Instituto Complutense de Análisis Económico

UNIVERSIDAD COMPLUTENSE

FACULTAD DE ECONOMICAS

Campus de Somosaguas

28223 MADRID

Teléfono 394 26 11 - FAX 294 26 13

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Alfonso Novales

Emilio Domínguez

Javier Pérez

Jesús Ruiz

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SOLVING NONLINEAR RATIONAL EXPECTATIONS MODELS BY EIGENVALUE-EIGENVECTOR DECOMPOSITIONS*

Alfonso Novales[1], Emilio Domínguez[2], Javier Pérez[3] and Jesús Ruiz[4]

ABSTRACT

We provide a summarized presentation of solution methods for rational expectations models, based on eigenvalue/eigenvector decompositions. These methods solve systems of stochastic linear difference equations by relying on the use of stability conditions derived from the eigenvectors associated to unstable eigenvalues of the coefficient matrices in the system. For nonlinear models, a linear approximation must be obtained, and the stability conditions are approximate. This is however, the only source of approximation error, since the nonlinear structure of the original model is used to produce the numerical solution. After applying the method to a baseline stochastic growth model, we explain how it can be used: i) to solve some identification problems that may arise in standard growth models, and ii) to solve endogenous growth models.

Keywords: Eigenvalue-eigenvector decompositions, numerical solutions, rational expectations.
JEL Classification: C63, E17.

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[1]Universidad Complutense de Madrid. Departamento de Economía Cuantitativa. Campus de Somosaguas. 28223 Madrid. Spain.

[2]Universidad Pública de Navarra. Departamento de Economía, Campus de Arrosadía. 31006 Pamplona (Navarra), Spain.

[3]Universidad Complutense de Madrid. Departamento de Economía Cuantitativa. Campus de Somosaguas. 28223 Madrid. Spain.

[4]Universidad Europea de Madrid. Departamento de Fundamentos del Análisis Económico. Madrid. Spain.

1 Introduction

We discuss in this chapter the main issues involved in practical applications of solution methods that have been proposed for rational expectations models, based on eigenvalue/eigenvector decompositions. Methods to solve linear stochastic difference equations under rationality of expectations go back to at least Blanchard and Kahn (1980) and have been studied by many authors ever since [for general surveys, see Whiteman (1983), the special issue of the Journal of Business and Economic Statistics (1990), Marcet (1993) or Danthine and Donaldson (1995)]. Our presentation relies heavily on Sims (1998), who has extended the existing practice in important directions that are discussed in this chapter. Although, strictly speaking, the methods apply exactly to systems of linear equations, the extension to compute an approximate solution to nonlinear rational expectations models is straightforward.

In this solution approach, each conditional expectation and the associated expectation error are treated as additional endogenous variables and an equation is added to the model, defining the expectation error. The numerical solution is in the form of a set of time series for all variables in the model economy, including all the conditional expectations and the associated expectations errors. Besides, as a by-product, an approximate characterization of the analytical dependence between expectations errors and structural shocks is obtained. Since it produces time series for the expectations errors, it allows for the possibility of multiple tests of the rationality hypothesis in the form of: i) lack of serial correlation in one-step ahead expectations errors, ii) a specific moving average structure for expectations errors of a given function at different horizons, iii) orthogonality between errors of expectations made at time t and variables in the information set available at that time, in the form of the accuracy test in den Haan and Marcet (1994). Numerical solutions to rational expectations errors are hardly ever tested along these directions. Precisely because so much emphasis has been paid on rationality as the benchmark when dealing with uncertainty in economic environments where agents solve optimization problems, it is quite surprising that so little attention has been paid to testing for the nature of the computed solution.

Since the solution method applies to any given set of (possibly nonlinear) stochastic difference equations, the method is not restricted to dealing with planning problems like we do in the applications in this chapter. It can equally well handle situations in which distortionary taxation, externalities, indivisibilities, public goods, etc., lead to decentralized allocation of resources which are inefficient [for applications of very different nature, see Sims (1994) and (1998)].

Economic models usually place bounds on the rates of growth of specific variables or linear combinations of variables. Well-known cases are standard planner's problems, in which state variables and their shadow prices cannot grow too fast for transversality conditions to hold and objective functions to be bounded. Methods based on eigenvalue/eigenvector decompositions rest on the use of stability conditions guaranteeing that the resulting solution satisfies the upper bounds on growth rates which may be implied by the underlying economic theory. In the simple applications we discuss, the stability conditions are obtained by imposing orthogonality between each eigenvector associated to an unstable eigenvalue in the decomposition of the linear system, and the vector of variables in it although in more complex models, stability conditions will adopt a different form [for a more general version of

stability conditions, see Sims(1998)]. The *stability conditions* link decision to state variables and exogenous shocks. They can sometimes be written to represent some decision variables as functions of states and exogenous variables. Together with other relations in the system, they characterize how optimal decisions are made, and can therefore be interpreted as *decision rules*. In some other cases, they will represent relationships between prices and states and exogenous variables, having therefore the interpretation of *pricing rules*.

Different solution methods for nonlinear models differ in i) the way they characterize the stable solution manifold, ii) the computation of the expectations in the model, and the amount of information they provide on them, and iii) the amount of nonlinearity that they preserve when computing the numerical solution. In our case, the application of the eigenvalue/eigenvector decomposition method to nonlinear models requires constructing a linear approximation to the model around steady state, from which to derive the stability conditions. They are then added to the original, nonlinear model, to compute a numerical solution. Even though the actual nonlinear structure of the model is used to produce the numerical solution, the set of stability conditions is obtained from a linear approximation, which introduces some numerical error. The approximation error that is introduced by the specific computational details of a particular solution approach will always end up being absorbed by the expectations errors, which is why testing for rationality should be considered a crucial component of a reported numerical solution.

The details we provide should be enough to design the application of the solution method to simple environments. As more interesting and complex models start being considered, rather more technical considerations are bound to arise, with independence of the solution method used. These more technical aspects emerge because stochastic, non-linear quadratic dynamic control problems under the assumption of rationality are hard to solve: rationality of expectations imposes very tight restrictions, which can either lead to nonexistence of solutions, or to a difficult computation process of the solutions, when they exist. In addition, the existence of state variables that accumulate over time will generally tend to produce unstable paths, that would violate the transversality conditions of the problem or the more general restrictions on growth rates that may exist. That motivates the consideration of stability conditions in this approach as a crucial piece of solving a model. The need to guarantee stability is also present in deterministic problems, as we review in Section 2, but it gets more complex in stochastic models. Solution methods will have to increasingly be able to accommodate these issues.

In Section 2 we review how a numerical solution can be derived from the standard deterministic Cass-Koopmans, Brock-Mirman economy, pointing out the relevance of stability conditions. In Section 3 we summarize the general structure used to solve linear rational expectations models and its extension to nonlinear models. In Section 4 we apply the solution method to Hansen's (1985) model of indivisible labor, which is also used as illustration in other chapters of this book. Comparisons with other solution approaches are discussed in Section 5. In Section 6 we show how the eigenvalue-eigenvector decomposition can help to separately identify variables of a similar nature, as it is the case when physical capital and inventories are inputs in an aggregate production technology. Section 7 shows how the solution method can be adapted to deal with endogenous growth models. The chapter closes with a summary.

2 Stability conditions and the initial choice of control variables in deterministic growth models

This Section is a reminder to the reader that: i) stability conditions are also needed in standard deterministic models to guarantee that transversality conditions will hold, and ii) as it is the case in applications of the solution method to stochastic setups, the stability conditions in deterministic models are given by the left eigenvectors corresponding to the unstable eigenvalues of the linear approximation to the model economy. A reader familiar with this discussion can safely skip this Section.

Let us consider the deterministic version of the standard Cass–Koopmans, Brock–Mirman planner's problem in an economy with decreasing returns to scale in physical capital and labour, but constant returns on the aggregate. In that economy, the only sustainable steady state is with zero growth for all per-capita variables. It is well known that the model has a *saddle point* structure, so that in the consumption/capital stock plane there is a single trajectory taking the economy towards its steady state. Given an initial stock of capital k_0, an initial choice of consumption other than the one corresponding to k_0 on the stable manifold will take the economy to diverge from its steady state. Besides, optimality requires staying on the stable manifold forever, so stability and optimality are in this simple model two sides of the same coin.

The model is usually formulated in continuous time, in which the specific issues dealing with time series generation do not arise. Let us suppose a constant relative risk aversion utility of consumption for the representative agent $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$, $\sigma > 0$. Labor is supplied inelastically, since leisure is not an argument in the utility function. Physical capital is subject to a depreciation rate of δ. Population growth could be easily incorporated to the model. The planner's problem in the Cass–Koopmans, Brock–Mirman economy is characterized by the intertemporal first order condition that links the marginal rate of substitution of consumption over time to the marginal product of capital, the law of motion of the capital stock, and the transversality condition:

$$\begin{aligned} \frac{c_t^{\sigma}}{c_{t-1}^{\sigma}} &= \beta\left(f'(k_{t-1}) + (1-\delta)\right), \qquad (1)\\ k_t &= (1-\delta)k_{t-1} + f(k_{t-1}) - c_t, \qquad (2)\\ \lim_{\tau\to\infty} \beta^{t+\tau} c_{t+\tau}^{-\sigma} k_{t+\tau} &= 0. \end{aligned}$$

The two first equations can be approximated around steady state values of consumption and capital, c_{ss} and k_{ss} :

$$\begin{pmatrix} k_t - k_{ss} \\ c_t - c_{ss} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} k_{t-1} - k_{ss} \\ c_{t-1} - c_{ss} \end{pmatrix} \qquad (3)$$

Using the standard decomposition of the A matrix of coefficients in the linear system (3): $A = \Gamma\Lambda\Gamma^{-1}$, where Λ has the eigenvalues of A along the diagonal and zeroes elsewhere, and

Γ has as columns the right–eigenvectors of A, and Γ^{-1} has as rows the left–eigenvectors of A, we can represent the dynamics of the solution from starting values k_0, c_0 as[1]:

$$\begin{aligned} \begin{pmatrix} k_t - k_{ss} \\ c_t - c_{ss} \end{pmatrix} &= \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} k_{t-1} - k_{ss} \\ c_{t-1} - c_{ss} \end{pmatrix} \\ &= \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} k_0 - k_{ss} \\ c_0 - c_{ss} \end{pmatrix} \qquad (4) \end{aligned}$$

That the model has a saddle point structure is reflected in the fact that one of the eigenvalues, λ_1, say, is greater than 1 in absolute value, while λ_2 is smaller[2] than 1.

The matrix product in the previous expression is:

$$\begin{pmatrix} k_t - k_{ss} \\ c_t - c_{ss} \end{pmatrix} = \begin{pmatrix} x_1\lambda_1^t(u_1(k_0 - k_{ss}) + v_1(c_0 - c_{ss})) + y_1\lambda_2^t(u_2(k_0 - k_{ss}) + v_2(c_0 - c_{ss})) \\ x_2\lambda_1^t(u_1(k_0 - k_{ss}) + v_1(c_0 - c_{ss})) + y_2\lambda_2^t(u_2(k_0 - k_{ss}) + v_2(c_0 - c_{ss})) \end{pmatrix}$$

and the transversality condition on the capital stock will hold only if the coefficient in the unstable eigenvalue, λ_1, is set equal to zero. But x_1 depends on the values of the structural parameters, and cannot be chosen to be zero. So, it is the bracket accompanying λ_1^t which will be zero. *That condition is the same for the capital stock and consumption equations*: $u_1(k_0 - k_{ss}) + v_1(c_0 - c_{ss}) = 0$, so that stability requires that initial consumption be chosen by:

$$c_0 - c_{ss} = -(k_0 - k_{ss})\frac{u_1}{v_1} = (k_0 - k_{ss})\frac{y_2}{y_1} = (k_0 - k_{ss})\frac{\lambda_2 - a_{11}}{a_{12}}$$

and it implies that, from then on:

$$\begin{aligned} k_t - k_{ss} &= y_1\lambda_2^t(u_2(k_0 - k_{ss}) + v_2(c_0 - c_{ss})), \\ c_t - c_{ss} &= y_2\lambda_2^t(u_2(k_0 - k_{ss}) + v_2(c_0 - c_{ss})) = \frac{y_2}{y_1}(k_t - k_{ss}) = \frac{\lambda_2 - a_{11}}{a_{12}}(k_t - k_{ss}), \end{aligned}$$

so that *the same condition between the deviations from steady state of the capital stock and consumption will hold at each point in time than at time* 0. That is the approximate

[1]The right eigenvectors are: $(x_1, x_2) = \left(1, \frac{\lambda_1 - a_{11}}{a_{12}}\right)$ and $(y_1, y_2) = \left(1, \frac{\lambda_2 - a_{11}}{a_{12}}\right)$, and the inverse matrix:

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}^{-1} = \frac{1}{x_1 y_2 - x_s y_1} \begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix}$$

[2]As we will see later on, the critical rate of growth below which the solution is stable is model–specific. The requirement for a well–defined solution to exist is that the objective function remains bounded, which will require upper bounds on its variable arguments. Those bounds will depend on the functional form of the objective function. Sometimes, transversality conditions take care of that. In other cases, transversality conditions may be needed for feasibility or optimality even when the objective function is bounded, so that extra upper bounds on growth rates will then need to be added, to guarantee that transversality conditions hold.

linear representation of the stable manifold in this problem. Precisely because the condition will actually hold for every t, the model can be solved using that condition and just one of the first order conditions (1), (2). *The condition which is not used will hold each period.* The stability condition above can be written as the inner product: $(y_2, -y_1)(k_0 - k_{ss}, c_0 - c_{ss})' = 0$, where $(y_2, -y_1)$, is the left eigenvector of A associated to the unstable root, λ_1.

Therefore, in deterministic models, the stability conditions can be seen as picking the *stable* initial values of the decision variables, as functions of the given initial values of the states. If we have less stability conditions than decision variables in the system[3], we will just be able to solve the model as a function of a given (arbitrary) starting value for one (or more) decision variables. In that case, given a vector of state variables, a whole continuum of initial decisions will take us to the steady state, and the solution is *indeterminate*, in the sense of Benhabib and Perli (1994) and Xie (1994). On the other hand, the system does not have a solution when there are more independent stability conditions than control variables need to be chosen. The stable subspace will then reduce to the steady state, if it exists, and the economy will be globally *unstable*, getting into divergent paths as soon as it experiences even minimum deviations from its steady state. Finally, the solution will be *unique* when the set of stability conditions can be used to represent all the control variables as functions of the state and exogenous variables, the system of equations having a unique solution.

The single stability condition we have described for the Cass–Koopmans, Brock–Mirman economy is very similar to the stability conditions we will compute in stochastic models in the next Sections to guarantee that the conditional expectations version of the transversality conditions will hold.

3 An overview of the solution strategy

Recently, Sims (1998) has generalized the work of Blanchard and Kahn (1980) in several directions, proposing a general discussion of the problem of solving stochastic, linear rational expectations models:

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + C + \Psi z_t + \Pi \eta_t \tag{5}$$

where C is a vector of constants, y_t is the vector of variables determined in the model, other than expectations errors, z_t is a vector of innovations in exogenous variables, and η_t is a vector of rational expectations errors, satisfying $E_t(\eta_{t+1}) = 0$.

Models with more lags can be accommodated by adding as new variables first order lags of already included variables as it is standard in dynamic representations. On the other hand, additional expectations variables can be introduced so that the resulting expectations errors are all one–period ahead. Models with more lags, lagged expectations, or expectations of more distant future values can be accommodated by defining variables at intermediate steps, and enlarging the y–vector.

[3]After using equations that involve only contemporaneous values of control variables to eliminate some control variables from the problem.

The core of the procedure consists on defining each conditional expectation as a new variable and adding to the model the associated expectations error and the equation defining the error. Taking arbitrary initial conditions y_0 and using (5) to generate a set of time series for the variables in y_t, conditional on sample realizations for z_t will generally lead to unstable paths, which will violate the transversality conditions unless stability conditions are added to the system. These conditions are defined by the eigenvectors associated to unstable eigenvalues of the matrices in (5), although the structure of the stability conditions is generally model–specific. When Γ_0 is invertible, we compute the eigenvalues of $\Gamma_0^{-1}\Gamma_1$, while when Γ_0 is singular, we need to compute the generalized eigenvalues of the pair (Γ_0, Γ_1).

The vector y_t includes the variables in the model with the more advanced subindeces, as well as the conditional expectations in the model, which are redefined as new variables. All of them are determined in the system. They may be decision variables for an economic agent, like consumption, the stock of capital, real balances, real debt, leisure, hours of work, etc., or variables which are determined as a function of them, like prices or interest rates. Also included in y_t are variables which are exogenous to the agents but follow laws of motion which have been added to the system, as it may happen with some policy variables or exogenous random shocks. The vector z_t contains variables which are determined outside the system, like policy variables which we have not endogeneized and do not show any serial correlation, or the innovations in policy variables or in the exogenous random shocks[4]. These can be either *demand shocks*, like those affecting the individual's preferences or Government expenditures, *supply shocks*, affecting the ability to produce commodities, or *errors of controlling* Government policy variables. When they are not white noise, the exogenous shocks themselves are included in y_t. For instance, the standard autoregression for a productivity shock: $\log(\theta_t) = \rho \log(\theta_{t-1}) + \epsilon_t$, will lead to a component of y_t being $\log(\theta_t)$, while ϵ_t will be a component of z_t. The vector η_t contains the rational expectations errors, which will be solved for endogenously, together with the state and decision variables in the model.

The solution method can also be applied to obtain approximate solutions to a set of stochastic, nonlinear difference equations, as in the applications we present in this chapter. To do so, we start by computing the linear approximation around steady state of the set of nonlinear equations so that, without loss of generality, we can consider the vector of constants C to be zero[5]. After appropriately redefining variables, the matrices Γ_0 and Γ_1 in the linear approximation to a nonlinear model contain: i) the partial derivatives of each equation of the system with respect to each of the variables in y_t, evaluated in steady state, and ii) rows of ones and zeroes, corresponding to intermediate variables which have been added to the system to make it a first–order autoregression in the presence of higher order lags, or higher order expectations. In this case, (5) will approximate the set of decision rules, budget constraints, policy rules and laws of motion for the exogenous variables, and all variables will be in deviations to their steady state values. The stability conditions are then obtained in this linear approximation, but the original, nonlinear model is used to generate the solution,

[4]Variables in z_t are independent: if two exogenous shocks are related, the linear approximation to their relationship will be added to the system; one of them will be in z_t while the other one will be included in y_t.

[5]In a later Section we will also consider the case when the levels of the variables are not constant in steady state, as it is the case in endogenous growth models.

in the form of a set of time series realization for all the variables in the economy, including the expectations that appear in the original system and the associated expectations errors.

Along this chapter, we describe how the method applies to relatively simple problems, and explain how to use it to simulate nonlinear rational expectations models emerging from optimizing behavior on the part of economic agents. The reader interested on a complete discussion of the technical and practical aspects of the solution method for linear models should read Sims (1998), which gives detailed account of the arguments that apply to a more general class problems than those we consider here. Sims' paper also contains a detailed explanation of a variety of unproven claims that we make along this chapter. When possible, we keep the same notation as in his paper to facilitate references to it.

The methods to characterize the stability conditions differ depending on whether or not the Γ_0 in (5) matrix is invertible. In general, however, a singular Γ_0 matrix might be obtained, and a slightly more general procedure will then be needed. We will examine both cases in the examples in the next sections.

4 Solving a standard, stochastic growth model

We start by describing some practical details of the implementation of the solution method to Hansen's (1985), (1997) model with indivisible labor, that was introduced to better capture some labor market features relative to the more basic version of the real business cycle model, and which is considered in other chapters of this volume [see, for instance, Uhlig (1998)]. In the linear approximation to this model, the Γ_0 matrix is invertible. Numerical solutions to the simpler growth model with productivity shocks but no labor/leisure decisions, the other benchmark used in this volume, as well as in the special issue of the *Journal of Business and Economic Statistics* (1990), can easily be derived as a special case of the discussion in this Section.

Given an initial value of the capital stock, k_0, let us assume that the representative household chooses sequences of consumption, employment and capital stock that solve the problem:

$$\max_{\{k_t,c_t,N_t\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{c_t^{1-\sigma}-1}{1-\sigma} - A_N N_t \right] \tag{6}$$

subject to

$$\begin{aligned} -c_t - k_t + (1-\delta)k_{t-1} + \theta_t k_{t-1}^{\alpha} N_t^{1-\alpha} &= 0 \\ -\log(\theta_t) + \rho\log(\theta_{t-1}) + \epsilon_t &= 0 \\ \text{given } k_0, \theta_0 & \end{aligned}$$

where N_t denotes the number of hours devoted to the production of the consumption commodity, A_N measures the relative disutility of working hours and the innovation ϵ_t in the productivity process is assumed to be $N(0,\sigma_\epsilon)$. After forming the Lagrangean and eliminating the Lagrange multipliers we get the equilibrium characterized by the set of equations:

$$\begin{aligned} c_t &= \theta_t k_{t-1}^{\alpha} N_t^{1-\alpha} - k_t + (1-\delta)k_{t-1} &(7)\\ c_t^{-\sigma} &= \beta E_t \left[c_{t+1}^{-\sigma} \left((1-\delta) + \alpha\theta_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} \right) \right] &(8)\\ A_N &= c_t^{-\sigma} \theta_t k_{t-1}^{\alpha} (1-\alpha) N_t^{-\alpha} &(9) \end{aligned}$$

and

$$\begin{aligned} \log(\theta_t) &= \rho\log(\theta_{t-1}) + \epsilon_t &(10)\\ \epsilon_t &\sim N(0,\sigma_\epsilon^2) \end{aligned}$$

plus the transversality condition $\lim_{\tau\to\infty} E_t \left[c_{t+\tau}^{-\sigma} k_{t+\tau} \beta^{\tau} \right] = 0$, and the initial conditions k_0, θ_0.

We now define a new variable W_t as equal to the conditional expectation in (8), and introduce the corresponding expectation error, η_t:

$$\begin{aligned} 0 &= -W_t + E_t \left[c_{t+1}^{-\sigma} \left((1-\delta) + \alpha\theta_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} \right) \right] &(11)\\ 0 &= -c_t^{-\sigma} + \beta W_t &(12)\\ 0 &= -W_{t-1} + c_t^{-\sigma} \left[(1-\delta) + \alpha\theta_t k_{t-1}^{\alpha-1} N_t^{1-\alpha} \right] - \eta_t, &(13)\\ & E_t[\eta_{t+1}] = 0. \end{aligned}$$

Treating the conditional expectations in the model as additional variables is distinctive of this method. It comes together with also adding as new variables the associated expectations errors, which will be solved for endogenously together with the rest of the variables, including the expectation.

The conditions characterizing the steady state are:

$$\begin{aligned} c_{ss} &= \theta_{ss} k_{ss}^{\alpha} N_{ss}^{1-\alpha} - k_{ss} + (1-\delta)k_{ss} \\ W_{ss} &= c_{ss}^{-\sigma} \left((1-\delta) + \alpha\theta_{ss} k_{ss}^{\alpha-1} N_{ss}^{1-\alpha} \right) \\ A_N &= c_{ss}^{-\sigma} \theta_{ss} k_{ss}^{\alpha} (1-\alpha) N_{ss}^{-\alpha} \\ 0 &= -c_{ss}^{-\sigma} + \beta W_{ss} \end{aligned}$$

where the steady state for technology is $\theta_{ss} = 1$. Then, we can solve for the steady state of all the variables of the economy,

$$\begin{aligned} \frac{k_{ss}}{N_{ss}} &= \left[\frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta \right) \right]^{\frac{1}{\alpha-1}} \\ c_{ss} &= \left(\frac{A_N}{1-\alpha} \right)^{\frac{-1}{\sigma}} \left(\frac{k_{ss}}{N_{ss}} \right)^{\frac{\alpha}{\sigma}} \end{aligned}$$

$$
\begin{aligned}
k_{ss} &= c_{ss}\left(\left(\frac{k_{ss}}{N_{ss}}\right)^{\alpha-1}-\delta\right)^{-1} \\
W_{ss} &= \frac{1}{\beta}c_{ss}^{-\sigma} \\
N_{ss} &= k_{ss}\left(\frac{k_{ss}}{N_{ss}}\right)^{-1}
\end{aligned}
$$

The system to be linearized is the one formed by the optimality conditions (7), (9), and (12), the definition of the expectation error (13), and the process for the exogenous shock (10). State variables are k_{t-1}, W_{t-1} and $\log(\theta_t)$, and decision variables are c_t, k_t and N_t. To linearize, we view each equation as a function: $f(c_t, N_t, W_t, k_t, \log(\theta_t), \eta_t, \epsilon_t) = 0$ and then, defining the vector $y_t = (c_t - c_{ss},\ N_t - N_{ss},\ W_t - W_{ss},\ k_t - k_{ss},\ \log(\theta_t))'$, the vector η_t, which contains the single expectation error denoted by the same letter, and the 1×1 vector z_t, containing the single exogenous innovation ϵ_t, the first order approximation around steady state is[6]:

$$\frac{\partial f}{\partial y_t}|_{ss}y_t + \frac{\partial f}{\partial y_{t-1}}|_{ss}y_{t-1} + \frac{\partial f}{\partial \eta_t}|_{ss}\eta_t + \frac{\partial f}{\partial \epsilon_t}|_{ss}\epsilon_t = 0$$

where steady state values of η_t and ϵ_t are equal to zero. Stacking these approximations, we can write the linearized system as:

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi\eta_t \tag{14}$$

where:

$$\Gamma_0 = \begin{pmatrix} 1 & -k_{ss}^{\alpha}(1-\alpha)N_{ss}^{-\alpha} & 0 & 1 & -k_{ss}^{\alpha}N_{ss}^{1-\alpha} \\ \sigma c_{ss}^{-\sigma-1}k_{ss}^{\alpha}(1-\alpha)N_{ss}^{-\alpha} & c_{ss}^{-\sigma}k_{ss}^{\alpha}(1-\alpha)\alpha N_{ss}^{-\alpha-1} & 0 & 0 & -c_{ss}^{-\sigma}k_{ss}^{\alpha}(1-\alpha)N_{ss}^{-\alpha} \\ -\sigma c_{ss}^{-\sigma-1} & 0 & -\beta & 0 & 0 \\ \sigma c_{ss}^{-\sigma-1}\left(\alpha k_{ss}^{\alpha-1}N_{ss}^{1-\alpha}+1-\delta\right) & -c_{ss}^{-\sigma}\alpha k_{ss}^{\alpha-1}(1-\alpha)N_{ss}^{-\alpha} & 0 & 0 & -\alpha c_{ss}^{-\sigma}k_{ss}^{\alpha-1}N_{ss}^{1-\alpha} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & \alpha k_{ss}^{\alpha-1}N_{ss}^{1-\alpha}+1-\delta & 0 \\ 0 & 0 & 0 & c_{ss}^{-\sigma}\alpha k_{ss}^{\alpha-1}(1-\alpha)N_{ss}^{-\alpha} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha(1-\alpha)c_{ss}^{-\sigma}k_{ss}^{\alpha-2}N_{ss}^{1-\alpha} & 0 \\ 0 & 0 & 0 & 0 & \rho \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

4.1 Characterizing the stability conditions

The constant term in (14) is zero, since variables in y_t are in deviations around their steady state values. As we already mentioned, for any sensible set of parameter values, Γ_0 is invertible in this model. Pre–multiplying by the inverse of Γ_0, we get a transformed system

[6]Obtaining the derivatives of the function f for the approximation is not necessarily hard work since one can use numerical or analytical differenciation with MATLAB, for example.

with an identity matrix of coefficients in y_t and, after appropriate redefinition of matrices ($\tilde{\Gamma}_1 = \Gamma_0^{-1}\Gamma_1$; $\tilde{\Psi} = \Gamma_0^{-1}\Psi$; $\tilde{\Pi} = \Gamma_0^{-1}\Pi$):

$$y_t = \Gamma_0^{-1}\Gamma_1 y_{t-1} + \Gamma_0^{-1}\Psi z_t + \Gamma_0^{-1}\Pi\eta_t = \tilde{\Gamma}_1 y_{t-1} + \tilde{\Psi}z_t + \tilde{\Pi}\eta_t \tag{15}$$

Matrix $\tilde{\Gamma}_1$ has a Jordan decomposition[7]: $\tilde{\Gamma}_1 = P\Lambda P^{-1}$,where P is the matrix of right–eigenvectors of $\tilde{\Gamma}_1$, P^{-1} is the matrix of left–eigenvectors, and Λ has the eigenvalues of $\tilde{\Gamma}_1$ in the diagonal, and zeroes elsewhere[8]. Multiplying the system by P^{-1} and defining $w_t = P^{-1}y_t$, we get:

$$w_t = \Lambda w_{t-1} + P^{-1}\left(\tilde{\Psi}z_t + \tilde{\Pi}\eta_t\right), \tag{16}$$

which is a system in linear combinations of the variables in the original vector y_t. We will have a corresponding equation for each eigenvalue λ_j of $\tilde{\Gamma}_1$:

$$w_{jt} = \lambda_{jj}w_{j,t-1} + P^{j\bullet}\left(\tilde{\Psi}z_t + \tilde{\Pi}\eta_t\right), \tag{17}$$

where $P^{j\bullet}$ denotes the j–th row of P^{-1}.

Economic models usually impose upper bounds on the rate of growth of some functions. Special, even if frequent cases, are standard planner's problems like the one we are considering, in which the product of state variables by their shadow prices cannot grow at a rate faster than β^{-1} for the transversality conditions to hold. Even though it is not necessary, this condition is usually imposed through the requirement that both, state variables and shadow prices, grow at a rate lower than $\beta^{-1/2}$. Besides, the quadratic approximation to the objective functions in an optimization problem will be bounded only if its variable arguments grow at a rate lower than $\beta^{-1/2}$, being β the time–discount factor. More general restrictions can be approximated by an upper bound φ on the rate of growth of a linear combination ϕy_t of the variables in the model. Using the relationship between y_t and w_t, a condition of the form: $\lim_{s\to\infty} E_t\left[\phi y_{t+s}\varphi^{-s}\right] = 0$ amounts to: $\phi P\lim_{s\to\infty} E_t\left[w_{t+s}\varphi^{-s}\right] = (\phi P)\lim_{s\to\infty}\left(\Lambda^s w_t\varphi^{-s}\right) = 0$, where we have set to zero current expectations of future z_t's and η_t's. Therefore, each of the w_j variables corresponding to a $|\lambda_{jj}| > \varphi$ and to a ϕP product different from zero, must be equal to its steady state value of zero for all t:

$$w_{jt} = P^{j\bullet}y_t = 0, \quad \forall t \tag{18}$$

producing a stability condition in the form of an orthogonality condition between an unstable left–eigenvector of the matrix product $\tilde{\Gamma}_1 = \Gamma_0^{-1}\Gamma_1$ and the vector of variables y_t, in deviations around steady state.

The resulting condition will be a linear relationship between decision variables, current and past states and exogenous variables, which could be interpreted either as a *decision rule*,

[7]The MATLAB function for doing this is: $\text{eig}(\Gamma_0, \Gamma_1)$

[8]We just consider the simpler case when all eigenvalues are different from each other. For cases with multiplicity of eigenvalues see Sims (1998).

if it is used to write one decision variable as a function of the other variables, or as a *pricing function*, if it is used to represent a mapping from states and decisions to prices.

In the special case when ϕP turns out to be zero, the upper bound on the growth rate of ϕy_t does not impose any obvious constraint, and the precise form of the associated stability condition needs to be worked out specifically.

4.2 Generating time series for a specific parameterization

For parameter values: $\sigma = 1.5, \delta = 0.025, \alpha = 0.36, \beta = 0.99, \rho = 0.95$, and an A_N value such that $N_{ss} = \frac{1}{3}$, we have the numerical estimates:

$$\Gamma_0 = \begin{pmatrix} 1 & -2.3706 & 0 & 1 & -1.2347 \\ 4.4026 & 2.9103 & 0 & 0 & -2.6947 \\ -1.8572 & 0 & -0.99 & 0 & 0 \\ 1.8759 & -0.0766 & 0 & 0 & -0.0399 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1.0101 & 0 \\ 0 & 0 & 0 & 0.0766 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.0020 & 0 \\ 0 & 0 & 0 & 0 & 0.95 \end{pmatrix}$$

and the matrix $\tilde{\Gamma}_1 = \Gamma_0 \Gamma_1^{-1}$ has a Jordan decomposition $\tilde{\Gamma}_1 = P \Lambda P^{-1}$, with

$$P = \begin{pmatrix} 1 & 0 & 0.0302 & 0.0317 & 0 \\ 0 & 1 & -0.0178 & -0.0158 & -0.0245 \\ 0 & 0 & -0.0566 & -0.0595 & 0 \\ 0 & 0 & 0.9978 & 0.9976 & -0.9997 \\ 0 & 0 & 0 & 0.0049 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9418 & 0 & 0 \\ 0 & 0 & 0 & 0.95 & 0 \\ 0 & 0 & 0 & 0 & 1.0725 \end{pmatrix},$$

and

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0.5331 & 0 & 0 \\ 0 & 1 & -0.7464 & -0.0245 & -0.8417 \\ 0 & 0 & -17.6576 & 0 & -213.9769 \\ 0 & 0 & 0 & 0 & 203.6377 \\ 0 & 0 & -17.6237 & -1.0003 & -10.3588 \end{pmatrix}$$

where the eigenvalues have been ordered increasingly along the diagonal of Λ, and the right–eigenvectors, the columns of P, have been ordered accordingly.

The stability condition is given by the last row of P^{-1}, which corresponds to the only eigenvalue above $0.99^{-1/2}$. We denote that row by $P^{5\bullet}$, $P^{5\bullet} = (0, 0, -17.6237, -1.0003, -10.3588)$, so the stability condition turns out to be:

$$w_{5t} = P^{5\bullet} y_t = 0 \ \forall t \ \Rightarrow W_t - W_{ss} + 0.0568(k_t - k_{ss}) + 0.5878 \log(\theta_t) = 0 \qquad (19)$$

which happens not to involve consumption or labor.

A single stability condition is what should be expected from the point of view of the discussion of the deterministic model in Section 2, since, even though there are two control

variables, consumption and labor, whose initial values need to be chosen, there is also a contemporaneous relationship (9) between them, so that we just need to figure out how to choose one of them to obtain a stable equilibrium. Besides, since there is a single expectations error in the model, a single stability condition is, in general, all that is needed to identify it.

The difference with the deterministic case is that now, the stability condition does not guarantee that the intertemporal, stochastic Euler equation (8) will hold in every period, since it incorporates the expectation error. The role of this equation, once it is written as (13), is precisely to provide us with the realization of the expectation error [9], which shows that the stability condition can also be seen as imposing an exact relationship between the rational expectation error and the innovation in the productivity shock, as it should be expected. Estimating the stability conditions allows us to also characterize numerically the relationships between expectations errors and innovations in structural processes, as we are about to see.

Stable solutions can be computed by adding the estimated stability condition (19) to the original, nonlinear model to have an enlarged system that can be solved for all the endogenous variables in the model, plus the expectations errors. Conditional on k_0 and θ_1, (7), (9) and (12) form a system in c_1, k_1, N_1, and W_1 which can be used to write the three latter variables as functions of c_1. Plugging those expressions into the stability condition (19), we obtain c_1. The optimal value for labor, N_1, is then obtained from (9), while from the budget constraint (7) we obtain physical capital, and the realization of the conditional expectation W_1 is obtained from (12). Then, the expectation error, η_1, can be obtained from (13). The process can be repeated every period.

It is clear from (16) that, as we said, setting up w_{jt} to zero each period when $|\lambda_{jj}| > \varphi$ and $\phi P \neq 0$ amounts to imposing an exact relationship between the vector of innovations in the structural shocks and the vector of expectations errors:

$$P^{j\bullet}\left(\tilde{\Psi} z_t + \tilde{\Pi} \eta_t\right) = 0 \ \ \forall t \qquad (20)$$

implying that expectations errors must fluctuate as functions of the structural innovations, in such a way that prevents any deviation of (20) from its steady state value of zero.

In this specific model, setting w_{jt} to zero each period in (17) implies:

$$P^{5\bullet}\left(\tilde{\Psi} z_t + \tilde{\Pi} \eta_t\right) = 0 \ \ \forall t \ \ \Rightarrow \ \ -1.6163\epsilon_t + \eta_t = 0 \ \ \forall t \qquad (21)$$

which is an exact relationship between the expectations error in the model and the innovation in the single structural shock. However, the expectations error we have computed from (13) depends in a nonlinear fashion from state and decision variables and hence, from exogenous shocks. It will not satisfy (21) exactly, which is a different approximation to the true, nonlinear relationship between expectation error and the innovation in the structural shock in productivity.

Problems for *existence* of a solution will tend to arise when there are more linearly independent stability conditions than conditional expectations in the model. The set of ex-

[9] The resulting expectations error is an approximation to the true expectations error, since it also incorporates the numerical error of the approximation to the stable manifold.

pectations errors cannot possibly adjust, in this case, so as to fully offset the fluctuations in the exogenous processes, in such a way that (20) holds, and there will not be a well defined relationship between expectations errors and structural innovations. If (20) cannot hold is because the stability conditions cannot all hold simultaneously. Hence, a stable solution will generally not exist. Unfortunately, an absolute result on existence cannot be produced out of a counting rule of unstable eigenvalues and expectations: it is conceivable that some stability conditions are redundant with the rest of the system in such a way that (20) can hold, even if the number of rows in $P^{j\bullet}$ exceeds the dimension of the vector η_t.

If there are as many stability conditions as expectations in the model none of them being redundant with the rest of the system, a unique solution will generally exist. In the simple models we present in this chapter, as well as in some more complex applications we have developed, this has always been the case[10]. Then, stable solutions may be obtained by combining the stability conditions with the rest of the (nonlinear) model. That system will provide us with a set of time series for all the variables in the original system, plus the variables we have defined as expectations, and the expectations errors. If there are less stability conditions than expectations in the model, we will generally have *sunspot equilibria*, since we could arbitrarily fix some expectation, and still solve for the rest in such a way that all the equations in the model hold. In this case we will have a continuum of equilibria.

5 Comparison to other solution methods

A variety of methods to solve nonlinear rational expectations models exist in the literature, so it is important to clearly understand the differences and similarities among them. In a specific situation, a method may be more accurate than other but also computationally more demanding, and the researcher should choose one or the other in terms of this trade-off.

Given the characteristics of rational expectations models, differences among solution methods may fall into:

- how much of the nonlinearity in the original model is preserved when actually computing a solution,
- how does a specific method guarantee that the obtained solution is stable,
- how does it deal with the expectations in the model: whether they are treated as an essential part of the model, and whether numerical values are obtained for them endogenously, as part of the solution,
- the way to handle the associated expectations errors: whether they are considered as an integral part of the model, and whether numerical values can be easily obtained for them. Precision in computing these errors should be considered as an important component of a solution to a rational expectations model, since these models impose a quite tight structure on the probability distribution of the expectations errors.

[10]In linear models, a rank condition for uniqueness can be found [see Sims(1998)] but it is not applicable to the nonlinear case. The condition has to do with the possibility that the model can be solved without having to condition on any endogenous expectation error.

These four characteristics are not independent from each other: departing from the original nonlinearity will make computation easier, but the functional form approximation error will be mostly captured by the expectations errors, which are generally computed residually, once the solution has been obtained for the rest of the variables. This numerical error will tend to show up in deviations from rationality, in the form of autocorrelated expectations errors, or as correlations between them and variables in the information set available to the agents when they made their decisions, for instance. These numerical deviations from rationality just reflect the fact that different amounts of the original nonlinearity will lead, other things equal, to a set of time series that will reflect more or less accurately the behavior of economic agents in the original model. That is why conducting thorough tests of rationality is so important. For this analysis to be feasible, we need to be able to generate time series for the conditional expectations in the model; it is clear that once we have them, we can produce time series for the expectations errors as differences to the realized values of the functions inside the expectations.

We compare in this Section how *eigenvalue–eigenvector decomposition methods* handle these issues, in relation to other methods based on linearization, *undetermined coefficients* and the *linear quadratic approximation*, as well as to a finite-element method, *parameterized expectations*.

Regarding *nonlinearity*, in the method based on the *eigenvalue/eigenvector decomposition* of the linear approximation to the standard, stochastic nonlinear growth model in Section 4, a single linear stability condition, with the variables either in levels or in logs[11], was added to the model. That is the degree of artificial linearity introduced in the solution since, other than that, the full structure of the original, nonlinear model, is used to generate the numerical solution. As a result, a nonlinear system of equations has to be solved each period to compute the solution.

Relative to this approach, the method of *undetermined coefficients* proposed by Uhlig [(1998), this volume] suggests taking a log–linear approximation to the set formed by the optimality conditions, the budget constraint, and the autoregressive process for the productivity shock. State and decision variables are then supposed to be linear functions of the initial states: $\tilde{k}_{t-1}, \tilde{\theta}_t$, where tilde denote now log-deviations from steady state:

$$\begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \\ \tilde{N}_t \end{bmatrix} = \begin{bmatrix} \nu_{kk} & \nu_{kz} \\ \nu_{ck} & \nu_{cz} \\ \nu_{Nk} & \nu_{Nz} \end{bmatrix} \begin{bmatrix} \tilde{k}_{t-1} \\ \tilde{\theta}_t \end{bmatrix} = \begin{bmatrix} 0.9418 & 0.1382 \\ 0.3930 & 0.3989 \\ -0.6376 & 1.1155 \end{bmatrix} \begin{bmatrix} \tilde{k}_{t-1} \\ \tilde{\theta}_t \end{bmatrix} \tag{22}$$

and the ν_{ij} parameters are obtained by plugging this linear representation into the set of optimality conditions, to identify the *undetermined coefficients*. Conditional expectations of single variables can be obtained through the linear representation above, but it would be hard to obtain the actual conditional expectations in the original model [see equation (8)]. In consistency with the log-linearization proposed in that method as a starting step, conditional expectations of nonlinear functions are approximated by linear combinations of conditional expectations of individual variables, so the representation above is all that

[11]Even though we obtain linear approximations around steady state, log-linear approximations could alternatively be used.

is needed to compute the approximated expectations. Expectations errors could then be computed, although they will incorporate a fair amount of numerical error, derived from the linear approximation to the equations in the model and to the conditional expectations in it.

Another popular solution approach [see Díaz (1998) in this volume] which is useful to solve stochastic, dynamic optimization problems, consists on building a *linear quadratic approximation* to the original model and apply the techniques of *dynamic programming*. The goal is to derive the value function, generally after eliminating some decision and state variables from the objective function by repeated substitutions of the available optimality conditions and constraints in their deterministic form. Then the linear solution to the problem of maximizing the resulting value function for the linear–quadratic approximation is obtained. In the analysis of Hansen's model, the budget constraint can be used to eliminate consumption from the objective function, and a linear–quadratic approximation to the return function $r(\log(\theta_t), k_{t-1}, k_t, N_t)$ can be obtained, showing that the solution to this model is in the form of a set of two decision rules, for physical capital and labor, as linear functions of the beginning-of-period capital stock and the productivity shock, the two state variables. For the parameter values used in the previous Section, they are:

$$\begin{aligned} k_t &= 0.7368 + 1.7499 \log(\theta_t) + 0.9418 k_{t-1} \\ N_t &= 0.5459 + 0.3718 \log(\theta_t) - 0.0168 k_{t-1} \end{aligned} \tag{23}$$

Once we have the optimal values of labor and the capital stock for time t, we obtain output from the production function, and consumption from the budget constraint. Relative to the two previous methods, we are in this case adding two linear relationships to the original model when computing the numerical solution, while the method of undetermined coefficients imposes linear dependence of the logs of all current state and decision variables on the logs of the state variables.

The method of *parameterized expectations* of den Haan and Marcet (1990) and Marcet and Lorenzoni (1988) computes time series for the conditional expectations using a proposed polynomial function. This function must be estimated, to minimize the size of the average error between each conditional expectation and the value of the nonlinear function of state and decision variables which is being forecasted. It fully preserves the nonlinearity in the original model, so that if the polynomial expectation function can be precisely estimated and the implied set of time series is stable, this is a convincing solution approach.

Summarizing, preserving nonlinearity will generally produce greater accuracy, to the cost of having to solve each period a nonlinear system of equations to obtain the realization of the time t-vector of variables, which is computationally demanding. The alternative of using some degree of linear approximation to the model to get around this difficulty will produce expectations errors with a generally more important deviation from rationality.

The second important issue is *stability of the solution* since Euler equations, by themselves, do not place enough restrictions to guarantee that a set of time series that satisfy them will also satisfy the transversality conditions in the model.

Methods based on the *eigenvalue/eigenvector decomposition* are designed precisely to take special care of characterizing the stable manifold of the system. In simple models, stability conditions are added to the model in the form of orthogonality between the left eigenvectors (generalized eigenvectors, in the next Section) associated to unstable eigenvalues and the

vector of variables in the model. As a result, these methods produce numerical solutions that satisfy the stochastic transversality conditions.

In the method of *undetermined coefficients*, computing the numerical values of the ν_{ij} coefficients reduces at some point to solving a second order equation in the coefficient associated to the capital stock ν_{kk}. The method of undetermined coefficients takes care of stability by choosing the stable eigenvalues when solving the second order equation in the coefficient associated to the capital stock ν_{kk}. For the parameterization considered here, produces the estimates shown in (22). This is not too far from methods based on eigenvalue/eigenvector decompositions, whose application to a baseline real business cycle we have described in the previous Section.

Relative to the latter, an important improvement in Sims (1998) with respect to Blanchard and Kahn (1980) is to propose a very general framework to characterize stability conditions for linear models, not necessarily linked to standard transversality conditions. In somewhat complex models that incorporate elaborate fiscal and monetary policy strategies, stability conditions may take some specific form different from the orthogonality condition we have just mentioned [see Sims (1994) for some examples]. This extension is quite relevant: having to choose the stable root in the second order equation for ν_{kk} in the undetermined coefficients method may not be much of a problem, but the difficulty to guarantee stability would quickly grow with the dimension of the state vector. This is why the general discussion on stability in Sims (1998) is so relevant.

The numerical solution derived from the *dynamic programming* approach, if it can be computed, it will also satisfy the transversality conditions, by construction. The difficulties here are computational, since we are then trying to simultaneously solve the problem of finding the optimal decision rules and the implied pricing equations as well as characterizing the stable manifold. Even with mildly complicated value functions, this approach might face serious computational difficulties.

All these methods based on linear approximations are subject to the limitation that the stability analysis will be valid so long as the economy is close to steady state, around which the linear approximation was computed so that, so long as fluctuations are reasonably sized, there will not be much problem with stability. On the other hand, that approximation may jeopardize the possibility of characterizing the transition to the steady state once the economy has been exposed to a structural change, leaving it *far* from the new steady state.

Stability is not explicitly analyzed when *parameterizing expectations*. However, minimizing the sum of squared residuals in the projection of the function inside the conditional expectation on its polynomial representation will tend to produce stability. Problems with stability may arise in setups which are not inherently stable, and they will show up as difficulty in reaching convergence in the algorithm estimating the parameters in the polynomial representations of expectations.

It can easily be understood how parameterizing expectations takes care of stability, at the same time than eigenvalue/eigenvector decompositions approximate parameterized expectations: a stability condition like (19) can be thought of as the linear approximation to a function: $W_t = a_0 k_t^{a_1} \theta_t^{a_2}$. This, in turn, can be seen as a polynomial representation of the conditional expectation W_t, with a_0, a_1 and a_2 being functions of the coefficients in (19) and the steady state values of the variables. If, rather than adding the stability condition (19), we had substituted W_t in Hansen's model by this polynomial expression, a stable solution would have been obtained. Besides, if the method of parameterized expectations were used to represent W_t as a function of k_t and θ_t, the resulting estimates, once convergence is achieved, should not differ very much from our estimates.

A third difference is the treatment of the *conditional expectations* in the model. Methods based on the *eigenvalue/eigenvector decomposition* handle expectations as any other endogenous variable in the model, producing time series both, for each of the conditional expectations and for the associated expectations errors. We have already pointed out the limitations of linear approximations in dealing with expectations: *a linear representation* of decision and state variables as linear functions of starting–of–period states, as in (22) and (23), provides the researcher with a very simple way to compute expectations, but they will be linear functions of the states. Even though the variables being used may be logged deviations from steady state, the representation of expectations is still rather limited. Besides, these will be expectations of individual variables, which will have to be used to approximate the conditional expectations of nonlinear functions in the model.

Incorporating the expectations as additional variables to the model is far from trivial: on the one hand, the dimension of the state vector and with it, computational requirements, increase. On the positive side, a higher dimensional state space may be quite useful when searching for a stable solution: in particular, expectations, treated as endogenous variables, play a central role in methods based on *eigenvalue/eigenvector decompositions*, as we have already described in the previous Section, and should become even clearer in the next one.

Since it is not based in any linear approximation, the method of *parameterizing expectations* might provide the more accurate realization for the conditional expectations in the model. However, reaching convergence in the algorithm that estimates the expectation function might take some effort: even when the algorithm works, thousands of artificial data are needed for convergence. As in any other method, the trade–off between computational simplicity and accuracy is quite evident.

The last issue has to do with the resulting expectations errors: in principle, *parameterizing expectations* may be the better suited method to produce *acceptable expectations errors*. The search for a good specification of the polynomial function used to represent expectations by a nonlinear least squares algorithm should produce good statistical properties: first, so long as there is noticeable autocorrelation in expectations errors, additional lags of the state variables will show significant explanatory power for the function being forecasted, and will be added to the expectations polynomial, generally reducing autocorrelation. On the other hand, this strategy will tend to produce collinearity in the polynomial function, and possible spurious dynamics in the solution. Second, the nonlinear least squares fit generates expectations errors which are uncorrelated with the gradient of the expectations function. That, in turn, will produce approximate lack of correlation with the variables included in the parameterized expectation. Since past decision variables will generally be each period continuous functions of available states, this property, together with lack of autocorrelation, will extend to any variable in the information set at time t. However, this positive aspect must once again be qualified by the need to reach a satisfactory solution to the problem of approximating the conditional expectations in the model.

Among the alternative solution strategies, we have already mentioned that models pre-

serving more nonlinearity will tend to produce less important deviations from rationality, which we assume is a basic premise imposed on the model. We have seen in the previous Section that methods based on *eigenvalue–eigenvector decompositions* provide additional evidence in terms of the relationship between rational expectations errors and the innovations in the structural shocks, which is an interesting characteristic of the model. With other methods, this type of relationships can be estimated through linear projections, although there is no guarantee that such a projection will be a well–specified model. For instance, one might find evidence of expectations errors responding not only to contemporaneous but to past endogenous innovations as well, which would obviously be a contradiction of rationality.

Finally, dealing with conditional expectations under rationality brings up additional issues under which solution methods will have to be increasingly scrutinized. One of them is how to impose the restrictions among expectations of a given function at different horizons, which are standard in theoretical rational expectations models. Another issue is how to impose in the solution strategy the restrictions that theoretical models sometimes impose among expectations under rationality, as it is the case in the model in the next Section. Not all the solution methods are similarly equipped to deal with these questions and we should expect to see increasing discussion on specific subjects like these, concerning the modelling of expectations under rationality.

Having discussed the implementation of the solution method in a simple baseline real growth model and having established some comparisons with alternative solution strategies, we now proceed to discuss its implementation in a more general setup.

6 Solving some identification issues: capital stock and inventories in the production function

Singular Γ_0 matrices appear often. Sometimes, singularity can be avoided by solving for some variables as functions of others and reducing system size, but that is not always feasible. A typical cause of singularity is that a subset of r variables appear in just q equations, $r > q$, being then impossible to solve for all of them and reflecting that identification of those variables is weak.

An interesting case in which this situation arises is Kydland and Prescott (1982) where physical capital, k_t, and inventories, i_t, play a very similar role: both accumulate and both are production inputs. In that paper, the technology shock, which is the only source of randomness in the economy, is assumed to have a complex stochastic structure that allows for identification of fixed investment and inventory investment apart from each other. The redundancy between physical capital and inventories shows up in that their contemporaneous values appear just in the budget constraint. We will see that, as a consequence, Γ_0 will be singular, producing an eigenvalue equal to infinity, and the associated eigenvector will allow for solving one variable apart from the other.

Let us consider the production technology:

$$F_t(\theta_t, k_{t-1}, i_{t-1}) = \theta_t \left[(1-\psi)k_{t-1}^{-\nu} + \psi i_{t-1}^{-\nu}\right]^{\frac{-\alpha}{\nu}} \tag{24}$$

where θ_t is an exogenous technology shock, as in previous sections. The marginal products of k_{t-1} and i_{t-1} are, at time t:

$$
\begin{aligned}
F_t^k &= \alpha(1-\psi)k_{t-1}^{-\nu-1}\theta_t\left[(1-\psi)k_{t-1}^{-\nu}+\psi i_{t-1}^{-\nu}\right]^{\frac{-\alpha}{\nu}-1} && (25)\\
F_t^i &= \alpha\psi i_{t-1}^{-\nu-1}\theta_t\left[(1-\psi)k_{t-1}^{-\nu}+\psi i_{t-1}^{-\nu}\right]^{\frac{-\alpha}{\nu}-1} && (26)
\end{aligned}
$$

Maintaining the assumption of a continuum of identical consumers, each endowed with a utility function with constant relative risk aversion: $U(c_t)=\frac{c_t^{1-\sigma}-1}{(1-\sigma)}$, $\sigma>0$, the optimality conditions are:

$$
\begin{aligned}
c_t+k_t-(1-\delta)k_{t-1}+i_t-i_{t-1}-F(\theta_t,k_{t-1},i_{t-1}) &= 0 && (27)\\
c_t^{-\sigma}-\beta E_t\left[(1-\delta+F_{t+1}^k)c_{t+1}^{-\sigma}\right] &= 0 && (28)\\
c_t^{-\sigma}-\beta E_t\left[(1+F_{t+1}^i)c_{t+1}^{-\sigma}\right] &= 0 && (29)\\
\log(\theta_t)-\rho\log(\theta_{t-1})-\epsilon_t &= 0 && (30)\\
\epsilon_t &\sim iid\,N(0,\sigma_\epsilon^2)
\end{aligned}
$$

where we have assumed that physical capital depreciates at a rate δ, $0<\delta<1$. Since they involve the realization of the productivity shock at time $t+1$, θ_{t+1}, the marginal products F_{t+1}^k, F_{t+1}^i are random variables when period t decisions, k_t and i_t, are made.

Additionally, two stability conditions must hold:

$$
\begin{aligned}
\lim_{\tau\to\infty} E_t\left[c_{t+\tau}^{-\sigma}k_{t+\tau}\beta^{-\tau}\right] &= 0\\
\lim_{\tau\to\infty} E_t\left[c_{t+\tau}^{-\sigma}i_{t+\tau}\beta^{-\tau}\right] &= 0
\end{aligned}
$$

Conditions (28) and (29) imply that the two conditional expectations of the cross–products of each marginal productivity by the marginal utility of future consumption are equal to each other at every point in time. However, it is convenient to maintain both of them in the model, and define new variables W_{k_t}, W_{i_t} equal to each expectation,

$$
\begin{aligned}
W_{k_t} &= E_t[(1-\delta+F_{t+1}^k)c_{t+1}^{-\sigma}] && (31)\\
W_{i_t} &= E_t[(1+F_{t+1}^i)c_{t+1}^{-\sigma}] && (32)
\end{aligned}
$$

as well as the associated one-step–ahead, serially uncorrelated, rational expectations errors η_t^k and η_t^i :

$$
\begin{aligned}
(1-\delta+F_t^k)c_t^{-\sigma}-W_{k_{t-1}}-\eta_t^k &= 0 && (33)\\
(1+F_t^i)c_t^{-\sigma}-W_{i_{t-1}}-\eta_t^i &= 0 && (34)
\end{aligned}
$$

With this, equations (28) and (29) become:

$$
\begin{aligned}
c_t^{-\sigma}-\beta W_{k_t} &= 0 && (35)\\
c_t^{-\sigma}-\beta W_{i_t} &= 0 && (36)
\end{aligned}
$$

The conditions characterizing steady state are:

$$
\begin{aligned}
W_{k_{ss}} &= \left(1-\delta+F_{ss}^k\right)c_{ss}^{-\sigma}\\
W_{k_{ss}} &= W_{i_{ss}}=\frac{1}{\beta}c_{ss}^{-\sigma}\\
W_{i_{ss}} &= \left(1+F_{ss}^i\right)c_{ss}^{-\sigma}\\
c_{ss} &= F(\theta_{ss},k_{ss},i_{ss})-\delta k_{ss}\\
F_{ss}^k &= \alpha(1-\psi)k_{ss}^{-\nu-1}\theta_{ss}F(\theta_{ss},k_{ss},i_{ss})^{-1}\\
F_{ss}^i &= \alpha\psi i_{ss}^{-\nu-1}\theta_{ss}F(\theta_{ss},k_{ss},i_{ss})^{-1}\\
\theta_{ss} &= 1
\end{aligned}
$$

from which we get the dependence of steady state values from structural parameters:

$$
\begin{aligned}
F_{ss}^k &= \frac{1}{\beta}-(1-\delta)\\
F_{ss}^i &= \frac{1}{\beta}-1\\
\frac{k_{ss}}{i_{ss}} &= \left[\frac{\psi}{1-\psi}\frac{1-\beta(1-\delta)}{1-\beta}\right]^{-\frac{1}{\nu+1}}\\
i_{ss} &= \left[F_{ss}^i\frac{1}{\alpha\psi}\left[(1-\psi)\left(\frac{k_{ss}}{i_{ss}}\right)^{-\nu}+\psi\right]^{-\frac{\alpha}{\nu}}\right]^{-\frac{1}{1+\alpha+\nu}}\\
k_{ss} &= \left(\frac{k_{ss}}{i_{ss}}\right)i_{ss}\\
F(\theta_{ss},k_{ss},i_{ss}) &= \theta_{ss}\left[(1-\psi)\left(\frac{k_{ss}}{i_{ss}}\right)^{-\nu}+\psi\right]^{-\frac{\alpha}{\nu}}i_{ss}^{\alpha}\\
c_{ss} &= -\delta k_{ss}+F(\theta_{ss},k_{ss},i_{ss})
\end{aligned}
$$

We can now compute the linear approximation to the system (27), (35), (36), (33), (34) and (30) around steady state:

$$
\Gamma_0 y_t=\Gamma_1 y_{t-1}+\Psi z_t+\Pi\eta_t \qquad (37)
$$

where vectors y_t, z_t, η_t are:

$$
\begin{aligned}
y_t &= [c_t - c_{ss}, k_t - k_{ss}, i_t - i_{ss}, W_{k_t} - W_{k_{ss}}, W_{i_t} - W_{i_{ss}}, \log(\theta_t)]' \qquad (38)\\
z_t &= \epsilon_t \\
\eta_t &= (\eta_t^k, \eta_t^i)'
\end{aligned}
$$

The matrices in the linear approximation are:

$$
\Gamma_0 = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & a_2^{-\frac{\alpha}{\nu}} k_{ss}^{\alpha} \\
-\sigma c_{ss}^{-\sigma-1} & 0 & 0 & -\beta & 0 & 0 \\
-\sigma c_{ss}^{-\sigma-1} & 0 & 0 & 0 & -\beta & 0 \\
-\sigma c_{ss}^{-\sigma-1}\left(1-\delta+\alpha(1-\psi)a_2^{-\frac{\alpha+\nu}{\nu}} k_{ss}^{\alpha-1}\right) & 0 & 0 & 0 & 0 & \alpha(1-\psi)a_2^{-\frac{\alpha+\nu}{\nu}} k_{ss}^{\alpha-1} c_{ss}^{-\sigma} \\
-\sigma c_{ss}^{-\sigma-1}\left(1+\alpha\psi a_2^{-\frac{\alpha+\nu}{\nu}} a_1^{-\nu-1} k_{ss}^{\alpha-1}\right) & 0 & 0 & 0 & 0 & \alpha\psi a_2^{-\frac{\alpha+\nu}{\nu}} a_1^{-\nu-1} k_{ss}^{\alpha-1} c_{ss}^{-\sigma} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
\Gamma_1 = \begin{pmatrix}
0 & (1-\delta)-\alpha(1-\psi)a_2^{-\frac{\alpha+\nu}{\nu}} k_{ss}^{\alpha-1} & 1+\alpha\psi a_2^{-\frac{\alpha+\nu}{\nu}} a_1^{-\psi-1} k_{ss}^{\alpha-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -F_{ss}^{kk} c_{ss}^{-\sigma} & -F_{ss}^{ki} c_{ss}^{-\sigma} & 1 & 0 & 0 \\
0 & -F_{ss}^{ik} c_{ss}^{-\sigma} & -F_{ss}^{ii} c_{ss}^{-\sigma} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho
\end{pmatrix},
$$

$$
\Psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},
$$

with:

$$
\begin{aligned}
a_1 &= \frac{\psi}{1-\psi}\frac{1-\beta(1-\delta)}{1-\beta} \\
a_2 &= (1-\psi)+\psi a_1^{-\nu} \\
a_3 &= \alpha(1-\psi)a_2^{-\frac{\alpha+\nu}{\nu}} \\
F_{ss}^{kk} &= \left[(\alpha+\nu)(1-\psi)\frac{1}{a_2}-\nu-1\right]\alpha(1-\psi)a_2^{-\frac{\alpha+\nu}{\nu}} a_1^{-\nu-2} k_{ss}^{\alpha-2}, \\
F_{ss}^{ii} &= \left[(\alpha+\nu)\psi\frac{1}{a_2}a_1^{-\nu}-\nu-1\right]\alpha\psi a_2^{-\frac{\alpha+\nu}{\nu}} a_1^{-\nu-2} k_{ss}^{\alpha-2}, \\
F_{ss}^{ik} &= \alpha(1-\psi)(\alpha+\nu)\psi a_2^{-\frac{\alpha+\nu}{\nu}} a_1^{-\nu-1} k_{ss}^{\alpha-2} = F_{ss}^{ki}
\end{aligned}
$$

6.1 Characterizing stability conditions

Each row of Γ_0 in (37) contains the partial derivatives of each equation in the system with respect to the components of the vector y_t. Since k_t and i_t just appear in (27), only the first element in the second and third columns of Γ_0 is nonzero. As a consequence, Γ_0 is singular and it is necessary to compute a QZ-decomposition[12] to obtain generalized eigenvalues: for any pair of square matrices like (Γ_0, Γ_1), there exist orthonormal matrices Q, Z ($QQ' = ZZ' = I$) and upper triangular matrices Λ and Ω such that[13]:

$$\Gamma_0 = Q'\Lambda Z', \; \Gamma_1 = Q'\Omega Z'.$$

Besides, Q and Z can be chosen so that all possible zeroes of Λ occur in the lower right corner and such that the remaining ratios $\frac{\omega_{ii}}{\lambda_{ii}}$ of diagonal elements in Ω and Λ, are non–decreasing in absolute value as we move down the diagonal. These ratios are the *generalized eigenvalues* of the pair (Γ_0, Γ_1).

Premultiplying the system by Q and replacing $Z'y_t$ with w_t, we get:

$$\Lambda w_t = \Omega w_{t-1} + Q\left(\Psi z_t + \Pi\eta_t\right), \qquad (39)$$

If we partition the set of generalized eigenvalues into those below and above the upper bound which is used as stability criterion (it could be $\beta^{-1/2}$), and order them decreasingly along the diagonal of Λ, we will have:

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix}\begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{pmatrix}\begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} + \begin{pmatrix} Q_{1\bullet} \\ Q_{2\bullet} \end{pmatrix}(\Psi z_t + \Pi\eta_t) \qquad (40)$$

where the second block of equations corresponds to the *unstable* eigenvalues. Some diagonal elements in Λ_{22}, but not in Λ_{11}, may be zero.

A zero element in the diagonal of Λ implies some lack of identification in the system, and an infinite generalized eigenvalue will arise. If Ω does not have a zero in the same position, the associated eigenvector will generally allow us to solve the identification problem, as we will see below. Γ_0 is then singular, but all the equations in the system are in this case linearly independent. If Ω has a zero in the same position, then there is an equation which is linear combination of the others so that, even though the system has as many equations as variables it is, in fact, incomplete.

In (40), let us denote the vector

$$x_t = Q\left(\Psi z_t + \Pi\eta_t\right) = \begin{pmatrix} Q_{1\bullet}\left(\Psi z_t + \Pi\eta_t\right) \\ Q_{2\bullet}\left(\Psi z_t + \Pi\eta_t\right) \end{pmatrix} = \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix}$$

Since the lower block of (40) corresponds to *unstable* eigenvalues, it must be solved towards the future, which makes w_{2t} depend on the whole future path of x_{2t}. Sims (1998) shows how

[12] The MATLAB command to perform a QZ–decomposition is $qz(\Gamma_0, \Gamma_1)$

[13] Q, Z, Λ and Ω could be complex, in which case, the transposition above has to be changed to transposition and complex conjugation. On the other hand, upper triangularity of Λ and Ω has to do with the possibility of repeated eigenvalues. When all eigenvalues are different from each other, both matrices are diagonal.

the discounted sum of future values of linear combinations in x_{2t} that defines w_{2t} must be equal to its conditional expectation, which yields as many stability conditions as variables there are in w_{2t}. Imposing those conditions we again get a set of relationships between the vector of rational expectations errors and the vector of innovations in the exogenous stochastic processes, similar to (20).

In the applications we discuss here, the vector x_t contains linear combinations of the innovations in the stochastic processes for the structural shocks and the expectations errors. Structural shocks, themselves, are included in the vector y_t, and it is just their innovations which are in x_t. Hence, $E_t(x_{2t+s}) = 0$ for all $s > 0$, and the stability conditions we have just described become:

$$w_{2t} = Z_{2\bullet}' y_t = 0, \forall t \tag{41}$$

where $Z_{2\bullet}'$ is the appropriate submatrix of Z. This set of conditions, taken to (40), amounts to having the relationships between rational expectations errors and structural innovations[14]:

$$Q_{2\bullet}(\Psi z_t + \Pi\eta_t) = 0 \;\Rightarrow Q_{2\bullet}\Psi z_t = -Q_{2\bullet}\Pi\eta_t \tag{42}$$

For reasonable parameterizations, there are two generalized eigenvalues in (39) with absolute size greater than $\beta^{-1/2}$. One of them is common to the version of the model without inventories (not analyzed in this chapter) so that it is associated to a standard stability condition, of the kind we saw in Hansen's model in Section 4. The other eigenvalue is equal to infinity.

The partition described in (40) leads in this model to an unstable block:

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z_{5\bullet}' y_t \\ z_{6\bullet}' y_t \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\begin{pmatrix} z_{5\bullet}' y_{t-1} \\ z_{6\bullet}' y_{t-1} \end{pmatrix}\begin{pmatrix} q_{5\bullet} \\ q_{6\bullet} \end{pmatrix}(\Psi z_t + \Pi\eta_t) \tag{43}$$

where $z_{5\bullet}'$, $z_{6\bullet}'$, $q_{5\bullet}$, $q_{6\bullet}$ denote the fifth and sixth rows of Z' and Q, i.e., $z_{5\bullet}'$ and $z_{6\bullet}'$ form the submatrix $Z_{2\bullet}'$ in (41), while $q_{5\bullet}$, $q_{6\bullet}$ form $Q_{2\bullet}$ in (42). The zero in the lower end of the diagonal in the first matrix shows the existence of a generalized eigenvalue equal to infinity, due to the weak identification of k_t and i_t. The other generalized eigenvalue is equal to b_{11}/a_{11}.

Written at time t, the last equation states: $b_{22} z_{6\bullet}' y_t = -q_{6\bullet}(\Psi z_{t+1} + \Pi\eta_{t+1})$. Taking expectations and noticing the lack of autocorrelation in ϵ_t, as well as in the two one-step-ahead forecast errors in η_t, this equation leads to: $z_{6\bullet}' y_t = 0$, which is a linear restriction among contemporaneous values of the conditional expectations and decision and state variables. Taken to the previous equation, we get:

$$a_{11} z_{5\bullet}' y_t = b_{11} z_{5\bullet}' y_{t-1} + q_{5\bullet}(\Psi z_t + \Pi\eta_t) \tag{44}$$

which is an explosive autoregression in $z_{5\bullet}' y_t$, since the generalized eigenvalue: $\frac{b_{11}}{a_{11}} > \beta^{-1/2}$, and the resulting trajectories for the variables in y_t will not satisfy the transversality

[14] However, we will impose (41), but not (42) when solving, since we will actually use the original, nonlinear model to compute an equilibrium realization, which will satisfy (42) only as an approximation. If we used the linear approximation (37) to the model to compute the solution, (42) would hold exactly.

conditions. Besides, the triangular structure of the system will transmit the explosiveness of (44) to the rest of the equations of the system. These explosive trajectories can be eliminated only if we impose $z_{5\bullet}' y_t = 0$. Together with the budget constraint and the remaining equations in the system, these two conditions will provide us with the time t values of decision variables, state variables and conditional expectations, c_t, k_t, i_t, W_{k_t}, and W_{i_t}.

6.2 Identifying capital stock and inventories apart from each other

With the parameterization: $\sigma = 1.5, \alpha = 0.36, \beta = 0.99, \delta = 0.025, \rho = 0.95, \nu = 4.0, \psi = 2.8\,10^{-6}$, [the two last parameters as in Kydland and Prescott (1982)], steady state values are: $c_{ss} = 2.7261, k_{ss} = 36.2067, i_{ss} = 3.6009, W_{k_{ss}} = 0.2244, W_{i_{ss}} = 0.2244, \theta_{ss} = 1$, and the numerical estimates of the Γ_0, Γ_1 matrices become:

$$\Gamma_0 = \begin{pmatrix} 1.0000 & 1.0000 & 1.0000 & 0 & 0 & -3.6313 \\ -0.1222 & 0 & 0 & -0.9900 & 0 & 0 \\ -0.1222 & 0 & 0 & 0 & -0.9900 & 0 \\ -0.1235 & 0 & 0 & 0 & 0 & 0.0078 \\ -0.1235 & 0 & 0 & 0 & 0 & 0.0022 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} 0 & 1.0101 & 1.0101 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0002 & -0.0003 & 1.0000 & 0 & 0 \\ 0 & -0.0003 & 0.0030 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9500 \end{pmatrix},$$

while Λ, Ω, ordered so that generalized eigenvalues increase in absolute size as we move down the diagonal of Λ are:

$$\Lambda = \begin{pmatrix} -1.0297 & -0.0002 & -1.3716 & -3.5179 & 0.3177 & -0.0112 \\ 0 & 0.0023 & -0.1487 & -0.2742 & 0.6340 & -0.7892 \\ 0 & 0 & -0.3106 & -0.7072 & 0.5853 & 0.3701 \\ 0 & 0 & 0 & 0.9945 & -0.0835 & 0.0478 \\ 0 & 0 & 0 & 0 & -0.6118 & -0.4668 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 0 & 0 & -1.3887 & 0.0219 & 0.1540 & 0.0020 \\ 0 & 0 & -0.1431 & 0.0023 & 0.0159 & 0.0002 \\ 0 & 0 & -0.3027 & -0.1069 & -0.7527 & 0.0004 \\ 0 & 0 & 0 & 0.9447 & -0.0744 & -0.0004 \\ 0 & 0 & 0 & 0 & -0.6342 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.0000 \end{pmatrix},$$

with a generalized eigenvalue equal to infinity. The finite eigenvalues are: $1.0366, 0.9500, 0.9745$, and there are two eigenvalues equal to zero. The Q, Z matrices of the QZ-decomposition are:

$$Q = \begin{pmatrix} 0.1187 & 0.1187 & -0.9711 & 0.1199 & 0.1199 & 0 \\ 0.1410 & -0.9848 & -0.1001 & 0.0124 & 0.0124 & 0 \\ -0.6112 & -0.0790 & -0.2166 & -0.5354 & -0.5354 & 0 \\ -0.0722 & -0.0093 & 0.0004 & 0.0391 & 0.0446 & -0.9956 \\ 0.7663 & 0.0990 & -0.0039 & -0.4442 & -0.4437 & -0.0939 \\ 0.0000 & 0.0000 & 0.0000 & -0.7071 & 0.7071 & 0.0039 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7071 & 0.7071 & 0.0009 & 0.0061 & 0.0003 \\ 0 & -0.7071 & 0.7071 & 0.0012 & 0.0081 & -0.0023 \\ 0 & -0.0003 & -0.0061 & 0.0968 & 0.7004 & 0.7071 \\ 0 & 0.0023 & -0.0082 & 0.1020 & 0.6997 & -0.7071 \\ 0 & 0 & 0 & -0.9901 & 0.1406 & -0.0037 \end{pmatrix}.$$

Since there are two generalized eigenvalues above $\beta^{-1/2}$, there are also two stability conditions needed for transversality conditions to hold, given by the two last columns of Z (two last rows of Z'):

$$0.0061\tilde{k}_t + 0.0081\tilde{i}_t + 0.7004\tilde{W}_{k_t} + 0.6997\tilde{W}_{i_t} + 0.1406\log(\theta_t) = 0 \quad (45)$$

$$0.0003\tilde{k}_t - 0.0023\tilde{i}_t - 0.0037\log(\theta_t) = 0 \quad (46)$$

where the first relationship happens not to involve consumption and the expectations $\tilde{W}_{k_t}$ and $\tilde{W}_{i_t}$ have dropped out of the second, since they are equal to each other. The second equation allowing us to identify k_t and i_t apart from each other. These two stability conditions, obtained from the unstable eigenvalues, impose a relationship between the structural innovation and the expectation errors, as in (42), $\eta_t = -(Q_{2\bullet}\Pi)^{-1}Q_{2\bullet}\Psi z_t$, which, under our parameterization, becomes:

$$\eta_t^k = -0.1029\epsilon_t, \quad \eta_t^i = -0.1085\epsilon_t,$$

which very clearly illustrates that the two expectations errors are an exact function of each other.

The actual mechanism to generate the set of time series that solve the model from initial values k_0, i_0, is as follows: first, a sample realization for the productivity shock θ_t is generated using (30). Then, initial values for W_{k_0}, W_{i_0}, c_0 come from (35), (36) and (45). Then, using the value of θ_1, (35), (36), (45), (46) and (27) form a complete system in $W_{k_1}, W_{i_1}, c_1, k_1, i_1$. This procedure can be iterated for each period. Having time series for all the variables, we can compute the expectations errors from (33) and (34) and run rationality tests on them, if desired.

A solution to the linear system (37) exists only if representation (42) is feasible, i.e., if the space spanned by the columns of $Q_{2\bullet}\Psi$ is included in the space spanned by the columns of $Q_{2\bullet}\Pi$. This condition becomes necessary and sufficient for the simpler cases in which $E_t(z_{t+1}) = 0$. In the specific model in this Section, $Q_{2\bullet}\Psi$ is a 2×1 vector, while $Q_{2\bullet}\Pi$ is a full rank 2×2 matrix, so that the condition is clearly satisfied. As an alternative, Sims (1998) suggests testing the condition for existence of a solution by regressing the columns of $Q_{2\bullet}\Psi$ on the columns of $Q_{2\bullet}\Pi$, to see if the resulting residuals are all equal to zero. In our example, the residual sum of squares turned out to be of order 10^{-34}, showing that the solution, in fact, exists.

6.3 Special case: zero depreciation rate

With zero depreciation on physical capital, there is no difference between the accumulation processes followed by the two inputs, and the equality of conditional expectations in (28) and (29) becomes:

$$E_t\left[F_{t+1}^i c_{t+1}^{-\sigma}\right] = E_t\left[F_{t+1}^k c_{t+1}^{-\sigma}\right]. \quad (47)$$

On the other hand, the marginal rate of transformation between physical capital, k_t, and inventories, i_t, which is in principle a random variable at time t is, with our technology:

$$RMT_{t+1}^{i,k} = \frac{F_{t+1}^i}{F_{t+1}^k} = \frac{\psi}{1-\psi}\left(\frac{i_t}{k_t}\right)^{-\nu-1} \quad (48)$$

which belongs to the information set available at time t. This feature of the model implies an exact relationship between two expectations:

$$E_t\left[F_{t+1}^i c_{t+1}^{-\sigma}\right] = E_t\left[RMT_{t+1}^{i,k} F_{t+1}^k c_{t+1}^{-\sigma}\right] = RMT_{t+1}^{i,k} E_t\left[F_{t+1}^k c_{t+1}^{-\sigma}\right] \quad (49)$$

so that, in the special case of zero depreciation, (47) and (49) imply:

$$RMT_{t+1}^{i,k} = 1 \quad \text{or} \quad k_t = \left(\frac{1-\psi}{\psi}\right)^{\frac{1}{1+\nu}} i_t \quad (50)$$

This particular form for the optimality condition eliminates the lack of identification between the optimal amounts of the two production inputs in the special case of zero depreciation. The infinite eigenvalue disappears and, with it, the stability condition (46) that we used to identify physical capital apart from inventories, which is no longer needed. That condition corresponds to the case of nonzero depreciation, which explains why the productivity shock appears in it. If we use $\delta = 0$ but ignore (50), the eigenvalue equal to infinity again arises, and the associated stability condition analogous to (46) becomes: $k_t = 12.8989\, i_t$, which is exactly equal to (50). Therefore, using stability conditions associated to infinite eigenvalues we can solve identification issues that only in special cases (here with zero depreciation) can also be solved analytically.

7 Solving endogenous growth models

Numerical solution methods must be applied with great care to endogenous growth models, since we need to distinguish between the lack of stability that can and should be eliminated through conditions like those in the previous Sections, and the lack of stationarity which is intrinsic to these models, even in steady state. In our particular approach, it looks as if the stability conditions could not possibly be obtained in endogenous growth models. Since they are derived from an approximation around the steady state, and the steady state levels of the variables change over time, it would seem necessary to compute the linear approximation for each single period, which would be clearly hopeless.

Luckily, the method described in the previous Sections can, in fact, be easily adapted to solve endogenous growth models. As an illustration, we will consider a planner's problem in an economy with aggregate constant returns to scale in physical and human capital, as in Uzawa (1965). Once the optimality conditions (including resources and technological constraints, as well as laws of motion for exogenous variables) have been obtained:

1. We transform the set of optimality conditions in ratios of the relevant variables, and compute the steady state values for the ratios, which will be uniquely defined.
2. Obtain the appropriate stability conditions for this transformed system. The stability conditions depend upon the approximation around the steady state for the model in ratios, which does not change over time. Hence, the conditions do not need to be revalued each period. Save these stability conditions.
3. Rewrite again the optimality conditions to make growth explicit for all those variables that experience nonzero growth in steady state, by multiplying and dividing each observation by the corresponding power of its growth rate.
4. Use the optimality conditions from 3.), together with the stability conditions from 2.), initial conditions for the state variables and sample realizations for the exogenous shocks, to generate time series for the variables in the economy in levels, excluding the deterministic growth components. These can be obtained separately.

Summarizing, the set of time series that solve the model are generated from the version of the model *in levels* in which deterministic growth has been made explicit. *That way, we can characterize whether the potential instability of the ultimately obtained time series for the original variables is purely due to their deterministic growth rate, or it rather reflects a more fundamental instability of the solution, which might be unacceptable.* The procedure we have just outlined guarantees that the nonstationarity of the solution can be fully represented by a single unit root, as it should be the case in any endogenous growth model, due to the presence of a unit eigenvalue in the coefficient matrix of its linear approximation.

We consider an economy with two sectors: in the first, output is produced from physical and human capital. In the second, human capital is produced from itself, without need of using physical capital. The unit of time which is available each period is split into both production activities. Output is obtained from a Cobb-Douglas technology in physical capital, k_t, and effective working hours, $u_t h_t$, the product of hours devoted to production, u_t, by human capital, h_t,. In the second sector, human capital is accumulated through a linear

technology, as a function of the amount of time devoted to this sector, $1 - u_t$. There are random productivity shocks θ_t, ξ_t in both sectors, following first order autoregressive structures. Physical and human capital depreciate at constant rates δ_k and δ_h each period. The representative consumer has a constant relative risk aversion utility function in its single argument, consumption, and discounts utility over time at a rate of $\beta, 0 < \beta < 1$. Population grows at a rate n, and the planner maximizes the aggregate utility:

$$\max_{\{\hat{c}_t, u_t, \hat{k}_t, \hat{h}_t\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} (\beta n)^{t-1} \left[\frac{\hat{c}_t^{1-\sigma} - 1}{1-\sigma} \right]$$

subject to

$$\begin{aligned}
n\hat{k}_t &= A\hat{k}_{t-1}^{\alpha}(u_t \hat{h}_{t-1})^{1-\alpha}\theta_t + (1-\delta_k)\hat{k}_{t-1} - \hat{c}_t && (51)\\
\hat{h}_t &= B(1-u_t)\hat{h}_{t-1}\xi_t + (1-\delta_h)\hat{h}_{t-1} && (52)\\
\log(\theta_t) &= \phi_\theta \log(\theta_{t-1}) + \epsilon_t^\theta && (53)\\
& \epsilon_t^\theta \sim N(0, \sigma_\theta^2) && \\
\log(\xi_t) &= \phi_\xi \log(\xi_{t-1}) + \epsilon_t^\xi && (54)\\
& \epsilon_t^\xi \sim N(0, \sigma_\xi^2) && \\
& \text{given } \hat{k}_0, \hat{h}_0, \theta_0, \xi_0 && \\
& u_t \in (0,1) && (55)\\
& \hat{c}_t, \hat{k}_t, \hat{h}_t \geq 0 &&
\end{aligned}$$

where variables with ^ present non–zero steady state growth, and we have the optimality conditions:

$$0 = -1 + E_t \left[\beta \left(\frac{\hat{c}_{t+1}}{\hat{c}_t} \right)^{-\sigma} \left(\alpha A \left(\frac{\hat{k}_t}{\hat{h}_t} \right)^{\alpha-1} u_{t+1}^{1-\alpha}\theta_{t+1} + 1 - \delta_k \right) \right] \quad (56)$$

$$0 = -\left(\frac{\hat{k}_{t-1}}{\hat{h}_{t-1}} \right)^{\alpha} u_t^{-\alpha} \frac{\theta_t}{\xi_t} + E_t \left[\beta n \left(\frac{\hat{c}_{t+1}}{\hat{c}_t} \right)^{-\sigma} \left(\frac{\hat{k}_t}{\hat{h}_t} \right)^{\alpha} u_{t+1}^{-\alpha} \frac{\theta_{t+1}}{\xi_{t+1}} (B\xi_{t+1} + 1 - \delta_h) \right] \quad (57)$$

together with (51) to (55).

Endogenous growth shows in the fact that this system can be solved for the steady state levels of the variables with zero steady state growth but only for ratios of variables with nonzero steady state growth. All ratios are referred in this case to human capital. Their steady state values can be obtained, but not those of the individual variables with steady state growth. On the other hand, we can also compute the steady state growth rate which is common, in this economy, to all variables with nonzero growth. Precisely because we can compute this growth rate, we cannot possibly solve for the steady state values of all variables, since we have the same number of equations than we would have in an exogenous growth model.

Denoting by ϖ_t^{ch} and ϖ_t^{kh} the ratios $\frac{\hat{c}_t}{\hat{h}_{t-1}}$ and $\frac{\hat{k}_t}{\hat{h}_t}$, defining each expectation as a new variable, and introducing the associated expectations errors, we have the system:

$$0 = -n\varpi_t^{kh}\left(B(1-u_t)\xi_t + 1 - \delta_h\right) + A\left(\varpi_{t-1}^{kh}\right)^{\alpha} u_t^{1-\alpha}\theta_t + (1-\delta_h)\varpi_{t-1}^{kh} - \varpi_t^{ch} \quad (58)$$

$$0 = -\left(\varpi_t^{ch}\right)^{-\sigma} + \beta W_t^1 \quad (59)$$

$$0 = -W_{t-1}^1 + \left[\left(\varpi_t^{ch}\right)^{-\sigma} \times \right.$$
$$\left.\left(B(1-u_{t-1})\xi_{t-1} + 1 - \delta_h\right)^{-\sigma}\left(\alpha A\left(\varpi_{t-1}^{kh}\right)^{\alpha-1} u_t^{1-\alpha}\theta_t + 1 - \delta_k\right)\right] - \eta_t^1 \quad (60)$$

$$0 = -\left(\varpi_{t-1}^{kh}\right)^{\alpha} u_t^{-\alpha}\frac{\theta_t}{\xi_t} + \beta n W_t^2 \quad (61)$$

$$0 = -W_{t-1}^2 + \left[\left(\frac{\varpi_t^{ch}}{\varpi_{t-1}^{ch}}\right)^{-\sigma} \times \right.$$
$$\left.\left(B(1-u_t)\xi_{t-1} + 1 - \delta_h\right)^{-\sigma}\left(\left(\varpi_{t-1}^{kh}\right)^{\alpha} u_{t-1}^{-\alpha}\frac{\theta_t}{\xi_t}\right)\left(B\xi_t + 1 - \delta_h\right)\right] - \eta_t^2 \quad (62)$$

together with the stochastic processes (53) and (54). These are seven equations in nine variables $(\varpi_t^{ch}, \varpi_t^{kh}, u_t, \theta_t, \xi_t, W_t^1, W_t^2, \eta_t^1, \eta_t^2)$, but the associated generalized eigenvalue problem produces two unstable eigenvalues[15].

With parameter values: $\sigma = 1.5, \beta = 0.99, A = 1, \alpha = 0.36,\ B = 0.0201, 1-\delta_k = 0.975, 1-\delta_h = 0.992, n = 1.0035, \phi_\theta = 0.95, \phi_\xi = 0.95$, we obtain as stability conditions:

$$-1.4126\left(\frac{c_t}{h_{t-1}} - \varpi_{ss}^{ch}\right) - 0.0806\left(\frac{k_t}{h_t} - \varpi_{ss}^{kh}\right) + 0.0205\tilde{u}_t - 1.6923\tilde{W}_t^1$$
$$+0.2811\tilde{W}_t^2 - 0.8823\log(\theta_t) = 0 \quad (63)$$

$$-5.2034\left(\frac{c_t}{h_{t-1}} - \varpi_{ss}^{ch}\right) + 0.0493\left(\frac{k_t}{h_t} - \varpi_{ss}^{kh}\right) - 0.1193\tilde{u}_t + 0.4284\tilde{W}_t^1$$
$$+1.0355\tilde{W}_t^2 + 0.3801\log(\theta_t) = 0 \quad (64)$$

which amount to the following relationships between expectations errors and structural innovations:

$$\eta_t^1 = 0.5924\epsilon_t^\theta - 0.1938\epsilon_t^\xi$$
$$\eta_t^2 = 0.1490\epsilon_t^\theta + 0.4184\epsilon_t^\xi$$

Once we have the two stability conditions, we turn to the original model, to rewrite it in a slightly different way. The steady state rate of growth in this model is: $\gamma = \frac{\hat{c}_{t+1}}{\hat{c}_t} = [\beta n(B + 1 - \delta_h)]^{\frac{1}{\sigma}}$. We now rewrite the optimality conditions (51), (52), (56) and (57) making explicit this growth rate ($x_t = \hat{x}_t\gamma^{-t}$, with $x_t = (c_t, k_t, h_t)$):

[15]The transformation in ratios eliminates the unit eigenvalue that arises in all endogenous growth models as a consequence of the steady state being a one–dimensional manifold.

$$0 = -c_t^{-\sigma} + E_t\left[\beta\gamma^{-\sigma}c_{t+1}^{-\sigma}\left(\alpha A\left(\frac{k_t}{h_t}\right)^{\alpha-1} u_{t+1}^{1-\alpha}\theta_{t+1} + 1 - \delta_k\right)\right] \quad (65)$$

$$0 = -\left(\frac{k_{t-1}}{h_{t-1}}\right)^{\alpha} u_t^{-\alpha}\frac{\theta_t}{\xi_t} + E_t\left[\beta n\gamma^{-\sigma}\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\left(\frac{k_t}{h_t}\right)^{\alpha} u_{t+1}^{-\alpha}\frac{\theta_{t+1}}{\xi_{t+1}}\left(B\xi_{t+1} + 1 - \delta_h\right)\right] \quad (66)$$

$$0 = -n\gamma k_t + Ak_{t-1}^{\alpha}\left(u_t h_{t-1}\right)^{1-\alpha}\theta_t + (1-\delta_k)k_{t-1} - c_t \quad (67)$$

$$0 = -\gamma h_t + B(1-u_t)h_{t-1}\xi_t + (1-\delta_h)h_{t-1} \quad (68)$$

It is not hard to show that the two conditional expectations in (65) and (66) are, precisely, W_t^1 and W_t^2 in (59) and (61). Therefore, their associated errors are the same η_t^1 and η_t^2 as in (60) and (62).

Together with the stochastic processes for the exogenous shocks, the definitions of the expectations errors (60) and (62), and the stability conditions from the model in ratios[16] (63), (64), this system has ten equations in as many variables $(c_t, k_t, h_t, u_t, \theta_t, \xi_t, W_t^1, W_t^2, \eta_t^1, \eta_t^2)$. Besides, the system has an structure that allows for a solution to the original, endogenous growth model in the levels of the variables, to be obtained, starting from a sample realization for the structural innovations, along the following lines. The global constraint of resources (67), the law of accumulation of human capital (68), the expectations equations (59), (61) and the two stability conditions (63), (64) form a nonlinear system in $k_1, h_1, c_1, u_1, W_1^1$ and W_1^2, as functions of k_0, h_0, θ_1 and ξ_1. By repeated substitutions, the stability conditions can be transformed into a system of two nonlinear equations in c_1, u_1 as functions of state variables and exogenous shocks. Then, we would obtain h_1 and k_1 from (67) and (68), and W_1^1 and W_1^2 from (59), (61), and the same procedure would be implemented to obtain optimal values for subsequent periods. Once we have produced time series for these variables, realizations for the expectations errors would be obtained from (60) and (62), and we could proceed to test for rationality, if desired.

The transformation of the model in ratios to human capital is time–invariant in steady state, because in this model all variables that grow in steady state experience the same growth rate. Hence, their ratio stays constant. However, even if the rates of growth were different, an appropriately defined ratio would still be constant in steady state, and the same procedure we have described above would lead to a stable solution.

Endogenous growth models can also be solved by parameterizing expectations or following Uhlig's approach, among other possible methods. They differ from our approach on the way to recover time series for the levels of the variables that experience nonzero steady state growth. Most methods would compute time series for ratios like $\hat{c}_t/\hat{h}_{t-1}$ or $\hat{k}_t/\hat{h}_t$ so that, to get time series for c_t, h_t and k_t, one would have to:

1. use the law of motion of physical capital: $\hat{h}_t = B(1-u_t)\xi_t\hat{h}_{t-1} + (1-\delta_h)\hat{h}_{t-1}$ and normalize variables to make growth explicit: $\hat{h}_t = h_t\gamma^t$, to have:

[16]Note that the ratios of consumption and physical capital to human capital are the same with and without the deterministic trend, which is why we can also use for the detrended variables the previously calculated stability conditions.

$$\frac{h_t}{h_{t-1}} = [B(1-u_t)\xi_t + (1-\delta_h)] \frac{1}{\gamma} \tag{69}$$

2. then, given an initial condition h_0 for human capital, we would compute:

$$h_t = \left[\Pi_{s=1}^{t} \left(\frac{B(1-u_s)\xi_s + (1-\delta_h)}{\gamma} \right)\right] h_0, \quad t = 1, 2, ..., T \tag{70}$$

3. and once we have the h_t-path, we get time series for physical capital and consumption from:

$$k_t = \varpi_t^{kh} h_t = \left(\frac{k_t}{h_t}\right) h_t, \quad t = 1, 2, ..., T$$

$$c_t = \varpi_t^{ch} h_{t-1} = \left(\frac{c_t}{h_{t-1}}\right) h_{t-1}, \quad t = 1, 2, ..., T$$

However, the numerical precision error involved in generating the u_t–time series, which in a single period may be arbitrarily small, will become sizeable when it is compounded over time as in (69). As a result, there will be some increasing error in the h_t-series for long horizons, which will translate through (70) into errors for some other endogenous variables. In our experience, these errors are not negligeable: for instance, in the situation known as the *exogenous growth* case in Caballé and Santos (1993) (with the exogenous shock fixed at their expected value of one), the numerical errors are large enough for the resulting time series not to return to the same steady state point where the economy was before undergoing an instantaneous shock, even though it is known theoretically that the economy should converge to that same initial state.

On the contrary, the approach we have proposed computes the values for the variables in the economy each time t by solving a nonlinear system of equations. As a result, precision errors do not accumulate over time, and remain small every single period. After experiencing an instantaneous perturbation, the resulting time series converge to exactly the same steady state point where the economy was before the shock.

Mendoza (1991) and Correia et al. (1995) propose stochastic, general equilibrium models of small and open economies in which some endogenous variables are integrated of order 1, $I(1)$, although with some cointegrating relationships among them. In that situation, whenever the model can be written in terms of the ratios of those variables with unit roots in such a way that the ratios are stationary, we will generally be able to find approximate stability conditions around a time invariant steady state. Using those stability conditions together with the optimality conditions as it has been described in this Section should allow us to obtain more accurate solutions for the integrated variables.

In particular, to be able to solve the model using the alternative approach of accumulating growth from an initial condition as in (69), it is necessary that ratios to state variables can be found that are stationary. That will not be possible if the only $I(1)$ variables are decision variables, as it is the case in Correia et al. (1995), where consumption foreign debt, the

balance of trade and the level of net foreign asset holdings are $I(1)$, the first two being cointegrated, while the stock of physical capital, the only state variable, is stationary. These authors worry about the numerical accuracy of their solution [see footnote 3], computed through a linear approximation as in King et al. (1988), which amounts to linearizing the Euler equations and solving numerically the resulting system of stochastic difference equations.

8 Conclusions

We have summarized in this chapter some of the practical details involved in the implementation of a solution strategy to produce stable solutions to rational expectations models, which is based on eigenvalue/eigenvector decompositions. We have taken as a base recent work by Sims (1998), who has produced a quite general discussion of the characterization of stable manifolds in linear models. He has extended the initial proposal of Blanchard and Kahn (1980), to accomodate a number of interesting generalizations. Even though the method is exact for linear models, it can also be applied to nonlinear models, starting from a linear approximation to the model around steady state, and we have discussed applications to some standard business cycle economies.

A distinctive feature of the method is the consideration of each conditional expectation, as well as the associated expectations error, as additional variables in the model. The addition of stability conditions, derived from the eigenvalue/eigenvector decomposition of the coefficient matrices in the linear system of stochastic difference equations, allows for generating a numerical solution, in the form of a set of time series for all the relevant variables, including the conditional expectations and the rational expectations errors.

The approach is similar in spirit to any other method based on linear–quadratic approximation, even though it fully exploits the nonlinear structure of the original model to produce a numerical solution. The stability conditions can be written as relationships between conditional expectations of (generally) nonlinear functions of future state and decision variables, and state variables known at the time the expectations were made. These functions could be compared to those emerging from the parameterized expectations method of den Haan and Marcet (1990) and Marcet and Lorenzoni [(1998) this volume], which does not explicitly consider stability conditions. On the other hand, the method based on the eigenvalue/eigenvector decomposition is quite close to the undetermined coefficients method proposed by Uhlig [(1998) this volume], to which it would look even more similar if we started from a log-linear, rather than from a linear approximation. A more developed set of rules to characterize the stable manifold in Uhlig (1998) would also approximate his proposal to the method we have described in this chapter.

After applying the method to a standard growth model, we have shown that it performs well in situations where identification is weak, as it is the case with physical capital and inventories as production inputs. We have also explained, in that same context, how the method will produce information on analytical restrictions among expectations in the model, that the researcher might not have perceived from the outset.

Finally, we have described how the method can easily be adapted to deal with endogenous growth models. In them, the steady state is not constant over time in the levels of the relevant variables, so that the standard linear approximation to the model cannot be obtained, and the method would not directly apply. However, extracting the deterministic trend from the variables and transforming the model in ratios of the relevant variables, allows for a stable solution to be obtained. The reason is that the stability conditions for the model in ratios, whose steady state is constant over time, can be used to solve for the variables in levels, once they have been normalized by their deterministic trend.

Solving for expectations errors is central to the solution of dynamic, stochastic economic models. Besides, the assumption of rationality imposes very tight conditions on the stochastic structure of these errors, which should be routinely analyzed as a standard part of any solution strategy. We have indicated that alternative solution methods for nonlinear models differ essentially in the amount of nonlinearity that preserve when computing the numerical solution. Even though it is arguable that diverse degrees of nonlinearity might produce strong differences in qualitative results, they may lead to expectations errors that fail to pass tests for rationality, in the form of autocorrelation or significant cross correlations between them or with variables contained in the information set which was available when expectations were made. Results from testing the implications of rationality on expectations errors should be an integral component of any reported numerical solution which, unfortunately, usually tend to focus more on the characteristics of the time series for the rest of the variables. The standard practice of numerically solving rational expectations models will have to evolve in this direction, since it might well be a criterion on which to base the choice of a particular solution method.

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