Journal of Mathematical Physics

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Cite as: J. Math. Phys. **61**, 043501 (2020); https://doi.org/10.1063/1.5134647
Submitted: 31 October 2019 . Accepted: 10 March 2020 . Published Online: 07 April 2020

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Journal of Mathematical Physics | ARTICLE | scitation.org/journal/jmp

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Submitted: 31 October 2019 • Accepted: 10 March 2020 •
Published Online: 7 April 2020

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ABSTRACT

We consider separatrix solutions of the differential equations for inflaton models with a single scalar field in a zero-curvature Friedmann–Lemaître–Robertson–Walker universe. The existence and properties of separatrices are investigated in the framework of the Hamilton–Jacobi formalism, where the main quantity is the Hubble parameter considered as a function of the inflaton field. A wide class of inflaton models that have separatrix solutions (and include many of the most physically relevant potentials) is introduced, and the properties of the corresponding separatrices are investigated, in particular, asymptotic inflationary stages, leading approximations to the separatrices, and full asymptotic expansions thereof. We also prove an optimal growth criterion for potentials that do not have separatrices.

I. INTRODUCTION

The theory of inflation is a candidate to solve several long-standing problems regarding the physical conditions in the early universe.[1–3] The present paper deals with single-field inflationary models defined by a potential function $v(\Phi)$ of the inflaton field Φ in a spatially flat Friedmann–Lemaître–Robertson–Walker spacetime.[4,5] The dynamical equation of these models is the nonlinear ordinary differential equation,

$$\ddot{\Phi} + 3H\dot{\Phi} + V'(\Phi) = 0, \tag{1}$$

where H is the Hubble parameter,

$$H^2 = \frac{1}{3\mathrm{M_{Pl}}^2}\left(\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\right), \tag{2}$$

with $\mathrm{M_{Pl}}$ being the reduced Planck mass, dots denote derivatives with respect to the cosmic time t, and primes denote derivatives of a function with respect to its argument. In turn, the Hubble parameter H is defined in terms of the scale factor a by $H = \dot{a}/a$. We consider models describing an expanding universe for which H is strictly positive at all times and the inflationary stage is characterized by the condition $\ddot{a} > 0$.

It is useful to rescale variables

$$\Phi(t) = \sqrt{\frac{2}{3}}\,\mathrm{M_{Pl}}\,\varphi(t), \quad V(\Phi) = \frac{\mathrm{M_{Pl}}^2}{3}v(\varphi), \quad H(t) = \frac{1}{3}h(t) \tag{3}$$

so that (1) and (2) can be written as

$$\ddot{\varphi} + h\dot{\varphi} + \frac{1}{2}v' = 0 \tag{4}$$

and

$$h = (\dot{\varphi}^2 + v(\varphi))^{1/2}, \tag{5}$$

respectively. In terms of h, the inflation condition $\ddot{a} > 0$ reads

$$h < \sqrt{\frac{3v}{2}}. \tag{6}$$

It is often claimed in the literature that many inflationary models exhibit "attractor solutions" of the differential equation (4) to which solutions evolve for wide sets of initial conditions.[6,7] This terminology has been contested by several authors,[7,8] since the notion of the attractor used in those contexts does not correspond to the mathematical notion of the attractor of a flow in the theory of dynamical systems, as defined, for example, in Ref. 9, and, in particular, does not satisfy Liouville's theorem on attractor behavior in Hamiltonian systems.[8] [On the other hand, the origin in the $(\varphi, \dot{\varphi})$ phase plane is an attractor in the mathematical sense, as is any point $(\varphi_0, 0)$, where φ_0 is a point of minimum of the potential.]

The solutions often called attractors are, in fact, *separatrices*,[6] and near these separatrices, occur the crucial inflationary stages of the solutions of (4).

There is an extensive body of work on dynamical systems theory as applied to cosmology and, in particular, to inflationary cosmology.[6,10–21] The standard approach to study (1) and (2) uses a Poincaré compactification of the $(\Phi, \dot{\Phi})$ phase plane, which resolves singular points at infinity and is specially suited to obtain global dynamical results and, therefore, all possible asymptotic behaviors. Particularly relevant to our work are the studies in Refs. 6, 12, and 18, which discuss orbits connecting critical points.

In this paper, however, we adopt the Hamilton–Jacobi formalism used by Salopek and Bond[22] to study the evolution of long-wavelength metric fluctuations and to find (in parametric form) the general isotropic solution of the exponential model known to drive power-law inflation, and by Liddle *et al.*[7] to generate explicit slow-roll expansions. More recently, the Hamilton–Jacobi formalism has also been successfully applied by Handley *et al.*[23] to study inflationary solutions of general inflationary models emerging from regions of kinetic dominance.

We stress that unlike the dynamical system approach, our straightforward implementation of the Hamilton–Jacobi method is not global (in particular, because of the lack of Poincaré compatification), but on the other hand, it permits using concepts and methods from the theory of ordinary differential equations such as super-solutions and isoclines[24,25] to characterize separatrices and generate efficiently their asymptotic expansions.

Independently of the formalism used, the study of separatrices is linked to the analysis of singularities. It follows from Eq. (4) that

$$\dot{h} = -\dot{\varphi}^2, \tag{7}$$

and, therefore, the Hubble parameter h is a positive monotonically decreasing function of t. This property implies that for smooth and positive potentials $v(\varphi)$, the solutions $\varphi(t)$ of (4) with arbitrary finite initial data do not have singularities forward in the cosmic time t. Furthermore, it can be proved that under mild conditions,[26–28] as $t \to \infty$, the solutions $\varphi(t)$ of (4) tend to critical points of $v(\varphi)$. However, due to (7), the function $h(t)$ may increase without bound backward in time so that $h(t)$ and $\varphi(t)$ may develop singularities. The presence of singularities backward in time can also be expected from the following heuristic argument: if the condition

$$\dot{\varphi}^2 \gg v(\varphi) \tag{8}$$

holds, then we may neglect v and v' in the inflaton equations (4) and (5) and approximate (4) by

$$\ddot{\varphi} + |\dot{\varphi}|\,\dot{\varphi} \sim 0. \tag{9}$$

Hence, we obtain two families of approximate solutions

$$\varphi(t) \sim -\log(t - t^*) + A, \quad t \to t^* \quad \text{for } \dot{\varphi} < 0, \tag{10}$$

and

$$\varphi(t) \sim \log(t - t^*) + A, \quad t \to t^* \quad \text{for } \dot{\varphi} > 0, \tag{11}$$

with $t > t^*$. These solutions depend on two arbitrary parameters t^* and A, which determine movable logarithmic singularities. The corresponding Hubble parameter (5) satisfies

$$h(t) \sim \frac{1}{t - t^*}, \quad t \to t^*. \tag{12}$$

For both approximate solutions (10) and (11), we have that $\dot{\varphi}^2 \sim \exp(2|\varphi|)$, and in view of (5), these singularities should not arise for confining potentials that grow faster than $\exp(2|\varphi|)$ as $|\varphi| \to \infty$.[23] Condition (8) defines what is called the kinetic dominance regime of inflation,[23,29,30] and the separatrices, if they exist, are boundaries of the regions in the $(\varphi, \dot{\varphi})$ phase space filled by the solutions with either asymptotic behavior (10) or (11).

In the Hamilton–Jacobi formulation of inflationary models,[5,7,22,31] the basic dependent variable is the Hubble parameter $h = h(\varphi)$ as a function of the inflaton field φ. Our main goal is to discuss the existence and characterization of separatrices from the large-φ behavior of $h(\varphi)$, which, in turn, will lead us to algorithms for the calculation of complete asymptotic expansions.

This paper is organized as follows: In Sec. II, we present the equations of the Hamilton–Jacobi formalism and its phase space R. We formulate sufficient conditions for the presence of solutions $\varphi(t)$ that blow up at a finite time. We also show that when restricted to R, solutions can be classified into two types that determine two non-overlapping regions filling the phase space R, and separatrices will be defined as boundaries between these two regions. Section III shows that if there exists a solution such that $h(\varphi)/\sqrt{v(\varphi)}$ is defined and bounded for large φ, then this solution is unique and is a (the) separatrix. The actual question of existence of separatrices is discussed in Sec. IV, where a wide class of potentials (which includes the standard potentials found in the literature) with separatrices is introduced. In addition, several examples of inflationary models beyond this class both with and without separatrices are also exhibited. Finally, Sec. V is devoted to the study of the asymptotics of separatrices.

Our main results are

1. We prove that the separatrix solutions $h_s = h_s(\varphi)$ separate bounded from unbounded trajectories in the phase space R. In fact, and because of a symmetry discussed below, it is enough to focus on unbounded trajectories in R as $\varphi \to \infty$, which correspond to the singular solutions (10).
2. We determine a wide class $\mathscr{C}_\alpha$ $(0 \le \alpha < 1)$ of potential functions that determine inflationary models with separatrices. In particular, the even monomial potentials, the Higgs potential, and the Starobinsky potential are members of $\mathscr{C}_0$. We also exhibit confining potentials without separatrices and potentials outside the class $\mathscr{C}_\alpha$ with a different type of separatrix solution for which $h(\varphi)/\sqrt{v(\varphi)}$ is not bounded as $\varphi \to \infty$.
3. We calculate the leading term of the asymptotic expansion as $\varphi \to \infty$ of separatrices for models of the class $\mathscr{C}_\alpha$. This result is applied to prove that backward inflation only occurs in the separatrix solutions of these models for $\alpha < 1/\sqrt{3}$.
4. For asymptotically divergent potentials with $\alpha = 0$, we obtain recursive relations that permit us to calculate explicitly as many terms as desired for the complete expansions of the separatrices in terms of differential polynomials of the square root $\sqrt{v}$. We also give asymptotic expansion for α-attractor E-models and models with hard potential walls.
5. We argue heuristically that these asymptotic expansions are likely to be Borel summable and discuss algorithms for an efficient approximate summation of these series.
6. Finally, we analyze the presence or absence of blow up in the inflaton fields $\varphi_s(t)$ corresponding to the separatrix solutions of several basic models.

II. THE HAMILTON–JACOBI FORMALISM

In this section, we assume that the scaled potential $v = v(\varphi)$ is a smooth positive function.

A. The Hamilton–Jacobi formalism and its phase space

The main physical quantity in the so-called Hamilton–Jacobi formalism for inflationary models is the scaled Hubble parameter $h(\varphi)$ considered as a function of the inflaton φ, which satisfies

$$(h')^2 = h^2 - v, \tag{13}$$

and gets its time dependence through the dependence of the inflaton φ on the cosmic time t. In essence and due to Eq. (5), this idea requires to consider $\dot{\varphi}$ as a function of φ, and therefore to deal separately with each strictly monotonic part of $\varphi(t)$.

To formalize this idea, consider the lower and upper half-planes in the $(\varphi, \dot{\varphi})$ phase plane,

$$D_- = \{(\varphi, \dot{\varphi}) \in \mathbb{R}^2 : \dot{\varphi} < 0\}, \quad D_+ = \{(\varphi, \dot{\varphi}) \in \mathbb{R}^2 : \dot{\varphi} > 0\}. \tag{14}$$

The map

$$(\varphi, \dot{\varphi}) \mapsto (\varphi, h) \tag{15}$$

defines two diffeomorphisms $T_\pm : D_\pm \to R$ from $D_\pm$ onto the open set

$$R = \{(\varphi, h) \in \mathbb{R}^2 : \sqrt{v(\varphi)} < h < +\infty\} \tag{16}$$

of the (φ, h) plane. It follows from Eq. (7) that

$$\dot{\varphi} = -h'(\varphi) \tag{17}$$

and that the parts of each solution $\varphi(t)$ of Eq. (4) in D_- (strictly decreasing) and in D_+ (strictly increasing) are described by the differential equations,

$$h' = \sqrt{h^2 - v}, \quad (\varphi, h) \in R, \tag{18}$$

and

$$h' = -\sqrt{h^2 - v}, \quad (\varphi, h) \in R, \tag{19}$$

respectively. The two first-order non-linear ordinary differential equations [(18) and (19)] along with Eq. (17) are referred to as the Hamilton–Jacobi formalism for inflationary models.[5,7,22,31]

A main advantage of this Hamilton–Jacobi formalism is that if we ignore the time dependence given by Eq. (17) and restrict our attention to the Hamilton–Jacobi phase plane R, Eqs. (18) and (19) are precisely the equations for the orbits of the dynamical system (4). Note that a given solution $\varphi(t)$ may consist of several (even infinitely many) strictly monotonic pieces, each piece in D_- and D_+ leading to a corresponding monotonic arc $h(\varphi)$ in the Hamilton–Jacobi phase space R (see Fig. 1) satisfying Eqs. (18) and (19), respectively. We call the curve $h = \sqrt{v(\varphi)}$, the lower boundary Γ of R, and note that the limiting value of $h'(\varphi)$ as $\varphi \to \Gamma$ is zero, i.e., if a solution $h(\varphi)$ reaches the lower boundary Γ, it does so with an horizontal half-tangent line.

An initial value problem in the Hamilton–Jacobi formalism is determined by a point $(\varphi(0), h(\varphi(0)))$ in R and by the sign of $\dot{\varphi}(0)$. For example, if $\dot{\varphi}(0)$ is negative, its value in terms of $\varphi(0)$ and $h(\varphi(0)))$ is given by Eq. (18),

$$\dot{\varphi}(0) = -h'(\varphi(0)) = -\sqrt{h(\varphi(0))^2 - v(\varphi(0))}, \tag{20}$$

and the solution $\varphi(t)$ is implicitly defined by

$$t = -\int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{h'(\varphi)} \tag{21}$$

in the corresponding interval, where $h'(\varphi) > 0$.

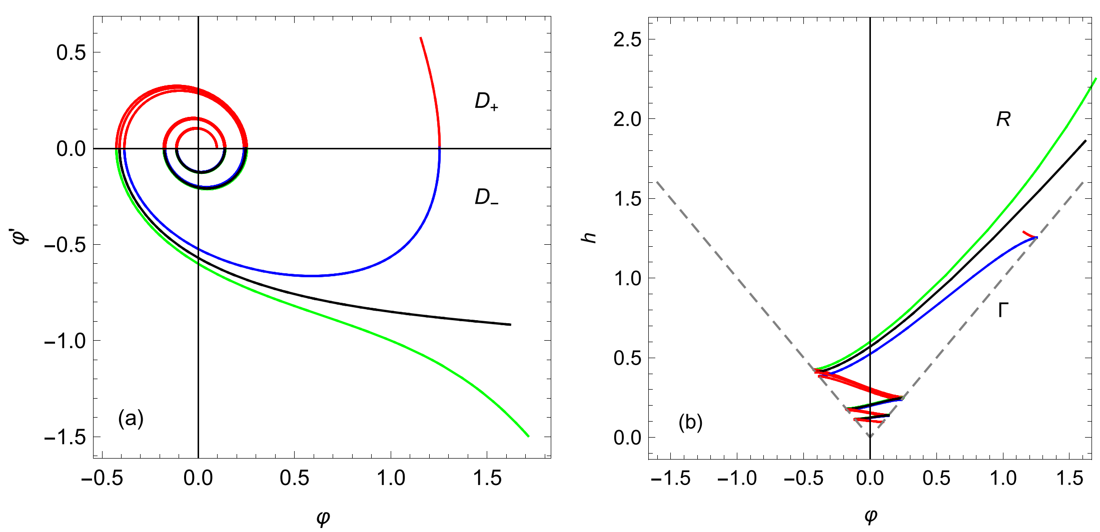

FIG. 1. Three trajectories in the $(\varphi, \dot{\varphi})$ phase plane of the quadratic model $v(\varphi) = \varphi^2$ exhibiting several monotonic pieces, and their corresponding arcs satisfying Eq. (18) or (19) in the phase space R.

The Hamilton–Jacobi formalism is particularly convenient for studying the possible blow up of solutions $\varphi(t)$. Indeed, the solution $\varphi(t)$ defined by (21) blows up at a given $t^* < 0$ [i.e., $\varphi(t) \to +\infty$ as $t \to t^*$] if and only if the integral

$$t^* = -\int_{\varphi(0)}^{\infty} \frac{d\varphi}{h'(\varphi)} \tag{22}$$

is well-defined and convergent on that interval. In this case, the orbit $(\varphi, \dot\varphi)$ tends to infinity in the fourth quadrant (e.g., the green orbit in Fig. 1). Note that from Eqs. (10) and (12), it follows that

$$h(\varphi) \sim e^{\varphi - A}, \quad \varphi \to \infty. \tag{23}$$

From (22), it is also clear that a necessary condition for a solution $h = h(\varphi)$ of (18) to determine a blow-up solution $\varphi = \varphi(t)$ is that

$$\lim_{\varphi\to\infty} \left(h(\varphi)^2 - v(\varphi)\right) = \infty. \tag{24}$$

Similar statements can be made for solutions with $\dot\varphi(0) > 0$ and Eq. (19). In particular, orbits $(\varphi, \dot\varphi)$ corresponding to blow-up solutions tend to infinity in the second quadrant. Since (19) reduces to (18) under the change of variable,

$$\widetilde{h}(\varphi) = h(-\varphi), \quad \widetilde{v}(\varphi) = v(-\varphi), \tag{25}$$

the analysis of (18) can also be applied to the strictly monotonic parts of the solutions $\varphi(t)$ of the inflaton equation (4) with trajectories in D_+. Therefore, hereafter, we will focus our attention on the differential equation (18). Thus, we will only deal with monotonic parts of solutions $\varphi(t)$ lying in the fourth quadrant of the plane $(\varphi, \dot\varphi)$.

B. The two types of possible orbits in the Hamilton–Jacobi formalism

Our main purpose is to characterize separatrices that extend to infinity in R. Therefore, it is enough to consider regions of the phase space R to the right of some appropriate (potential-dependent) value φ_0, i.e., regions of the form

$$R_0 = \{(\varphi, h) \in \mathbb{R}^2 : \varphi_0 \le \varphi < +\infty, \sqrt{v(\varphi)} < h < +\infty\}, \tag{26}$$

where we assume that the scaled potential v and its first derivative v' are smooth and strictly positive for $\varphi \ge \varphi_0$. Note that this stipulation extends the validity of our analysis to potentials not necessarily monotonic on the whole real line.

The next proposition states that the solutions $h(\varphi)$ of Eq. (18) in R_0 do not blow up at finite values of φ.

Proposition 1. The solution of (18) with initial value $h(\varphi_0) = h_0$ satisfies

$$h(\varphi) < h_0 e^{\varphi - \varphi_0}, \quad \forall \varphi > \varphi_0. \tag{27}$$

Proof. Taking into account that $\sqrt{h^2 - v} < h$, the solutions $h_{\sup}(\varphi) = h_0 e^{\varphi - \varphi_0}$ of the differential equation,

$$h'_{\sup} = h_{\sup}, \tag{28}$$

are super solutions[24,25] of Eq. (18). Hence, given solutions h of (18) and $h_{\sup}$ of (28) with the same initial data $(\varphi_0, h_0) \in R_0$, $h(\varphi) < h_{\sup}(\varphi)$ for $\varphi > \varphi_0$, i.e., any solution $h(\varphi)$ is bounded by a suitable exponential and, therefore, cannot blow up at a finite value of φ. □

The next proposition is a straightforward consequence of Proposition 1 and shows that there are only two possible behaviors for the orbits, which we call type-A and type-B solutions of Eq. (18).

Proposition 2. Given that a solution $h = h(\varphi)$ of (18) with initial value $h(\varphi_0) = h_0$, either it leaves R_0 by reaching the lower boundary Γ with a horizontal half-tangent line (type-A solution) or it exists and remains in R_0 for all $\varphi > \varphi_0$ (type-B solution).

Some comments are in order.

1. Since $\sqrt{h^2 - v}$ is smooth and positive on R_0, the solution $h(\varphi)$ obtained by integrating Eq. (18) backward from any point $(\varphi_1, \sqrt{v(\varphi_1)})$ of Γ can be continued all the way back to the vertical line $\varphi = \varphi_0$ of R_0. Since this solution has zero slope at φ_1 [because the differential equation (18) implies $h'|_\Gamma = 0$], by monotonicity, it cannot end at another point $(\varphi_2, \sqrt{v(\varphi_2)})$ of Γ with $\varphi_2 < \varphi_1$.
2. The set of solutions that end at the lower boundary Γ (type-A solutions) is always nonempty and fills a subregion of R_0 that may or may not be the whole phase space R_0. Again by monotonicity, it follows that there is an r with $\sqrt{v(\varphi_0)} < r \le \infty$ such that the solutions

starting at $h(\varphi_0) < r$ are type-A solutions. If $r \neq \infty$, the solutions with $h(\varphi_0) > r$ (type-B solutions) are global, i.e., they are defined for all $\varphi \geq \varphi_0$. As we will prove below for a wide class of models (the class $\mathscr{C}_\alpha$ introduced below), these type-B solutions grow exponentially as $\varphi \to \infty$. Furthermore, the solution corresponding to $h(\varphi_0) = r$, which separates the two types of solutions, will be shown to be globally defined, although its rate of growth as $\varphi \to \infty$ depends on the potential of the model. This solution, when it exists, is what we call the separatrix of the model.

C. Separatrices

We are now able to formulate a precise definition of a separatrix of Eq. (18):

Definition 1. If Eq. (18) has both type-A and type-B solutions, then $r < \infty$ and the solution corresponding to the initial condition $h(\varphi_0) = r$ is called a separatrix.

In Sec. IV, we will see that many potentials have separatrices for which $h(\varphi)/\sqrt{v(\varphi)}$ is bounded. It will be shown that separatrices of this kind have special properties.

III. PROPERTIES OF SEPARATRICES IN THE HAMILTON–JACOBI FORMALISM

To gain a better understanding of the solutions of Eq. (18) and of the existence of separatrices, it is useful to introduce the modified Hubble parameter,

$$\mathfrak{h}(\varphi) \equiv \frac{h(\varphi)}{\sqrt{v(\varphi)}}. \tag{29}$$

In terms of $\mathfrak{h} = \mathfrak{h}(\varphi)$, the orbit equation (18) reads

$$\mathfrak{h}' = \sqrt{\mathfrak{h}^2 - 1} - \mathfrak{v}\,\mathfrak{h}, \tag{30}$$

where

$$\mathfrak{v} \equiv (\log\sqrt{v})'. \tag{31}$$

Note that the function $\mathfrak{h}$ is related to the Hubble normalized potential $V/3H^2$: in fact,

$$\frac{1}{\mathfrak{h}^2} = \frac{v}{h^2} = \frac{V}{3H^2}, \tag{32}$$

where we have set $\mathrm{M}_{\mathrm{Pl}} = 1$. Similarly, for positive, strictly increasing potentials $V(\Phi)$, the function $\mathfrak{v}(\varphi)$ is proportional to the function $\lambda(\Phi)$ defined by

$$\lambda(\Phi) = \frac{V'(\Phi)}{V(\Phi)}. \tag{33}$$

Using the scaled variables (3), it follows that

$$\lambda(\Phi) = \sqrt{\frac{3}{2}}\frac{v'(\varphi)}{v(\varphi)} = \sqrt{6}\mathfrak{v}(\varphi), \tag{34}$$

again with $\mathrm{M}_{\mathrm{Pl}} = 1$. [Note that in the application of dynamical systems theory to ΛCDM cosmology, it is customary to use strictly decreasing potentials $V(\Phi)$ and to define $\lambda(\Phi)$ with an additional minus sign.[19]]

The phase space of (30) is given by $\mathfrak{R}_0$,

$$\mathfrak{R}_0 = \{(\varphi, \mathfrak{h}) \in \mathbb{R}^2 : \varphi_0 \leq \varphi < +\infty,\ 1 \leq \mathfrak{h} < +\infty\}. \tag{35}$$

Incidentally, the function,

$$y \equiv \log \mathfrak{h}(\varphi), \tag{36}$$

verifies the so-called *master equation* introduced by Handley *et al.* in Ref. 23,

$$y' = \sqrt{1 - e^{-2y}} - \mathfrak{v}. \tag{37}$$

The equivalent of Proposition 2 for the modified Hubble parameter is the following:

Proposition 3. A solution $\mathfrak{h} = \mathfrak{h}(\varphi)$ of (30) with an initial value $\mathfrak{h}(\varphi_0) = \mathfrak{h}_0$ either leaves $\mathfrak{R}_0$ by reaching the finite boundary $\mathfrak{h} = 1$ with a negative slope $-\mathfrak{v}$ (type-A solution) or exists and remains in $\mathfrak{R}_0$ for all $\varphi > \varphi_0$ (type-B solution).

We illustrate this modified phase space $\mathfrak{R}_0$ and Proposition 3 in Fig. 2, where we plot the solutions $\mathfrak{h}(\varphi)$ of Eq. (30) with $\mathfrak{v}(\varphi) = 1/\varphi$ corresponding to the curves $h(\varphi)$ of Fig. 3.

Equation (30) can be rewritten in the integral form as

$$\mathfrak{h}(\varphi) = \frac{\sqrt{v(\varphi_0)}}{\sqrt{v(\varphi)}}\mathfrak{h}(\varphi_0)\exp\int_{\varphi_0}^{\varphi}\sqrt{1 - \frac{1}{\mathfrak{h}(x)^2}}\,dx. \tag{38}$$

Thus, given two solutions $\mathfrak{h}_i(\varphi)$ $(i = 1, 2)$ of (30), we have

$$\frac{\mathfrak{h}_2(\varphi)}{\mathfrak{h}_1(\varphi)} = \frac{\mathfrak{h}_2(\varphi_0)}{\mathfrak{h}_1(\varphi_0)}\exp\int_{\varphi_0}^{\varphi}\left(\sqrt{1 - \frac{1}{\mathfrak{h}_2(x)^2}} - \sqrt{1 - \frac{1}{\mathfrak{h}_1(x)^2}}\right)dx. \tag{39}$$

Note that the potential v does not appear in the identity (39). This property is very convenient to analyze the behavior of the solutions of (18).

Proposition 4. Given two solutions $h_i(\varphi)(i = 1, 2)$ of (18) such that $h_2(\varphi_0) > h_1(\varphi_0)$, the corresponding functions $\mathfrak{h}_i(\varphi)$ $(i = 1, 2)$ satisfy

$$\mathfrak{h}_2(\varphi) > \mathfrak{h}_1(\varphi) \quad \text{for all } \varphi \geq \varphi_0, \tag{40}$$

and

$$\frac{\mathfrak{h}_2(\varphi)}{\mathfrak{h}_1(\varphi)} > \frac{\mathfrak{h}_2(\varphi')}{\mathfrak{h}_1(\varphi')} \quad \text{for all } \varphi > \varphi' > \varphi_0. \tag{41}$$

Proof. The first statement (40) is a consequence of the fact that the functions $\mathfrak{h}_i(\varphi)$ $(i = 1, 2)$ are solutions of the ordinary differential equation (30) with initial conditions satisfying $\mathfrak{h}_2(\varphi_0) > \mathfrak{h}_1(\varphi_0)$. The second statement (41) follows at once from (40) and the identity (39). □

The next theorem is a reformulation of several results proved by Handley *et al.*[23]

Theorem 1. *Let $\mathfrak{h}_s = \mathfrak{h}_s(\varphi)$ be a solution of (30) defined and bounded for all $\varphi \geq \varphi_0$ in $\mathfrak{R}_0$. Then,*

1. *$\mathfrak{h}_s$ is the only solution of (30) defined and bounded for all $\varphi \geq \varphi_0$.*

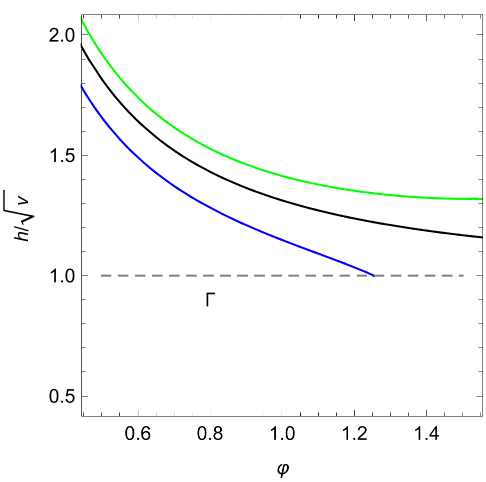

FIG. 2. Solutions $\mathfrak{h}(\varphi)$ of Eq. (30) in the phase space $\mathfrak{R}_0$ corresponding to the curves $h(\varphi)$ of Fig. 3.

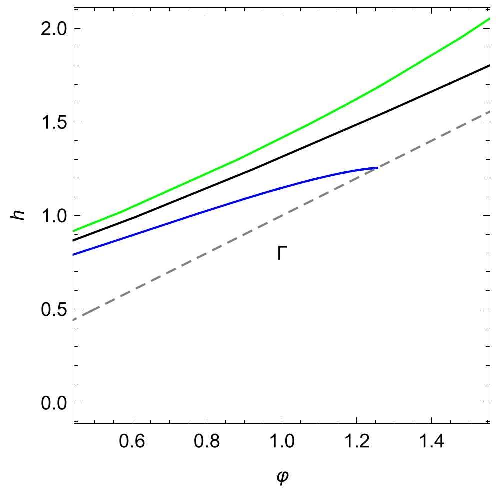

FIG. 3. Three solutions $h(\varphi)$ of Eq. (18) in the phase space R_0 for the quadratic model $v(\varphi) = \varphi^2$: the blue line is a type-A solution, the green line is a type-B solution, and the black line is the separatrix (itself a type-B solution). The gray line is the lower boundary Γ of R_0.

2. *If a solution $\mathfrak{h} = \mathfrak{h}(\varphi)$ of (30) is such that $\mathfrak{h}(\varphi_0) > \mathfrak{h}_s(\varphi_0)$, then $\mathfrak{h}$ is a type-B solution and grows exponentially as $\varphi \to \infty$.*
3. *If a solution $\mathfrak{h} = \mathfrak{h}(\varphi)$ of (30) is such that $\mathfrak{h}(\varphi_0) < \mathfrak{h}_s(\varphi_0)$, then $\mathfrak{h}$ is a type-A solution.*

Therefore, $\mathfrak{h}_s(\varphi)$ is the separatrix.

Proof. Let $\mathfrak{h}_s$ be a solution of (30) defined and bounded for all $\varphi \geq \varphi_0$, and let $\mathfrak{h}$ be a solution of (30) such that $\mathfrak{h}(\varphi_0) > \mathfrak{h}_s(\varphi_0)$, then according to (41), we have that

$$\frac{\mathfrak{h}(\varphi)}{\mathfrak{h}_s(\varphi)} > \Delta \equiv \frac{\mathfrak{h}(\varphi_0)}{\mathfrak{h}_s(\varphi_0)} > 1 \quad \text{for all } \varphi > \varphi_0. \tag{42}$$

Consequently,

$$\sqrt{1 - \frac{1}{\mathfrak{h}(x)^2}} - \sqrt{1 - \frac{1}{\mathfrak{h}_s(x)^2}} > \sqrt{1 - \frac{1}{\Delta^2 \mathfrak{h}_s(x)^2}} - \sqrt{1 - \frac{1}{\mathfrak{h}_s(x)^2}}. \tag{43}$$

Furthermore, the following function of $\mathfrak{h}_s$,

$$f(\mathfrak{h}_s) \equiv \sqrt{1 - \frac{1}{\Delta^2 \mathfrak{h}_s^2}} - \sqrt{1 - \frac{1}{\mathfrak{h}_s^2}}, \tag{44}$$

is positive and decreasing so that $f(\mathfrak{h}_s) \geq C$, where $C = f(\mathfrak{h}_*)$, with $\mathfrak{h}_*$ being the supremum of $\mathfrak{h}_s(\varphi)$ on the interval $\varphi \geq \varphi_0$. Therefore, from (39) and (43), we obtain that

$$\frac{\mathfrak{h}(\varphi)}{\mathfrak{h}_s(\varphi)} > \Delta\, \exp(C(\varphi - \varphi_0)) \quad \text{for all } \varphi \geq \varphi_0, \tag{45}$$

so that $\mathfrak{h}(\varphi)$ grows exponentially as $\varphi \to \infty$. This proves the statements 1 and 2. Regarding statement 3, it is clear that given that a solution $\mathfrak{h} = \mathfrak{h}(\varphi)$ of (30) such that $\mathfrak{h}(\varphi_0) < \mathfrak{h}_s(\varphi_0)$, it would be a bounded solution defined for all $\varphi \geq \varphi_0$ unless it leaves $\mathfrak{R}_0$ by crossing the lower boundary $\mathfrak{h} = 1$ of $\mathfrak{R}_0$ at a finite value of φ. □

From statement 2 of Theorem 1, it follows that type B solutions $h = h(\varphi)$ of (18) determine blow-up solutions $\varphi = \varphi(t)$ of the inflaton equation (4).

IV. EXISTENCE OF SEPARATRIX SOLUTIONS

A. A class $\mathscr{C}_\alpha$ of potentials with separatrix solutions

Definition 2. Given $0 \le \alpha < 1$, we define $\mathscr{C}_\alpha$ as the set of all the C^∞ real functions $v = v(\varphi)$ such that

1. *Both $v = v(\varphi)$ and its derivative $v' = v'(\varphi)$ are strictly positive for all φ larger than some φ_0.*
2. *The function $\mathfrak{v}$ defined in* Eq. (31) *satisfies*

$$\lim_{\varphi\to\infty} \mathfrak{v}(\varphi) = \alpha. \tag{46}$$

For convenience, we will take φ_0 such that $\mathfrak{v}(\varphi) < 1$ for all $\varphi > \varphi_0$.

Note that using Eq. (34) for potentials of the class $\mathscr{C}_\alpha$, condition (46) is equivalent to

$$\lim_{\Phi\to\infty} \lambda(\Phi) = \sqrt{6}\alpha. \tag{47}$$

For later applications, also note that the 0-isocline of Eq. (30) in $\mathfrak{R}_0$ is generally given by

$$\mathfrak{h}_{\rm iso}(\varphi) = \frac{1}{\sqrt{1-\mathfrak{v}(\varphi)^2}}, \tag{48}$$

which, for potentials $v \in \mathscr{C}_\alpha$, has the finite limit,

$$\lim_{\varphi\to\infty} \mathfrak{h}_{\rm iso}(\varphi) = \frac{1}{\sqrt{1-\alpha^2}}. \tag{49}$$

In the following example, we present a particular family of exponential potentials belonging to the class $\mathscr{C}_\alpha$, which will be used in the Proof of Theorem 2.

Example 1. The family of exponential potential functions,[10,22,32]

$$v_\alpha(\varphi) = (1-\alpha^2)\, e^{2\alpha\varphi}, \quad 0 < \alpha < 1, \tag{50}$$

belong to the class $\mathscr{C}_\alpha$. They determine a constant function $\mathfrak{v}(\varphi)$,

$$\mathfrak{v}(\varphi) = \alpha. \tag{51}$$

Their corresponding Hamilton–Jacobi equations *(18) have an explicit solution given by*

$$h_s(\varphi) = e^{\alpha\varphi}, \tag{52}$$

which according to Theorem 1 is a separatrix. Furthermore, in this case, the modified Hubble parameter $\mathfrak{h}_s(\varphi)$ (29) coincides with the 0-isocline solution (48) and it is given by the constant value,

$$\mathfrak{h}_s(\varphi) = \mathfrak{h}_{\rm iso}(\varphi) = \frac{1}{\sqrt{1-\alpha^2}}. \tag{53}$$

Theorem 2. *If $v \in \mathscr{C}_\alpha$, then the differential* equation *(18) has a separatrix solution $h_s(\varphi)$.*

Proof. Let us consider the differential equation (30) for the modified Hubble parameter $\mathfrak{h}$ determined by $\mathfrak{v} = (\log\sqrt{v})'$. From Proposition 3, we have that the phase space of (30),

$$\mathfrak{R}_0 = \{(\varphi, \mathfrak{h}) \in \mathbb{R}^2 \,:\, \varphi_0 \le \varphi < +\infty,\, 1 \le \mathfrak{h} < +\infty\}, \tag{54}$$

contains two possible types of solutions, those leaving $\mathfrak{R}_0$ by crossing the lower finite boundary $\mathfrak{h} = 1$ (type-A solutions) and those staying in $\mathfrak{R}_0$ for all $\varphi > \varphi_0$ (type-B solutions). Note that $\mathfrak{h}'(\varphi) = -\mathfrak{v}(\varphi) < 0$ at any point $(\varphi, 1)$ in the lower boundary.

Let us denote by I_A and I_B the subsets of real numbers $\mathfrak{h}_0 = \mathfrak{h}(\varphi_0) > 1$ corresponding to the initial data of the solutions $\mathfrak{h} = \mathfrak{h}(\varphi)$ of (30) of type A and type B, respectively. From Proposition 3, we have that

$$I_A \cap I_B = \emptyset, \quad I_A \cup I_B = (1, +\infty). \tag{55}$$

The subset I_A is nonempty [see comment (i) after Proposition 2]. To prove that the subset I_B is also nonempty, we use the model associated with a potential v_{α_0} of form (50) with $\alpha < \alpha_0 < 1$. Then, since $v \in \mathscr{C}_\alpha$ and due to (46), we can always choose a φ_0 such that

$$\mathfrak{v}(\varphi) < \alpha_0 \quad \text{for all } \varphi > \varphi_0. \tag{56}$$

Hence, it follows that

$$\sqrt{\mathfrak{h}^2 - 1} - \mathfrak{v}\,\mathfrak{h} > \sqrt{\mathfrak{h}^2 - 1} - \alpha_0\,\mathfrak{h} \quad \text{for all } (\varphi, \mathfrak{h}) \in \mathfrak{R}_0. \tag{57}$$

This means that the solutions of the differential equation (30) corresponding to $\mathfrak{v}$ are super solutions of the differential equation (30) corresponding to v_{α_0}.

We know that the constant line

$$\mathfrak{h}_{\text{iso}}(\varphi) = \frac{1}{\sqrt{1 - \alpha_0^2}} \tag{58}$$

is a 0-isocline of the differential equation (30) corresponding to v_{α_0}, and because of Eq. (57), the solutions of the differential equation (30) corresponding to v that cross this line will do it with a positive value of $\mathfrak{h}'$. Consequently, they cannot come back below the line (58), they are type-B solutions, and I_B is nonempty.

If we denote by r the real number defining both the supremum of the set I_A and the infimum of the set I_B, then the solution $\mathfrak{h}_r(\varphi)$ of (30) such that $\mathfrak{h}_r(\varphi_0) = r$ is of type B. Otherwise, it would hit a point φ_r of the boundary $\mathfrak{h} = 1$ and the backward solution $\widetilde{\mathfrak{h}}(\varphi)$ corresponding to another point $\widetilde{\varphi} > \varphi_r$ of the boundary $\mathfrak{h} = 1$ would verify $\widetilde{\mathfrak{h}}(\varphi_0) > r$, which is a contradiction. Furthermore, it cannot cross the line (58). Indeed, if it hits that line at a point $(\widetilde{\varphi}_0, (1 - \alpha_0^2)^{-1/2})$, then the backward solution $\widetilde{\mathfrak{h}}(\varphi)$ corresponding to another point of the line (58) to the right of $(\widetilde{\varphi}_0, (1 - \alpha_0^2)^{-1/2})$ will verify $\widetilde{\mathfrak{h}}(\varphi_0) < r$, which is a contradiction. Finally, it is clear that $\mathfrak{h}_r$ is a bounded solution as it is defined for all $\varphi \geq \varphi_0$ and remains inside the region bounded by the lines $\mathfrak{h} = 1$ and $\mathfrak{h} = (1 - \alpha_0^2)^{-1/2}$. Therefore, $\mathfrak{h}_r(\varphi)$ is a separatrix solution. □

In Fig. 4, we illustrate this argument for the quadratic potential $v(\varphi) = \varphi^2$, where we have taken $\alpha_0 = \sqrt{5}/3$ so that the horizontal dashed line corresponds to the 0-isocline $\mathfrak{h}_{\text{iso}}(\varphi) = 3/2$, while the slopes plotted on top of this line correspond to Eq. (30) with $\mathfrak{v}(\varphi) = 1/\varphi$ and φ_0 has been taken slightly greater than $1/\alpha_0$.

Example 2. The model with the Higgs potential,[33]

$$v(\varphi) = (\varphi^2 - 1)^2, \tag{59}$$

belongs to the extended class of potentials introduced in Sec. II B *for* $\varphi_0 \geq 1$ *and has an explicit separatrix solution of (18) given by*

$$h_s(\varphi) = \varphi^2 + 1. \tag{60}$$

Note that $\mathfrak{h}_s(\varphi) = (\varphi^2 + 1)/(\varphi^2 - 1) \to 1$ *as* $\varphi \to \infty$.

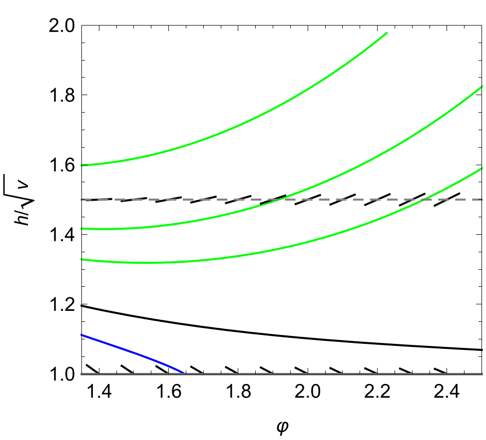

FIG. 4. Illustration of the argument leading to Theorem 2 for the case of the quadratic potential $v(\varphi) = \varphi^2$ and $\alpha_0 = \sqrt{5}/3$. The horizontal dashed line is the 0-isocline $\mathfrak{h}_{\text{iso}}(\varphi) = 3/2$, the slopes on top of this line correspond to Eq. (30) with $\mathfrak{v}(\varphi) = 1/\varphi$ (as do the half-slopes at $\mathfrak{h} = 1$), and φ_0 has been taken slightly greater than $1/\alpha_0$. The region between the horizontal lines $\mathfrak{h} = 1$ and $\mathfrak{h}_{\text{iso}}(\varphi) = 3/2$ is a backward-invariant region for the flow generated by Eq. (30). The separatrix is the black, continuous curve.

B. A class of potentials without a separatrix

The next proposition establishes an exponential lower bound for models $v \notin \mathscr{C}_\alpha$ without separatrix solutions.

Proposition 5. If the potential $v(\varphi)$ grows faster than $\exp(2\varphi)$ *as $\varphi \to \infty$, then all the solutions of* Eq. (18) *are type-A solutions and there is no separatrix.*

Proof. As a consequence of Proposition 1, the functions $h_{\sup}(\varphi) = h_0 e^{\varphi-\varphi_0}$ are super solutions of (18). Hence, under our assumption on $v(\varphi)$, these super solutions cross the boundary $h(\varphi) = \sqrt{v(\varphi)}$ at finite values of φ, and so do the solutions of (18), which lie between $\sqrt{v(\varphi)}$ and $h_{\sup}(\varphi)$. □

The sharp character of this bound is shown by the next example.

Example 3. Equation (30) *for the potential function,*

$$v(\varphi) = e^{2\varphi}, \tag{61}$$

reads

$$\mathfrak{h}' = \sqrt{\mathfrak{h}^2 - 1} - \mathfrak{h}. \tag{62}$$

The right-hand side of (62) is upper bounded by $-1/(2\mathfrak{h})$ *so that the solutions,*

$$\mathfrak{h}_{\sup}(\varphi) = \sqrt{\mathfrak{h}_0^2 + \varphi_0 - \varphi}, \tag{63}$$

of the differential equation,

$$\mathfrak{h}'_{\sup} = -\frac{1}{2\mathfrak{h}_{\sup}}, \tag{64}$$

are super solutions of (62). Hence, if $\mathfrak{h}$ and $\mathfrak{h}_{\sup}$ are solutions of (62) and (64), respectively, with the same initial data $(\varphi_0, \mathfrak{h}_0) \in \mathfrak{R}_0$, then $\mathfrak{h}(\varphi) < \mathfrak{h}_{\sup}(\varphi)$ for $\varphi > \varphi_0$. Thus, all the solutions of (62) leave $\mathfrak{R}_0$ at a finite value of φ. Therefore, all the solutions are of type-A and, consequently, there is no separatrix solution for the model (61).

In Ref. 19, Alho and Uggla used a dynamical systems analysis to show that global and asymptotic bounds should be imposed on $-V'(\Phi)/V(\Phi)$ to obtain a viable cosmological model that continuously deforms ΛCDM cosmology. Particularly relevant to the present work are the bounds on that magnitude, which for our positive strictly increasing potentials and our sign convention, translate to $0 \le \lambda < \sqrt{6}$, or, using (34), to $0 \le \mathfrak{v} < 1$. Hence, Eq. (47) implies that $\alpha \le 1$, and the models belong to $\mathscr{C}_\alpha$ if $\alpha < 1$. Models with $\lambda \ge \sqrt{6}$, i.e., $\alpha \ge 1$ ("steep-enough potentials"), do not belong to our class $\mathscr{C}_\alpha$ and exhibit an oscillatory behavior toward the past.[11]

C. Beyond the class $\mathscr{C}_\alpha$: Potentials with unbounded separatrices in $\mathfrak{R}_0$

Although it is true for potentials in the class $\mathscr{C}_\alpha$ (Theorem 2), the notion of the separatrix as given in Definition 1 does not imply the boundedness of $\mathfrak{h}_s(\varphi) = h_s(\varphi)/\sqrt{v(\varphi)}$ in $\mathfrak{R}_0$.

For example, let us consider the potential function,

$$v(\varphi) = \frac{2\varphi - 1}{\varphi^4}\, e^{2\varphi}, \tag{65}$$

with $\varphi_0 > 1/2$. It is an exponential function of the type discussed in Proposition 5 modulated by a decaying rational function. It has been generated by imposing the following solution of (18):

$$h(\varphi) = \frac{e^{\varphi}}{\varphi}. \tag{66}$$

In this case, $\lim_{\varphi\to\infty} \mathfrak{v}(\varphi) = 1$ so that v is outside the class $\mathscr{C}_\alpha$ with $0 \le \alpha < 1$. Thus, despite the fact that the function $\mathfrak{h} = h/\sqrt{v} \sim \sqrt{\varphi/2}$ is unbounded as $\varphi \to \infty$, we will show that it may be considered to be a separatrix.

In order to analyze what happens near $h(\varphi)$, we introduce the variable $w \equiv \mathfrak{h}(\varphi)/\sqrt{\varphi}$. We expect w to be bounded away from zero for the separatrix solution. The equation for w is

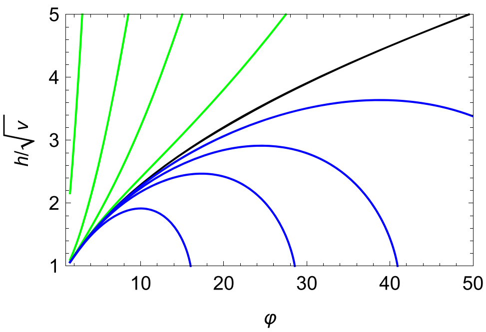

FIG. 5. Unbounded separatrix $\mathfrak{h}_s(\varphi) = \varphi/\sqrt{2\varphi - 1}$ (black curve) and a few trajectories in $\mathfrak{R}_0$ for the potential $v(\varphi) = (2\varphi - 1)e^{2\varphi}/\varphi^4$ of Eq. (65).

$$w'\sqrt{\varphi} + w\frac{1}{2\sqrt{\varphi}} = \sqrt{\varphi w^2 - 1} - \mathfrak{v}(\varphi)\sqrt{\varphi}\,w, \tag{67}$$

or

$$w' = \sqrt{w^2 - \frac{1}{\varphi}} - (1 + b(\varphi))w, \tag{68}$$

where

$$b(\varphi) \equiv (\mathfrak{v}(\varphi) + \frac{1}{2\varphi}) - 1 \sim -\frac{1}{\varphi}, \quad \varphi \to \infty. \tag{69}$$

Thus, for $\varphi \gg 1$, we get

$$2ww' \sim \frac{2w^2 - 1}{\varphi}. \tag{70}$$

An explicit solution is $w_c \sim 1/\sqrt{2}$ that corresponds to our original function h. The other solutions either go to $+\infty$, if they lie above w_c, or to $-\infty$ infinite, if they lie below w_c. Indeed, integrating Eq. (70) gives

$$|2w^2 - 1| \sim C\varphi^2. \tag{71}$$

For the solutions above (respectively, below) the constant solution w_c, we have $w \sim C\varphi$ (respectively, $w \sim -C\varphi$) as $\varphi \to \infty$. Hence, we have $\mathfrak{h} \sim w\sqrt{\varphi} \sim C\varphi^{3/2}$ so that $h = \mathfrak{h}\sqrt{v} \sim e^{\varphi}$, the maximal divergence that can be found. The approximations used in this heuristic argument can be justified because the terms left out in the expansions are of lower order. We illustrate these results in Fig. 5, where we plot the unbounded separatrix,

$$\mathfrak{h}_s(\varphi) = \varphi/\sqrt{2\varphi - 1}, \tag{72}$$

and a few trajectories in $\mathfrak{R}_0$ for the potential (65).

V. ASYMPTOTIC EXPANSIONS OF SEPARATRICES

A. Leading term as $\varphi \to \infty$: Separatrices with backward inflation

The next result gives the leading term of the asymptotic expansion of the separatrix $h_s(\varphi)$ as $\varphi \to \infty$ for potentials in the class $\mathscr{C}_\alpha$. In particular, for $\alpha = 0$, this leading term coincides with the slow-roll approximation[4,5] to h_s.

Theorem 3. *The leading asymptotic behavior of the separatrix $h_s(\varphi)$ for potentials $v \in \mathscr{C}_\alpha$ is*

$$h_s(\varphi) \sim \frac{\sqrt{v(\varphi)}}{\sqrt{1 - \alpha^2}}, \quad \varphi \to \infty. \tag{73}$$

Proof. For potentials $v \in \mathscr{C}_\alpha$ and $\varphi > \varphi_0$, the function $\mathfrak{h}_s(\varphi)$ is bounded from below [by (1)], bounded from above, and satisfies the integral equation (38),

$$\mathfrak{h}_s(\varphi) = \sqrt{\frac{v(\varphi_0)}{v(\varphi)}}\mathfrak{h}_s(\varphi_0)\exp\left(\int_{\varphi_0}^{\varphi}\sqrt{1-\frac{1}{\mathfrak{h}_s(x)^2}}\,dx\right). \tag{74}$$

If the potential $v(\varphi)$ diverges as $\varphi \to \infty$, then from (74), it follows that the condition $\mathfrak{h}_s(\varphi) > 1$ can be satisfied for all $\varphi > \varphi_0$ only if the integral in (74) also diverges as $\varphi \to \infty$. As a consequence, applying the L'Hôpital rule in (74) yields

$$\mathfrak{h}_s(\varphi) \sim \frac{\mathfrak{h}_s(\varphi)}{\mathfrak{v}(\varphi)}\sqrt{1-\frac{1}{\mathfrak{h}_s(\varphi)^2}}, \quad \varphi \to \infty. \tag{75}$$

Hence,

$$\lim_{\varphi\to\infty}\mathfrak{h}_s(\varphi) = \frac{1}{\sqrt{1-\alpha^2}}, \tag{76}$$

and (73) follows.

If the potential $v(\varphi) \to$ constant as $\varphi \to \infty$, then $\alpha = 0$. Indeed, according to Definition 2 $v'(\varphi) > 0$ for $\varphi \geq \varphi_0$, and from the elementary equation,

$$v(\varphi) = v(\varphi_0) + \int_{\varphi_0}^{\varphi} v'(x)\,dx, \tag{77}$$

we have that $v'(\varphi) \to 0$ as $\varphi \to \infty$. Therefore, from (74), we deduce that $\mathfrak{h}_s(\varphi)$ is bounded only if the integral in (74) is convergent as $\varphi \to \infty$, and this requires that $\mathfrak{h}_s(\varphi) \to 1$ ($h_s(\varphi) \sim \sqrt{v(\varphi)}$) as $\varphi \to \infty$. □

From (6) and (73), it follows that the separatrices of the models for potentials $v \in \mathscr{C}_\alpha$ support inflation as $\varphi \to \infty$ (backward inflation) provided that

$$\alpha < \frac{1}{\sqrt{3}}. \tag{78}$$

In particular, this means that in case the separatrix blows up at a given cosmic time t^*, inflation takes place in a neighborhood of the singularity t^*. Again, using (47), condition (78) for accelerated expansion is in agreement with the result $\lambda < \sqrt{2}$ of Ref. 19.

B. The asymptotic expansion of the separatrix for divergent potentials with $\alpha = 0$

For potentials such that $\lim_{\varphi\to\infty} v(\varphi) = \infty$ and $\alpha = 0$ (e.g., monomial potentials), $v' = o(v)$ and we can go beyond the slow-roll approximation and find asymptotic expansions of the form

$$h_s(\varphi) \sim u + \sum_{n=1}^{\infty}\frac{h_n[u]}{u^n}, \quad \varphi \to \infty, \tag{79}$$

where the coefficients $h_n[u] = h_n(u', u'', \ldots, u^{(n)})$ are differential polynomials in the derivatives $u^{(j)} (j \geq 1)$ of the function,

$$u \equiv \sqrt{v(\varphi)}. \tag{80}$$

Indeed, if we substitute (79) into (13) and identify coefficients of powers of $1/u^n$, $(n \geq -2)$, we obtain the recursion relation,

$$\sum_{j+k=n;\, j,k\geq -1}\Big(\big(h_j' - (j-1)h_{j-1}u'\big)\big(h_k' - (k-1)h_{k-1}u'\big) - h_j h_k\Big) + \delta_{n,-2} = 0, \tag{81}$$

where h_n' stands for the total derivative of the differential polynomial h_n with respect to φ, and $h_{-2} \equiv 0$. The explicit recursive character of (81) is exhibited by the equivalent relation,

$$h_{n+1} = \frac{1}{2}\sum_{j+k=n;\, j,k\geq 0}\Big(\big(h_j' - (j-1)h_{j-1}u'\big)\big(h_k' - (k-1)h_{k-1}u'\big) - h_j h_k\Big), \quad n \geq -1. \tag{82}$$

Thus, we find that $h_{-1} = 1$, $h_0 = 0$, and, to the third order in $1/u$,

$$h_s(\varphi) \sim u + \frac{(u')^2}{2u} + \frac{u''(u')^2}{u^2} + \left(u'''(u')^3 + \frac{5}{2}(u''u')^2 - \frac{5}{8}(u')^4\right)\frac{1}{u^3} + \cdots, \quad \varphi \to \infty. \tag{83}$$

Example 4. For the even monomial potentials,

$$v(\varphi) = \varphi^{2p}, \tag{84}$$

Eq. (83) *reduces to a power series expansion,*

$$h_s(\varphi) \sim \varphi^p + \sum_{n=1}^{\infty} b_n \varphi^{p-2n}, \tag{85}$$

where the coefficients b_n satisfy the recurrence relation,

$$b_1 = \frac{p^2}{2}, \qquad b_{n+1} = p(p-2n)b_n - \frac{1}{2}\sum_{j+k=n+1} b_j b_k + \frac{1}{2}\sum_{j+k=n}(p-2j)(p-2k)b_j b_k. \tag{86}$$

Thus, the first terms of these expansion are

$$h_s(\varphi) \sim \varphi^p + \frac{p^2}{2}\varphi^{p-2} + \frac{1}{8}p^3(3p-8)\varphi^{p-4} + \frac{1}{16}p^4(5p^2-40p+72)\varphi^{p-6} + \cdots. \tag{87}$$

In particular, for the quadratic potential ($p = 1$), we find

$$h_s(\varphi) \sim \varphi + \frac{1}{2\varphi} - \frac{5}{8\varphi^3} + \frac{37}{16\varphi^5} - \frac{1773}{128\varphi^7} + \cdots, \tag{88}$$

and for the quartic potential ($p = 2$),

$$h_s(\varphi) \sim \varphi^2 + 2 - \frac{2}{\varphi^2} + \frac{12}{\varphi^4} - \frac{122}{\varphi^6} + \cdots. \tag{89}$$

Example 5. For the Higgs potential $v(\varphi) = (\varphi^2 - a^2)^2$, the function $u = \varphi^2 - a^2$, whose inverse powers can, in turn, be re-expanded in powers of $1/\varphi^2$ to give again an asymptotic power series whose first terms are

$$h_s(\varphi) \sim \varphi^2 + 2 - a^2 + \frac{2(a^2-1)}{\varphi^2} + \cdots. \tag{90}$$

C. Educated match summation of asymptotic expansions in inverse powers of the inflaton

Similar asymptotic expansions (in particular, for the square of the Hubble parameter) have been derived by different procedures, and there is some interest in the numerical summation of these series, which is typically performed by Padé approximants.[18] In this brief section, we point out how the recently developed educated match summation method[34] can be used to advantage for this purpose.

For concreteness, let us consider the asymptotic expansion of the separatrix for the quadratic potential (88). From the recursion relation (86) with $p = 1$, it follows that

$$\frac{b_n}{b_{n-1}} \sim -2n, \quad n \to \infty, \tag{91}$$

i.e., in addition to the alternating sign, there is a factorial growth of the coefficients. These two facts lead to conjecture that the series might be Borel summable and that the method of educated match, wherein the series to be summed is matched to the known, Borel-summable asymptotic expansion of (in general, a linear combination of scaled versions of) the confluent hypergeometric function $\Phi(z) = z^{-a}U(a, 1+a-b, 1/z)$. We proceed in close analogy to the calculation of Sec. 3.3 in Ref. 34: because of the pattern of signs in Eq. (88), we pull out a factor $1/\varphi$. The simplest approximant requires only the first two coefficients, $b_0 = 1/2$ and $b_1 = -5/8$, and we choose $a = 1/2$ and $b = 1$ to impose the regularity of the approximant at $\varphi = 0$. With these values of the parameters, the confluent hypergeometric function can be written in terms of the complementary error function erfc [see Eq. (7.1.2) in Ref. 35] and we find

$$h_s(\varphi) \sim \varphi + \sqrt{\frac{\pi}{10}}\, e^{2\varphi^2/5}\operatorname{erfc}(\sqrt{2/5}\,\varphi). \tag{92}$$

This simple, analytic approximation to the separatrix $h_s(\varphi)$ is surprisingly accurate on all the range $\varphi \geq 0$. For example, it gives the maximum error at $h_s(0) \approx \sqrt{\pi/10} = 0.560\,499\ldots$, while the value obtained by (unstable) numerical integration is $h_s(0) = 0.569\,172\,64\ldots$, i.e., an error of less than 1.53%. The error decreases monotonically and quickly as $\varphi \to \infty$. As an illustration, in Fig. 6, we plot the result of a numerical integration of the corresponding differential equation using a shooting strategy to find the appropriate initial condition at $\varphi = 0$, the graph of the approximant (92), and, in the dashed line, the result of a $[1, 1]$ Padé approximant, which uses one more term of the expansion (88). The

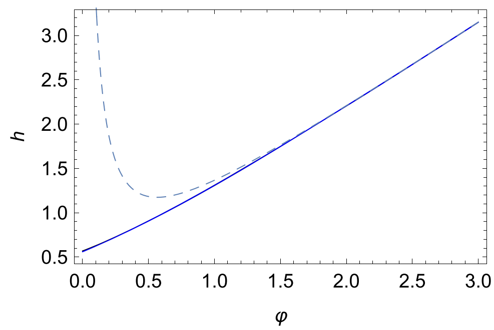

FIG. 6. Separatrix of the quadratic potential $v(\varphi) = \varphi^2$ obtained by numerical integration of the differential equation, by the analytic approximant (92) (continuous lines, actually superimposed in this figure), and by a Padé approximant (dashed line). The value at the origin given by the numerical integration is $h_s(0) = 0.569\,172\,64\ldots$, while the value given by the analytic approximant is $h_s(0) \approx \sqrt{\pi/10} = 0.560\,499\ldots$, with an error of less than 1.53%.

first two graphs are in effect superimposed, while the Padé approximant diverges at the origin. Higher-order approximants are increasingly accurate.

Likewise, for the quartic potential, we find

$$h_s(\varphi) \sim 1 + \varphi^2 + \sqrt{\frac{\pi}{2}}\varphi e^{\varphi^2/4}\operatorname{erfc}(\varphi/2). \tag{93}$$

This analytic approximation gives the maximum error at $h_s(0) \approx 1$, while the value obtained by (unstable) numerical integration is $h_s(0) = 0.954\,931\ldots$, i.e., an error of less than 4.72%.

D. α-attractor E-models and the Starobinsky potential

Single-field inflationary models with potentials of the form

$$v(\varphi) = (1 - e^{-\beta\varphi})^{2p} \quad (\beta > 0), \tag{94}$$

are called α-attractor E-models.[36–38] In particular, the case $p = 1$ is the well-known Starobinsky model,[1,39] which fits nicely the most recent experimental results.[40]

Theorem 2 shows that these models do have separatrices, but since the function $u = \sqrt{v(\varphi)}$ is bounded as $\varphi \to \infty$, the ansatz (79) does not apply. However, there is a unique asymptotic series of the form

$$h_s(\varphi) \sim \sum_{n=0}^{\infty} b_n e^{-n\beta\varphi}. \tag{95}$$

Indeed, if we substitute this expansion into Eq. (13) and identify the coefficients of the exponentials $e^{-n\beta\varphi}$, we get the recurrence relation,

$$\sum_{j+k=n} (\beta^2 jk - 1) b_j b_k = \begin{cases} (-1)^{n-1}\binom{2p}{n}, & n = 0, 1, \ldots, 2p, \\ 0, & n > 2p. \end{cases} \tag{96}$$

The solution to this recurrence relation has to be given independently for three ranges of n. Concretely, the first three coefficients are

$$b_0 = 1, \quad b_1 = -p, \quad b_2 = \frac{p(2p-1)}{2} + \frac{1}{2}(\beta^2 - 1)p^2, \tag{97}$$

for $n = 3, \ldots, 2p$,

$$b_n = -\big(\beta^2(n-1) - 1\big)p b_{n-1} + \frac{1}{2}\sum_{j+k=n,j,k\geq 2} (\beta^2 jk - 1) b_j b_k + \frac{(-1)^n}{2}\binom{2p}{n}, \tag{98}$$

while for $n > 2p$,

$$b_n = -\big(\beta^2(n-1)-1\big)pb_{n-1} + \frac{1}{2}\sum_{j+k=n, j,k\geq 2}(\beta^2 jk-1)b_j b_k. \tag{99}$$

Moreover,

$$\frac{b_n}{b_{n-1}} \sim -\beta^2 np, \quad n\to\infty, \tag{100}$$

and it is also a reasonable conjecture that the series (95) might be Borel-summable in the variable $e^{-\beta\varphi}$.

By the way of example, the first three terms of the asymptotic expansion for the separatrix of the general α-attractor E-model are

$$h_s(\varphi) \sim 1 - pe^{-\beta\varphi} + \left(\frac{p(2p-1)}{2} + \frac{1}{2}(\beta^2-1)p^2\right)e^{-2\beta\varphi} + \cdots, \tag{101}$$

which, for the Starobinsky potential,[1,39] reduce to

$$h_s(\varphi) \sim 1 - e^{-\beta\varphi} + \frac{\beta^2}{2}e^{-2\beta\varphi} + \cdots. \tag{102}$$

We may also apply this method for negative values of φ by considering the potentials $\tilde{v}(\varphi) = v(-\varphi)$, which belong to the class $\mathscr{C}_\alpha$ with $\alpha = p\beta$, provided that $p\beta < 1$. This condition is equivalent to the condition $\bar{\lambda} < 1$ used in Ref. 20 to study the global dynamics of E-models.

E. Exponentially steep potential well

As a final example, we briefly study the steep exponential potential,

$$v(\varphi) = e^{2\beta\varphi} + e^{-2\beta\varphi}, \tag{103}$$

where β is a positive constant. Foster[11] showed that this potential has a separatrix if $\beta < 1$. This result also follows from our Theorem 2, since this potential belongs to the class $\mathscr{C}_\alpha$ with $\alpha = \beta$.

It is easily seen that the asymptotic expansion for the separatrix can be written as

$$h_s(\varphi) \sim \frac{e^{\beta\varphi}}{\sqrt{1-\beta^2}} + \sum_{n=1}^{\infty} b_n e^{-\beta(4n-1)\varphi}. \tag{104}$$

Substitution of (104) into (18) leads to the recurrence relation,

$$b_n = \frac{\sqrt{1-\beta^2}}{2\big((4n-1)\beta^2+1\big)}\sum_{j+k=n}\big((4j-1)(4k-1)\beta^2-1\big)b_j b_k, \tag{105}$$

and the first terms in the expansion of $h_s(\varphi)$ as $\varphi\to\infty$ are

$$h_s(\varphi) \sim \frac{e^{\beta\varphi}}{\sqrt{1-\beta^2}} + \frac{\sqrt{1-\beta^2}}{2(3\beta^2+1)}e^{-3\beta\varphi} + \frac{(1-\beta^2)^{3/2}(9\beta^2-1)}{8(3\beta^2+1)^2(7\beta^2+1)}e^{-7\beta\varphi} + \frac{(1-\beta^2)^{5/2}(9\beta^2-1)(21\beta^2-1)}{16(3\beta^2+1)^3(7\beta^2+1)(11\beta^2+1)}e^{-11\beta\varphi} + \cdots. \tag{106}$$

F. Separatrices with or without blow up inflaton field

Finally, we will use the asymptotic result (83) to make a brief digression on the behavior of the inflaton field $\varphi_s(t)$ corresponding to a separatrix $h_s(\varphi)$ as a function of the cosmic time. These solutions $\varphi_s(t)$ may be either defined for all t or blow up at a finite negative value of t. For potentials that diverge as $\varphi\to\infty$ and with $\alpha = 0$, we can use the expansion (83) to obtain

$$h_s' = \sqrt{h_s^2 - v} \sim (\sqrt{v})', \quad \varphi\to\infty. \tag{107}$$

Hence, Eq. (22) shows that $\varphi_s(t)$ blows up if and only if $1/(\sqrt{v})'$ is integrable as $\varphi\to\infty$.

In particular, the separatrices of inflaton models with even monomial potentials $v = \varphi^{2p}$ for $p = 1, 2$ as well as Higgs potentials $v(\varphi) = (\varphi^2 - a^2)^2$ have separatrix fields $\varphi_s(t)$ without blow up. For instance, the inflaton or the separatrix of the Higgs model with $a = 1$ is given explicitly by

$$\varphi_s(t) = \varphi_0 e^{-2t}. \tag{108}$$

Likewise, the expansion (102) of the separatrix for the Starobinsky model (102) shows that $1/\sqrt{h^2 - v} \sim e^{\beta\varphi}/\beta$. Therefore, the integral (22) is divergent and the separatrix determines a solution $\varphi_s(t)$ of (4) without blow-up.

The case $\varphi \to -\infty$ for the Starobinsky model can be analyzed with the change of variables (25), which reduces the problem to the case $\varphi \to \infty$ for the potential $\tilde{v}(\varphi) = (e^{\beta\varphi} - 1)^2$. It can be proved that a separatrix exists only for $\beta < 1$.

Finally, the separatrices $h_s(\varphi) = e^{\alpha\varphi}$ of the exponentially increasing potentials v_α (50) of class $\mathscr{C}_\alpha$ are associated with the explicit inflaton fields,

$$\varphi_s(t) = -\frac{1}{\alpha}\log\left(\alpha^2 t + e^{-\alpha\varphi_0}\right), \tag{109}$$

which blow up at a finite time.

ACKNOWLEDGMENTS

The financial support of the Spanish Ministerio de Economía y Competitividad under Project Nos. FIS2015-63966-P, PGC2018-094898-B-I00, and PGC2018-098440-B-I00 is gratefully acknowledged. J.L.V. would like to thank the Departamento de Análisis Matemático y Matemática Aplicada of the Universidad Complutense de Madrid for his appointment as Honorary Professor.

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