Risk transference constraints in optimal reinsurance

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Abstract. This paper will deal with the optimal reinsurance problem and will involve the goals of both insurer and reinsurer. In particular, the study will incorporate the initial (before reinsurance) risk that the reinsurer uses in order to diversify (or hedge) the risk ceded by the insurer, general methods to prevent the reinsurer moral hazard will be extended, and a new constraint will have to be satisfied by the selected reinsurance contract, namely, "the reinsurer increment of risk will have to be lower than the contract premium". Simultaneously, since the contract must be attractive to the insurer too, "the contract premium will have to be lower than the insurer risk reduction". Integrating both ideas, "the contract premium will have to be higher than the reinsurer risk growth and lower than the insurer risk mitigation". Bearing in mind both requirements, that is, the protection against the moral hazard and the spread containing the contract premium, the optimal reinsurance problem will be studied under very general conditions about the risk measures and premium principles involved, general solutions will be provided, and a particular case with the conditional value at risk will be presented in more detail.

Key words Optimal reinsurance, risk measure, moral hazard, marginal indemnity, relationships between premium and transferred risk.

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1 Introduction

The seminal papers by Borch (1960) and Arrow (1963) gave rise to the optimal reinsurance problem. Since then, many important extensions have been presented. Some recent ones involve dispersion risk measures (Tan and Weng, 2014), the coherent risk measures of Artzner *et al.* (1999) (Cheung *et al.*, 2019) or even the omega ratio (Balbás *et al.,* 2021*a*). Other approaches are related to the existence of multivariate risks (Cheung *et al.*, 2014). Interesting surveys/overviews may be found in Centeno and Simoes (2009) and Albrecher *et al.* (2017), among others. Though the traditional approach focuses on the insurer viewpoint, the reinsurer one is also deserving of the attention of recent literature (Cai *et al.*, 2016, Jiang *et al.,* 2018, or Zhang *et al.*, 2018, to name a few). However, most of these studies deal with the risk that the insurer transfers to the reinsurer without integrating this risk with that provoked by the sale of other reinsurance contracts. This seems to be the first novelty of this paper, that is, the extra risk received by the reinsurer becomes a small part of a much larger set of policies. Our approach is not similar to other interesting ones analyzing risk sharing strategies (Hamm *et al.,* 2020), since we just recover the optimal reinsurance problem and involve the initial (before reinsurance) risk that the reinsurer uses in order to diversify (or hedge) the risk ceded by the insurer. Furthermore, without loss of generality, we will suppose that this initial reinsurer risk may be understood as the reinsurer background risk.[1] This could imply that this risk is non-insurable, which might be false in general, but it may be viewed as a non-insurable risk at the time the portfolio of reinsurance contracts is formed without considering the corporate risk management tools put in place after. Anyway, beyond semantic discussions, the reader may select "reinsurer background risk" or "reinsurer risk before reinsurance", and the theoretical findings of this paper will remain the same.

The paper outline is as follows. Section 2 is devoted to presenting the general framework. The insurer manages the risk by means of a coherent and expectation bounded (Rockafellar *et al.*, 2006) co-monotone additive risk measure. This is a general setting including most of the classical risk measures such as the conditional value at risk ($CV@R$, Rockafellar *et al.*, 2006), the weighted conditional value at risk ($WCV@R$, Cherny, 2006), many versions of the weighted value at risk ($WV@R$, Cherny, 2006), the spectral risk measures, many risk measures given by distortion functions and some consistent risk measures (Goovaerts *et al.,* 2004). Similarly, the reinsurer manages its risk by means of a risk measure with analogous properties, though the co-monotone additivity is not imposed, and consequently the ambiguous (or robust) risk measures (Balbás *et al.,* 2015) are included too. Finally, the reinsurer premium principle is given by a sub-linear (sub-additive and positively homogeneous) operator dominating the expectation. Obviously, the expected value premium principle ($EVPP$) is included, but the framework is much more general and also contains most of the premium principles given by deviations, distortion risk measures or coherent risk measures.

Section 3 presents three main constraints that the optimal reinsurance problem may incorporate but, nevertheless, are frequently ignored in the literature. The first and second constraints involve the reinsurer viewpoint, while the third one is related to the insurer purposes. Briefly, the first constraint protects the reinsurer against the moral hazard, the second one imposes that the reinsurer risk increment must be compensated by the contract premium, and the third one imposes that the premium must be compensated by the insurer risk reduction. Integrating the second and third restrictions in a simple sentence, the contract premium must be higher than the reinsurer risk increment and lower than the

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[1] In general, the background risk may play very important roles in several actuarial problems (Ping and Zanjani, 2015, or Chi and Wei, 2020).

insurer risk reduction. It seems that a contract whose price lies within this spread of risk modifications will be very attractive to both companies. Indeed, the reinsurer will improve the expected wealth after signing the contract and simultaneously will see how the new contract makes the global risk decrease if the premium is involved in the risk computation. Besides, the insurer will see how the reserve (or capital requirement) mitigation is higher than the paid premium, which also allows it to reduce the price of its policies and become more competitive in the insurance market. With regard to the reinsurer moral hazard protection, we will follow the approach of Balbás *et al.* *(*2015) and will deal with a decision variable which is not the ceded risk, but the sensitivity (or mathematical derivative) of the ceded risk with respect to the total one. Nevertheless, there are two important differences with respect to Balbás *et al.* *(*2015). Firstly, we do not impose an essentially bounded global risk, though this innovation was already partially incorporated in Balbás *et al.* *(*2021*a*). Secondly, a real novelty is that we extend the discussion so as to integrate the ceded risk with the reinsurer background risk.

The optimal reinsurance problem is addressed in Section 4. As said above, the decision variable is adapted so as to prevent the moral hazard, but the spread containing the contract premium is not considered for two reasons. On the one hand, since the optimization problem will be modeled by the insurer (the client in a reinsurance contract), the incorporation of the premium lower bound implicitly implies that the insurer has some knowledge about the reinsurer background risk, which could be unrealistic. On the other hand, it will be seen later that a correct approach to the problem will imply the fulfillment of the restriction about the premium upper bound, and therefore the restriction does not have to be a priori imposed. The optimization problem involves two objectives, namely, the insurer expected wealth and the insurer risk, and therefore one will have to look for Pareto-efficient strategies.[2] Theorem 7 and its corollaries are the most important results of this section and will provide us with explicit expressions of the optimal contract, which often saturates the restrictions affecting the sensitivities of the ceded risk with respect to the total one.

As indicated above, the restriction about the premium upper bound is always satisfied by the optimal contract, and this is proved in Theorem 13 (and its remark), the most important result of Section 5.1. Then, Section 5.2 is devoted to dealing with the premium lower bound. Example 15 is a simple counter-example illustrating that the lower bound is not always respected and, actually, as pointed out in Remark 16, if the reinsurer portfolio is ignored, then it is not possible to find “a reasonable harmony” between the insurer purposes and the reinsurer ones. In other words, if the reinsurer portfolio is taken as “zero”, then the reinsurer risk increment will not be compensated by the contract premium. Some more counter-examples are given in order to point out the role of the reinsurer risk parameters in its risk mitigation, and these examples suggest that the

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[2] Optimal reinsurance problems involving more than one objective and Pareto-efficiency are common (Dimitrova and Kaishev, 2010, Asimit *et al.*, 2017, Jiang *et al.,* 2018, Zhang *et al.*, 2018, etc.).

variance of the global indemnity to be paid by the reinsurer is much more important than the expected indemnity. This intuition inspires three propositions providing us with sufficient conditions guaranteeing that the optimal contract premium also satisfies the required lower bound.

To sum up, there might be three main contributions in this paper. Firstly, a method to prevent the reinsurer moral hazard is extended. Secondly, a correct approach to the insurer problem leads to a reserve reduction, since the contract price is lower than the risk mitigation. Thirdly, the reinsurer global risk is important, and it is frequently possible to predict whether the optimal solution will satisfy the reinsurer requirements and will have a price that reduces the reinsurer global risk and improves its expected wealth. Furthermore, explicit expressions for the optimal solution are given and all of these achievements are reached in a very general setting with respect to the involved risk measures and premium principles. The important particular case involving the $CV@R$ is presented in Section 6. Some concluding remarks are presented in Section 7, and all of the proofs are placed in the appendix.

2 Background and notation

2.1 Preliminaries

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. The $\mathbb{R}-$valued non-negative random variables Y and $\mathcal{Y}$ will represent indemnities to be paid by a direct insurer and a reinsurer during a planning period $[0, T]$. We will assume that the insurer is interested in a reinsurance contract so that $Y = r(Y) + c(Y)$ can be divided into the retained risk $r(Y)$ and the ceded one $c(Y)$, while the reinsurer will deal with $\mathcal{Y}$ in order to hedge (or diversify) the additional received risk $c(Y)$. In other words, after the contract signature, the risk of both companies will become $(r(Y), c(Y) + \mathcal{Y})$, rather than $(Y, \mathcal{Y})$.

As usual, denote by $[0, \infty)^2$ the non-negative convex cone of $\mathbb{R}^2$. The probability measure $\mathbb{P}$ induces another probability measure

$$\mathbb{P}\left(\omega \in \Omega; \left(Y(\omega), \mathcal{Y}(\omega)\right) \in B\right)$$

for every Borel set $B \subset [0, \infty)^2$. Though it is an abuse of language, we will still denote by $\mathbb{P}$ this induced probability measure, and without loss of generality one can replace $(\Omega, \mathcal{F}, \mathbb{P})$ with the probability space $\left([0, \infty)^2, \mathcal{B}_2, \mathbb{P}\right)$, $\mathcal{B}_2$ denoting the Borel $\sigma-$algebra of $[0, \infty)^2$. Consequently, Y and $\mathcal{Y}$ become the first and second orthogonal projections on the real line, respectively. Moreover, $\mathbb{P}$ induces the (marginal) probability measures $\mathbb{P}_Y(B) = \mathbb{P}(B \times [0, \infty))$ and $\mathbb{P}_{\mathcal{Y}}(B) = \mathbb{P}([0, \infty) \times B)$ for every $B \in \mathcal{B}_1$, $\mathcal{B}_1$ denoting the Borel $\sigma-$algebra of $[0, \infty)$. Obviously, Y and $\mathcal{Y}$ will be independent if and only if $\mathbb{P}$ becomes the tensor product $\mathbb{P}_Y \otimes \mathbb{P}_{\mathcal{Y}}$ of $\mathbb{P}_Y$ and $\mathbb{P}_{\mathcal{Y}}$,[3] though this assumption will only be imposed

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[3] That is, $\mathbb{P}(B \times B') = \mathbb{P}_Y(B)\,\mathbb{P}_{\mathcal{Y}}(B')$ holds for every $B, B' \in \mathcal{B}_1$.

if it is clearly indicated. Lastly, Y may be identified with the identity map from $([0,\infty), \mathcal{B}_1, \mathbb{P}_Y)$ on $\mathbb{R}$, and $\mathcal{Y}$ may be identified with the identity map from $([0,\infty), \mathcal{B}_1, \mathbb{P}_{\mathcal{Y}})$ on $\mathbb{R}$.

If $L^2(\mathbb{P})$, $L^2(\mathbb{P}_Y)$ and $L^2(\mathbb{P}_{\mathcal{Y}})$ denote the natural Hilbert spaces of square-integrable random variables (Kopp, 1984), then we will assume that $Y \in L^2(\mathbb{P}_Y)$ and $\mathcal{Y} \in L^2(\mathbb{P}_{\mathcal{Y}})$.[4] In other words, Y and $\mathcal{Y}$ have finite expectation and finite variance. Suppose that $U \in L^2(\mathbb{P}_Y)$. Then, $[0,\infty)^2 \ni (t,s) \to U(t) \in \mathbb{R}$ is a random variable belonging to $L^2(\mathbb{P})$. Though it is again an abuse of language, this new random variable will be often denoted by U, but sometimes $I_Y(U)$ will be preferred because U and $I_Y(U)$ will have to be distinguished.[5] A similar idea applies for every $\mathcal{U} \in L^2(\mathbb{P}_{\mathcal{Y}})$.

The rest of this section will be devoted to recalling some properties of convex functions and presenting the involved risk measures and premium principles. Some assumptions imposed on the risk measures could be relaxed and several results would remain true. Actually, these risk measures will have to be optimized by duality-linked methods, and all of these methods may be extended to continuous and convex functions (Balbás *et al.,* 2021*b*). Nevertheless, dealing with coherent and/or expectation bounded risk measures one can simplify the mathematical exposition and simultaneously provide a general enough framework, as will be pointed out with the analyzed examples. The optimal reinsurance literature often deals with risk measures that are coherent or expectation bounded (or both).

2.2 Background about sub-linear functionals

Let us summarize some notions and properties that will play a critical role in this paper. A deeply study of them all may be found in Zalinescu (2002).

A function $\varphi : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be sub-linear if it is sub-additive ($\varphi(U_1 + U_2) \leq \varphi(U_1) + \varphi(U_2)$) and positively homogeneous ($\varphi(\lambda U) = \lambda \varphi(U)$ if $\lambda \geq 0$). $\varphi : L^2(\mathbb{P}) \to \mathbb{R}$ is sub-linear and continuous if and only if

$$\partial_{\varphi} = \left\{ Z \in L^2(\mathbb{P}) ; \mathbb{E}(ZU) \leq \varphi(U) \ \ \forall U \in L^2(\mathbb{P}) \right\} \tag{1}$$

is a convex and weakly-compact set such that

$$\varphi(U) = Max \left\{ \mathbb{E}(ZU) ; Z \in \partial_{\varphi} \right\} \tag{2}$$

holds for every $U \in L^2(\mathbb{P})$ (Zeidler, 1995).[6] Important examples of sub-linear and continuous functions are many deviations such as the absolute deviation $\varphi(U) = \mathbb{E}(|U - \mathbb{E}(U)|)$, where $\mathbb{E}$ denotes "mathematical expectation", the

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[4] $L^2(\mathbb{P})$, $L^2(\mathbb{P}_Y)$ and $L^2(\mathbb{P}_{\mathcal{Y}})$ may be replaced by infinitely many other Banach spaces of random variables, but we have made this choice in order to simplify the mathematical exposition.

[5] Notice that both random variables have the same expectation, the same variance, the same norm and the same cumulative distribution function.

[6] This is the continuous and sub-linear functions representation theorem.

standard deviation $\varphi(U) = \sqrt{\mathbb{E}\left((U - \mathbb{E}(U))^2\right)}$, the down-side standard semi-deviation $\varphi(U) = \sqrt{\mathbb{E}\left(\left((\mathbb{E}(U) - U)^+\right)^2\right)}$, and the up-side standard semi-deviation $\varphi(U) = \sqrt{\mathbb{E}\left(\left((U - \mathbb{E}(U))^+\right)^2\right)}$.

If $\varphi : L^2(\mathbb{P}) \to \mathbb{R}$ is sub-linear, translation-invariant ($\varphi(U + \lambda) = \varphi(U) - \lambda$ if $\lambda \in \mathbb{R}$) and decreasing ($\varphi(U_2) \leq \varphi(U_1)$ if $U_2 \geq U_1$), then φ is said to be a coherent risk measure (Artzner *et al.,* 1999). If $\varphi : L^2(\mathbb{P}) \to \mathbb{R}$ is sub-linear, translation-invariant and mean-dominating ($\varphi(U) \geq -\mathbb{E}(U)$), then φ is said to be an expectation bounded risk measure (Rockafellar *et al.*, 2006). A function $\varphi : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be law-invariant if $\gamma(U_1) = \gamma(U_2)$ when U_1 and U_2 have the same cumulative distribution function. A function $\varphi : L^2(\mathbb{P}) \to \mathbb{R}$ is said to be co-monotone additive if $\gamma(U_1 + U_2) = \gamma(U_1) + \gamma(U_2)$ holds when U_1 and U_2 are co-monotone.[7] Important examples of continuous, coherent and expectation bounded risk measures which are also law-invariant and co-monotone additive are $CV@R$ and $WCV@R$. The value at risk ($V@R$) does not satisfy all the properties above. Actually, it is only positively homogeneous, translation-invariant, decreasing, law-invariant and co-monotone additive, but it is not continuous, it is not sub-additive and it is not mean-dominating either.

2.3 The insurer risk measure

Let us suppose that the insurer deals with the risk measure $\gamma : L^2(\mathbb{P}) \to \mathbb{R}$ in order to control and manage its risk. We will assume that γ is continuous, coherent, expectation bounded and co-monotone additive. If for expositional reasons we implement a minor modification of the sign in (1) and (2), then it becomes obvious that

$$\partial_\gamma = \left\{Z \in L^2(\mathbb{P}) ; -\mathbb{E}(ZU) \leq \gamma(U) \ \forall U \in L^2(\mathbb{P})\right\} \tag{3}$$

is weakly-compact and

$$\gamma(U) = Max\left\{-\mathbb{E}(ZU) ; Z \in \partial_\gamma\right\} \tag{4}$$

holds for every $U \in L^2(\mathbb{P})$. Moreover, since γ is coherent and expectation bounded, one has that (Artzner *et al.*, 1999, and Rockafellar *et al.*, 2006)

$$\mathbb{P}(Z \geq 0) = 1 \tag{5}$$

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[7] Recall that the random variables $U_1, U_2, ...U_n$ are said to be co-monotone if their joint distribution is given by the Fréchet–Hoeffding copula

$$c(u_1, u_2, ..., u_n) = Min\ \{u_1, u_2, ...u_n\}$$

for $0 \leq u_i \leq 1$, $i = 1, 2, ..., n$. In particular, if there exist a random variable U whose range is (almost surely) included in an interval (a, b) with $-\infty \leq a < b \leq +\infty$, and a family of (non necessarily strictly) increasing functions $f_i : (a, b) \to \mathbb{R}$ such that $U_i = f_i(U)$, $i = 1, 2, ..., n$, then $U_1, U_2, ...U_n$ are co-monotone (Dhaene *et al.*, 2002).

and

$$\mathbb{E}(Z) = 1 \tag{6}$$

hold for every $Z \in \partial_\gamma$, and

$$\mathbb{P}(Z = 1) = 1 \Longrightarrow Z \in \partial_\gamma. \tag{7}$$

Though the assumptions above are not always satisfied,[8] many important examples of risk measure may be considered. Among them, the $CV@R$, the $WCV@R$ and the spectral risk measures. Another interesting example is the mathematical expectation with the negative sign, that is, $\gamma(U) = -\mathbb{E}(U)$.

2.4 The reinsurer risk measure

Let us assume that the reinsurer risk measure $\Gamma : L^2(\mathbb{P}) \to \mathbb{R}$ is continuous, coherent and expectation bounded.[9] Accordingly,

$$\partial_\Gamma = \left\{ Z \in L^2(\mathbb{P}) ; -\mathbb{E}(ZU) \leq \Gamma(U) \;\; \forall U \in L^2(\mathbb{P}) \right\} \tag{8}$$

is weakly-compact and (4), (5), (6) and (7) hold if γ and ∂_γ are properly substituted by Γ and ∂_Γ.

2.5 The reinsurer premium principle

The continuous and sub-linear function $\Pi : L^2(\mathbb{P}_Y) \to \mathbb{R}$ will provide us with the reinsurer premium principle, that is, $\Pi(U)$ will be the price of U. As above, the continuity and sub-linearity of Π imply that

$$\partial_\Pi = \left\{ Z \in L^2(\mathbb{P}_Y) ; \mathbb{E}(ZU) \leq \Pi(U) \;\; \forall U \in L^2(\mathbb{P}_Y) \right\}$$

is weakly-compact and

$$\Pi(U) = Max \left\{ \mathbb{E}(ZU) ; Z \in \partial_\Pi \right\} \tag{9}$$

for every $U \in L^2(\mathbb{P}_Y)$. Furthermore, we will impose Π to satisfy the existence of a loading rate $k > 0$ such that

$$\Pi(U) \geq (1 + k)\, \mathbb{E}(U) \tag{10}$$

holds for every $U \in L^2(\mathbb{P}_Y)$. Accordingly,

$$\mathbb{P}_Y(Z = 1 + k) = 1 \Longrightarrow Z \in \partial_\Pi.$$

There are many examples of premium principles satisfying the conditions above.[10] The classical $EVPP$, that is, $\Pi(U) = (1 + k)\, \mathbb{E}(U)$ with $k > 0$, is a

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[8] For instance, the entropic risk measure (Kupper and Schachermayer, 2011) and the ambiguous risk measures of Balbás *et al.* (2015) do not satisfy all of these properties.

[9] *I.e.*, γ and Γ satisfy quite similar properties, though we do not impose Γ being comonotone additive. Accordingly, the ambiguous risk measures of Balbás *et al.* (2015) are included.

[10] Gerber (1979), Haezendonck and Goovaerts (1982), and Goovaerts *et al.* (1984) present a very interesting and complete list of actuarial premium principles.

particular case such that $\partial_{\Pi} = \{1+k\}$ becomes a singleton. If $\rho : L^2(\mathbb{P}_Y) \to \mathbb{R}$ is a continuous and expectation bounded risk measure, $k_1 > 0$ and $k_2 \geq 0$, then

$$\Pi(U) = (1+k_1)\,\mathbb{E}(U) + k_2\rho(-U) \tag{11}$$

also satisfies the imposed properties. If $\sigma : L^2(\mathbb{P}_Y) \to \mathbb{R}$ is a continuous deviation measure (Rockafellar *et al.*, 2006),[11] $k_1 > 0$ and $k_2 \geq 0$, then $\Pi(U) = (1+k_1)\,\mathbb{E}(U) + k_2\sigma(U)$ satisfies the required properties.

3 Natural constraints

Let us justify three "natural constraints" that the optimal reinsurance literature frequently omits. The first and second ones are related to the reinsurer protection, whereas the third one is related to the insurer objective and guarantees a balance between the ceded risk and the contract premium.

3.1 Moral hazard prevention

Let us deal with the Balbás *et al.* (2015) approach in order to prevent the reinsurer moral hazard. Consider that the marginal indemnity of Zhuang *et al.* (2016) equals one for "large realizations of Y", that is, the mathematical derivative $d(c(Y))/dY$ equals one for Y "large enough". Then, the insurer might lose every incentive to detect frauds once Y achieves some thresholds, since this detection could imply a cost and the reinsurer will finally pay every indemnification. This is a common problem in the optimal reinsurance literature, since the optimal contract is frequently closely related to a stop-loss one.[12] Balbás *et al.* (2015) address this problem by considering the Lebesgue measure $\mathbb{L}$ on $\mathcal{B}_1$, the space $L^\infty(\mathbb{L})$ of essentially bounded $\mathbb{R}-$valued measurable functions endowed with the norm given by the essential supremum, and the functional

$$j(x)(t) = \int_0^t x(s)\,ds \tag{12}$$

for $x \in L^\infty(\mathbb{L})$ and $t \in [0,\infty)$. The variable $x \in L^\infty(\mathbb{L})$ may be interpreted as the marginal indemnification of the contract $j(x)$. For instance, for a stop-loss reinsurance with the deductible $d > 0$, one can take

$$x = x(t) = \begin{cases} 0, & t < d \\ 1, & t > d \end{cases} \tag{13}$$

and (12) leads to

$$j(x) = \begin{cases} 0, & t \leq d \\ t-d, & t \geq d \end{cases}$$

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[11] The absolute deviation, the standard deviation, the down-side standard semi-deviation, etc.

[12] Under quite general conditions, Balbás *et al.* (2009) did not impose $c(Y)$ and $r(Y)$ to be co-monotone and nevertheless they proved that a stop-loss contract solves the optimal reinsurance problem.

that is, this contract is perfectly identified by means of its marginal indemnification (13).

Remark 1 *Let us consider* $x \in L^{\infty}(\mathbb{L})$ *as the decision variable in the optimal reinsurance problem. Since the insurer global risk becomes the identity map on* $[0, \infty)$*, that is, the insurer global risk equals* $j(1)$*, one can consider that* $j(x)$ *is the ceded risk and* $j(1-x)$ *is the retained one, and* $Y = c(Y) + r(Y)$ *is obvious because* $j(1) = j(x) + j(1-x)$ *is obvious too. The reinsurer can fix* $H \in L^{\infty}(\mathbb{L})$ *with* $0 \leq H \leq 1$ *as the maximal marginal indemnity that it will accept paying. For instance, if* $H = 0.5$*, then the reinsurer will never pay more than* 50% *per claim. If there is a threshold* $d > 0$ *such that*

$$H = H(t) = \begin{cases} 1, & t < d \\ 0.8, & t > d \end{cases}$$

then the reinsurer accepts paying up to 100% *per claim if the global indemnity is lower than d, but it will never pay more than* 80% *per claim once the threshold is reached. Consequently, once the reinsurer selects* H*, the insurer will have to impose the constraint*

$$0 \leq x \leq H \tag{14}$$

in the optimal reinsurance problem. Notice that (14) and $H \leq 1$ *also guarantee that* $j(1)$*,* $j(1-x)$ *and* $j(x)$ *are co-monotone, since they are (non necessarily strictly) increasing functions of* $t \in [0, \infty)$*.*[13] □

In order to study the properties of j, recall that $L^{\infty}(\mathbb{L})$ is the dual space of $L^{1}(\mathbb{L})$, and $L^{1}(\mathbb{L})$ is contained in the dual of $L^{\infty}(\mathbb{L})$ (Zeidler, 1995). The usual bilinear product of $f \in L^{\infty}(\mathbb{L})$ and an element g in the dual $(L^{\infty}(\mathbb{L}))'$ of $L^{\infty}(\mathbb{L})$ will be denoted by $\langle f, g \rangle$.[14] If $g \in L^{1}(\mathbb{L})$, then

$$\langle f, g \rangle = \int_{0}^{\infty} f(s)\, g(s)\, ds. \tag{15}$$

Proposition 2 below is presented without proof because it can be found in Balbás *et al.* (2019).

Proposition 2 $j(x) \in L^{2}(\mathbb{P}_Y)$ *for every* $x \in L^{\infty}(\mathbb{L})$*, and* $j : L^{\infty}(\mathbb{L}) \rightarrow L^{2}(\mathbb{P}_Y)$ *is linear and continuous when* $L^{\infty}(\mathbb{L})$ *and* $L^{2}(\mathbb{P}_Y)$ *are endowed with both the norm topologies and the weak*$^{*}-$*topologies. The adjoint operator* j^{*} $:L^{2}(\mathbb{P}_Y) \rightarrow (L^{\infty}(\mathbb{L}))'$ *is* $L^{1}(\mathbb{L})-$*valued,*[15] *and* $j^{*} : L^{2}(\mathbb{P}_Y) \rightarrow L^{1}(\mathbb{L})$ *is linear*

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[13] Notice that the reinsurer can also force a strictly increasing retained risk by choosing $h \in (0, 1)$ and $H \leq h$.

In general, if H equals 1 in an interval $I \subset \mathbb{R}$, then the moral hazard could be not totally eliminated, since the incentives of the insurer to prevent frauds and incur in the associated cost might vanish within this interval. This is clearer if $I = (d, \infty)$ for some $d \in \mathbb{R}$, *i.e.*, for the stop-loss reinsurance. Nevertheless, we will impose the condition $H \leq 1$ rather than $H < 1$, since this is a less restricted scenario with respect to the reinsurer choice about H.

[14] Similar notations will be used in similar situations.

[15] Recall that j^{*} is characterized by $\mathbb{E}(j(f)G) = \langle f, j^{*}(G) \rangle$ for every $f \in L^{\infty}(\mathbb{L})$ and every $G \in L^{2}(\mathbb{P}_Y)$ (Zeidler, 1995).

and continuous for both the norm and the weak−topologies. Lastly, j^* *is given by*

$$j^*(Z)(t) = \int_{[t,\infty)} Z(s)\,\mathbb{P}_Y(ds)$$

for every $Z \in L^2(\mathbb{P}_Y)$ *and every* $t \in [0,\infty)$. $\square$

As indicated in Section 2, given $j(x) \in L^2(\mathbb{P}_Y)$ one can easily extend this random variable to a new one $I_Y(j(x)) \in L^2(\mathbb{P})$. Then, one has:

Proposition 3 *Consider* $J(x) = I_Y(j(x))$ *for every* $x \in L^\infty(\mathbb{L})$. *Then,* $J : L^\infty(\mathbb{L}) \to L^2(\mathbb{P})$ *is linear and continuous when* $L^\infty(\mathbb{L})$ *and* $L^2(\mathbb{P})$ *are endowed with both the norm topologies and the weak***−topologies. The adjoint operator* $J^* : L^2(\mathbb{P}) \to (L^\infty(\mathbb{L}))'$ *is* $L^1(\mathbb{L})$*−valued, and* $J^* : L^2(\mathbb{P}) \to L^1(\mathbb{L})$ *is linear and continuous for both the norm and the weak−topologies. Lastly,* J^* *is given by*

$$J^*(Z)(t) = \int_{R_t} Z(s)\,\mathbb{P}(ds)$$

for every $Z \in L^2(\mathbb{P})$ *and every* $t \in [0,\infty)$*, where* $R_t = [t,\infty) \times [0,\infty)$. $\square$

3.2 Reinsurer risk diversification

As said in the second section, the reinsurer may deal with its portfolio $\mathcal{Y}$ in order to diversify (or hedge) the ceded risk $j(x)$. If agreement with the insurer is reached, then the reinsurer indemnification $\mathcal{Y}$ may be replaced by $\mathcal{Y} + J(x) - \Pi(j(x))$. Obviously, the reinsurer expected profit will increase, that is,

$$\Pi(j(x)) - \mathbb{E}(\mathcal{Y} + J(x)) \geq -\mathbb{E}(\mathcal{Y}), \tag{16}$$

since (10) leads to

$$\left\{ \begin{array}{l} \Pi(j(x)) - \mathbb{E}(J(x)) = \Pi(j(x)) + \mathbb{E}(-j(x)) \geq \\ (1+k)\,\mathbb{E}(j(x)) - \mathbb{E}(j(x)) = k\mathbb{E}(j(x)) \geq 0, \end{array} \right. \tag{17}$$

and the inequality becomes strict unless $x = 0$ (absence of contract).[16] Nevertheless, the new risk $\Gamma(\Pi(j(x)) - J(x) - \mathcal{Y})$ may be higher than the initial one $\Gamma(-\mathcal{Y})$. It is obvious that the contract will become much more attractive for the reinsurer if $\mathcal{Y}$ really diversifies the ceded risk $j(x)$, that is, if

$$\Gamma(\Pi(j(x)) - J(x) - \mathcal{Y}) \leq \Gamma(-\mathcal{Y}) \tag{18}$$

holds.[17] Since Γ is translation-invariant one has that $\Gamma(\Pi(j(x)) - J(x) - \mathcal{Y}) = \Gamma(-J(x) - \mathcal{Y}) - \Pi(j(x))$, so (18) is equivalent to $\Gamma(-J(x) - \mathcal{Y}) - \Gamma(-\mathcal{Y}) \leq$

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[16] Notice that similar expressions to (16) and (17) still hold if the reinsurer sells multiple reinsurance contracts at the same time, that is, the reinsurer's expected profit does not decrease even if multiple contracts are sold for multiple insurers. The reason is the fulfillment of (10).

[17] Alternatively, one can assume that the increment of the expected profit will compensate the increment of risk, that is, $\Gamma(\Pi(j(x)) - J(x) - \mathcal{Y}) - \Gamma(-\mathcal{Y}) \leq \mathbb{E}(\Pi(j(x)) - j(x))$. Nevertheless, we will deal with the stronger restriction (18) because it seems to be more realistic from a practical viewpoint.

$\Pi(j(x))$. Besides, (14) leads to the chain of implications $x \geq 0 \Longrightarrow J(x) \geq 0 \Longrightarrow -J(x) - \mathcal{Y} \leq -\mathcal{Y}$, and therefore $\Gamma(-J(x) - \mathcal{Y}) \geq \Gamma(-\mathcal{Y})$ because Γ is decreasing. In other words, (18) holds if and only if

$$0 \leq \Gamma(-J(x) - \mathcal{Y}) - \Gamma(-\mathcal{Y}) \leq \Pi(j(x)), \tag{19}$$

holds, that is, the additional received risk is compensated by the received premium. To sum up, (16) implies that the reinsurer improves the expected profit, so the contract would become very attractive if it made the global risk decrease, that is, if (18) (or (19)) held.

3.3 Balance between price and ceded risk

(16) (or (17)) implies that the insurer expected profit will decrease if the contract is signed, since the premium to pay is higher than the expected income (the expected indemnification that the reinsurer will pay). Nevertheless, the insurer may be interested in the contract if the ceded risk is higher than the amount to be paid, *i.e.*,

$$\Pi(j(x)) \leq \gamma(-j(x)). \tag{20}$$

Indeed, (20) and the co-monotone additivity of γ lead to

$$\left\{ \begin{array}{l} \gamma(-j(1-x) - \Pi(j(x))) = \gamma(-j(1-x)) + \Pi(j(x)) = \\ \gamma(-j(1)) - \gamma(-j(x)) + \Pi(j(x)) \leq \gamma(-j(1)), \end{array} \right. \tag{21}$$

i.e., the insurer risk after signing the contract is lower than its initial risk even if the contract price is involved. For instance, if $\gamma(-j(1))$ is interpreted as an initial capital requirement (or initial reserve) making the global risk vanish, and $\gamma(-j(1-x) - \Pi(j(x)))$ is the capital requirement after signing the contract, (21) shows that (20) implies a reserve mitigation, or, alternatively, the insurer may reduce the price of the policies that are sold.[18]

Remark 4 *To sum up the ideas of Section* 3*, (14) protects the reinsurer against the moral hazard, (18) (or (19)) implies that the reinsurer global risk decreases despite the fact that its expected profit increases, and (20) implies that the insurer risk decreases in such a manner that its policies may become cheaper or its reserve may become lower (or both). Furthermore, taking* $x = 0$*, it becomes evident that the simultaneous fulfillment of (14), (18) and (20) is feasible. Nevertheless, it is interesting to show this feasibility with a strict inequality in (20). Indeed, If (18) and (20) become equalities, then the reinsurer may be still interested, since its expected profit increases and its global risk remains the same, but the insurer should not be satisfied, because the expected profit decreases and the needed reserve remains the same.*

In general, the strict inequality in (20) cannot be guaranteed, and it is easy to give a counter-example. For instance, suppose that $\mathcal{Y} = \lambda Y$ *with* $\lambda > 0$*,*

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[18] In some sense, this idea is related to that of Zhuang *et al.* (2017), who proposed that the presence of reinsurance affects the insurance strategies.

$\Gamma = \gamma$,[19] *and* $\Pi(U) = (1+k)\,\mathbb{E}(U)$ *for every* $U \in L^2(\mathbb{P}_Y)$. *Then,* Y, $J(x)$ *and* $\mathcal{Y}$ *are co-monotone for every* x *satisfying (14). Hence, (18) and (20), along with the co-monotone additivity of* $\Gamma = \gamma$, *lead to*

$$\begin{array}{l}\Gamma(-\mathcal{Y}) \geq \Gamma(\Pi(j(x)) - J(x) - \mathcal{Y}) = \Gamma(-J(x) - \mathcal{Y}) - \Pi(j(x)) = \\ \Gamma(-J(x)) + \Gamma(-\mathcal{Y}) - \Pi(j(x)) \geq \Pi(j(x)) + \Gamma(-\mathcal{Y}) - \Pi(j(x)) = \Gamma(-\mathcal{Y}),\end{array}$$

and therefore all the inequalities in the chain become equalities. In particular, $\Gamma(-\mathcal{Y}) = \Gamma(\Pi(j(x)) - J(x) - \mathcal{Y})$, *and* $\Gamma(-J(x)) + \Gamma(-\mathcal{Y}) - \Pi(j(x)) = \Pi(j(x)) + \Gamma(-\mathcal{Y}) - \Pi(j(x)) \Longrightarrow \gamma(-j(x)) = \Pi(j(x))$, *that is, both (18) and (20) become equalities.* □

In order to overcome the caveat pointed out in the counter-example of Remark 4, in future sections we will need some more results guaranteeing the fulfillment of (18) or (20).

4 Optimal reinsurance problem

The optimal reinsurance problem may become very complex if (18) and (20) are involved. Moreover, the incorporation of (18) implicitly implies that the insurer has some knowledge about the reinsurer risk, which could be unrealistic. Consequently, we are going to present a more classical approach, and the fulfillment of both constraints will be addressed in Sections 5.1 and 5.2, respectively.

Let us assume that the insurer is interested in both the risk minimization and the expected wealth maximization. Since the problem involves the optimization of two convex objectives, the Pareto-efficient solutions may be computed by the scalarization method (Nakayama *et al.*, 1985). Accordingly, fix $H \in L^\infty(\mathbb{L})$ with $0 < h \leq H \leq 1$ (selected by the reinsurer),[20] and fix $\alpha \geq 0$ (selected by the insurer).[21] The insurer objective becomes

$$\begin{array}{l}\gamma(-j(1-x) - \Pi(j(x))) - \alpha\mathbb{E}((-j(1-x) - \Pi(j(x)))) = \\ \gamma(-j(1-x)) + \alpha\mathbb{E}(j(1-x)) + (1+\alpha)\,\Pi(j(x)),\end{array}$$

and the insurer problem becomes

$$\left\{\begin{array}{l} Min\ \gamma(-j(1-x)) + \alpha\mathbb{E}(j(1-x)) + (1+\alpha)\,\Pi(j(x)) \\ 0 \leq x \leq H \end{array}\right. \tag{22}$$

$x \in L^\infty(\mathbb{L})$ being the decision variable.[22]

Problem (22) may be simplified because $\gamma(-j(1-x))$ may equal a linear expression.

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[19] *I.e.*, Γ is co-monotone additive and $\Gamma(I_Y(U)) = \gamma(U)$ for every $U \in L^2(\mathbb{P}_Y)$.

[20] If $h \in \mathbb{R}$ did not not exist, then H could be replaced by $H_n(\omega) = Max\{H(\omega), 1/n\}$, $\omega \in [0,\infty)$ and $n \in \mathbb{N}$. Then, (22) could be analyzed as a limit case of a sequence of problems satisfying the required conditions. Nevertheless, we will not address this topic in order to shorten the exposition.

[21] α indicates the "relative importance" assigned to the second objective (wealth) with respect to the first one (risk). The higher the α, the higher the importance.

[22] It is worth noting the existence of several analogies between this optimization problem and the non restricted one presented by Cheung and Lo (2017) in their "cost-benefit approach".

Lemma 5 *a) There exists* $\tilde{Z}_\gamma \in \partial_\gamma$ *such that* $\gamma\left(-j\left(1\right)\right) = \mathbb{E}\left(j\left(1\right)\tilde{Z}_\gamma\right)$ *holds. Furthermore,* $\gamma\left(-j\left(x\right)\right) = \mathbb{E}\left(j\left(x\right)\tilde{Z}_\gamma\right)$ *and* $\gamma\left(-j\left(1-x\right)\right) = \mathbb{E}\left(j\left(1-x\right)\tilde{Z}_\gamma\right)$ *hold for every* $x \in L^\infty\left(\mathbb{L}\right)$ *such that* $0 \le x \le 1$.

b) Suppose that (11) holds and ρ *is co-monotone additive. There exists* $\tilde{Z}_\Pi \in \partial_\Pi$ *such that* $\Pi\left(-j\left(1\right)\right) = \mathbb{E}\left(j\left(1\right)\tilde{Z}_\Pi\right)$ *holds. Moreover,* $\Pi\left(-j\left(x\right)\right) = \mathbb{E}\left(j\left(x\right)\tilde{Z}_\Pi\right)$ *and* $\Pi\left(-j\left(1-x\right)\right) = \mathbb{E}\left(j\left(1-x\right)\tilde{Z}_\Pi\right)$ *hold for every* $x \in L^\infty\left(\mathbb{L}\right)$ *such that* $0 \le x \le 1$. □

Remark 6 *Henceforth, let us fix* $\tilde{Z}_\gamma \in \partial_\gamma$ *such that* $\gamma\left(-j\left(1\right)\right) = \mathbb{E}\left(j\left(1\right)\tilde{Z}_\gamma\right)$. *Obviously, Lemma* 5*a) implies that (22) is equivalent to*

$$\left\{\begin{array}{l} Min\ \mathbb{E}\left(j\left(1-x\right)\tilde{Z}_\gamma\right) + \alpha\mathbb{E}\left(\left(j\left(1-x\right)\right)\right) + \left(1+\alpha\right)\Pi\left(j\left(x\right)\right) \\ 0 \le x \le H \end{array}\right. \tag{23}$$

$x \in L^\infty\left(\mathbb{L}\right)$ *being the decision variable.*

If Π *is given by (11) and* ρ *is co-monotone additive, then Lemma* 5*b) enables us to fix* $\tilde{Z}_\Pi \in \partial_\Pi$ *such that* $\Pi\left(-j\left(1\right)\right) = \mathbb{E}\left(j\left(1\right)\tilde{Z}_\Pi\right)$. *Consequently, (22) and (23) become equivalent to*

$$\left\{\begin{array}{l} Min\ \mathbb{E}\left(j\left(1-x\right)\tilde{Z}_\gamma\right) + \alpha\mathbb{E}\left(j\left(1-x\right)\right) + (1+\alpha)\mathbb{E}\left(j\left(x\right)\tilde{Z}_\Pi\right) \\ 0 \le x \le H \end{array}\right. \tag{24}$$

which is a linear problem. A very important particular case arises under the EVPP. □

Theorem 7 below will be given without proof because a very similar result may be found in Balbás *et al.* (2015). Nevertheless, it is interesting to present this theorem because it will allow us to prove important new results. In particular, Corollary 8 will provide us with new methods to solve (23), Corollaries 9 and 10 will present a explicit solution of (23), and Theorem 13 will guarantee the fulfillment of the insurer requirement (20).

Theorem 7 *a) (23) is bounded and solvable, that is, this problem attains a finite optimal value.*

b) The dual problem of (23) becomes

$$\left\{\begin{array}{l} Max\ \mathbb{E}\left(j\left(1\right)\left(\tilde{Z}_\gamma + \alpha\right)\right) - \langle H, m_H\rangle \\ J^*\left(\left(1+\alpha\right)Z_\Pi - \tilde{Z}_\gamma - \alpha\right) - m_0 + m_H = 0 \\ m_0 \ge 0,\ m_H \ge 0,\ Z_\Pi \in \partial_\Pi \end{array}\right. \tag{25}$$

$\left(Z_\Pi, m_0, m_H\right) \in L^2\left(\mathbb{P}_Y\right) \times L^1\left(\mathbb{L}\right) \times L^1\left(\mathbb{L}\right)$ *being the decision variable. (25) is solvable and its optimal value equals the optimal value of (23).*

c) Suppose that $\tilde{x}$ is (23)-feasible and $\left(\tilde{Z}_{\Pi}, m_0, m_1\right)$ is (25)-feasible. They solve the corresponding problem if and only if

$$\left\{ \begin{array}{l} \mathbb{E}\left(j\left(\tilde{x}\right)\tilde{Z}_{\Pi}\right) \geq \mathbb{E}\left(j\left(\tilde{x}\right)Z_{\Pi}\right), \quad \forall Z_{\Pi} \in \partial_{\Pi} \\ m_0\tilde{x} = m_H\left(H - \tilde{x}\right) = 0 \end{array} \right. \tag{26}$$

d) If $\tilde{x}$ solves (23), then $\tilde{x}$ solves the linear problem

$$\left\{ \begin{array}{l} Min\ \mathbb{E}\left(\tilde{Z}_{\gamma} j\left(1-x\right)\right) + \alpha\mathbb{E}\left(j\left(1-x\right)\right) + \left(1+\alpha\right)\mathbb{E}\left(\tilde{Z}_{\Pi} j\left(x\right)\right) \\ 0 \leq x \leq H \end{array} \right. \tag{27}$$

for every $\left(\tilde{Z}_{\Pi}, m_0, m_H\right)$ solving the linear problem (25), $x \in L^{\infty}\left(\mathbb{L}\right)$ being the decision variable.[23] *Furthermore, (23) and (27) have the same achievable optimal value.* □

Corollary 8 *Suppose that $\left(\tilde{Z}_{\Pi},, m_0, m_H\right)$ solves (25). Then,*

$$\left\{ \begin{array}{l} m_0 = J^*\left(\left(1+\alpha\right)\tilde{Z}_{\Pi} - \tilde{Z}_{\gamma} - \alpha\right)^{+} \\ m_H = J^*\left(\left(1+\alpha\right)\tilde{Z}_{\Pi} - \tilde{Z}_{\gamma} - \alpha\right)^{-} \end{array} \right. \tag{28}$$

□

Corollary 9 *Suppose that $\left(\tilde{Z}_{\Pi},, m_0, m_H\right)$ solves (25), $\tilde{x}$ solves (23), and $\tilde{x}$ is the unique solution of (27). Then, (28) holds and $\tilde{x} = H\chi_{m_H>0}$, where $\chi_{m_H>0}$ is the classical indicator*

$$\chi_{m_H>0}\left(t\right) = \left\{ \begin{array}{ll} 1, & if\ m_H\left(t\right) > 0 \\ 0, & otherwise \end{array} \right.$$

□

Corollary 10 *Suppose that (11) holds with ρ co-monotone additive. Take the element $\tilde{Z}_{\Pi} \in \partial_{\Pi}$ of Lemma 5b) and Problem (24). Take m_0 and m_H as in (28). Then, $\left(\tilde{Z}_{\Pi}, m_0, m_H\right)$ solves (25), and (24) (or (22) and (23)) is solved by $\tilde{x} = H\chi_{m_H>0}$. In particular, all of these properties remain true if Π is given by the $EVPP$.* □

Remark 11 *There are very efficient algorithms to solve finite- and infinite-dimensional linear optimization problems (Anderson and Nash, 1987), so they may be useful to solve (25). Moreover, only the first component $\tilde{Z}_{\Pi}$ of the dual solution is required, since (28) provides us with the rest of components. Besides, Corollaries 9 and 10 will frequently apply, since they only add weak*

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[23] Notice the analogy between (24) and (27).

assumptions. Hence, once $\tilde{Z}_\Pi$ is known, (28) gives m_H, and $\tilde{x} = H\chi_{m_H>0}$ is the solution of the optimal reinsurance problem. $H\chi_{m_H>0}$ is the marginal indemnification, while the global indemnification becomes $j\left(H\chi_{m_H>0}\right)$, which depends on the realized value of Y and becomes

$$j\left(H\chi_{m_H>0}\right)(Y) = \int_0^Y H(t)\, m_H(t)\, dt.$$

For instance, if $Y = 1$, that is, the realized cost for the insurer equals one, then $\int_0^1 H(t)\, m_H(t)\, dt$ will be the amount to be paid by the reinsurer. Section 6 will yield an illustrative example. □

5 On the contract premium spread

5.1 Verifying the insurer additional requirement

Let us fix a solution $\tilde{x}$ of (23). Fix also a dual solution $\left(\tilde{Z}_\Pi, m_0, m_H\right)$, and let us show that the insurer requirements will be always satisfied, that is, the paid premium is never higher than the ceded risk.

Lemma 12 *Inequality*

$$\mathbb{E}\left(j(\tilde{x})\, Z_\Pi\right) \leq \frac{1}{1+\alpha}\left(\mathbb{E}\left(J(\tilde{x})\left(\tilde{Z}_\gamma + \alpha\right)\right) - \langle H, m_H\rangle\right) \tag{29}$$

holds for every $Z_\Pi \in \partial_\Pi$, with equality if $Z_\Pi = \tilde{Z}_\Pi$. □

Theorem 13 *$\tilde{x}$ satisfies (20). Moreover, (20) holds as a strict inequality if at least one of the assertions below holds:*
i) $m_H \neq 0$.
ii) $\gamma(-j(\tilde{x})) > \mathbb{E}(j(\tilde{x}))$ and $\alpha > 0$. □

Remark 14 *Condition ii) in Theorem 13 is not at all restrictive. On the one hand, $\alpha > 0$ only means that the insurer is not indifferent to its expected wealth. On the other hand, for the most important expectation bounded risk measures (CV@R, WCV@R, etc.), $\gamma(-j(\tilde{x})) > \mathbb{E}(j(\tilde{x}))$ will hold unless $j(\tilde{x})$ has a null variance (or becomes constant). Nevertheless, (12) and $\tilde{x} \geq 0$ show that a null variance of $j(\tilde{x})$ implies that $\tilde{x} = 0$, that is the whole risk is retained. To sum up, ii) will hold for the most important risk measures if the insurer is not indifferent to its expected wealth and the optimal ceded risk does not vanish. Consequently, the inclusion of the expected wealth in the insurer objective seems to be an adequate decision.* □

5.2 On the reinsurer requirement fulfillment

Problems (22) and (23) do not involve the reinsurer risk $\mathcal{Y}$, so their solution $\tilde{x}$ will not depend on $\mathcal{Y}$ either. As already said, it may be realistic to assume that the insurer does not have enough knowledge about $\mathcal{Y}$, and therefore it will deal with a reinsurance problem only involving the available information. Consequently, the question "can $\mathcal{Y}$ diversify the received extra-risk $J(\tilde{x})$?" can only be answered after solving Problem (23), that is, once $J(\tilde{x})$ is known. This idea will inspire this section. First, it will be shown that (18) (or (19)) may fail, and then Propositions 20, 21 and 22 will yield sufficient and necessary and sufficient conditions guaranteeing the fulfillment of (18) and (19).

Throughout this section let us fix a solution $\tilde{x}$ of (23) and a dual solution $\left(\tilde{Z}_\Pi, m_0, m_H\right)$ of (25). In general, (18) may be false, as pointed out in Example 15 below.

Example 15 *Suppose that $\tilde{x}$ solves (23) and satisfies (20) as a strict inequality (for instance, the conditions of Theorem* 13 *hold). Suppose also that $\Gamma = \gamma$ and $\mathcal{Y} = 0$. Then, (18) becomes $\gamma\left(\Pi\left(j\left(\tilde{x}\right)\right) - J\left(\tilde{x}\right)\right) \leq 0$, that is, $\gamma\left(-j\left(\tilde{x}\right)\right) \leq \Pi\left(j\left(\tilde{x}\right)\right)$, while (20) implies that $\Pi\left(j\left(\tilde{x}\right)\right) < \gamma\left(-j\left(x\right)\right)$. One has an obvious contradiction.* ☐

Remark 16 *Example* 15 *shows the very important role of the reinsurer risk $\mathcal{Y}$. If it is ignored or, equivalently, it is taken as $\mathcal{Y} = 0$, then it is not possible to find "a reasonable harmony" between the insurer purposes and the reinsurer ones.* ☐

Example 17 *Consider again Example* 15*, but replace $\mathcal{Y} = 0$ with $\mathcal{Y} = M$, with $M \in \mathbb{R}$ strictly positive and "as large as desired". Since $\gamma\left(-\mathcal{Y}\right) = M$ and $\gamma\left(\Pi\left(j\left(\tilde{x}\right)\right) - J\left(\tilde{x}\right) - \mathcal{Y}\right) = \gamma\left(-J\left(\tilde{x}\right)\right) - \Pi\left(j\left(\tilde{x}\right)\right) + M$, (18) is still equivalent to $\gamma\left(-j\left(\tilde{x}\right)\right) \leq \Pi\left(j\left(\tilde{x}\right)\right)$, and the caveat of Example* 15 *still applies. In other words, "a very large expected value of $\mathcal{Y}$" does not overcome the problem.* ☐

Example 18 *Suppose in this example that Γ is law-invariant. Then, it is known that $\Gamma\left(-Z\right) = \mathbb{E}\left(Z\right) + \Gamma_0 \sigma\left(Z\right)$ holds for every normal distribution $Z \sim \mathcal{N}\left(\mathbb{E}\left(Z\right), \sigma\left(Z\right)\right)$ with expectation $\mathbb{E}\left(Z\right)$ and standard deviation $\sigma\left(Z\right)$, Γ_0 being the risk of a $\mathcal{N}\left(0,1\right)$ random variable. Suppose that $\Gamma_0 > 0$. Consider two independent normal distributions Z_1 and Z_2 such that $\mathbb{E}\left(Z_2\right) > 0$, and let us analyze whether*

$$\Gamma\left(-Z_1 - Z_2\right) - \Gamma\left(-Z_1\right) \leq \left(1+k\right)\mathbb{E}\left(Z_2\right) \tag{30}$$

holds, where $k > 0$ (see (10)).[24] *The inequality is equivalent to*

$$\mathbb{E}\left(Z_1 + Z_2\right) + \Gamma_0 \sigma\left(Z_1 + Z_2\right) - \mathbb{E}\left(Z_1\right) - \Gamma_0 \sigma\left(Z_1\right) \leq \left(1+k\right)\mathbb{E}\left(Z_2\right).$$

Since Z_1 and Z_2 are independent, one has

$$\Gamma_0\left(\sqrt{\sigma\left(Z_1\right)^2 + \sigma\left(Z_2\right)^2} - \sigma\left(Z_1\right)\right) \leq k\mathbb{E}\left(Z_2\right).$$

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[24] Notice that (30) is analogous to (19), though $\mathcal{Y}$ and $J\left(\tilde{x}\right)$ are not normal distributions (they cannot achieve negative values).

Multiplying numerator and denominator by $\sqrt{\sigma(Z_1)^2+\sigma(Z_2)^2}+\sigma(Z_1)$, *one has*

$$\Gamma_0\frac{\sigma(Z_2)^2}{\sqrt{\sigma(Z_1)^2+\sigma(Z_2)^2}+\sigma(Z_1)}\leq k\mathbb{E}(Z_2).$$

Evidently, the latter expression holds if $\sigma(Z_1)\to\infty$, *that is, (30) is guaranteed if* $\sigma(Z_1)$ *is "large enough".* $\square$

Remark 19 *Examples* 17 *and* 18, *along with the intuition, suggest that the fulfillment of (18) (or (19)) may be very closely related to the variance of* $\mathcal{Y}$. *A large variance, along with a low correlation between* Y *and* $\mathcal{Y}$, *will facilitate this fulfillment, while a "tiny variance" will make it quite difficult to reduce the initial risk of the reinsurer. Nevertheless, if one draws on the variance of* $\mathcal{Y}$, *then the analysis will become a little bit more restrictive (see Footnote* 4*), so we will prefer to deal with both the norm of* $\mathcal{Y}$,[25] *and the geometric properties of the continuous convex function* Γ. *In general, a convex function is never lower than its linear approximations (or sub-gradients). Actually, if one generalizes our framework, Propositions* 20 *and* 21 *below may be easily extended from* $L^2(\mathbb{P})$ *to general Banach spaces of random variables, while Proposition* 22 *may be extended to general Banach lattices (Meyer-Nieburg, 1991).* $\square$

Proposition 20 *Consider* $\tilde{Z}\in\partial_\Gamma$ *with* $\Gamma(-J(\tilde{x})-\mathcal{Y})=\mathbb{E}\left((J(\tilde{x})+\mathcal{Y})\tilde{Z}\right)$.[26]

a) (19) holds (respectively, holds as a strict inequality) if and only if

$$\mathbb{E}\left((J(\tilde{x})+\mathcal{Y})\tilde{Z}\right)\leq\Gamma(-\mathcal{Y})+\mathbb{E}\left(J(\tilde{x})\tilde{Z}_\Pi\right)\tag{31}$$

holds (respectively, (31) holds as a strict inequality).

b) If

$$\mathbb{E}\left(J(\tilde{x})\tilde{Z}\right)\leq\mathbb{E}\left(J(\tilde{x})\tilde{Z}_\Pi\right)\tag{32}$$

then $J(\tilde{x})$ *satisfies (18) and (19). Furthermore, if (32) holds as strict inequality, then so do (18) and (19).*

c) If $\tilde{Z}\leq I_Y\left(\tilde{Z}_\Pi\right)$, *then (18) and (19) hold.*

d) (32) holds (respectively, holds as a strict inequality) if

$$\mathbb{E}\left(J(\tilde{x})\tilde{Z}\right)\leq(1+k)\,\mathbb{E}(J(\tilde{x}))\tag{33}$$

holds (respectively, (33) holds as a strict inequality). $\square$

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[25] Recall that the variance σ^2 and the norm $\|.\|$ are closely related in a L^2-space, since $\sigma(Z)=\|Z-\mathbb{E}(Z)\|$.

[26] Recall that such a $\tilde{Z}$ does exist. It is worth emphasizing that $\tilde{Z}$ often relies heavily on $J(\tilde{x})+\mathcal{Y}$.

Proposition 21 *Suppose that* $\Pi(j(\tilde{x})) > 0$.[27] *Consider a family*

$$\left\{R^{(v}; v \geq 0\right\} \subset \mathcal{B}_2$$

such that $R^{(0} = [0,\infty)^2$, $\mathbb{P}\left(R^{(v}\right) > 0$ *for every* $v \geq 0$, $R^{(v} \supset R^{(w}$ *if* $v \leq w$, *and*

$$\bigcap_{n=1}^{\infty} R^{(n} = \varnothing. \tag{34}$$

There exists $v_0 > 0$ *such that (32) holds for every* $Z \in \partial_\Gamma$ *such that* $Z(t,s) = 0$ *if* $(t,s) \notin R^{(v_0}$. *In particular, if* $\tilde{Z} \in \partial_\Gamma$ *satisfies;*

i) $\Gamma(-J(\tilde{x}) - \mathcal{Y}) = \mathbb{E}\left((J(\tilde{x}) + \mathcal{Y})\tilde{Z}\right)$.

ii) $\tilde{Z}(t,s) = 0$ *if* $(t,s) \notin R^{(v_0}$.

Then, (18) and (19) hold. □

Proposition 22 *Suppose that* $\mathcal{Y} \neq 0$ *and* $\Pi(j(\tilde{x})) > 0$. *Consider* D *such that* $\|\mathcal{Y}\| < D$. *Then, there exists* $d > 0$ *such that*

$$\Gamma(Z) \geq \Gamma\left(\frac{-J(\tilde{x}) - \mathcal{Y}}{\|\mathcal{Y}\|}\right) - \frac{\Pi(j(\tilde{x}))}{D} \tag{35}$$

if $\|Z - (-J(\tilde{x}) - \mathcal{Y})/\|\mathcal{Y}\|\| \leq d$. *Furthermore, if* $\|Y\| \leq d\|\mathcal{Y}\|$, *then (18) and (19) hold.*[28] □

6 Involving the CV@R

6.1 Main ingredients

Since the $CV@R$ has been often used in optimal reinsurance problems, let us particularize the results of previous sections to this important particular case. First of all, recall that for the $CV@R_{1-\mu}$ with a confidence level $1-\mu \in (0,1)$ one has that $\partial_{CV@R_{1-\mu}}$ is composed of those $Z \in L^\infty(\mathbb{P}) \subset L^2(\mathbb{P})$ such that (6) holds and

$$0 \leq Z \leq 1/\mu \tag{36}$$

(Rockafellar *et al.*, 2006). Consider $\left\{\mu_\gamma, \mu_\Gamma, \mu_\rho\right\} \subset (0,1)$, $k_1 > 0$, $k_2 \geq 0$, $\gamma = CV@R_{1-\mu_\gamma}$, $\Gamma = CV@R_{1-\mu_\Gamma}$, $\rho = CV@R_{1-\mu_\rho}$ and Π given by (11). According to Remark 6, the optimal reinsurance problem becomes linear and is given by (24). According to Lemma 5, Remark 6 and Corollary 10, the primal and dual solutions may be easily found if one knows the element $\tilde{Z}_\Pi \in \partial_\Pi$ such

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[27] Usually, Π is strictly increasing, and therefore $\Pi(j(\tilde{x})) = 0$ is equivalent to $j(\tilde{x}) = 0$, that is, absence of agreement between insurer and reinsurer.

[28] It is known that (4) (for Γ rather than γ) implies the lower semi-continuity of Γ when $L^2(\mathbb{P})$ is endowed with the weak topology. Thus, the existence of d may be replaced by the weaker existence of an appropriate weak neighborhood of 0. (19) will hold if $Y/\|\mathcal{Y}\|$ belongs to this neighborhood.

that $\rho\left(-\tilde{Z}_\Pi y\right) = \mathbb{E}\left(\tilde{Z}_\Pi y\right)$. In order to simplify the exposition, suppose that $\mathbb{P}$ is atomless and is given by a density function which is strictly positive on $(0,\infty)^2$. Then, it is easy to see the existence of $\{t_\gamma, s_\Gamma, t_\rho\} \subset (0,\infty)$ such that

$$\begin{cases} \mathbb{P}_y\left(t_\gamma,\infty\right) = \mu_\gamma \\ \mathbb{P}_y\left(t_\rho,\infty\right) = \mu_\rho \\ \mathbb{P}_Y\left(s_\Gamma,\infty\right) = \mu_\Gamma \end{cases} \tag{37}$$

Consequently, it is also easy to see that

$$\begin{cases} \tilde{Z}_\gamma = \left(1/\mu_\gamma\right)\chi_{(t_\gamma,\infty)\times(0,\infty)} \\ \tilde{Z}_\rho = \left(1/\mu_\rho\right)\chi_{(t_\rho,\infty)} \\ \tilde{Z}_\Pi = 1 + k_1 + k_2\tilde{z}_\rho \\ \tilde{Z}_\Gamma = \left(1/\mu_\Gamma\right)\chi_{(0,\infty)\times(s_\Gamma,\infty)} \end{cases} \tag{38}$$

are elements of ∂_γ, ∂_ρ, ∂_Π and ∂_Γ such that

$$\begin{cases} \gamma\left(-J\left(1\right)\right) = \mathbb{E}\left(J\left(1\right)\tilde{Z}_\gamma\right) \\ \rho\left(-j\left(1\right)\right) = \mathbb{E}\left(j\left(1\right)\tilde{Z}_\rho\right) \\ \Pi\left(-j\left(1\right)\right) = \mathbb{E}\left(j\left(1\right)\tilde{Z}_\Pi\right) \\ \Gamma\left(-\mathcal{Y}\right) = \mathbb{E}\left(\mathcal{Y}\tilde{Z}_\Gamma\right). \end{cases} \tag{39}$$

6.2 The insurer point of view

(38), (39) and Corollary 10 imply that (25) is solved by

$$\begin{cases} \tilde{Z}_\Pi = 1 + k_1 + \left(k_2/\mu_\rho\right)\chi_{(t_\rho,\infty)} \\ m_0 = J^*\left(\left(1+\alpha\right)\tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha\right)^+ \\ m_H = J^*\left(\left(1+\alpha\right)\tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha\right)^- \end{cases} \tag{40}$$

whereas (24) is solved by $\tilde{x} = H\chi_{m_H>0}$. Obviously, $m_H > 0$ if and only if $J^*\left(\left(1+\alpha\right)\tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha\right) < 0$, which is equivalent to

$$J^*\left(\tilde{Z}_\Pi\right) < \frac{J^*\left(\tilde{Z}_\gamma\right)}{1+\alpha} + \frac{\alpha J^*\left(1\right)}{1+\alpha}. \tag{41}$$

Let us suppose that $\alpha > 0$. (37), (38) and Propositions 2 and 3 show that

$$\begin{cases} J^*\left(\tilde{Z}_\gamma\right)(t) = \begin{cases} \left(1/\mu_\gamma\right)\mathbb{P}_y\left(t,\infty\right), & t \geq t_\gamma \\ 1, & t \leq t_\gamma \end{cases} \\ J^*\left(\tilde{Z}_\Pi\right)(t) = \begin{cases} \left(1+k_1+k_2/\mu_\rho\right)\mathbb{P}_y\left(t,\infty\right), & t \geq t_\rho \\ \begin{cases} \left(1+k_1\right)\mathbb{P}_y\left(t,t_\rho\right) + \\ \left(1+k_1+k_2/\mu_\rho\right)\mathbb{P}_y\left(t_\rho,\infty\right), \end{cases} & t \leq t_\rho \end{cases} \\ J^*\left(1\right)(t) = \mathbb{P}_y\left(t,\infty\right) \end{cases} \tag{42}$$

In particular, $J^*(1) \leq J^*\left(\tilde{Z}_\gamma\right)$ and $J^*(1) < J^*\left(\tilde{Z}_\Pi\right)$. Obviously, $1 < 1 + k_1 \leq 1 + k_1 + k_2/\mu_\rho$, so let us verify whether $J^*\left(\tilde{Z}_\Pi\right) < J^*\left(\tilde{Z}_\gamma\right)$ holds. The fulfillment of this inequality critically depends on the relationships among $1+k_1$, $1 + k_1 + k_2/\mu_\rho$ and $1/\mu_\gamma$. Since all the cases are essentially similar from an analytical point of view, let us assume that

$$1 + k_1 + k_2/\mu_\rho < 1/\mu_\gamma \tag{43}$$

holds, and let us leave the rest of cases.[29] (42) shows that

$$\begin{cases} \lim\limits_{t\to 0} J^*\left(\tilde{Z}_\gamma\right)(t) = J^*\left(\tilde{Z}_\gamma\right)(0) = 1, \\ \lim\limits_{t\to 0} J^*\left(\tilde{Z}_\Pi\right)(t) = J^*\left(\tilde{Z}_\Pi\right)(0) \geq 1 + k_1 > 1. \end{cases}$$

Thus, since both $J^*\left(\tilde{Z}_\Pi\right)$ and $J^*\left(\tilde{Z}_\gamma\right)$ are decreasing functions, $J^*\left(\tilde{Z}_\Pi\right)$ is strictly decreasing, and $J^*\left(\tilde{Z}_\Pi\right)(t) < J^*\left(\tilde{Z}_\gamma\right)(t)$ holds for "t large enough", it is easy to see that there is a unique $t_0 \in (0, \infty)$ such that

$$\begin{cases} J^*(1) < J^*\left(\tilde{Z}_\Pi\right)(t) < J^*\left(\tilde{Z}_\gamma\right)(t), & t > t_0 \\ J^*(1) < J^*\left(\tilde{Z}_\Pi\right)(t_0) = J^*\left(\tilde{Z}_\gamma\right)(t_0) & \\ J^*(1) < J^*\left(\tilde{Z}_\gamma\right)(t) < J^*\left(\tilde{Z}_\Pi\right)(t), & t < t_0 \end{cases} \tag{44}$$

which is characterized by

$$J^*\left(\tilde{Z}_\Pi\right)(t_0) = 1. \tag{45}$$

The right hand side of (41) is a weighted average of $J^*(1)$ and $J^*\left(\tilde{Z}_\gamma\right)$, and recall that (41) is equivalent to $m_H > 0$. Hence, (44) implies that $m_H(t) = 0$ for $t \leq t_0$. For $t \geq t_\gamma$ (41) holds if (see (42))

$$\left(\frac{1}{\mu_\gamma(1+\alpha)} + \frac{\alpha}{1+\alpha}\right)\mathbb{P}_y(t,\infty) > \left(1 + k_1 + k_2/\mu_\rho\right)\mathbb{P}_y(t,\infty)$$

which is obvious if α is small enough because $1 < 1 + k_1 + k_2/\mu_\rho < 1/\mu_\gamma$ and therefore $1+k_1+k_2/\mu_\rho$ is a weighted average of 1 and $1/\mu_\gamma$. In order to shorten the exposition, let us assume that α is smaller than the solution of

$$\frac{1}{\mu_\gamma(1+\alpha)} + \frac{\alpha}{1+\alpha} = 1 + k_1 + k_2/\mu_\rho, \tag{46}$$

---

[29] This is the most natural case. Indeed, suppose for instance that $\gamma = \rho$ and the loading rates are $k_1 = k_2 = 10\%$. Then, $1 + k_1 + k_2/\mu_\rho < 1/\mu_\rho$ if and only if $\mu_\rho < 0.9/1.1 \approx 0.818181818$, that is $1 - \mu_\rho > 0.1819$ or, equivalently, the confidence level for γ and ρ is higher than 18.19%.

and one will have that $m_H(t) > 0$ for $t \geq t_\gamma$.[30] Finally, if $t_0 < t \leq t_\gamma$, (41) holds if

$$\frac{1}{(1+\alpha)} + \frac{\alpha}{1+\alpha} \mathbb{P}_y(t,\infty) > \left(1 + k_1 + k_2/\mu_\rho\right) \mathbb{P}_y(t,\infty). \tag{47}$$

Since $\alpha/(1+\alpha) < 1 < 1 + k_1$, (47) will hold in an interval $(t_\alpha, t_\gamma]$, which might also be empty. To sum up, if $1 + k_1 + k_2/\mu_\rho < 1/\mu_\gamma$ and $\alpha > 0$ is lower than the solution of (46), then $m_H(t) > 0$ if and only if $t > t_\alpha$ (out of a Lebesgue-null sets) where $t_\alpha \in (t_0, t_\gamma]$, t_0 is characterized by (45), and t_γ is characterized by (37). Whence, the solution of (24) becomes

$$\tilde{x} = H\chi_{(t_\alpha,\infty)}, \tag{48}$$

that is, the marginal optimal indemnity $H\chi_{(t_\alpha,\infty)}$ equals zero if the realized value of Y is lower than t_α, and it equals H if Y becomes higher than t_α. (48) satisfies Condition *ii*) in Theorem 13, so (20) always holds, and the premium paid by the insurer is always strictly lower than the ceded risk. Table$-I$ below summarizes the additional assumptions and findings of Section 6.2.

$$\left(\begin{array}{l}
\textbf{Table} - \textbf{I}.\ \textit{Additional assumptions and findings of Section } 6.2. \\
\left\{\begin{array}{l}
\textbf{Additional assumptions} \\
\alpha > 0\ (\textit{so the insurer requirement holds}). \\
\alpha\ \textit{is lower than the solution of}\ (46). \\
1 + k_1 + k_2/\mu_\rho < 1/\mu_\gamma.
\end{array}\right. \\
\left\{\begin{array}{l}
\textbf{Findings} \\
(37), (38)\ \textit{and}\ (40)\ \textit{give the dual solution}. \\
\textit{The primal solution takes the form}\ \tilde{x} = H\chi_{(t_\alpha,\infty)}. \\
\textit{Contract};\ \mathrm{Re}\textit{insurer\_indemnity} = \left\{\begin{array}{l}
0,\ \textit{if}\ Y \leq t_\alpha \\
\int_{t_\alpha}^{y} H(t)\,dt,\ \textit{if}\ Y > t_\alpha.
\end{array}\right.
\end{array}\right.
\end{array}\right)$$

6.3 The reinsurer point of view

Once the solution (48) of (23) has been found and the fulfillment of (20) has been guaranteed, let us focus on the fulfillment of (18) (or (19)). In order to shorten and simplify the exposition, let us suppose that $H = h \in (0,1]$ remains constant.[31] (12) and (48) imply that

$$j(\tilde{x}) = h\,(t - t_\alpha)^{+}. \tag{49}$$

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[30] As indicated above, there are many cases that depend on the parameters of the problem and their relationships. A main purpose of this section is to illustrate the paper's theoretical findings, so let us focus on the main ideas only.

[31] Though the details are more tedious, things are essentially similar if

$$H = \sum_{j=1}^{n} h_j \chi_{(a_j,b_j)} + h_{n+1}\chi_{(a_{n+1},\infty)}$$

is a simple function.

(6) and (36) show that $\Gamma\left(-\mathcal{Y}-J\left(\tilde{x}\right)\right)=\Gamma\left(-\mathcal{Y}-h\left(t-t_{\alpha}\right)^{+}\right)$ equals the optimal value of the linear problem in the $Z-$variable

$$Max\ \mathbb{E}\left(Z\left(\mathcal{Y}+h\left(t-t_{\alpha}\right)^{+}\right)\right)\left\{\begin{array}{l}\mathbb{E}\left(Z\right)=1\\ 0\leq Z\leq 1/\mu_{\Gamma}\end{array}\right. \tag{50}$$

Its dual becomes (Anderson and Nash, 1987)

$$Min\ \lambda_{1}+\frac{1}{\mu_{\Gamma}}\mathbb{E}\left(\lambda_{2}\right)\left\{\begin{array}{l}\lambda_{1}+\lambda_{2}-\lambda_{3}=\mathcal{Y}+h\left(t-t_{\alpha}\right)^{+}\\ \lambda_{2},\lambda_{3}\geq 0\\ \left(\lambda_{1},\lambda_{2},\lambda_{3}\right)\in\mathbb{R}\times L^{2}\left(\mathbb{P}\right)\times L^{2}\left(\mathbb{P}\right)\end{array}\right. \tag{51}$$

where the real number λ_1 and the random variables λ_2 and λ_3 are the decision variables of the optimization problem. According to Anderson and Nash (1987), if the feasible elements $\tilde{Z}$ and $\left(\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}_3\right)$ satisfy the complementary slackness condition

$$\tilde{Z}\tilde{\lambda}_{3}=\left(1/\mu_{\Gamma}-\tilde{Z}\right)\tilde{\lambda}_{2}=0, \tag{52}$$

then they will solve (50) and (51), respectively. Consider the set

$$R_{1}^{(v}=\left\{\left(t,s\right)\in\left[0,\infty\right)^{2};s<Min\left\{v,v+h\left(t_{\alpha}-t\right)\right\}\right\}. \tag{53}$$

It is easy to see that $R_1^{(0}=\varnothing$, $\mathbb{P}\left(R_1^{(v)}\right)<1$ for every $v\geq 0$, $R_1^{(v}\subset R_1^{(w}$ if $v\leq w$, and

$$\bigcup_{n=1}^{\infty}R_{1}^{(n}=\left[0,\infty\right)^{2},$$

i.e., the family of complementary sets $R^{(v}=\left[0,\infty\right)^{2}\setminus R_1^{(v}$ satisfies the assumptions of Proposition 21. Moreover, there exists a unique $w>s_{\Gamma}$ such that $\mathbb{P}\left(R^{(w)}\right)=\mu_{\Gamma}$. Consider

$$\left\{\begin{array}{l}\tilde{Z}=\left(1/\mu_{\Gamma}\right)\chi_{R^{(w}}\\ \tilde{\lambda}_{1}=w\\ \tilde{\lambda}_{2}\left(t,s\right)=\left(s+h\left(t-t_{\alpha}\right)^{+}-w\right)^{+}\\ \tilde{\lambda}_{3}\left(t,s\right)=\left(s+h\left(t-t_{\alpha}\right)^{+}-w\right)^{-}\end{array}\right. \tag{54}$$

Since the feasibility is evident, if (52) holds then they will solve (50) and (51). (53) implies that $(t,s)\in R^{(w}$ if and only if $s\geq w$ or $s+h\left(t-t_{\alpha}\right)-w\geq 0$. Hence, $\tilde{Z}\tilde{\lambda}_3=0$ because $\tilde{Z}=0$ if $(t,s)\in R_1^{(w}$ and $\tilde{\lambda}_3=0$ if $(t,s)\in R^{(w}$. Besides, $\left(1/\mu_{\Gamma}-\tilde{Z}\right)\tilde{\lambda}_2=0$ because $\tilde{Z}=1/\mu_{\Gamma}$ if $(t,s)\in R^{(w}$ and $\tilde{\lambda}_2=0$ if $(t,s)\in R_1^{(w}$.

Once it is known that (54) solves (50) and (51), Proposition 20 may apply. In particular, bearing in mind (31), (32), (38), (39) and (49), the necessary and sufficient condition guaranteeing (19) becomes

$$\left\{ \begin{array}{l} (1/\mu_\Gamma)\, \mathbb{E}\left(\left(h\,(t-t_\alpha)^+ + s\right)\chi_{R^{(w}}\right) \le \\ (1/\mu_\Gamma)\, \mathbb{E}\left(s\chi_{(0,\infty)\times(s_\Gamma,\infty)}\right) + h\mathbb{E}\left((t-t_\alpha)^+\left(1+k_1+k_2\tilde{Z}_\rho\right)\right), \end{array} \right. \tag{55}$$

while a sufficient condition becomes

$$(1/\mu_\Gamma)\, \mathbb{E}\left((t-t_\alpha)^+\chi_{R^{(w}}\right) \le \mathbb{E}\left((t-t_\alpha)^+\left(1+k_1+k_2\tilde{Z}_\rho\right)\right). \tag{56}$$

Table$-II$ below summarizes the additional assumptions and findings of Section 6.3.

$$\left(\begin{array}{l} \textbf{Table} - \textbf{II}.\ \textit{Additional assumptions and findings of Section } 6.3. \\ \left\{ \begin{array}{l} \textbf{Additional assumption} \\ H = h \textit{ remains constant and } 0 < h \le 1. \end{array} \right. \\ \left\{ \begin{array}{l} \textbf{Findings} \\ \textit{For } v > 0,\ \textit{consider the set } R_1^{(v} \textit{ of } (53) \\ \textit{Equation } \mathbb{P}\left(R_1^{(v}\right) = 1 - \mu_\Gamma \textit{ has a unique solution } v = w. \\ \textit{The reinsurer requirement holds if and only if } (55) \textit{ holds} \\ \textit{If } (56) \textit{ holds, then the reinsurer requirement holds.} \end{array} \right. \end{array} \right)$$

6.4 Numerical experiment

Let us present a numerical example illustrating the ideas above of Table$-I$ and Table$-II$. In general, the fulfillment of both (55) and (56) may be easily checked in practice. Table$-III$ and Table$-IV$ below summarize some numerical experiments that have been implemented under several assumptions, namely, Y has a exponential distribution with $\mathbb{E}(Y) = 1$, $\mathcal{Y}$ has a exponential distribution with $\mathbb{E}(\mathcal{Y}) = \lambda$, $h = 0.9 = 90\%$, $k_1 = 0.1 = 10\%$, and $\gamma = \Gamma = \rho = CV@R_{95\%}$. (43) becomes $k_2 < 0.945 = 94.5\%$, and α will be lower than the solution of (46) if and only if $0 < \alpha < (18.9 - 20k_2)\,/\,(0,1 + 20k_2)$. In particular, we have taken $k_2 = 0.1 = 10\%$, and therefore $0 < \alpha < 8.047619048$. (37) and (38) lead to

$$\left\{ \begin{array}{l} t_\gamma = t_\rho = -\log(0.05) \approx 2.995732274 \\ s_\Gamma = -\lambda \log(0.05) \\ \tilde{Z}_\gamma = 20\chi_{(-\log(0.05),\infty)\times(0,\infty)} \\ \tilde{Z}_\Pi = 1.1 + 2\chi_{(-\log(0.05),\infty)} \\ \tilde{Z}_\Gamma = 20\chi_{(0,\infty)\times(-\lambda\log(0.05),\infty)} \end{array} \right.$$

Hence, the reinsurer risk is given by

$$\begin{cases} \Gamma\left(-\mathcal{Y}\right) = 20\mathbb{E}\left(\chi_{(0,\infty)\times(-\lambda\log(0.05),\infty)}s\right) = \\ \dfrac{20}{\lambda}\displaystyle\int\limits_{-\lambda\log(0.05)}^{\infty} se^{-s/\lambda}ds = \lambda\left(1-\log\left(0.05\right)\right), \end{cases} \tag{57}$$

the price of the optimal contract (49) becomes

$$\begin{cases} \mathbb{E}\left(\left(1.1+2\chi_{(-\log(0.05),\infty)}\right)h\left(t-t_\alpha\right)^{+}\right) = \\ \mathbb{E}\left(\left(0.99+1.8\chi_{(-\log(0.05),\infty)}\right)\left(t-t_\alpha\right)^{+}\right) = \\ = 0.99\displaystyle\int\limits_{t_\alpha}^{\infty}\left(t-t_\alpha\right)e^{-t}dt + 1.8\displaystyle\int\limits_{\tau_\alpha}^{\infty}\left(t-t_\alpha\right)e^{-t}dt, \end{cases} \tag{58}$$

with $\tau_\alpha = Max\left\{t_\alpha, -\log\left(0.05\right)\right\}$, and the ceded risk is given by

$$\begin{cases} \gamma\left(-h\left(t-t_\alpha\right)^{+}\right) = \\ 18\mathbb{E}\left(\chi_{(-\log(0.05),\infty)}\left(t-t_\alpha\right)^{+}\right) = 18\displaystyle\int\limits_{\tau_\alpha}^{\infty}\left(t-t_\alpha\right)e^{-t}dt. \end{cases} \tag{59}$$

(42) leads to

$$\begin{cases} J^{*}\left(\tilde{Z}_\gamma\right)(t) = \begin{cases} 20e^{-t}, & t \geq -\log\left(0.05\right) \\ 1, & t \leq -\log\left(0.05\right) \end{cases} \\ J^{*}\left(\tilde{Z}_\Pi\right)(t) = \begin{cases} 3.1e^{-t}, & t \geq -\log\left(0.05\right) \\ 1.1\left(e^{-t}-0.05\right)+0.155, & t \leq -\log\left(0.05\right) \end{cases} \\ J^{*}\left(1\right)(t) = e^{-t} \end{cases}$$

so (41) holds if and only if $t > t_\alpha$ with t_α solving

$$1.1\left(e^{-t}-0.05\right)+0.155 = \left(1+\alpha e^{-t}\right)/\left(1+\alpha\right),$$

i.e.,

$$t_\alpha = -\log\left(\frac{0.9-0.1\alpha}{1.1+0.1\alpha}\right). \tag{60}$$

One also has that $(t,s) \in R_1^{(v}$ becomes equivalent to

$$\begin{cases} 0 < s < v \\ 0 < t < t_\alpha + \left(v-s\right)/0.9. \end{cases} \tag{61}$$

Thus, if one assumes that Y and $\mathcal{Y}$ are independent, then

$$\mathbb{P}\left(R^{(v)}\right) = 1 - \mathbb{P}\left(R_1^{(v}\right) = 1 - \frac{1}{\lambda}\int_0^v e^{-s/\lambda}\left(\int_0^{t_\alpha + (v-s)/0.9} e^{-t}dt\right)ds$$

$$= 1 - \frac{1}{\lambda}\int_0^v e^{-s/\lambda}\left(1 - e^{-t_\alpha - (v-s)/0.9}\right)ds$$

$$= 1 - \frac{1}{\lambda}\left(\int_0^v e^{-s/\lambda}ds - \int_0^v e^{-s/\lambda - t_\alpha - (v-s)/0.9}ds\right),$$

that is,

$$\left\{\begin{array}{l} \mathbb{P}\left(R^{(v)}\right) = \\ Exp\left(-\frac{v}{\lambda}\right) + \frac{0.9}{\lambda - 0.9}\left(Exp\left(-\frac{v + \lambda t_\alpha}{\lambda}\right) - Exp\left(-\frac{v + 0.9t_\alpha}{0.9}\right)\right), \end{array}\right. \tag{62}$$

so the value w (or $w(\alpha, \lambda)$) such that $\mathbb{P}\left(R^{(w)}\right) = 0.05$ must be computed by a numerical method. Since $(1/\mu_\Gamma)\,\mathbb{E}\left(s\chi_{(0,\infty)\times(s_\Gamma,\infty)}\right) = \Gamma(-\mathcal{Y}) = \lambda(1 - \log(0.05))$ has been given in (57), after estimating $w(\alpha, \lambda)$ one is able to estimate

$$\mathbb{E}\left(\left(h(t - t_\alpha)^+ + s\right)\chi_{R^{(w}}\right) \tag{63}$$

and

$$\mathbb{E}\left((t - t_\alpha)^+ (1 + k_1 + k_2\tilde{z}_\rho)\right)$$

too, so that one can check whether (55) (and therefore (19)) holds or fails. It has been done, and Table$-III$ below reports the result. The second row gives the optimal contract, the third one gives the selected values for α, the fourth one gives t_α (see (60)), the fifth row shows the risk ceded by the insurer (according to (59)), the contract premium is shown in the sixth row (see (58)), and the rest of rows present the reinsurer risk increment provoked by the contract, which is given by

$$\begin{array}{l} \Gamma\left(-\mathcal{Y} - h(t - t_\alpha)^+\right) - \Gamma(-\mathcal{Y}) = \\ \mathbb{E}\left(\left(h(t - t_\alpha)^+ + s\right)\chi_{R^{(w}}\right) - \lambda(1 - \log(0.05)). \end{array} \tag{64}$$

and depends on λ. All of these quantities have been rounded to the third decimal place. As can be seen, the constraints imposed in Section 3 are satisfied in all of the selected cases. The insurer global risk without reinsurance equals 3.996.

Table − III. *Numerical results if Y and $\mathcal{Y}$ are independent*

Optimal contract		$0.9\left(Y-t_{\alpha}\right)^{+}$				
	$\alpha=$	0.4,	0.8,	1,	1.25,	2.5
	$t_{\alpha}=$	0.282,	0.364,	0.405,	0.458,	0.731
Ceded risk		3.342,	3.269,	3.231,	3.184,	2.938
Premium		1.081,	1.015,	0.983,	0.945,	0.771
Reinsurer	$\lambda=10$	0.726,	0.671,	0.644,	0.613,	0.472
risk	$\lambda=50$	0.701,	0.646,	0.618,	0.582,	0.451
growth	$\lambda=100$	0.715,	0.664,	0.639,	0.609,	0.468

Notice that the results of Table−III are quite natural. Indeed, as the importance of the insurer expected wealth increases (α increases), the contract deductible increases and, consequently, ceded risk, premium and reinsurer risk increment decrease. Besides, according to Examples 15, 17 and 18, Remark 16 and, more importantly, Remark 19, the reinsurer risk growth will become lower that the contract premium if the variance (or norm) of $\mathcal{Y}$ becomes much higher than the variance (or norm) of Y, and for that reason λ has been selected much larger than the standard deviation of Y.

Beyond the relationship between the variances of Y and $\mathcal{Y}$, and according to Remark 19, the dependency structure of both risks may also play a critical role. Thus, suppose that Y and $\mathcal{Y}$ do not have to be independent. (62) does not have to hold, and one does not have any general expression for $\mathbb{P}\left(R^{(v)}\right)$. Suppose that the joint distribution of Y and $\mathcal{Y}$ is given by the copula $c:[0,1]^{2}\rightarrow[0,1]$, $F(t,s)=c\left(1-e^{t},1-e^{s/\lambda}\right)$ is the cumulative distribution function of $(Y,\mathcal{Y})$, and

$$f(t,s)=\frac{\partial^{2}F}{\partial s\partial t}$$

is the joint density function. The unknown w (or $w(\alpha,\lambda)$) of the equation $\mathbb{P}\left(R^{(w)}\right)=0.05$ also solves

$$\int_{0}^{w}\left(\int_{0}^{t_{\alpha}+(w-s)/0.9}f(t,s)\,dt\right)ds=0.95. \tag{65}$$

The integral of the left hand side is a continuous and increasing function of w converging to zero as w converges to zero and converging to one as w diverges to $+\infty$. Furthermore, this integral can be estimated by Monte Carlo simulation for a large enough sample of the w variable, which allows us to solve (65). Once $w(\alpha,\lambda)$ is known, (63) can be estimated, and the reinsurer risk growth (64) can be known too. These steps have been addressed for the Clayton copula whose parameter equals one, that is, $c(0,0)=0$ and

$$c(u,v)=\frac{uv}{u+v-uv}$$

for $0 \leq u \leq 1$, $0 \leq v \leq 1$, and $(u,v) \neq (0,0)$. Table$-IV$ below reports the numerical results. Obviously, the joint distribution of $(Y,\mathcal{Y})$ only affects the reinsurer risk growth, so the non provided rows remain the same as they were in Table$-III$.

Table − IV. *Numerical results under a Clayton copula*

Re*insurer*	$\lambda = 10$	0.000,	0.000,	0.000,	0.000,	0.000
risk	$\lambda = 50$	0.000,	0.000,	0.000,	0.000,	0.000
*g*row*th*	$\lambda = 100$	0.001,	0.001,	0.001,	0.001,	0.000

Notice that the reinsurer results become better than they were in Table$-III$. The reason is that the selected copula implies more dependence in the left hand side than in the right hand one. Therefore, if the realized value of $\mathcal{Y}$ becomes small, then it will be expected a small realized value of Y, and consequently the ceded risk $c(Y)$ will not deteriorate the good results of the reinsurer. In contrast, for a large realized value of $\mathcal{Y}$, the realized value of Y will show a much higher degree of independence, and the reinsurer risk global indemnity $c(Y)+\mathcal{Y}$ will benefit from the usual diversification effect.

Finally, it is very easy to find a copula provoking high dependency between Y and $\mathcal{Y}$, as well as the failure of the reinsurer requirement. Indeed, consider the Fréchet–Hoeffding copula $c(u,v) = Min\{u,v\}$. Then, Y and $\mathcal{Y}$ become co-monotone (Dhaene *et al.*, 2002), and it is easy to see that $\mathcal{Y} = \lambda Y$. One is under the conditions of the counter-example of Remark 4, so the fulfillment of both (18) and (20) implies that they will become equalities. Since (20) is fulfilled as a strict inequality (Table$-III$), (18) cannot hold.

7 Conclusion

A reinsurance contract must be attractive to both insurer and reinsurer. Consequently, the reinsurer initial (before reinsurance) risk matters, since it will be used to diversify (or hedge) the risk ceded by the insurer. More particularly, the growth of this initial risk provoked by the reinsurance contract must be lower than the contract premium, since this inequality implies a global risk reduction if the premium effect is involved. The incorporation of this initial risk in the optimal reinsurance problem could be unrealistic, since it would implicitly imply that the insurer has some knowledge about the reinsurer risk. Thus, it could be more rational to deal with a more classical approach and analyze whether the obtained solution satisfies the reinsurer requirement. Besides, since the contract must be interesting to the insurer too, the price cannot be "too expensive", and a reasonable upper bound has been proposed, namely, the premium must be lower than the insurer risk reduction, since this inequality implies that the insurer may become more competitive in the insurance market and reduce the initial reserve. Nevertheless, it has not been necessary to impose this constraint, since it is automatically fulfilled under a correct approach to the problem.

The reinsurer risk incorporation has required integrating this risk and the insurer one in a single two-dimensional random variable, which has facilitated the extension of previous approaches attempting to prevent the reinsurer moral hazard. More particularly, the two-dimensional random variable has allowed us to deal with the marginal indemnity as a decision variable, instead of the indemnity itself. Then, the problem has been presented in a very general framework with respect to both the risk measurement criteria and the premium principle, and general expressions of the optimal solution have been given. The particular case involving the conditional value at risk has been studied.

Appendix (proofs)

Proof of Proposition 3. $J = I_Y \circ j$ implies that $J^* = j^* \circ I_Y^*$ (Zeidler, 1995), so all the assertions above trivially follow from Proposition 2 except the given formula for $J^*(Z)(t)$. Consider $x \in L^\infty(\mathbb{L})$. Fubini´s theorem implies that

$$\begin{aligned}&\mathbb{E}(J(x)Z) = \int_{R_0} Z(t,w)\, J(x)(t)\, \mathbb{P}(dt,dw) = \\ &\int_{R_0} Z(t,w)\left(\int_0^t x(v)\,dv\right)\mathbb{P}(dw_1,dw_2) = \int_0^\infty x(t)\left(\int_{R_t} Z(s)\,\mathbb{P}(ds)\right)dt \\ &= \left\langle x, \int_{R_t} Z(s)\,\mathbb{P}(ds)\right\rangle\end{aligned}$$

due to (15). □

Proof of Lemma 5. *a*) Notice that the existence of $\tilde{Z}_\gamma$ trivially follows from (4) and the *weak*−compactness of (3). Since $-j(1)$, $-j(x)$ and $-j(1-x)$ are co-monotone, we have that $\gamma(-j(1)) = \gamma(-j(x)) + \gamma(-j(1-x))$. Besides,

$$\gamma(-j(x)) \geq \mathbb{E}\left(j(x)\,\tilde{Z}_\gamma\right) \tag{66}$$

and

$$\gamma(-j(1-x)) \geq \mathbb{E}\left(j(1-x)\,\tilde{Z}_\gamma\right) \tag{67}$$

follow from (3). Thus,

$$\begin{aligned}&\gamma(-j(1)) = \gamma(-j(x)) + \gamma(-j(1-x)) \geq \\ &\mathbb{E}\left(j(x)\,\tilde{Z}_\gamma\right) + \mathbb{E}\left(j(1-x)\,\tilde{Z}_\gamma\right) = \mathbb{E}\left(j(1)\,\tilde{Z}_\gamma\right) = \gamma(-j(1)),\end{aligned}$$

so the inequality in the chain must become a equality. Obviously, if (66) or (67) were a strict inequality, then one would be facing a contradiction.

b) Since ρ is co-monotone additive, one has that $\Pi(-j(1)) = \Pi(-j(x)) + \Pi(-j(1-x))$. Thus, the rest of the proof is similar to the proof of *a*). □

Proof of Corollary 8. It is evident that

$$\left(\tilde{Z}_\Pi, J^*\left((1+\alpha)\,\tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha\right)^+, J^*\left((1+\alpha)\,\tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha\right)^-\right)$$

is (25)-feasible. If (28) failed, then $m_H \geq J^* \left((1+\alpha) \tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha \right)^-$ and $m_H \neq J^* \left((1+\alpha) \tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha \right)^-$ (Meyer-Nieburg, 1991). Hence, $0 < h \leq H$ would lead to

$$\langle H, m_H \rangle > \left\langle H, J^* \left((1+\alpha) \tilde{Z}_\Pi - \tilde{Z}_\gamma - \alpha \right)^- \right\rangle$$

and the objective function of (25) would become strictly higher if $\left(\tilde{Z}_\Pi, , m_0, m_H \right)$ were replaced by the given feasible solution. In other words, we would have a contradiction. □

Proof of Corollary 9. It is evident that $\tilde{x}$ is (23)-feasible and (27)-feasible. The dual of (27) may be computed with usual methods in linear programming (Anderson and Nash, 1987) and becomes identical to (25) if one replaces $Z_\Pi \in \partial_\Pi$ with $Z_\Pi = \tilde{Z}_\Pi$. Moreover, standard linear programming methods also lead to the second condition in (26). Thus, (26) implies the optimality of $\left(\tilde{Z}_\Pi, , m_0, m_H \right)$ and $H \chi_{m_H > 0}$ for (27) and its dual, and $\tilde{x} = H \chi_{m_H > 0}$ is provoked by the uniqueness of the solution of (27) and Theorem 7*d*). □

Proof of Corollary 10. It is obvious that the $EVPP$ is a particular case arising if $k_2 = 0$, so let us focus on the general case. It is evident that $\tilde{x}$ is (23)-feasible. As in the proof of Corollary 9, the dual of (24) becomes similar to (25) if one replaces $Z_\Pi \in \partial_\Pi$ with $Z_\Pi = \tilde{Z}_\Pi$. Moreover, (26) are also necessary and sufficient optimality conditions. Thus, the optimality of $\left(\tilde{Z}_\Pi, , m_0, m_H \right)$ and $H \chi_{m_H > 0}$ follows from Theorem 7*c*). □

Proof of Lemma 12. The first condition in (26) leads to $\mathbb{E} \left(j \left(\tilde{x} \right) Z_\Pi \right) \leq \mathbb{E} \left(j \left(\tilde{x} \right) \tilde{Z}_\Pi \right)$. The equality constraint of (25) implies that

$$\mathbb{E} \left(j \left(\tilde{x} \right) \tilde{Z}_\Pi \right) = \left\langle \tilde{x}, j^* \left(\tilde{Z}_\Pi \right) \right\rangle = \frac{\langle \tilde{x}, m_0 \rangle - \langle \tilde{x}, m_H \rangle + \mathbb{E} \left(J \left(\tilde{x} \right) \tilde{Z}_\gamma \right) + \alpha \mathbb{E} \left(J \left(\tilde{x} \right) \right)}{1+\alpha}.$$

Therefore, (15) and the second condition in (26) lead to (29). □

Proof of Theorem 13. Lemma 5 and Remark 6 show that $\gamma \left(-j \left(\tilde{x} \right) \right) = \mathbb{E} \left(J \left(\tilde{x} \right) \tilde{Z}_\gamma \right)$, and therefore (29) in Lemma 12 is equivalent to

$$\mathbb{E} \left(j \left(\tilde{x} \right) Z_\Pi \right) \leq \frac{\gamma \left(-j \left(\tilde{x} \right) \right) + \alpha \mathbb{E} \left(j \left(x \right) \right)}{1+\alpha} - \frac{\langle H, m_H \rangle}{1+\alpha} \tag{68}$$

for every $Z_\Pi \in \partial_\Pi$. Since $\gamma \left(-j \left(\tilde{x} \right) \right) \geq \mathbb{E} \left(j \left(\tilde{x} \right) \right)$ (recall that γ is expectation bounded), (68) leads to

$$\mathbb{E} \left(j \left(\tilde{x} \right) Z_\Pi \right) \leq \frac{\gamma \left(-j \left(\tilde{x} \right) \right) + \alpha \gamma \left(-j \left(\tilde{x} \right) \right)}{1+\alpha} - \frac{\langle H, m_H \rangle}{1+\alpha} = \gamma \left(-j \left(\tilde{x} \right) \right) - \frac{\langle H, m_H \rangle}{1+\alpha}$$

for every $Z_\Pi \in \partial_\Pi$, and (9) leads to

$$\Pi\left(j\left(\tilde{x}\right)\right) \leq \gamma\left(-j\left(\tilde{x}\right)\right) - \frac{\langle H, m_H \rangle}{1+\alpha}$$

that is, $\tilde{x}$ satisfies (20). Besides, since $H \geq h > 0$ and $m_H \geq 0$, $\langle H, m_H \rangle > 0$ if *i*) holds (see (15)), which implies that $\Pi\left(j\left(\tilde{x}\right)\right) < \gamma\left(-j\left(\tilde{x}\right)\right)$ and $\tilde{x}$ satisfies (20) as a strict inequality. Lastly, suppose that *ii*) holds. Lemma 12 shows that (29) becomes an equality if $Z_\Pi = \tilde{Z}_\Pi$. Therefore, $\gamma\left(-j\left(\tilde{x}\right)\right) = \mathbb{E}\left(J\left(\tilde{x}\right)\tilde{Z}_\gamma\right)$ implies that (68) becomes a equality if $Z_\Pi = \tilde{Z}_\Pi$. Since (26) implies that $\Pi\left(j\left(\tilde{x}\right)\right) = \mathbb{E}\left(j\left(\tilde{x}\right)Z_\Pi\right)$, one has

$$\Pi\left(j\left(\tilde{x}\right)\right) = \frac{\gamma\left(-j\left(\tilde{x}\right)\right) + \alpha\mathbb{E}\left(j\left(\tilde{x}\right)\right)}{1+\alpha} - \frac{\langle H, m_H \rangle}{1+\alpha}.$$

Whence,

$$\Pi\left(j\left(\tilde{x}\right)\right) < \frac{\gamma\left(-j\left(\tilde{x}\right)\right) + \alpha\gamma\left(-j\left(\tilde{x}\right)\right)}{1+\alpha} - \frac{\langle H, m_H \rangle}{1+\alpha} \leq \gamma\left(-j\left(\tilde{x}\right)\right)$$

because *ii*) holds. □

Proof of Proposition 20. The assumption is

$$\Gamma\left(-J\left(\tilde{x}\right) - \mathcal{Y}\right) = \mathbb{E}\left(\left(J\left(\tilde{x}\right) + \mathcal{Y}\right)\tilde{Z}\right).$$

(9) and (26) lead to $\Pi\left(j\left(\tilde{x}\right)\right) = \mathbb{E}\left(J\left(\tilde{x}\right)\tilde{Z}_\Pi\right)$. Thus, *a*) becomes evident.

b), *c*) *and d*). (8) leads to

$$\Gamma\left(-J\left(\tilde{x}\right) - \mathcal{Y}\right) = \mathbb{E}\left(\left(J\left(\tilde{x}\right) + \mathcal{Y}\right)\tilde{Z}\right) =$$
$$\mathbb{E}\left(J\left(\tilde{x}\right)\tilde{Z}\right) + \mathbb{E}\left(\mathcal{Y}\tilde{Z}\right) \leq \mathbb{E}\left(J\left(\tilde{x}\right)\tilde{Z}\right) + \Gamma\left(-\mathcal{Y}\right).$$

Thus, if (32) holds (respectively, holds as a strict inequality), then (9) and (26) lead to $\Pi\left(j\left(\tilde{x}\right)\right) = \mathbb{E}\left(J\left(\tilde{x}\right)\tilde{Z}_\Pi\right)$, and $\Gamma\left(-J\left(\tilde{x}\right) - \mathcal{Y}\right) \leq \Pi\left(j\left(\tilde{x}\right)\right) + \Gamma\left(-\mathcal{Y}\right)$ (respectively, $\Gamma\left(-J\left(\tilde{x}\right) - \mathcal{Y}\right) < \Pi\left(j\left(\tilde{x}\right)\right) + \Gamma\left(-\mathcal{Y}\right)$). Besides, (10) shows that (33) implies (32). □

Proof of Proposition 21. (34) implies that

$$\lim_{n\to\infty} \mathbb{P}\left(R^{(n}\right) = 0. \tag{69}$$

Besides,

$$J\left(\tilde{x}\right)\partial_\Gamma = \left\{J\left(\tilde{x}\right)Z;\ Z \in \partial_\Gamma\right\} \subset L^1\left(\mathbb{P}\right)$$

is uniformly integrable because ∂_Γ is weakly-compact (Kopp, 1984), so (69) implies the existence of $n_0 \in \mathbb{N}$ such that

$$Sup\left\{\int_{R^{(n}} J\left(\tilde{x}\right)(t,s)\,Z\left(t,s\right)\mathbb{P}\left(dt,ds\right);\ z \in \partial_\Gamma\right\} < \mathbb{E}\left(J\left(\tilde{x}\right)\tilde{Z}_\Pi\right) = \Pi\left(j\left(\tilde{x}\right)\right)$$

for $n \geq n_0$. Hence, for $Z \in \partial_\Gamma$, (32) holds for $v_0 = n_0$ because

$$\mathbb{E}\left(J\left(\tilde{x}\right)Z\right) = \int\limits_{R^{(0}} J\left(\tilde{x}\right)\left(t,s\right)Z\left(t,s\right)\mathbb{P}\left(dt,ds\right) = \int\limits_{R^{(n_0}} J\left(\tilde{x}\right)\left(t,s\right)Z\left(t,s\right)\mathbb{P}\left(dt,ds\right)$$

if $Z\left(t,s\right) = 0$ for $\left(t,s\right) \notin R^{(v_0}$. Moreover, if i) and ii) hold, then (18) and (19) are implied by Proposition 20. $\square$

Proof of Proposition 22. The existence of d trivially follows from the continuity (and therefore lower semi-continuity) of Γ at $\left(-J\left(\tilde{x}\right) - \mathcal{Y}\right)/\left\|\mathcal{Y}\right\|$. Suppose that $\left\|Y\right\| \leq d\left\|\mathcal{Y}\right\|$. Since $j\left(\tilde{x}\right) \leq j\left(1\right) = Y \Longrightarrow \left\|j\left(\tilde{x}\right)\right\| \leq \left\|Y\right\| \Longrightarrow \left\|j\left(\tilde{x}\right)\right\|/\left\|\mathcal{Y}\right\| \leq \left\|Y\right\|/\left\|\mathcal{Y}\right\| \leq d$, one has that $\left\|-\mathcal{Y}/\left\|\mathcal{Y}\right\| - \left(-J\left(\tilde{x}\right) - \mathcal{Y}\right)/\left\|\mathcal{Y}\right\|\right\| = \left\|J\left(\tilde{x}\right)/\left\|\mathcal{Y}\right\|\right\| \leq d$, and (35) leads to

$$\Gamma\left(-\frac{\mathcal{Y}}{\left\|\mathcal{Y}\right\|}\right) \geq \Gamma\left(\frac{-J\left(\tilde{x}\right) - \mathcal{Y}}{\left\|\mathcal{Y}\right\|}\right) - \frac{\Pi\left(j\left(\tilde{x}\right)\right)}{D} \geq \Gamma\left(\frac{-J\left(\tilde{x}\right) - \mathcal{Y}}{\left\|\mathcal{Y}\right\|}\right) - \frac{\Pi\left(j\left(\tilde{x}\right)\right)}{\left\|\mathcal{Y}\right\|},$$

so (19) becomes obvious because Γ is positively homogeneous. $\square$

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