_i^* \otimes E_j)\,(1 - g + \deg(E_i^* \otimes E_j)) \\ =& n_i n_j (1-g) + (n_i n_j - n_j n_i) d_1 = n_i n_j (1-g), \end{aligned} \tag{5.5}$$

so $\chi(\bigoplus_{i=1}^b S_i^{a_i^1}, S_j) = \sum_{i=1}^b a_i^1 \chi(S_i, S_j) = n_j(\bar{n}_1 - 1)(1-g)$ where $\bar{n}_1 = \sum_{i=1}^b a_i^1 n_i$. Then

$$\dim \operatorname{Ext}^1(\tilde{T}_1, S_j) = n_j\mu - n_j(\bar{n}_1 - 1)(1-g).$$

The dimension of the last cohomology group is computed with the same tools than the previous paragraph: using the long exact sequence arisen from (5.4) we have

$$\begin{array}{c} 0 \longrightarrow \operatorname{Hom}(\bigoplus_{i=1}^b S_j^{a_j^2}, S_k) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1(\tilde{T}_1, S_k) \longrightarrow \operatorname{Ext}^1(\tilde{T}_2, S_k) \longrightarrow \operatorname{Ext}^1(\bigoplus_{i=1}^b S_j^{a_j^2}, S_k) \longrightarrow 0 \end{array}$$

thus $\dim \operatorname{Ext}^1(\tilde{T}_2, S_k) = -\chi(\tilde{T}_2, S_k) = -(\chi(\tilde{T}_1, S_j) + \chi(S_j, S_k))$ and

$$\dim \operatorname{Ext}^1(\tilde{T}_2, S_k) = n_k\mu - (\bar{n}_1 + \bar{n}_2 - 1)n_k(1-g),$$

where $\bar{n}_2 = \sum_{j=1}^b a_j^2 n_j$.

3 Strata of length 1

The construction of these strata are labeled by some words. The intention is to guide the reading, and mark the steps of each construction.

■ **Stratum No. 1** This first stratum X_1 is done in whole generality in section 7.4.

Basis The basis is $U = \big((\operatorname{Jac} C)^3 \setminus \Delta\big) \times \operatorname{Jac} C$ where $\Delta \subset (\operatorname{Jac} C)^3$ is the union of all diagonals. The degree of these line bundles is $\frac{d_1}{3}$, except for the line bundles parametrized by the last factor whose degree is d_2.
We denote $\bar{U} = (\operatorname{Jac} C)^4$.
Recall the action of the symmetric group $\mathfrak{S}_3$ which permutes the bundles E_1, E_2 and E_3.

Fibre The bundle $\tilde{X}_1$ over U has fibre

$$\prod_{i=1}^{3} \mathbb{P}\operatorname{Ext}^1(T', S_i), \text{ where } \begin{cases} T' = (0, L, 0) \text{ (the } \textit{initial triple}) \\ S_1 = (E_i, 0, 0) \text{ where } \operatorname{rk} E_i = 1 \end{cases}$$

Extending The bundle $\tilde{X}_1$ can be extended to $\bar{U}$. The extended bundle shall be denoted by $\bar{X}_1$.

Splitting The bundle $\bar{X}_1$ can be written as a product of bundles. For $i=1,2,3$ define $X_{1,i}$ to be the projective bundle over $(\operatorname{Jac} X)^2$ with fibres isomorphic to $\mathbb{P}\operatorname{Ext}^1(T', S_i)$. Hence,

$$\bar{X}_1 = \prod_{i=1,\mathcal{J}}^{3} X_{1,i},$$

where $\mathcal{J}$ is a shorthand for the Jacobian variety $\operatorname{Jac} C$ and the symbol $\prod_{i=1,\mathcal{J}}^{3}$ denotes the fibred product over $\mathcal{J}$. Notice that the group $\mathfrak{S}_3$ acts by permutting factors.

Diagonals The difference $\bar{X}_1 \setminus \tilde{X}_1$ is the restriction on $\bar{U} \setminus U$ of $\bar{X}_1$, where $\bar{U} \setminus U$ are the union of all diagonals of $\bar{U}$. For any subset $S \subset \{1,2,3\}$ consisting of two elements, we assign the diagonal

$$X_S = X_{1,j} \times_{\mathcal{J}} \prod_{i \in S, \mathcal{J}^2} X_{1,i}$$

where $j \notin S$.

There is another diagonal, the small diagonal, which corresponds to $S = \{1,2,3\}$ and it is

$$X_S = \prod_{i \in S, \mathcal{J}^2} X_{1,i}.$$

Quotient

- For the case $\bar{X}_1$, the quotient is

$$\operatorname{Sym}^3_{\mathcal{J}} X_{1,i};$$

- For the case $\#S = 2$, the quotient by $\mathfrak{S}_3$ is

$$X_{1,j} \times_{\mathcal{J}} \operatorname{Sym}^2_{\mathcal{J}^2} X_{1,\bullet};$$

- For the case $\#S = 3$ (the smallest diagonal) is

$$\operatorname{Sym}^3_{\mathcal{J}^2} X_{1,\bullet}.$$

The Class Summing up all these quotients, the class of this stratum is

$$[X_1] = \lambda^3_{\mathcal{J}}(X_{1,\bullet}) - \lambda^2_{\mathcal{J}^2}(X_{1,\bullet})[X_{1,\bullet}] - \lambda^3_{\mathcal{J}^2}(X_{1,\bullet}),$$

where $X_{1,\bullet}$ is the class of $X_{1,i} = \frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\mathbb{J}$ for $i=1,2,3$. We have

$$\begin{aligned}
\lambda^2_{\mathbb{J}}(X_{1,\bullet}) &= \frac{1}{6}\Bigg\{\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^3 \mathbb{J}^3 + 3\delta^2\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\mathbb{J}\right)\cdot\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\mathbb{J}+ \\
&\quad + 2\delta^3\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\mathbb{J}\right)\Bigg\}\mathbb{J} \\
&= \frac{1}{6}\frac{(\mathbb{L}^{\mu+1-g}-1)^3}{(\mathbb{L}-1)^3}\mathbb{J}^4 + \frac{1}{2}\frac{(\mathbb{L}^{2(\mu+1-g)}-1)(\mathbb{L}^{\mu+1-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\delta^2(\mathbb{J})\mathbb{J}^2+ \\
&\quad + \frac{1}{3}\frac{\mathbb{L}^{3(\mu+1-g)}-1}{\mathbb{L}^3-1}\delta^3(\mathbb{J}), \\
\lambda^2_{\mathbb{J}^2}(X_{1,\bullet})[X_{1,\bullet}] &= \frac{1}{2}\left\{\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^2 + \delta^2\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)\right\}\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\mathbb{J}^3 \\
&= \frac{(\mathbb{L}^{\mu+1-g}-1)^2(\mathbb{L}^{\mu+2-g}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\mathbb{J}^3, \\
\lambda^3_{\mathbb{J}}(X_{1,\bullet}) &= \frac{1}{6}\Bigg\{\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^3 + 3\delta^2\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}+ \\
&\quad + 2\delta^3\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)\Bigg\}\mathbb{J}^2 \\
&= \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+2-g}-1)(\mathbb{L}^{\mu+3-g}-1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\mathbb{J}^2.
\end{aligned}$$

Therefore the class of this stratum is

$$
\begin{aligned}
[X_1/\mathfrak{S}_3] = &\frac{1}{6}\frac{(\mathbb{L}^{\mu+1-g}-1)^3}{(\mathbb{L}-1)^3}\mathbb{J}^4 - \frac{(\mathbb{L}^{\mu+1-g}-1)^2(\mathbb{L}^{\mu+2-g}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\mathbb{J}^3 - \\
&- \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+2-g}-1)(\mathbb{L}^{\mu+3-g}-1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\mathbb{J}^2 + \\
&+ \frac{1}{2}\frac{(\mathbb{L}^{2(\mu+1-g)}-1)(\mathbb{L}^{\mu+1-g}-1)}{(\mathbb{L}^2-1)(\mathbb{L}-1)}\delta^2(\mathbb{J})\mathbb{J}^2 + \\
&+ \frac{1}{3}\frac{\mathbb{L}^{3(\mu+1-g)}-1}{\mathbb{L}^3-1}\delta^3(\mathbb{J}).
\end{aligned}
$$

arising from the equality $[X_{1,\bullet}] = \mathbb{J}\cdot\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}$.

■ **Stratum No. 2** For the second stratum, we have the following

Basis The basis is

$$U = ((\operatorname{Jac} C)^2 \setminus \Delta) \times \operatorname{Jac} C$$

where Δ is the diagonal of $(\operatorname{Jac} C)^2$.

Fibre The stratum X_2 is a bundle over U with fibres

$$\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_1)) \times \mathbb{P}\operatorname{Ext}^1(T', S_2)$$

where $T' = (0, L, 0)$ is the initial triple, and $S_i = (E_i, 0, 0)$ for $i = 1, 2$.

Class The group action is trivial, and X_2 is a Zariski trivial bundle over U, so

$$
\begin{aligned}
[X_2] =&[\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_1))][\mathbb{P}\operatorname{Ext}^1(T', S_2)]\left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
=&\frac{(\mathbb{L}^{\mu+1-g}-1)^2(\mathbb{L}^{\mu-g}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}
$$

■ **Stratum No. 3** In this stratum it appears a bundle of degree 2, so in principle we have to distinguish between even and odd degree. The construction of this stratum follows similar arguments to the precedent paragraph:

Basis The basis is

$$U = \left(M^s(2, \tfrac{2}{3}d_1) \times \operatorname{Jac} C\right) \times \operatorname{Jac} C$$

where the degree of the line bundle parametrized by the first factor $\operatorname{Jac} C$ is $\frac{d_1}{3}$. This means that the bundle only appears when d_1 is multiple of 3.

Fibre The stratum X_3 is a bundle over U with fibres isomorphic to

$$\mathbb{P}\operatorname{Ext}^1(T', S_1) \times \mathbb{P}\operatorname{Ext}^1(T', S_2).$$

The Class In our case d_1 is multiple of 3, so the stratum X_3 is not a locally Zariski trivial bundle over U — notice that $\frac{2}{3}d_1$ is multiple of 2 and hence the universal bundle over $M^s(2, \frac{2}{3}d_1)$ is not locally trivial. Hence, its class is written as follows

$$
\begin{aligned}
[X_3] =&[\mathbb{P}\operatorname{Ext}^1(T', S_1)][\mathbb{P}\operatorname{Ext}^1(T', S_2)][M^s(2, \tfrac{2}{3}d_1)][\operatorname{Jac} C]^2 \\
=&\frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{2(\mu+1-g)}-1)}{(\mathbb{L}-1)^2}[M^s(2, \tfrac{2}{3}d_1)]\mathbb{J}^2.
\end{aligned}
$$

Remark 5.1. *The expression obtained for the class of this stratum is done in $\bar{K}_0(\mathfrak{Var}_{\mathbb{C}})$. This because to the universal bundle of $M^s(2,0)$ is not a Zariski locally trivial bundle.*

■ Stratum No. 4 The stratum X_4 is a bundle over $U = \operatorname{Jac} C \times \operatorname{Jac} C$ (the first factor parametrizes line bundles of degree $\frac{d_1}{3}$) with fibre $\operatorname{Gr}(3, \mathbb{P}\operatorname{Ext}^1(T', S_1))$. The class is

$$\begin{aligned}[X_4] =&[\operatorname{Gr}(3, \mathbb{P}\operatorname{Ext}^1(T', S_1))][\operatorname{Jac} C]^2\\ =&\frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu-g}-1)(\mathbb{L}^{\mu-1-g}-1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\mathbb{J}^2.\end{aligned}$$

■ Stratum No. 5 The stratum X_5 is a bundle over $M^s(3, d_1) \times \operatorname{Jac} C$ and the fibre is $\mathbb{P}\operatorname{Ext}^1(T', S_1)$, so its class is

$$\begin{aligned}[X_5] =&[\mathbb{P}\operatorname{Ext}^1(T', S_1)][M^s(3, d_1)][\operatorname{Jac} C]\\ =&\frac{\mathbb{L}^{3(\mu+1-g)}-1}{\mathbb{L}^3-1}[M^s(3, d_1)]\mathbb{J}.\end{aligned}$$

4 Strata of length 2

We begin to deal with the set of stratum with two steps. The first two strata are a particular case for a general construction done in previous chapters, so we do not offer details, only a scheme of this construction.

■ Stratum No. 6 The first stratum is the most complicated of this list. A generalized construction is developed in Section 7.4.

Basis The basis is

$$U = \big((\operatorname{Jac} C)^3 \setminus \Delta\big) \times \operatorname{Jac} C,$$

where Δ is the union of all diagonals of $(\operatorname{Jac} C)^3$, which parametrizes the line bundles E_1, E_2 and E_3 of degree $\frac{d_1}{3}$.

Step 1 Let consider the following bundle: let Y denote the bundle over $(\operatorname{Jac} C)^2 \times \operatorname{Jac} C$ with fibres isomorphic to $\mathbb{P}\operatorname{Ext}^1(T', S_1) \times \mathbb{P}\operatorname{Ext}^1(T', S_2)$. We include the case $E_1 \cong E_2$: such diagonal will be removed later.

Step 2 We complete the basis for the second step attaching the third factor $\operatorname{Jac} C$ to the bundle Y. That is, we shall consider the cartesian product

$$B = Y \times \operatorname{Jac} C \setminus ((Y \times_{\mathcal{J}_1} \operatorname{Jac} C) \cup (Y \times_{\mathcal{J}_2} \operatorname{Jac} C))$$

where each fibre product identifies E_3 with E_i for $i = 1, 2$. Over this space, $\bar{X}_6$ is the bundle with fibre

$$F = \mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3) \setminus \big(\mathbb{P}\operatorname{Ext}^1(T_1^1, S_3) \cup \mathbb{P}\operatorname{Ext}^1(T_1^2, S_3)\big)$$

where T_1^i is the corresponding extension in $\operatorname{Ext}^1(T', S_i)$ given by the push-forward of $S_1 \oplus S_2 \to S_i$ for each $i = 1, 2$

$$\operatorname{Ext}^1(T', S_1 \oplus S_2) \to \operatorname{Ext}^1(T', S_i).$$

It is easy to check the equality

$$\mathbb{P}\operatorname{Ext}^1(T_1^1, S_3) \cap \mathbb{P}\operatorname{Ext}^1(T_1^2, S_3) = \mathbb{P}\operatorname{Ext}^1(T', S_3),$$

as subspaces of $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3)$.

Split I The bundle Y defined in the Step 1 can be written as a product: let Y_i be the bundle over $\operatorname{Jac} C \times \operatorname{Jac} C$ with fibres isomophic to $\mathbb{P}\operatorname{Ext}^1(T', S_i)$ for $i = 1, 2$. Clearly, $Y = Y_1 \times_{\mathcal{J}} Y_2$, the fibre product over the Jacobian variety which parametrizes the initial triple T' (recall that $\mathcal{J}$ is a shorthand for $\operatorname{Jac} C$).

We use this splitting for the basis B of $\bar{X}_6$: let

$$B_i = (Y_i \times \operatorname{Jac} C) \setminus (Y_i \times_{\mathcal{J}} \operatorname{Jac} C)$$

for $i = 1, 2$. Then, $B = B_1 \times_{\mathcal{J}} B_2$ where the fibre product identifies their second factor of $\operatorname{Jac} C$, that is, the Jacobian variety parametrizing the E_3 bundle.

Diagonal I The case $E_1 \cong E_2$ corresponds to doing the fibre product of the bundles B_1 and B_2 in the following way:

$$B^{\Delta} = B_1 \times_{\mathcal{J}^2 \setminus \mathcal{J}} B_2.$$

There is a natural inmersion $B^{\Delta} \hookrightarrow B$ and the restriction $\bar{X}_6$ to B^{Δ} will be denoted by X_6^{Δ}. Therefore

$$X_6 = \bar{X}_6 \setminus X_6^{\Delta}$$

where X_6 denotes the stratum *before* quotienting by the finite group $\mathfrak{S}_2$.

Fibre The action on the fibre of $X_2 \to Y \times \operatorname{Jac} C$ is worked out next

- First, the map

$$\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3) \setminus \mathbb{P}\operatorname{Ext}^1(T', S_3) \to \mathbb{P}\operatorname{Ext}^1(S_1 \oplus S_2, S_3)$$

is a locally Zariski trivial vector bundle with fibres isomorphic to $\operatorname{Ext}^1(T', S_3)$. This follows from the general fact that for any eqpimorphism between two vector spaces $f : V \to W$, we have the natural isomorphism of alegraic varieties

$$\mathbb{P}V \setminus \mathbb{P}\ker f \cong \ker f \otimes U^*_{\mathbb{P}W}$$

where $U_{\mathbb{P}W} \to \mathbb{P}W$ denotes the tautological bundle over a projective space (this is a particular case of the Lemma 7.23).

- Second, there is a natural projection

$$\mathbb{P}\operatorname{Ext}^1(S_1 \oplus S_2, S_3) \setminus \big(\mathbb{P}\operatorname{Ext}^1(S_1, S_3) \sqcup \mathbb{P}\operatorname{Ext}^1(S_2, S_3)\big) \to \mathbb{P}\operatorname{Ext}^1(S_1, S_3) \times \mathbb{P}\operatorname{Ext}^1(S_2, S_3)$$

which is a principal $\mathbb{C}^{\times}$-bundle. This also comes from the general fact

$$\mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}V_1 \sqcup \mathbb{P}V_2) \cong \big(U^*_{\mathbb{P}V_1} \otimes U_{\mathbb{P}V_2}\big) \setminus \{0\}.$$

(see Lemma 7.24).

- Third, let F denote the fibre of X_6 over $Y \times \operatorname{Jac} C$, and let

$$\tilde{F} = \operatorname{Ext}^1(T_1, S_3) \setminus \big[\operatorname{Ext}^1(T_1^1, S_3) \cup \operatorname{Ext}^1(T_1^2, S_3)\big]$$

denote the same space of F before quotienting by $\mathbb{C}^{\times}$. We have the following commutative diagram

$$\begin{tikzcd}
\tilde{F} \arrow[r, "\mathbb{C}^{\times}"] \arrow[d, "\operatorname{Ext}^1(T', S_3)"'] & F \arrow[d, "\mathbb{C}^{\times} \times \operatorname{Ext}^1(T', S_3)"] \\
(\operatorname{Ext}^1(S_1, S_3) \setminus \{0\}) \times (\operatorname{Ext}^1(S_2, S_3) \setminus \{0\}) \arrow[r, "\mathbb{C}^{\times} \times \mathbb{C}^{\times}"'] & \mathbb{P}\operatorname{Ext}^1(S_1, S_3) \times \mathbb{P}\operatorname{Ext}^1(S_2, S_3)
\end{tikzcd} \tag{5.6}$$

where next to the arrows we show the fibre of the corresponding map.

Action I The action of $\mathfrak{S}_2$ on F is also codified in this diagram. The group $\mathfrak{S}_2$ acts trivially on the fibre $\operatorname{Ext}^1(T', S_3)$ (first vertical map in (5.6)). The action on $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ (the fibre of the bottom horizontal map) switches both factors. Meanwhile, the $\mathbb{C}^{\times}$ factor of the fibre on top horizontal map and second vertical map has the action $x \mapsto x^{-1}$.

Split II We define $\tilde{Y}$ to be the bundle over $Y \times \operatorname{Jac} C$ with fibres

$$\left(\operatorname{Ext}^1(S_1, S_3) \setminus \{0\}\right) \times \left(\operatorname{Ext}^1(S_2, S_3) \setminus \{0\}\right),$$

It splits as

$$\tilde{Y} = \left(\tilde{Y}_1 \times_{\mathcal{J}} \tilde{Y}_2\right) \setminus \left(\tilde{Y}_1 \times_{\mathcal{J}^2} \tilde{Y}_2\right) \tag{5.7}$$

where $\tilde{Y}_i$ is a bundle over B_i (here $E_3 \ncong E_i$) with fibre $\operatorname{Ext}^1(S_i, S_3) \setminus \{0\}$ for $i = 1, 2$. We define $\tilde{X}_6$ to be the bundle over $\tilde{Y}$ with fibre $\operatorname{Ext}^1(T', S_3)$, which is induced by the first vertical map of the diagram (5.6), that is, the fibre of the map $\tilde{X}_6 \to B$ is isomorphic to $\tilde{F}$. Hence, the first horizontal map of the mentioned diagram is, indeed, the projection $\tilde{X}_6 \to \bar{X}_6$ making it as a principal $\mathbb{C}^\times$-bundle.

Diagonal II Following similar arguments, we have the corresponding diagonal subsets for $\tilde{Y}$ and $\tilde{X}_6$: let $\tilde{Y}^\Delta = \tilde{Y}_1 \times_{\mathcal{J}^2 \setminus \mathcal{J}} \tilde{Y}_2$ and $\tilde{X}_6^\Delta$ the restriction of $\tilde{X}_6$ over $\tilde{Y}^\Delta$.

Action II The action of $\mathfrak{S}_2$ permutes the factor of $\tilde{Y}$. Let us check that this group acts trivially on the fibre of $\tilde{X}_6 \to \bar{X}_6$. The $\mathbb{C}^\times$ factor of the second column of the diagram (5.6) comes from the projection

$$\begin{array}{ccc} \mathbb{C}^\times \times \mathbb{C}^\times & \longrightarrow & \mathbb{C}^\times \\ (z_1, z_2) & \longmapsto & z_1 \cdot z_2^{-1} \end{array} \tag{5.8}$$

so the fibre of the map $\tilde{X}_6 \to \bar{X}_6$ is isomorphic to the kernel of (5.8), that is the diagonal of $\mathbb{C}^\times \times \mathbb{C}^\times$, where the action of $\mathfrak{S}_2$ becomes trivial. A consequence of this is the equality

$$[\tilde{X}_6/\mathfrak{S}_2] = \frac{[\bar{X}_6/\mathfrak{S}_2]}{\mathbb{L} - 1}.$$

Similar arguments show that

$$[\tilde{X}_6^\Delta/\mathfrak{S}_2] = \frac{[X_6^\Delta/\mathfrak{S}_2]}{\mathbb{L} - 1}.$$

Quotient I The quotient $\tilde{X}_6/\mathfrak{S}_2$ is now easy: since the fibre of $\tilde{X}_6 \to \tilde{Y}$ is a vector space with trivial action, then

$$[\tilde{X}_6/\mathfrak{S}_2] = [\operatorname{Ext}^1(T', S_3)][\tilde{Y}/\mathfrak{S}_2].$$

The quotient of $\tilde{Y}/\mathfrak{S}_2$ is obtained using the above splitting:

$$[\tilde{Y}/\mathfrak{S}_2] = \lambda^2_{\mathcal{J}}(\tilde{Y}_\bullet)[\operatorname{Jac} C].$$

Quotient II The quotient of the diagonal $\tilde{X}_6^\Delta$ by $\mathfrak{S}_2$ uses the same arguments than the case of $\tilde{X}_6$. So the quotient is

$$[\tilde{X}_6^\Delta] = [\operatorname{Ext}^1(T', S_3)][\tilde{Y}^\Delta/\mathfrak{S}_2], \text{ and}$$
$$[\tilde{Y}^\Delta/\mathfrak{S}_2] = \lambda^2_{\mathcal{J}^2 \setminus \mathcal{J}}(\tilde{Y}_\bullet)[\operatorname{Jac} C].$$

The class Summing up, the class of the stratum is

$$\begin{aligned} [X_6/\mathfrak{S}_2] =& \frac{1}{\mathbb{L} - 1}\left([\tilde{X}_6/\mathfrak{S}_2] - [\tilde{X}_6^\Delta/\mathfrak{S}_2]\right) \\ =& \frac{\mathbb{L}^{\mu+1-g}}{\mathbb{L} - 1}\left([\tilde{Y}/\mathfrak{S}_2] - [\tilde{Y}^\Delta/\mathfrak{S}_2]\right). \end{aligned} \tag{5.9}$$

Now we have

$$\begin{aligned} [\tilde{Y}_\bullet] &= \left([\operatorname{Ext}^1(S_i, S_3)] - 1\right)[\mathbb{P}\operatorname{Ext}^1(T', S_i)]\left([\operatorname{Jac} C]^2 - [\operatorname{Jac} C]\right) \\ &= \left(\mathbb{L}^{g-1} - 1\right)\frac{\mathbb{L}^{\mu+1-g} - 1}{\mathbb{L} - 1}(\mathbb{J}^2 - \mathbb{J}). \end{aligned}$$

so

$$
\begin{aligned}
[\tilde{Y}/\mathfrak{S}_2] &= \lambda^2_{\mathcal{J}^2}(\tilde{Y}_\bullet) = \lambda^2\left((\mathbb{L}^{g-1}-1)\mathbb{P}^{\mu-g}(\mathbb{J}-1)\right)\mathbb{J}^2,\\
&= \frac{1}{2}\Bigg\{(\mathbb{L}^{g-1}-1)^2\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^2(\mathbb{J}-1)^2+\\
&\quad + (\mathbb{L}^{2(g-1)}-1)\frac{\mathbb{L}^{2(\mu+1-g)}-1}{\mathbb{L}-1}(\delta^2(\mathbb{J})-1)\Bigg\}\mathbb{J}^2,\\
[\tilde{Y}^\Delta/\mathfrak{S}_2] &= \lambda^2_{\mathcal{J}^3\setminus\mathcal{J}^2}(\tilde{Y}_\bullet) = \lambda^2\left((\mathbb{L}^{g-1}-1)\mathbb{P}^{\mu-g}\right)(\mathbb{J}-1)\mathbb{J}^2\\
&= \frac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+1}-\mathbb{L}^{\mu+1-g}-\mathbb{L}^{g+1}+\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)}(\mathbb{J}-1)\mathbb{J}^2.
\end{aligned}
$$

We plug these equalities in (5.9) to get the class of $[X_6/\mathfrak{S}_2]$:

$$
\begin{aligned}
[X_6/\mathfrak{S}_2] = \frac{\mathbb{L}^{\mu+1-g}}{\mathbb{L}-1}\Bigg\{&\frac{1}{2}(\mathbb{L}^{g-1}-1)^2\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^2(\mathbb{J}-1)^2-\\
&-\frac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+1}-\mathbb{L}^{\mu+1-g}-\mathbb{L}^{g+1}+\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)}(\mathbb{J}-1)+\\
&+(\mathbb{L}^{2(g-1)}-1)\frac{\mathbb{L}^{2(\mu+1-g)}-1}{\mathbb{L}-1}(\delta^2(\mathbb{J})-1)\Bigg\}.
\end{aligned}
$$

■ **Stratum No. 7** Although this is a case in some sonse symmetric to the previous stratum, its construction is fairly simple. This stratum is one of the family that shall be constructed in Section 7.4.

Basis As in the previous stratum the basis is

$$U = \left((\operatorname{Jac} C)^3 \setminus \Delta\right) \times \operatorname{Jac} C.$$

Step 1 We can suppose that the first step is a bundle Y over $(\operatorname{Jac} C)^2$ whose fibres are isomorphic to $\mathbb{P}\operatorname{Ext}^1(T', S_1)$.

Step 2 (Basis) We “attach” to Y the other two Jacobian varieties of U:

$$\tilde{Y} = \left(Y \times \left((\operatorname{Jac} C)^2 \setminus \Delta\right)\right) \setminus \left[Y \times_{\mathcal{J}} \left((\operatorname{Jac} C)^2 \setminus \Delta\right) \sqcup Y \times_{\mathcal{J}} \left((\operatorname{Jac} C)^2 \setminus \Delta\right)\right],$$

where the fibre products over $\mathcal{J} = \operatorname{Jac} C$ are on each factor of $(\operatorname{Jac} C)^2 \setminus \Delta$. Thus, the class of the generic fibre of $\tilde{Y} \to Y$ is $(\mathbb{J}^2 - \mathbb{J}) - 2(\mathbb{J}-1) = \mathbb{J}^2 - 3\mathbb{J} + 2$.

Step 2 (Fibre) The stratum $\tilde{X}_7$ (before quotienting by the finite group $\mathfrak{S}_2$) is a bundle over $\tilde{Y}$ with fibre

$$\left(\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) \times \mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3)\right) \setminus \left(\mathbb{P}\operatorname{Ext}^1(T', S_2) \times \mathbb{P}\operatorname{Ext}^1(T', S_3)\right).$$

Split Let $X_{7,i}$ (resp. $Z_{7,i}$) be the bundle over

$$B_i = (Y \times \operatorname{Jac} C) \setminus (Y \times_{\mathcal{J}} \operatorname{Jac} C), \text{ where } i = 2, 3,$$

with fibre $\mathbb{P}\operatorname{Ext}^1(T_1, S_i)$ (resp. $\mathbb{P}\operatorname{Ext}^1(T', S_i)$) for $i = 2, 3$. Define $\bar{X}_{7,i}$ to be the extension of the bundle $X_{7,i}$ over the diagonal corresponding to $L_2 \cong L_3$. It is easy to check the equality

$$\bar{X}_7 = (X_{7,2} \times_Y X_{7,3}) \setminus (Z_{7,2} \times_Y Z_{7,3}).$$

Notice that $L_1 \not\cong L_2$ and $L_1 \not\cong L_3$ in $\bar{X}_7$ automatically.

Diagonal We remove the case $L_2 \cong L_3$ corresponding to the difference $X_7^{\Delta} = \bar{X}_7 \setminus X_7$. Indeed, this is the restriction of $\bar{X}_7$ to the diagonal of the basis, that is,

$$X_7^{\Delta} = (X_{7,2} \times_{B_\bullet} X_{7,3}) \setminus (Z_{7,2} \times_{B_\bullet} Z_{7,3})$$

where $B_\bullet \cong B_i$ for some $i = 2, 3$ (this makes sense because of $B_2 \cong B_3$).
The action of $\mathfrak{S}_2$ permutes the fibre of X_7^{Δ} over $B_\bullet$.

Quotient Then, we use the equality $[\tilde{X}_7/\mathfrak{S}_2] = [\bar{X}_7/\mathfrak{S}_2] - [X_7^{\Delta}/\mathfrak{S}_2]$. On one hand, we have

$$[\bar{X}_7/\mathfrak{S}_2] = \lambda^2_Y(X_{7,\bullet}) - \lambda^2_Y(Z_{7,\bullet}).$$

On the other hand, the diagonal has a similar expression:

$$[X_7^{\Delta}/\mathfrak{S}_2] = \lambda^2_{B_\bullet}(X_{7,\bullet}) - \lambda^2_{B_\bullet}(Z_{7,\bullet}).$$

The Class Then, the class of $[X_7] = [\tilde{X}_7/\mathfrak{S}_2]$ is the difference

$$[\tilde{X}_7/\mathfrak{S}_2] = \lambda^2_Y(X_{7,\bullet}) - \lambda^2_Y(Z_{7,\bullet}) - \left(\lambda^2_{B_\bullet}(X_{7,\bullet}) - \lambda^2_{B_\bullet}(Z_{7,\bullet})\right),$$

and each part is

$$\begin{aligned}
\lambda^2_Y(X_{7,j}) &= \lambda^2\left(\frac{\mathbb{L}^{\mu}-1}{\mathbb{L}-1}(\mathbb{J}-1)\right)\frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1}\mathbb{J}^2 \\
&= \frac{1}{2}\left\{\left(\frac{\mathbb{L}^{\mu}-1}{\mathbb{L}-1}\right)^2(\mathbb{J}-1)^2 + \delta^2\left(\frac{\mathbb{L}^{\mu}-1}{\mathbb{L}-1}(\mathbb{J}-1)\right)\right\}\frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1}\mathbb{J}^2 \\
&= \left\{\frac{1}{2}\left(\frac{\mathbb{L}^{\mu}-1}{\mathbb{L}-1}\right)^2(\mathbb{J}^2-2\mathbb{J}) + \frac{(\mathbb{L}^{\mu}-1)(\mathbb{L}^{\mu}-\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)} + \right. \\
&\quad \left. + \frac{\mathbb{L}^{2\mu}-1}{\mathbb{L}^2-1}\delta^2(\mathbb{J})\right\}\frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1}\mathbb{J}^2.
\end{aligned}$$

The case $\lambda^2_Y(Z_{7,j})$ is similar: replace μ by $\mu+1-g$ inside the λ^2 operator

$$\begin{aligned}
\lambda^2_Y(Z_{7,j}) &= \left\{\frac{1}{2}\left(\frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)^2(\mathbb{J}^2-2\mathbb{J}) + \frac{(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+1-g}-\mathbb{L})}{(\mathbb{L}-1)(\mathbb{L}^2-1)} + \right. \\
&\quad \left. + \frac{\mathbb{L}^{2(\mu+1-g)}-1}{\mathbb{L}^2-1}\delta^2(\mathbb{J})\right\}\frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1}\mathbb{J}^2.
\end{aligned}$$

On the other hand we have the diagonal part, corresponding to the case $L_2 \cong L_3$

$$\begin{aligned}
\lambda^2_{B_\bullet}(X_{7,j}) &= \lambda^2\left(\frac{\mathbb{L}^{\mu}-1}{\mathbb{L}-1}\right)\frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1}(\mathbb{J}-1)\mathbb{J}^2 \\
&= \frac{(\mathbb{L}^{\mu}-1)(\mathbb{L}^{\mu+1}-1)(\mathbb{L}^{\mu+1-g}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}(\mathbb{J}-1)\mathbb{J}^2.
\end{aligned}$$

and,

$$\lambda^2_{B_\bullet}(Z_{7,j}) = \frac{(\mathbb{L}^{\mu+1-g}-1)^2(\mathbb{L}^{\mu-g+2}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}(\mathbb{J}-1)\mathbb{J}^2.$$

So the class of the stratum X_7 is

$$\begin{aligned}
[X_7/\mathfrak{S}_2] &= \frac{\mathbb{L}^{\mu+1-g}(\mathbb{L}^{g-1}-1)}{\mathbb{L}-1}\left\{\frac{\mathbb{L}^{\mu}+\mathbb{L}^{\mu+1-g}-2}{\mathbb{L}-1}(\mathbb{J}-1)^2 - \right. \\
&\quad \left. - \frac{\mathbb{L}^{\mu+1}+\mathbb{L}^{\mu+2-g}-\mathbb{L}-1}{(\mathbb{L}-1)(\mathbb{L}^2-1)}(\mathbb{J}-1) + \frac{\mathbb{L}^{\mu+1-g}(\mathbb{L}^{g-1}-1)}{2(\mathbb{L}+1)}(\delta^2(\mathbb{J})-1)\right\}\times \\
&\quad \times \frac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\mathbb{J}^2
\end{aligned}$$

Stratum No. 8

From this point on, we have no more strata with non trivial finite group.

Basis In this case, the basis is $U = \big((\operatorname{Jac} X)^2 \setminus \Delta\big) \times \operatorname{Jac} C$.

Step 1 The first step is a bundle Y over U whose fibre is isomorphic to $\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_1))$. As always, T' is the so-called initial triple and $S_1 = (E_1, 0, 0)$ with degree $\frac{d_1}{3}$.

Step 2 The stratum X_8 is a bundle over Y with fibre a subset of $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)$: look at the short exact sequence

$$0 \to \operatorname{Ext}^1(T', S_2) \to \operatorname{Ext}^1(\tilde{T}_1, S_2) \to \operatorname{Ext}^1(S_1^2, S_2) \to 0 :$$

the image of any element of $\operatorname{Ext}^1(\tilde{T}_1, S_2)$ must be live in

$$\mathcal{V}(2, \operatorname{Ext}^1(S_1, S_2)) \subset \operatorname{Ext}^1(S_1^2, S_2)$$

So the class of the fibre of $X_2 \to Y$ is

$$[\operatorname{Ext}^1(T', S_1)] \frac{[\mathcal{V}(2, \operatorname{Ext}^1(S_1, S_2)]}{\mathbb{L} - 1}.$$

The class Summing up, the class of the stratum is

$$\begin{aligned}
[X_8] =&[\operatorname{Ext}^1(T', S_2)] \frac{[\mathcal{V}(2, \operatorname{Ext}^1(S_1, S_2)]}{\mathbb{L} - 1} [\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_1))] \left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
=&\mathbb{L}^{\mu+1-g} \frac{(\mathbb{L}^{g-1} - 1)(\mathbb{L}^{g-2} - 1)}{\mathbb{L} - 1} \frac{(\mathbb{L}^{\mu+1-g} - 1)(\mathbb{L}^{\mu-g} - 1)}{(\mathbb{L} - 1)(\mathbb{L}^2 - 1)} \left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}$$

Stratum No. 9

In some sense, this is the mirrored stratum of the previous one.

Basis The basis is the same as the previous stratum: $U = \big((\operatorname{Jac} C)^2 \setminus \Delta\big) \times \operatorname{Jac} C$.

Step 1 This step is a bundle Y over U with fibres $\mathbb{P}\operatorname{Ext}^1(T', S_1)$.

Step 2 The stratum X_9 is a bundle over Y whose fibres are subsets of $\operatorname{Gr}(2, \operatorname{Ext}^1(\tilde{T}_1, S_2))$. By the short exact sequence

$$0 \to \operatorname{Ext}^1(T', S_2^2) \to \operatorname{Ext}^1(\tilde{T}_1, S_2^2) \to \operatorname{Ext}^1(S_1, S_2^2) \to 0$$

we have to remove $\operatorname{Ext}^1(T', S_2^2)$ (the kernel of the last map in the exact sequence). Then, the fibre is

$$\operatorname{Gr}(2, \operatorname{Ext}^1(\tilde{T}_1, S_2)) \setminus \operatorname{Gr}(2, \operatorname{Ext}^1(T', S_2)).$$

Class Summing up this, the class of this stratum is

$$\begin{aligned}
[X_9] =& \Big([\operatorname{Gr}(2, \operatorname{Ext}^1(\tilde{T}_1, S_2))] - [\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_2))]\Big) [\mathbb{P}\operatorname{Ext}^1(T', S_1)] \left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
=& \frac{(\mathbb{L}^{\mu} - 1)(\mathbb{L}^{\mu-1} - 1) - (\mathbb{L}^{\mu+1-g} - 1)(\mathbb{L}^{\mu-g} - 1)}{(\mathbb{L} - 1)(\mathbb{L}^2 - 1)} \frac{\mathbb{L}^{\mu+1-g} - 1}{\mathbb{L} - 1} \left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}$$

Stratum No. 10

Basis The basis is $U = \big((\operatorname{Jac} C)^2 \setminus \Delta\big) \times \operatorname{Jac} C$ where the degree of the line bundles are $\frac{d_1}{3}$.

Step 1 Clearly, the first step is a bundle Y over U with fibres isomophic to

$$\mathbb{P}\operatorname{Ext}^1(T', S_1) \times \mathbb{P}\operatorname{Ext}^1(T', S_2)$$

where T' is the initial triple.

Step 2 The second step, and the stratum, is a bundle X_{10} over Y where the fibre is an open subset of $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_1)$ where $\tilde{T}_1 \in Y$. Such a subset consists in triples T such that the image of the surjection

$$q : \operatorname{Ext}^1(\tilde{T}_1, S_1) \to \operatorname{Ext}^1(S_1 \oplus S_2, S_1)$$

satisfies that $q(T) \in \left(\operatorname{Ext}^1(S_1, S_1) \setminus \{0\}\right) \times \left(\operatorname{Ext}^1(S_2, S_1) \setminus \{0\}\right)$. The generic fibre of q is isomorphic to $\operatorname{Ext}^1(T', S_1)/\operatorname{Hom}(S_1, S_1)$ where it is easy to check that $\operatorname{Hom}(S_1, S_1) \cong \mathbb{C}$. Then, the class of the fibre is

$$\frac{[\operatorname{Ext}^1(T', S_1)]([\operatorname{Ext}^1(S_1, S_1)] - 1)([\operatorname{Ext}^1(S_2, S_1)] - 1)}{\mathbb{L}(\mathbb{L} - 1)},$$

and the class of the stratum is

$$\begin{aligned}
[X_{10}] =& \frac{[\operatorname{Ext}^1(T', S_1)]([\operatorname{Ext}^1(S_1, S_1)] - 1)([\operatorname{Ext}^1(S_2, S_1)] - 1)}{\mathbb{L}(\mathbb{L} - 1)} \times \\
& \times [\mathbb{P}\operatorname{Ext}^1(T', S_1)][\mathbb{P}\operatorname{Ext}^1(T', S_2)] \left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
=& \mathbb{L}^{\mu-g} \frac{(\mathbb{L}^g - 1)(\mathbb{L}^{g-1} - 1)}{\mathbb{L} - 1} \left(\frac{\mathbb{L}^{\mu+1-g} - 1}{\mathbb{L} - 1}\right)^2 \left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}$$

■ Stratum No. 11

A case in some sense the symmetric to the case X_{10}.

Basis As in previous cases, the basis is $U = \left((\operatorname{Jac} C)^2 \setminus \Delta\right) \times \operatorname{Jac} C$.

Step 1 This is a bundle Y over U where the fibre is isomorphic to $\mathbb{P}\operatorname{Ext}^1(T', S_1)$.

Step 2 The stratum X_{11} is a bundle over Y with fibres the space $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_1) \times \mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)$ minus the kernel of the sujection

$$\operatorname{Ext}^1(\tilde{T}_1, S_1 \oplus S_2) \to \operatorname{Ext}^1(S_1, S_1 \oplus S_2)$$

which is isomorphic to $\operatorname{Ext}^1(T', S_1 \oplus S_2)/\operatorname{Hom}(S_1, S_1)$. That is, the fibre is

$$\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_1) \times \mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) \setminus \left[\left(\mathbb{P}\left(\operatorname{Ext}^1(T', S_1)/\operatorname{Hom}(S_1, S_1)\right) \times \mathbb{P}\operatorname{Ext}^1(T', S_2)\right)\right].$$

Then the class of the stratum is

$$\begin{aligned}
[X_{11}] =& \Big\{ \left[\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_1) \times \mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)\right] - \\
& - \left[\mathbb{P}\left(\operatorname{Ext}^1(T', S_1)/\operatorname{Hom}(S_1, S_1)\right) \times \mathbb{P}\operatorname{Ext}^1(T', S_2)\right] \Big\} \times \\
& \times \left[\mathbb{P}\operatorname{Ext}^1(T', S_1)\right] \left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
=& \frac{(\mathbb{L}^\mu - 1)^2 - (\mathbb{L}^{\mu+1-g} - 1)(\mathbb{L}^{\mu-g} - 1)}{(\mathbb{L} - 1)^2} \frac{\mathbb{L}^{\mu+1-g} - 1}{\mathbb{L} - 1} \left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}$$

■ Stratum No. 12

The class of $M^s(2, d)$ appears again in the following two strata.

Basis The basis is $U = \operatorname{Jac} C \times M^s(2, \frac{2}{3}d_1) \times \operatorname{Jac} C$ where $\frac{d_1}{3}$ is integer. Notice that $\frac{2}{3}d_1$ is an even integer number (see Remark 5.1)

Step 1 This is a bundle Y over $\mathbb{P}\operatorname{Ext}^1(T', S_1)$ where $S_1 = (E_1, 0, 0)$ and $\operatorname{rk} E_1 = 1$.

Step 2 The stratum X_{12} is a bundle over Y with fibres at the point $\tilde{T}_1 \in Y$ isomorphic to $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) \setminus \mathbb{P}\operatorname{Ext}^1(T', S_2)$ where $S_1 = (E_2, 0, 0)$ and $\operatorname{rk} E_2 = 2$.

The class The class of the stratum is

$$\begin{aligned}
[X_{12}] =& \left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)] - [\mathbb{P}\operatorname{Ext}^1(T', S_2)]\right) [\mathbb{P}\operatorname{Ext}^1(T', S_1)][\operatorname{Jac} C]^2[M^s(2, \tfrac{2}{3}d_1)] \\
=& \frac{(\mathbb{L}^{2\mu} - \mathbb{L}^{2(\mu+1-g)})(\mathbb{L}^{\mu+1-g} - 1)}{(\mathbb{L} - 1)^2} \mathbb{J}^2 \mathbb{M}_2^{\text{even}}
\end{aligned}$$

Stratum No. 13 It is very similar to the previous stratum.

Basis The basis is $U = \operatorname{Jac} C \times M^s(2, \frac{2}{3}d_1) \times \operatorname{Jac} C$.

Step 1 Y is a bundle over U with fibres isomorphic to $\mathbb{P}\operatorname{Ext}^1(T', S_1)$ where $S_1 = (E_1, 0, 0)$ with $\operatorname{rk} E_1 = 2$.

Step 2 The stratum X_{13} is a bundle over Y with fibres

$$\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) \setminus \mathbb{P}\operatorname{Ext}^1(T', S_2)$$

The class Hence the class is

$$\begin{aligned}[X_{13}] &= \left([\mathbb{P}\operatorname{Ext}^1(T_1, S_2)] - [\mathbb{P}\operatorname{Ext}^1(T', S_2)]\right)[\mathbb{P}\operatorname{Ext}^1(T', S_1)][\operatorname{Jac} C]^2[M^s(2, \tfrac{2}{3}d_1)]\\ &= \frac{(\mathbb{L}^{\mu+g-1} - \mathbb{L}^{\mu+1-g})(\mathbb{L}^{\mu+1-g} - 1)}{(\mathbb{L}-1)^2}\mathbb{J}^2\mathbb{M}_2^{\text{even}}.\end{aligned}$$

Stratum No. 14 The following group of strata finishes the set of strata whose triples has length of 2.

Basis The basis of this stratum is $U = \operatorname{Jac} C \times \operatorname{Jac} C$, where the degree of the line bundle is $\frac{d_1}{3}$.

Step 1 The first step is a bundle Y over U whose fibres are isomorphic to $\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_1))$.

Step 2 The stratum X_{14} is a bundle over Y whose fibres are an open subset of $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_1)$ where $\tilde{T}_1 \in Y$. The triples belonging to an open subset satisfying the property that their image by the natural map

$$\operatorname{Ext}^1(\tilde{T}_1, S_1) \to \operatorname{Ext}^1(S_1^2, S_1)$$

belongs to $\mathcal{V}(2, \operatorname{Ext}^1(S_1, S_1))$. Since the kernel of this map is isomorphic to $\operatorname{Ext}^1(T', S_1)/\operatorname{Hom}(S_1^2, S_1)$ the class of the fibre is

$$\frac{[\operatorname{Ext}^1(T', S_1)][\mathcal{V}(2, \operatorname{Ext}^1(S_1, S_1))]}{\mathbb{L}^2(\mathbb{L}-1)}$$

The class Summing up all, we have that the class of this stratum is

$$\begin{aligned}[X_{14}] &= \frac{[\operatorname{Ext}^1(T', S_1)][\mathcal{V}(2, \operatorname{Ext}^1(S_1, S_1))]}{\mathbb{L}^2(\mathbb{L}-1)}[\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_1))][\operatorname{Jac} C]^2\\ &= \mathbb{L}^{\mu-1-g}\frac{(\mathbb{L}^g - 1)(\mathbb{L}^g - \mathbb{L})(\mathbb{L}^{\mu+1-g} - 1)(\mathbb{L}^{\mu-g} - 1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\mathbb{J}^2.\end{aligned}$$

Stratum No. 15

Basis It has the same basis as stratum No. 14: $U = \operatorname{Jac} C \times \operatorname{Jac} C$.

Step 1 The first step is a bundle Y over U where the fibre at (T', S_1) is $\mathbb{P}\operatorname{Ext}^1(T', S_1)$.

Step 2 The stratum X_{15} is a bundle over Y with fibre

$$\operatorname{Gr}(2, \operatorname{Ext}^1(\tilde{T}_1, S_1)) \setminus \left(\operatorname{Gr}(2, \operatorname{Ext}^1(T', S_1))/\operatorname{Hom}(S_2, S_1)\right).$$

The class This yields the class:

$$\begin{aligned}[X_{15}] &= \left([\operatorname{Gr}(2, \operatorname{Ext}^1(\tilde{T}_1, S_1))] - \left[\operatorname{Gr}(2, \frac{\operatorname{Ext}^1(T', S_1)}{\operatorname{Hom}(S_2, S_1)})\right]\right)[\mathbb{P}\operatorname{Ext}^1(T', S_1)][\operatorname{Jac} C]^2\\ &= \frac{(\mathbb{L}^\mu - 1)(\mathbb{L}^{\mu-1} - 1) - (\mathbb{L}^{\mu-g} - 1)(\mathbb{L}^{\mu-1-g} - 1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\frac{\mathbb{L}^{\mu+1-g} - 1}{\mathbb{L}-1}\mathbb{J}^2\end{aligned}$$

5 Strata of length 3

■ **Stratum No. 16** This stratum will be the model for the rest of following strata. We include details here, and the next cases will be covered only in general lines.

Basis The basis is $\left((\operatorname{Jac} C)^3 \setminus \Delta\right) \times \operatorname{Jac} C$ where the degree of the line bundles are $\frac{d_1}{3}$, and the diagonal Δ is

$$[\Delta] = 3 \cdot \left[(\operatorname{Jac} C)^2 \setminus [\operatorname{Jac} C]\right] + [\operatorname{Jac} C] = 3[\operatorname{Jac} C]^2 - 2[\operatorname{Jac} C].$$

Here the $\operatorname{Jac} C$ is the smallest diagonal.

Step 1 The first step is a bundle Y_1 over U with fibres isomorphic to $\operatorname{Ext}^1(T', S_1)$.

Step 2 The second step is a bundle Y_2 over Y_1 whose fibres are an open subset of $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)$ where $\tilde{T}_1 \in Y_1$. This open subset is defined to be the set of triples which do not belong to the kernel of the natural map

$$\operatorname{Ext}^1(\tilde{T}_1, S_2) \to \operatorname{Ext}^1(S_1, S_2). \tag{5.10}$$

Such kernel is isomorphic to $\operatorname{Ext}^1(T', S_2)$, so the fibres are $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) \setminus \mathbb{P}\operatorname{Ext}^1(T', S_2)$.

Step 3 The stratum X_{16} is a bundle over Y_2 where the fibre is a subset of triples of $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3)$ consisting of elements which does not live in the kernel of the natural map

$$\operatorname{Ext}^1(\tilde{T}_2, S_3) \to \operatorname{Ext}^1(S_2, S_3). \tag{5.11}$$

As in the previous step, this kernel is isomorphic to $\operatorname{Ext}^1(\tilde{T}_1, S_3)$, so the fibre is $\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3) \setminus \mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3)$.

The class Gather this information and the class of the stratum is

$$\begin{aligned}
[X_{16}] = &\Big([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3)] - [\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3)]\Big) \cdot \Big([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)] - [\mathbb{P}\operatorname{Ext}^1(T', S_2)]\Big) \times \\
&\times [\mathbb{P}\operatorname{Ext}^1(T', S_1)] \left([\operatorname{Jac} C]^4 - 3[\operatorname{Jac} C]^3 + 2[\operatorname{Jac} C]^2\right) \\
= &\frac{(\mathbb{L}^{\mu+g-1} - \mathbb{L}^{\mu})(\mathbb{L}^{\mu} - \mathbb{L}^{\mu+1-g})(\mathbb{L}^{\mu+1-g} - 1)}{(\mathbb{L}-1)^3} \left(\mathbb{J}^4 - 3\mathbb{J}^3 + 2\mathbb{J}^2\right).
\end{aligned}$$

■ **Stratum No. 17**

Basis The basis is $U = \left((\operatorname{Jac} C)^2 \setminus \Delta\right) \times \operatorname{Jac} C$.

Step 1 This step is the same bundle Y_1 as in the stratum No. 16.

Step 2 The construction of this step follows the same arguments shown in the previous stratum, but there is a change: the kernel of the map (5.10) is now $\operatorname{Ext}^1(T', S_2)/\operatorname{Hom}(S_1, S_2)$ (here $S_2 \cong S_1$).

Step 3 The stratum X_{17} is a bundle over the last step and has the same fibre as the stratum No. 16 at step 3.

The class The class of the stratum is

$$\begin{aligned}
[X_{17}] = &\Big([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3)] - [\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3)]\Big) \left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)] - \left[\mathbb{P}\left(\frac{\operatorname{Ext}^1(T', S_2)}{\operatorname{Hom}(S_1, S_2)}\right)\right]\right) \times \\
&\times [\mathbb{P}\operatorname{Ext}^1(T', S_1)] \left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
= &\frac{(\mathbb{L}^{\mu+g-1} - \mathbb{L}^{\mu})(\mathbb{L}^{\mu} - \mathbb{L}^{\mu-g})(\mathbb{L}^{\mu+1-g} - 1)}{(\mathbb{L}-1)^3} \left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}$$

■ **Stratum No. 18** The construction of this stratum is, in practice, the same as in stratum No. 16. There is only one difference: the basis U is in this case

$$U = \left((\operatorname{Jac} C)^2 \setminus \Delta\right) \times \operatorname{Jac} C.$$

The class of the stratum is

$$
\begin{aligned}
[X_{18}] = &\left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3)] - [\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_3)]\right) \cdot \left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)] - [\mathbb{P}\operatorname{Ext}^1(T', S_2)]\right) \times \\
&\times [\mathbb{P}\operatorname{Ext}^1(T', S_1)] \left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
= &\frac{(\mathbb{L}^{\mu+g-1} - \mathbb{L}^{\mu})(\mathbb{L}^{\mu} - \mathbb{L}^{\mu+1-g})(\mathbb{L}^{\mu+1-g} - 1)}{(\mathbb{L}-1)^3} \left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}
$$

■ Stratum No. 19

This stratum is the same as the previous case stratum No. 17. The only difference lies in the kernel of the maps (5.10) and (5.11). In the first map, the kernel is isomorphic to $\operatorname{Ext}^1(T', S_2)$, but the kernel of the second map (5.11) is

$$
\operatorname{Ext}^1(\tilde{T}_1, S_3)/\operatorname{Hom}(S_2, S_3)
$$

where $S_2 \cong S_3$. So the class of this stratum is

$$
\begin{aligned}
[X_{19}] = &\left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3)] - \left[\mathbb{P}\left(\frac{\operatorname{Ext}^1(\tilde{T}_1, S_3)}{\operatorname{Hom}(S_2, S_3)}\right)\right]\right) \cdot \left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)] - [\mathbb{P}\operatorname{Ext}^1(T', S_2)]\right) \times \\
&\times [\mathbb{P}\operatorname{Ext}^1(T', S_1)] \left([\operatorname{Jac} C]^3 - [\operatorname{Jac} C]^2\right) \\
= &\frac{(\mathbb{L}^{\mu+g-1} - \mathbb{L}^{\mu-1})(\mathbb{L}^{\mu} - \mathbb{L}^{\mu+1-g})(\mathbb{L}^{\mu+1-g} - 1)}{(\mathbb{L}-1)^3} \left(\mathbb{J}^3 - \mathbb{J}^2\right).
\end{aligned}
$$

■ Stratum No. 20

Basis This basis is $U = \operatorname{Jac} C \times \operatorname{Jac} C$, where the degree of the first line bundle is $\frac{d_1}{3}$ and the second one is d_2.

Step 1 Y_1 is the same as in any stratum of this group.

Step 2 The kernel of the map (5.10) is $\operatorname{Ext}^1(T', S_2)/\operatorname{Hom}(S_1, S_2)$ where $S_2 \cong S_1$, so the fibre of $Y_2 \to Y_1$ is

$$
\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) \setminus \mathbb{P}\left(\operatorname{Ext}^1(T', S_2)/\operatorname{Hom}(S_1, S_2)\right).
$$

Step 3 The kernel of the map (5.11) is now $\operatorname{Ext}^1(\tilde{T}_1, S_3)/\operatorname{Hom}(S_2, S_3)$, so the fibre of $X_{20} \to Y_2$ is

$$
\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3) \setminus \mathbb{P}\left(\operatorname{Ext}^1(\tilde{T}_1, S_3)/\operatorname{Hom}(S_2, S_3)\right).
$$

The class The class of this stratum is

$$
\begin{aligned}
[X_{20}] = &\left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_2, S_3)] - \left[\mathbb{P}\left(\frac{\operatorname{Ext}^1(\tilde{T}_1, S_3)}{\operatorname{Hom}(S_2, S_3)}\right)\right]\right) \times \\
&\times \left([\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2)] - \left[\mathbb{P}\left(\frac{\operatorname{Ext}^1(T', S_2)}{\operatorname{Hom}(S_1, S_2)}\right)\right]\right) \cdot [\mathbb{P}\operatorname{Ext}^1(T', S_1)][\operatorname{Jac} C]^2 \\
= &\frac{(\mathbb{L}^{\mu+g-1} - \mathbb{L}^{\mu-1})(\mathbb{L}^{\mu} - \mathbb{L}^{\mu-g})(\mathbb{L}^{\mu+1-g} - 1)}{(\mathbb{L}-1)^3}\mathbb{J}^2.
\end{aligned}
$$

6 Another stratificaton for the flip loci

Before computing the class of $M^s(3,0)$ in $\bar{K}(\mathfrak{Var}_{\mathbb{C}})$, we discuss another way to compute the class of the flip locus $\mathcal{S}^+_{\sigma_m}$. Such a way consists in parametrizing separately the components of a triple $T=(E,L,\phi)\in\mathcal{S}^+_{\sigma_m}$: on one hand the bundle E via the standard filtration, and on the other hand the morphism $\phi: L\to E$.

We begin by parametrizing the bundle E of the triple $T=(E,L,\phi)\in\mathcal{S}^+_{\sigma_m}$. Recall the standard filtration: for $T\in\mathcal{S}^+_{\sigma_m}$, we have the filtration

$$0=T_0\subset T_1\subset\ldots\subset T_{r-1}\subset T_r=T,$$

where $\bar{T}_i=T_{i+1}/T_i$ is the maximal σ_m-polystable triple of $\tilde{T}_i=T/T_i$. This gives the short exact sequences

$$0\to T_{i+1}/T_i\to T/T_i\to T/T_{i+1}\to 0.$$

Moreover, since the initial triple is $T/T_{r-1}=(0,L,0)$, the standard filtration induces a filtration on $E\cong T_{r-1}$, that is, we have

$$0=E_0\subset E_1\subset E_2\subset\cdots\subset E_{r-2}\subset E_{r-1}=E,$$

with the property that $\bar{E}_i=E_{i+1}/E_i$ is the maximal polystable bundle of $\tilde{E}_i=E/E_i$, and also we have the short exact sequence of bundles

$$0\to E_{i+1}/E_i\to E/E_i\to E/E_{i+1}\to 0.$$

The conditions imposed on extensions of triples holds for extensions of bundles. That is, for any $i=1,\ldots,r-1$, we write $E_{i+1}/E_i=\bigoplus_{j=1}^b F_j^{a_i^j}$ where F_j is a stable bundle for any j and

1) the extension E/E_i belongs to $\prod_{i=1}^b\mathcal{V}(a_i^j,\mathrm{Ext}^1(E/E_{i+1},F_j))$;
2) the map $q: E/E_i\to E/E_{i+1}$ induces a map on extensions $q^*:\mathrm{Ext}^1(E/E_{i+1},E_i/E_{i+1})\to\mathrm{Ext}^1(E_{i+1}/E_{i+2},E_i/E_{i+1})$; then $q^*(E/E_i)$ lives in $\prod_{j=1}^b\mathcal{V}(a_{i+1}^j,\mathrm{Ext}^1(F_j,E_i/E_{i+1}))$.

On the other hand, any bundle E gives a triple by adding a map $\varphi: L\to E$. That is, we consider all maps of $\mathrm{Hom}(L,E)$. Nevertheless, not every map $\phi: L\to E$ induces a σ_m^+-stable triple: if there exists a subbundle $F\subset E$ where $\varphi(L)\subset F$, then (F,L,φ) becomes a desestabilizing triple. It is easy to check this: since $\mu_{\sigma_m}(T)=\frac{d_1}{n_1}$ for any triple T of type (n_1,n_2,d_1,d_2), then $\mu_{\sigma_m^+}(T)=\frac{d_1}{n_1}+\epsilon\frac{1}{n_1+n_2}$: for the case $T'=(F,L,\varphi)$,

$$\mu_{\sigma_m^+}(T')=\frac{d_F}{n_F}+\frac{\epsilon}{n_F+1}>\frac{d_1}{n_1}+\frac{\epsilon}{n_1+1}=\mu_{\sigma_m^+}(T),$$

since $\mu(F)=\mu(E)$ and $\frac{1}{n_F+1}>\frac{1}{n_1+1}$.

Now, there are several questions to answer in order to be ready to construct any strata in such a way:

1) First, we have to determine the maximal subbundles for a bundle E determined by the maximal polystable triples E_{i+1}/E_i.
2) There are several extensions that give the same bundle E/E_i, so we have to find a group whose action on $\mathrm{Ext}^1(E/E_{i+1},E_{i+1}/E_i)$ identifies the classes producing the same bundle.
3) Also, we have to identify maps $\varphi: L\to E$ which yield isomorphic triples. We look for a group whose action on $\mathrm{Hom}(L,E)$ makes this identification.

■ **Partial Splitting** Before we give an answer to the second point we have to write some words about extensions.

First, an extension of E' by E'' is a 3-upla (p, E, q) —up to automorphism— consisting of three objects: a bundle E with two maps $p : E'' \to E$ and $q : E \to E'$. Recall that (p_1, E_1, q_1) and (p_2, E_2, q_2) are equivalent extensions if there exists a map $\Psi : E \to E$ such that the diagram

$$\begin{tikzcd} E'' \arrow[r, "p_1"] \arrow[d, equal] & E_1 \arrow[r, "q_1"] \arrow[d, "\Psi"] & E' \arrow[d, equal] \\ E'' \arrow[r, "p_2"'] & E_2 \arrow[r, "q_2"'] & E' \end{tikzcd} \tag{5.12}$$

is commutative

Recall that a short exact sequence

$$0 \longrightarrow E'' \xrightarrow{p} E \xrightarrow{q} E' \longrightarrow 0$$

is *split* if $E \cong E'' \oplus E'$, and this is equivalent to the existence of so called section $s : E' \to E$ (resp. $s : E \to E''$) such that $q \circ s = 1_{E'}$ (resp. $s \circ p = 1_{E''}$). Sometimes we have partial splittings:

Definition 5.2. *Let*

$$0 \longrightarrow E'' \xrightarrow{p} E \xrightarrow{q} E' \longrightarrow 0 \tag{5.13}$$

be a short exact sequence of bundles. We say that (5.13) is left (resp. right) partially split if there exists a section $s : E' \to E$ (resp. $s : E \to E''$) such that $q \circ s : E' \to E'$ (resp. $s \circ p : E'' \to E''$) is a projector.

In this case, E is not isomorphic to $E' \oplus E''$, but it satisfies the following.

Proposition 5.3. *Let (p, E, q) be an extension fo E' by E''. If there exists a right partial section $s : E' \to E$ then E admits a splitting*

$$E \cong (E'/\ker(q \circ s)) \oplus F$$

where $F \in \mathrm{Ext}^1(\ker(q \circ s), E'')$ does not admit a partial split.

Proof. Consider the short exact sequence

$$0 \to \ker(q \circ s) \to E' \to E'/\ker(q \circ s) \to 0. \tag{5.14}$$

The map s induces a map $\bar{s} : E'/\ker(q\circ s) \to E$ because of $\ker(q\circ s) = \ker s$. So $q\circ\bar{s} : E'/\ker(q\circ s) \to E'$ is a section of (5.14) and hence $E' \cong \ker(q \circ s) \oplus (E'/\ker(q \circ s))$.

On the other hand, the pullback of the map $q \circ \bar{s}$ on E gives a split extension:

$$\begin{tikzcd} E'' \arrow[r] \arrow[d, equal] & (q \circ \bar{s})^* E \arrow[r] \arrow[d] & E'/\ker(q \circ s) \arrow[l, bend right, dotted, "\tilde{s}"'] \arrow[d, "q\circ\bar{s}"] \\ E'' \arrow[r, "p"'] & E \arrow[r, "q"'] & E' \arrow[l, bend right, "s"'] \end{tikzcd}$$

Then, the partial section s induces another section which splits the top short exact sequence in the above diagram $\tilde{s} : E'/\ker(q \circ s) \to (q \circ \bar{s})^*E$. Then, $(q \circ \bar{s})^*E \cong E'' \oplus (E/\ker(q \circ s))$.

Now, consider the exact sequence in cohomology

$$\mathrm{Ext}^1(\ker(q \circ s), E'') \to \mathrm{Ext}^1(E', E'') \to \mathrm{Ext}^1(E'/\ker(q \circ s), E'').$$

The above fact implies that the image of $[p, E, q]$ on $\mathrm{Ext}^1(E'/(\ker(q \circ s)), E'')$ is 0, so there exists $F \in \mathrm{Ext}^1(\ker(q \circ s), E'')$ giving a commutative diagram

$$\begin{tikzcd}
 & E'/\ker(q\circ s) \arrow[r, equal] \arrow[d] & E'/\ker(q\circ s) \arrow[d] \\
E'' \arrow[r, "p"] \arrow[d, equal] & E \arrow[r, "q"] \arrow[d] & E' \arrow[d] \\
E'' \arrow[r] & F \arrow[r] & \ker(q\circ s)
\end{tikzcd}$$

where the splitted short exact sequence (the third column) induces a splitting in the second column in a natural way: the natural section $E' \to E'/\ker(q \circ s)$ —the quotient map—, composed by with q gives a section of the short exact sequence of the second column. From this we have the proposition. □

This result reduces the problem of finding the endomorphims of an extension E which is partially split to the case non-splitting (the above bundle F). So, the vector space of endomorphisms $\mathrm{End}(E)$ splits as

$$\mathrm{End}(E) = \mathrm{End}(E'/\ker(q \circ s)) \oplus \mathrm{End}(F) \oplus \mathrm{Hom}(E'/\ker(q \circ s), F) \oplus \mathrm{Hom}(F, E'/\ker(q \circ s)).$$

This means that we have to study the automorphisms of an extension without any partial section.

Remark 5.4. *An analogous result holds for left partial splittings.*

■ Maximal subbundles

As before, let

$$0 \longrightarrow E'' \xrightarrow{p} E \xrightarrow{q} E' \longrightarrow 0 \tag{5.15}$$

be a short exact sequence of bundles where E'' is a polystable bundle and does not admit any partial splitting.

Under these asumptions we have:

Lemma 5.5. *The maximal subbundles of E (understanding maximality by the order induced by the inclusion) of the same slope come from the maximal subbundles of E'. That is, there is a correspondence one-to-one between maximal bundles of E and of E'.*

Proof. Consider the short exact sequence (5.15). Let $F' \subset E'$ a maximal subbundle of the same slope. Define $F = q^{-1}(F')$ the pullback. If this bundle is not a maximal subbundle of the same slope for E, let $\bar{F}$ be the maximal one containing F. Then $q(\bar{F}) \subset E'$ contadicts the maximality of F'.

On the other hand, if F is a maximal subbundle of E, the quotient bundle $q(F) \subset E'$ must be the maximal subbundle: on the contrary, we have on the contrary, the previous paragraph contradicts the maximality of $F \subset E$. Nevertheless, it can occur that $q(F) = E'$. In such a case, the quotient map $E \to E/F$ gives a partial section of the short exact sequence (5.15). □

■ Group action on the Ext group

We come back to the diagram (5.12). It is possible to obtain two extensions with isomorphic bundles E. We have to identify these extensions by the action of a group. There are two ways to get non-equivalent extension with same bundle E:

- An automorphism ϕ of E' gives another extension by pullback

$$\begin{tikzcd}
E'' \arrow[r, "p"] \arrow[d, equal] & \phi^* E \arrow[r, "\phi^* q"] \arrow[d, "q^*\phi"'] & E' \arrow[d, "\phi"] \\
E'' \arrow[r, "p"'] & E \arrow[r, "q"'] & E'
\end{tikzcd}$$

- An automorphism ψ of E'' gives another extension by pushforward the extension

$$\begin{tikzcd} E'' \arrow[r, "p"] \arrow[d, "\psi"'] & E \arrow[r, "q"] \arrow[d, "p_*\psi"] & E' \arrow[d, equal] \\ E'' \arrow[r, "\psi_* p"'] & \psi_* E \arrow[r, "q"'] & E' \end{tikzcd}$$

Then, the automorphism is $\mathrm{Aut}(E') \times \mathrm{Aut}(E'')/\sim$ where two pairs (ϕ, ψ) and (ϕ', ψ') are equivalent if and only if $\phi^*\psi_*E$ and $\phi'^*\psi'_*E$ are equivalent extensions.

The question now is when an automorphism of E' or E'' gives a non-equivalent bundle.

Let $0 \longrightarrow E'' \xrightarrow{p} E \xrightarrow{q} E' \longrightarrow 0$ be a short exact sequence of bundles such that E'' is a polystable bundle. We suppose that E lives in $\prod_{j=1}^{b} \mathcal{V}(a_j, \mathrm{Ext}^1(E', F_j))$ where $E'' = \bigoplus_{j=1}^{b} F_j^{a_j}$. Furthermore, for any subbundle $\rho : F' \hookrightarrow E'$, ρ^*E is a non-trivial extension of F' by E''.

Recall for $(\alpha, \beta) \in \mathrm{Aut}(E'') \times \mathrm{Aut}(E')$, we have the map $E \mapsto \alpha_*\beta^*E = [p \circ \beta^{-1}, E, \alpha \circ q]$. We want to know when $\alpha_*\beta^*E \cong E$. In that case, we have the following diagram

$$\begin{tikzcd} E'' \arrow[r, "p\circ\alpha^{-1}"] \arrow[d, equal] & E \arrow[r, "\beta\circ q"] \arrow[d, "\Psi"] & E' \arrow[d, equal] \\ E'' \arrow[r, "p"'] & E \arrow[r, "q"'] & E' \end{tikzcd} \tag{5.16}$$

where Ψ is an automorphism. Then,

Lemma 5.6. *Under the assumption of the previous paragraphs, let (α, β) be a pair of $\mathrm{Aut}(E'') \times \mathrm{Aut}(E')$. Suppose there exists a map $\Psi \in \mathrm{Aut}(E)$ satisfying the diagram. Then $\Psi = \lambda 1_E$ and hence $\alpha = \lambda 1_{E''}$ and $\beta = \lambda 1_{E'}$.*

Proof. Since $E'' = \bigoplus_i F_i^{a_i}$ where F_i are stable bundles for $i = 1, \ldots, r$, we have the decomposition

$$\mathrm{Ext}^1(E', E'') = \bigoplus_{i=1}^{r} \mathrm{Ext}^1(E', F_i)^{a_i}.$$

Thus, $(p, E, q) \sim (p \circ \alpha, E, \beta \circ q)$ if and only if their projection onto the factors $\mathrm{Ext}^1(E', F_i)$ are equal. So we can assume that E'' is a stable bundle and in this assumption $\alpha = \lambda 1_{E''}$. Let us consider $\Psi - \lambda 1_E$. By the first isomorphism theorem $E/\ker(\Psi - \lambda 1_E) \cong \mathrm{Im}(\Psi - \lambda 1_E) \subset E$. Since $E'' \subseteq \ker(\Psi - \lambda 1_E)$, this isomorphism gives a partial section $s : E' \to E$. To see this, consider $s' = \overline{(\Psi - \lambda 1_E)} : \mathrm{coker}(\Psi - \lambda 1_E) \to \mathrm{Im}(\Psi - \lambda 1_E)$. The commutativity of (5.16) shows that $q \circ s' \circ \beta^{-1}$ is a projector. Hence $s = s' \circ \beta^{-1}$ is a partial section contradicting the hypothesis unless $\beta = \lambda 1_{E''}$. □

Corollary 5.7. *The group $\mathrm{Aut}(E') \times_{\mathbb{C}^\times} \mathrm{Aut}(E'')$ identifies extensions which have the same bundle E. Furthermore, this action on $\mathrm{Ext}^1(E', E'')$ is effective.*

Remark 5.8. *For the case where the assumption on pullbacks ρ^*E could be split, the extension E is equivalent to an extension F of $\mathrm{Ext}^1(F', E'')$, so the action will be given by $\mathrm{Aut}(F') \times_{\mathbb{C}^\times} \mathrm{Aut}(E'')$.*

■ The group of automorphisms

The third question, the group which identifies maps of $\mathrm{Hom}(L, E)$ which gives isomorphic triples is $\mathrm{Aut}(E) \otimes_{\mathbb{C}^\times} \mathrm{Aut}(L) \cong \mathrm{Aut}(E)$.

At this point, it seems necessary to compute the group $\mathrm{Aut}(E)$.

Let us consider the following setting: fix a bundle E in a the short exact sequence

$$0 \longrightarrow E' \xrightarrow{p} E \xrightarrow{q} E'' \longrightarrow 0.$$

Furthermore, we suppose that E' is a polystable bundle, that is, $E' = \bigoplus_{i=1}^{b} F_i^{a_i}$ where F_i are stable bundles for $i = 1, \dots, b$, and there exist no right and left partial splitting, that is, the extension class of E lives in

$$\prod_{i=1}^{n} \mathcal{V}(a_i, \mathrm{Ext}^1(E'', F_i)),$$

and for any subbundle $i : S \hookrightarrow E''$, the image of $i^* : \mathrm{Ext}^1(E'', E') \to \mathrm{Ext}^1(S, E')$ is non-zero.

Our aim is to compute the group $\mathrm{Aut}(E) \subset \mathrm{Hom}(E, E)$. To do this, let us study carefully the maps in 1.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \mathrm{Hom}(E'',E'') \arrow[d, "\delta"'] \\
0 \arrow[r] & \mathrm{Hom}(E'',E') \arrow[r, "q^*"] \arrow[d, "p_*"'] & \mathrm{Hom}(E,E') \arrow[r, "p^*"] \arrow[d, "p_*"'] & \mathrm{Hom}(E',E') \arrow[r, "\delta"] \arrow[d, "p_*"'] & \mathrm{Ext}^1(E'',E') \arrow[d, "p_*"'] \\
0 \arrow[r] & \mathrm{Hom}(E'',E) \arrow[r, "q^*"] \arrow[d, "q_*"'] & \mathrm{Hom}(E,E) \arrow[r, "p_*"] \arrow[d, "q_*"'] & \mathrm{Hom}(E',E) \arrow[r, "\delta"] \arrow[d, "q_*"'] & \mathrm{Ext}^1(E'',E) \arrow[d, "q_*"'] \\
0 \arrow[r] & \mathrm{Hom}(E'',E'') \arrow[r, "q^*"] \arrow[d, "\delta"'] & \mathrm{Hom}(E,E'') \arrow[r, "p^*"] \arrow[d, "\delta"'] & \mathrm{Hom}(E',E'') \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathrm{Ext}^1(E'',E'') \arrow[d, "\delta"'] \\
\mathrm{Hom}(E',E') \arrow[r, "\delta"] & \mathrm{Ext}^1(E'',E') \arrow[r, "q^*"] \arrow[d, "p_*"'] & \mathrm{Ext}^1(E,E') \arrow[r, "p^*"] \arrow[d, "p_*"'] & \mathrm{Ext}^1(E',E') \arrow[r, "\delta"] \arrow[d, "p_*"'] & \mathrm{Ext}^2(E'',E') \\
 & {} & {} & {} &
\end{tikzcd}$$

FIG. 1: A commutative diagram

- First, the map $p^* : \mathrm{Hom}(E, E') \to \mathrm{Hom}(E', E')$ is the zero map. If any map $s : E \to E'$ has a $p^* s = s \circ p$ non-zero, automatically it gives a *partial split*, this contradicts the assumptions on the class of the extension of E. With similar arguments is easy to proof that the map $q_* : \mathrm{Hom}(E'', E) \to \mathrm{Hom}(E'', E'')$ is zero. We conclude that the maps $q^* : \mathrm{Hom}(E'', E') \to \mathrm{Hom}(E, E')$ and $p_* : \mathrm{Hom}(E'', E') \to \mathrm{Hom}(E'', E)$ are isomorphisms. Also, we get the injectivity of $\delta : \mathrm{Hom}(E', E') \to \mathrm{Ext}^1(E'', E')$ and $\delta : \mathrm{Hom}(E'', E'') \to \mathrm{Ext}^1(E'', E')$.
- Second, the connecting map $\delta : \mathrm{Hom}(E', E'') \to \mathrm{Ext}^1(E'', E'')$ is injective. Take any map $\sigma : E' \to E''$, then we have

$$\begin{tikzcd}
E' \arrow[r] \arrow[d, "\sigma"'] & E \arrow[r] \arrow[d] & E'' \arrow[d, equal] \\
E'' \arrow[r] & \delta(\sigma) = \sigma^* E \arrow[r] & E''
\end{tikzcd}$$

By assumption, $\delta(\sigma)$ is not the trivial extension, so $\delta(\sigma) \neq 0$ unless $\sigma = 0$. In other words, δ is injective.

In a similar way, the map $\delta : \mathrm{Hom}(E', E'') \to \mathrm{Ext}^1(E', E')$ is injective. The injectivity of these maps implies that the maps $p^* : \mathrm{Hom}(E, E'') \to \mathrm{Hom}(E', E'')$ and $q_* : \mathrm{Hom}(E', E) \to \mathrm{Hom}(E', E'')$ are zero.

Now, the 2 synthesizes the situation where we have dropped the maps when they are zero.
From this diagram, we deduce the short exact sequence

$$0 \to \mathrm{Hom}(E'', E') \to \mathrm{Hom}(E, E) \to \ker\big(\mathrm{Hom}(E', E) \to \mathrm{Ext}^1(E'', E)\big) \to 0.$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \mathrm{Hom}(E'',E'') \arrow[d, hook] \\
0 \arrow[r] & \mathrm{Hom}(E'',E') \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Hom}(E,E') \arrow[d, hook] & \mathrm{Hom}(E',E') \arrow[r, hook] \arrow[d, "\sim"'] & \mathrm{Ext}^1(E'',E') \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(E'',E) \arrow[r, hook] & \mathrm{Hom}(E,E) \arrow[r] \arrow[d] & \mathrm{Hom}(E',E) \arrow[r] & \mathrm{Ext}^1(E'',E) \arrow[d, two heads] \\
0 \arrow[r] & \mathrm{Hom}(E'',E'') \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Hom}(E,E'') \arrow[d] & \mathrm{Hom}(E',E'') \arrow[r, hook] \arrow[d, hook] & \mathrm{Ext}^1(E'',E'') \\
\mathrm{Hom}(E',E') \arrow[r, hook] & \mathrm{Ext}^1(E'',E') \arrow[r] \arrow[d] & \mathrm{Ext}^1(E,E') \arrow[r, two heads] \arrow[d] & \mathrm{Ext}^1(E',E') \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

FIG. 2: The simplified version for the commutative diagram in 1

which is a mere traslation of the exact sequence of the second row of 2. So, it remains to determined the last term of the short exact sequence. Looking at the 2, we see that it is isomorphic to

$$\ker\left(\mathrm{Hom}(E',E') \to \mathrm{Ext}^1(E'',E')/\mathrm{Hom}(E'',E'')\right)$$

that is, a map $\alpha : E' \to E'$ lives in this kernel if there exists another map $\beta : E'' \to E''$ such that $\alpha_* E \cong \beta^* E$.

This case leads to the hypothesis of Lemma 5.6: recall that $\alpha_* E$ is $[p \circ \alpha^{-1}, E, q]$, and $\beta^* E$ is $[p, E, \beta^{-1} \circ q]$, so if they are equivalent, there exists $\Psi : E \to E$ such that the following diagram

$$\begin{tikzcd}
E'' \arrow[r, "p\circ\alpha^{-1}"] \arrow[d, equal] & E \arrow[r, "q"] \arrow[d, "\Psi"] & E' \arrow[d, equal] \\
E'' \arrow[r, "p"'] & E \arrow[r, "\beta^{-1}\circ q"'] & E''
\end{tikzcd}$$

gives the same properties for α and β of the mentioned lemma, that is

$$\begin{cases} \Psi \circ p = p \circ \alpha, \\ q \circ \Psi = \beta \circ q. \end{cases}$$

So we conclude that $\Psi = \lambda 1_E$ and hence $\alpha = \lambda 1_{E''}$ and $\beta = \lambda 1_{E'}$. This implies the isomorphism

$$\ker\left(\mathrm{Hom}(E',E') \to \mathrm{Ext}^1(E'',E')/\mathrm{Hom}(E'',E'')\right) \cong \mathbb{C} \cdot 1_{E'}.$$

We have proved the following statement.

Theorem 5.9. *Let E be a bundle in a short exact sequence*

$$0 \longrightarrow E' \xrightarrow{p} E \xrightarrow{q} E'' \longrightarrow 0$$

which does not admit any partial section (left or right). Then, the group $\mathrm{End}(E)$ *satisfies the short exact sequence*

$$0 \to \mathrm{Hom}(E'',E') \to \mathrm{End}(E) \to \mathbb{C} \cdot 1_E \to 0.$$

TABLE 1: AUTORMORPHISM FOR $F \to E \to L$

Case	Short exact sequence	Endomorphism	Automorphism	Matrix
2.a	$F \xrightarrow{p} E \xrightarrow{q} L_1$	$E \xrightarrow{1_E} E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$
2.b	$L_1 \oplus L_2 \xrightarrow{p} E \xrightarrow{q} L_2$	$1_E : E \longrightarrow E$ $E \xrightarrow{q} L_2 \hookrightarrow L_2 \oplus L_1 \xrightarrow{p} E$	$\mathbb{C}^\times \times \mathbb{C}$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \alpha \\ 0 & 0 & \lambda \end{pmatrix}$
2.b.1	+ section $s : E \to L_2$ $L_2 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_2)$	$1_F : F \to F$ $1_{L_2} : L_2 \to L_2$ $F \xrightarrow{q'} L_2 \xrightarrow{\sim} L_2$	$\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}$	$\begin{pmatrix} \mu & 0 & \alpha \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
2.b.2	+ section $s : E \to L_1$ $L_1 \oplus (L_2 \xrightarrow{p'} F \xrightarrow{q'} L_2)$	$1_F : F \to F$ $1_{L_1} : L_1 \to L_1$ $F \xrightarrow{q'} L_2 \xrightarrow{\sim} L_2 \xrightarrow{p'} F$	$\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}$	$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & \alpha \\ 0 & 0 & \lambda \end{pmatrix}$
2.c	$L_1 \oplus L_2 \xrightarrow{p} E \xrightarrow{q} L_3$	$1_E : E \to E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
2.c.1	+ section $s : E \to L_2$ $L_2 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_3)$	$1_F : F \to F$ $1_{L_2} : L_2 \to L_2$	$\mathbb{C}^\times \times \mathbb{C}^\times$	$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
2.c.2	+ section $s : E \to L_1$ $L_1 \oplus (L_2 \xrightarrow{p'} F \xrightarrow{q'} L_3)$	$1_F : F \to F$ $1_{L_1} : L_1 \to L_1$	$\mathbb{C}^\times \times \mathbb{C}^\times$	$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
2.d	$L_1 \oplus L_1 \xrightarrow{p} E \xrightarrow{q} L_2$	$E \xrightarrow{1_E} E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
2.d.1	+ section $s : E \to L_1$ $L_1 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_2)$	$F \xrightarrow{1_F} F$ $L_1 \xrightarrow{1_{L_1}} L_1$ $L_1 \xrightarrow{\sim} L_1 \xrightarrow{p'} F$	$\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}$	$\begin{pmatrix} \mu & 0 & 0 \\ \alpha & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
2.e	$L_1 \oplus L_1 \xrightarrow{p} E \xrightarrow{q} L_1$	$E \xrightarrow{1_E} E$ $E \xrightarrow{q} L_1 \xrightarrow{i} L_1 \oplus L_1 \xrightarrow{p} E$	$\mathbb{C}^\times \times \mathbb{C} \times \mathbb{C}$	$\begin{pmatrix} \lambda & 0 & \alpha \\ 0 & \lambda & \beta \\ 0 & 0 & \lambda \end{pmatrix}$
2.e.1	+ section $s : E \to L_1$ $L_1 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_1)$	$1_F : F \to F$ $1_{L_1} : L_1 \to L_1$ $F \xrightarrow{q'} L_1 \xrightarrow{\sim} L_1$ $F \xrightarrow{q'} L_1 \xrightarrow{\sim} L_1 \xrightarrow{p'} F$ $L_1 \xrightarrow{\sim} L_1 \xrightarrow{p} F$	$\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^3$	$\begin{pmatrix} \mu & 0 & \alpha \\ \gamma & \lambda & \beta \\ 0 & 0 & \lambda \end{pmatrix}$

Furthermore, the group $\mathrm{Aut}(E)$ *is isomorphic to* $(\mathbb{C}^\times \cdot 1_E) \times \mathrm{Hom}(E'', E')$.

■ **Table of automorphisms of extensions** Here, we provide a table with all possibilities for the group of automorphisms of the current extension.

The Table 1 runs along all possibilities for the case $0 \to F \to E \to L \to 0$ where F is of rank 2 (and hence $\operatorname{rk} L = 1$).

To read the information properly, we have to remember this:

- The first column is the "label" of the case (we start at 2, we leave the label 1 for the case E polystable); here, the subcases (e. g. 2.c.1 or 2.c.2) are the partially split cases, so we give the section and the splitting in the next column.
- The second column is the short exact sequence.
- The third column are the generators for $\mathrm{End}(E)$.
- The fourth column is the multiplicative group of automorphisms.
- The fifth column is the matrix representation for an autormorphism, under the assumption that E splits completely (which does as a differential real bundle); the parameters λ and μ are the coefficient for identity maps, while α, β and γ are coefficients for maps which comes from $\mathrm{Hom}(E', E'')$.

In cases 2.d.1 and 2.e.1, there are several possibilities for sections $s : E \to L_1$, more concretely, such possibilities are parametrized by $\mathbb{P}^1$.

Table 2 computes all cases for $0 \to L \to E \to F \to 0$ where, as before, F is a polystable bundle of rank 2, and L is a line bundle. The possibilities for this case are in a one-to-one correspondence with cases 2.x. Such a correspondence becomes an equivalence in the cases where the short exact sequence has a partial splitting.

Table 2: Autormorphism for $L \to E \to F$

Case	Short exact sequence	Endomorphism	Automorphism	Matrix
3.a	$L_1 \xrightarrow{p} E \xrightarrow{p} F$	$E \xrightarrow{1_E} E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$
3.b	$L_1 \xrightarrow{p} E \xrightarrow{q} L_1 \oplus L_2$	$1_E : E \longrightarrow E$ $E \xrightarrow{q} L_1 \oplus L_2 \longrightarrow L_1 \xrightarrow{p} E$	$\mathbb{C}^\times \times \mathbb{C}$	$\begin{pmatrix} \lambda & \alpha & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
3.b.1	+ section $s : L_1 \to E$ $L_1 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_2)$	**Equivalent to case 2.b.1**		
3.b.2	+ section $s : L_2 \to E$ $L_2 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_2)$	**Equivalent to case 2.b.2**		
3.c	$L_1 \xrightarrow{p} E \xrightarrow{q} L_2 \oplus L_3$	$1_E : E \longrightarrow E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
3.c.1	+ section $s : L_2 \to E$ $L_2 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_3)$	**Equivalent to case 2.c.1**		
3.c.2	+ section $s : L_3 \to E$ $L_3 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_2)$	**Equivalent to case 2.c.2**		
3.d	$L_2 \xrightarrow{p} E \xrightarrow{q} L_1 \oplus L_1$	$1_E : E \longrightarrow E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
3.d.1	+ section $s : L_1 \oplus L_1 \to E$ $L_1 \oplus (L_2 \xrightarrow{p'} F \xrightarrow{q'} L_1)$	**Equivalent to case 2.d.1**		
3.e	$L_1 \xrightarrow{p} E \xrightarrow{q} L_1 \oplus L_1$	$1_E : E \longrightarrow E$ $E \xrightarrow{q} L_1 \oplus L_1 \xrightarrow{\pi} L_1 \xrightarrow{p} E$	$\mathbb{C}^\times \times \mathbb{C}^2$	$\begin{pmatrix} \lambda & \alpha & \beta \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
3.e.1	+ section $s : L_1 \oplus L_1 \to E$ $L_1 \oplus (L_1 \xrightarrow{p'} F \xrightarrow{q'} L_1)$	**Equivalent to case 2.e.1**		

Finally, the Table 3 has the last cases where the bundle E is in a short exact sequence $0 \to L_1 \to E \to F \to 0$ where F also is in a short exact sequence $0 \to L_2 \to F \to L_3 \to 0$. In these cases, sections are not allowed, since any split would makes it as a direct sum.

Table 3: Autormorphism for $L_3 \to E \to F$ with $L_1 \to F \to L_2$

Case	Short exact sequence	Endomorphism	Automorphism	Matrix
4.a	$L_3 \xrightarrow{p_2} E \xrightarrow{q_2} F$ $L_1 \xrightarrow{p_1} F \xrightarrow{q_1} L_2$	$1_E : E \longrightarrow E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
4.b	$L_1 \xrightarrow{p_2} E \xrightarrow{q_2} F$ $L_1 \xrightarrow{p_1} F \xrightarrow{q_1} L_2$	$1_E : E \longrightarrow E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
4.c	$L_2 \xrightarrow{p_2} E \xrightarrow{q_2} F$ $L_1 \xrightarrow{p_1} F \xrightarrow{q_1} L_2$	$1_E : E \longrightarrow E$ $E \xrightarrow{q_2} F \xrightarrow{q_1} L_2 \xrightarrow{\sim} L_2 \xrightarrow{p_2} E$	$\mathbb{C}^\times \times \mathbb{C}$	$\begin{pmatrix} \lambda & 0 & \alpha \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
4.d	$L_2 \xrightarrow{p_2} E \xrightarrow{q_2} F$ $L_1 \xrightarrow{p_1} F \xrightarrow{q_1} L_1$	$1_E : E \longrightarrow E$	$\mathbb{C}^\times$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$
4.e	$L_1 \xrightarrow{p_2} E \xrightarrow{q_2} F$ $L_1 \xrightarrow{p_1} F \xrightarrow{q_1} L_1$	$1_E : E \longrightarrow E$ $E \xrightarrow{q_2} F \xrightarrow{q_1} L_1 \xrightarrow{\sim} L_1 \xrightarrow{p_2} E$	$\mathbb{C}^\times \times \mathbb{C}$	$\begin{pmatrix} \lambda & 0 & \alpha \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

■ **Some remarks** This new stratification is finer, so we get more strata. Moreover, it is grounded in simple facts than the *standard filtration*.

We fix the critical value σ_m and define $\sigma_m^+ = \sigma_m + \epsilon$ for $\epsilon > 0$ enough small so that there is no another critical value between σ_m and σ_m^+. First, the σ_m-slope of a triple $T = (E, L, \phi)$ is equal to the slope $\mu(E)$ of the bundle E.

Fact 5.10. *The bundle E must be semistable or polystable.*

An unstable bundle E has a subbundle $F \subset E$ with $\mu(F) > \mu(E)$. Hence the subtriple $T' = (F, 0, 0)$ is a σ_m^+-desestabilizing bundle for T.

Fact 5.11. *A σ_m^+-desestabilizing triple T' for $T = (E, L, \phi)$ is of the form (F, L, ϕ), where F is a proper subbundle with $\mu(F) = \mu(E)$.*

This is due the formula of σ_m^+-slope for these triples:

$$\mu(T') = \mu(F) + \epsilon \frac{1}{\mathrm{rk}(F) + 1} > \mu(E) + \epsilon \frac{1}{\mathrm{rk}(E) + 1} = \mu(T).$$

Both facts gives a description for σ_m^+-stable triples:

Lemma 5.12. *A σ_m^+-stable $T = (E, L, \phi)$ consists of a semistable (or polystable) bundle E where the image of L by the map ϕ is not included in any proper subbundle $F \subset E$ of the same slope.*

By this lemma, we parametrize the semistable bundles and we add a map $\phi : L \to E$ but taking care that the image of such map is not included in a subbundle of E of the same slope.

This stratification is equivalent to the former. Nonetheless, we have to precise how to parametrize these semistable bundles. To do this, we try to write E by a sequence of short exact sequences of polystable bundles as we have done with triples. This is easy: we parametrize E by the maximal polystable proper subbundle of the same slope. Then, we define a stratum by the following data:

- the ranks and degrees of the components of such maximal polystable bundles of the same slope. Since slopes are fixed, ranks and degrees are determined by each other.
- a set of partial sections of the short exact sequence.

This defines an unique stratum after taking into account that partial sections give partial splittings and this gives an equivalence between strata. This fact has been notice already in the table 2 (case 3.x).

■ The stratum for general cases

Using this list, it is easy to compute the class of the stratum in the most of cases.

Case 2 non-split We compute the space parametrizing the bundles E. The general situation is $0 \to F \to E \to L_3 \to 0$. We distinguish cases depending of the polystability of F. We denote by $\mathcal{E}$ this space. There exists a projection $\mathcal{E} \to B$ where

$$B = \begin{cases} M^s(2, \frac{2}{3}d_1) \times \operatorname{Jac} C & \text{if } F \text{ stable,} \\ (\operatorname{Jac} C)^k \setminus \Delta & \text{if } F = L_1 \oplus L_2, \end{cases}$$

where k denotes the number of non-isomorphic line bundles and Δ is the union of diagonals. The fibre $\operatorname{Fibre}(\mathcal{E} \to B)$ of the projection map $\mathcal{E} \to B$ is isomorphic to:

$$\begin{cases} F \text{ stable} & \Longrightarrow \operatorname{Fibre}(\mathcal{E} \to B) \cong \mathbb{P}\operatorname{Ext}^1(L_3, F) \\ F = L_1 \oplus L_2 \Longrightarrow & \begin{cases} \operatorname{Fibre}(\mathcal{E} \to B) \cong \mathbb{P}\operatorname{Ext}^1(L_3, L_1) \times \mathbb{P}\operatorname{Ext}^1(L_3, L_2) & \text{if } L_1 \ncong L_2 \\ \operatorname{Fibre}(\mathcal{E} \to B) \cong \operatorname{Gr}(2, \operatorname{Ext}^1(L_3, L_\bullet)) & \text{if } L_\bullet \cong L_1 \cong L_2 \end{cases} \end{cases}$$

Here $\operatorname{Fibre}(E \to X)$ denotes the fibre of the map $E \to X$.

Now, triples are given by maps $\phi : L \to E$. Let $\mathcal{F}$ be the space parametrizing such maps. There exists a projection $\mathcal{F} \to \mathcal{E} \times \operatorname{Jac} C$ sending (E, L, ϕ) to the pair (E, L). The fibre consists of all possible maps ϕ which give a stable triple, equivalently the image of ϕ does sit in not any subbundle of E of the same slope. The maximal subbundle is the image of F by p, so $\phi \in \operatorname{Hom}(L, E) \setminus \operatorname{Hom}(L, F)$. The group $\operatorname{Aut}(E) \times_{\mathbb{C}^\times} \operatorname{Aut}(L) \cong \operatorname{Aut}(E)$ identifies maps which produce isomorphic triples. Taking into account the following two facts:

- the space $\operatorname{Hom}(L, E) \setminus \operatorname{Hom}(L, F)$ projects onto $\operatorname{Hom}(L, L_3) \setminus \{0\}$ with fibres $\operatorname{Hom}(L, F)$, and
- $\operatorname{Aut}(E) \cong \operatorname{Hom}(L_3, F) \times (\mathbb{C}^\times \cdot 1_E)$, the subgroup $\operatorname{Hom}(L_3, F) \times \{1_E\}$ gives an action of the basis $\operatorname{Hom}(L, L_3) \setminus \{0\}$ to the fibre $\operatorname{Hom}(L, F)$,

the fibre of $\mathcal{F} \to \mathcal{E} \times \operatorname{Jac} C$ is isomorphic to

$$\left(\frac{\operatorname{Hom}(L, F)}{\operatorname{Hom}(L_3, F)} \right) \otimes U^*_{\mathbb{P}\operatorname{Hom}(L, L_3)}$$

where $U_{\mathbb{P}^n}$ denotes the tautological bundle over the projective space $\mathbb{P}^n$.

We leave the case 2.c aside for later, since this case has an action of $\mathbb{Z}_2$ which permutes L_1 with L_2.

Table 4 collects the class for every case in this group of strata.

Case 3. Non-split. Here, the bundle E satisfies the short exact sequence $0 \longrightarrow L_1 \xrightarrow{p} E \xrightarrow{q} F \longrightarrow 0$. This is exactly the case symmetric case 2. Hence arguments used in the previous case work here. In particular, the description of B, and $\mathcal{E}$ is the same, that is

$$\begin{cases} F \text{ stable} & \Longrightarrow \operatorname{Fibre}(\mathcal{E} \to B) \cong \mathbb{P}\operatorname{Ext}^1(F, L_1) \\ F = L_2 \oplus L_3 \Longrightarrow & \begin{cases} \operatorname{Fibre}(\mathcal{E} \to B) \cong \mathbb{P}\operatorname{Ext}^1(L_2, L_1) \times \mathbb{P}\operatorname{Ext}^1(L_3, L_1) & \text{if } L_1 \ncong L_2 \\ \operatorname{Fibre}(\mathcal{E} \to B) \cong \operatorname{Gr}(2, \operatorname{Ext}^1(L_\bullet, L_1)) & \text{if } L_\bullet \cong L_2 \cong L_3 \end{cases} \end{cases}$$

TABLE 4: CLASS FOR STRATUM $2.x$

Stratum	**Class**
2.a	$\mathbb{L}^{2(\mu-g+1)}\dfrac{(\mathbb{L}^{\mu-g+1}-1)(\mathbb{L}^{2(g-1)}-1)}{(\mathbb{L}-1)^2}\mathbb{M}_2^{\text{even}}\mathbb{J}^2$
2.b	$\mathbb{L}^{2(\mu-g)}\dfrac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{g}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}$
2.c	$\mathbb{L}^{2(\mu-g+1)}\dfrac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1}\left\{\lambda^2\left(\dfrac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}(\mathbb{J}-1)\right)-\lambda^2\left(\dfrac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)(\mathbb{J}-1)\right\}\mathbb{J}^2$
2.d	$\mathbb{L}^{2(\mu-g+1)}\dfrac{(\mathbb{L}^{\mu-g+1}-1)(\mathbb{L}^{g-1}-1)(\mathbb{L}^{g-2}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}$
2.e	$\mathbb{L}^{2(\mu-g-1)}\dfrac{(\mathbb{L}^{\mu-g+1}-1)(\mathbb{L}^{g}-1)(\mathbb{L}^{g-1}-1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\mathbb{J}^2$

Nevertheless, the fibre of $\mathcal{F}\to\mathcal{E}\times\operatorname{Jac}C$ is different. Here, the maximal proper subbundle is not unique as in previous case: any maximal proper subbundle of F gives a maximal proper subbundle of E by pullback, and this depends on F:

F stable	Unique maximal proper subbundle is $\{0\}$,
$F=L_2\oplus L_3$ and $L_2\ncong L_3$	Two maximal proper subbundles: L_2 and L_3,
$F=L_2\oplus L_3$ and $L_2\cong L_3$	A pencil of maximal proper subbundles corresponding to any inclusion $L_2\hookrightarrow L_2^{\oplus 2}$.

So, the image of $\phi: L\to E$ by the natural projection $\operatorname{Hom}(L,E)\to\operatorname{Hom}(L,F)$ induced by q lives in

F stable	$q_*(\phi)\in\operatorname{Hom}(L,F)\setminus\{0\}$
$F=L_2\oplus L_3$ and $L_2\ncong L_3$	$q_*(\phi)\in(\operatorname{Hom}(L,L_2)\setminus\{0\})\times(\operatorname{Hom}(L,L_3)\setminus\{0\})$
$F=L_2\oplus L_3$ and $L_\bullet\cong L_2\cong L_3$	$q_*(\phi)\in\mathcal{V}(2,\operatorname{Hom}(L,L_\bullet))$

Therefore, the fibre of the projection $\mathcal{F}\to\mathcal{E}\times\operatorname{Jac}C$ is

F stable	$\mathbb{P}\operatorname{Hom}(L,E)\setminus\mathbb{P}\operatorname{Hom}(L,L_1)$
$F=L_2\oplus L_3$ and $L_2\ncong L_3$	$\left(\dfrac{\operatorname{Hom}(L,L_1)}{\operatorname{Hom}(F,L_1)}\right)\otimes U^*_{\mathbb{P}\operatorname{Hom}(L,L_2\oplus L_3)}\setminus\left[\coprod_{i=2,3}\left(\dfrac{\operatorname{Hom}(L,L_1)}{\operatorname{Hom}(F,L_1)}\right)\otimes U^*_{\mathbb{P}\operatorname{Hom}(L,L_i)}\right]$
$F=L_2\oplus L_3$ and $L_\bullet\cong L_2\cong L_3$	$\left(\dfrac{\operatorname{Hom}(L,L_1)}{\operatorname{Hom}(F,L_1)}\right)\otimes U^*_{\mathcal{V}(2,\operatorname{Hom}(L,L_\bullet))/\mathbb{C}^\times}$

Table 5 gives all subcases for this case. The case 3.c has an action of a finite group, and this will be covered later.

Subcases 2 & 3. Partially split. All cases follow the pattern $E=L_1\oplus F$ where F satisfies the short exact sequence

$$0\longrightarrow L_2\xrightarrow{p}F\xrightarrow{q}L_3\longrightarrow 0.$$

The fibre of the map $\mathcal{E}\to B$ is the same in any case: $\operatorname{Fibre}(\mathcal{E}\to B)\cong\mathbb{P}\operatorname{Ext}^1(L_3,L_2)$. On the other hand, the fibre of $\mathcal{F}\to\mathcal{E}\times\operatorname{Jac}C$ is

$$\mathbb{P}\left(\frac{\operatorname{Hom}(L,L_1)}{\operatorname{Hom}(L_3,L_1)}\right)\times\left[\left(\frac{\operatorname{Hom}(L,L_2)}{\operatorname{Hom}(L_1\oplus L_3,L_2)}\right)\otimes U^*_{\mathbb{P}\operatorname{Hom}(L,L_3)}\right].$$

TABLE 5: CLASS FOR STRATUM $3.x$

Stratum	**Class**
3.a	$\mathbb{L}^{\mu-g+1}\dfrac{(\mathbb{L}^{2(g-1)}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^2}\mathbb{J}^2\mathbb{M}_2^{\text{even}}$
3.b	$\mathbb{L}^{\mu-g}\left(\dfrac{\mathbb{L}^{2(\mu+1-g)}-1}{\mathbb{L}-1}-2\dfrac{\mathbb{L}^{\mu+1-g}-1}{\mathbb{L}-1}\right)\dfrac{(\mathbb{L}^g-1)(\mathbb{L}^{g-1}-1)}{(\mathbb{L}-1)^2}(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}$
3.c	$\dfrac{\mathbb{L}^{\mu-g+1}}{\mathbb{L}-1}\left\{\lambda^2\left((\mathbb{L}^{\mu-g+1}-1)\dfrac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}(\mathbb{J}-1)\right)-\lambda^2\left((\mathbb{L}^{\mu-g+1}-1)\dfrac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)(\mathbb{J}-1)\right\}\cdot\mathbb{J}^2$
3.d	$\mathbb{L}^{\mu-g+1}\dfrac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{g-2}-1)(\mathbb{L}^{\mu-g+1}-1)(\mathbb{L}^{\mu-g+1}-\mathbb{L})}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}(\mathbb{J}^2-\mathbb{J})\mathbb{J}$
3.e	$\mathbb{L}^{\mu-1-g}\dfrac{(\mathbb{L}^g-1)(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu+1-g}-1)(\mathbb{L}^{\mu+1-g}-\mathbb{L})}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\mathbb{J}^2$

To prove this, it is easy to check that the two maximal proper subbundles of E are F and $L_1\oplus L_2$, so, by the splitting $\operatorname{Hom}(L,E)\cong\operatorname{Hom}(L,L_1)\oplus\operatorname{Hom}(L,F)$, we have that the map ϕ lives in

$$(\operatorname{Hom}(L,L_1)\setminus\{0\})\times(\operatorname{Hom}(L,F)\setminus\operatorname{Hom}(L,L_2)).$$

Now, look at the group action of $\operatorname{Aut}(E)$: here we have several components:

- The group $\operatorname{Aut}(L_1)\cong\mathbb{C}^\times$ acts on the factor $\operatorname{Hom}(L,L_1)\setminus\{0\}$ projectivizing it.
- The group generated by 1_F also projectivizing the factor $\operatorname{Hom}(L,F)\setminus\operatorname{Hom}(L,L_2)$. This gives the space
$$\mathbb{P}\operatorname{Hom}(L,F)\setminus\mathbb{P}\operatorname{Hom}(L,L_2)\cong\operatorname{Hom}(L,L_2)\otimes U^*_{\mathbb{P}\operatorname{Hom}(L,L_3)}.$$
- The other subgroups mix up the factors:
  - $\operatorname{Hom}(L_1,F)\cong\operatorname{Hom}(L_1,L_2)$ makes an action of $\operatorname{Hom}(L,L_1)$ over $\operatorname{Hom}(L,L_2)$;
  - $\operatorname{Hom}(F,L_1)\cong\operatorname{Hom}(L_3,L_1)$ becomes an action of the fibre $\operatorname{Hom}(L,L_3)$ to $\operatorname{Hom}(L,L_1)$;
  - and finally $\operatorname{Hom}(L_3,L_2)$ is an action of the basis to the fibre of $\operatorname{Hom}(L,L_2)\otimes U^*_{\mathbb{P}\operatorname{Hom}(L,L_3)}$.

Table 6 has the list of the classes for every stratum of the splitted case.

Case 4. This case follows the pattern: E satisfies a short exact sequence

$$0\longrightarrow L_3\xrightarrow{p_2}E\xrightarrow{q_2}F\longrightarrow 0,$$

where F also satisfies

$$0\longrightarrow L_2\xrightarrow{p_1}F\xrightarrow{q_1}L_1\longrightarrow 0.$$

Both short exact sequences cannot be partially split (in that case, we come back to cases 2 and 3).

The space parametrizing the bundle E, denoted by $\mathcal{E}$, must be constructed in two steps: the first step yields a bundle $\mathcal{E}_1$ over B whose fibres are isomorphic to $\mathbb{P}\operatorname{Ext}^1(L_1,L_2)$. The second step is a bundle $\mathcal{E}$ over $\mathcal{E}_1$ whose fibres are isomorphic to $\mathbb{P}\operatorname{Ext}^1(F,L_3)\setminus\mathbb{P}\left(\frac{\operatorname{Ext}^1(L_1,L_3)}{\operatorname{Hom}(L_2,L_3)}\right)$. We remove the subspace $\operatorname{Ext}^1(L_2,L_3)$ because the extensions living in this subspace are partially split by the exact sequence in cohomology

$$\cdots\to\operatorname{Hom}(L_2,L_3)\to\operatorname{Ext}^1(L_1,L_3)\to\operatorname{Ext}^1(F,L_3)\to\operatorname{Ext}^1(L_2,L_3)\to 0.$$

Table 6: Class for stratum $2.x$ or $3.x$ for partially split sequences

Stratum	Class
2.b.1	$\mathbb{L}^{\mu-g+1}\frac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu-g}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}(\mathbb{J}^2-\mathbb{J})\mathbb{J}$
2.b.2	$\mathbb{L}^{\mu-g}\frac{(\mathbb{L}^{g}-1)(\mathbb{L}^{\mu-g+1}-1)^2}{(\mathbb{L}-1)^3}(\mathbb{J}^2-\mathbb{J})\mathbb{J}$
2.c.1/2	$\mathbb{L}^{\mu-g+1}\frac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu-g+1}-1)^2}{(\mathbb{L}-1)^3}(\mathbb{J}^3-3\mathbb{J}^2+2\mathbb{J})\mathbb{J}$
2.d.1	$\mathbb{L}^{\mu-g}\frac{(\mathbb{L}^{g}-1)(\mathbb{L}^{\mu-g}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}(\mathbb{J}^2-\mathbb{J})\mathbb{J}$
2.e.1	$\mathbb{L}^{\mu-g-1}\frac{(\mathbb{L}^{g}-1)(\mathbb{L}^{\mu-g}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}\mathbb{J}^2$

Parametrizing the space of maps $\varphi : L \to E$, $\mathcal{F}$ is easier: the maximal proper subbundle of E is p_1^*E. So the fibre of the map $\mathcal{F} \to \mathcal{E} \times \operatorname{Jac} C$ is given by $\operatorname{Hom}(L, E) \setminus \operatorname{Hom}(L, p_1^*E)$. By the short exact sequence

$$0 \to p_1^*E \to E \to L_1 \to 0$$

we have the space $\operatorname{Hom}(L, E) \setminus \operatorname{Hom}(L, p_1^*E)$ projects onto $\operatorname{Hom}(L, L_1) \setminus \{0\}$ with fibres isomorphic to $\operatorname{Hom}(L, p_1^*E)$. Now, $\operatorname{Aut}(E)$ acts on this space. The quotient is

$$\left(\frac{\operatorname{Hom}(L, p_1^*E)}{\operatorname{Hom}(L_1, L_3)}\right) \otimes U^*_{\mathbb{P}\operatorname{Hom}(L,L_1)}.$$

Table 7 gives a detailed description for the class of every stratum.

Table 7: Class for stratum $4.x$

Stratum	Class
4.a	$\mathbb{L}^{2\mu-g+1)}\frac{(\mathbb{L}^{g-1}-1)^2(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}(\mathbb{J}^3-3\mathbb{J}^2+2\mathbb{J})\mathbb{J}$
4.b	$\mathbb{L}^{2\mu-g}\frac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{g}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}(\mathbb{J}^2-\mathbb{J})\mathbb{J}$
4.c	$\mathbb{L}^{2\mu-g+1}\frac{(\mathbb{L}^{g-1}-1)(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}(\mathbb{J}^2-\mathbb{J})\mathbb{J}$
4.d	$\mathbb{L}^{2\mu-g+1}\frac{(\mathbb{L}^{g}-1)(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}(\mathbb{J}^2-\mathbb{J})\mathbb{J}$
4.e	$\mathbb{L}^{2\mu-g}\frac{(\mathbb{L}^{g}-1)(\mathbb{L}^{g-1}-1)(\mathbb{L}^{\mu-g+1}-1)}{(\mathbb{L}-1)^3}\mathbb{J}^2$

This finishes the computation of all stratum except the cases 2.c and 3.c which deserve a greater space.

The non trivial strata In cases 2.c and 3.c, we have to rewrite bundles $\mathcal{E}$ and $\mathcal{F}$ in an explicit way so that the action $\mathbb{Z}_2$ which permutes L_1 with L_2 (or L_2 with L_3), permutes factors in the fibres of these bundles.

Since in both cases we have the same basis for $\mathcal{E}$, the space parametrizing the stable bundle $F(3, \operatorname{Jac} C) \times \operatorname{Jac} C$, we rearrange such a basis to make clear the action of $\mathbb{Z}_2$ which permutes L_1 with L_2. The correct way is the following: take $(L_1, L_3) \in F(2, \operatorname{Jac} C)$ and $(L_2, L_3) \in F(2, \operatorname{Jac} C)$, then

$$F(3, \operatorname{Jac} C) \cong (F(2, \operatorname{Jac} C) \times_{\operatorname{Jac} C} F(2, \operatorname{Jac} C)) \setminus \left(F(2, \operatorname{Jac} C) \times_{F(2, \operatorname{Jac} C)} F(2, \operatorname{Jac} C)\right),$$

where the first fibre product identifies the common factor L_3.

Case 2.c. For $i = 1, 2$ denote by $\mathcal{E}_i$ the bundle over $F(2, \operatorname{Jac} C)$ —which parametrizes pairs (L_i, L_3)— with fibres isomorphic to $\mathbb{P}\operatorname{Ext}^1(L_3, L_i)$. Then, easily one check the equality

$$\mathcal{E} \cong (\mathcal{E}_1 \times_{\operatorname{Jac} C} \mathcal{E}_2) \setminus \left(\mathcal{E}_1 \times_{F(2, \operatorname{Jac} C)} \mathcal{E}_2\right).$$

Here, clearly $\mathbb{Z}_2$ permutes these factors.

The bundle $\mathcal{F}$ is easier to be written down in factors. Consider a bundle over $B_1 \to \mathcal{E}_i \times \operatorname{Jac} C$ with fibres $\mathbb{P}\operatorname{Hom}(L, L_3)$. Over this space B_i, it is possible to define a bundle $\mathcal{F}_i$ with fibres isomorphic to $\operatorname{Hom}(L, L_i) \otimes U^*_{\operatorname{Hom}(L, L_3)}$.

Both $\mathcal{F}_i$ project onto a common space: consider a bundle $\mathcal{C} \to (\operatorname{Jac} \mathcal{C})^{\in}$ (parametrizing pairs (L_3, L)) with fibres isomorphic to $\mathbb{P}\operatorname{Hom}(L, L_3)$. Also, consider an extension $\bar{\mathcal{C}}$ of $\mathcal{C}$ where the basis is now $F(2, \operatorname{Jac} C) \times \operatorname{Jac} C$ (here, the basis parametrizes (L_i, L_3, L)). Then, we have projections $\mathcal{F}_i \to \mathcal{C}$ and $\mathcal{F}_i \to \bar{\mathcal{C}}$, and the stratum is

$$\mathcal{F}_1 \times_{\mathcal{C}} \mathcal{F}_2 \setminus \mathcal{F}_1 \times_{\bar{\mathcal{C}}} \mathcal{F}_2.$$

The group $\mathbb{Z}_2$ permutes the fibres of $\mathcal{F}_i \to \mathcal{C}$ and $\mathcal{F}_i \to \bar{\mathcal{C}}$. Then, the quotient is

$$\operatorname{Sym}^2_{\mathcal{C}} \mathcal{F}_\bullet \setminus \operatorname{Sym}^2_{\bar{\mathcal{C}}} \mathcal{F}_\bullet.$$

The class of this stratum is

$$\mathbb{L}^{2(\mu-g+1)} \frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1} \left[\lambda^2 \left(\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}(\mathbb{J}-1)\right) - \lambda^2 \left(\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)(\mathbb{J}-1)\right]\mathbb{J}^2.$$

Rearranging this equality by using the formula for $\lambda^2(\mathbb{P}^n[X])$ we get

$$\begin{aligned}
\frac{1}{2}\mathbb{L}^{2(\mu-g+1)} \frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1} &\left[\left(\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)^2 (\mathbb{J}-1)^2 - \left(\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)^2 (\mathbb{J}-1) + \right.\\
&\left. + \frac{\mathbb{L}^{2(g-1)}-1}{\mathbb{L}^2-1}\left(\delta^2(\mathbb{J})-1\right) - \frac{\mathbb{L}^{2(g-1)}-1}{\mathbb{L}^2-1}(\mathbb{J}-1)\right]\mathbb{J}^2 = \\
&= \frac{1}{2}\mathbb{L}^{2(\mu-g+1)} \frac{\mathbb{L}^{\mu-g+1}-1}{\mathbb{L}-1}\left[\left(\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)^2 \left((\mathbb{J}-1)^2 - (\mathbb{J}-1)\right) + \right.\\
&\left. + \frac{\mathbb{L}^{2(g-1)}-1}{\mathbb{L}^2-1}\left(\delta^2(\mathbb{J})+\mathbb{J}\right)\right]\mathbb{J}^2.
\end{aligned}$$

Case 3.c. Now, to write conveniently the bundle $\mathcal{E}$, we use the same notation as in case 2.c. Now, $\mathcal{E}_i$ is a bundle over $F(2, \operatorname{Jac} C)$ with fibres isomorphic to $\mathbb{P}\operatorname{Ext}^1(L_i, L_1)$ where $i = 2, 3$. So,

$$\mathcal{E} \cong (\mathcal{E}_2 \times_{\operatorname{Jac} C} \mathcal{E}_3) \setminus \left(\mathcal{E}_2 \times_{F(2, \operatorname{Jac} C)} \mathcal{E}_3\right).$$

The fibre of $\mathcal{F} \to \mathcal{E}$ is now isomorphic to

$$\operatorname{Hom}(L, L_1) \otimes U^*_{\mathbb{P}\operatorname{Hom}(L, L_2 \oplus L_3) \setminus [\mathbb{P}\operatorname{Hom}(L, L_2) \sqcup \mathbb{P}\operatorname{Hom}(L, L_3)]}.$$

To write this conveniently, observe that the action of $\mathrm{Aut}(E)$ commutes with the $\mathbb{Z}_2$ action: this means that we can consider the space of maps $\phi \in \mathrm{Hom}(L, E)$ which produce a σ_m^+-stable triple as a bundle over

$$(\mathrm{Hom}(L, L_2) \setminus \{0\}) \times (\mathrm{Hom}(L, L_3) \setminus \{0\})$$

whose fibre is $\mathrm{Hom}(L, L_1)$ and the projection onto the fibre $\mathrm{Fibre}(\mathcal{F} \to \mathcal{E} \times \mathrm{Jac}\, C)$ is a principal $\mathbb{C}^\times$-bundle where the action of $\mathbb{Z}_2$ is trivial in such group fibres. Then, consider $\tilde{\mathcal{F}}_i$ a bundle over $\mathcal{E}_i \times \mathrm{Jac}\, C$ with fibres $\mathrm{Hom}(L, L_i) \setminus \{0\}$ for $i = 2, 3$. As before, consider

$$\tilde{\mathcal{F}}' = \tilde{\mathcal{F}}_2 \times_{(\mathrm{Jac}\, C)^2} \tilde{\mathcal{F}}_3 \setminus \tilde{\mathcal{F}}_2 \times_{F(2, \mathrm{Jac}\, C) \times \mathrm{Jac}\, C} \tilde{\mathcal{F}}_3$$

where we also identify the component $\mathrm{Jac}\, C$ which parametrizes the bundle L. Over this bundle, we define $\tilde{\mathcal{F}}$ to be the bundle over $\tilde{\mathcal{F}}'$ with fibres $\mathrm{Hom}(L, L_1)$. Clearly, $\tilde{\mathcal{F}}$ is a $\mathbb{C}^\times$-bundle over $\mathcal{F}$. The action of $\mathcal{F}$ permutes the factors of $\tilde{\mathcal{F}}'$. From this, the quotient is computed in terms of symmetric spaces.

Therefore, the class of this stratum is

$$\begin{aligned}
[\mathcal{F}/\mathfrak{S}_2] &= \frac{[\tilde{\mathcal{F}}/\mathfrak{S}_2]}{\mathbb{L}-1} = \frac{\mathbb{L}^{\mu+1-g}}{\mathbb{L}-1}[\tilde{\mathcal{F}}'/\mathfrak{S}_2] \\
&= \frac{\mathbb{L}^{\mu-g+1}}{\mathbb{L}-1}\left\{\lambda^2\left((\mathbb{L}^{\mu-g+1}-1)\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}(\mathbb{J}-1)\right) - \lambda^2\left((\mathbb{L}^{\mu-g+1}-1)\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)(\mathbb{J}-1)\right\}\cdot \mathbb{J}^2.
\end{aligned}$$

Using the formula $\lambda^2(X) = \frac{1}{2}(X^2 + \delta^2(X))$, we have

$$\begin{aligned}
[\mathcal{F}/\mathfrak{S}_2] = \frac{1}{2}\frac{\mathbb{L}^{\mu+1-g}}{\mathbb{L}-1}\Bigg\{&(\mathbb{L}^{\mu+1-g}-1)^2\left(\frac{\mathbb{L}^{g-1}-1}{\mathbb{L}-1}\right)^2(\mathbb{J}^2-3\mathbb{J}+2)+ \\
&+(\mathbb{L}^{2(\mu+1-g)}-1)\frac{\mathbb{L}^{2(g-1)}-1}{\mathbb{L}-1}(\delta^2(\mathbb{J})-\mathbb{J})\Bigg\}.
\end{aligned}$$

The sum of all these classes is lengthy and annoying to write. To do this, we use Mathematica. It is more interesting (and useful) to write down the outline of the Mathematica sheet than to write the intermediate steps of such a computation.

7 The Mathematica sheet

To make the computation as simple as possible, we set $\mu = 2g - 1$.

The class of $\mathbb{L}$ shall be denoted by `t` in Mathematica. First, we define the classes of the projective space, the Stiefel Variety, the Grassmanian variety and the Moduli of bundles of even and odd degree:

```
P[g_]:=(t^(g+1)-1)/(t-1)
V[k_,N_]:=Product[(1-t^(N-i))/(1-t^(i+1)),{i,0,k-1}]
Gr[k_,N_]:=Product[(1-t^(N-i))/(1-t^(i+1)),{i,0,k-i}]
M2[0]:=1/(2*(1-t)*(1-t^2))*
       (2*J[1,1]*J[1,t]-J[1,1]^2*(1+2*t^(g+1)-t^2)-J[2,-1]*(1-t)^2)
M2[1]:=(J[1,1]*J[1,t]-t^g*J[1,1]^2)/((1-t)*(1-t^2))
```

Notice that `J[1,1]` is the class of the Jacobian variety $\mathbb{J}$ or $\zeta_{\bar{C}}(1)$. The other class are (as one can guess) `J[1,t^k]` is $\zeta_{\bar{C}}(\mathbb{L}^k)$ for any k, `J[2,-1]` is $\delta^2(\zeta_{\bar{C}}(1))$, and `J[3,1]` is $\delta^3(\zeta_{\bar{C}}(1))$.

Now, we write the operators λ^2 and λ^3 and a weird operator denoted by `Sym2Cx` which corresponds with the twisting quotient by $\mathfrak{S}_2$ of stratum No. 6 or case 3.c.

```
Sym2[f_]:=Simplify[1/2*(f^2+(f/.{t->t^2,J[k_,t_]->J[2*t,-(t)^2]}))]
Sym3[f_]:=Symplify[1/6*(f^3+3*(f/.{t->t^2,J[k_,t_]->J[2*t,-(t)^2]})*f+
          2*(f/.{t->t^2,J[k_,t_]->J[2*k,-(t)^2]}))]
Sym2Cx[f_]:=Simplify[1/2((t-1)*f^2+(t+1)*
            (f/.{t->t^2,J[k_,t_]->J[2*t,-(t)^2]}))]
```

Below, we write all strata. Since we have computed these classes in two differents ways, we denote `S[i]` the stratum computed in the first way, and `SS[i]` the other result of this computation. Clearly, if you put the following input

```
Simplify[S[i]-SS[i]]
```

The result should be `0`. In the following lines we write the input of some classes, as an example:

- In the fisrt stratum there is the quotient by $\mathfrak{S}_3$. Here is an example how to introduce this in Mathematica:

```
S[1] = Refine[(Sym3[P[m+1-g]*J[1,1]] -
       Sym2[P[m+1-g]]*P[m+1-g]*(J[1,1]^2-J[1,1]) -
       Sym3[P[m+1-g]]*J[1,1])*N1, t > 0];
```

- The stratum 6 uses the operator `Sym2Cx`:

```
S[6] = Refine[
       t^(m+1-g)/(t-1)*(Sym2[(t^(g-1)-1)*P[m+1-g]*(J[1,1]-1)] -
       Sym2[(t^(g-1)-1)*P[m+1-g]]*(J[1,1]-1))*J[1,1]*N1,
       t > 0];
```

- The stratum 7 should be

```
S[7] = Refine[( (Sym2[P[m]*(J[1,1]-1)]*P[m+1-g]*J[1,1] -
       Sym2[P[m]]*(J[1,1]-1)*P[m+1-g]*J[1,1]) -
       (Sym2[P[m+1-g]*(J[1,1]-1)]*P[m+1-g]*J[1,1]-
       Sym2[P[m+1-g]]*(J[1,1]-1)*P[m+1-g]*J[1,1]))*J[1,1],t>0]//Simplify
```

If we use the `Refine` command of Mathematica is due to the program does not simplify "properly"; in our case, Mathematica does not simplify $(t^2)^{\frac{1}{2}d}$ to t^d. We have to make explicit that the `t` variable is positive. Notice this is only necessary for these strata which we use `Sym2` and `Sym3` operators.

Once one has written all strata in Mathematica using these defined operators, we define the variable `Suma` to be the sum of all strata except the fourth stratum. Our next task is to compute the class of $\mathcal{N}^s_{\sigma_m^+}(3,1,d_1,d_2)$ for an adequate d_1 and d_2 (we choose $d_1 = 6g-3$ and $d_2 = 0$). It is not difficult to compute residues using Mathematica. We define the residue of the polynomial

$$F(x) = \frac{\zeta_{\bar{C}}(\mathbb{L}^k)}{\prod_{i=1}^r (1-\mathbb{L}^{k_i}x)x^{\mathtt{NN}}},$$

whose poles are in $x = \mathbb{L}^{-k_i}$. The code is the following

```
Resf[i_,k_,NN_]:=If[k[[i]]<0,
   -1/t^k[[i]]*J[1,t^(-k[[i]])]*t^(Sum[k[[i]]-k[[j]],{j,1,i-1}])/
   (t^(-k[[i]]*(NN+1))*Product[t^(k[[i]]-k[[j]])-1,{j,i+1,Length[k]}]),
   -1/t^k[[i]]*J[1,t^(k[[i]]-1)]*t^(k[[i]]*(NN+1))*
   t^(Sum[k[[i]]-k[[j]],{j,1,i-1}])/Product[i^k[[i]]-k[[j]])-1,{j,1,i-1}]*
   Product[1-t^(k[[j]]-k[[i]]),{j,i+1,Length[k]}]*t^(-(2*(-k[[i]])+1)*J)]
```

Recall this operator computes the residue of $F(x)$ at the i-th pole $x = \mathbb{L}^{-k_i}$. In other words,

$$\mathtt{Resf[i\_,k\_,NN\_]} = \operatorname{Res}_{x=\mathbb{L}^{-k_i}} \frac{1}{\prod_{i=1}^r (1-\mathbb{L}^{k_i}x)x^{\mathtt{NN}+1}} = \operatorname*{coeff}_{x^0} F(x).$$

Remember the input data of this operator are:

- `i` represents the number of the pole, that is, $x = \mathbb{L}^{-k_i}$ (in the natural order of $\{k_i\}$);
- `k` is a vector or a list of the exponents of all $(1-\mathbb{L}^{k_i}x)$ in the denominator.
- `NN` is the exponent of $\mathbb{L}$, in this case is $2g-1$.

Now, we apply this to the functions of three or four poles. To do this, we have to compute n_0 and $\bar{n}_0$. The number n_0 is the lowest integer number strictly greater than $\frac{\sigma_c+d_1+d_2}{2}$, so $n_0 = \frac{2}{3}d_1+1$ (recall $\sigma_c = \sigma_m = \frac{d_1}{3} - d_2$ and —this is very important— $\frac{d_1}{3}$ is an integer number). The integer number $\bar{n}_0$ is the even number such that $n_0 \le \bar{n}_0 \le n_0+1$, so $\bar{n}_0 = 2\left(\frac{1}{3}d_1+1\right)$. Then, $d_1-d_2-n_0 = \mu-1$ and $d_1-d_2-\bar{n}_0 = \mu-2$. The following commands compute the residue of $F(x)$:

```
F3[a_] := Sum[-Resf[i,Sort[{a,0,1}],m-1],{i,1,3}]//Simplify
F4[a_,b_] := Sum[-Resf[i,Sort[{a,b,0,1}],m-2],{i,1,4}]//Simplify
```

where the mathematical meaning is

$$\texttt{F3[a\_]} = \operatorname*{coeff}_{x^0} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^a x)x^{\mu-1}},$$
$$\texttt{F4[a\_,b\_]} = \operatorname*{coeff}_{x^0} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)(1-\mathbb{L}^a x)(1-\mathbb{L}^b x)x^{\mu-2}},$$

So, the class of $\mathcal{N}^s_{\sigma_m^+}(3,1,6g-3,0)$ can be computed with the following command:

```
N31s = Simplify[J[1,1]^2*(J[1,t]-t^g*J[1,1])/((1-t)^2*(1-t^2))*(
         t^(2*d1-2*d2-2*n)*F3[-2]-t^(2*g-2-2*d1+3*n)*F3[3])
       +J[1,1]^3*t^(g-1)/((1-t)^2*(1+t))*(
         t^(2*d1-2*d2-2*(n+1)+1)*F4[-2,-1]
         +t^(2*g-2-2*d1+3*(n+1))*F4[2,3]
         -(1+t)*t^(g-1-d2+(n+1)/2)*F4[-1,2])
       /.{d1->3*(m+d2),d2->d2,n->2*(m+d2)+1},
       Assumptions->g>2]
```

where we obtained the class

$$[\mathcal{N}^s_{\sigma_m^+}(3,1,d_1,d_2)] = \left\{ \mathbb{L}^{g-1}\frac{\mathbb{L}^{2g+2}+\mathbb{L}^{2\mu}+\mathbb{L}^{1+g+\mu}}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\mathbb{J}^2 - \right.$$
$$- \frac{2\mathbb{L}^{2g+1}+\mathbb{L}^{\mu+g-1}+\mathbb{L}^{2+g+\mu}+2\mathbb{L}^{2\mu}}{(\mathbb{L}-1)(\mathbb{L}^3-1)}\zeta_{\bar{C}}(\mathbb{L})\,\mathbb{J} -$$
$$\left. - \frac{\mathbb{L}^{2(\mu+1-g)}+\mathbb{L}^{\mu+1-g}+1}{(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\zeta_{\bar{C}}(\mathbb{L}^2)\,\mathbb{J}\right\} \frac{\mathbb{L}^{\mu+1-g}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)}\mathbb{J}^2$$

From now, it is possible to obtain the class of $M^s(3,0)$ (as soon as the reader don't make any mistake writting any strata!). One approach is the following:

```
CoefficientList[N31s-Suma,{J[3,1],J[2,-1],J[1,1],J[1,t],J[1,t^2]}]//Simplify
```

The output of this command it not easy to understand but it gives us the coefficients of `J[1,1]^4`, `J[1,1]^3*J[1,t]`, `J[1,1]^2*J[1,t]*J[1,t^2]`, and finally `J[1,1]^2*J[1,t]*J[3,1]`. The rest of the coefficients are `0`. So this difference is

$$\texttt{N31s-Suma} = \frac{\mathbb{L}^{3(\mu+1-g)}-1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\zeta_{\bar{C}}(\mathbb{L})\,\zeta_{\bar{C}}(\mathbb{L}^2)\,\mathbb{J}^2 - \frac{(\mathbb{L}^{3(\mu+1-g)}-1)(2\mathbb{L}^{2g+1}-\mathbb{L}^3+1)}{(\mathbb{L}-1)^3(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\zeta_{\bar{C}}(\mathbb{L})\,\mathbb{J}^3 -$$
$$- \frac{\mathbb{L}^{3(\mu+1-g)}-1}{3(\mathbb{L}^3-1)}\delta^3(\mathbb{J})\mathbb{J} + \frac{(\mathbb{L}^{3(\mu+1-g)}-1)(3\mathbb{L}^{3g+1}-3(\mathbb{L}^2-1)\mathbb{L}^{g+1}+(\mathbb{L}^2-1)^2)}{3(\mathbb{L}-1)^3(\mathbb{L}^2-1)^2}\mathbb{J}^4.$$

Finally, we get the main result of this chapter:

Theorem 5.13. *The class in $\bar{K}(\mathfrak{Var}_{\mathbb{C}})$ of $M^s(3,0)$ is*

$$[M^s(3,0)] = \frac{1}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\left\{\left(\frac{\mathbb{L}^{3g+1}}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)^2} - \frac{\mathbb{L}^{g+1}}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} + \frac{1}{(\mathbb{L}-1)^2}\right)\mathbb{J}^3 - \right.$$
$$- \frac{2\mathbb{L}^{2g+1}-\mathbb{L}^3+1}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\zeta_{\bar{C}}(\mathbb{L})\,\mathbb{J}^2 +$$
$$\left. + \frac{1}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\zeta_{\bar{C}}(\mathbb{L})\,\zeta_{\bar{C}}(\mathbb{L}^2)\,\mathbb{J}\right\} - \frac{1}{\mathbb{L}^2+\mathbb{L}+1}\delta^3(\mathbb{J}).$$

Remark 5.14. *Compare this result with the obtained one on $M^s(3,1)$:*

$$[M^s(3,1)] = \frac{\mathbb{L}^{3g-1}}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)^2}\mathbb{J}^3 - \frac{\mathbb{L}^{2g-1}}{(\mathbb{L}-1)^3(\mathbb{L}^3-1)}\mathbb{J}^2\zeta_{\bar{C}}(\mathbb{L}) + \\ + \frac{1}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\mathbb{J}\zeta_{\bar{C}}(\mathbb{L})\,\zeta_{\bar{C}}\big(\mathbb{L}^2\big)\,.$$

The output of the last command is quite obscure, it is difficult to associate each class with its corresponding coefficient. To make a clearer computation, one can do this step by step. First, we compute the coefficients of `J[3,1]`: we only have two.

```
C1=CoefficientList[N31-Suma,J[3,1]]
```

Now, `C1` is a list of two entries. The entry `C1[[2]]` is easy (the coefficient of $\delta^3(\mathbb{J})$), so we look for other coefficients in `C1[[1]]`:

```
C2=CoefficientList[C1[[1]],J[2,-1]]//Factor
```

Again, the coefficient of `J[2,-1]` is 0. Again, we compute the other coefficient on `C2[[1]]`:

```
C3=CoefficientList[C2[[1]],J[1,t^2]]//Simplify
```

This gives two entries, where the second entry, `C3[[2]]` is the coefficient of $\zeta_{\bar{C}}\big(\mathbb{L}^2\big)$. We look at the entry `C3[[1]]` and compute the coefficient of $\zeta_{\bar{C}}(\mathbb{L})$:

```
C4=CoefficientList[C3[[1]],J[1,t]]//Simplify
```

Now, both entries has a good look: the first one, `C4[[1]]` is the coefficient of $\mathbb{J}^4$ as one can check; the second one, `C4[[2]]` the coefficient of $\mathbb{J}^3\zeta_{\bar{C}}(\mathbb{L})$.

Chapter 6

Rank 4

This chapter is devoted to obtain the class of the moduli space of pairs of rank 4 in $K_0'(\mathfrak{Var})$ or $K_0'(\mathfrak{Mot})$. The classes of the strata that stratifies the flip loci $\mathcal{S}^{\pm}_{\sigma_c}$ for σ_c critical value are analogous to the corresponding classes obtained in $M^s(3,0)$ and $\mathcal{N}^s_\sigma(2,1,d_1,d_2)$ in their respective chapters. This makes the work easier and we do not need to repeat the arguments; we adapt the equalities to these cases. Unfortunately, the class of $\mathcal{N}_\sigma(4,1,d_1,d_2)$ is very large and can take many lines. Instead of to presenting the class, we prefer to give a different solution: a worksheet of Mathematica for a painless computation —similar to $M^s(3,0)$ one—. This points out a fairly simple way to get the class of $\mathcal{N}_\sigma(4,1,d_1,d_2)$ in a reasonable presentation to deal with it without spending a great amount of effort.

1 Critical values

We compute the critical values for the moduli space of pairs of rank 4. Let $T=(E,L,\phi)\in\mathcal{N}^{\mathrm{ss}}_{\sigma_c}$ a semistable triple. Such a triple belongs to a set, or $\mathcal{N}_{\sigma_c^+}$ (so it is a σ_c^--stable triple) either $\mathcal{N}_{\sigma_c^-}$ (so it is a σ_c^+-stable triple).

If $\sigma_c\in\mathbb{R}$ is a critical value for $T=(E,L,\phi)$, a properly σ_c-semistable triple of type $(4,1,d_1,d_2)$, there exists a σ_c-desestabilizing subtriple $T'\subset T$. Then, we have two cases

1) The triple T is a σ_c^+-stable triple (that is, $T\in\mathcal{N}_{\sigma_c^+}(4,1,d_1,d_2)$) where $\sigma_c^+=\sigma_c+\epsilon$ for $\epsilon>0$.
2) The triple T is a σ_c^--stable triple where $\sigma_c^-=\sigma_c-\epsilon$ for $\epsilon>0$.

We start with the first case. Let $T'\subset T$ be a σ_c-desestabilizing triple, we claim that T' is of type $(n',0,d',0)$, that is $T'=(F,0,0)$. In this case we have

$$\mu_{\sigma_c}(T)=\mu_{\sigma_c}(T'),\ \text{and}\ \mu_{\sigma_c^+}(T)>\mu_{\sigma_c^+}(T').$$

Let $\lambda(T)=\frac{n_2}{n_1+n_2}$ be denote the coefficient of σ in $\mu_\sigma(T)=\frac{d_1+d_2}{n_1+n_2}+\sigma\frac{n_2}{n_1+n_2}$. The above equality implies $\lambda(T)>\lambda(T')$, so

$$\frac{1}{n_1+1}>\frac{n_2}{n'+n_2}$$

and hence $n_2=0$. This shows our claim. In this case, we have the equality

$$\frac{d_1+d_2}{5}+\frac{1}{5}\sigma_c=\frac{d}{n},\ \text{thus}\ \sigma_c=\frac{5}{n}d-d_1-d_2$$

for $d\in\mathbb{Z}$ and $n=1,\cdots,4$. So that $\sigma_c=5d-d_1-d_2$ for $d\in\frac{1}{4}\mathbb{Z}\cup\frac{1}{3}\mathbb{Z}$.

For the second case we claim that the desestabilizing $T'\subset T$ has type $(n',1,d',d_2)$. As before, consider the slope $\lambda(T)=\frac{n_2}{n_1+n_2}$, the equalities

$$\begin{cases}\mu_{\sigma_c}(T)=\mu_{\sigma_c}(T),\\ \mu_{\sigma_c-\epsilon}(T)>\mu_{\sigma_c-\epsilon}(T).\end{cases}$$

gives that $\lambda(T)<\lambda(T')$. This inequality implies our claim on the type of T'.

Now, let $Q=T/T'$ be the quotient triple. Clearly, $Q=(F,0,0)$ and $\mu(Q)=\mu_{\sigma_c}(T)$ since it satisfies the short exact sequence $0\to T'\to T\to Q\to 0$. The equality of their slopes implies that

$$\sigma_c=\frac{5}{n}d-d_1-d_2$$

for $n=1,\cdots,4$ and $d\in\mathbb{Z}$.

Finally, the condition $\mu_m \le \sigma \le \mu_M$ gives the inequalities

$$\frac{d_1}{4} - d_2 \le \sigma_c \le \frac{2}{3}\left(d_1 - 4d_2\right).$$

Hence the bounds for the quotient $\frac{d}{n}$ are

$$\frac{d_1}{4} \le \frac{d}{n} \le \frac{1}{3}\left(d_1 - d_2\right).$$

Proposition 6.1. *The critical values for the moduli space $\mathcal{N}_\sigma(4,1,d_1,d_2)$ are*

$$\sigma_m = \frac{5}{6}m - d_1 - d_2,$$

for $m \in 2\mathbb{Z} \cup 3\mathbb{Z}$ and $\frac{3}{2}d_1 \le m \le 2(d_1 - d_2)$.

Proof. Summarizing the above dissertation: Since $n = 1, \ldots, 4$, $6\frac{d}{n}$ is an integer number and

$$6\frac{d}{n} \in \begin{cases} 2\mathbb{Z} & \text{if } n = 3, \\ 3\mathbb{Z} & \text{if } n = 2, 4, \\ 6\mathbb{Z} & \text{if } n = 1. \end{cases}$$

On the other hand, the case $n = 4$ occurs only when the subtriple $(E, 0, 0)$ is a desestabilizing triple of $T = (E, L, \phi)$, and this occurs only for $\sigma_c = \sigma_m$ or σ_M. Taking $m = 6d$, this finishes the proof. □

Remark 6.2. *The degree of the bundles of the triple S_i are given by the equality*

$$\mu(S_i) = \mu(T_i)$$

so $\frac{d_1+d_2+\sigma_m}{5} = \frac{d}{n}$, which is equivalent to $d = m \cdot \frac{n}{6}$. This means that not all strata appear on any critical value: a stratum appears if and only if any bundle involved in the stratum has integer "degree" $m\frac{n}{6}$.

■ The dimension of cohomology group

Along this chapter we have to compute the dimension of several cohomology groups. Such dimensions of $\mathrm{Ext}^1(T, S_i)$ or $\mathrm{Ext}^1(T, S_i)$ follow some patterns. To compute this, recall the formula of $\chi(T'', T')$ in Proposition 2.18. To simplify this computations, we suggest to write down this in Mathematica to find the pattern. This goes as follows:

```
(* \[Chi](T'',T') *)
    \[Chi] = (1-g)*(nn[1]*n[1]+nn[2]*n[2]-nn[2]*n[1])+nn[1]*d[1]-
             n[1]*dd[1]+nn[2]*d[2]-n[2]*dd[2]-nn[2]*d[1]+n[1]*dd[2];
    a[N_, k_] := -\[Chi]/.{nn[1]->N-k,nn[2]->1,
                  dd[1]->d1-(4-N-k)*m/6,dd[2]->d2,n[1]->k,
                  d[1]->k*m/6,d[2]->0,n[2]->0}//Simplify
    b[N_, k_] := -\[Chi]/.{n[1]->N-k,n[2]->1,
                  d[1]->d1-(4-N-k)*m/6,d[2]->d2,nn[1]->k,
                  dd[1]->k*m/6,dd[2]->0,nn[2]->0}//Simplify
```

Here, `N` is the rank of the bundle E in the triple $T = (E, L, \phi)$ where $T \in \mathrm{Ext}^1(T'', T')$ and k is the rank of T' in case `a[N,k]` (resp. T'' in case `b[N,k]`).

To find that pattern, we make a table with these values

```
Table[Table[a[N,k],{k,1,N-1}],{N,2,4}]
Table[Table[b[N,k],{k,1,N-1}],{N,2,4}]
```

If we run these two lines, we find that the patterns are

$$\begin{aligned} \texttt{a[N,k]} &= k(d_1 - d_2 - \frac{m}{2}) + k(N-k-1)(g-1), \\ \texttt{b[N,k]} &= k(g - 1 - d_1 + \frac{2}{3}m) + k(N-k-1)(g-1). \end{aligned}$$

so we define $a = d_1 - d_2 - \frac{m}{2}$ and $b = g - 1 - d_1 + \frac{2}{3}m$. Furthermore, we define in Mathematica the dimensions as follows:

```
a[N_, k_] = k*\[Alpha] + k*(N - k - 1)*(g - 1);
b[N_, k_] = k*\[Beta] + k*(N - k - 1)*(g - 1);
(* \[Alpha]=a[2,1]=d1-d2-m/2
\[Beta]=b[2,1]=g-1+2*m/3-d1 *)
```

We write these constants in terms of α and β (in the text code are written as `\[Alpha]` and `\[Beta]`, respectively) which represents the constants a and b respectively. In this text we write a and b.

These two tables synthetize the dimensions of these cohomology groups:

`a[N,k]`	**2**	**3**	**4**
1	$d_1 - d_2 - \frac{m}{2}$	$d_1 - d_2 - \frac{m}{2} + g - 1$	$d_1 - d_2 - \frac{m}{2} + 2(g-1)$
2	$*$	$2\left(d_1 - d_2 - \frac{m}{2}\right)$	$2\left(d_1 - d_2 - \frac{m}{2} + g - 1\right)$
3	$*$	$*$	$3\left(d_1 - d_2 - \frac{m}{2}\right)$

`b[N,k]`	**2**	**3**	**4**
1	$g - 1 + \frac{2}{3}m - d_1$	$2(g-1) + \frac{2}{3}m - d_1$	$3(g-1) + \frac{2}{3}m - d_1$
2	$*$	$2\left(g - 1 - \frac{2}{3}m - d_1\right)$	$2\left(2(g-1) + \frac{2}{3}m - d_1\right)$
3	$*$	$*$	$3\left(g - 1 + \frac{2}{3}m - d_1\right)$

Here, the first row shows the value for `N`, while the first column is the value for `k`.

Remark 6.3. *It is true that for pairs of rank n for any positive integer number n: the dimension of* `a[N,k]` *and* `b[N,k]` *is written in terms of* `a[2,1]` *and* `b[2,1]` *as follows:*

$$\begin{cases} \texttt{a[N,k]} = k\texttt{a[2,1]} + k(N-k-1)(g-1), \\ \texttt{b[N,k]} = k\texttt{b[2,1]} + k(N-k-1)(g-1), \end{cases}$$

where `a[2,1]` *(resp.* `b[2,1]`*) is the dimension of* $\mathrm{Ext}^1(T', S_1)$ *(respectively* $\dim \mathrm{Ext}^1(S_1, T')$*) where* $T' \in \mathcal{N}^s_{\sigma_c}(1, 1, d_1 - (n-1)d_0, d_2)$ *and* $S_1 = (L, 0, 0)$ *where* $\deg L = d_0$ *and* $rk L = 1$*. The value for* d_0 *is determined in each case for* n *and depends on the index of the critical values* σ_k*. Note these values strongly depend on* n*: for instance, for* $n = 2$, $d_0 = k$*; for* $n = 3$, $d_0 = \frac{k}{2}$ *and for* $n = 4$ *is* $d_0 = \frac{k}{6}$*.*

Certainly, this simplifies (and unifies) the computation for all ranks n*.*

2 The first set of strata

This set consists of all strata with initial triple T' of type $(3, 1, d_1 - \frac{m}{6}, d_2)$. This is a very simple class because it is formed by one stratum: the unique possibility is represented by the following short exact sequence:

$$0 \to S_1 \to T \to T' \to 0 \text{ (or } 0 \to T' \to T \to S_1 \to 0),$$

where $S_1 = (L_1, 0, 0)$ with L_1 a line bundle of degree $\frac{m}{6}$. So the class is

$$[X^+_{3-1}] = \frac{\mathbb{L}^{a+2(g-1)} - 1}{\mathbb{L} - 1}\mathbb{J}\left[\mathcal{N}^s_{\sigma_{k/3}}(3, 1, d_1 - \frac{k}{6}, d_2)\right],$$

$$[X^-_{3-1}] = \frac{\mathbb{L}^{b+2(g-1)} - 1}{\mathbb{L} - 1}\mathbb{J}\left[\mathcal{N}^s_{\sigma_{k/3}}(3, 1, d_1 - \frac{k}{6}, d_2)\right].$$

Notice that we take the stable part of $\mathcal{N}^s_{\sigma_c}(3, 1, d_1 - \frac{m}{2}, d_2)$. The critical value is $\sigma_m = \frac{5}{6}m - d_1 - d_2$ and this is a critical value for pairs of rank 3:

$$\sigma_n = \sigma_m \iff 2n - \left(d_1 - \frac{m}{6}\right) - d_2 = \frac{5}{6}m - d_1 - d_2,$$

so $n = \frac{1}{3}k$. Since $m \in 6\mathbb{Z}$, then $n \in 2\mathbb{Z}$, so $\sigma_{m/3}$ is a critical value for pairs of rank 3.

Recall that the class of $\mathcal{N}^s_{\sigma_k}(3, 1, d_1, d_2)$ has been computed in Proposition 4.1.

3 The second set of strata

This set consists of all strata whose initial triple T' is of type $(2, 1, d_1 - \frac{m}{3}, d_2)$. This set is analogous (except the last stratum) to the set of strata found to compute the class of $\mathcal{N}_{\sigma_c}(3, 1, d_1, d_2)$ (see Chapter 4). The details of the construction of these strata can be found in those pages.

Stratum No. 2-1 This is described by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and the class is (in both cases)

$$\begin{aligned}
[X^+_{2-1}] &= \Big(\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) - \mathbb{P}\operatorname{Ext}^1(T', S_2)\Big)\mathbb{P}\operatorname{Ext}^1(T', S_1)(\mathbb{J}^2 - \mathbb{J})[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \left(\frac{\mathbb{L}^{a+2(g-1)} - \mathbb{L}^{a+g-1}}{\mathbb{L}-1}\right)\frac{\mathbb{L}^{a+g-1}-1}{\mathbb{L}-1}(\mathbb{J}^2 - \mathbb{J})[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)], \\
[X^-_{2-1}] &= \Big(\mathbb{P}\operatorname{Ext}^1(S_2, \tilde{T}_1) - \mathbb{P}\operatorname{Ext}^1(S_2, T')\Big)\mathbb{P}\operatorname{Ext}^1(S_1, T')(\mathbb{J}^2 - \mathbb{J})[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \left(\frac{\mathbb{L}^{b+2(g-1)} - \mathbb{L}^{b+g-1}}{\mathbb{L}-1}\right)\frac{\mathbb{L}^{b+g-1}-1}{\mathbb{L}-1}(\mathbb{J}^2 - \mathbb{J})[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)].
\end{aligned}$$

Stratum No. 2-2 This is given by the matrix $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 \end{pmatrix}$. The class is

$$\begin{aligned}
[X^+_{2-2}] &= \left(\mathbb{P}\operatorname{Ext}^1(\tilde{T}_1, S_2) - \mathbb{P}\left(\frac{\operatorname{Ext}^1(T', S_2)}{\operatorname{Hom}(S_1, S_2)}\right)\right)\mathbb{P}\operatorname{Ext}^1(T', S_1)\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \left(\frac{\mathbb{L}^{a+2(g-1)} - \mathbb{L}^{a+g-1}}{\mathbb{L}-1}\right)\frac{\mathbb{L}^{a+g-2}-1}{\mathbb{L}-1}\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)], \\
[X^-_{2-2}] &= \left(\mathbb{P}\operatorname{Ext}^1(S_2, \tilde{T}_1) - \mathbb{P}\left(\frac{\operatorname{Ext}^1(T', S_2)}{\operatorname{Hom}(S_1, S_2)}\right))\right)\mathbb{P}\operatorname{Ext}^1(S_1, T')\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \left(\frac{\mathbb{L}^{b+2(g-1)} - \mathbb{L}^{b+g-1}}{\mathbb{L}-1}\right)\frac{\mathbb{L}^{b+g-1}-1}{\mathbb{L}-1}\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)].
\end{aligned}$$

Stratum No. 2-3 This is the strata given by the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}$. The classes are

$$\begin{aligned}
[X^+_{2-3}] &= \big(\lambda^2(\mathbb{P}\operatorname{Ext}^1(T', S_1)\mathbb{J}) - \lambda^2(\mathbb{P}\operatorname{Ext}^1(T', S_1))\mathbb{J}\big)[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \left(\lambda^2\left(\frac{\mathbb{L}^{a+g-1}-1}{\mathbb{L}-1}\mathbb{J}\right) - \lambda^2\left(\frac{\mathbb{L}^{a+g-1}-1}{\mathbb{L}-1}\right)\mathbb{J}\right)[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)], \\
[X^+_{2-3}] &= \big(\lambda^2(\mathbb{P}\operatorname{Ext}^1(S_1, T')\mathbb{J}) - \lambda^2(\mathbb{P}\operatorname{Ext}^1(S_1, T'))\mathbb{J}\big)[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \left(\lambda^2\left(\frac{\mathbb{L}^{b+g-1}-1}{\mathbb{L}-1}\mathbb{J}\right) - \lambda^2\left(\frac{\mathbb{L}^{b+g-1}-1}{\mathbb{L}-1}\right)\mathbb{J}\right)[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)].
\end{aligned}$$

Stratum No. 2-4 This strata corresponds to the matrix $\mathbf{A} = \begin{pmatrix} 2 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 \end{pmatrix}$, that is, $S_1 = L_1^{\oplus 2}$. The classes are

$$\begin{aligned}
[X^+_{2-4}] &= \operatorname{Gr}(2, \operatorname{Ext}^1(T', L))\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \frac{(\mathbb{L}^{a+g-1}-1)(\mathbb{L}^{a+g-2}-1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)], \\
[X^-_{2-4}] &= \operatorname{Gr}(2, \operatorname{Ext}^1(L, T'))\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)] \\
&= \frac{(\mathbb{L}^{b+g-1}-1)(\mathbb{L}^{b+g-2}-1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\mathbb{J}[\mathcal{N}^s_{\sigma_{m/6}}(2, 1, d_1 - \frac{m}{3}, d_2)].
\end{aligned}$$

■ **Stratum No. 2-5** This is the last stratum of this set. In this case, there is only one step defined by $S_1 = (M, 0, 0)$ where $\operatorname{rk} M = 2$: we used to denote this stratum by $\mathbf{A} = (\,1_2\,)$ and $\mathbf{n} = (\,2\,)$. The classes are

$$\begin{aligned}
[X^+_{2-5}] &= \mathbb{P}\operatorname{Ext}^1(T', S_1)\mathbb{M}_2^{\frac{m}{3}}[\mathcal{N}^s_{\sigma_{m/6}}(2,1,d_1-\frac{m}{3},d_2)]\\
&= \frac{\mathbb{L}^{2(a+g-1)}-1}{\mathbb{L}-1}\mathbb{M}_2^{\frac{m}{3}}[\mathcal{N}^s_{\sigma_{m/6}}(2,1,d_1-\frac{m}{3},d_2)],\\
[X^-_{2-5}] &= \mathbb{P}\operatorname{Ext}^1(S_1, T')\mathbb{M}_2^{\frac{m}{3}}[\mathcal{N}^s_{\sigma_{m/6}}(2,1,d_1-\frac{m}{3},d_2)]\\
&= \frac{\mathbb{L}^{2(b+g-1)}-1}{\mathbb{L}-1}\mathbb{M}_2^{\frac{m}{3}}[\mathcal{N}^s_{\sigma_{m/6}}(2,1,d_1-\frac{m}{3},d_2)].
\end{aligned}$$

Notice that the class $\mathbb{M}_2^{\frac{m}{3}}$ depends on if $\frac{m}{3}$ being even or odd.

The last question is if the critical value $\sigma_m = \frac{5}{6}m - d_1 - d_2$ is a critical value for pairs of rank 2. Recall that the critical value for those pairs are $\sigma_n = 3n - d_1 - d_2$, so from the equality

$$\sigma_n = \sigma_m \iff 3n - \left(d_1 - \frac{m}{3}\right) - d_2 = \frac{5}{6}m - d_1 - d_2$$

we have $\frac{1}{2}k = 3n$, that is $n \in \mathbb{Z}$, so $\sigma_{m/6}$ is a critical value for pairs of rank 2 whenever $m \in 6\mathbb{Z}$.

Remark 6.4. *The class of $\mathcal{N}^s_{\sigma_{m/6}}(2,1,d_1,d_2)$ is exactly the same as Lemma 4.4, that is,*

$$\left[\mathcal{N}^s_{\sigma_{m/6}}(2,1,d_1-\frac{m}{6},d_2)\right] = \frac{\mathbb{J}^2[\mathcal{N}(1,1,d_1-\frac{m}{2},d_2)]}{\mathbb{L}-1}\cdot \operatorname*{coeff}_{x^{d_1-d_2-m/6}}\left(\frac{\mathbb{L}^{b+2}x}{1-\mathbb{L}^2x} - \frac{\mathbb{L}^a}{(1-\frac{x}{\mathbb{L}})} + 1\right)$$

4 The third set of strata

We break this set into four groups:

- The first group consists of strata where the triple T is obtained by one extension
$$0 \to T' \to T \to S_1 \to 0 \text{ (or } 0 \to S_1 \to T \to T' \to 0),$$
where S_1 is a polystable triple of type $(F, 0, 0)$ where $\operatorname{rk} F = 3$.
- The second group consists of strata where the triple T is in a short exact sequence
$$0 \to T_1 \to T \to S_2 \to 0 \text{ (or } 0 \to S_2 \to T \to T_1 \to 0),$$
where S_1 is a polystable triple of type $(F, 0, 0)$ where F has rank 2 and T_1 is in
$$0 \to T' \to T_1 \to S_1 \to 0 \text{ (or } 0 \to S_1 \to T_1 \to T' \to 0),$$
where $S_1 = (L_1, 0, 0)$ where $\operatorname{rk} L_1 = 1$.
- The third group is formed by strata where the triple T is in an analogous short exact sequence but the rôle of S_1 and S_2 are swapped. Thus, the triple T is in a short exact sequence
$$0 \to T_1 \to T \to S_2 \to 0 \text{ (or } 0 \to S_2 \to T \to T_1 \to 0),$$
but $S_2 = (L_2, 0, 0)$ where $\operatorname{rk} L_2 = 1$, and T_1 satisfies
$$0 \to T' \to T_1 \to S_1 \to 0 \text{ (or } 0 \to S_1 \to T_1 \to T' \to 0),$$
where $S_1 = (F, 0, 0)$ polystable triple where $\operatorname{rk} F = 2$.
- The triples T which belong to the fourth group satisfy the following set of short exact sequences
$$\begin{cases} 0 \to T_2 \to T \to S_3 \to 0 \text{ (or } 0 \to S_3 \to T \to T_2 \to 0),\\ 0 \to T_1 \to T \to S_2 \to 0 \text{ (or } 0 \to S_2 \to T \to T_1 \to 0),\\ 0 \to T' \to T \to S_1 \to 0 \text{ (or } 0 \to S_1 \to T \to T' \to 0),\end{cases}$$
where $S_i = (L_i, 0, 0)$ and $\operatorname{rk} L_i = 1$ for $i = 1, 2, 3$.

The outline of the construction of these strata is written in page (see Chapter 5). So we do not offer any explanation. Also, we list these class in four tables — group by group. Furthermore, we simplify the notation:

- When we write $T'S_i$, we understand $\mathrm{Ext}^1(T', S_i)$ or its class $[\mathrm{Ext}^1(T', S_i)]$, depending on the context.
- Recall that $\mathcal{N}^s_{\sigma_c}(1, 1, d_1, d_2) \cong \mathrm{Sym}^{d_1-d_2} X \times \mathrm{Jac}^{d_2} X$. To translate the list of the Chapter 5 to this list, we have to remove one $\mathbb{J}$ class and put the class $[\mathcal{N}^s_{\sigma_c}(1, 1, d_1, d_2)]$ which is $\lambda^{d_1-d_2}(X)\mathbb{J}$.
- Indeed, we have two groups, the stratum $[X^+_{3,i}]$ and $[X^-_{3,i}]$. Both classes are closely related: while in $X^+_{3,i}$ we work with $\mathrm{Ext}^1(T, S_i)$, in $X^-_{3,i}$ we have $\mathrm{Ext}^1(S_i, T')$. We simply swap these triples.

Case	Matrix	Class
3-1.1	$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	$\big(\lambda^3(\mathbb{P}T'S_\bullet \cdot \mathbb{J}) - \lambda^2(\mathbb{P}T'S_\bullet) \cdot \mathbb{P}T'S_\bullet \cdot (\mathbb{J}^2 - \mathbb{J}) - \lambda^3(\mathbb{P}T'S_\bullet) \cdot \mathbb{J}\big) \cdot \mathbb{J} \cdot \lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-1.2	$\begin{pmatrix}2 & 1\end{pmatrix}$	$\mathrm{Gr}(2, T'S_1) \cdot \mathbb{P}T'S_2 \cdot (\mathbb{J}^2 - \mathbb{J}) \cdot \mathbb{J} \cdot \lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-1.3	$\begin{pmatrix}1_2 & 1\end{pmatrix}$	$\mathbb{P}T'S_1 \cdot \mathbb{P}T'S_2 \cdot \mathbb{M}_2^{\frac{m}{3}} \cdot \mathbb{J}\lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-1.4	$\begin{pmatrix}3\end{pmatrix}$	$\mathrm{Gr}(3, T'S_\bullet) \cdot \mathbb{J} \cdot \mathbb{J}\lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-1.5	$\begin{pmatrix}1_3\end{pmatrix}$	$\mathbb{P}T'S\mathbb{M}_3^{\frac{m}{2}}\mathbb{J}\lambda^{d_1 - \frac{m}{2} - d_2}(X)$

Table 1: The class of the first group of strata

Case	Matrix	Class
3-2.1	$\begin{pmatrix}0 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$	$\dfrac{T'S_3}{\mathbb{L}-1}\Big\{\lambda^2\left(\left(S_\bullet S_3 - 1\right) \cdot \mathbb{P}T'S_\bullet \cdot (\mathbb{J}-1)\right)\mathbb{J} - {}$ $-\lambda^2\left((S_\bullet S_3 - 1)\mathbb{P}T'S_\bullet\right) \cdot (\mathbb{J}-1)\mathbb{J}\Big\} \cdot \mathbb{J} \cdot \lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-2.2	$\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}$	$T'S_2 \cdot \dfrac{\mathcal{V}(2, S_1S_2)}{\mathbb{L}-1}\,\mathrm{Gr}(2, T'S_1) \cdot (\mathbb{J}^2 - \mathbb{J}) \cdot \mathbb{J} \cdot \lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-2.3	$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$	$T'S_1 \cdot \dfrac{(S_1S_1 - 1)}{\mathbb{L}} \cdot \dfrac{S_2S_1 - 1}{\mathbb{L}-1} \cdot \mathbb{P}T'S_1 \cdot \mathbb{P}T'S_2 \cdot (\mathbb{J}^2 - \mathbb{J}) \cdot \mathbb{J} \cdot \lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-2.4	$\begin{pmatrix}0 & 1\\ 1_2 & 0\end{pmatrix}$	$(\mathbb{P}\tilde{T}_1S_2 - \mathbb{P}T'S_2) \cdot \mathbb{P}T'S_1 \cdot \mathbb{J} \cdot \mathbb{M}_2^{\frac{m}{3}} \cdot \mathbb{J} \cdot \lambda^{d_1 - \frac{m}{2} - d_2}(X)$
3-2.5	$\begin{pmatrix}1\\ 2\end{pmatrix}$	$T'S_1 \cdot \dfrac{\mathcal{V}(2, S_1S_1)}{\mathbb{L}^2(\mathbb{L}-1)} \cdot \mathrm{Gr}(2, T'S_2) \cdot \mathbb{J} \cdot \mathbb{J} \cdot \lambda^{d_1 - \frac{m}{2} - d_2}(X)$

Table 2: The class of the second group of strata

We have to do some comments about these tables. First, as we have learned before, not every stratum appears for each critical value σ_m. Most of them appear for $m \in 6\mathbb{Z}$, and only a selected group of them appears for $m \in 3\mathbb{Z}$ or $m \in 2\mathbb{Z}$. The stratum 2-5 appears in case $m \in 3\mathbb{Z}$, and no more strata appears at such integers numbers. Moreover, the strata 3-1.3 appears in case $m \in 2\mathbb{Z}$; in that case, notice that for $m \in 6\mathbb{Z}$, the class of $\mathbb{M}_3^{\frac{m}{2}}$ changes since $\frac{m}{2}$ is multiple of 3.

The dimension of projective and grassmanian varieties are computed using the code explained some pages above. Left to compute the dimension of $\mathrm{Ext}^1(S_i, S_j)$, that is, $\mathrm{Ext}^1(L_i, L_j)$ where L_i and

Case	Matrix	Class
3-3.1	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\Big\{\Big(\lambda^2(\mathbb{P}\tilde{T}_1 S_\bullet \cdot (\mathbb{J}-1)) - \lambda^2(\mathbb{P}\tilde{T}_1 S_\bullet)\cdot(\mathbb{J}-1)\Big) -$ $- \big(\lambda^2(\mathbb{P}T' S_\bullet \cdot (\mathbb{J}-1)) - \lambda^2(\mathbb{P}T' S_\bullet)\cdot(\mathbb{J}-1)\big)\Big\}\times$ $\times \mathbb{P}T'S_1\mathbb{J}\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-3.2	$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$	$\Big(\mathrm{Gr}(2,\tilde{T}_1S_2) - \mathrm{Gr}(2,T'S_2)\Big)\cdot \mathbb{P}T'S_1\cdot(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-3.3	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\left(\mathbb{P}\tilde{T}_1S_1\cdot\mathbb{P}\tilde{T}_1S_2 - \mathbb{P}\left(\frac{T'S_1}{\mathbb{L}}\right)\cdot\mathbb{P}T'S_2\right)\cdot\mathbb{P}T'S_1\cdot(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-3.4	$\begin{pmatrix} 0 & 1_2 \\ 1 & 0 \end{pmatrix}$	$(\mathbb{P}\tilde{T}_1S_2 - \mathbb{P}T'S_2)\cdot\mathbb{P}T'S_1\mathbb{J}\cdot\mathbb{M}_2^{\frac{m}{3}}\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-3.5	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\left(\mathrm{Gr}(2,\tilde{T}_1S_1) - \mathrm{Gr}\left(2,\frac{T'S_1}{\mathbb{L}}\right)\right)\cdot\mathbb{P}T'S_1\cdot\mathbb{J}\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$

Table 3: The class of the third group of strata

Case	Matrix	Class
3-4.1	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$(\mathbb{P}\tilde{T}_2S_3 - \mathbb{P}\tilde{T}_1S_3)\cdot(\mathbb{P}\tilde{T}_1S_2 - \mathbb{P}T'S_2)\cdot\mathbb{P}T'S_1\times$ $\times(\mathbb{J}^3-3\mathbb{J}^2+2\mathbb{J})\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-4.2	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbb{P}\tilde{T}_2S_2 - \mathbb{P}\tilde{T}_1S_2)\cdot\left(\mathbb{P}\tilde{T}_1S_1 - \mathbb{P}\left(\frac{T'S_1}{\mathbb{L}}\right)\right)\cdot\mathbb{P}T'S_1\cdot(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-4.3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$	$(\mathbb{P}\tilde{T}_2S_1 - \mathbb{P}\tilde{T}_1S_1)\cdot(\mathbb{P}\tilde{T}_1S_2 - \mathbb{P}T'S_2)\cdot\mathbb{P}T'S_1\cdot(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-4.4	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\left(\mathbb{P}\tilde{T}_2S_2 - \mathbb{P}\left(\frac{\tilde{T}_1S_2}{\mathbb{L}}\right)\right)\cdot(\mathbb{P}\tilde{T}_1S_2 - \mathbb{P}T'S_2)\cdot\mathbb{P}T'S_1\cdot(\mathbb{J}^2-\mathbb{J})\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$
3-4.5	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\left(\mathbb{P}\tilde{T}_2S_1 - \mathbb{P}\left(\frac{\tilde{T}_1S_1}{\mathbb{L}}\right)\right)\cdot\left(\mathbb{P}T_1S_1 - \mathbb{P}\left(\frac{T'S_1}{\mathbb{L}}\right)\right)\cdot\mathbb{P}T'S_1\cdot\mathbb{J}\cdot\mathbb{J}\cdot\lambda^{d_1-\frac{m}{2}-d_2}(X)$

Table 4: The class of the fourth group of strata

L_j are stable bundles. This is computed in (5.5):

$$\mathrm{Ext}^1(L_i,L_j) = n_in_j(g-1)+\epsilon,\ \text{where } \epsilon = \begin{cases} 1 & \text{if } L_i\cong L_j,\\ 0 & \text{otherwise.}\end{cases}$$

5 The Mathematica show

In the previous section, we gave a complete list of all strata which form the flip loci $\mathcal{S}^{\pm}_{\sigma_n}$. From this, the interested reader might obtain the class of $\mathcal{N}_\sigma(4,1,d_1,d_2)$ for σ a non-critical value. As we have said at the beginning of this chapter, we solve the computation by making a worksheet in Mathematica. Recall the convention of variables that we use in the previous Mathematica worksheet in section 5.7.

```
P[g_] := (t^(g)-1)/(t-1)

(*Stiefel variety*)
V[k_,N_] := Product[t^N-t^i,{i,0,k-1}]

(*Grassmanian*)
Gr[k_,N_] := Product[(1-t^(N-i))/(1-t^(i+1)),{i,0,k-1}]

(*Moduli stable bundles rk=2 degree even*)
M2[0] = 1/(2*(1-t)*(1-t^2))*(2*J[1,1]*J[1,t]
        -J[1,1]^2*(1+2*t^(g+1)-t^2)-J[2,-1]*(1-t)^2);

(*Moduli stable bundles rk=2 degree odd*)
M2[1]=(J[1,1]*J[1,t]-t^g*J[1,1]^2)/((1-t)*(1-t^2))//Simplify;

M3[1] = (J[1, 1]*(t^(3*g)*(1+t+t^2)*J[1, 1]^2+
        J[1, t]*(-t^(2*g)*(1+t)^2*J[1, 1]+t J[1, t^2])))
/((-1+t)^4*t*(1+t)^2*(1+t+t^2));

M3[0] = 1/(3(-1+t)^5(1+t)^2(1+t+t^2))((1+t+t^2)(3t^(1+3g)-
        3t^(1+g)(-1+t^2)+(-1+t^2)^2)J[1,1]^3+3J[1,1]^1J[1,
        t]((1+t)(-1+t^3-2t^(1+2g))J[1,1]+J[1,t^2])-(-1+t)^4(1+t)^2J[3,1]);
```

Now, we define the same symmetric operator on classes:

```
Sym2[f_] := Simplify[1/2*(f^2+(f/.{t->t^2,J[k_,t_]->J[2*k,-(t)^2]}))]
Sym3[f_] := Simplify[1/6*(f^3+3*(f/.{t->t^2,J[k_,t_]->J[2*k,-(t)^2]})*f+
            2*(f/.{J[k_,t_]->J[3*k,t^3],t->t^3}))]
```

To write automatically the dimension of Ext^1 groups, we insert in our worksheet the dimension of these cohomology groups as we have defined before:

```
a[N_, k_] = k*\[Alpha] + k*(N - k - 1)*(g - 1);
b[N_, k_] = k*\[Beta] + k*(N - k - 1)*(g - 1);
```

In this case, we have

$$\begin{cases} a = \alpha = \texttt{a[2,1]} = d_1 - d_2 - \dfrac{m}{2}, \\ b = \beta = \texttt{b[2,1]} = g - 1 - d_1 + \dfrac{2}{3}m, \end{cases}$$

where we use letters a and b in the text to denote such constants in the `Mathematica` worksheet to avoid the conflict with the definition of `a[k,N]` and `b[k,N]`. Recall that in the text we use α and β to denote such constants.

Recall that `N` is related to the rank of the extended triple $T \in \mathrm{Ext}^1(T'', T')$ —not T' or T''!—, and k is the rank of the stable bundle "encapsulated" in the triple S_i. More concretely,

- If $T' = (E', 0, 0)$ and $T'' = (E'', L, \phi'')$, then

$$\texttt{a[N,k]} = \dim \mathrm{Ext}^1(T'', T')$$

  where the variable `N` is the rank of E, and `k` is the rank of E'. These groups of extensions appear in the stratum X^+.

- Conversely, if $T'' = (E'', 0, 0)$ and $T' = (E', L, \phi')$, then

$$\texttt{b[N,k]} = \dim \operatorname{Ext}^1(T'', T')$$

where the variable `N` is the rank of E, and `k` is the rank of E''. Such dimensions of extension groups appear in the stratum X^-.

■ **Part 1** We write down in our worksheet the class of the first set of strata. Despite of its simplicity —this set consists of one stratum—, the class of $\mathcal{N}^s_{\sigma_{m/3}}(3, 1, d_1 - \frac{m}{6}, d_2)$ has a quite long expression. Mathematica is not so smart: if we do not split this class, this software will mix a lot the output giving a weird and unuseful output. Thus, to get rid of it, we rearrange this class by splitting into several pieces according to the number of factors $(1 - \mathbb{L}^k x)$ in the denominator.

At the first stage, we compute the coefficients of the class of $\mathcal{N}^s_{\sigma_{m/3}}(3, 1, d_1 - \frac{m}{6}, d_2)$ from the equality in Proposition 4.8:

$$\begin{aligned} d_1 - d_2 - n_0 &= \left(d_1 - \frac{m}{6}\right) - d_1 - \frac{m}{3} = d_1 - d_2 - \frac{m}{2} = a, \\ g - 1 - d_1 + \frac{3}{2} n_0 &= g - 1 - \left(d_1 - \frac{m}{6}\right) + \frac{3}{2} \cdot \frac{m}{3} = g - 1 - d_1 - \frac{2}{3} m = b. \end{aligned}$$

The class of $\mathcal{N}_{\sigma^+_{m/3}}(3, 1, d_1 - \frac{m}{6}, d_2)$ should be the sum `XX + XXb`, where $\sigma^+_{m/3} = \sigma_{m/3} + \epsilon$ for a sufficiently small ϵ, and where

```
XX = N1*J[1, 1]^2/((-1+t)^2*(1+t))*(t^(2*\[Beta])/((-1+t^2x)*(-1+t^3 x))
     +(t+1)*t^(\[Alpha]+\[Beta])/((1-x/t)*(-1+t^2*x))
     +t^(2*\[Alpha])/((1-x/t)*(1-x/t^2)));

XXb = N1*J[1, 1]*(t^g*J[1, 1]-J[1, t])/((-1+t)^3*(1+t))*
      (t^(2*\[Beta])/(-1+t^3*x)+t^(2*\[Alpha])/(1-x/t^2));
```

To write the stable part of $\mathcal{N}^s_{\sigma_{m/3}}(3, 1, d_1 - \frac{m}{6}, d_2)$, we use the `Mathematica` worksheet where we compute this class and we export the definitions:

```
XXStra = Sum[SS[2, k], {k, 1, 5}] // FullSimplify;
XXStrb = SS[3, 1];
XXStr = Sum[SS[2, k], {k, 1, 5}] + SS[3, 1] // FullSimplify;
XXbStr = SSb[2, 5];
```

so that the class of $\mathcal{N}^s_{\sigma_{m/3}}(3, 1, d_1 - \frac{m}{3}, d_2)$ is `(XX - XXStra - XXStrb) + (XXb - XXbStr)` when $m \in 6\mathbb{Z}$ and `XX + (XXb - XXbStr)` when $m \notin 6\mathbb{Z}$. Notice that each piece fulfil the following

- `XX` has two factors of $(1 - \mathbb{L}^k x)$,
- `XXb` and `XXStrb` has one factor of this type,
- `XXStra` has no factors.

So we write

```
Sb[3, 1] = P[a[4, 1]]*J[1, 1]*(XXb - XXStrb);
SSb[3, 1] = P[b[4, 1]]*J[1, 1]*(XXb - XXStrb);
Sa[3, 1] = P[a[4, 1]]*J[1, 1]*(XX);
SSa[3, 1] = P[b[4, 1]]*J[1, 1]*(XX);
SbStr[3, 1] = -P[a[4, 1]]*J[1, 1]*(XXbStr + XXStra);
SSbStr[3, 1] = -P[b[4, 1]]*J[1, 1]*(XXbStr + XXStra);

TheSuma[3] = SSa[3, 1] - Sa[3, 1] // FullSimplify
TheSumb[3] = SSb[3, 1] - Sb[3, 1] // FullSimplify
TheSumStr[3] = SSbStr[3, 1] - SbStr[3, 1] // FullSimplify
```

The output gives these three classes

$$A_1 = \frac{\mathbb{L}^{2(g-1)}(\mathbb{L}^b - \mathbb{L}^a)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} \left(\frac{\mathbb{L}^{2b}}{(1-\mathbb{L}^2 x)(1-\mathbb{L}^3 x)} - \frac{\mathbb{L}^{a+b}(1+\mathbb{L})}{(1-\mathbb{L}^2 x)(1-\frac{x}{\mathbb{L}})} + \right.$$
$$\left. + \frac{\mathbb{L}^{2a}}{(1-\frac{x}{\mathbb{L}})(1-\frac{x}{\mathbb{L}^2})} \right) [\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)],$$
$$A_2 = \frac{\mathbb{L}^{2(g-1)}(\mathbb{L}^b - \mathbb{L}^a)}{\mathbb{L}-1} \left(\frac{\mathbb{L}^g \mathbb{J} - \zeta_{\bar{C}}(\mathbb{L})}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} \left(\frac{\mathbb{L}^{2a}}{1-\frac{x}{\mathbb{L}^2}} - \frac{\mathbb{L}^{2b}}{1-\mathbb{L}^3 x} \right) \mathbb{J}^2 - \right.$$
$$\left. - \frac{\mathbb{L}^{b+g-1}-1}{(\mathbb{L}-1)^2} \left(\frac{\mathbb{L}^{b+1}x}{1-\mathbb{L}^2 x} - \frac{\mathbb{L}^{a-1}}{1-\frac{x}{\mathbb{L}}} \right) \mathbb{J}^2 \right) [\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)],$$
$$A_3 = \frac{\mathbb{L}^{2(g-1)}(\mathbb{L}^b - \mathbb{L}^a)}{(\mathbb{L}-1)^3(\mathbb{L}^2-1)} \left((\mathbb{L}^{2b+g-1} - \mathbb{L}^{g+1} + \mathbb{L}^2 - 1)\mathbb{J} - (\mathbb{L}^{2b}-1)\zeta_{\bar{C}}(\mathbb{L}) \right) \mathbb{J}[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)]$$

■ Part 2

To write the class of $\mathcal{N}^s_{\sigma_{m/6}}(2,1,d_1 - \frac{m}{3}, d_2)$ we rewrite this class in terms of a and b:

$$d_1 - d_2 - d_M = d_1 - \frac{m}{6} - d_2 - \frac{m}{3} = d_1 - d_2 - \frac{m}{2} = a,$$
$$g + 1 - d_1 + 2d_M = g + 1 - \left(d_1 - \frac{m}{3}\right) + 2\frac{m}{6} = g + 1 - d_1 + \frac{2}{3}m = b + 2.$$

so this class is the sum of the following two variables:

```
N2a = N1*J[1, 1]/(t-1);
N2b = N2a*(-(t^\[Alpha]/(1-x/t)) + (t^(\[Beta]+2)*x)/(1-t^2 x));
```

As before, we try to compute separately classes depending on whether they involve x variable or not: this gives the 'part A' and the 'part B'.

We write separately all strata where the initial triple is a pair of rank 2. The 'part A' is done by the code:

```
S[2, 1] = (P[a[4, 1]]-P[a[3, 1]])*P[a[3, 1]]*(J[1, 1]^2-J[1, 1])*N2a;
SS[2, 1] = (P[b[4, 1]]-P[b[3, 1]])*P[b[3, 1]]*(J[1, 1]^2-J[1, 1])*N2a;
(* 1 / 1 *)
S[2, 2] = (P[a[4, 1]] - P[a[3, 1] - 1])*P[a[3, 1]]*J[1, 1]*N2a;
SS[2, 2] = (P[b[4, 1]] - P[b[3, 1] - 1])*P[b[3, 1]]*J[1, 1]*N2a;
(* 1 1 *)
S[2, 3] = Refine[(Sym2[P[a[3, 1]]*J[1, 1]]-Sym2[P[a[3, 1]]]*J[1, 1])*N2a,
          t > 0] // Simplify;
SS[2, 3] = Refine[(Sym2[P[b[3, 1]]*J[1, 1]] - Sym2[P[b[3, 1]]]*J[1, 1])*N2a,
          t > 0] // Simplify;
(* 2 *)
S[2, 4] = Gr[2, a[3, 1]] *J[1, 1]*N2a;
SS[2, 4] = Gr[2, b[3, 1]]*J[1, 1]*N2a;
(* 1_ 2 *)
S[2, 5] = P[a[4, 2]]*(M2[0] - M2[1])*N2a;
SS[2, 5] = P[b[4, 2]]*(M2[0] - M2[1])*N2a;
TheSuma[2] = Sum[SS[2, k] - S[2, k], {k, 1, 5}];
Sb[2, 5] = P[a[4, 2]]*M2[1]*N2a;
SSb[2, 5] = P[b[4, 2]]*M2[1]*N2a;
```

The 'part B' is the same code sustituing `N2a` by `N2b`. This assures that `Mathematica` does not mix the part without explicit x variable and the part with explicit variable x.

We write the class `S[2,5]` and `SS[2,5]` in a slightly different way. This strata appears for $m \in 3\mathbb{Z}$ and in order to make easier the sum along the parameter m, we modify this stratum so that the sum along m becomes:

$$\sum_{m\in 3\mathbb{Z}} (\texttt{SSb[2,5]-Sb[2,5]}) + \sum_{m\in 6\mathbb{Z}} (\texttt{SS[2,5]-S[2,5]}).$$

We obtain the following equalities:

$$\begin{aligned} A_4 = \texttt{TheSuma[2]} &= \mathbb{L}^{2(g-1)} \frac{(\mathbb{L}^b - \mathbb{L}^a)(\mathbb{L}^{a+g-1} + \mathbb{L}^{b+g-1} - \mathbb{L} - 1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} \mathbb{J}^2[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)], \\ A_5 = \texttt{TheSumb[2]} &= \mathbb{L}^{2(g-1)} \frac{(\mathbb{L}^b - \mathbb{L}^a)(\mathbb{L}^{a+g-1} + \mathbb{L}^{b+g-1} - \mathbb{L} - 1)}{(\mathbb{L}-1)^2(\mathbb{L}^2-1)} \times \\ &\times \left(\frac{\mathbb{L}^a}{1 - \frac{x}{\mathbb{L}}} - \frac{\mathbb{L}^{b+1}}{1 - \mathbb{L}^2 x} \right) \mathbb{J}^2 \Big[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)\Big], \end{aligned}$$

■ **Part 3** This part corresponds to the classes of the third set of strata. We do not write all classes, we just write some examples to indicate how to apply these tools to compute the class of the moduli space $\mathcal{N}^s_\sigma(4,1,d_1,d_2)$.

- The class of the stratum determined by the constants $\mathbf{A} = (1 \quad 1 \quad 1)$ and $\mathbf{n} = (1 \quad 1 \quad 1)$:

```
S[1, 1] = Refine[(Sym3[P[a[2,1]]*J[1,1]]-
          Sym2[P[a[2,1]]]*P[a[2,1]]*(J[1,1]^2-J[1,1])-
          Sym3[P[a[2,1]]]*J[1,1])*N1,t>0]//Simplify
```

  and the other class (the corresponding to the stratum in $\mathcal{S}^-_{\sigma_m}$:

```
SS[1, 1] = Refine[(Sym3[P[b[2,1]]*J[1,1]]-
           Sym2[P[b[2,1]]]*P[b[2,1]]*(J[1,1]^2-J[1,1])-
           Sym3[P[b[2,1]]]*J[1,1])*N1,t>0]//Simplify
```

  Here, `N1` is a shorthand for $\mathcal{N}^s_{\sigma_{m/2}}(1,1,d_1 - \frac{m}{2}, d_2)$.
- This is the class of the stratum

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{n} = (1 \quad 1 \quad 1):$$

```
S[1, 6] = Refine[t^a[2,1]/(t-1)*
          (Sym2[(t^(g-1)-1)*P[a[2,1]]*(J[1,1]-1)]*J[1,1]-
          Sym2[(t^(g-1)-1)*P[a[2,1]]]*(J[1,1]-1)*J[1,1])*N1, t>0]//Simplify
```

  and

```
SS[1, 6] = Refine[t^b[2,1]/(t-
           1)*(Sym2[(t^(g-1)-1)*P[b[2,1]]*(J[1,1]-1)]*J[1,1]-
           Sym2[(t^(g-1)-1)*P[b[2,1]]]*(J[1,1]-1)*J[1,1])*N1,
           t>0]//Simplify
```

- And the class of the stratum

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{n} = (1 \quad 1 \quad 1):$$

```
S[1, 11] = Refine[(Sym2[P[a[3,2]]*(J[1,1]-1)]-
           Sym2[P[a[3,2]]]*(J[1,1]-1)
           -(Sym2[P[a[2,1]]*(J[1,1]-1)]-
           Sym2[P[a[2,1]]]*(J[1,1]-1)))*P[a[2,1]]*J[1,1]*N1,
           t>0]//Simplify
```

  and the other stratum is getting by substituing `a[N,k]` by `b[N,k]`.
- An important case is the stratum $\mathbf{A} = (1_3)$ (where $\mathbf{n} = (3)$) which corresponds with the stratum `S[1,5]` and `SS[1,5]`. Since the class $M^s(3, \frac{m}{2})$ depends on $m \in 2\mathbb{Z}$, we modify this definition as follows:

```
S[1, 5] = P[a[4, 3]]*N1*(M3[0] - M3[1]);
Sb[1, 5] = P[a[4, 3]]*N1*M3[1];
```

So, we have

$$\sum_{m\in 2\mathbb{Z}} \left(S^-_{1,5} - S^+_{1,5}\right) = \sum_{m\in 2\mathbb{Z}} (\texttt{SSb[1,5]-Sb[1,5]}) + \sum_{m\in 6\mathbb{Z}} (\texttt{SS[1,5]-S[1,5]}).$$

- A simple case is the following. Observe the use of the integer functions `a[N,k]`:

```
S[1, 14] = (P[a[4,2]]-P[a[2,1]])*P[a[2,1]]*J[1,1]*M2[0]*N1//Simplify
```

This class corresponds to the stratum

$$\mathbf{A} = \begin{pmatrix} 0 & 1_2 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{n} = (\,1 \quad 2\,).$$

The class $[\mathcal{S}^-_{\sigma_c}] - [\mathcal{S}^+_{\sigma_c}]$ is obtained with the command

```
TheSum[1] = Sum[SS[1, k] - S[1, k], {k, 1, 20}]
```

As before, to compute the class of the difference $[\mathcal{S}^-_{\sigma_m}] - [\mathcal{S}^+_{\sigma_m}]$, we compute coefficient by coefficient. This works quite nice with Mathematica. We obtain the coefficient form the "oddest" terms (like `J[3,1]`$= \delta^3(\zeta_{\bar{C}}(1)))$ to the simplest (like `J[1,1]`$= \zeta_{\bar{C}}(1) = \mathbb{J}$). Run the following commands to obtain the first coefficient:

```
C31 = CoefficientList[TheSum[1], J[3, 1]];
C31[[2]] // Simplify
```

and we get

$$A_6 = \frac{(\mathbb{L}-2)(\mathbb{L}^{3b} - \mathbb{L}^{3a})}{3(\mathbb{L}-1)(\mathbb{L}^3-1)} \left[\mathcal{N}^s_{\sigma_m}(1, 1, d_1 - \frac{m}{2}, d_2)\right] \delta^3(\mathbb{J}).$$

The following coefficient computed is the `J[2,-1]`$= \delta^2(\zeta_{\bar{C}}(1))$:

```
C21 = CoefficientList[C31[[1]], J[2, -1]]
C21[[2]] // Simplify // Factor
```

and surprisingly this gives the nice quantity 0.

The next is the class of `J[1,t^2]`$= \zeta_{\bar{C}}(\mathbb{L}^2)$, whose coefficient is

```
C1t2 = CoefficientList[C21[[1]], J[1, t^2]];
C1t2[[2]] // Simplify
```

which gives

$$A_7 = -\frac{(\mathbb{L}-2)(\mathbb{L}^{3b} - \mathbb{L}^{3a})}{(\mathbb{L}-1)^3(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)} \mathbb{J}\zeta_{\bar{C}}(\mathbb{L})\,\zeta_{\bar{C}}(\mathbb{L}^2)\,[\mathcal{N}^s_{\sigma_m}(1, 1, d_1 - \frac{m}{2}, d_2)]$$

The next coefficient is not so nice as before. To compute the coefficient of `J[1,t]`$= \zeta_{\bar{C}}(\mathbb{L})$ we run the commands:

```
C1t1 = CoefficientList[C1t2[[1]], J[1, t]];
C1t1[[2]]
```

The output of this code is truly weird. Computing the coefficient of $t^a =$ `t^\[Alpha]` and and $t^b =$ `t^\[Beta]` simplifies a lot:

```
CoefficientList[C1t1[[2]], t^\[Alpha]] // FullSimplify // Apart
CoefficientList[C1t1[[2]], t^\[Beta]] // FullSimplify // Apart
```

and we obtain the class

$$A_8 = \left\{ \frac{\mathbb{L}^{2(g-1)}}{(\mathbb{L}^{-}1)^3(\mathbb{L}^2-1)} \left[(\mathbb{L}^{3b} - \mathbb{L}^{2b} - \mathbb{L}^b) - (\mathbb{L}^{3a} - \mathbb{L}^{2a} - \mathbb{L}^a)\right] - \right.$$
$$\left. - \frac{\mathbb{L}-2}{(\mathbb{L}-1)^4(\mathbb{L}^2-1)}(\mathbb{L}^{3b} - \mathbb{L}^{3a}) + \frac{\mathbb{L}^{2(g-1)}(1 - 2\mathbb{L} - 2\mathbb{L}^3 + \mathbb{L}^4)}{(\mathbb{L}-1)^4(\mathbb{L}^2-1)(\mathbb{L}^3-1)} \right\} [\mathcal{N}(1, 1, d_1 - \frac{m}{2}, d_2)]\mathbb{J}^2\zeta_{\bar{C}}(\mathbb{L})$$

(Here, we manipulate a bit the output of these commands).

Finally, the class of `J[1,1]`$= \zeta_{\bar{C}}(1) = \mathbb{J}$ is not so easy to write. We run the following command:
`C1 = CoefficientList[C1t1[[1]], J[1, 1]];`

We have three non zero coefficients: `C1[[2]]`, `C1[[3]]` and `C1[[4]]` corresponding to the coefficient of $\zeta_{\bar{C}}(1)$, $\zeta_{\bar{C}}(1)^2$ and $\zeta_{\bar{C}}(1)^3$ respectively. The output of the two first coefficients invite us to join them:

$$A_9 = \frac{\mathbb{L}^{g-3}\left(\mathbb{L}^g - 1\right)^2\left(\mathbb{L}^{2a} - \mathbb{L}^{3a} - \mathbb{L}^{2b} + \mathbb{L}^{3b}\right)}{(\mathbb{L}-1)^3}(\mathbb{J}^3 - 2\mathbb{J}^2).$$

The coefficient of $\mathbb{J}^4$ is:

$$A_{10} = \left\{\frac{\mathbb{L}^{2(g-1)}}{(\mathbb{L}-1)^3(\mathbb{L}^2-1)}\left[(\mathbb{L}^{3b} - \mathbb{L}^{2b} - \mathbb{L}^b) - (\mathbb{L}^{3a} - \mathbb{L}^{2a} - \mathbb{L}^a)\right] + \right.$$
$$\left. + \frac{\mathbb{L}-2}{(\mathbb{L}-1)^4}\left(\frac{\mathbb{L}^{3g-1}}{(\mathbb{L}^2-1)^2} - \frac{\mathbb{L}^{g-1}}{\mathbb{L}^2-1} - \frac{1}{3}\right)(\mathbb{L}^{3b} - \mathbb{L}^{3a})\right\}\mathbb{J}^4[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)].$$

At this time, the class $[\mathcal{S}^-_{\sigma_c}] - [\mathcal{S}^+_{\sigma_c}]$ is the sum $A_1 + A_2 + \cdots + A_{10}$. With these pieces in a clearer form, it should be easy (but noisy) to rearrange with the aid of the Mathematica. We leave this to the reader.

■ **Sum along critical values** Indeed, we do not need to rearrange these classes to obtain a complete expression for $[\mathcal{N}_\sigma(4,1,d_1,d_2)]$. We sum along $m \in 6\mathbb{Z}$ from $m = m_0$ to the greatest bound $2(d_1 - d_2)$. Nevertheless, for $m > 2(d_1 - d_2)$

$$\operatorname*{coeff}_{x^0} \frac{\zeta_{\bar{C}}(x)}{(1-x)(1-\mathbb{L}x)x^{d_1 - d_2 - \frac{m}{2}}}\mathbb{J} = 0$$

so, the good way is to compute the sum of these pieces along $m \in 6\mathbb{Z}$ from $m = m_0$ to infinity. We simply substitute each fraction $\mathbb{L}^{ka}[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)]$ or $\mathbb{L}^{kb}[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)]$ by

$$\sum_{m=m_0, m\in 6\mathbb{Z}}^{\infty} \mathbb{L}^{ka}[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)] = \frac{\mathbb{L}^{ka_0}}{1 - \left(\frac{x}{\mathbb{L}^k}\right)^3}[\mathcal{N}(1,1,d_1 - \frac{m_0}{2}, d_2)],$$
$$\sum_{m=m_0, m\in 6\mathbb{Z}}^{\infty} \mathbb{L}^{kb}[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)] = \frac{\mathbb{L}^{kb_0}}{1 - \mathbb{L}^{4k}x^3}[\mathcal{N}(1,1,d_1 - \frac{m_0}{2}, d_2)],$$
$$\sum_{m=m_0, m\in 6\mathbb{Z}}^{\infty} \mathbb{L}^{ia+jb}[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)] = \frac{\mathbb{L}^{ia_0+jb_0}}{1 - \mathbb{L}^{4j-3i}x^3}[\mathcal{N}(1,1,d_1 - \frac{m_0}{2}, d_2)]$$

where $k = 1,2,3$, $a_0 = d_1 - d_2 - \frac{m_0}{2}$ and $b_0 = g - 1 - d_1 + \frac{3}{2}m_0$.

■ **Part 4** Let us go to the last situations: when $m \notin 6\mathbb{Z}$. We have two cases.

If $m \in 3\mathbb{Z}$, we have that the class of $\mathcal{S}^-_{\sigma_m} \setminus \mathcal{S}^+_{\sigma_m}$ is

```
Sb[2, 5] = P[a[4, 2]]*M2[1]*N2a;
SSb[2, 5] = P[b[4, 2]]*M2[1]*N2a;
```

and

```
Sb[2, 5] = P[a[4, 2]]*M2[1]*N2b;
SSb[2, 5] = P[b[4, 2]]*M2[1]*N2b;
```

where we obtain the classes

$$\frac{\mathbb{L}^{2(g-1)}(\mathbb{L}^{3b} - \mathbb{L}^{3a})}{(\mathbb{L}-1)^3(\mathbb{L}^2-1)}(\zeta_{\bar{C}}(\mathbb{L}) - \mathbb{L}^g\mathbb{J})\mathbb{J}[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)],$$
$$\frac{\mathbb{L}^{2g-1}(\mathbb{L}^{2b} - \mathbb{L}^{2a})}{(\mathbb{L}-1)^3(\mathbb{L}^2-1)}\left(\frac{\mathbb{L}^{a-1}}{1 - \frac{x}{\mathbb{L}}} - \frac{\mathbb{L}^b x}{1 - \mathbb{L}^2 x}\right)(\mathbb{L}^g\mathbb{J} - \zeta_{\bar{C}}(\mathbb{L}))\mathbb{J}^2[\mathcal{N}(1,1,d_1 - \frac{m}{2}, d_2)],$$

[Rank 4 — 112]

As before, we sum from $m = m_0$ to infinity using the equality

$$\sum_{m=m_0, m\in 3\mathbb{Z}}^{\infty} \mathbb{L}^{ia+jb}[\mathcal{N}(1,1,d_1-\frac{m}{2},d_2)] = \frac{\mathbb{L}^{ia_0+jb_0}}{1-\mathbb{L}^{2j-\frac{3}{2}i}x^{\frac{3}{2}}}[\mathcal{N}(1,1,d_1-\frac{m_0}{2},d_2)].$$

- **Part 5** If $m \in 2\mathbb{Z}$, we have that $[\mathcal{S}^-_{\sigma_m}] - [\mathcal{S}^+_{\sigma_m}]$ is the difference of these classes

```
Sb[1, 5] = P[a[4, 3]]*N1*(M3[1]);
SSb[1, 5] = P[b[4, 3]]*N1*(M3[1]);
```

that is

$$\frac{(\mathbb{L}^{3a}-\mathbb{L}^{3b})}{(\mathbb{L}-1)^2}\left(\frac{\mathbb{L}^{3g-1}}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2}\mathbb{J} - \frac{\mathbb{L}^{2g-1}}{(\mathbb{L}-1)^2(\mathbb{L}^3-1)}\mathbb{J}\zeta_{\bar{C}}(\mathbb{L}) + \frac{\zeta_{\bar{C}}(\mathbb{L})\,\zeta_{\bar{C}}(\mathbb{L}^2)}{(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\right)\mathbb{J}[\mathcal{N}(1,1,d_1-\frac{m}{2},d_2)]$$

Here, the sum along even integer numbers from $m = m_0$ to infinity can be done almost automatically using the equality

$$\sum_{m=m_0, m\in 2\mathbb{Z}}^{\infty} \mathbb{L}^{ai+bj}[\mathcal{N}(1,1,d_1-\frac{m}{2},d_2)] = \frac{\mathbb{L}^{a_0i+b_0j}}{1-\mathbb{L}^{\frac{4}{3}j-i}x}[\mathcal{N}(1,1,d_1-\frac{m_0}{2},d_2)]$$

Notice that $j = 3$ in our case: then the exponent $\frac{4}{3}j - i$ is an integer number.

Making this substitution in every sum, we obtain the class of the moduli of pairs of rank 4. We remember that this substitution can be done using the power of the mathematical software.

Chapter 7

The Hodge conjecture for pairs

The main aim of this chapter is to prove the Hodge Conjecture for the moduli of pairs of rank less than 4. At the first stage, we give some reference about the Hodge Conjecture. In the next subsection, we give a generalization of some family of strata. One gives the bricks of the proof, we begin to prove the main theorem using the Arapura-Kang functor. Nevertheless, the fact that $M^s(2,0)$ and $M^s(3,0)$ does not have a locally Zariski trivial universal bundle makes us to consider the Brauer group of these classes to obtain the desired results.

1 The Generalized Hodge Conjecture

Let X be a smooth projective variety of dimension n. We have the following well-known result due to Hodge:

Hodge decomposition Theorem 7.1. *Let X be a compact Kähler manifold. Let $H^{p,q}(X)$ be the space of cohomology classes whose harmonic representative is of type (p,q). There is a direct sum decomposition*

$$H^m_{\mathrm{DR}}(X) \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q}(X)$$

Moreover $H^{q,p}(X) = \overline{H^{p,q}(X)}$.

This inspire the abstract idea of Hodge structure:

Definition 7.2. *A* pure R-Hodge structure *of weight m is a R-module V (generally, $R = \mathbb{Z}$ or $\mathbb{Q}$) with a decreasing filtration F on $V_{\mathbb{C}} = V \otimes_R \mathbb{C}$ of length m satisfying*

$$F^pV \cap \overline{F^qV} = 0 \text{ if } p+q = m+1.$$

Equivalently, a pure Hodge structure of weight m is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=m} V^{p,q}$ such that $V^{q,p} = \overline{V^{p,q}}$. In this case, $V^{p,q} = F^pV \cap \overline{F^qV}$.

Definition 7.3. *The* level *of a Hodge structure $H = (V,F)$ of weight m is the largest difference $|p-q|$ such that $V^{p,q} \neq 0$.*

Then, we can reformulate the Hodge decomposition Theorem to say

The real vector space $H^m_{\mathrm{DR}}(X)$ of any compact Kähler manifold (in particular, a smooth projective variety) X admits a natural pure Hodge structure of weight m.

Indeed, (pure) Hodge structures form a category $\mathfrak{hs}$ defining its maps as follows:

Definition 7.4. *Let $H = (V,F)$ and $H' = (V',F')$ be two R-Hodge structures of weight m. A map $f : H \to H'$ is a R which preserves the Hodge structures, that is, $f_{\mathbb{C}} : F^pV_{\mathbb{C}} \hookrightarrow F'^pV'_{\mathbb{C}}$.*

One can define several operations and objects on this category: Hodge substructures, direct sum, tensor product, etc. Then, another formulation of the Hodge decomposition theorem is the following:

There exists a functor $\mathfrak{H}^m : \mathfrak{Var}_{\mathbb{C}} \to \mathfrak{hs}$ such that $\mathfrak{H}^m(X) = (H^m(X,\mathbb{Q}),F)$ where X is a smooth projective variety. A map between varieties gives a map between its Hodge structures. We shall see that this map descends to Grothendieck groups.

Now, let $Z \subset X$ be a complex subvariety of codimension c. We have the long exact sequence

$$\cdots \to H^m(X,Z) \to H^m(X) \to H^m(X \setminus Z) \to \cdots$$

Definition 7.5. *A Hodge $\mathbb{Q}$-substructure $H \subset H^m(X)$ is* supported *on $Z \subset X$ if*

$$H \subset \mathrm{Im}\left(H^m(X, Z, \mathbb{Q}) \to H^m(X, \mathbb{Q})\right).$$

or equivalently, $H \subset \ker(H^m(X, \mathbb{Q}) \to H^m(X \setminus Z, \mathbb{Q}))$.

The codimension of the subvarieties gives a filtration:

Definition 7.6. *The* coniveau filtration *is the union of all Hodge $\mathbb{Q}$-substructures supported by some $Z \subset X$ of codimension $\geq c$. In other words,*

$$\mathcal{N}^c H^m(X, \mathbb{Q}) = \bigcup_{\operatorname{codim} Z \geq c} \mathrm{Im}\left(H^m(X, Z, \mathbb{Q}) \to H^m(X, \mathbb{Q})\right).$$

Recall the Poincaré-Lefschetz duality: $H^m(X, Z, \mathbb{Q}) \cong H^{m-2c}(Z)[c]$, so $\mathcal{N}^c H^m(X, \mathbb{Q})$ has level $m - 2c$. Hence,

$$\mathcal{N}^c H^m(X, \mathbb{Q}) \subseteq F^c H^m(X) \cap H^m(X, \mathbb{Q}),$$

since $F^c H^m(X) \cap H^m(X, \mathbb{Q})$ "has level $m - 2c$".

Remark 7.7. *The coniveau filtration can be extended to singular projective varieties.*

We might think the conversely is true, that is,

$$F^c H^m(X) \cap H^m(X, \mathbb{Q}) \subseteq \mathcal{N}^c H^m(X, \mathbb{Q}).$$

This would be the Hodge conjecture except this implies the equality and, as Grothendieck shown, $F^c H^m(X) \cap H^m(X, \mathbb{Q})$ is not a Hodge substructure. To correct this gap, Grothendieck himself gives a new version of the Hodge conjecture:

Generalized Hodge Conjecture 7.8. *Let X be a smooth projective variety. The largest rational Hodge substructure of $F^c H^m(X) \cap H^m(X, \mathbb{Q})$ is the union of all rational Hodge substructures supported on codimension $\geq c$ subvarieties of X.*

Alternatively, for every Hodge $\mathbb{Q}$-substructure H' of $H^m(X, \mathbb{Q})$ of level at most $m - 2c$, there exists a subvariety $Z \subset X$ of codimension $\geq c$ such that H' is supported on Z.

2 The Standard Hodge Conjecture

First, we recall some well-known facts on cohomology. The pairing $H^k_{\mathrm{DR}}(X, \mathbb{C}) \times H^{2n-k}_{\mathrm{DR}}(X, \mathbb{C}) \to \mathbb{C}$ is defined to be

$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

where α and β are representatives of the cohomology classes $[\alpha] \in H^k_{\mathrm{DR}}(X, \mathbb{C})$ and $[\beta] \in H^{2n-k}_{\mathrm{DR}}(X, \mathbb{C})$ respectively. Stokes theorem assures us that this map does not depend on the choosen representatives of the cohomology classes. Poincaré proves that this pairing is non-degenerate. This means that

Theorem 7.9. [Poincaré isomorphism] *Let X be an algebraic variety of dimension n, then*

$$H^{2n-k}_{\mathrm{DR}}(X, \mathbb{C})^\vee \cong H^k_{\mathrm{DR}}(X, \mathbb{C}).$$

Joininig this results with the isomorphism $H^{2n-k}_{\mathrm{DR}}(X, \mathbb{C})^\vee \cong H_{2n-k}(X, \mathbb{C})$ (the singular homology), the above pairing rewrites as follows

$$\begin{aligned} H^k_{\mathrm{DR}}(X, \mathbb{C}) \times H_k(X, \mathbb{C}) &\longrightarrow \mathbb{C} \\ ([\alpha], [Z]) &\longmapsto \textstyle\int_Z \alpha \end{aligned}$$

Hence, for any algebraic subvariey $Z \subset X$ there is a natural dual class

$$\int_{Z^*} : H^{2n-2k}_{\mathrm{DR}}(X, \mathbb{C}) \to \mathbb{C} : \alpha \mapsto \int_{Z^*} \alpha$$

(where $Z^* = Z \setminus Z_{\mathrm{sing}}$) and this gives, by Poincaré duality, a class $[\beta_Z] \in H^{2k}_{\mathrm{DR}}(X, \mathbb{C})$ such that the map

$$\begin{tikzcd} H^{2n-2k}_{\mathrm{DR}}(X, \mathbb{C}) \arrow[r] & \mathbb{C} \\ \alpha \arrow[r, mapsto] & \int_X \alpha \wedge \beta_Z \end{tikzcd}$$

is the map $\int_{Z^*}$. Then, we define $[Z]$ to be the cohomology class $[\beta_Z] \in H^{2k}_{\mathrm{DR}}(X, \mathbb{C})$. Indeed, this class belongs to $H^{k,k}(X, \mathbb{Z}) = H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X)$. The correspondence "subvariety"−"cohomology class" can be written as a group map: let $C^k(X)$ be the free abelian group generated by subvarieties of codimension k in X, we define the map

$$\begin{tikzcd} [\ \ ] : C^k(X) \otimes \mathbb{Q} \arrow[r] & H^{k,k}(X, \mathbb{Q}) \\ Z \arrow[r, mapsto] & {[Z] = [\beta_Z]} \end{tikzcd}$$

extended by linearity.

Standard Hodge Conjecture 7.10. *Let X be an smooth projective algebraic variety over $\mathbb{C}$. The map $[\ \ ] : C^k(X) \otimes \mathbb{Q} \to H^{k,k}(X, \mathbb{Q})$ is surjective for all k.*

The Generalized Hodge Conjecture implies the Standard Hodge Conjecture: this is the special case $m = 2k$ for the $\mathbb{Q}$-Hodge substructure $H = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$ of $H^{2k}(X, \mathbb{Q})$ where the conjecture claims that there exists a subvariety Z of X of codimension $\geq k$ such that H is supported on Z.

3 The Arapura-Kang functor

Given a smooth projective variety X, the cohomology groups $H^k(X)$ have a natural Hodge structure. There are two filtrations:

- The *coniveau* filtration which is descending:

$$N^p H^k(X) = \sum_{\mathrm{codim}\, S \geq p} \ker\left(H^k(X) \xrightarrow{H^k(i)} H^k(X \setminus S) \right)$$

  where $i : X \setminus S \hookrightarrow X$ is the natural inclusion.
- Denote by $\mathcal{F}^p H^k(X)$ to be the largest rational Hodge substructure contained in $F^p H^k(X)$

By definition, a smooth projective variety X satisfies the Generalized Hodge Conjecture (known as GHC) if the equality $\mathcal{F}^p H^k(X) = N^p H^k(X)$ holds. Notice that the inclusion $N^p H^k(X) \subseteq \mathcal{F}^p H^k(X)$ is always true.

Let us introduce the following two categories: the category of Hodge structure $\mathfrak{hs}$ which becomes an abelian variety with the tensor product. Furthermore, $\mathfrak{fhs}$ denotes the category of filtered Hodge structures. The two previous filtrations induces two functors in these categories:

$$\begin{tikzcd} \Gamma : \mathfrak{hs} \arrow[r] & \mathfrak{fhs} & \Phi : \mathfrak{fhs} \arrow[r] & \mathfrak{hs} \\ H^*(X) \arrow[r, mapsto] & (H^*(X), \mathcal{F}^*) & (H^*(X), N^*) \arrow[r, mapsto] & H^*(X) \end{tikzcd}$$

These two maps induce well-defined homomorphisms of groups

$$\gamma : K_0(\mathfrak{hs}) \to K_0(\mathfrak{fhs}),$$
$$\phi : K_0(\mathfrak{fhs}) \to K_0(\mathfrak{hs}).$$

where the Grothendieck groups of these categories are built by the equivalence relation

$$[H \oplus H'] \sim [H] + [H'] \text{ and } [H \otimes H'] \sim [H][H'].$$

The link between these Grothendieck groups are the maps λ and ν:

$$\begin{array}{rclcrcl} \lambda : K_0^{\mathrm{bl}}(\mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}) & \longrightarrow & K_0(\mathfrak{fhs}) & \qquad & \nu : K_0^{\mathrm{bl}}(\mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}}) & \longrightarrow & K_0(\mathfrak{fhs}) \\ [X] & \longmapsto & [H^*(X), N^*] & & [X] & \longmapsto & \gamma([H^*(X)]) = [H^*(X), \mathcal{F}^*] \end{array}$$

The largest part of the paper [AK06] is devoted to prove that these maps are indeed homomorphisms of groups. More concretely, they preserve the sum operation $+$.

This gives the main result of [AK06].

Theorem 7.11. [AK06] *Let X be a smooth projective variety. Then, the following statements are equivalent:*

1) GHC holds for X.
2) $[X] \in \ker(\nu - \lambda)$.
3) The equality $FP_{\lambda([X])}(t,u) = FP_{\nu([X])}(t,u)$ of the filtered Poincaré polynomials holds.

Nevertheless, λ and μ does not preserve the product, but it is true that it does for some special classes. Let $\mathbb{L} = [\mathbb{P}^1] - [\infty] \in K_0^{\mathrm{bl}}(\mathrm{SP}\mathfrak{V}\mathrm{ar})$ the Lefschetz class and $\mathbb{K} = [\mathbb{Q}(-1), N^*] \in K_0(\mathfrak{fhs})$ where $N^1/N^2 = \mathbb{Q}(-1)$. It is well-known that $\lambda(\mathbb{L}) = \nu(\mathbb{L}) = \mathbb{K}$ and

Lemma 7.12. [AK06] *For any class $\eta \in K^{\mathrm{bl}}(\mathrm{SP}\mathfrak{V}\mathrm{ar}_{\mathbb{C}})$,*

$$\lambda(\mathbb{L} \cdot \eta) = \mathbb{K} \cdot \lambda(\eta),$$
$$\nu(\mathbb{L} \cdot \eta) = \mathbb{K} \cdot \nu(\eta).$$

■ Hodge Conjecture

Let $M = (X, p)$ be an effective motive. Then the cohomology of M is

$$H^k(M) = \mathrm{Im}(p^* : H^k(X) \to H^k(X)),$$

for each integer k. This produces a map $K(\mathfrak{Mot}) \to K(\mathfrak{hs})$, $M \mapsto \sum H^k(M)$.

Let $\mathfrak{fhs}$ be the category of filtered Hodge structures. For X smooth projective, the level filtration $\mathcal{F}^p H^k(X)$ gives an element $\mathcal{F}(X)$ in $\mathfrak{fhs}$. Therefore, for $M = (X, p)$, we have an element

$$\mathcal{F}(M) = \sum \mathcal{F}^\bullet H^k(M) \in K_0(\mathfrak{fhs}).$$

Hence, there is a map

$$\lambda : K(\mathfrak{Mot}) \to K(\mathfrak{fhs}).$$

Also, the coniveau filtration produces a map

$$\nu : K(\mathfrak{Mot}) \to K(\mathfrak{fhs}).$$

Let

$$\Theta = \lambda - \nu : K(\mathfrak{Mot}) \to K(\mathfrak{fhs}).$$

An important remark is that for M an effective motive, $\Theta(M)$ is an effective Hodge structure. We have the following two important properties:

1) for any *smooth projective* variety X, $\Theta(X) = 0$ if and only if the Hodge Conjecture holds for X.
2) $\Theta(\mathbb{L} \cdot x) = \mathbb{K} \cdot \Theta(x)$, where $\mathbb{K}$ is the Hodge structure of $\mathbb{A}^1$ (the one-dimensional Hodge structure with weight $(1,1)$).

Unfortunately, the map Θ is not known to be a multiplicative map. However, the important feature is that the kernel of Θ is an abelian subgroup stable by multiplication by $\mathbb{L}$ and $\mathbb{L}^{-1}$, which allows us to consider the map on the completions

$$\Theta : \hat{K}(\mathfrak{Mot}) \to \hat{K}(\mathfrak{fhs})\,. \tag{7.1}$$

Lemma 7.13. *If M and N are effective motives, then*

$$\Theta(M \oplus N) = 0 \Longleftrightarrow \Theta(M) = \Theta(N) = 0\,.$$

Proof. In the left direction, it is obvious. In the right direction, just note that Θ sends effective motives to effective filtered Hodge structures. □

For $X \in \mathfrak{Var}_{\mathbb{C}}$, we write $\Theta(X) := \Theta(h(X))$.

The following result is well-known (cf. [Ara06]). For convenience of the reader, we include a proof. The genericity of the curve appearing in the proposition means that it belongs to the complement of a countable union of some closed sets of the moduli of curves (see [BP02, Section 1] or [Ara06, Proposition 6.5]).

Proposition 7.14. *Let C be a generic curve. Then*

1) $\Theta(C^k) = 0$*, for any* $k \geq 0$.
2) $\Theta(\operatorname{Sym}^k C) = 0$*, for any* $k \geq 0$.
3) $\Theta(\operatorname{Jac} C) = 0$.
4) $\Theta(\mathcal{R}_C) = 0$.

Proof. Let $A = \operatorname{Jac} C$, which is a polarised abelian variety. The Hodge group of A, $\operatorname{Hg}(A)$, is defined as the group of all linear automorphisms of $V = H^1(A, \mathbb{Q})$ which leave invariant all Hodge cycles of the varieties $A \times \cdots \times A$. Let E denote the polarisation of A. Then $\operatorname{Hg}(A) = \operatorname{Sp}(V, E)$. This can be proved with the arguments of [BP02]: the Hodge group of $\operatorname{Jac} C$ for general C contains the Hodge group of any degeneration of C, in particular, degenerating C to a reducible nodal curve consisting of the union of two curves of genus a and $g - a$, we see that $\operatorname{Sp}(2g) \supset \operatorname{Hg}(A) \supset \operatorname{Sp}(2a) \times \operatorname{Sp}(2g - 2a)$. As the general Jacobian has Neron-Severi group equal to $\mathbb{Z}$, the argument in [BP02, Theorem 5] proves that $\operatorname{Hg}(A) = \operatorname{Sp}(2g)$.

By [BL04, Proposition 17.3.4], $\operatorname{End}_{\mathbb{Q}}(A) = \operatorname{End}(V)^{\operatorname{Hg}(A)} = \mathbb{Q}$. Now the Lefschetz group of A is

$$Lf(A) = \{g \in \operatorname{Sp}(V, E)\ ;\ g \circ f = f \circ g, \forall f \in \operatorname{End}_{\mathbb{Q}}(A)\}_0 = \operatorname{Sp}(V, E) = \operatorname{Hg}(A)\,.$$

By [Tan81] or [BL04, Exercise 13], $H^*_{Hodge}(A^k)$ is generated by divisors, so the Hodge Conjecture holds for A^k, for all $k \geq 1$. This is written, in our terminology, as $\Theta((\operatorname{Jac} C)^k) = 0$.

Now $h^1(C)$ is a submotive of $h(\operatorname{Jac} C)$, so Lemma 7.13 implies that $\Theta(h^1(C)^k) = 0$, and therefore $\Theta(C^k) = 0$ for any $k \geq 1$. From this, using (1.12) and Lemma 7.13, we get $\Theta(\operatorname{Sym}^k C) = 0$, for any $k \geq 1$. Finally, $\mathcal{R}_C$ is generated by varieties obtained by taking iterated products and symmetric products of the curve C. Such a manifold is isomorphic to a quotient C^k/F, where $k \geq 1$ and $F \subset \mathfrak{S}_k$ is a subgroup of the permutation group of the factors of C^k. Using (1.12) and Lemma 7.13 again, it follows that $\Theta(C^k/F) = 0$. □

4 Some family of strata

Now we move on to the issue of giving an explicit description for the strata $X^{\pm}(\mathbf{A}, \mathbf{n})$ corresponding to a critical value σ_c and a type $\mathbf{n}(n_i)$ and $\mathbf{A}$, using Proposition 2.26 and Proposition 2.27. Recall that each n_i determines the corresponding d_i by (2.9). Our aim is to prove that $[X^{\pm}(\mathbf{A}, \mathbf{n})] \in \mathcal{R}_C$, where the ring $\mathcal{R}_C$ is defined in Section 1.2.

■ The simplest case

We start with a simple case.

Proposition 7.15. *Let σ_c be any critical value (possibly $\sigma_c = \sigma_m, \sigma_M$). Let $n \geq 1$, and $\mathbf{n} = (n_i)$ and $\mathbf{A} = (a_i^j)_{i,j=1}^{r,b}$ be a type such that $\mathfrak{S}_{\mathbf{n}} = \{1\}$. If $\sigma_c = \sigma_m$, we assume $\mathbf{n} \neq \mathbf{n}_0$. Suppose that $[M^s(n'', d'')]$ and $[\mathcal{N}_{\sigma_c}^s(n', 1, d', d_o)]$ are in $\mathcal{R}_C$, for any $n', n'' < n$, $\gcd(n'', d'') = 1$. Assume also that $\gcd(n_i, d_i) = 1$, for every $i = 1, \ldots, b$. Then $[X^{\pm}(\mathbf{A}, \mathbf{n})] \in K(\mathfrak{Mot})$ belongs to $\mathcal{R}_C$.*

Proof. If the group $\mathfrak{S}_{\mathbf{n}}$ is trivial, then by [Mn10, Theorem 6.1], the fibration

$$F \to \tilde{\tilde{X}}^{\pm}(\mathbf{A}, \mathbf{n}) \to \mathcal{M}(\mathbf{n})$$

has fiber F which is *affinely stratified* (AS, for short). That means that it is an iterated fiber bundle of spaces which are an affine space minus a linear subspace. It is easy to see that a Zariski locally trivial fibration whose base and fibre are both AS has AS total space. Therefore, $[F] = P(\mathbb{L})$ for some polynomial P. Then

$$F/\operatorname{GL}(\mathbf{n}) \to X^{\pm}(\mathbf{A}, \mathbf{n}) \to \mathcal{M}(\mathbf{n}) \tag{7.2}$$

has fiber such that $[F/\operatorname{GL}(\mathbf{n})] = P(\mathbb{L})/[\operatorname{GL}(\mathbf{n})]$, which is a series in $\mathbb{L}$.

As $\gcd(n_i, d_i) = 1$, there are universal bundles $\mathcal{E}_i \to M^s(n_i, d_i) \times C$. Analogously, there is a universal triple $\mathcal{T}' \to \mathcal{N}' \times C$ since there are universal bundles over any moduli space of σ-stable triples of type $(n, 1)$. The bundle (7.2) is constructed iteratively by taking bundles of relative Ext^1-groups of the $\mathcal{E}_i$'s and $\mathcal{T}'$. All these bundles are then Zariski locally trivial, since the dimension of the Ext^1-groups is constant, because $\operatorname{Ext}^0 = \operatorname{Ext}^2 = 0$ in all these cases, as it is proven in [Mn10]. This means that (7.2) is locally trivial in the Zariski topology, and hence $[X^{\pm}(\mathbf{A}, \mathbf{n})] = [\mathcal{M}(\mathbf{n})]\,[F/\operatorname{GL}(\mathbf{n})]$.

The basis of the fibration is the space

$$\mathcal{M}(\mathbf{n}) = \tilde{U}(\mathbf{n}) \times \mathcal{N}_{\sigma_c}^s(n', 1, d', d_o) = \left(\prod_{i=1}^{b} M^s(n_i, d_i) \setminus \Delta\right) \times \mathcal{N}_{\sigma_c}^s(n', 1, d', d_o)\,,$$

where Δ is a union of some diagonals, each of which is a product of some moduli spaces $M^s(n_j, d_j)$. Therefore

$$[X^{\pm}(\mathbf{A}, \mathbf{n})] = [\tilde{U}(\mathbf{n})]\,[\mathcal{N}_{\sigma_c}^s(n', 1, d', d_o)]\,P(\mathbb{L})/[\operatorname{GL}(\mathbf{n})] \in \mathcal{R}_C.$$

□

■ A family of strata with one non-trivial step

Now we consider the strata $X^{+}(\mathbf{A}, \mathbf{n}) \subset \mathcal{S}_{\sigma_c^+}$ where the standard filtration has only one nontrivial step, hence it is of the form $0 \subset T_1 \subset T_2 = T$. So $r = 1$, and we only have $\mathbf{a}_1 = (a_1, a_2, \ldots, a_b)$, where all $a_i > 0$. We write the type as a matrix

$$\mathbf{A} = (\, a_1 \quad a_2 \quad \cdots \quad a_b \,)\,, \qquad \mathbf{n} = (\, n_1 \quad n_2 \quad \cdots \quad n_b \,) \tag{7.3}$$

To describe geometrically the strata $X^{+}(\mathbf{A}, \mathbf{n})$ in this and in the following section, we will make use of partitions of sets. If

$$[b] := \{1, \ldots, b\}$$

denotes the set of the first b positive integer numbers, we define a *partition* π to be a collection of disjoint subsets of $[b]$ whose union is $[b]$. An element of a partition is called a *brick*. We can define a partition by an equivalence relation, for which the bricks are the equivalence classes.

There is a natural partial order on the set of partitions of $[b]$. If π, π' are two such partitions, then we say that $\pi \leq \pi'$ if any $\beta \in \pi$ is a subset of some $\beta' \in \pi'$.

Let π' and π'' be two partitions on $[b]$. Then the *intersection* $\pi' \wedge \pi''$ of the partitions π' and π'' is defined as

$$\pi' \wedge \pi'' = \{\beta' \cap \beta'' \subset [b]\,;\, \beta' \in \pi' \text{ and } \beta'' \in \pi''\}.$$

Obviously, $\pi' \wedge \pi'' \leq \pi'$ and $\pi' \wedge \pi'' \leq \pi''$.

Proposition 7.16. *Let σ_c be any critical value. Let $n \geq 1$, and $\mathbf{n}$ be given by (7.3), and assume that $\mathfrak{S}_{\mathbf{n}}$ is non-trivial. Suppose that $[M^s(n'', d'')]$ and $[\mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)]$ are in $\mathcal{R}_C$, for any $n', n'' < n$, $\gcd(n'', d'') = 1$. Assume also that $\gcd(n_i, d_i) = 1$, for every $i = 1, \ldots, b$. Then $[X^{\pm}(\mathbf{A}, \mathbf{n})] \in \mathcal{R}_C$.*

Proof. We will only do the case of $X^+(\mathbf{A}, \mathbf{n})$, the other one being analogous. Given $\mathbf{n}$, we have two partitions of $[b]$: π_0 defined by the equivalence relation

$$i \sim j \iff n_i = n_j, \tag{7.4}$$

and π_1 defined by

$$i \sim j \iff n_i = n_j \text{ and } a_i = a_j.$$

Of course, we have $\pi_1 \leq \pi_0$. With the notations of Section 2.6,

$$\tilde{U}(\mathbf{n}) = \prod_{\beta \in \pi_0} \left(M^s(n_\beta, d_\beta)^{|\beta|} \setminus \Delta_\beta \right),$$

where Δ_β stands for the big diagonal, and we write $n_\beta := n_i$ and $d_\beta := d_i$ for any $i \in \beta$. The space $\tilde{X}^+(\mathbf{A}, \mathbf{n})$ is a bundle over $\tilde{U}(\mathbf{n}) \times \mathcal{N}'$, where $\mathcal{N}' = \mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)$, whose fiber over $(S_1, \ldots, S_b, T')$ is

$$F = \prod_{\beta \in \pi_0} \prod_{\substack{\gamma \in \pi_1 \\ \gamma \subset \beta}} \mathrm{Gr}(a_\gamma, \mathrm{Ext}^1(T', S_\gamma))^{|\gamma|} = \prod_{\gamma \in \pi_1} \mathrm{Gr}(a_\gamma, \mathrm{Ext}^1(T', S_\gamma))^{|\gamma|}. \tag{7.5}$$

Step 1. We extend this bundle to a bundle $\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta$ over the union of all diagonals, that is, we have the bundle

$$\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta \to \prod_{\beta \in \pi_0} M^s(n_\beta, d_\beta)^{|\beta|} \times \mathcal{N}' = \prod_{\gamma \in \pi_1} M^s(n_\gamma, d_\gamma)^{|\gamma|} \times \mathcal{N}',$$

with fiber (7.5).

We want to write the total space $\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta$ as a product of bundles. For each $\gamma \in \pi_1$, we have a Zariski locally trivial bundle

$$\mathcal{E}_\gamma \to M^s(n_\gamma, d_\gamma) \times \mathcal{N}' \tag{7.6}$$

with fiber $\mathrm{Gr}(a_\gamma, \mathrm{Ext}^1(T', S_\gamma))$ over $(S_\gamma, T') \in M^s(n_\gamma, d_\gamma) \times \mathcal{N}'$. We can consider $\mathcal{E}_\gamma$ as a bundle over $\mathcal{N}'$. Its fiber over T' is the Zariski locally trivial bundle $\mathcal{E}_{\gamma, T'}$ with basis $M^s(n_\gamma, d_\gamma)$ and fiber $\mathrm{Gr}(a_\gamma, \mathrm{Ext}^1(T', S_\gamma))$. Then $\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta$ is the fiber product of the bundles $(\mathcal{E}_\gamma)^{|\gamma|}$ over $\mathcal{N}'$:

$$\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta = \prod_{\substack{\gamma \in \pi_1 \\ \mathcal{N}'}} (\mathcal{E}_\gamma)^{|\gamma|}.$$

Here we are thinking of $\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta$ as a bundle over $\mathcal{N}'$, as we just did for each $\mathcal{E}_\gamma$.

Let us take the quotient of $\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta$ by the finite group $\mathfrak{S}_{\mathbf{n}}$. We use partitions to describe these groups. For any partition π of $[b]$, let

$$\mathfrak{S}_\pi = \prod_{\beta \in \pi} \mathfrak{S}_\beta.$$

So $\mathfrak{S}_{\mathbf{n}} = \mathfrak{S}_{\pi_1}$. Then

$$\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta / \mathfrak{S}_{\mathbf{n}} = \prod_{\substack{\gamma \in \pi_1 \\ \mathcal{N}'}} (\mathcal{E}_\gamma)^{|\gamma|} / \mathfrak{S}_\gamma = \prod_{\substack{\gamma \in \pi_1 \\ \mathcal{N}'}} \mathrm{Sym}^{|\gamma|}(\mathcal{E}_\gamma),$$

where $\mathrm{Sym}^{|\gamma|}(\mathcal{E}_\gamma)$ is the bundle over $\mathcal{N}'$ whose fiber over T' is the symmetric product $\mathrm{Sym}^{|\gamma|}(\mathcal{E}_{\gamma, T'})$. Therefore, since by assumption $[M^s(n_\gamma, d_\gamma)], [\mathcal{N}'] \in \mathcal{R}_C$, it follows that $\mathcal{E}_{\gamma, T'}$ is motivated by C and, by definition, the same holds for $\mathrm{Sym}^{|\gamma|}(\mathcal{E}_{\gamma, T'})$. So, we have

$$[\tilde{X}^+(\mathbf{A}, \mathbf{n})^\Delta / \mathfrak{S}_{\mathbf{n}}] \in \mathcal{R}_C\,.$$

Step 2. Now we deal with the diagonals. Consider the partition π_0 defined by (7.4), and let β be a subset of a brick of π_0. We define

$$\mathcal{E}_\beta^\Delta = \prod_{\substack{i\in\beta \\ M^s(n_\beta,d_\beta)\times\mathcal{N}'}} \mathcal{E}_i\,,$$

as the fiber product over $M^s(n_\beta, d_\beta)\times\mathcal{N}'$ of the bundles $\mathcal{E}_i \to M^s(n_\beta, d_\beta)\times\mathcal{N}'$ given in (7.6). So there is a fibration

$$\prod_{\substack{\gamma\in\pi\wedge\pi_1 \\ \gamma\subset\beta}} \mathrm{Gr}(a_\gamma, \mathrm{Ext}^1(T', S_\gamma))^{|\gamma|} \to \mathcal{E}_\beta^\Delta \to M^s(n_\beta, d_\beta)\times\mathcal{N}'\,.$$

We have the natural inclusion $\mathcal{E}_\beta^\Delta \hookrightarrow (\mathcal{E}_\beta)^{|\beta|}$ as the smallest diagonal. Clearly, there is a equivalence between partitions $\pi\le\pi_0$ and diagonals: for any $\pi\le\pi_0$, let

$$\mathcal{E}_\pi^\Delta = \prod_{\beta\in\pi}\mathcal{E}_\beta^\Delta \to \mathcal{N}'\,,$$

where we have the obvious inclusion $\mathcal{E}_\pi^\Delta \subset \tilde{X}^+(\mathbf{A},\mathbf{n})^\Delta$. Hence

$$\tilde{X}^+(\mathbf{A},\mathbf{n}) = \tilde{X}^+(\mathbf{A},\mathbf{n})^\Delta \setminus \left(\bigcup_{\pi\le\pi_0}\mathcal{E}_\pi^\Delta\right).$$

Since we know that $[\tilde{X}^+(\mathbf{A},\mathbf{n})^\Delta/\mathfrak{S}_\mathbf{n}]\in\mathcal{R}_C$, the proof is complete as long as we show that

$$\left(\bigcup_{\pi\le\pi_0}\mathcal{E}_\pi^\Delta\right)/\mathfrak{S}_\mathbf{n}$$

lies in $\mathcal{R}_C$. This is a stratified space, so we check that each stratum is motivated by C. So, fix $\pi\le\pi_0$ and consider $\mathfrak{S}_\mathbf{n}\cdot\pi = \{g\cdot\pi\,;\, g\in\mathfrak{S}_\mathbf{n}\}$ the orbit of π under $\mathfrak{S}_\mathbf{n}$. The corresponding stratum in the quotient is

$$\left(\bigcup_{\pi'\in\mathfrak{S}_\mathbf{n}\cdot\pi}\mathcal{E}_{\pi'}^\Delta\right)/\mathfrak{S}_\mathbf{n} = \mathcal{E}_\pi^\Delta/\mathrm{Stab}_{\mathfrak{S}_\mathbf{n}}(\pi)\,, \tag{7.7}$$

where

$$\mathrm{Stab}_{\mathfrak{S}_\mathbf{n}}(\pi) = \{g\in\mathfrak{S}_\mathbf{n}\,;\, g\cdot\pi=\pi\}.$$

So it is enough to see that (7.7) is motivated by C.

The action of $\mathfrak{S}_{\pi_1}$ on diagonals are not so simple as the extended space $\tilde{X}(\mathbf{A},\mathbf{n})^\Delta$. We introduce the difficulty with the following examples.

Example. Consider $\pi_0=[4]$ and $\pi_1=\{\{1,2\},\{3,4\}\}$. The diagonal associated to the partition

$$\pi = \{\{1,3\},\{2,4\}\}$$

has non trivial group of symmetry. In this case, the element $\sigma=(1\ 2)(3\ 4)\in\mathfrak{S}_{\pi_1}$ preserves π and hence leaves invariant the diagonal $\mathcal{E}_\pi^\Delta$. Notice σ permutes the bricks of π.

In the previous example, the element σ generates the subgroup which leaves the diagonal $\mathcal{E}_\pi^\Delta$ invariant. The quotient by this subgroup is

$$\mathrm{Sym}^2_{\mathcal{N}'}\mathcal{E}_\gamma^\Delta,\ \text{where } \gamma\in\pi \text{ and } \mathcal{N}'=\mathcal{N}(n',1,d',d_2).$$

Example. A more dramatic example appears in the following setting. Let $\pi_0=[8]$ and $\pi_1=\{\{1,2,3,4\},\{5,6,7,8\}\}$. Consider the diagonal described by the partition

$$\pi = \{\{1,2,5,6\},\{3,4,7,8\}\}.$$

The following element

$$\sigma = (1\ 3\ 2\ 4)(5\ 7\ 6\ 8) \in \mathfrak{S}_{\pi_1}$$

leaves π invariant but permutes its bricks. In contrast to the former example, σ has order four, so not only permutes bricks, but σ^2 moves inside bricks.

We need to study closely the subgroup which leaves invariant a given diagonal. Let $\pi \leq \pi_0$ be a partition of $[s]$. Consider the subgroup which leaves the diagonal $\mathcal{E}_\pi^\Delta$ invariant

$$\mathrm{Stab}_{\pi_1}(\pi) = \{g \in \mathfrak{S}_{\pi_1} \ : \ g \cdot \pi = \pi\}.$$

We have the subgroup which leaves the bricks of π invariant

$$\mathrm{Stab}^\circ_{\pi_1}(\pi) = \{g \in \mathfrak{S}_{\pi_1} \ : \ g \cdot \gamma = \gamma \text{ for all } \gamma \in \pi\}.$$

Lemma 7.17. *We have the isomorphism*

$$\mathrm{Stab}^\circ_{\pi_1}(\pi) \cong \mathfrak{S}_{\pi \wedge \pi_1}.$$

Proof. Any element of $\sigma \in \mathrm{Stab}^\circ_{\pi_1}(\pi)$ belongs to $\mathfrak{S}_\pi$, but also to $\mathfrak{S}_{\pi_1}$ by definition. Thus

$$\mathrm{Stab}^\circ_{\pi_1}(\pi) \subset \mathfrak{S}_\pi \cap \mathfrak{S}_{\pi_1} = \mathfrak{S}_{\pi \wedge \pi_1}.$$

On the other hand, there is a natural inclusion

$$\mathfrak{S}_{\pi \wedge \pi_1} \hookrightarrow \mathrm{Stab}^\circ_{\pi_1}(\pi)$$

since any element of $\mathfrak{S}_{\pi \wedge \pi_1}$ leaves invariant bricks. This gives the other inclusion. □

Furthermore, we have the following

Lemma 7.18. *The subgroup* $\mathrm{Stab}^\circ_{\pi_1}(\pi)$ *of* $\mathrm{Stab}_{\pi_1}(\pi)$ *is a normal subgroup.*

Proof. Let $\sigma \in \mathrm{Stab}^\circ_{\pi_1}(\pi)$ and $\tau \in \mathrm{Stab}_{\pi_1}(\pi)$. Let $\gamma \in \pi$ be any brick. Notice $\tau(\gamma) \in \pi$ is another brick. Since σ does not move bricks, then $\sigma \circ \tau(\gamma) = \tau(\gamma)$. Then

$$\tau^{-1} \circ \sigma \circ \tau(\gamma) = \tau^{-1} \circ \tau(\gamma) = \gamma$$

and then

$$\tau^{-1} \circ \sigma \circ \tau \in \mathrm{Stab}^\circ_{\pi_1}(\pi). \ \square$$

This allows us to consider the short exact sequence between groups

$$0 \to \mathfrak{S}_{\pi \wedge \pi_1} \to \mathrm{Stab}_{\pi_1}(\pi) \to Q \to 0.$$

The classes of the quotient group Q acts permuting bricks of π. More concretely, the equivalence relation defining Q

$$\sigma \sim \tau \iff \sigma^{-1}\tau \text{ does not move bricks}$$

translate into

$$\sigma \sim \tau \iff \sigma(\gamma) = \tau(\gamma), \text{ for all } \gamma \in \pi.$$

It is possible to give a precise presentation for Q.

Firstly, we use the fact that any element $g \in \mathfrak{S}_s$ decomposes uniquely into disjoint cycles $g = g_1 \cdots g_k$ up to order. On the other hand, if $g \in \mathrm{Stab}_{\pi_1}(\pi) \subset \mathfrak{S}_{\pi_1}$, then $g_i \subseteq \delta$ considering g_i as subset of $[b]$, for some $\delta \in \pi_1$. We have the following lemma

Lemma 7.19. *Let* $g \in \mathrm{Stab}_{\pi_1}(\pi)$ *be an element and a decomposition* $g = g_1 \cdots g_k$ *into disjoint cycles. Then* $|g_i \cap \gamma|$ *does not depend on* $\gamma \in \pi$.

Proof. It easy to see $g_i(|g_i \cap \gamma|) \subset \gamma'$ for some $\gamma \in \pi$ since $g(\beta) \in \pi$. From this fact, it follows that $|g_i \cap \gamma|$ does not depend on a given γ. □

From this simple lemma we deduce a nice expression for Q:

Corollary 7.20. *Let $\bar{\pi}$ be the partition of the set π defined by the equivalence relation*

$$\gamma_1 \sim \gamma_2 \iff |\gamma_1 \cap \delta| = |\gamma_2 \cap \delta| \text{ for all } \delta \in \pi_1.$$

Then we have the isomorphism

$$Q \cong \mathfrak{S}_{\bar{\pi}}.$$

The quotient (7.7) is then rewritten as

$$\mathcal{E}_\pi^\Delta / \mathrm{Stab}_{\mathfrak{S}_\mathbf{n}}(\pi) = (\mathcal{E}_\pi^\Delta / \mathfrak{S}_{\pi\wedge\pi_1}) / \mathfrak{S}_{\bar{\pi}}\,. \tag{7.8}$$

From the definition of $\mathcal{E}_\pi^\Delta$, we have

$$\mathcal{E}_\pi^\Delta / \mathfrak{S}_{\pi\wedge\pi_1} = \prod_{\beta\in\pi} \mathcal{E}_\beta^\Delta / \mathfrak{S}_{\pi\wedge\pi_1,\beta}\,, \tag{7.9}$$

where $\mathfrak{S}_{\pi\wedge\pi_1,\beta} = \prod_{A\in\pi\wedge\pi_1} \mathfrak{S}_{A\cap\beta}$. Therefore the quotient

$$\mathcal{E}_\beta^\Delta / \mathfrak{S}_{\pi\wedge\pi_1,\beta} \to M^s(n_\beta, d_\beta) \times \mathcal{N}' \tag{7.10}$$

is a fiber bundle with fiber

$$\prod_{\substack{\gamma\in\pi\wedge\pi_1\\ \gamma\subset\beta}} \mathrm{Sym}^{|\gamma|}\,\mathrm{Gr}(a_\gamma, \mathrm{Ext}^1(T', S_\gamma))\,.$$

Fix $\delta \in \bar{\pi}$. So δ is a subset of π and, from the definition of $\bar{\pi}$, it follows that for any $\beta \in \delta$, the space (7.10) is the same. Denote it by $\mathcal{E}_\delta$ and consider it, as before, as a bundle $\mathcal{E}_\delta$ over $\mathcal{N}'$. Thus the quotient of (7.9) by $\mathfrak{S}_{\bar{\pi}}$ is

$$\left(\prod_{\beta\in\pi} \mathcal{E}_\delta\right) / \mathfrak{S}_{\bar{\pi}} = \left(\prod_{\delta\in\bar{\pi}} (\mathcal{E}_\delta)^{|\delta|}\right) / \mathfrak{S}_{\bar{\pi}} = \prod_{\delta\in\bar{\pi}} \mathrm{Sym}^{|\delta|}\,\mathcal{E}_\delta$$

as a bundle over $\mathcal{N}'$. This is the quotient (7.8) and clearly its class in $K(\mathfrak{Mot})$ lies in $\mathcal{R}_C$.

This finishes the proof of the proposition. □

Directly from this proof we obtain the class of this family of strata.

Corollary 7.21. *The class of the stratum given by the constants*

$$\mathbf{A} = (\,a_1 \quad a_2 \quad \ldots \quad a_b\,), \text{ and } \mathbf{n} = (\,n_1 \quad n_2 \quad \ldots \quad n_b\,)$$

is given by

$$\Bigg[\prod_{\gamma\in\pi_1} \lambda^{|\gamma|}\,\{\mathrm{Gr}(a_\gamma, T'S_\gamma)\cdot[M^s(n_\gamma,d_\gamma)]\} -$$

$$- \sum_{\pi\in P/\sim} \sum_{\delta\in\bar{\pi}} \lambda^{|\delta|} \left\{ \prod_{\substack{\gamma\in\pi\wedge\pi_1\\ \gamma\subset\delta}} \Big(\lambda^{|\gamma|}(\mathrm{Gr}(a_\gamma, T'S_\gamma))\Big)\cdot[M^s(n_\gamma,d_\gamma)] \right\}\Bigg][\mathcal{N}']$$

where $P = \{\pi \text{ partition} : \pi \le \pi_0\}$, $\bar{\pi}$ is the partition of π defined by the equivalence relation

$$\beta_1 \sim \beta_2 \iff |\beta_1 \cap \delta| = |\beta_2 \cap \delta| \text{ for all } \delta \in \pi_1,$$

and $\sim$ is the equivalence relation on P defined by the induced action of $\mathfrak{S}_{\pi_1}$ on partitions.

A family of strata with two non-trivial steps: Part I In this section we want to study some strata $X^+(\mathbf{A},\mathbf{n}) \subset \mathcal{S}_{\sigma_c^+}$ for which the standard filtration has $r = 2$, i.e., it is of the form $0 \subset T_1 \subset T_2 \subset T_3 = T$. First we analyse the case where the type is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} n_1 & n_2 & \cdots & n_{b-1} & n_b \end{pmatrix} \tag{7.11}$$

for some n_i, $b \geq 3$, and non-trivial $\mathfrak{S}_{\mathbf{n}}$. Although we shall only need the case $b = 3$, $n_i = 1$ for Theorem Theorem 7.33, we will work out the general case.

Recall that the basis of this stratum is

$$\tilde{U}(\mathbf{n}) = \left(\prod_{i=1}^{b} M^s(n_i, d_i) \setminus \Delta \right), \tag{7.12}$$

where $\Delta = \{(E_1, \dots, E_b)\,;\, E_i \cong E_j, \text{ for some } i, j\}$ is the big diagonal.

Proposition 7.22. *Let σ_c be any critical value. Let $n \geq 1$, and $\mathbf{n}$ be given by (7.11). Suppose that $[M^s(n'', d'')]$ and $[\mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)]$ are in $\mathcal{R}_C$, for any $n', n'' < n$, $\gcd(n'', d'') = 1$. Assume also that $\gcd(n_i, d_i) = 1$, for every $i = 1, \dots, b$. Then $[X^{\pm}(\mathbf{A},\mathbf{n})] \in \mathcal{R}_C$.*

Proof. Again we only consider the case of $X^+(\mathbf{A},\mathbf{n})$. By Proposition 2.26, we have to construct a two-step iterated fibration, with basis $\tilde{U}(\mathbf{n}) \times \mathcal{N}'$ where $\mathcal{N}' = \mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)$, and then take the quotient by the symmetric group $\mathfrak{S}_{\mathbf{n}}$ which is a subgroup of the permutation group on the first $b-1$ factors of (7.12).

The first step is a bundle $X_1^+ \to X_0^+ := \tilde{U}(\mathbf{n}) \times \mathcal{N}'$ with fibers

$$\prod_{i=1}^{b-1} \mathbb{P}\operatorname{Ext}^1(T', S_i).$$

Recall that we write S_i for the triple $(E_i, 0, 0)$ with $E_i \in M^s(n_i, d_i)$.

The second step is a bundle $X_2^+ \to X_1^+$. If $\widetilde{T} \in X_1^+$ is such that

$$0 \to S_1 \oplus \cdots \oplus S_{b-1} \to \widetilde{T} \to T' \to 0,$$

then, from Proposition 2.26, the fiber of X_2^+ over $\widetilde{T}$ is

$$\mathbb{P}\operatorname{Ext}^1(\widetilde{T}, S_b) \setminus \bigcup_{i=1}^{b-1} \mathbb{P}\operatorname{Ext}^1(\widetilde{T}_i, S_b) \tag{7.13}$$

where, for each i, $\widetilde{T}_i$ is the triple fitting in the natural exact sequence

$$0 \to S_i \to \widetilde{T} \to \widetilde{T}_i \to 0$$

so that

$$0 \to \operatorname{Ext}^1(\widetilde{T}_i, S_b) \to \operatorname{Ext}^1(\widetilde{T}, S_b) \to \operatorname{Ext}^1(S_i, S_b) \to 0.$$

We extend the fibration to a basis larger than (7.12). Consider for each $1 \leq i \leq b-1$, the space

$$Y_i := (M^s(n_i, d_i) \times M^s(n_b, d_b) \setminus \Delta_i) \times \mathcal{N}',$$

$\Delta_i = \{(E_i, E_b)\,;\, E_i \cong E_b\} \subset M^s(n_i, d_i) \times M^s(n_b, d_b)$, so that the fiber product

$$\bar{X}_0^+ = \prod_{\substack{1 \leq i \leq b-1 \\ M^s(n_b, d_b) \times \mathcal{N}'}} Y_i$$

consists of bundles $(E_1,\dots,E_{b-1},E_b)$, where $E_i\not\cong E_b$, for $i\neq b$. The fibration $X_1^+\to X_0^+$ extends to a fibration $\bar X_1^+\to\bar X_0^+$. The dimension of $\mathrm{Ext}^1(\widetilde T,S_b)$ stays constant as we move over each Y_i, so the fibration $X_2^+\to X_1^+$ extends to a fibration $\bar X_2^+\to\bar X_1^+$ with fibers $\mathbb{P}\,\mathrm{Ext}^1(\widetilde T,S_b)$, also compactifying the fibers (7.13).

The action of $\mathfrak{S}_{\mathbf{n}}$ extends to $\bar X_1^+$. To work out the quotient, consider again the partition π_0 of $[b-1]$ given by $i\sim j\iff n_i=n_j$. Let $\mathcal{E}_i$ be the bundle

$$\mathbb{P}\,\mathrm{Ext}^1(T',S_i)\to\mathcal{E}_i\to Y_i$$

and consider it as a bundle over $B=M^s(n_b,d_b)\times\mathcal{N}'$. Then

$$\bar X_1^+=\prod_B\mathcal{E}_i=\prod_{\substack{\alpha\in\pi_0\\ B}}(\mathcal{E}_\alpha)^{|\alpha|}\,.$$

The quotient by $\mathfrak{S}_{\mathbf{n}}$ is

$$\bar X_1^+/\mathfrak{S}_{\mathbf{n}}=\prod_{\substack{\alpha\in\pi_0\\ B}}\mathrm{Sym}^{|\alpha|}\,\mathcal{E}_\alpha\,,\tag{7.14}$$

and $\bar X_2^+/\mathfrak{S}_{\mathbf{n}}\to\bar X_1^+/\mathfrak{S}_{\mathbf{n}}$ is a projective bundle with fibers $\mathbb{P}\,\mathrm{Ext}^1(\widetilde T,S_b)$ (this family is locally trivial in the Zariski topology since it is the projectivization of a vector bundle).

We can construct a family of triples parametrized by (7.14) using that $\gcd(n_i,d_i)=1$, for all i. This family is clearly Zariski locally trivial. The family of S_b over $M^s(n_b,d_b)$ is also Zariski locally trivial since n_b and d_b are coprime. So

$$[\bar X_2^+/\mathfrak{S}_{\mathbf{n}}]=[\bar X_1^+/\mathfrak{S}_{\mathbf{n}}]\cdot[\mathbb{P}^N]\in\mathcal{R}_C\,.$$

Now we will deal with the diagonals and the sub-fibrations. We have fibrations

$$\mathbb{P}\,\mathrm{Ext}^1(\widetilde T,S_b)\to\bar X_2^+\to\bar X_1^+\qquad\text{and}\qquad\bar X_1^+\to\prod_{\substack{1\le i\le b-1\\ M^s(n_b,d_b)\times\mathcal{N}'}}Y_i.$$

The space $\bar X_2^+$ is thus a stratified space, where the strata are given according to the various diagonals inside Δ in (7.12) and according to the sub-fibrations with fibers $\mathbb{P}\,\mathrm{Ext}^1(\widetilde T_I,S_b)$ (cf. (7.13)), where $I\subset[b-1]$, $S_I:=\bigoplus_{i\in I}S_i$ and

$$0\to S_I\to\widetilde T\to\widetilde T_I\to 0.$$

Note that from this we have an extension $0\to S_{I^c}\to\widetilde T_I\to T'\to 0$, with $I^c:=[b-1]\setminus I$.

If all strata induced in the quotient $\bar X_1^+/\mathfrak{S}_{\mathbf{n}}$ are in $\mathcal{R}_C$, then the main open set $X_1^+/\mathfrak{S}_{\mathbf{n}}$, which is the stratum $X^+(\mathbf{A},\mathbf{n})$ we are dealing with, lies also in $\mathcal{R}_C$. Let us prove that every stratum in $\bar X_1^+/\mathfrak{S}_{\mathbf{n}}$ is motivated by C, thus completing the proof.

The diagonals of (7.12) are labeled by partitions $\pi\le\pi_0$ and the sub-fibrations are labeled by sets $I\subset[b-1]$. Let π_I be the partition $\{I,I^c\}$. The stratum $S_{(\pi,I)}$ associated to the pair (π,I) is constructed as follows. For each $\beta\in\pi$, let

$$Y_\beta=(M^s(n_\beta,d_\beta)\times M^s(n_b,d_b)\setminus\Delta_\beta)\times\mathcal{N}'$$

where $n_\beta=n_i$ and $d_\beta=d_i$, for any $i\in\beta$. Then we have a fibration X_π^+ over the basis

$$\prod_{\substack{\beta\in\pi\\ M^s(n_b,d_b)\times\mathcal{N}'}}Y_\beta\tag{7.15}$$

whose fiber over $((E_\beta)_{\beta\in\pi},E_b,T')\in Y_\beta$ is

$$\prod_{\beta\in\pi}\mathbb{P}\,\mathrm{Ext}^1(T',S_\beta)^{|\beta|}$$

and then a second fibration over X_π^+ with fiber $\mathbb{P}\operatorname{Ext}^1(\widetilde{T}_I, S_b)$ over $\widetilde{T}_I \in X_\pi^+$.

The group $\mathfrak{S}_\mathbf{n}$ acts on the pairs (π, I), with orbit $\mathfrak{S}_\mathbf{n} \cdot (\pi, I)$ and the corresponding stratum in the quotient is

$$\left(\bigcup_{(\pi', I') \in \mathfrak{S}_\mathbf{n} \cdot (\pi, I)} S_{(\pi', I')} \right) / \mathfrak{S}_\mathbf{n} = S_{(\pi, I)} / \operatorname{Stab}_{\mathfrak{S}_\mathbf{n}}(\pi, I)$$

where $\operatorname{Stab}_{\mathfrak{S}_\mathbf{n}}(\pi, I)$ is the stabilizer of the pair (π, I). Hence we have to prove that

$$[S_{(\pi,I)} / \operatorname{Stab}_{\mathfrak{S}_\mathbf{n}}(\pi, I)] \in \mathcal{R}_C .$$

Now, there is an exact sequence

$$1 \to \mathfrak{S}_{\pi \wedge \pi_I} \to \operatorname{Stab}_{\mathfrak{S}_\mathbf{n}}(\pi, I) \to \mathfrak{S}_P \to 1,$$

where P is the partition of the set π given by

$$\gamma_1 \sim \gamma_2 \iff |\gamma_1 \cap I| = |\gamma_2 \cap I|, \ |\gamma_1 \cap I^c| = |\gamma_2 \cap I^c|,$$

for $\gamma_1, \gamma_2 \in \pi$, and $\mathfrak{S}_P = \prod_{p \in P} \mathfrak{S}_p$. Then

$$X_\pi^+ / \mathfrak{S}_{\pi \wedge \pi_I} \tag{7.16}$$

is a fibration over (7.15) with fiber

$$\prod_{\delta \in \pi \wedge \pi_I} \operatorname{Sym}^{|\delta|} \left(\mathbb{P}\operatorname{Ext}^1(T', S_\delta) \right) .$$

Let $\beta \in \pi$.

- If $\beta \subset I$, then we have a fibration $\operatorname{Sym}^{|\beta|}(\mathbb{P}\operatorname{Ext}^1(T', S_\beta)) \to \mathcal{E}_\beta \to X_\beta$.
- If $\beta \subset I^c$, then we have a fibration $\operatorname{Sym}^{|\beta|}(\mathbb{P}\operatorname{Ext}^1(T', S_\beta)) \to \mathcal{E}_\beta \to X_\beta$.
- If $\beta = \beta_1 \cup \beta_2$, where $\beta_1 = \beta \cap I$, $\beta_2 = \beta \cap I^c$ are both non-trivial, then the fibration is $\operatorname{Sym}^{|\beta_1|}(\mathbb{P}\operatorname{Ext}^1(T', S_\beta)) \times \operatorname{Sym}^{|\beta_2|}(\mathbb{P}\operatorname{Ext}^1(T', S_\beta)) \to \mathcal{E}_\beta \to X_\beta$.

Then (7.16) can be rewritten as

$$\prod_{\substack{\beta \in \pi \\ B}} \mathcal{E}_\beta = \prod_{\substack{p \in P \\ B}} (\mathcal{E}_p)^{|p|}$$

(recall $B = M^s(n_b, d_b) \times \mathcal{N}'$) and

$$(X_\pi^+ / \mathfrak{S}_{\pi \wedge \pi_I}) / \mathfrak{S}_P = \prod_{\substack{p \in P \\ B}} \operatorname{Sym}^{|p|} \mathcal{E}_p \, .$$

So

$$[X_\pi^+ / \operatorname{Stab}_{\mathfrak{S}_\mathbf{n}}] \in \mathcal{R}_C .$$

Finally, the second fibration has as fiber a projective space, and this is a product at the level of K-theory. □

■ **More on this family** This family of strata can be enlarged with some variations. Our first variation is the following:

$$\mathbf{n} = \begin{pmatrix} n_1 & n_2 & \cdots & n_b \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & a \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

where $a \geq 1$.

Clearly, the only difference lies in the fibre F of $X_2^+ \to X_1^+$ of the previous stratum. If we have to prove an analogous proposition for these family, we only have to care with this fibre F.

This is the plan:

1) Find an adequate expression for the fibre F to the action of $\mathfrak{S}_\mathbf{n}$.
2) Write some geometric results to deal with F.
3) Prove an analogous theorem for this family.

Let us begin.

Recall that the fibre is a subset of

$$\mathrm{Gr}(a, \mathrm{Ext}^1(\tilde{T}_1, S_b))$$

determined by the condition: the image of $\xi \in \mathcal{V}(a, \mathrm{Ext}^1(\tilde{T}_1, S_b))$ by the map

$$q : \mathrm{Ext}^1(\tilde{T}_1, S_b)^a \to \mathrm{Ext}^1(S_1 \oplus S_{b-1}, S_1)^a$$

belongs to

$$\prod_{i=1}^{b-1} \left(\mathrm{Ext}^1(S_i, S_1)^a \setminus \{0\}\right).$$

To deal with this space, we prove some auxiliar results on the Grassmanian variety.

Let V and W vector spaces and a linear projection $q : V \to W$. We define the space

$$\mathrm{Gr}^{(k)}(r, V) = \{H \in \mathrm{Gr}(r, V) \,:\, \dim q(H) = k\},$$

for any $0 < k \leq r$ integers numbers. Then,

Lemma 7.23. *There exists a natural projection*

$$\begin{array}{rcl} \mathrm{Gr}^{(k)}(r, V) & \longrightarrow & \mathrm{Gr}(k, W) \times \mathrm{Gr}(r-k, K) \\ H & \longmapsto & (q(H), H \cap K) \end{array}$$

where $K = \ker q$ which makes $\mathrm{Gr}^{(k)}(r, V)$ as a affine fibration.

Proof. For a pair $(\bar{H}, H_K) \in \mathrm{Gr}(k, W) \times \mathrm{Gr}(r-k, K)$ we define the space of sections

$$\gamma(q) = \{s : \bar{H} \to V/H_K \text{ linear } : \bar{q} \circ s : \bar{H} \hookrightarrow W \text{ is the natural inclusion}\}$$

where $\bar{q} : V/H_K \to W$ fits into the following commutative diagram

$$\begin{tikzcd} V \arrow[d] \arrow[dr, "q"] & \\ V/H_K \arrow[r, "\bar{q}"'] & W \end{tikzcd}$$

The space $\gamma(q)$ can be endowed with a affine structure with underlying vector space $\mathrm{Hom}(\bar{H}, K/H_K)$: this is straightforward from the fact the difference of two sections $s, s' \in \gamma(q)$ belongs to the vector space $\mathrm{Hom}(\bar{H}, K/H_K)$ because of $\bar{q} \circ (s - s') = 0$.

Let the natural fibration

$$\gamma(q) \hookrightarrow \Gamma(q) \to \mathrm{Gr}(k, W) \times \mathrm{Gr}(r-k, K)$$

There exists an isomorphism

$$\begin{array}{rcl} \Gamma(q) & \longrightarrow & \mathrm{Gr}^{(k)}(r, V) \\ s & \longmapsto & \mathrm{Im}(s) + H_K \end{array}$$

The surjectivity is clear. The inverse of this map is the inverse of the induced map (an isomorphism) $\bar{q} : V/H_K \to \bar{H}$ by q. This proves the injectivity and hence the lemma. $\square$

The space $\mathrm{Gr}(a, V)$ decomposes into disjoint pieces as follows:

$$\mathrm{Gr}(a, V) = \bigsqcup_{k=0}^{a} \mathrm{Gr}^{(k)}(a, V). \tag{7.17}$$

Now, let us consider $V = V_1 \times \cdots \times V_b$ a vector space which is a product of b of them. Let $r > 0$ be an integer number and let $\vec{k} = (k_1, \cdots, k_s)$ be a vector of integer numbers such that $0 \leq k_i < r$ for any $i = 1, \cdots, s$. We define

$$\mathrm{Gr}^{(\vec{k})}(r, V) = \{H \in \mathrm{Gr}(r, V) \,:\, \dim q_i(H) = k_i\},$$

and

$$\mathcal{V}^{(\vec{k})}(r, V) = \{f : \mathbb{C}^r \to V \,:\, \dim \ker(q_i \circ f) = r - k_i,\ f \text{ injective}\},$$

where $q_i : V \to V_i$ is the natural projection on factors. Clearly,

$$\mathcal{V}^{(\vec{k})}(r, V)/\,\mathrm{GL}(r) = \mathrm{Gr}^{(\vec{k})}(r, V).$$

There exists a projection

$$\begin{aligned} \mathcal{V}^{(\vec{k})}(r, V) &\longrightarrow \textstyle\prod_{i=1}^{s} \left(\mathrm{Gr}(r - k_i, \mathbb{C}^r) \times \mathrm{Gr}(k_i, V_i)\right) \\ f &\longmapsto (\ker(q_i \circ f), \mathrm{Im}(q_i \circ f))_{i=1}^{s} \end{aligned}$$

This map is not surjective: observe that $f : \mathbb{C}^r \to V$ is injective if and only if $\bigcap_{i=1}^{s} \ker(q_i \circ f) = \{0\}$.

To recover $f_i = q_i \circ f$ from its kernel and its image we need a map

$$\bar{f}_i : \mathbb{C}^r/\ker(q_i \circ f) \to \mathrm{Im}(q_i \circ f)$$

which lives in $\mathrm{GL}(k_i, \mathbb{C})$. Then we have

Lemma 7.24. *Consider V, r and $\vec{k}$ as above. Then $\mathcal{V}^{(\vec{k})}(r, V)$ is a principal $\prod_{i=1}^{s} \mathrm{GL}(k_i, \mathbb{C})$-bundle over*

$$\prod_{i=1}^{s} \left(\mathrm{Gr}(r - k_i, \mathbb{C}^r) \times \mathrm{Gr}(k_i, V_i)\right).$$

The basis of this principal bundle is the subset

$$\left\{(K_i, \bar{H}_i)_{i=1}^{b} \in \prod_{i=1}^{b} \left(\mathrm{Gr}(r - k_i, \mathbb{C}^r) \times \mathrm{Gr}(k_i, V_i)\right) \,:\, \bigcap_{i=1}^{s} K_i = \{0\}\right\}.$$

To complete the description, we introduce the following notation. For $j = 0, \cdots, n$, consider the following subspace

$$\left[\prod_{i=1}^{b} \mathrm{Gr}(k_i, V)\right]^{(j)} = \{(H_i)_{i=1}^{s} \in \prod_{i=1}^{s} \mathrm{Gr}(k_i, V) \,:\, \dim\left(\cap_{i=1}^{b} H_i\right) = j\}.$$

With this notation, the basis of the principal bundle of the Lemma 7.24 is

$$\prod_{i=1}^{s} \mathrm{Gr}(k_i, V_i) \times \left[\prod_{i=1}^{s} \mathrm{Gr}(r - k_i, \mathbb{C}^r)\right]^{(0)}.$$

This invites us to study such subspaces. For $0 < j \leq \min\{k_i : i = 1, \ldots, s\}$ we have the projection

$$\begin{aligned} \left[\textstyle\prod_{i=1}^{s} \mathrm{Gr}(k_i, V)\right]^{(j)} &\longrightarrow \mathrm{Gr}(j, V) \\ (H_1, \cdots, H_s) &\longmapsto \cap_i H_i \end{aligned}$$

and the fibre at $H \in \mathrm{Gr}(k,V)$ is isomorphic to

$$\left[\prod_{i=1}^{s} \mathrm{Gr}(k_i - j, V/H)\right]^{(0)}.$$

Then,

Lemma 7.25. *For $j > 0$ there exists the isomorphism*

$$\left[\prod_{i=1}^{s} \mathrm{Gr}(k_i, V)\right]^{(j)} \cong \left[\prod_{i=1}^{s} \mathrm{Gr}(k_i - j, V/U_{\mathrm{Gr}(j,V)})\right]^{(0)}. \tag{7.18}$$

where $U_{\mathrm{Gr}(j,V)}$ denotes the universal bundle over $\mathrm{Gr}(j,V)$.

We do not have a direct expression for $\left[\prod_{i=1}^{s} \mathrm{Gr}(k_i, V)\right]^{(0)}$, but

$$\begin{aligned}\left[\prod_{i=1}^{s} \mathrm{Gr}(k_i, V)\right]^{(0)} &= \prod_{i=1}^{s} \mathrm{Gr}(k_i, V) \setminus \left(\bigsqcup_{j=1}^{\max_i\{k_i\}} \left[\prod_{i=1}^{s} \mathrm{Gr}(k_i, V)\right]^{(j)} \right) \\ &= \prod_{i=1}^{s} \mathrm{Gr}(k_i, V) \setminus \left(\bigsqcup_{j=1}^{\max_i\{k_i\}} \left[\prod_{i=1}^{s} \mathrm{Gr}(k_i - j, V/U_{\mathrm{Gr}(j,V)})\right]^{(0)} \right).\end{aligned} \tag{7.19}$$

Now, we are ready to show the following proposition:

Proposition 7.26. *Let $b \geq 1$ $a > 0$ be integer numbers. Consider the stratum given by the constants*

$$\mathbf{n} = \begin{pmatrix} n_1 & n_2 & \cdots & n_b \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{pmatrix}.$$

Then, the class $[X(\mathbf{A}, \mathbf{n})] \in \hat{K}_0(\mathfrak{Var})$ belongs to the subring $\mathcal{R}_C$.

Proof. This follows the same arguments as in Proposition 7.22. The difference lies in the computation of the fibre F of the affinely stratified bundle $X_2^+ \to X_1^+$: we have to assure that we can rewrite F so that the action of $\mathfrak{S}_{\mathbf{n}}$ is the permutation of factors. Hence, the quotient can be written in terms of Sym^k of spaces.

By (7.17), the space

$$\mathrm{Gr}(a, \mathrm{Ext}^1(\tilde{T}_1, S_b))$$

decomposes into $a + 1$ pieces $\mathrm{Gr}^{(k)}(a, \mathrm{Ext}^1(\tilde{T}_1, S_b))$ for $k = 0, \dots, a$, taking into account the projection map $q : \mathrm{Ext}^1(\tilde{T}_1, S_b^a) \to \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b^a)$. Notice that any extension living in $\mathrm{Gr}^{(0)}(\mathrm{Ext}^1(\tilde{T}_1, S_b))$ does not belongs to the fibre F, so we forget this subspace. This decomposition chops the space X_2^+ into several pieces.

Fix $k = 1, \dots, a$. By Lemma 7.23, the subspace $\mathrm{Gr}^{(k)}(a, \mathrm{Ext}^1(\tilde{T}_1, S_b))$ projects onto the products

$$\mathrm{Gr}(k, \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b)) \times \mathrm{Gr}(a - k, \mathrm{Ext}^1(T', S_b))$$

where we identify $\ker q \cong \mathrm{Ext}^1(T', S_b)$. The action of $\mathfrak{S}_{\mathbf{n}}$ acts non-trivially on $\mathrm{Gr}(k, \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b))$. We focus on this grassmanian: we chop this space in several pieces to turn the action of $\mathfrak{S}_{\mathbf{n}}$ in a permutation or identification of pieces. To do this, we consider the $\mathrm{GL}(k,\mathbb{C})$-bundle $\mathcal{V}^{(\vec{k})}\left(k, \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b)\right)$. Notice that the $\mathrm{GL}(k,\mathbb{C})$-action commutes with the $\mathfrak{S}_{\mathbf{n}}$-action, so the action on the fibre is trivial and

$$[\mathrm{Gr}^{(\vec{k})}(k, \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b)/\mathfrak{S}_{\mathbf{n}}] = \frac{[\mathcal{V}^{(\vec{k})}(k, \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b)/\mathfrak{S}_{\mathbf{n}}]}{[\mathrm{GL}(k,\mathbb{C})]}.$$

Let $q_i : \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b) \to \mathrm{Ext}^1(S_i, S_b)$ be the natural projection for $i = 1, \dots, b-1$. For any vector $\vec{k} = (k_1, \dots, k_{b-1})$, Lemma 7.24 claims that the space

$$\mathcal{V}^{(\vec{k})}(k, \mathrm{Ext}^1(S_1 \oplus \cdots \oplus S_{b-1}, S_b))$$

projects onto

$$\prod_{i=1}^{b-1} \mathrm{Gr}(k_i, \mathrm{Ext}^1(S_i, S_b)) \times \left[\prod_{i=1}^{b-1} \mathrm{Gr}(r - k_i, \mathbb{C}^r) \right]^{(0)},$$

making it a principal $\prod_{i=1}^{b-1} \mathrm{GL}(k_i, \mathbb{C})$-bundle. The action of $\mathfrak{S}_{\mathbf{n}}$ on this space is clear. Now, let us consider the factor

$$\left[\prod_{i=1}^{b-1} \mathrm{Gr}(r - k_i, \mathbb{C}^r) \right]^{(0)}.$$

We apply here the last equality of (7.19). There are more pieces but the rank of the grassmanians are lower than previous the one. We repeat this process on each piece like, and an integer $j_r > 0$ is added each time giving a sequence of integers $j_1, j_2, \dots, j_r$ with $j_i > 0$ for all $i = 1, \dots, r$. Such a sequence determines a piece

$$\left[\prod_{i=1}^{b-1} \mathrm{Gr}(r - k_i - j_1 - j_2 - \ldots - j_r, \mathbb{C}^r) \right]^{(0)}.$$

This process finishes when $r - k_i - j_1 - j_2 - \ldots - j_r = 0$ for each $i = 1, \dots, b-1$, and then

$$\left[\prod_{i=1}^{b-1} \mathrm{Gr}(r - k_i - j_1 - j_2 - \ldots - j_r, \mathbb{C}^r) \right]^{(0)} = \prod_{i=1}^{b-1} \mathrm{Gr}(r - k_i - j_1 - j_2 - \ldots - j_r, \mathbb{C}^r).$$

Because $j_i > 0$ for all $i = 1, \dots, r$, this is always achieved.

Once this process is finishes, we get a collection of pieces where the action of $\mathfrak{S}_{\mathbf{n}}$ is either identification of pieces or permutation of factors. Therefore, this splitting finishes the proof. □

■ A family of strata with two non-trivial steps: Part II

Now we move to the case

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \mathbf{n} = (\, n_1 \quad n_2 \quad \cdots \quad n_{b-1} \quad n_b \,), \tag{7.20}$$

for some n_i, $b \geq 3$, and non-trivial $\mathfrak{S}_{\mathbf{n}}$. This case is slightly easier than the previous one. For Theorem 7.33, we only need $b = 3$ and $n_i = 1$, but again we do the general case. The basis of the stratum is also given by (7.12), where Δ is the big diagonal.

Proposition 7.27. *Let σ_c be any critical value. Let $n \geq 1$, and $\mathbf{n}$ be given by (7.20). Suppose that $[M^s(n'', d'')]$ and $[\mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)]$ are in $\mathcal{R}_C$, for any $n', n'' < n$, $\gcd(n'', d'') = 1$. Assume also that $\gcd(n_i, d_i) = 1$, for every $i = 1, \dots, b$. Then $[X^{\pm}(\mathbf{A}, \mathbf{n})] \in \mathcal{R}_C$.*

Proof. We only work out $X^+(\mathbf{A}, \mathbf{n})$. By Proposition 2.26, we have to construct a two-step iterated fibration, with base $\tilde{U}(\mathbf{n}) \times \mathcal{N}'$ where $\mathcal{N}' = \mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)$, and then take the quotient by the symmetric group $\mathfrak{S}_{\mathbf{n}}$ which is a subgroup of the permutation group on the first $b-1$ factors of (7.12).

The first step is a bundle $X_1^+ \to \tilde{U}(\mathbf{n}) \times \mathcal{N}'$ with fibers

$$\mathbb{P}\,\mathrm{Ext}^1(T', S_b),$$

and the second step is a bundle $X_2^+ \to X_1^+$ whose fibers are the spaces

$$\prod_{i=1}^{b-1} \mathbb{P}\,\mathrm{Ext}^1(\widetilde{T}, S_i) \setminus \prod_{i=1}^{b-1} \mathbb{P}\,\mathrm{Ext}^1(T', S_i),$$

where $\widetilde{T}$ is the triple corresponding to the point in X_1^+ determined by the extension

$$0 \to S_b \to \widetilde{T} \to T' \to 0 \,.$$

We compactify the fibration to a fibration with fibers

$$\prod_{i=1}^{b-1} \mathbb{P}\,\mathrm{Ext}^1(\widetilde{T}, S_i).$$

The quotient by $\mathfrak{S}_{\mathbf{n}}$ is done in exactly the same fashion as that carried out in the second point of this Section, changing the role of $\mathcal{N}'$ to that of X_1^+.

The other stratum is worked out in the same fashion, as a fibration over X_1^+ with fiber

$$\prod_{i=1}^{b-1} \mathbb{P}\,\mathrm{Ext}^1(T', S_i).$$

So finally $[X_2^+/\mathfrak{S}_{\mathbf{n}}] = [X^+(\mathbf{A}, \mathbf{n})] \in \mathcal{R}_C$. □

5 Some cases with non-coprime rank and degree

So far, we have analysed cases where $\gcd(n_i, d_i) = 1$, for all $i = 1, \dots, b$. If $\gcd(n, d) = 1$, then there exists a universal bundle $\mathcal{E} \to M^s(n, d) \times C$, and hence the projective bundle $\mathcal{U}_m \to M^s(n, d)$ is locally trivial in the Zariski topology. In the case where $\gcd(n, d) > 1$ such a universal bundle does not exist, and we have to circumvent the situation in another way.

Fix the values n, and d_1 and d_2 so that $\gcd(n, d) > 1$. For the critical value$\sigma_c = \sigma_m$, the Lemma 2.25 assures that there exists only a stratum $X^+(\mathbf{A}, \mathbf{n}) \subset \mathcal{S}^+_{\sigma_m} = \mathcal{N}_{\sigma_m^+}(n, 1, d_1, d_2)$ such that it is a projective fibration on $M^s(n, d_1) \times \mathrm{Jac}\, C$. We denote this stratum by $\mathcal{U}_m(n, 1, d_1, d_2)$. The union of the rest of strata will be denoted by

$$\mathcal{D}_m(n, 1, d_1, d_2) = \mathcal{N}^s_{\sigma_m}(n, 1, d_1, d_2) \setminus \mathcal{U}_m(n, 1, d_1, d_2). \tag{7.21}$$

Proposition 7.28. *Let σ_c be any critical value and let $\mathbf{n}$ be a type with $r = 1$, $\gcd(n_1, d_1) > 1$, $a_1 = 1$, and $\gcd(n_i, d_i) = 1$, for $i = 2, \dots, b$. Assume that $\mathfrak{S}_{\mathbf{n}} = \{1\}$. Define $\mathcal{D}_m(\tilde{n}, 1, \tilde{d}, d_o)$ as in (7.21). Suppose that $[M^s(n'', d'')]$, $[\mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)]$ and $[\mathcal{D}_m(\tilde{n}, 1, \tilde{d}, d_o)]$ are in $\mathcal{R}_C$, for any $n', n'', \tilde{n} < n$, $\gcd(n'', d'') = 1$. Then $[X^{\pm}(\mathbf{A}, \mathbf{n})] \in \mathcal{R}_C$.*

Proof. We deal with the case $X^+(\mathbf{A}, \mathbf{n})$, the other one being similar. The space $X^+(\mathbf{A}, \mathbf{n})$ is a fibration over

$$\mathcal{M}(\mathbf{n}) = M^s(n_1, d_1) \times \left(\prod_{i=2}^{b} M^s(n_i, d_i) \setminus \Delta \right) \times \mathcal{N}' \,,$$

Δ denoting a suitable diagonal, with fiber

$$\mathbb{P}\,\mathrm{Ext}^1(T', S_1) \times \prod_{i=2}^{b} \mathrm{Gr}(a_i, \mathrm{Ext}^1(T', S_i)) \,.$$

As $\gcd(n_i, d_i) = 1$, for $i \geq 2$, we know that there are universal bundles over $M^s(n_i, d_i)$. The same happens for $\mathcal{N}' = \mathcal{N}_{\sigma_c}(n', 1, d', d_o)$. So the bundle over $\mathcal{M}(\mathbf{n})$ with fiber $\prod_{i=2}^{b} \mathrm{Gr}(a_i, \mathrm{Ext}^1(T', S_i))$ is Zariski locally trivial, and hence the motives are multiplicative. So we shall assume from now on that $b = 1$.

For $T' \in \mathcal{N}'$, we consider the subset $X^+_{T'} \subset X^+(\mathbf{A}, \mathbf{n})$ corresponding to fixing T'. Clearly $[X^+(\mathbf{A}, \mathbf{n})] = [X^+_{T'}]\,[\mathcal{N}']$, so we have to see that $X^+_{T'}$ is motivated by C.

The map $X^+_{T'} \to M^s(n_1, d_1)$ is a fibration with fiber $\mathbb{P}\operatorname{Ext}^1(T', S_1)$. This projective fibration defines a Brauer class

$$\mathrm{cl}(X^+_{T'}) \in \mathrm{Br}(M^s(n_1, d_1)).$$

Now consider the fibration $\mathcal{U}_m = \mathcal{U}_m(n_1, 1, d_1, L_e) \to M^s(n_1, d_1)$, for some line bundle L_e with $\deg L_e = e \ll 0$. It has fiber $\mathbb{P}\operatorname{Hom}(L_e, L_1)$. The same argument as in the proof of Proposition 3.2 in [BLMn12] shows that the Brauer class

$$\mathrm{cl}(\mathcal{U}_m) \in \mathrm{Br}(M^s(n_1, d_1))$$

satisfies

$$\mathrm{cl}(\mathcal{U}_m) = \pm\, \mathrm{cl}(X^+_{T'}). \tag{7.22}$$

Now consider the pull-back diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & X^+_{T'} \arrow[d, "g"] \\ \mathcal{U}_m \arrow[r, "f"'] & M^s(n_1, d_1)\,, \end{tikzcd}$$

where all maps are projective fibrations. Consider the fibration $X^+_{T'} \to M^s(n_1, d_1)$. It has Brauer class $c_1 = \mathrm{cl}(X^+_{T'}) \in \mathrm{Br}(M^s(n_1, d_1))$. Then the fibration $\mathcal{A} \to \mathcal{U}_m$ is the pull-back under $f : \mathcal{U}_m \to M^s(n_1, d_1)$, so it has Brauer class $f^*c_1 \in \mathrm{Br}(\mathcal{U}_m)$. Now, by [Gab81, p. 193], there is an exact sequence

$$\mathbb{Z} \cdot \mathrm{cl}(\mathcal{U}_m) \longrightarrow \mathrm{Br}(M^s(n_1, d_1)) \xrightarrow{f^*} \mathrm{Br}(\mathcal{U}_m) \longrightarrow 0.$$

From this, and using (7.22), it follows that $f^*c_1 = 0$. This implies that $\mathcal{A} \to \mathcal{U}_m$ is Zariski locally trivial. Similarly, using the pull-back under g, the Brauer class of $\mathcal{A} \to X^+_{T'}$ is also trivial, so the fibration is Zariski locally trivial as well.

The above implies that the motives satisfy $[\mathcal{A}] = [\mathcal{U}_m]\,[\mathbb{P}^a] = [X^+_{T'}]\,[\mathbb{P}^b]$, for some $a, b \geq 0$. Hence

$$[X^+_{T'}] = [\mathcal{U}_m]\,[\mathbb{P}^a]\,[\mathbb{P}^b]^{-1}\,.$$

By our assumptions on $\mathcal{D}_m$ and since $n_1 < n$, it follows that $[\mathcal{U}_m] \in \mathcal{R}_C$. Hence $[X^+_{T'}] \in \mathcal{R}_C$, as required. □

Consider now the case

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathbf{n} = (\, n_1 \quad n_2 \quad \cdots \quad n_b \,)\,. \tag{7.23}$$

Proposition 7.29. *Let σ_c be any critical value and let $\mathbf{n}$ be given by (7.23), such that $\gcd(n_1, d_1) > 1$ and $\gcd(n_i, d_i) = 1$, for $i = 2, \ldots, b$. Assume that $\mathfrak{S}_{\mathbf{n}} = \{1\}$. Suppose that $[M^s(n'', d'')]$ and $[\mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)]$ are in $\mathcal{R}_C$, for any $n', n'' < n$, $\gcd(n'', d'') = 1$. Then $[X^{\pm}(\mathbf{A}, \mathbf{n})] \in \mathcal{R}_C$.*

Proof. Since $r = 2$, we have two steps. The first step X^+_1 sits as the total space of a fibration whose basis is

$$\mathcal{M}(\mathbf{n}) = M^s(n_1, d_1) \times \left(\prod_{i=2}^{b} M^s(n_i, d_i) \setminus \Delta \right) \times \mathcal{N}'\,,$$

where $\mathcal{N}' = \mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)$, and whose fiber is

$$\mathbb{P}\operatorname{Ext}^1(T', S_1).$$

Then X^+_1 is a space in the situation covered by Proposition 7.28, hence it lies in $\mathcal{R}_C$. The second step is a bundle $X^+_2 \to X^+_1$ whose fiber is

$$\prod_{i=2}^{b} \mathbb{P}\operatorname{Ext}^1(\widetilde{T}, S_i) \setminus \prod_{i=2}^{b} \mathbb{P}\operatorname{Ext}^1(T', S_i), \tag{7.24}$$

where $\widetilde{T}$ is given by the extension $0 \to S_1 \to \widetilde{T} \to T' \to 0$. There is a universal bundle parametrizing triples over $\mathcal{N}'$, another one parametrizing triples $\widetilde{T}$ (because the triples in X_1^+ are all σ_c^+-stable), and another universal bundle over $M^s(n_i, d_i)$, for $i = 2, \dots, b$. So the bundle over $X_2^+ \to X_1^+$ with fiber (7.24) is Zariski locally trivial, and hence the motives are multiplicative. So $[X^+(\mathbf{A}, \mathbf{n})] = [X_2^+] \in \mathcal{R}_C$.

The case of $[X^-(\mathbf{A}, \mathbf{n})]$ is similar. $\square$

Finally, we have to look also to the case

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} n_1 & n_2 & \cdots & n_b \end{pmatrix}. \tag{7.25}$$

Proposition 7.30. *Let σ_c be any critical value and let $\mathbf{n}$ be given by (7.25), such $\gcd(n_1, d_1) > 1$ and $\gcd(n_i, d_i) = 1$, for $i = 2, \dots, b$. Assume that $\mathfrak{S}_{\mathbf{n}} = \{1\}$. Suppose that $[M^s(n'', d'')]$ and $[\mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)]$ are in $\mathcal{R}_C$, for any $n', n'' < n$, $\gcd(n'', d'') = 1$. Then $[X^\pm(\mathbf{A}, \mathbf{n})] \in \mathcal{R}_C$.*

Proof. Since $r = 2$, we have two steps. The first step X_1^+ sits as the total space of a fibration whose basis is

$$\mathcal{M}(\mathbf{n}) = M^s(n_1, d_1) \times \left(\prod_{i=2}^{b} M^s(n_i, d_i) \setminus \Delta \right) \times \mathcal{N}',$$

where Δ is the suitable diagonal, $\mathcal{N}' = \mathcal{N}^s_{\sigma_c}(n', 1, d', d_o)$, and whose fiber is

$$\prod_{i=2}^{b} \mathbb{P}\operatorname{Ext}^1(T', S_i).$$

Then X_1^+ is a space in the situation covered by Proposition 7.28, hence it lies in $\mathcal{R}_C$.

The second step is a bundle $X_2^+ \to X_1^+$. If $\widetilde{T} \in X_1^+$ is given by

$$0 \to S_2 \oplus \cdots \oplus S_b \to \widetilde{T} \to T' \to 0,$$

then the fiber of $X_2^+ \to X_1^+$ over $\widetilde{T}$ is

$$\mathbb{P}\operatorname{Ext}^1(\widetilde{T}, S_b) \setminus \bigcup_{i=1}^{b-1} \mathbb{P}\operatorname{Ext}^1(\widetilde{T}_i, S_b) \tag{7.26}$$

where, for each i, $\widetilde{T}_i$ is the triple fitting in the natural exact sequence

$$0 \to S_i \to \widetilde{T} \to \widetilde{T}_i \to 0$$

so that

$$0 \to \operatorname{Ext}^1(\widetilde{T}_i, S_b) \to \operatorname{Ext}^1(\widetilde{T}, S_b) \to \operatorname{Ext}^1(S_i, S_b) \to 0.$$

Now, concerning the fibration $X_2^+ \to X_1^+$, we are in a situation similar to that of Proposition 7.28, the only difference being that instead of having a projective fibration, we have a projective fibration minus some sub-fibrations (which are also projetive). So, we conclude again that $X_2^+ \to X_1^+$ is Zariski locally trivial, and since X_1^+ is motivated by C, we deduce using (7.26) that $[X^+(\mathbf{A}, \mathbf{n})] = [X_2^+] \in \mathcal{R}_C$.

The case of $[X^-(\mathbf{A}, \mathbf{n})]$ is similar. $\square$

6 Proof of the main results

Here, we prove the main results of this chapter. Firstly, we deal with the coprime case.

Proposition 7.31. *Let σ_c be any critical value. Suppose $n \leq 4$, and let $\mathbf{n}$ be any type except $\mathbf{n} = \mathbf{n}_0 = ((n),(1))$, $\sigma_c = \sigma_m$. Suppose that $[\mathcal{N}_\sigma(n',1,d',d_o)]$ are in $\mathcal{R}_C$, for any $n' < n$. Then the stratum $[X^{\pm}(\mathbf{A},\mathbf{n})] \in K(\mathfrak{Mot})$ lies in $\mathcal{R}_C$.*

Proof. Assume first that $\gcd(n_i, d_i) = 1$, for all $i = 1, \ldots, b$. If the group $\mathfrak{S}_{\mathbf{n}}$ is trivial, then Proposition 7.15 gives the result. Now suppose that the group $\mathfrak{S}_{\mathbf{n}}$ is non-trivial. If $r = 1$, then the general result we prove in Proposition 7.16 implies that $[X^{\pm}(\mathbf{A},\mathbf{n})] \in \mathcal{R}_C$. As $n \leq 4$, the remaining cases are just

- $r = 2$, $b = 3$, $n' = 1$ and $(n_i) = (1,1,1)$, $\mathbf{a}_1 = (0,0,1)$, $\mathbf{a}_2 = (1,1,0)$;
- $r = 2$, $b = 3$, $n' = 1$ and $(n_i) = (1,1,1)$, $\mathbf{a}_1 = (1,1,0)$, $\mathbf{a}_2 = (0,0,1)$.

These situations are included in Proposition 7.22 and Proposition 7.27.

Assume now that there exists some pair (n_i, d_i) with n_i and d_i not coprime. Since $n \leq 4$, this situation can only occur in one of the following cases (where we always have $\mathfrak{S}_{\mathbf{n}}$ trivial):

- $r = 1$, $b = 2$, $n' = 1$ and $(n_i) = (2,1)$, $\mathbf{a}_1 = (1,1)$;
- $r = 1$, $b = 1$, $n' = 2$ and $(n_1) = (2)$, $\mathbf{a}_1 = (1)$;
- $r = 1$, $b = 1$, $n' = 1$ and $(n_1) = (3)$, $\mathbf{a}_1 = (1)$;
- $r = 2$, $b = 2$, $n' = 1$ and $(n_i) = (2,1)$, $\mathbf{a}_1 = (0,1)$, $\mathbf{a}_2 = (1,0)$;
- $r = 2$, $b = 2$, $n' = 1$ and $(n_i) = (2,1)$, $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (0,1)$.

The first three possibilities are covered by Proposition 7.28, the fourth by Proposition 7.29 and the last by Proposition 7.30. $\square$

Taking the isomorphism
$$\mathcal{N}_\sigma(n,1,d,0) \cong \mathfrak{M}_\tau(n,d) \times \mathrm{Jac}$$
into account, where $\mathfrak{M}_\tau(n,d)$ is the moduli space of polystable triples, we have

Theorem 7.32. *Let $\tau \in I$ be non-critical and $n \leq 4$. Suppose that C is a very general curve. Then the moduli spaces $\mathfrak{M}_\tau(n,d)$ satisfy the Hodge Conjecture.*

Theorem 7.33. *Let $n \leq 4$. For any σ, the moduli spaces $\mathcal{N}^s_\sigma(n,1,d,d_o)$ lie in $\mathcal{R}_C$. The same holds for $M^s(n,d)$ whenever $\gcd(n,d) = 1$.*

Proof. By definition, $[\mathrm{Sym}^k C] \in \mathcal{R}_C$, for any $k \geq 1$. Also, by (1.13) and (1.14), we have $[\mathrm{Jac}\, C] \in \mathcal{R}_C$. So the result is true for $n = 1$.

Now we proceed inductively. Let $n' < n$ and assume that the motives of $M^s(n',d')$ and $\mathcal{N}_\sigma(n',1,d',d_o)$ lie in $\mathcal{R}_C$. Then Proposition 7.31 says that $X^{\pm}(\mathbf{A},\mathbf{n})$ is also motivated by C (except for the case $\sigma_c = \sigma_m$, $\mathbf{n} = \mathbf{n}_0$). So if $\sigma_c > \sigma_m$,
$$[\mathcal{S}_{\sigma_c^{\pm}}] = \Big[\bigsqcup_{\mathbf{n}} X^{\pm}(\mathbf{A},\mathbf{n})\Big] \in \mathcal{R}_C.$$

In particular, since $\mathcal{S}_{\sigma_M^-} = \mathcal{N}_{\sigma_M^-}$, we see that $\mathcal{N}_{\sigma_M^-}$ is motivated by C. From Lemma 2.13, we conclude that $[\mathcal{N}^s_\sigma(n,1,d,d_o)] \in \mathcal{R}_C$, for any $\sigma > \sigma_m$.

On the other hand, for $\sigma_c = \sigma_m$ we have
$$[\mathcal{D}_m] = \Big[\bigsqcup_{\mathbf{n} \neq \mathbf{n}_0} X^{+}(\mathbf{A},\mathbf{n})\Big] \in \mathcal{R}_C.$$

Therefore also
$$\mathcal{U}_m = \mathcal{N}^s_{\sigma_m^+}(n,1,d,d_o) \setminus \mathcal{D}_m$$
is in $\mathcal{R}_C$.

Finally, if $\gcd(n,d)=1$, using the projection in Proposition 2.5 and considering, $d/n-d_o>2g-1$, we deduce that $M^s(n,d)$ also lies in $\mathcal{R}_C$. □

Note that Theorem 7.33 also applies to critical values $\sigma=\sigma_c$. Also, it is worth noticing that the result that the moduli spaces $M^s(n,d)$, for any n,d coprime, are motivated by C, has already been proved by Del Baño in Theorems 4.5 and 4.11 of [dB01].

Corollary 7.34. *For a generic curve C, for generic σ, and for $n\leq 4$, $\mathcal{N}_\sigma(n,1,d,d_o)$ satisfies the Hodge conjecture.*

Proof. Recall that for generic σ, $\mathcal{N}^s_\sigma(n,1,d,d_o)=\mathcal{N}_\sigma(n,1,d,d_o)$ is smooth and projective. Thus the result follows by applying the map Θ defined in (7.1), and noting that $\mathcal{R}_C\subset\ker\Theta$ by Proposition 7.14. □

The proof of the Hodge conjecture for $M(n,d)$, where n,d are coprime and C is generic, is given in [dB01, Corollary 5.9].

A comment about the restriction $n\leq 4$ is in order. The main obstacle to give the result for arbitrary rank is the geometrical analysis of the flip loci.

It is clear that $\tilde{X}^{\pm}(\mathbf{A},\mathbf{n})$ is in $\mathcal{R}_C$ by induction on the rank. However, we have to quotient by the finite group $\mathfrak{S}_{\mathbf{n}}$. For a *smooth projective* variety X, the motive $h(X/\mathfrak{S}_{\mathbf{n}})$ is — see (1.12) — a sub-motive of $h(X)$, and with this it would follow that if $[X]\in\mathcal{R}_C$ then $[X/\mathfrak{S}_{\mathbf{n}}]\in\mathcal{R}_C$. However, the same statement does not hold for quasi-projective varieties (our spaces $\tilde{X}^{\pm}(\mathbf{A},\mathbf{n})$ are smooth but quasi-projective). That is why we have to carry a finer analysis of the flip loci.

An alternative route would be to work with stacks in the spirit of [GPHS11], to prove that the motive of $\mathcal{N}_\sigma$ is in $\mathcal{R}_C$. This misses the geometrical description, but it might be applied for arbitrary rank. Indeed, very recently this has been undertaken in [Moz13].

Chapter 8

Toric Variety and Equivariant Bundles

1 Toric Variety

We denote by $\mathcal{T}^r$ the non-compact algebraic torus $(\mathbb{C}^\times)^r$. This is an algebraic group with the usual product in complex numbers. We drop the superscript and we write $\mathcal{T}$ whenever the context determines the dimension of the torus.

Definition 8.1. *A* toric variety *of dimension r is an algebraic variety which has an embedding of $\mathcal{T}^r$ as a Zariski open subset where*

- *the complementary of the torus is a crossing normal divisor;*
- *and the action of $\mathcal{T}^r$ extends to the whole variety.*

There is a lot of literature about toric varieties because of the increasing popularity of this topic. The most recent book (and most complete) is the Cox et. al. [CLS11] with a modern approach. A more classical point of view are the beatiful introduction suggested in Milnor's book [Ful93], and the synthetic and impressive Oda's book [Oda12].

The most part of this chapter, we do not need the full characterization of the toric varieties r-dimensional in term of fans in $\mathbb{R}^r$ explained below. The important issue is the stratification induced by the action of $\mathcal{T}^r$:

- The open subset $U \subset X$ where the $\mathcal{T}^r$-action is free is isomorphic to $\mathcal{T}^r$;
- the action isn't free on the complementary subset $X \setminus U$, but this action stratifies this subset.

We shall see that such stratification induced by the action is the same as the stratification induced by the cones of the associated fan of the toric variety X. Moreover, recall that a complete toric variety means that the union of the cones of the related fan must be $\mathbb{R}^n$.

We deal with equivariant bundles. An *equivariant bundle* is a bundle E over X toric variety is a bundle $\pi : E \to X$ where the action of T^r lifts to E and π becomes an equivariant map.

2 What is a connection? The Atiyah Sequence

Definition 8.2. *Let E be a bundle over an algebraic variety X. A* connection *on E is a linear map*

$$\nabla : \Gamma(X, E) \to \Omega^1(X) \otimes \Gamma(X, E),$$

satisfying the Leibniz rule

$$\nabla f \cdot s = f \nabla s + s \cdot \mathrm{d}f,$$

where s is a section of E and $f : X \to \mathbb{C}$ is a function.

We define the extension $\nabla^k : \Omega^k(X) \otimes \Gamma(X, E) \to \Omega^{k+1}(X) \otimes \Gamma(X, E)$ by the Leibniz rule

$$\nabla (\omega \otimes s) = \omega \wedge \nabla s + (-1)^k \mathrm{d}\omega \cdot s.$$

The composition of two consecutive ∇ gives the curvature of a connection:

Definition 8.3. *Let (E, ∇) be a bundle over X with a connection ∇. We define the* curvature Θ *to be*

$$\Theta = \nabla^1 \circ \nabla : \Gamma(X, E) \to \Omega^2(X) \otimes \Gamma(X, E).$$

This is a tensor, that is, $\Theta(f \cdot s) = f \cdot \Theta(s)$ and this is due to $\mathrm{d} \circ \mathrm{d}f = 0$ for any $f : X \to \mathbb{C}$ function.

Definition 8.4. *A connection ∇ on a bundle E over X is* flat *or* integrable *if its curvature vanishes, that is, $\Theta = 0$.*

There is a different approach for connections. This is the geometric point of view enclosed in Ehresmann spirit (it is known as the Ehresmann connection). We give a short mention:

The connection was introduced as a way to differentiate sections of a bundle E. Recall that a vector $v \in TX$ tangent bundle is indeed a derivative: given a funcion $f : X \to \mathbb{C}$, let $v.f$ denote the derivative of f along the direction of v. Observe that the operator $v.$ satisfies the Leibniz rule. We want to write something like $v.s$ where $s : X \to E$ is a section. Notice that a function $f : X \to \mathbb{C}^k$ is a particular case of a section, by considering $E = X \times \mathbb{C}^k$ the trivial bundle of rank k. Moreover, we define $\mathrm{d}f$ a 1-form defined by $\mathrm{d}f(v) = v.f$. So we want to define something like $\mathrm{d}s$ (or $\mathrm{d}_\nabla s$ or ∇s). It is important to remember the property that a function is *constant* if and only if $v.f = 0$ for all $v \in TX$: this property shall gives the notion of *flat sections* and the notion of a connection.

To do this, we work on a principal bundle. It is well known that a bundle E gives a principal $\mathrm{GL}(V)$-bundle where V is the fibre of $E \to X$. Conversely, a principal G-bundle P with a representation $\rho : G \to \mathrm{GL}(V)$ gives a vector bundle $E_\rho = (P \times V)/G$ where the G-action is defined to be $g \cdot (p, v) = (p \cdot g, \rho(g)^{-1} v)$.

To define the flatness of a section, we define the following short exact sequence called the *Atiyah exact sequence* (we follow the Appendix §3 of [Mac87] or §3.2, Definition 3.2.6 of [Mac05]). Let $\pi : P \to X$ be a principal G-bundle over an algebraic bundle X. Let $T^\pi P$ denote the vertical subbundle of TP, i. e., the kernel of the map $\mathrm{d}\pi : TP \to TX$. It is well-known that $T^\pi P \cong P \times \mathfrak{g}$. The G-action on P induces a non trivial action on TP and $T^\pi P$ (not on TX where it is trivial) by the adjoint action of G on $\mathfrak{g}$. So we have the short exact sequence on bundles *over* X

$$0 \longrightarrow T^\pi P/G \xrightarrow{i} TP/G \xrightarrow{\mathrm{d}\pi} TX \longrightarrow 0,$$

or better,

$$0 \longrightarrow P \times_G \mathfrak{g} \xrightarrow{i} TP/G \xrightarrow{\mathrm{d}\pi} TX \longrightarrow 0. \tag{8.1}$$

The bundle $P \times_G \mathfrak{g}$ is known as the *adjoint bundle*, and TP/G as the *Atiyah bundle*. They are usually denoted by $\mathrm{ad}(P)$ and $\mathrm{At}(P)$ respectively. (Remark: some authors define the Atiyah bundle as TP^G).

We have identified $T^\pi P \cong P \times \mathfrak{g}$ as the vertical bundle, and clearly TX is the corresponding horizontal component of TP. A split of (8.1) gives an horizontal component as subbundle of TP in a G-invariant way. So, we define a connection to be a map $\gamma : TP/G \to P \times_G \mathfrak{g}$. Such a map γ must be a G-invariant 1-form of $\Omega^1_P(\mathfrak{g})$, so its restriction on fibres is the Maurer-Cartan form*. Notice that to give a section γ is equivalent to a section $\beta : TX \to TP/G$ by the equations:

$$\begin{cases} \mathrm{d}\pi \circ \beta = 1_{TX} \\ \gamma \circ \beta = 0 \\ \gamma \circ i = 1_{P \times_G \mathfrak{g}} \end{cases}$$

It is well-known that a connection γ on a principal G-bundle P induces an affine connection on the vector bundle E_ρ where $\rho : G \to \mathrm{GL}(V)$ is a representation. Nevertheless, we give a brief introduction. Recall that a section $s : X \to E_\rho$ is equivalent to a G-invariant map $\rho : P \to V$, that is, $f(pg) = \rho(g)^{-1}(f(g))$. We define the affine connection to be

$$\nabla_v s_f = \tilde{v}.f + \mathrm{d}\rho(\gamma(\tilde{v})).f(p)$$

where $\tilde{v} \in T_p P$ is any lift of $v \in T_x X$ at any point $p \in P$ such that $\pi(p) = x$. This is well-defined because of the G-invariance of f and this formula depends on $\tilde{v}$ modulo the vertical tangent space at the point p.

---

* A *Maurer-Cartan form* of G is a 1-form with values in $\mathfrak{g}$, that is, $\omega \in \Omega^1_G(\mathfrak{g})$ such that for $v \in T_g G$, $\omega(g) = \mathrm{d}L_{g^{-1}} v$. Indeed, if we regard G as a trivial G-bundle over a point, the Maurer-Cartan form is a connection of this principal bundle.

We are interested in a special case: the log-connection. Let X be an algebraic variety and let $U \subset X$ be a Zariski open subset such that $D = X \backslash U$ is a crossing normal divisor. The bundle $\mathcal{O}_X(-D)$ is the set of section where we let simple zeroes on D. Now, $TX(-\log D)$ is the subsheaf of TX generated by all locally defined holomorphic vector fields that preserves the subsheaf $\mathcal{O}_X(-D) \subset \mathcal{O}_X$. Then, a *log-connection* of X is a connection on $TX(-\log D)$, that is, is a section is a splitting for the Atiyah exact sequence

$$0 \longrightarrow \mathrm{ad}(P)(-\log D) \longrightarrow \mathrm{At}(P)(-\log D) \longrightarrow TX(-\log D) \longrightarrow 0.$$

3 The log-connection on G-pairs

Definition 8.5. *A G-*pair *is a pair (X, D) where X is an algebraic variety endowed with a G-action and $D \subset X$ is an invariant divisor.*

Definition 8.6. *Let X be an algebraic variety equipped with an action of a group G. Let E be a fibre bundle over X. We say E is* equivariant *if $g^*E \cong E$ for any $g \in G$.*

The action of G defines the map $op : \mathfrak{g} \to H^0(X, TX)$ defined as follows: the map op sends $v \in \mathfrak{g}$ to $\frac{\mathrm{d}}{\mathrm{d}t}\phi_v(t) \cdot x \in H^0(X, TX)$, where ϕ_v is the flow of the field v over X. In sheaves, it defines $op_X : \mathcal{O}_X \otimes \mathfrak{g} \to TX$. Some properties about this map can be found in [Bri07].

Definition 8.7. *A* log-parallelizable *G-pair is a G-pair (X, D) where the map op_X is an isomorphism.*

Let E be an equivariant bundle over a log-parallelizable G-pair (X, D). We see there is a link between *equivariance* and the existence of a natural integrable logarithmic connection on E. The underlying idea which connects both concepts is the action G over any variety defines a notion of horizontality when such an action lifts to any bundle (like TX).

Let (X, D) be a log-parallelizable G-pair and let E be an equivariant holomorphic vector bundle. The same definition of the map op_X can be apply to TP_E where P_E is the associated $\mathrm{GL}(r, \mathbb{C})$-principal bundle of E, so we have the map

$$op_X : \mathcal{O}_X \otimes \mathfrak{g} \to TP_E.$$

It follows from the definition that the image of this map is contained in $TP_E(-\log D)$, because of D is invariant by the action of G.

Recall that a connection is equivalent to a section of the Atiyah exact sequence (8.1). A candidate for such section is the map

$$\gamma = q \circ op_X \circ op_{X,D}^{-1} : TX(-\log D) \to TP_E(-\log D)/G,$$

where $q : TP_E(-\log D) \to TP_E(-\log D)/G$ is the quotient map. But this is fairly simply to see. Observe the following diagram

$$\begin{tikzcd} TP_E(-\log D) \arrow[r, "\mathrm{d}\pi"] & TX(-\log D) \\ & \mathcal{O}_X \otimes \mathfrak{g} \arrow[ul, "op_X"] \arrow[u, "op_{X,D}"', "\sim"] \end{tikzcd}$$

where op_X is an isomorphism by definition of log-parallelizable G-pair. This is commutative because of op_X map arisen from the action of G, whose action on P_E comes from X, and $\mathrm{d}\pi$ is an equivariant map (or —maybe better— by the fact that TP_E is an equivariant bundle).

On the other hand, on $TX(-\log D)$ and $TP_E(-\log D)/G$ is defined a Lie bracket operator (see details on [Mac05]). Clearly, the maps op_X and $op_{X,D}$ preserves the Lie algebra structures, so the logarithmic connection on E is integrable.

Then, we have the following proposition:

Theorem 8.8. *Let (X, D) be a log parallelizable G-pair. Let E be an equivariant holomorphic vector bundle. Then, there exists a natural integrable log connection singular over D.*

A particular case of this case is the following:

Lemma 8.9. *Let X be a complete toric variety and let $U \subset X$ be the open Zariski subset where the action of the non-compact complex torus $\mathcal{T} = (\mathbb{C}^\times)^r$ is free. Then (X, D) is a log-parallelizable $\mathcal{T}$-pair.*

Proof. We have to see that $\beta = op_X$ is an isomorphism. Recall that $TX(-\log D)$ is holomorphically trivial by [Win07, Main Theorem]. It is clear that β is surjective on $U = X \setminus D$. Consider the homomorphism $\bigwedge^r \beta : \bigwedge^r \mathcal{O}_X \otimes \mathfrak{t} \to \bigwedge^r TX(-\log D)$ where r is the rank of both bundles. Then, $\bigwedge^r \beta$ defines a non-zero holomorphic section on $\mathrm{Hom}(\mathcal{O}_X \otimes \mathfrak{t}, TX(-\log D)) \cong \mathcal{O}_X$. Since $\bigwedge^r \beta$ is holomorphic, it is nowhere vanishing, hence it is an isomorphism. This implies that β is an isomorphism over X. □

Remark 8.10. *It follows from the definition of the op map: a section on $\mathcal{O}(-D)$ vanishes on D, and for any $v \in \mathfrak{g}$, $op(v).f$ vanishes on D since $op(v)$ is always tangent to D.*

From this lemma we have,

Corollary 8.11. *An equivariant holomorphic vector bundle E over a toric variety X has a natural integrable log connection singular over D.*

We can take this result and try the following equivalence:

Theorem 8.12. *Assume that G is a simple connected Lie group. Let (X, D) be a log parallelizable G-pair. Let E be a holomorphic vector bundle on X. The following statements are equivalent:*

1) The vector bundle E admits an equivariant structure.
2) The vector bundle E admits an integrable logarithmic connection singular over D.

Proof. To do this equivalence, consider the following group: let $\mathcal{G}_E$ denote the set of pairs (ϕ, f) where $f : X \to X$ automorphism and $\phi : E \to E$ satisfying the following commutative square

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

There is an obvious projection map $p : \mathcal{G}_E \to \mathrm{Aut}(X)$. On the other hand, we have the natural map $\rho : G \to \mathrm{Aut}(X)$. If the image of p is contained in the image of ρ, then we have proved the theorem. To do this, it is necessary to give some claims. Let us look at the differential of p, $\mathrm{d}p : \mathrm{Lie}(\mathcal{G}_E) \to \mathrm{Lie}(\mathrm{Aut}(X))$.

First claim. *The Lie algebra of the group* $\mathrm{Aut}(X)$ *is* $H^0(X, TX)$.

And a similar assertion is

Second claim. *The Lie algebra of the group* $\mathcal{G}_E$ *is* $H^0(X, TP)^G$.

A section $\mathcal{X} \in H^0(X, TP)^G$ defines a flow $\phi_t : I \times P \to P$ where $I \subset \mathbb{R}$ is an interval around the 0 with $\phi_0 = 1_P$, or better, $\phi_0 = (1_X, 1_P)$. The invariance of G assures that the map ϕ_t leaves invariant the fibres of $\pi : P \to X$ for any $t \in I$, so this is an element of $\mathcal{G}_E$. Then, the flow is, indeed, a curve $\phi : I \to \mathcal{G}_E$ at the identity, and hence gives an element of $T_e\mathcal{G}_E$, the Lie algebra of $\mathcal{G}_E$. The conversely is trivial, since any invariant field of $\mathcal{G}_E$ produces a map $\phi_t : I \to \mathrm{Aut}(P)$, easily identified with a field $\mathcal{X} \in H^0(X, TP)^G$.

Then, the differential of the map $p : \mathcal{G}_E \to \mathrm{Aut}(X)$ is a map

$$\mathrm{d}p : H^0(X, TP)^G \to H^0(X, TX).$$

Now, it's time to introduce the section of this map.

Third claim. *The map* $H^0(\nabla) : H^0(X, TX(-\log D)) \to H^0(X, TP(-\log D)/G)$ *is a section for* $\mathrm{d}p$.

This is true, mainly due to that the projection p is compatible with the projection $\pi : P \to X$. Now, the connection ∇ is a section for $\mathrm{d}\pi$, so it is also for $\mathrm{d}p$. Moreover, the composition $\mathrm{d}p \circ H^0(\nabla) : H^0(X, TX(-\log D)) \hookrightarrow H^0(X, TX)$ is the inclusion map

With these three assertions, the proof of this theorem is easy: by the second claim we rewrites the map $H^0(\nabla) : \mathfrak{g} \to \mathrm{Lie}(\mathcal{G}_E)$. This integrates to the map

$$\tilde{\nabla} : G \to \mathcal{G}_E$$

because of G is simply connected. Furthermore, this is a section for the projection $p : \mathcal{G}_E \to \mathrm{Aut}(X)$ by the third assertion, so the image of p includes automorphisms induced by the action of G. This proves the theorem.

Now, recall that the converse was proved in Theorem 8.8, so the theorem is proved. □

The case for toric varieties is a bit different. Recall that the non-compact complex torus $\mathcal{T}$ is connected but not simply connected group, so we cannot use the last argument of the proof.

To prove the image of $p : \mathcal{G}_E \to \mathrm{Aut}(X)$ contains any element of the image of $\rho : \mathcal{T} \to \mathrm{Aut}(X)$, recall the map $op_{X,D} : \mathcal{O}_X \otimes \mathfrak{t} \to TX(-\log D)$ is an isomorphism by Lemma 8.9. Althought $\mathcal{T}$ is not simply-connected Lie group, the simply connected Lie group associated to $\mathfrak{t}$ is $(\mathbb{C}^r, +)$. The exponential map $\exp : \mathbb{C}^r \to \mathcal{T}$ is surjective; indeed it is the quotient map

$$0 \to \mathbb{Z}^r \hookrightarrow \mathbb{C}^r \to \mathcal{T} \to 0.$$

Hence, the lift map of ∇, i.e.,

$$\tilde{\nabla} : \mathrm{Aut}(\tilde{X}) \to \mathcal{G}_E,$$

the group of automorphisms of X is the set of automorphisms invariant by $\mathbb{Z}^r$, so there exists a map

$$\tilde{\nabla}' : \mathrm{Aut}(X) \to \mathcal{G}_E$$

which is a section of $p : \mathcal{G}_E \to \mathrm{Aut}(X)$. This proves an implication of the following theorem for toric varieties, which gives a deeper result:

Theorem 8.13. *Let E be a holomorphic vector bundle over the toric variety X. The following three statements are equivalent:*

1) *The vector bundle E admits an equivariant structure.*
2) *The vector bundle E admits an integrable logarithmic connection singular over D.*
3) *The vector bundle E admits a logarithmic connection singular over D.*

Proof. We have proved that the first statement implies the second statement. It is obvious that the second statement implies the third one. Finally, the last implication was proved early, so we have the theorem. □

4 A counterexample

Here, we give an example where the hypothesis of G being a simply connected group is essential in the general case of G-pairs.

Let $G = \mathrm{PGL}(2, \mathbb{C})$ be the projective group which come from the quotient by $\mathrm{GL}(2, \mathbb{C})$ by its center. Let $X = \mathbb{P}(\mathfrak{M}(2, \mathbb{C})) \cong \mathbb{CP}^3$ be the projective space of the vector space of 2×2 matrices. Clearly, the $\mathrm{GL}(2, \mathbb{C})$-action on $\mathfrak{M}(2, \mathbb{C})$ induces an action of the group G on X. Furthermore, this induces a natural inclusion $\mathrm{PGL}(2, \mathbb{C}) \hookrightarrow \mathbb{P}(\mathfrak{M}(2, \mathbb{C}))$. Define $D = X \setminus G$ to be the boundary divisor. Clearly, D is defined by the determinant map $\det A = a_{11}a_{22} - a_{12}a_{21}$ in $\mathrm{GL}(2, \mathbb{C})$, so its degree is 2.

Claim. *The vector bundle $TX(-\log D)$ is holomorphically trivial.*

Consider the short exact sequence of coherent sheaves on X:

$$0 \to TX(-\log D) \to TX \to i_* N_X D \to 0, \tag{8.2}$$

where $i : D \hookrightarrow X$ is the natural inclusion and $N_X D$ is the normal bundle on D as subvariety of X. By the Poincaré adjunction formula we have the isomorphism

$$i_* N_X D = i_* \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, i^{-1}\mathcal{O}_X) = \operatorname{Hom}(i_*\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) \cong \mathcal{O}_X(D)|_D,$$

and $\deg(\mathcal{O}_X(D)|_D) = 4$ because of $\deg D = 2$. By the short exact sequence (8.2) we have $\deg TX(-\log D) = 0$.

Now, look at the map $op_{X,D} : \mathcal{O}_X \otimes \mathfrak{sl}(2, \mathbb{C}) \to H^0(X, TX(-\log D))$ induced by the action of $\mathrm{PGL}(2, \mathbb{C})$. Its determinant map

$$\bigwedge^3 op_{X,D} : \bigwedge^3 \mathcal{O}_X \otimes \mathfrak{sl}(2, \mathbb{C}) \to \bigwedge^3 (TX(-\log D))$$

is a no where vanishing homomorphism between line bundles. Furthermore, since $\bigwedge^3 \mathcal{O}_X \otimes \mathfrak{sl}(2, \mathbb{C}) \cong \mathcal{O}_X$, $\bigwedge^3 op_{X,D}$ is a non vanishing section and hence $\bigwedge^3 (TX(-\log D)) \cong \mathcal{O}_X$. So $TX(-\log D)$ is holomorphically trivial vector bundle.

Consequence. *(X, D) is a log parallelizable $\mathrm{PGL}(2, \mathbb{C})$-pair.*

Claim. *The tautological bundle $L = \mathcal{O}_X(-1)$ has an integrable logarithmic connection ∇ singular over D.*

To prove this claim, consider $L \otimes L \cong \mathcal{O}_X(-D)$. This bundle has a tautological integrable logarithmic connection singular over D given by the de Rham differential $\alpha \mapsto \mathrm{d}\alpha$.

A connection over any line bundle ξ induces a connection on $\xi^{\otimes n}$ and it inherits integrability and singularity over a divisor. Furthermore, there is a one-to-one correspondence between logarithmic connection on ξ and on $\xi^{\otimes n}$, so the above connection on $L \otimes L$ induces a logarithmic connection on L which is integrable and singular over D.

Claim. *The line bundle L does not admit an equivariant structure.*

The projection $\pi : \mathrm{GL}(2, \mathbb{C}) \to \mathrm{PGL}(2, \mathbb{C})$ is nothing but the projection $\pi : L|_{X \setminus D} \to X \setminus D$, so a non-vanishing section of $L|_{X \setminus D}$ is a section for the exact sequence of groups

$$0 \to \mathbb{C}^{\times} \to \mathrm{GL}(2, \mathbb{C}) \to \mathrm{PGL}(2, \mathbb{C}) \to 0$$

which is non split, so this contradicts our assumptions and we have proved the claim.

This finishes the counterexample.

Remark 8.14. *The pullback $R_h^* L$ for $h \in \mathrm{PGL}(2, \mathbb{C})$ is holomorphically isomorphic to L. This means that equivariance is not equivalent to the assertion $R_h^* L \cong L$ on log parallelizable G-pairs.*

5 Cones and Fans on Toric Varieties

Let N be a free $\mathbb{Z}$-module of rank n and let $M = \operatorname{Hom}(N, \mathbb{Z})$ denote the dual lattice. We consider the real vector spaces associated to these lattices $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 8.15. *A subset $\sigma \subset N_{\mathbb{R}}$ is called a* strongly convex rational polyhedral cone *if there exists a finite number of elements $n_1, n_2, \dots, n_k \in N$ such that*

$$\sigma = \mathbb{R}_+ n_1 + \mathbb{R}_+ n_2 + \cdots + \mathbb{R}_+ n_k,$$

and that $\sigma \cap (-\sigma) = \{0\}$. Here, we denote $\mathbb{R}_+$ the set of nonnegative real numbers.

To avoid such large name, we understand *cone* for strongly convex rational polyhedral cone.
There are two ways to associate an affine toric variety to a cone $\sigma \subset N_{\mathbb{R}}$:

1) Let us $S_\sigma = M \cap \sigma^\vee$ consider a semigroup where

$$\sigma^\vee = \{y \in M_{\mathbb{R}} : \langle x, y\rangle \geq 0 \text{ for all } x \in \sigma\}.$$

Then $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ is the toric variety (in this case, we have an affine variety) associated to σ cone.
2) Let S_σ be as above, then $U_\sigma = \operatorname{Hom}_{\text{s.g.}}(S_\sigma, \mathbb{C}^\times) \subset \operatorname{Hom}(\mathcal{T}, \mathbb{C}^\times)$ where $\operatorname{Hom}(\mathcal{T}, \mathbb{C}^\times)$ is the group of characters of the non-compact complex toric group, and $\operatorname{Hom}_{\text{s.g.}}(S_\sigma, \mathbb{C}^\times)$ is the set of homomorphism as semigroups. Indeed, the subset $\operatorname{Hom}_{\text{s.g.}}(S_\sigma, \mathbb{C}^\times)$ is the set of characters $\chi : (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ which extend to U_σ.

The associated non-compact complex torus of U_σ is $\mathcal{T} = N \otimes \mathbb{C}^\times \cong \operatorname{Hom}(M, \mathbb{C}^\times)$. There is a natural inclusion:

$$\operatorname{Hom}(M, \mathbb{C}^\times) \hookrightarrow \operatorname{Hom}(S_\sigma, \mathbb{C}^\times)$$

where the map sends the character $t : M \to \mathbb{C}^\times$ to its restriction to $S_\sigma \subset M$; the image of this map will be denoted by $U \subset U_\sigma$ and it is isomorphic to $\mathcal{T}$. The action of $\mathcal{T}$ on U_σ is the product of characters.

Remark 8.16. *The variety* $\operatorname{Hom}(S_\sigma, \mathbb{C}^\times)$ *is the set of closed points of* $\operatorname{Spec} \mathbb{C}[S_\sigma]$.

The smoothness of the affine toric variety U_σ is reflected in σ.

Theorem 8.17. *Let $\sigma \subset N_{\mathbb{R}}$ a cone. The affine toric variety U_σ is smooth if and only if the semigroup $N \cap \sigma$ is generated by s elements where $s = \dim \sigma$.*

If U_σ is a non-singular variety, then

$$U_\sigma \cong (\mathbb{C}^\times)^{n-s} \times \mathbb{C}^s, \tag{8.3}$$

where $s = \dim \sigma$ where $\dim \sigma = \dim_{\mathbb{R}}(\sigma + (-\sigma))$ as a vector space.

Definition 8.18. *A subset $\tau \subset \sigma$ such that there exists $m \in M$ with*

$$\tau = \sigma \cap \langle m\rangle^\perp = \{x \in \sigma : \langle m, x\rangle = 0\}$$

shall be called a face *and shall be denoted by $\tau < \sigma$.*

Clearly, for any $\tau < \sigma$ face we have the inclusion $U_\tau \subset U_\sigma$.
The following definition gives the way to construct a non-affine toric variety.

Definition 8.19. *A* fan *is a nonempty collection of cones in $N_{\mathbb{R}}$ satisfying the following conditions:*

1) *every face of any $\sigma \in \Delta$ is contained in Δ,*
2) *for any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.*

Then, the toric variety associated to a fan is the union of $\{U_\sigma\}_{\sigma\in\Delta}$ glued one each other by the following relation: U_σ and U_τ are glueing by the common subset $U_{\sigma\cap\tau} \hookrightarrow U_\sigma, U_\tau$. This defines a toric variety $X(\Delta)$ which is a irreducible and normal variety of dimension $n = \dim N_{\mathbb{R}}$. Note that the inclusion $\mathcal{T} \hookrightarrow U_\sigma$ as a Zariski open subset for any $\sigma \in \Delta$ guarranties the inclusion $\mathcal{T} \hookrightarrow X(\Delta)$.

Theorem 8.20. *Let X be a toric variety. Then, there exists a fan Δ and an equivariant isomorphism $\phi : X \to X(\Delta)$.*

Indeed, fans forms a category (roughly speaking, a map between fans $f : (N, \Delta) \to (M, \Delta')$ is a map of $\mathbb{Z}$-modules $f : N \to M$ which preserves cones). A refined result of previous statement claims that there is a equivalence of categories between fans and toric varieties.

For any cone $\sigma \in \Delta$, we define the distinguished point x_σ defined to be

$$x_\sigma(m) = \begin{cases} 1 & \text{if } m \in \sigma^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Under the identification (8.3), the point x_σ has coordinates $(1,\overset{(n-s)}{\cdots},1,0,\overset{(s)}{\cdots},0)$. Let $\mathcal{O}_\sigma$ denote the orbit of x_σ by the action of $\mathcal{T}$. In the non-singular case, this orbit is isomorphic to $(\mathbb{C}^\times)^{n-s}$ where s denotes the dimension of the cone, so in particular $\mathcal{O}_{\{0\}} = U_{\{0\}}$ is the maximal torus $\mathcal{T}$. Notice that the subgroup $\mathcal{T}_\sigma = \ker(\mathcal{T}_\sigma \to \mathrm{Aut}(\mathcal{O}_\sigma) = \mathcal{O}_\sigma)$ is the stabilizer of the point x_σ. This set of open subsets stratifies the toric variety as follows

$$X(\Delta) = \bigsqcup_{\sigma\in\Delta} \mathcal{O}_\sigma$$

and it is preserved by the action of $\mathcal{T}$. In particular, $U_\sigma = \bigsqcup_{\tau\leq\sigma} \mathcal{O}_\sigma$. Let X_σ denote the closure of $\mathcal{O}_\sigma$ in $X(\Delta)$. This subset inherits the stratification

$$X_\sigma = \bigsqcup_{\tau\geq\sigma} \mathcal{O}_\tau,$$

where we understand by $\tau \geq \sigma$ the cases $\sigma < \tau$ (σ face of τ) or $\tau = \sigma$. By this stratification, $X_\sigma \cap X_\tau = X_{\sigma\cap\tau}$. If σ is a cone of dimension 1, then X_σ is an invariant divisor and the complementary of the union of these invariant divisors coincides with the maximal torus $\mathcal{T}$. Moreover, the subvariety $D = X(\Delta) \setminus \mathcal{T}$ is a normal crossing divisor.

Any element of M and N corresponds to character or a one-parameter group respectively.

- An element $m \in M$ corresponds to a character $\chi_m : X(\Delta) \to \mathbb{C}^\times$: for any $\sigma \in \Delta$, we define $\chi_m : U_\sigma \to \mathbb{C}^\times$ to be $\chi_m(x) = x(m)$ where a point $x \in U_\sigma$ is viewed as a map $x : S_\sigma \to \mathbb{C}^\times$.
- Fix $\sigma \in \Delta$, an element $n \in N$ coresponds to an one-parameter group $\gamma_n : \mathbb{C}^\times \to U_\sigma$ defined by the map $\gamma_n(z) : m \mapsto \langle m, n\rangle$.

It is relatively easy to check that $\lim_{z\to 0} \gamma_n(z) = x_\sigma$ where $n \in N$ is in the relative interior of σ, whenever such cone exists in Δ — if does not exists such cone, then $\lim_{z\to 0} \gamma_n(z)$ does not exists in $X(\Delta)$. Supported by this fact, the intuition gives the following results

Theorem 8.21. *$X(\Delta)$ is a complete variety if and only if $\bigcup_{\sigma\in\Delta} \sigma = N_\mathbb{R}$.*

6 Chern clases of equivariant bundles

Let X be a smooth complete complex toric variety of dimension d so that the boundary is a simple normal crossing divisor. In [Kly90, Theorem 3.2.1], the Chern classes of an equivariant vector bundle $E \longrightarrow X$ are computed through a resolution. In this section we will compute the Chern classes of an equivariant vector bundle E through the natural logarithmic connection on E. As the Newton classes $N_p(E)$ (the sum over the p-th powers of the Chern roots of E) can be expressed in terms of the residues of an integrable logarithmic connection (see [EV86, Corollary B.3]), we use such formula to compute the Chern character in our case.

We begin with introducing some notation. The fan associated to X in $N \cong \mathbb{Z}^d$ will be denoted by Δ. The complex torus acting on X is $\mathcal{T} = T_N = N \otimes \mathbb{C}^*$. As before, let $\mathcal{U} \subset X$ be the dense open orbit of $\mathcal{T}$, and denote

$$D = X \setminus \mathcal{U}\,,$$

which is a normal crossing divisor.

For any cone $\sigma \in \Delta$, we have an open subset

$$\mathcal{U}_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{d-k} \tag{8.4}$$

where $k = \dim\sigma$. This open subset has a distinguished point $x_\sigma \in \mathcal{U}_\sigma$ and its coordinates under the identification (8.4) are $x_\sigma = (0,\overset{(k)}{\cdots},0,1,\overset{(d-k)}{\cdots},1)$. The orbit of x_σ is denoted by $\mathcal{O}_\sigma = \mathcal{T}\cdot x_\sigma$. This is a torus of dimension $d-k$. The stabilizer of such a point is the subgroup

$$T_\sigma = \ker(T_N \to \mathcal{O}_\sigma) \subset \mathcal{T}_N\,.$$

The closure of $\mathcal{O}_\sigma$ in X will be denoted by X_σ; it is also a toric variety. For one dimensional cone $\delta \in \Delta_1$, such subvarieties X_δ correspond to divisors, and

$$D = \bigcup_{\delta\in\Delta_1} X_\delta\,.$$

Let E be an equivariant vector bundle of rank r on X, and let ∇ be the integrable logarithmic connection on E constructed in Theorem 8.8. For any cone $\sigma \in \Delta$, the equivariant structure is given locally by a representation $\rho_\sigma : T_\sigma \longrightarrow \mathrm{GL}\,(V)$ which extends to T_N. Indeed, the action of $\mathcal{T} = T_N$ on E produces an action of T_N on $E|_{X_\sigma}$, reducing to an action of T_σ on the fixed fiber, i.e., on V.

The linear representation ρ_σ splits into isotypical components

$$V = \bigoplus_{\chi\in M} V^\sigma(\chi)\,, \tag{8.5}$$

where $\rho_\sigma(t)v = \chi(t)\cdot v$ for $v \in V^\sigma(\chi)$. Here $M = \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ is the dual lattice of N.

Fix a 1-dimensional cone $\delta \in \Delta$ generated by some $n \in N$. As a result of (8.5) there is a basis $v_1,\ldots,v_r \in V$ where $v_i \in V(\chi_i)$ and $\chi_i = \chi_{m_i}$, $m_i \in M$, $i = 1,\ldots,r$. The sections $v_i(t) = \rho_\delta(t)\cdot v_i \in \Gamma(\mathcal{U},E)$ satisfy the condition $\nabla v_i = 0$ and extend to $\mathcal{U}_\delta = \mathcal{U}\cup\mathcal{X}_\delta$; here we are using the alternative description of a logarithmic connection. These extended sections are denoted by v_i. The section $s_i(t) = \chi_i(t)^{-1}v_i(t) \in \Gamma(\mathcal{U}\cup X_\delta, E)$ is a non-vanishing one and

$$\nabla s_i = \frac{\mathrm{d}\chi_i^{-1}}{\chi_i^{-1}}\, s_i = -\frac{\mathrm{d}\chi_{m_i}}{\chi_{m_i}}\, s_i\,. \tag{8.6}$$

gives the following local expression for ∇

$$\omega_\delta = -\mathrm{diag}\left(\frac{\mathrm{d}\chi_{m_i}}{\chi_{m_i}}\,;\, i = 1,\ldots,r\right). \tag{8.7}$$

Let $\mathrm{Res}_\delta : E(-\log D) \longrightarrow E|_{X_\delta}$ be the Poincaré residue map, where $E(-\log D) = E\otimes TX(-\log D)$, and let

$$\Gamma_\delta = \mathrm{Res}_\delta \circ \nabla$$

be the function defined on [EV86, Appendix B]. Clearly $\Gamma_\delta(s_i) = -\langle m_i, n\rangle\,(s_i|_{X_\delta})$.

Let $\Delta_1 = \{\delta_1,\ldots,\delta_s\}$ be the set of one-dimensional faces where the i-th face is generated by a primitive element $n_i \in N$. Recall the isotypical decomposition (8.5) associated to ρ_{δ_j}

$$V = \bigoplus_{i=1}^{r} V^j(\chi_{m_{i,j}}).$$

Then,

$$\Gamma_{\delta_1}^{\alpha_1}\circ\cdots\circ\Gamma_{\delta_s}^{\alpha_s} = (-1)^{\alpha_1+\ldots+\alpha_s}\mathrm{diag}\left(\prod_{j=1}^{s}\langle m_{i,j}, n_j\rangle^{\alpha_j} \;:\; i = 1,\ldots,r\right), \tag{8.8}$$

for $\alpha_j \geq 0$, $j = 1,\ldots,s$. Now we apply the formula in [EV86, Cor. B.3],

$$N_p(E) = (-1)^p \sum_{\alpha_1+\ldots+\alpha_s=p} \frac{p!}{\alpha_1!\cdots\alpha_s!}\mathrm{Tr}\left(\Gamma_{\delta_1}^{\alpha_1}\circ\cdots\Gamma_{\delta_s}^{\alpha_s}\right)[X_{\delta_1}]^{\alpha_1}\cdots[X_{\delta_s}]^{\alpha_s},$$

to compute the Newton classes $N_p(E)$ of E, i.e, the sum of p-th powers of the Chern roots.

Using (8.8) we have

$$\begin{aligned}
N_p(E) &= \sum_{\alpha_1+\ldots+\alpha_s=p} \frac{p!}{\alpha_1!\cdots\alpha_s!}\left(\sum_{i=1}^{r}\left(\prod_{j=1}^{s}\langle m_{i,j},n_j\rangle^{\alpha_j}\right)[X_{\delta_1}]^{\alpha_1}\cdots[X_{\delta_s}]^{\alpha_s}\right) \\
&= \sum_{i=1}^{r}\sum_{\alpha_1+\ldots+\alpha_s=p}\frac{p!}{\alpha_1!\cdots\alpha_s!}\left((\langle m_{i,1},n_1\rangle[X_{\delta_1}])^{\alpha_1}\cdots(\langle m_{i,s},n_s\rangle[X_{\delta_s}])^{\alpha_s}\right) \\
&= \sum_{i=1}^{r}\left(\sum_{j=1}^{s}\langle m_{i,j},n_j\rangle[X_j]\right)^{p}.
\end{aligned}$$

Set

$$d_{nm} = \dim V^{\delta}(\chi_m)\,, \tag{8.9}$$

where $\delta = \mathbb{R}_+ \cdot n$. Then we enlarge the rank of summation as follows:

$$N_p(E) = \sum_{m\in M}\left(\sum_{n\in\Delta_1} d_{nm}\langle m,n\rangle[X_n]\right)^{p}. \tag{8.10}$$

where X_n is the divisor associated to the one-dimensional face $\delta = \mathbb{R}_+ \cdot n$.

This gives the Chern character

$$\mathrm{ch}\,(E) = \sum_i \exp(\xi_i) = \sum_{p=0}^{\infty}\frac{1}{p!}N_p(E)\,,$$

where ξ_i are the Chern roots. Here

$$\begin{aligned}
\mathrm{ch}(E) &= \sum_{p=0}^{\infty}\sum_{m\in M}\frac{1}{p!}\left(\sum_{n\in\Delta_1} d_{nm}\langle m,n\rangle[X_n]\right)^{p} \\
&= \sum_{m\in M}\exp\left(\sum_{n\in\Delta_1} d_{nm}\langle m,n\rangle[X_n]\right) \\
&= \sum_{m\in M}\prod_{n\in\Delta_1}\exp(d_{nm}\langle m,n\rangle[X_n]).
\end{aligned}$$

From (8.10), one easily compute the Chern classes. For $m \in M$, define

$$L_m := \sum_{n\in\Delta_1} d_{nm}\langle m,n\rangle\,[X_n]\,.$$

Thus, $N_p(E) = \sum_{m\in M} L_m^p$ is the sum of p-powers over M which are by definition the Newton polynomials (note that this is a finite sum). Hence L_m are the Chern roots. Now it is straightforward to check that

$$c_k(E) = \sum_{\substack{m_1,\ldots,m_k\in M\\ m_i\neq m_j}}\prod_{i=1}^{k} L_{m_i} = \sum_{\substack{m_1,\ldots,m_k\in M\\ m_i\neq m_j}}\prod_{i=1}^{k}\left(\sum_{n\in\Delta_1} d_{nm_i}\langle m_i,n\rangle\,[X_n]\right). \tag{8.11}$$

For $k = 1$, the class $c_1(E)$ coincides with the one in [Kly90]. This is done in [Kly90], (3.2.4).

7 Semistability and restriction to invariant curves

Let X be a complete toric variety over an algebraically closed field k. We consider pairs (C, f) where C is an irreducible smooth projective curve over k and $f : C \to X$ separable morphism such that $f(C)$ is preserved by the action of the torus on X. If C is also equivariant, hence $C \cong \mathbb{P}^1_k$.

Proposition 8.22. *Let E be an equivariant vector bundle of rank r over X. The following two statements are equivalent:*

1) For every invariant curve (C, f), the pullback f^*E is semistable.
2) There is a line bundle L over X such that the vector bundle E is isomorphic to $L^{\oplus r}$.

Proof. The second statement in the proposition evidently implies the first statement.

We prove the direction (1) ⇒ (2). By a classical result due to Grothendieck [Gro57], the bundle f^*E over $C \cong \mathbb{P}^1_k$ decomposes in a direct sum of bundles. Semistability implies that $f^*E \cong \xi^{\oplus r}$ where ξ is a line bundle over the projective line. Thus, $f^* \operatorname{End}(E) = \operatorname{End}(f^*E)$ is trivial for any invariant pair (C, f), so by [HtaP10] we have that $\operatorname{End}(E)$ is trivial on X.

Now, for any global sections $\phi \in H^0(X, \operatorname{End}(E)) \cong k^{r^2}$, the coefficients of its characteristic polynomial are constant functions, due X is a complete variety and hence any function is constant. In other words, the set of eigenvalues of ϕ_x does not depend on $x \in X$. Take ϕ such section such that the eigenvalues of ϕ are distincts. The eigenspace decomposition produces a decomposition

$$E = \bigoplus_{i=1}^{r} L_r$$

so that

$$\operatorname{End}(E) = \bigoplus_{i,j=1}^{r} L_i \otimes L_i^*,$$

but $\operatorname{End}(E) = \mathcal{O}_X^{\oplus r^2}$, so by [Ati56] we conclude that $L_j \otimes L_i^* \cong \mathcal{O}_X$ for $1 \leq i, j \leq r$. Therefore, the second statement in the proposition holds. □

Remark 8.23. *The isomorphism in the statement* (2) *is not necessary equivariant. Let us consider this example: let $X = \mathbb{CP}^1$ be the complex projective line and $L = \mathbb{CP}^1 \times \mathbb{C}$ the trivial bundle. We endowed two different equivariant structure:*

- *we denote by L_1 the trivial bundle with the torus action $t \cdot (x, y) = (tx, y)$.*
- *we denote by L_2 the trivial bundle with the torus action $t \cdot (x, y) = (tx, ty)$.*

The direct sum $E = L_1 \oplus L_2$ satisfies the second statement of the theorem. However, there is no equivariant bundle L' such that $E \cong L' \oplus L'$.

8 Equivariant Krull-Schmidt decomposition

Definition 8.24. *An equivariant vector bundle on a smooth complete toric variety X is called* decomposable *if it is a direct sum of two equivariant vector bundles of positive rank. An equivariant vector bundle is called* indecomposable *if it is not decomposable.*

It is well-known that E is decomposable if and only if the $\mathrm{GL}(V)$-principal bundle P_E admits a reduction of its group to $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ where $V = V_1 \oplus V_2$. In particular, E is indecomposable if and only if the maximal torus of $\operatorname{Aut}^{\mathcal{T}}(E)$ is of dimension one, that is, is isomorphic to $\mathbb{C}^\times$. Indeed, the dimension of the maximal torus $T \subset \operatorname{Aut}^{\mathcal{T}}(E)$ determines the number of equivariant bundles in which E decomposes.

Then, let $\operatorname{Aut}^{\mathcal{T}}(E)$ denote the group of all holomorphic equivariant automorphisms of the vector bundle E. According to [BP06], any two maximal tori of $\operatorname{Aut}^{\mathcal{T}}(E)$ differ by an inner automorphism of

$\mathrm{Aut}^{\mathcal{T}}(E)$. Then, fixing a maximal torus $T_0 \subset \mathrm{GL}(V)$, let H_0 be the centralizer of T_0 in $\mathrm{GL}(V)$ which is a Levi subgroup. Then, [BP06, Theorem 4.1] can be applied to this case and yields:

1) The associated principal bundle P_E admits an $\mathcal{T}$-equivariant reduction to H_0 and does not admit any $\mathcal{T}$-equivariant reduction to $H_0' \subsetneq H_0$ proper Levi subgroup.
2) If $H \subset G$ is a Levi subgroup of $\mathrm{GL}(V)$ such that P_E admits an $\mathcal{T}$-equivariant reduction of the structure group to H, but P_E does not admit any $\mathcal{T}$-equivariant reduction of the structure group to any proper Levi subgroup $H' \subsetneq H$, then H is conjugate to H_0.
3) Therefore, two $\mathcal{T}$-equivariant reductions of the structure group $P_{E_{H_0}}$ and $P'_{E_{H_0}}$ to H_0, so there is an $\mathcal{T}$-equivariant $\mathcal{T}$-equivariant automorphism $\phi \in \mathrm{Aut}^{\mathcal{T}}(P_E)$ of P_E such that $\phi(P_{E_{H_0}}) = P'_{E_{H_0}}$.

   This has a direct translation in terms of decomposability.

Corollary 8.25. *Let E be an equivariant vector bundle over a smooth complete complex toric variety X. Let*

$$E = \bigoplus_{i=1}^{m} E_i \text{ and } E = \bigoplus_{i=1}^{n} F_i$$

be two decompositions of E into direct sum of indecomposable equivariant vector bundles. Then $m = n$, and there is a permutation σ of $\{1, \ldots, m\}$ such that the equivariant vector bundle E_i is isomorphic to the equivariant vector bundle $F_{\sigma(i)}$ for every $i \in \{1, \ldots, m\}$.

Bibliography

[AK06] Donu Arapura and Su-Jeong Kang. Coniveau and the grothendieck group of varieties. *Michigan Math. J.*, 54(3):611–622, 2006.

[Ara06] Donu Arapura. Motivation for hodge cycles. *Adv. Math.*, 207(2):762–781, 2006.

[Ati56] Michael F. Atiyah. On the krull-schmidt theorem with application to sheaves. *Bulletin de la Société Mathématique de France*, 84:307–317, 1956.

[BD91] Steven B. Bradlow and Georgios D. Daskalopoulos. Moduli of stable pairs for holomorphic bundles over riemann surfaces. *International Journal of Mathematics*, 02(05):477–513, 1991.

[BGP96] Steven B. Bradlow and Oscar García-Prada. Stable triples, equivariant bundles and dimensional reduction. *Mathematische Annalen*, 304(1):225–252, 1996.

[BGP97] Olivier Biquard and Oscar García-Prada. Parabolic vortex equations and instantons of infinite energy. *Journal of Geometry and Physics*, 21(3):238 – 254, 1997.

[BGPG04] Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen. Moduli spaces of holomorphic triples over compact riemann surfaces. *Mathematische Annalen*, 328(1-2):299–351, 2004.

[Bit04] Franziska Bittner. The universal euler characteristic for varieties of characteristic zero. *Compositio Mathematica*, 140(04):1011–1032, 2004.

[BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.

[BLMn12] Indranil Biswas, Marina Logares, and Vicente Muñoz. Brauer group of moduli spaces of pairs. *Comm. Algebra*, 40(5):1605–1617, 2012.

[BP02] Indranil Biswas and Kapil H. Paranjape. The Hodge conjecture for general Prym varieties. *J. Algebraic Geom.*, 11(1):33–39, 2002.

[BP06] Indranil Biswas and A. J. Parameswaran. On the equivariant reduction of structure group of a principal bundle to a levi subgroup. *Journal de Mathematiques Pures et Appliques*, 85(1):54–70, January 2006.

[BR94] Indranil Biswas and S. Ramanan. An infinitesimal study of the moduli of hitchin pairs. *Journal of the London Mathematical Society*, 49(2):219–231, 1994.

[Bra91] Steven B. Bradlow. Special metrics and stability for holomorphic bundles with global sections. *Journal of Differential Geometry*, 33(1):169–213, 1991.

[Bra95] et al. Bradlow, Steven. *Stable augmented bundles over Riemann surfaces.* Cambridge University Press, 1995.

[Bri07] Michel Brion. Log homogeneous varities. *Proceedings of the XIVth Latin American Algebra Colloquium, Rev. Mat. Iberoamericana*, pages 1–39, 2007.

[CLS11] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties.* Graduate Studies in Mathematics. American Mathematical Society, 2011.

[dB01] Sebastian del Baño. On the Chow motive of some moduli spaces. *J. Reine Angew. Math.*, 532:105–132, 2001.

[dBRNA98] Sebastian del Baño Rollin and Vicente Navarro Aznar. On the motive of a quotient variety. *Collect. Math.*, 49(2-3):203–226, 1998. Dedicated to the memory of Fernando Serrano.

[Del71a] Pierre Deligne. Théorie de Hodge. I. Actes Congr. Internat. Math. 1970, 1, 425-430 (1971)., 1971.

[Del71b] Pierre Deligne. Théorie de hodge : Ii. *Publications Mathématiques de l'IHÉS*, 40:5–57, 1971.

[Del74] Pierre Deligne. Théorie de hodge : Iii. *Publications Mathématiques de l'IHÉS*, 44:5–77, 1974.

[Don85] S. K. Donaldson. Anti self-dual yang-mills connections over complex algebraic surfaces and stable vector bundles. *Proceedings of the London Mathematical Society*, s3-50(1):1–26, 1985.

[Don87] S. K. Donaldson. Infinite determinants, stable bundles and curvature. *Duke Mathematical Journal*, 54(1):231–247, 1987.

[EV86] Hélène Esnault and Eckart Viehweg. Logarithmic de rham complexes and vanishing theorems. *Inventiones mathematicae*, 86(1):161–194, 1986.

[Fuj74] Akira Fujiki. On resolution of cyclic quotient singularities. *Publ. RIMS, Kyoto Univ.*, 10:293–328, 1974.

[Ful93] William Fulton. *Introduction to Toric Varieties.* Annals of Mathematics Studies. Princeton University Press, 1993.

[Gab81] Ofer Gabber. Some theorems on Azumaya algebras. In *The Brauer group (Sem., Les Plans-sur-Bex, 1980)*, volume 844 of *Lecture Notes in Math.*, pages 129–209. Springer, Berlin, 1981.

[Got01] L. Gottsche. On the motive of the hilbert scheme of points on a surface. *Math. Res. Lett.*, 8(no. 5-6):613–627, 2001.

[GP94] Oscar García-Prada. Dimensional reduction of stable bundles, vortices and stable pairs. *International Journal of Mathematics*, 05(01):1–52, 1994.

[GPHS11] O. García-Prada, J. Heinloth, and A. Schmitt. On the motives of moduli of chains and higgs bundles. 2011.

[Gro57] A. Grothendieck. Sur la classification des fibres holomorphes sur la sphère de Riemann. *Am. J. Math.*, 79:121–138, 1957.

[GZLMH04a] Sabir M Gusein-Zade, Ignacio Luengo, and Alejandro Melle-Hernández. A power structure over the grothendieck ring of varieties. *Mathematical Research Letters*, 11(1):49–58, 2004.

[GZLMH04b] Sabir M Gusein-Zade, Ignacio Luengo, and Alejandro Melle-Hernandez. Power structure over the grothendieck ring of varieties and generating series of hilbert schemes of points. *arXiv preprint math/0407204*, 2004.

[HS97] P. J. Hilton and U. Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[HtaP10] Milena Hering, Mircea Mustaţă, and Sam Payne. Positivity properties of toric vector bundles. *Annales de l institut fourier*, 60(2):607–640, 2010.

[Jan94] Uwe Jannsen. Mixed motives, motivic cohomology, and ext-groups. In *Proceedings of the International Congress of Mathematicians*, volume 1, page 2. Zürich, Switzerland, 1994.

[Kly90] Alexander A. Klyachko. Equivariant bundles on toral varieties. *Mathematics of the USSR-Izvestiya*, 35(2):337, 1990.

[Kob82] Shoshichi Kobayashi. Curvature and stability of vector bundles. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 58(4):158–162, 1982.

[Mac87] K. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1987.

[Mac05] K.C.H. Mackenzie. *General Theory of Lie Groupoids and Lie Algebroids*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2005.

[Man68] Ju I. Manin. Correspondences, Motifs and Monoidal Transformations. *Mathematics of The Ussr-sbornik*, 6:439–470, 1968.

[Mn08] Vicente Muñoz. Hodge polynomials of the moduli spaces of rank 3 pairs. *Geometriae Dedicata*, 136(1):17–46, 2008.

[Mn10] Vicente Muñoz. Hodge structures of the moduli spaces of pairs. *Internat. J. Math.*, 21(11):1505–1529, 2010.

[Moz13] M. Mozgovoy, S; Reineke. Moduli spaces of stable pairs and non-abelian zeta functions of curves via wall-crossing. 2013.

[Mum63] David Mumford. Projective invariants of projective structures and applications. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 526–530. Inst. Mittag-Leffler, Djursholm, 1963.

[NS65] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact riemann surface. *Annals of Mathematics*, 82(3):pp. 540–567, 1965.

[Oda12] Tadao Oda. *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties*. Ergebnisse der Mathematik und Ihrer Grenzgebiete, 3 Folge/A Series of Modern Surveys in Mathematics Series. Springer London, Limited, 2012.

[Poo02] Bjorn Poonen. The grothendieck ring of varieties is not a domain. *Mathematical Research Letters*, 9:493–497, 2002.

[Sch94] A. J. Scholl. Classical motives. In *Proc. Symp. Pure Math. 55.1*, pages 163–187, 1994.

[Sch08] Alexander H. W. Schmitt. *Geometric Invariant Theory and Decorated Principal Bundles (Zurich Lectures in Advanced Mathematics)*. European Mathematical Society, 2008.

[Ser91] Jean-Pierre Serre. Motifs. *Astérisque*, 198(200):333–349, 1991.

[Tan81] Tankeev. On algebraic cycles on surfaces and abelian varieties. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(2):398–434, 1981.

[Tha94] Michael Thaddeus. Stable pairs, linear systems and the verlinde formula. *Inventiones mathematicae*, 117(1):317–353, 1994.

[VMnVG07] Daniel Ortega Vicente Muñoz and María-Jesús Vázquez-Gallo. Hodge polynomials of the moduli spaces of pairs. *International Journal of Mathematics*, 18(06):695–721, 2007.

[VMnVG09] Daniel Ortega Vicente Muñoz and María-Jesús Vázquez-Gallo. Hodge polynomials of the moduli spaces of triples of rank (2,2). *Quarterly Journal of Mathematics*, 60(2):235–272, 2009.

[Win07] J. Winkelmann. On manifolds with trivial logarithmic tangent bundle. *Osaka Jour. Math.*, 41:473–484, 2007.