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On character varieties of singular manifolds



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Abstract

In this paper, we construct a lax monoidal Topological Quantum Field Theory that computes virtual classes, in the Grothendieck ring of algebraic varieties, of G -representation varieties over manifolds with conic singularities, which we call nodefolds. This construction is valid for any algebraic group G , in any dimension and also in the parabolic setting. In particular, this TQFT allows us to compute the virtual classes of representation varieties over complex singular planar curves. In addition, in the case $G = \mathrm{SL}_2(k)$, the virtual class of the associated character variety over a nodal closed orientable surface is computed both in the non-parabolic and parabolic scenarios.

Keywords: Cone singularities, Character variety, TQFT, Grothendieck ring

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1 Introduction

Let X be a topological space with finitely generated fundamental group and let G be an algebraic group over an algebraically closed field k . The *representation variety* of X is the set of representations

$$\mathfrak{X}_G(X) = \{\rho : \pi_1(X) \rightarrow G\}.$$

This set inherits a natural structure of an algebraic variety from the one on G . Moreover, there is an action of G on $\mathfrak{X}_G(X)$ by conjugation, which corresponds to identifying isomorphic representations. This action is far from being free and, indeed, it has varying stabilizers on the subvariety of reducible representations according to the Jordan–Holder filtration of the $k[G]$ -module. For this reason, the orbit space $\mathcal{R}_G(X) = \mathfrak{X}_G(X)/G$ is no longer an algebraic variety. To overcome this issue, if G is reductive, we can consider the Geometric Invariant Theory (GIT) quotient

$$\mathcal{R}_G(X) = \mathfrak{X}_G(X) // G,$$

usually referred to as the *character variety* of X into G also known as the moduli space of isomorphism classes of representations of $\pi_1(X)$ into G .

Character varieties have been widely studied due to their tight relation with other important gauge theoretic moduli spaces, in a celebrated correspondence known as the non-abelian Hodge correspondence. Through this correspondence, if $G = \mathrm{GL}_n(\mathbb{C})$ (resp.

$G = \mathrm{SL}_n(\mathbb{C})$ and X is a smooth projective algebraic curve, then $\mathcal{R}_G(X)$ turns out to be biholomorphic, but not algebraically isomorphic, to the moduli space of flat connections on rank n vector bundles (resp. and fixed determinant) [58,59] and diffeomorphic to the moduli space of degree 0 and rank n Higgs bundles (resp. and fixed determinant) [15,57]. Thanks to these maps, $\mathcal{R}_G(X)$ is naturally endowed with a non-trivial hyperkähler structure [38,39].

Even a more striking correspondence can be obtained if we endow the character variety with a parabolic structure Q . This is a finite collection of punctures $p_1, \dots, p_s \in X$ that we remove from X , together with a collection of G -conjugacy classes $\lambda_1, \dots, \lambda_s \subseteq G$, usually referred to as the *holonomies*. In this way, the *parabolic representation variety*, $\mathfrak{X}_G(X, Q)$, is the collection of representations $\rho : \pi_1(X - \{p_1, \dots, p_s\}) \rightarrow G$ such that if γ_i is the positively oriented loop around p_i then $\rho(\gamma_i) \in \lambda_i$ (see Sect. 2.1 for a precise definition). Analogously, the *parabolic character variety* is $\mathcal{R}_G(X, Q) = \mathfrak{X}_G(X, Q) // G$. The parabolic structure allows us to bend the non-abelian Hodge correspondence in such a way that, for some holonomies, $\mathcal{R}_G(X, Q)$ is isomorphic to the moduli space of rank n flat connections with logarithmic singularities around p_1, \dots, p_s (and the monodromy around these points is determined by the holonomies of Q) and diffeomorphic to the moduli space of rank n parabolic Higgs bundles (and the weights of the parabolic structure are determined by the holonomies of Q) [56].

The limit for possible generalizations of the non-abelian Hodge correspondence in this direction is achieved when allowing nodal singularities on X . In this case, Bhosle proved in [6] that if the curve X is singular, with only nodes as singularities, this correspondence is no longer bijective. Recently, in [7] Bhosle studies the situation for more general singularities including cusps and ordinary nodes of any order and describes a bijective correspondence between the character variety for the singular curve and an open dense subset of the moduli space of Higgs bundles on the normalization of the singular curve.

The fact of the non-abelian Hodge correspondence not being bijective over the whole moduli space of Higgs bundles is crucial when studying the topology of character varieties. The topology of these character varieties, such as their Betti numbers, is usually studied by means of the above correspondence (see the foundational paper by Hitchin [38]), by computing the Betti numbers of the moduli space of Higgs bundles and using that the isomorphism in the correspondence is in fact a diffeomorphism. As the correspondence extends to the parabolic case, this is also the method to compute Betti numbers of parabolic character varieties. But the work by Bhosle [7] shows that this method does not extend beyond the parabolic situation.

The study of character varieties gets a boost from string theoretic ideas. Being the moduli spaces of Higgs bundles an example of SYZ mirror symmetry [36], and under the light of non-abelian Hodge theory, it is natural to ask whether this is also the case for character varieties. This idea originated a whole string of works dealing with the computation of invariants related to the Hodge structures of character varieties which we explain below. The first invariant to consider, after Betti numbers, was the so-called E -polynomial since it possesses some convenient calculability properties, such as preservation of the union and product of varieties, and generalizes the Poincaré polynomial. The next step is then to determine the class in the Grothendieck ring of algebraic varieties, $[\mathcal{R}_G(X)] \in \mathrm{KVar}_k$, also known as the virtual class. The importance of the virtual class is that it extends any other algebraic invariant that preserves the disjoint union and product of varieties. To be

precise, if $\chi : \mathbf{Var}_k \rightarrow R$ is any semi-ring homomorphism with values in a certain ring R that is invariant under isomorphism, then there exists a unique ring homomorphism $\tilde{\chi} : \mathbf{KVar}_k \rightarrow R$ such that $\chi(X) = \tilde{\chi}([X])$ for any algebraic variety X , where $[X] \in \mathbf{KVar}_k$ is the virtual class of X . Under this point of view, virtual classes should be understood as the most general invariant that can be considered up to ‘cut-and-paste’.

Moreover, the rationale behind considering virtual classes goes beyond this algebraic property. For instance, it is well-known that the Grothendieck ring of algebraic varieties is the correct ring to formulate the celebrated Kontsevich’s motivic integral [41], a very sophisticated measure-theoretic extension of the p -adic integral developed to provide direct proofs of the fact that any two birational smooth Calabi–Yau manifolds have the same Betti numbers, a deep theorem already proven by Batyrev using arithmetic techniques [4]. This motivic integration in the Grothendieck ring of varieties also plays an important role in singularity theory, particularly to analyse Hodge numbers of crepant resolutions and, in general, in relation to Kapranov’s zeta function [16, 42, 61]. In this later setting, motivic integration provides the appropriate setting to formulate the topological mirror symmetry conjecture for singular manifolds [5].

Due to the aforementioned importance, the problem of computing virtual classes of character varieties has been intensively studied. In the seminal paper [35], the authors proposed a method for computing the E -polynomial of $\mathcal{R}_G(X)$, a coarser invariant than the virtual class made by alternating sums of Hodge numbers, by means of counting points of the character variety on finite fields, in the spirit of the Weil conjectures. The aim of this work was precisely to compute these E -polynomials in a family of algebraic varieties to ponder what kind of symmetries in the Hodge diamond should be expected for mirror-symmetric character varieties. This work has been subsequently extended in [33, 48, 49] to the parabolic generic setting and other groups. However, despite the insight obtained with these techniques into the arithmetic structure of these varieties and their E -polynomial, the calculation of the subtler virtual class in the Grothendieck ring of varieties remains completely elusive to these methods. Another approach was initiated in [45], based on looking for geometric hands-on decompositions of the character variety according to the stabilizers of the adjoint action. This work was extended in [44, 47] to give rise to very explicit formulas for the E -polynomial in the case of rank 2.

Finally, a third approach was initiated in [26], based on the construction of a Topological Quantum Field Theory (TQFT) that computes the virtual classes of representation varieties over closed manifolds of arbitrary dimension into any group. Using this technique, the virtual classes of parabolic character varieties were computed for $G = \mathrm{SL}_2(k)$ in [23] for Jordan-type holonomies and in [25] for arbitrary holonomies. This method has also been used for automatizing the calculations and reaching higher rank for G the subgroup $\mathcal{U}_r \subseteq \mathrm{GL}_r(k)$ of upper-triangular matrices for $r = 2, 3, 4$, for both orientable [32] and non-orientable surfaces [62].

However, very few works have addressed the problem of computing the virtual class of character varieties over topological spaces different from closed surfaces or closed surfaces with punctures, mainly due to the intrinsic difficulties derived from analysing an invariant that cannot be captured with arithmetic techniques. One of the most studied cases arises when we consider an embedded link $K \subseteq \mathbb{R}^3$ and we take $X = \mathbb{R}^3 - K$. These are the so-called knot character varieties and have been studied, for instance, in [27, 53, 54] for torus knots or in [37] for the figure eight knot. In addition, in [21] the E -polynomial is

computed for the complement of unknotted links (i.e. free fundamental group), and in [22] for tori $X = S^1 \times \dots \times S^1$ (i.e. free abelian fundamental group).

In this paper, we study character varieties over singular manifolds. Specifically, we focus on an analogue of topological manifolds that we call *nodefolds*, in which nodal singularities are allowed. More precisely, instead of being locally Euclidean, as topological manifolds are, the nodefolds are locally a cone over a closed manifold. In particular, nodefolds generalize topological manifolds since discs, the local models of topological manifolds, can be seen as cones over the sphere.

However, the importance of this notion is that it encompasses much more general spaces, like any plane complex projective curve (possibly singular) and, in general, complex hypersurfaces. Indeed, classical results about the structure of the Milnor fibration [50] show that, from a purely topological point of view, a singularity in a plane curve is either nodal-like or cusp-like. In the former case, locally the curve looks like a cone over a bunch of circles, which we call a bouquet of cones, which justifies the name *nodefold*. Notice that, in the later case of a cusp-like singularity, the singular point is non-smooth but topologically the curve is locally Euclidean (the homeomorphism is, of course, non-differentiable). In this vein, cusp-like singularities remain undetected through the eyes of the fundamental group of the curve and, eventually, the character variety is the same as if the curve was non-singular.

Coming back to the motivation provided by mirror symmetry to the study of character varieties, a whole program by Chuang, Diaconescu, Donagi, Pantev and collaborators has been developed in the last years to give a string theoretic approach to the cohomology of character varieties (see, for instance, [20] and [14]). This program explores and uses the relations between different moduli spaces and different invariants such as moduli spaces of Higgs bundles, flat connections and refined Chern–Simons invariants of torus knots. Our hope is that by computing the virtual class of character varieties of curves with nodal singularities, we shed some light into the possibility of expanding this program and by reverse thinking understand these relations for nodal curves.

In order to study these character varieties over nodefolds, in this paper we extend the construction of the TQFT of González-Prieto et al. [26] to representation varieties over nodefolds. More precisely, let \mathbf{NBdp}_n be the category of n -dimensional nodefold bordisms with a finite number of basepoints, that is, the analogy of the usual category of smooth bordisms but on which conic singularities are allowed. Also, let $\mathbf{KVar}_k\text{-Mod}$ be the usual category of modules over the ground ring \mathbf{KVar}_k . We prove in Sect. 3 the following result.

Theorem (Theorem 3.9) *Let G be an algebraic group. For any $n \geq 1$, there exists a lax monoidal symmetric functor*

$$Z_G : \mathbf{NBdp}_n \rightarrow \mathbf{KVar}_k\text{-Mod}$$

such that, for any closed connected n -nodefold X and any basepoint $\star \in X$, we have $Z_G(X, \star)(1) = [\mathfrak{X}_G(X)]$, where $1 \in \mathbf{KVar}_k$ is the unit of the ring.

This construction extends naturally to the parabolic setting. In this framework, we equip bordisms with parabolic structures. In dimension 2, as aforementioned, the parabolic structure is given by a collection of punctures and holonomies but, in higher dimension, punctures should be replaced by codimension 2 co-oriented submanifolds with a choice of holonomy for the meridians around them (see Sect. 2.1 for further details). In this way, we

also obtain a functor $Z_G : \mathbf{NBdp}_n(\Lambda) \rightarrow \mathbf{KVar}_k\text{-Mod}$, where $\mathbf{NBdp}_n(\Lambda)$ is the category of nodefolds bordisms with parabolic data with holonomies in Λ .

Furthermore, we also show that this functor is actually a functor of 2-categories in a natural way. Suppose that X, X' are two n -dimensional nodefolds (i.e. two morphisms of \mathbf{NBdp}_n). We say that a map $f : X \rightarrow X'$ is a *conic degeneration* if, roughly speaking, it is a homeomorphism outside the singular points of X and X' . It captures the idea of “collapsing points” in a nodefold, in which several points of X are collapsed via f into a single point of X' in such a way that f becomes a ramified covering. In this manner, the category \mathbf{NBdp}_n is endowed with a 2-category structure by taking the conic degenerations a 2-morphism. As we see in detail below, these 2-morphisms in \mathbf{NBdp}_n turn into a natural 2-structure in $\mathbf{KVar}_k\text{-Mod}$ whose 2-morphisms between homomorphisms are called *twists*. This 2-structure arose naturally in the original construction [26], but in the singular setting it plays a more important role since it controls the effect of conic degenerations on the representation variety in a very precise way.

In the context of representation varieties over nodefolds, conic degenerations play a very important role to relate the virtual class of a representation variety over node surfaces (i.e. nodefolds of dimension 2) and over smooth surfaces. Given a node surface X , a *normalization* is a conic degeneration $f : \Sigma \rightarrow X$, where Σ is a smooth surface. Normalizations always exist and are unique up to homeomorphism. As byproduct of the analysis of conic degenerations through the TQFT, we obtain the following.

Theorem (Theorem 3.15) *Let G be an algebraic group. Let X be a node surface with l singular points which, locally, are cones over bunches of r_1, r_2, \dots, r_l circles, respectively; and let Q be a parabolic structure on X . Then, iff $f : \Sigma \rightarrow X$ is the normalization of X , and Q' is the parabolic structure induced in Σ via f , we have*

$$[\mathfrak{X}_G(X, Q)] = [\mathfrak{X}_G(\Sigma, Q')] \times [G]^{r_1 + \dots + r_l - 1}.$$

While this result can be obtained more easily through the usual Seifert–Van Kampen theorem and the Hurwitz formula for ramified coverings, the proof given in this work provides a deeper insight into the structure of the representation variety than the usual proof. The Seifert–Van Kampen theorem (in its version for fundamental groupoids) actually underlies the whole construction of the TQFT so, in some sense, the proof by means of the TQFT and conic degenerations encodes all the needed topological ingredients in a very effective way. In particular, the same kind of techniques used in this paper could be used to understand the effect of wilder singularities than nodefolds (namely, singularities arising in complex hypersurfaces) at the level of the representation variety. We postpone this analysis to future work.

Finally, in this paper we study how to transfer the results about the $SL_r(k)$ -virtual classes of representation varieties over node surfaces into the ones for the associated character varieties. For this purpose, we use the theory of pseudo-quotients, as developed in [24]. This theory allows us to chop the representation variety into strata on which the action can be understood more easily. This chopping is not, in general, compatible with the GIT quotient but it is compatible with a weaker version called pseudo-quotient, as introduced in [24]. Pseudo-quotients may not agree with GIT quotients on the nose, but their virtual classes agree. In this way, we can reassemble the virtual class of the quotient just by adding up the virtual classes of the quotients of each stratum.

In our case, roughly speaking the idea is to decompose the representation variety into its loci of reducible and irreducible representations as

$$\mathfrak{X}_{\mathrm{SL}_r(k)}(X) = \mathfrak{X}_{\mathrm{SL}_r(k)}^r(X) \sqcup \mathfrak{X}_{\mathrm{SL}_r(k)}^{\mathrm{ir}}(X).$$

On $\mathfrak{X}_G^{\mathrm{ir}}(X)$, the stabilizer of the action is the set of multiples of the identity, so the action of $\mathrm{PGL}_r(k)$ is free there. In this way, the GIT quotient on $\mathfrak{X}_{\mathrm{SL}_r(k)}^{\mathrm{ir}}(X)$ reduces to an orbit space and, thus $[\mathfrak{X}_{\mathrm{SL}_r(k)}^{\mathrm{ir}}(X) // \mathrm{SL}_r(k)] = [\mathfrak{X}_{\mathrm{SL}_r(k)}^{\mathrm{ir}}(X)] / [\mathrm{PGL}_r(k)]$. On the other hand, each orbit of the reducible locus, $\mathfrak{X}_{\mathrm{SL}_r(k)}^r(X)$, has in its closure an orbit of a semi-simple representation (a situation called a core). In this way, the GIT quotient $\mathfrak{X}_{\mathrm{SL}_r(k)}^r(X) // \mathrm{SL}_r(k)$ is equivalent to a product of lower rank representations, and the action reduces to the action of the symmetric group permuting representations of the same rank.

In particular, in the rank 2 case, we have that the semi-simple representations of $\mathfrak{X}_{\mathrm{SL}_2(k)}^r(X)$ are just the diagonal representations, and this diagonal form is unique up to permutation of the eigenvalues. Moreover, due to the unit determinant condition, the two eigenvalues are inverse, so we finally get that

$$\mathfrak{X}_{\mathrm{SL}_2(k)}^r(X) = (k^*)^N / \mathbb{Z}_2.$$

Here, N is a positive integer that depends on the number of generators in a presentation of $\pi_1(X)$ and the number of punctures on the parabolic structure Q , and the action of \mathbb{Z}_2 is given by $(\lambda_1, \dots, \lambda_N) \mapsto (\lambda_1^{-1}, \dots, \lambda_N^{-1})$. Therefore, using González-Prieto [24, Theorem 4.5] we get that

$$\begin{aligned} [\mathcal{R}_{\mathrm{SL}_2(k)}(X)] &= [\mathfrak{X}_{\mathrm{SL}_2(k)}^r(X) // \mathrm{SL}_2(k)] + [\mathfrak{X}_{\mathrm{SL}_2(k)}^{\mathrm{ir}}(X) // \mathrm{SL}_2(k)] \\ &= [(k^*)^N / \mathbb{Z}_2] + \frac{[\mathfrak{X}_{\mathrm{SL}_2(k)}^{\mathrm{ir}}(X)]}{[\mathrm{PGL}_2(k)]} = [(k^*)^N / \mathbb{Z}_2] + \frac{[\mathfrak{X}_{\mathrm{SL}_2(k)}(X)] - [\mathfrak{X}_{\mathrm{SL}_2(k)}^r(X)]}{[\mathrm{PGL}_2(k)]}. \end{aligned}$$

Notice that the last equality is particularly useful, since the computation of $[\mathfrak{X}_{\mathrm{SL}_2(k)}^{\mathrm{ir}}(X)]$ may be quite involved since it is an open set of the whole character variety. However, the virtual class of its complement, $\mathfrak{X}_{\mathrm{SL}_2(k)}^r(X) \subseteq \mathfrak{X}_{\mathrm{SL}_2(k)}(X)$, is much easier to get since it is a very small closed set whose counting reduces to lower rank cases. Since the virtual class of the whole variety is obtained through the TQFT, we have all the needed ingredients.

Analogous arguments can be carried out in the parabolic setting, but now taking into account that the holonomies of the punctures may restrict the possible reducible representations. Conducting these analyses, we obtain the main result of this paper, whose proof is a compilation of the calculations of Sect. 4. In the following result, we are working on the localization of the Grothendieck ring KVar_k by $q, q + 1$ and $q - 1$, where $q = [\mathbb{A}^1] \in \mathrm{KVar}_k$ is the virtual class of the affine line.

Theorem (Theorem 4.4) *Let X be the closed node surface whose singular points have a total of b branches, and whose normalization is a closed orientable genus g surface. Let Q be a parabolic structure with $r > 0$ punctures with holonomy of Jordan type with trace 2 and $s > 0$ punctures with diagonalizable holonomies with arbitrary eigenvalues $\xi_1, \dots, \xi_s \in k - \{0, \pm 1\}$. Then, the virtual class of the $\mathrm{SL}_2(k)$ -character variety is*

$$\begin{aligned} [\mathcal{R}_{\mathrm{SL}_2(k)}(X, Q)] &= q^{2g+s-2} (q-1)^{2g+r-2} (2^{2g+s-1} - 2^s + (q+1)^{2g+r+s-2}) (q^3 - q)^{b-1} \\ &\quad + \mathcal{I}_r(\xi_1, \dots, \xi_s) (q^3 - q)^{b-2}, \end{aligned}$$

where the so-called interaction term $\mathcal{I}_r(\xi_1, \dots, \xi_s)$ depends on the arithmetic of the chosen eigenvalues and is given by

$$\begin{aligned} \mathcal{I}_r(\xi_1, \dots, \xi_s) = & q^{2g+s-1}(q-1)^{2g+r-1}(\alpha_+ + \alpha_-) \left(2^{2g} + 2^{2g}q - 2q - 2 \right. \\ & \left. + (q+1)^{2g+r} + (q+1) \left(1 - 2^{2g-1} - \frac{1}{2}(q+1)^{2g+r-1} \right) \right) \\ & + q^{2g+s-1}(q-1)^{2g+r}(q+1)\alpha_+. \end{aligned}$$

Here, α_+ (resp. α_-) denotes one half of the number of tuples $(\epsilon_1, \dots, \epsilon_s) \in \{\pm 1\}^s$ such that $\xi_1^{\epsilon_1} \dots \xi_s^{\epsilon_s} = 1$ (resp. such that $\xi_1^{\epsilon_1} \dots \xi_s^{\epsilon_s} = -1$).

For a full statement of the theorem containing the cases $s = 0$ or $r = 0$, see Theorem 4.4. The remaining cases of other parabolic structures can be easily computed from these ones, as explained in Sect. 4.4.

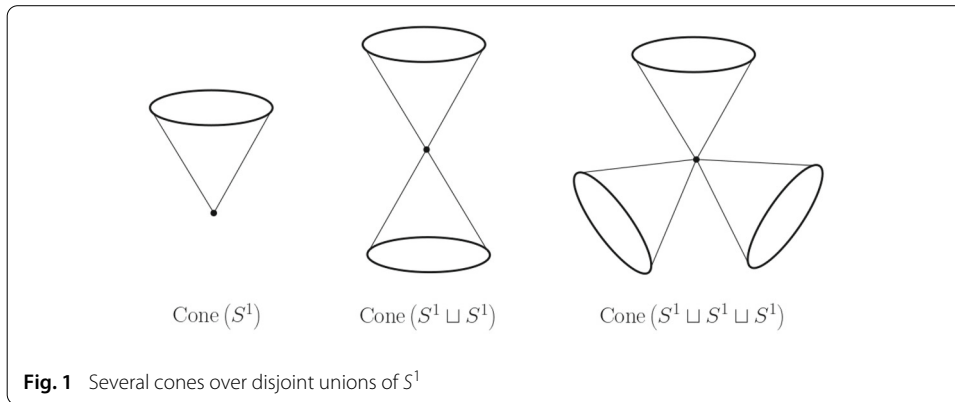
The structure of this paper is as follows. In Sect. 2 we introduce nodefolds and conic degenerations, as well as representation and character varieties over them and their parabolic counterparts. Section 3 is devoted to the construction of the lax monoidal TQFT computing virtual classes of representation varieties over nodefolds. In particular, in Sect. 3.4 we shall exploit the properties of conic degenerations under the built TQFT to relate the virtual classes of representation varieties over a nodefold and over its normalization. Section 4 is focused on the analysis of the GIT quotient to get the virtual class of the associated character variety in the $G = \text{SL}_2(k)$ case. Section 4.1 is devoted to the non-parabolic case, while Sects. 4.2 and 4.3 focus on the parabolic setting, with holonomies of Jordan and semi-simple type, respectively. Finally, in Sect. 4.4 we show how these results can be put together to obtain the virtual class of parabolic $\text{SL}_2(k)$ -character varieties with an arbitrary parabolic structure.

2 Nodefolds and representation varieties

In the following we introduce a new class of topological spaces that look like topological manifolds but with some special points around which they are no longer locally euclidean but locally a cone or even a bouquet of cones as in Fig. 1. We call this type of topological spaces nodefolds because they serve us to describe Riemann surfaces corresponding to nodal curves.

Singular manifolds are omnipresent in mathematical physics and geometry. They play a crucial role in string theory, particularly in Klebanov and Strassler’s models of $\mathcal{N} = 4$ superconformal field theories [30, 31, 40], where they used models of the form $X \times S$, where X is an 5-dimensional anti-de Sitter space and S is a singular manifold with a single conic singularity. The desingularization procedure also appears in Vafa’s models [28], where A-models in the singular conifold are shown to correspond to closed A-models on the resolved singular manifold.

Singularities also have a prominent role in moduli space theory of Calabi–Yau (CY) threefolds. It is known that the moduli spaces of CY corresponding to the different topological types of the underlying threefold form a web, and it is possible to follow finite length paths (in the so-called Zamolodchikov metric) between them joining non-homeomorphic CYs [11, 12, 29]. The intersection points of these moduli spaces are precisely singular CYs that serve as bridges between these moduli spaces, with the branches corresponding to



the possible desingularizations of the conic singularity in an operation known as *geometric transition*. It is expected that mirror symmetry should map a pair of smooth CY manifolds related by a geometric transition to another pair also connected through another geometric transition (see [51,55]). These ideas have also been used to construct new examples of orbifolds with G_2 holonomy [1].

Definition 2.1 A *nodefold* X is a second countable, paracompact topological space that is locally conic. The later means that, for any $p \in X$, there exists an open neighbourhood $U \subseteq X$, a non-empty closed manifold B and a homeomorphism $\varphi : U \rightarrow \text{Cone}(B)$ with $\varphi(p) = v_B$. Here

$$\text{Cone}(B) = (B \times [0, 1]) / (B \times \{0\})$$

denotes the (topological) cone over B and $v_B = [B \times \{0\}] \in \text{Cone}(B)$ is the vertex of the cone. The dimension of the nodefold is then $\dim X = \dim B + 1$.

Example 2.2 • Observe that, if $B = S^{n-1}$, then $\text{Cone}(S^{n-1})$ is homeomorphic to the n -dimensional open ball. In that case, we say that the point p with $\varphi(p) = v_B$, is a *smooth point*. Otherwise, we call p a *conic point*. In this manner, all the (topological) manifolds have a natural nodefold structure. Moreover, observe that all the points of $\text{Cone}(B) - \{v_B\}$ are smooth so the set of conic points of a nodefold X is a discrete set, denoted CX .

- On the other hand, B may not be connected with r connected components. In that case, $\text{Cone}(B)$ is genuinely a cone with r branches. For instance, if B is the disjoint union of r copies of S^{n-1} , then $\text{Cone}(B)$ is the result of gluing r cones along their vertices, as shown in Fig. 1.

Analogously, a *nodefold with boundary* X is a second countable, paracompact topological space such that each point is either locally conic or a boundary point (i.e. locally homeomorphic to $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$). From now on, the boundary points of X will be denoted by ∂X . Observe that ∂X is a compact $(n - 1)$ -dimensional manifold. In this manner, all the conic points of X belong to the interior $X - \partial X$.

Definition 2.3 Let X_1 and X_2 be nodefolds and let $f : X_1 \rightarrow X_2$ be a continuous map. We say that f is a *conic degeneration* if there exists a finite subset $S \subseteq CX_2$ such that $f^{-1}(s)$ is finite for all $s \in S$ and $f : X_1 - f^{-1}(S) \rightarrow X_2 - S$ is a homeomorphism.

2.1 Representation varieties

As with usual topological manifolds, we can consider representation varieties over a node-fold. Fix an algebraic group G over a certain algebraically closed field k . Suppose that X is a compact nodefold (maybe with boundary). Then, the representation variety over X , $\mathfrak{X}_G(X)$, is the set of group representations

$$\rho : \pi_1(G) \rightarrow G.$$

Recall that this set is naturally endowed with the structure of an algebraic variety as follows. Since X is compact, we have a presentation of $\pi_1(G)$ with finitely many generators $\pi_1(G) = \langle \gamma_1, \dots, \gamma_n \mid R_\alpha(\gamma_1, \dots, \gamma_n) = 1 \rangle$. This defines an injective map $\Psi : \mathfrak{X}_G(X) \rightarrow G^n$ given by $\Psi(\rho) = (\rho(\gamma_1), \dots, \rho(\gamma_n))$. The image of Ψ is an algebraic set of G^n , so we can put in $\mathfrak{X}_G(X)$ the algebraic structure induced by Ψ . In other words, we have

$$\mathfrak{X}_G(X) = \{ (g_1, \dots, g_n) \in G^n \mid R_\alpha(g_1, \dots, g_n) = 1 \}.$$

Representation varieties over singular manifolds represent an active research area due to their connection with moduli spaces of Higgs bundles over singular curves in a potential non-abelian Hodge correspondence, as studied in [6] and [8], among others. For a survey on Higgs bundles nodal curves, see [43].

We can extend the aforementioned definition to consider several basepoints. Let $A \subseteq X$ be a finite set and let $\Pi(X, A)$ be the fundamental groupoid of X with basepoints in X . The G -representation variety of (X, A) , $\mathfrak{X}_G(X, A)$, is the set of groupoid homomorphisms $\rho : \Pi(X, A) \rightarrow G$. It can be reduced to usual representation varieties as follows. Pick points $a_1, \dots, a_m \in A$ in different connected components of X and such that any other point of A shares a connected component with some of the points a_i . In this way, a groupoid representation $\rho : \Pi(X, A) \rightarrow G$ is completely determined by the induced group representations $\pi_1(X, a_i) \rightarrow G$ together with the images of any $|A| - m$ joining the points of A with the basepoint in the same component. These later images are unrestricted, so they can be any element of G , which gives rise to a natural identification

$$\mathfrak{X}_G(X, A) = \prod_{i=1}^m \mathfrak{X}_G(X, a_i) \times G^{|A|-m}. \tag{1}$$

Observe that each of the factors $\mathfrak{X}_G(X, a_i)$ is a usual representation variety, so $\mathfrak{X}_G(X, A)$ is naturally endowed with the structure of an algebraic variety too. For further details check [23, 26]

This construction inherits the functoriality properties from the fundamental groupoid. In this way, if $f : (X, A) \rightarrow (X', A')$ is a continuous map with $f(A) \subseteq A'$, it induces a regular morphism

$$\mathfrak{X}_G(f) : \mathfrak{X}_G(X', A') \rightarrow \mathfrak{X}_G(X, A).$$

In particular, if $X \subseteq X'$, the inclusion induces a restriction map $\mathfrak{X}_G(X', A') \rightarrow \mathfrak{X}_G(X, A' \cap X)$.

Finally, we can also consider parabolic structures on the representation variety. In analogy with González-Prieto [23], a parabolic structure on a nodefold X (maybe with boundary) with basepoints A is a finite set $Q = \{ (S_1, \lambda_1), \dots, (S_r, \lambda_r) \}$ such that

- $S_i \subseteq X$ are pairwise disjoint co-oriented smooth submanifolds of codimension 2 with $S_i \cap CX = S_i \cap A = \emptyset$ and such that all the submanifolds S_i intersect ∂X transversally.

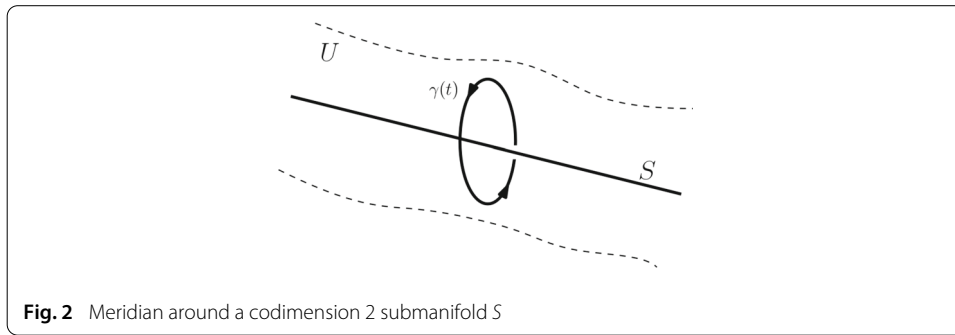


Fig. 2 Meridian around a codimension 2 submanifold S

- $\lambda_i \subseteq G$ are algebraic subvarieties that are invariant under the action of G by conjugation. Typically, we take $\lambda_i = [g_i]$ the conjugacy classes of some elements $g_i \in G$.

Remark 2.4 If Q is a parabolic structure on a nodefold (X, A) with boundary, there is naturally induced parabolic structure on ∂X , denoted $Q|_{\partial X}$, given by the collection of pairs $(S_i \cap \partial X, \lambda_i)$ for $(S_i, \lambda_i) \in Q$ with $S_i \cap \partial X \neq \emptyset$. Observe that it is crucial at this point that S_i intersects ∂X transversally, as imposed above. In a similar vein, if $M \subseteq \partial X$ is the disjoint union of some connected components of ∂X , we can also consider $Q|_M$.

Let us denote by $S = \bigcup_i S_i$ the total collection of submanifolds of the parabolic structure Q . Given a loop $\gamma \in \Pi(X - S, A)$, we say that γ is *around* S_i if $\gamma \in \text{Ker } \iota_i$, where $\iota_i : \Pi(X - S, A) \rightarrow \Pi(X - (S - S_i), A)$ is the induced map by the inclusion. These loops can be easily described as those lying in the complement of the zero section of the normal bundle νS_i to S_i (embedded into X as a tubular neighbourhood) that vanish when the zero section is added.

In other words, as proven in [60], the loops around S_i are the normal subgroup generated by any meridian (as normal subgroup). The meridians are, given a point $x \in S_i$, the usual generators of the fundamental group of the fibre $\nu_x S_i - \{0\} \sim S^1$ (see Fig. 2). With this description, we say that a meridian around S_i is positive if it is positively oriented with respect to the orientation of the holed plane $\nu_x S_i - \{0\}$ given by the co-orientation of S_i .

Given such parabolic structure Q , we can consider the associated parabolic G -representation variety, denoted $\mathfrak{X}_G(X, A, Q)$. Then, $\mathfrak{X}_G(X, A, Q)$ is the set of groupoid homomorphisms $\rho : \Pi(X - S, A) \rightarrow G$ such that $\rho(\gamma) \in \lambda_i$ if γ is a positive meridian around S_i . Again, choosing wisely the generators of $\Pi(X, A)$ (see [23, Sect. 4]), it is possible to give a natural decomposition of $\mathfrak{X}_G(X, A, Q)$ into a product of algebraic varieties, in the spirit of Eq. (1). In this manner, $\mathfrak{X}_G(X, A, Q)$ is also an algebraic variety.

Remark 2.5 Observe that, since the collection of loops around S_i are the normal subgroup generated by any positive meridian γ , the condition $\rho(\gamma) \in \lambda_i$ actually restricts the image of all these loops. Moreover, any two meridians are conjugated so, since the subvariety λ_i is closed by conjugation, the parabolic condition does not depend on the chosen meridian.

Finally, suppose that $f : (X, A) \rightarrow (X', A')$ is a conic degeneration. Then, given a parabolic structure $Q' = \{(S'_i, \lambda_i)\}$ in (X', A') , f induces a parabolic structure $f^*Q' = \{(f^{-1}(S'_i), \lambda_i)\}$ in (X, A) . Moreover, if Q is another parabolic structure on (X, A) such that $Q \subseteq f^*Q'$, the induced map of representation varieties actually preserves the parabolic structures giving rise to a map

$$\mathfrak{X}_G(f) : \mathfrak{X}_G(X', A', Q') \rightarrow \mathfrak{X}_G(X, A, Q).$$

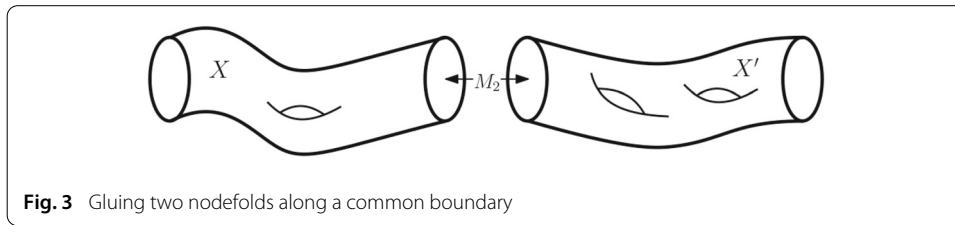


Fig. 3 Gluing two nodefolds along a common boundary

2.2 Gluing nodefolds

In this section, we discuss some gluing properties of nodefolds along boundaries and how they affect the associated representation varieties. In analogy with [26], this allow us to define the field theory part of a TQFT (see Sect. 3).

Consider two nodefolds with boundary X and X' and suppose that their boundaries share a common component, so that we can decompose the boundaries as $\partial X = M_1 \sqcup M_2$ and $\partial X' = M_2 \sqcup M_3$, as depicted in Fig. 3. We can use this common boundary to glue X and X' together to give rise to a topological space $X \cup_{M_2} X'$. It is straightforward to check that $X \cup_{M_2} X'$ actually inherits a nodefold structure, and that the conic points satisfy $C(X \cup_{M_2} X') = CX \sqcup CX'$. Moreover, the new nodefold has boundary $\partial(X \cup_{M_2} X') = M_1 \sqcup M_3$.

Proposition 2.6 *Suppose that $A \subseteq X$ and $A' \subseteq X'$ are finite subsets with $A \cap M_2 = A' \cap M_2$. Moreover, suppose that $A \cap M_2$ meets all the connected components of M_2 . Then, the following commutative diagram of groupoids induced by the natural inclusions*

$$\begin{array}{ccc}
 \Pi(X \cup_{M_2} X', A \cup A') & \longleftarrow & \Pi(X', A') \\
 \uparrow & & \uparrow \\
 \Pi(X, A) & \longleftarrow & \Pi(M_2, A \cap M_2)
 \end{array}$$

is a pushout.

Proof The proof is essentially Seifert–van Kampen theorem for fundamental groupoids, as proven in [10]. Take open collarings $U \subseteq X$ and $U' \subseteq X'$ around M_2 , i.e. such that $U, U' \cong M_2 \times [0, 1)$. Shrinking if necessary, we can also suppose that $A \cap U = A \cap M_2$ and $A' \cap U' = A' \cap M_2$.

Then, using Seifert–van Kampen theorem with the open sets $X \cup_{M_2} U'$ and $X' \cup_{M_2} U$ of $X \cup_{M_2} X'$, we get that the following diagram is a pushout

$$\begin{array}{ccc}
 \Pi(X \cup_{M_2} X', A \cup A') & \longleftarrow & \Pi(X' \cup_{M_2} U, A') \\
 \uparrow & & \uparrow \\
 \Pi(X \cup_{M_2} U', A) & \longleftarrow & \Pi(U \cup_{M_2} U', A \cap M_2)
 \end{array}$$

Now, the result follows using that $(X \cup_{M_2} U', A)$ has the homotopy type of (X, A) , $(X' \cup_{M_2} U, A')$ has the homotopy type of (X', A') , and $(U \cup_{M_2} U', A \cap M_2)$ has the homotopy type of $(M_2, A \cap M_2)$. □

Furthermore, since the functor $\text{Hom}(-, G)$ is co-continuous, we can extend this result to representation varieties.

Corollary 2.7 *Under the hypotheses of Proposition 2.6, for any algebraic group G , we have that the representation variety of the gluing is the fibred product*

$$\mathfrak{X}_G(X \cup_{M_2} X', A \cap A') = \mathfrak{X}_G(X, A) \times_{\mathfrak{X}_G(M_2, A \cap M_2)} \mathfrak{X}_G(X', A').$$

These results can be easily extended to the parabolic setting. Suppose that we fix an algebraic group G and that Q and Q' are parabolic structures on two nodefolds (X, A) and (X', A') . Suppose also that we want to glue X and X' along a common boundary M_2 as above. This gluing can be extended to the parabolic structure provided that the induced parabolic structures agree $Q|_{M_2} = Q'|_{M_2}$, giving rise to a new parabolic structure on $X \cup_{M_2} X'$, denoted $Q \cup_{M_2} Q'$. Roughly speaking, it is given by gluing together the submanifolds S_i, S'_j along the boundary M_2 , and labelling the result with $\lambda_i = \lambda'_j$.

At the level of representation varieties, observe that this construction is compatible with the description of $\mathfrak{X}_G(X \cup_{M_2} X', Q \cup_{M_2} Q')$ as a product variety, in the spirit of Eq. (1). In this way, we obtain also the following result.

Corollary 2.8 *In the aforementioned hypothesis, for any algebraic group G , the parabolic G -representation variety of the gluing is the fibred product*

$$\mathfrak{X}_G(X \cup_{M_2} X', A \cup A', Q \cup_{M_2} Q') = \mathfrak{X}_G(X, A, Q) \times_{\mathfrak{X}_G(M_2, A \cap M_2, Q|_{M_2})} \mathfrak{X}_G(X', A', Q').$$

3 TQFTs for representation varieties over nodefolds

In this section, we construct a lax monoidal Topological Quantum Field Theory (TQFT) computing the virtual class in the Grothendieck ring of algebraic varieties, of representation varieties over nodefolds. This construction extends the TQFT built in [26] for topological manifolds.

3.1 Grothendieck ring of algebraic varieties

From now on, we work on a fixed algebraically closed field k . Let \mathbf{Var}_k be the category of algebraic varieties over k , that is of integral separated schemes of finite type over k , with regular morphisms between them. Together with the disjoint union of varieties and their Cartesian product (more precisely, their fibred product over k), the isomorphism classes of objects of \mathbf{Var}_k form a semi-ring.

From it we can construct the so-called *Grothendieck ring* of algebraic varieties, or K -theory ring, denoted $K\mathbf{Var}_k$. Explicitly, $K\mathbf{Var}_k$ is the ring generated by the isomorphism classes of algebraic varieties, denoted $[Z]$ for $Z \in \mathbf{Var}_k$ and usually referred to as the *virtual class* of Z , subject to the relations

$$[Z_1 \sqcup Z_2] = [Z_1] + [Z_2], \quad [Z_1 \times Z_2] = [Z_1][Z_2].$$

Indeed, by combining both relations, we can get a slightly more general multiplicativity property.

Proposition 3.1 *Suppose that $\pi : Z \rightarrow B$ is a locally trivial fibration of algebraic varieties in the Zariski topology with fibre F . Then, we have $[Z] = [B][F]$.*

Proof Choose an open cover $\{U_i\}$ of B such that $\pi|_{U_i}$ is trivial. Then, by standard multiplicativity, we have that $[\pi^{-1}(U_i)] = [U_i \times F] = [U_i][F]$. In this way, coming back to the

global situation we have

$$[Z] = \left[\bigsqcup_i \pi^{-1}(U_i) \right] = \sum_i [\pi^{-1}(U_i)] = \sum_i [U_i] [F] = [F] \sum_i [U_i] = [F][B].$$

□

An important element in \mathbf{KVar}_k is the class of the affine line, typically denoted $q = [\mathbb{A}_k^1]$ and called the *Lefschetz motive*. The notation $\mathbb{L} = [\mathbb{A}^1]$ is also very standard, especially in motivic theory, but we avoid it in this paper.

Remark 3.2 Despite its simple definition, very little is known about the ring structure of \mathbf{KVar}_k , even in the complex case. For almost fifty years, it was expected that it is an integral domain. Finally, the answer is no but, more strikingly, the Lefschetz motive is a zero divisor [9]. Note that this captures very subtle properties of algebraic varieties, namely that there exist non-isomorphic algebraic varieties $Z_1, Z_2 \in \mathbf{KVar}_k$ such that $Z_1 \times k$ and $Z_2 \times k$ are isomorphic.

Throughout this paper, we need to divide by $q, q + 1$ and $q - 1$ several times. For this reason, instead of working in \mathbf{KVar}_k , we work on the localization of this ring by the multiplicative set generated by $q, q + 1$ and $q - 1$. The requirement of this localization should be clear from the context, so, to lighten the notation, we denote this localization also by \mathbf{KVar}_k . Observe that this localization has the effect that, all the zero divisor partners of $q, q + 1$ and $q - 1$ are annihilated.

In the complex case $k = \mathbb{C}$, $\mathbf{KVar}_{\mathbb{C}}$ extends some Hodge-theoretic invariants called the E -polynomial (a.k.a. Deligne–Hodge polynomial). Given a complex algebraic variety Z , its rational compactly supported cohomology, $H_c^k(Z, \mathbb{Q})$, is endowed with a natural mixed Hodge structure given by two filtrations: an increasing weight filtration W_{\bullet} and a decreasing Hodge filtration F^{\bullet} [17, 18]. From them, we can consider the so-called *Hodge numbers* as

$$h_c^{k;p,q}(Z) = \dim \operatorname{Gr}_F^p \left(\left(\operatorname{Gr}_{p+q}^W H_c^k(Z, \mathbb{Q}) \right) \otimes_{\mathbb{Q}} \mathbb{C} \right),$$

where $\operatorname{Gr}(\cdot)$ denoted the graded complex of a filtration. These numbers can be collected in the *E-polynomial*

$$e(Z)(u, v) = \sum_k (-1)^k h_c^{k;p,q}(Z) u^p v^q,$$

which is a polynomial in $\mathbb{Z}[u, v]$. By additivity of alternating sums of complexes and Künneth isomorphism, we get that the E -polynomial defines a semi-ring homomorphism $e : \mathbf{Var}_k \rightarrow \mathbb{Z}[u, v]$, so it can be extended to a ring homomorphism $e : \mathbf{KVar}_k \rightarrow \mathbb{Z}[u, v]$.

In particular, under this morphism we have that $e(q) = e([k]) = uv$. Moreover, if $[Z] \in \mathbf{KVar}_k$ lies in the subring generated by the Lefschetz motive, that is, $[Z] = P(q)$ for some polynomial $P \in \mathbb{Z}[q]$, then the previous computation shows that $e([Z]) = P(uv)$. In this vein, for algebraic varieties whose virtual class is generated by q , the virtual class extends the E -polynomial.

Remark 3.3 If $[Z]$ is generated by q , then $e(Z)$ is a polynomial in the variable uv , which means that $h_c^{k;p,q}(Z) = 0$ whenever $p \neq q$. In general, the varieties with such vanishing of Hodge numbers are called *balanced*. In this situation, it is customary to write the E -polynomial in the variable $q = uv$, which justifies our notation for the Lefschetz motive.

The aim of this paper is to construct a TQFT computing the virtual classes of G -representation varieties over nodefolds. Moreover, we focus on the case $G = \text{SL}_2(k)$ where we perform all the calculations explicitly. For this reason, it is important to understand the virtual classes of some related groups.

- $[\text{GL}_2(k)] = q^4 - q^3 - q^2 + q$. This follows by noticing that we have a locally trivial fibration in the Zariski topology $\text{GL}_2(k) \rightarrow k^2 - \{(0, 0)\}$ given by $A \mapsto Ae_1$ where e_1 is the first vector of the canonical basis of k^2 . The fibre of this map is the set of vectors of k^2 that do not lie in the line spanned by Ae_1 . In this way

$$[\text{GL}_2(k)] = [k^2 - \{(0, 0)\}] [k^2 - k] = (q^2 - 1)(q^2 - q).$$

Similar expressions for the virtual class of $[\text{GL}_n(k)]$ for $n \geq 2$ can be obtained with a similar argument.

- $[\text{PGL}_2(k)] = q^3 - q$. Consider the quotient map $\text{GL}_n(k) \rightarrow \text{PGL}_n(k) = \text{GL}_n(k)/k^*$, with $k^* = k - \{0\}$. This is a locally trivial fibration in the Zariski topology with fibre k^* , so $[\text{PGL}_n(k)] = [\text{GL}_n(k)]/(q - 1)$. Using the formula above for $[\text{GL}_2(k)]$ the result follows.
- $[\text{SL}_2(k)] = q^3 - q$. The argument is similar to the one of $\text{PGL}_2(k)$ but, now, instead of considering the quotient map, we consider the map $\text{GL}_n(k) \rightarrow \text{SL}_n(k)$ that, to a matrix $A \in \text{GL}_n(k)$ assigns the same matrix but with the first column divided by $\det(A) \neq 0$. Again, this is a locally trivial map in the Zariski topology with fibre k^* .

3.2 Functoriality of Grothendieck rings

The previous construction of the Grothendieck ring of algebraic varieties can be performed relative to a base variety. In this scenario, we discover new functoriality properties arising which are very useful for our construction of a TQFT.

Given an algebraic variety S , let us denote by \mathbf{Var}_S the category of algebraic varieties over S . Explicitly, this category has, as objects regular morphisms $Z \rightarrow S$ and its morphisms are regular maps $Z \rightarrow Z'$ preserving the base maps. In particular, if $S = \star$ is the singleton variety then $\mathbf{Var}_\star = \mathbf{Var}_k$ is the usual category of algebraic varieties.

Again, together with the disjoint union \sqcup of algebraic varieties, and the fibred product \times_S over S , we may consider its associated Grothendieck ring \mathbf{KVar}_S . The element of \mathbf{KVar}_S induced by a morphism $h : Z \rightarrow S$ is denoted as $[(Z, h)]_S \in \mathbf{KVar}_S$, or just $[Z]_S$ when the map is clear from the context. In this notation, the unit of \mathbf{KVar}_S is $\mathbb{1}_S = [S, \text{Id}_S]_S$.

This construction exhibits some important functoriality properties that are extremely useful in our construction. Suppose that $f : S_1 \rightarrow S_2$ is a regular morphism. It induces:

- A ring homomorphism $f^* : \mathbf{KVar}_{S_2} \rightarrow \mathbf{KVar}_{S_1}$ given by $f^*[Z]_{S_2} = [Z \times_{S_2} S_1]_{S_1}$. In particular, taking the projection map $i : S \rightarrow \star$ we get a ring homomorphism $i^* : \mathbf{KVar}_k \rightarrow \mathbf{KVar}_S$ that endows the rings \mathbf{KVar}_S with a natural structure of \mathbf{KVar}_k -module that corresponds to the Cartesian product.
- A \mathbf{KVar}_k -module homomorphism $f_! : \mathbf{KVar}_{S_1} \rightarrow \mathbf{KVar}_{S_2}$ given by $f_![(Z, h)]_{S_1} = [(Z, f \circ h)]_{S_2}$. In general $f_!$ is not a ring homomorphism but, for $[Z_1] \in \mathbf{KVar}_{S_1}$ and $[Z_2] \in \mathbf{KVar}_{S_2}$, the projection formula $f_!([Z_2] \times_{S_2} f^*[Z_1]) = f_![Z_2] \times_{S_1} [Z_1]$ holds, which implies that $f_!$ is a \mathbf{KVar}_k -module homomorphism.

The induced morphisms are functorial, in the sense that $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_! = g_! \circ f_!$. In particular, if $i : T \hookrightarrow S$ is an inclusion, then $i^*f^* = f|_T^*$.

3.3 Construction of the TQFT

The notion of TQFT is very useful for computing algebraic invariants attached to manifolds. Recall from Atiyah [3] that a n -dimensional Topological Quantum Field Theory (TQFT) is a monoidal symmetric functor

$$Z : \mathbf{Bd}_n \rightarrow R\text{-Mod}.$$

Here, \mathbf{Bd}_n is the category of n -dimensional bordisms, that is, the category whose objects are closed manifolds of dimension $n - 1$ and a morphism $X : M_1 \rightarrow M_2$ is a bordism class between M_1 and M_2 i.e. a compact n -dimensional manifold with $\partial X = M_1 \sqcup M_2$ up to boundary-preserving homeomorphism. It becomes a monoidal category with monoidal product the disjoint union (of manifolds and morphisms). The target category, $R\text{-Mod}$, is the usual category of R -modules and homomorphisms, with monoidal structure given by tensor product. There exists in the literature many constructions of TQFT that had provided deep insight in different areas, like [13, 19, 34, 52].

Remark 3.4 Due to the physical inception of many of these TQFTs, it is customary to impose that the objects of \mathbf{Bd}_n are smooth oriented manifolds and the bordisms between them are also smooth and oriented, in such a way that the orientation agrees with the ones at the boundaries. However, we do not use these extra structures in this paper.

In [23] (see also [26]), it was constructed a sort of TQFT to compute virtual classes of representation varieties over closed manifolds. Here, we extend such construction to nodefolds. For that purpose, first we need to enlarge the category \mathbf{Bd}_n to also include nodefolds, as well as some extra pieces of data needed for defining representation varieties.

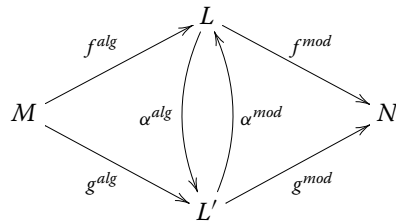
From now on, we fix an algebraic group G and a collection Λ of subvarieties of G invariant under conjugation (typically, Λ is a collection of conjugacy classes).

Definition 3.5 Let $n \geq 1$. We define the *category of n -dimensional pairs of nodefold bordisms with parabolic data* Λ , $\mathbf{NBdp}_n(\Lambda)$, as the 2-category given by:

- **Objects:** The objects are triples (M, A, Q) , where M is a closed manifold of dimension $n - 1$, $A \subseteq M$ is a finite set of points meeting each connected component of M and Q is a G -parabolic structure on (M, A) . The empty manifold, \emptyset , is also an object.
- **Morphisms:** A morphism $(M_1, A_1, Q_1) \rightarrow (M_2, A_2, Q_2)$ is a triple (X, A, Q) where X is a n -dimensional nodefold with $\partial X = M_1 \sqcup M_2$, $A \subseteq X$ is a finite set of points meeting each connected component of X and with $A \cap M_1 = A_1$ and $A \cap M_2 = A_2$, and Q is a parabolic structure on X with $Q|_{M_1} = Q_1$ and $Q|_{M_2} = Q_2$. Morphisms are defined up to homotopy equivalence, that is, (X, A, Q) and (X', A', Q') are declared to be equal if there exists homotopy equivalences $f : (X, A) \rightarrow (X', A')$ and $g : (X', A') \rightarrow (X, A)$ such that $f^*Q' = Q$, $g^*Q = Q'$ and the homotopies $g \circ f \sim \text{Id}_X$ and $f \circ g \sim \text{Id}_{X'}$ also preserve Q and Q' . In addition, we allow an object (M, A, Q) to be seen as a morphism $(M, A, Q) : (M, A, Q) \rightarrow (M, A, Q)$ (as a very thin bordism), and we decree it as the identity morphism. Composition of morphisms is given by gluing.
- **2-Morphisms:** A 2-morphism between bordisms (X, A, Q) and (X', A', Q') is given by a conic degeneration $f : X \rightarrow X'$ with $Q \subseteq f^*Q'$. Vertical composition is given by composition of the degenerations.

With this definition, $\mathbf{NBdp}_n(\Lambda)$ also has a natural monoidal structure given by disjoint union.

Another important category is the category of algebras with twists. Let R be a fixed commutative and unitary ring. Let M and N be two R -algebras and suppose that $f, g : M \rightarrow N$ are two homomorphisms as R -modules. We say that g is an *immediate twist* of f if there exists R -algebras L and L' , an algebra homomorphism $\alpha^{alg} : L \rightarrow L'$ and a R -module homomorphism $\alpha^{mod} : L' \rightarrow L$ such that the following diagram commutes.



Here, f^{alg} and g^{alg} are R -algebra homomorphisms, f^{mod} and g^{mod} are R -modules homomorphisms and $f = f^{mod} \circ f^{alg}$ and $g = g^{mod} \circ g^{alg}$.

In general, given $f, g : M \rightarrow N$ two R -module homomorphisms, we say that g is a *twist* of f if there exists a finite sequence $f = f_0, f_1, \dots, f_r = g : M \rightarrow M$ of homomorphisms such that f_{i+1} is an immediate twist of f_i for $0 \leq i \leq r - 1$.

In that case, we define the *category of R -algebras with twists*, $R\text{-Mod}_t$, as the category whose objects are R -algebras, its 1-morphisms are R -modules homomorphisms and, given homomorphisms f and g , a 2-morphism $f \Rightarrow g$ is a twist from f to g . Composition of 2-cells is juxtaposition of twists. With this definition, $R\text{-Mod}_t$ has a 2-category structure. Moreover, it is a monoidal category with the usual tensor product.

Definition 3.6 Fix a commutative ring with unit R . A lax monoidal 2-functor

$$Z : \mathbf{NBdp}_n(\Lambda) \rightarrow R\text{-Mod}_t$$

is called a conic lax monoidal TQFT.

Remark 3.7 Recall that lax monoidality means that Z preserves the unit of the monoidal structure, $Z(\emptyset) = R$, but we only have a morphism

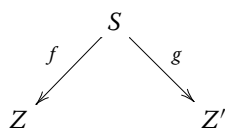
$$Z(M, A, Q) \otimes_R Z(M', A', Q') \rightarrow Z((M, A, Q) \sqcup (M', A', Q'))$$

which is no longer an isomorphism.

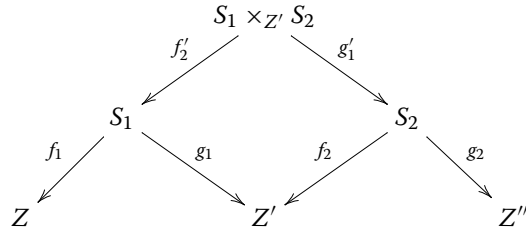
The aim of this section is to develop a conic lax monoidal TQFT computing virtual classes of representation varieties, which is constructed as a composition of two 2-functors

$$\mathbf{NBdp}_n(\Lambda) \xrightarrow{\mathcal{F}_G} \mathbf{Span}(\mathbf{Var}_k) \xrightarrow{\mathcal{Q}} \mathbf{KVar}_k\text{-Mod}.$$

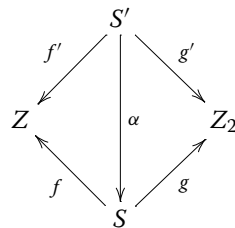
Remark 3.8 Recall that $\mathbf{Span}(\mathbf{Var}_k)$ is the 2-category of spans of algebraic varieties. More precisely, the objects of this category are algebraic varieties and a morphism between algebraic varieties Z and Z' is a span of regular maps of the form



Given two spans $(S_1, f_1, g_1) : Z \rightarrow Z'$ and $(S_2, f_2, g_2) : Z' \rightarrow Z''$, its composition is given by pullback. Explicitly, we define $(Z_2, f_2, g_2) \circ (Z_1, f_1, g_1) = (S_1 \times_{Z'} S_2, f_2' \circ f_1', g_2 \circ g_1')$, where f_2', g_1' are the morphisms in the pullback diagram



A 2-morphisms of $\mathbf{Span}(\mathbf{Var}_k)$ between spans $(f, g, S), (f', g', S') : Z_1 \rightarrow Z_2$ is given by a regular morphism $\alpha : S' \rightarrow S$ such that the following diagram commutes



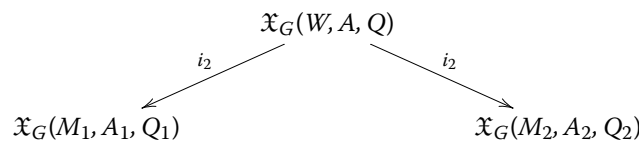
Finally, $\mathbf{Span}(\mathbf{Var}_k)$ also exhibits a natural monoidal structure given by the usual cartesian product of varieties and regular morphisms.

The functor $\mathcal{Q} : \mathbf{Span}(\mathbf{Var}_k) \rightarrow \mathbf{KVar}_k\text{-Mod}_t$ is called the *quantization* of the TQFT. It is a lax monoidal functor that was constructed in [23]. Roughly speaking, it is the motivic version of a Fourier–Mukai transform with identity kernel that assigns:

- Objects: given an algebraic variety $X \in \mathbf{Span}(\mathbf{Var}_k)$, it assigns $\mathcal{Q}(X) = \mathbf{KVar}_X$, the Grothendieck ring of the category of algebraic varieties over X , seen as a \mathbf{KVar}_k -module.
- Morphisms: given a span $(f, g, S) : X \rightarrow Y$, it associates the \mathbf{KVar}_k -module homomorphism $g_! \circ f^* : \mathbf{KVar}_X \rightarrow \mathbf{KVar}_Y$.
- 2-Morphisms: to a 2-morphism $\alpha : (f, g, S) \Rightarrow (f', g', S')$, it assigns the twisting immediate twisting $(\alpha^* = \alpha^{alg}, \alpha_! = \alpha^{mod}) : g_! \circ f^* \Rightarrow g'_! \circ f'^*$.

With respect to the functor $\mathcal{F}_G : \mathbf{NBdp}_n(\Lambda) \rightarrow \mathbf{Span}(\mathbf{Var}_k)$, usually referred to as the *field theory*, the construction works as follows:

- Objects: to an object (M, A, Q) , it assigns the associated parabolic representation variety $\mathfrak{X}_G(M, A, Q)$.
- Morphisms: given a nodefold bordism $(X, A, Q) : (M_1, A_1, Q_1) \rightarrow (M_2, A_2, Q_2)$, let us denote by $j_1 : (M_1, A_1, Q_1) \hookrightarrow (X, A, Q)$ and $j_2 : (M_2, A_2, Q_2) \hookrightarrow (X, A, Q)$ the inclusion maps as boundaries. Then we associate to this situation the span



where i_1, i_2 are the maps induce by the inclusions j_1, j_2 at the level of representation varieties.

- 2-Morphisms: To a 2-morphism $(X, A, Q) \rightarrow (X', A', Q')$ given by a conic degeneration $f : X \rightarrow X'$, we associate the regular map $\mathfrak{X}_G(f) : \mathfrak{X}_G(X', A', Q') \rightarrow \mathfrak{X}_G(X, A, Q)$. Observe that, by construction, this map intertwines with the inclusion morphisms.

By its very definition, the assignment \mathcal{F}_G commutes with vertical composition. Moreover, by Corollary 2.8, \mathcal{F}_G also commutes with horizontal composition. Finally, since $\mathfrak{X}_G((X, A, Q) \sqcup (X', A', Q')) = \mathfrak{X}_G((X, A, Q) \times \mathfrak{X}_G(X', A', Q'))$, the functor \mathcal{F}_G is monoidal.

Set $Z_G = \mathcal{Q} \circ \mathcal{F}_G : \mathbf{NBdp}_n(\Lambda) \rightarrow \mathbf{KVar}_k\text{-Mod}$, which is a lax monoidal 2-functor. Let $(X, A, Q) : \emptyset \rightarrow \emptyset$, which is given by an n -dimensional compact nodefold X without boundary. Observe that $\mathfrak{X}_G(\emptyset) = \star$ is the singleton variety, so denoting by $c : \mathfrak{X}_G(X, A, Q) \rightarrow \star$ the projection map, we have that

$$Z_G(X, A, Q) = \mathcal{Q} \circ \mathcal{F}_G(X, A, Q) = \mathcal{Q} \left(\star \xleftarrow{c} \mathfrak{X}_G(X, A, Q) \xrightarrow{c} \mathfrak{X}_G(\star) = 1 \right) = c_! c^*.$$

In particular, if we apply this map to the unit $[\star]_\star \in \mathbf{KVar}_k$, and using that c^* is a ring homomorphism, we have that

$$Z_G(X, A, Q)([\star]_\star) = c_! c^* 1_\star = c_! [\mathfrak{X}_G(X, A, Q)]_{\mathfrak{X}_G(X, A, Q)} = [\mathfrak{X}_G(X, A, Q)]_\star.$$

This is nothing but the virtual class of the associated parabolic representation variety in \mathbf{KVar}_k . Therefore, putting all together we have proven the following result.

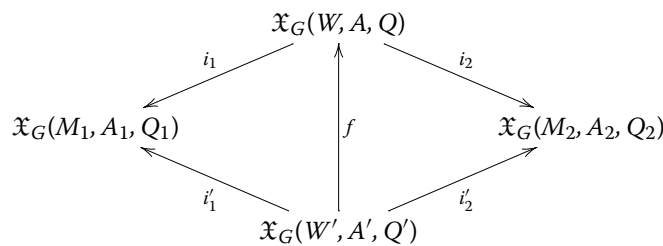
Theorem 3.9 *For any algebraic group G and any $n \geq 1$, there exists a conic lax monoidal TQFT*

$$Z_G : \mathbf{NBdp}_n(\Lambda) \rightarrow \mathbf{KVar}_k\text{-Mod}$$

computing the virtual classes of parabolic representation varieties over nodefolds.

3.4 Effect of conic degenerations

In order to get some flavour about the behaviour of Z_G with respect to degenerations, suppose that $(W, A, Q), (W', A', Q') : (M_1, A_1, Q_1) \rightarrow (M_2, A_2, Q_2)$ are two nodefold bordisms and that $f : (W, A, Q) \rightarrow (W', A', Q')$ is a degeneration. Hence, under the field theory, we have a commutative diagram of spans



where we have simplified the notation $\mathfrak{X}_G(f) = f : \mathfrak{X}_G(W', A', Q') \rightarrow \mathfrak{X}_G(W, A, Q)$. Therefore, we have that $Z_G(W, A, Q) = (i_2)_!(i_1)^*$ and $Z_G(W', A', Q') = (i'_2)!f^*(i'_1)^*$. In this way, the endomorphism $f_! f^* : \mathbf{KVar}_k / \mathfrak{X}_G(W, A, Q) \rightarrow \mathbf{KVar}_k / \mathfrak{X}_G(W', A', Q')$ gives the corresponding immediate twist $Z_G(W, A, Q) \Rightarrow Z_G(W', A', Q')$.

A particular example of this phenomenon appears when W is a genuine compact smooth manifold with a degeneration to a nodefold $f : W \rightarrow W'$, a situation that is called a *normalization* of the nodefold W' .

Lemma 3.10 *Let $f : W \rightarrow W'$ be a normalization with W of dimension n . Then, around any conic point $p' \in W'$, W' is locally homeomorphic to $\text{Cone}(\bigsqcup_r S^{n-1})$. The number $r > 1$ is called the number of branches of W' at p' .*

Proof Pick $p \in f^{-1}(p')$. Since W is a manifold, there exists an open set $U \subseteq W$, with $U \cap f^{-1}(CW') = \{p\}$, homeomorphic to the n -dimensional open ball $B^n = \text{Cone}(S^{n-1})$. Hence, since f is a degeneration, $f : U - \{p\} \rightarrow f(U) - \{p'\}$ is a homeomorphism so $f|_U : U \rightarrow f(U)$ is a continuous bijective map. Thus, maybe restricting U , we find that $f|_U$ is a homeomorphism.

Therefore, locally around p' , W' is homeomorphic to the gluing of $r = |f^{-1}(p')|$ open balls along a common interior point, which is $\text{Cone}(S^{n-1} \sqcup \dots \sqcup S^{n-1})$. \square

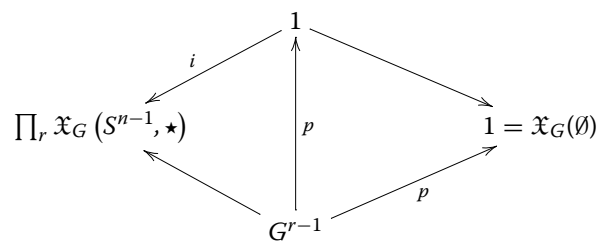
Now, let $f : W \rightarrow W'$ be a normalization with W a closed connected n -dimensional manifold. For simplicity, we suppose that W' as a unique conic point $p' \in W'$ and let $f^{-1}(p') = \{p_1, \dots, p_r\}$. By the previous proposition, there exist open neighbourhoods $U_i \subseteq W$ of p_i homeomorphic to open balls and an open set $U' \subseteq W'$ of p' such that f gives a homeomorphism of U' with the gluing of $U_1 \cup \dots \cup U_r$ along their centre.

In this way, setting $\tilde{W} = W - U_1 - \dots - U_r$ and $\tilde{W}' = W' - U'$ we obtain decompositions of the bordisms $W, W' : \emptyset \rightarrow \emptyset$ as

$$W = \left(\bigsqcup_r D^n \right) \circ \tilde{W}, \quad W' = \text{Cone} \left(\bigsqcup_r S^{n-1} \right) \circ \tilde{W}'.$$

Here D^n denotes the closed n -dimensional ball and we are seeing $\bigsqcup_r D^n$ and $\text{Cone}(\bigsqcup_r S^{n-1})$ both as bordisms $\bigsqcup_r S^{n-1} \rightarrow \emptyset$.

Now, let us choose $A' \subseteq W'$ with $p' \notin A'$ such that it contains a single point on each of the r copies of the boundaries $S^{n-1} \subseteq W'$ and does not meet the interior of the cone. Let us also denote $A = f^{-1}(A') \subseteq W$. Observe that we have $\mathfrak{X}_G(\bigsqcup_r D^n, A) = \prod_r \mathfrak{X}_G(\star) = 1$ and $\mathfrak{X}_G(\text{Cone}(\bigsqcup_r S^{n-1}), A') = \mathfrak{X}_G(\text{Cone}(\bigsqcup_r S^{n-1})) \times G^{r-1} = \mathfrak{X}_G(\star) \times G^{r-1} = G^{r-1}$. Therefore, under the field theory \mathcal{F}_G , we obtain the commutative diagram of spans



Here, the uppermost span is the corresponding one to $(\bigsqcup_r D^n, A) : \bigsqcup_r(S^{n-1}, \star) \rightarrow \emptyset$ and the left arrow is the inclusion of $1 = (1, \dots, 1)$ into $\prod_r \mathfrak{X}_G(S^{n-1}, \star)$. The lowermost span is the corresponding to $(\text{Cone}(\bigsqcup_r S^{n-1}), A') : \bigsqcup_r(S^{n-1}, \star) \rightarrow \emptyset$ and the right and vertical arrows are the projection $p : G^{r-1} \rightarrow 1$.

Remark 3.11 If $n > 2$, then S^{n-1} is simply connected and, thus, the map i is the identity. On the other hand, if $n = 2$, then $\mathfrak{X}_G(S^{n-1}, \star) = G$ so $\prod_r \mathfrak{X}_G(S^{n-1}, \star) = G^r$. However, even in that case, the left lowermost map $G^{r-1} \rightarrow \prod_r \mathfrak{X}_G(S^{n-1}, \star) = G^r$ is not an inclusion, but the projection onto $1 \in G^r$.

Therefore, we obtain that

$$Z_G(\bigsqcup_r D^n, A) = i^*, \quad Z_G(\text{Cone}(\bigsqcup_r S^{n-1}), A) = p_! p^* i^* = [G]^{r-1} \times i^*.$$

Hence, we get that

$$\begin{aligned} [\mathfrak{X}_G(W')] \times [G]^{|A|-1} &= Z_G(W', A)(1) = Z_G(\text{Cone}(\bigsqcup_r S^{n-1}), A) \circ Z_G(\tilde{W}, A)(1) \\ &= i^* Z_G(\tilde{W}, A)(1) \times [G]^{r-1} = Z_G(W, A)(1) \times [G]^{r-1} = [\mathfrak{X}_G(W)] \times [G]^{r+|A|-2}. \end{aligned}$$

Therefore, if we work the localization of KVar_k by $[G]$, we get that $[\mathfrak{X}_G(W')] = [\mathfrak{X}_G(W)] \times [G]^{r-1}$.

Proceeding analogously in the case of $l > 1$ conic points in W' with r_1, \dots, r_l branches, we finally obtain that

$$[\mathfrak{X}_G(W')] = [\mathfrak{X}_G(W)] \times [G]^{r_1 + \dots + r_l - 1}. \tag{2}$$

Analogous results can be obtained if W and W' are not closed or in the case of parabolic structures.

A very important example of nodefold comes from plane curves, as the following result shows.

Proposition 3.12 *Let X be a complex projective irreducible plane curve, maybe with singularities. Then X is a nodefold whose conic points $p \in X$ are those singular points with $r > 1$ branches. In that case, p has an open neighbourhood homeomorphic to $\text{Cone}(S^1 \sqcup \dots \sqcup S^1)$.*

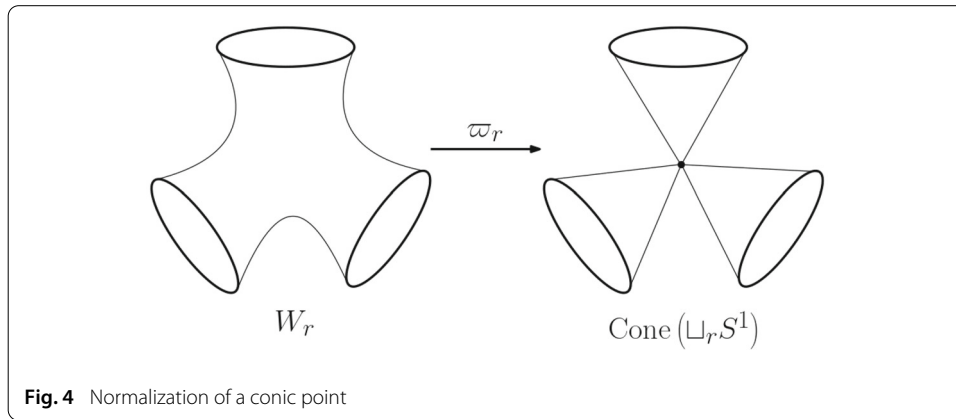
Proof If $p \in X$ is a regular point, then X around p is a topological manifold. Hence, we can suppose that $p \in X$ is a singular point with $r \geq 1$ branches. Let $B_\epsilon(p) \subseteq \mathbb{C}^2$ be a small open ball of radius $\epsilon > 0$ with centre at p and let $S_\epsilon(p)$ be its boundary sphere. By [50, Corollary 2.9], for ϵ small enough, $S_\epsilon(p) \cap X$ is a (smooth) link in $S_\epsilon(p) \cong S^3$ so it is homeomorphic to $B = S^1 \sqcup \dots \sqcup S^1$, where r is the number of branches of X around p . Moreover, by Milnor [50, Theorem 2.10], $B_\epsilon(p) \cap X$ is homeomorphic to $\text{Cone}(B)$, proving that X is locally conic at p of the claimed form. Finally, observe that if $r = 1$, $\text{Cone}(S^1)$ is homeomorphic to an open ball, so p is smooth. \square

Remark 3.13 Indeed, Theorem 2.10 of [50] holds for any irreducible hypersurface. Hence, in general, we actually have that any irreducible hypersurface is a nodefold. However, for real dimension of the hypersurface $n \geq 4$, there is a variety of compact manifolds of dimension $n - 1$ so, for higher dimensions, it is not possible to identify the local structure of a conic point so easily.

Example 3.14 If $p \in X$ is a nodal point with $r > 1$ branches, then it has an open neighbourhood homeomorphic to the cone over $S^1 \sqcup \dots \sqcup S^1$. A cusp point is a smooth point (in the nodefold sense) since it has $r = 1$ branches.

The previous result fully classifies complex projective plane curves topologically. First observe that, for each $r > 0$, there exists a smooth compact surface W_r with boundary $\partial W_r = \bigsqcup_r S^1$ that gives rise to a degeneration $\varpi_r : W_r \rightarrow \text{Cone}(\bigsqcup_r S^1)$ ramified at the vertex of the cone. W_r is obtained by “blowing-up” the cone $\text{Cone}(\bigsqcup_r S^1)$ at its vertex, as shown in Fig. 4.

Now, let X be a complex projective plane curve that, according to Proposition 3.12, topologically is a node surface with p_1, \dots, p_l conic points with r_1, \dots, r_l branches each. By removing small neighbourhoods of X around the conic points p_i and replacing them



with W_{r_i} , we obtain a smooth closed surface Σ and a normalization $\varpi : \Sigma \rightarrow X$. Hence, using Eq. (2) we get that

$$[\mathfrak{X}_G(X)] = [\mathfrak{X}_G(\Sigma)] \times [G]^{r_1+\dots+r_l-1}. \tag{3}$$

This shows that Σ is uniquely determined by X up to homeomorphism since, for $G = \text{GL}_1(k)$, $[\mathfrak{X}_{\text{GL}_1(k)}(\Sigma)] = [\text{GL}_1(k)]^{2g} = (q-1)^{2g}$, where g is the genus of Σ . In this way, X is characterized by the genus g of its normalization and the tuple of branches (r_1, \dots, r_l) .

The above discussion together with remark 2.4 proves the following.

Theorem 3.15 *Let G be an algebraic group. Let X be a node surface with l singular points which, locally, are cones over bunches of r_1, r_2, \dots, r_l circles, respectively; and let Q be a parabolic structure on X . Then, if $f : \Sigma \rightarrow X$ is the normalization of X , and Q' is the parabolic structure induced in Σ via f , we have*

$$[\mathfrak{X}_G(X, Q)] = [\mathfrak{X}_G(\Sigma, Q')] \times [G]^{r_1+\dots+r_l-1}.$$

Remark 3.16 Obviously, a simpler proof of the fact that the normalization is determined by X can be obtained by applying the Hurwitz formula to the branched covering ϖ . However, we present this proof here since it is in line with the so-called Torelli-like theorems, aiming to characterize the underlying manifold by means of the moduli spaces on it.

Moreover, the previous computation shows that we can focus on a very particular node surface. Fix $g, b \geq 1$ and let $\Sigma_{g,b}$ be the node surface whose normalization is Σ_g , the closed smooth surface of genus g , and with a unique conic point with b branches. By (3), we have that $[\mathfrak{X}_G(X)] = [\mathfrak{X}_G(\Sigma_{g,r_1+r_2+\dots+r_l})]$. For this reason, from now on we focus on the node surfaces $\Sigma_{g,b}$.

4 Character varieties over nodefolds

Once we have computed the virtual class of representation varieties over node surfaces, in this section, we compute the virtual class of the associated character varieties. Recall that, given a complex algebraic group G and a nodefold X , the character variety is the GIT quotient

$$\mathcal{R}_G(X) = \mathfrak{X}_G(X) // G.$$

Here, the action of G on $\mathfrak{X}_G(X)$ is given by conjugation i.e. $(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$ for $g \in G, \rho \in \mathfrak{X}_G(X)$ and $\gamma \in \pi_1(X)$.

In order to understand this quotient, we use the theory of pseudo-quotients as developed in [24], especially Sects. 3–5. Let us denote by $\mathfrak{X}_G^r(X)$ and $\mathfrak{X}_G^{ir}(X)$ the subvarieties of $\mathfrak{X}_G(X)$ of reducible and irreducible representations, respectively, and by $\mathcal{R}_G^r(X) = \mathfrak{X}_G^r(X) // G$ and $\mathcal{R}_G^{ir}(X) = \mathfrak{X}_G^{ir}(X) // G$ the corresponding character varieties. If G is a linear algebraic group (so in particular it is affine), then, by [24, Proposition 6.4] we have that if G^0 denotes the centre of G , then $\mathfrak{X}_G^{ir}(X) \rightarrow \mathfrak{X}_G^{ir}(X) // (G/G^0) = \mathcal{R}_G^{ir}(X)$ is a free geometric quotient. Applying now [24, Theorem 5.4] we get that $[\mathcal{R}_G^{ir}(X)] [G/G^0] = [\mathfrak{X}_G^{ir}(X)]$.

Now, let us suppose that $G = GL_n(k)$. Consider a partition τ of n , that is a multiset of the form $\tau = [r_1^{a_1} r_2^{a_2} \dots r_s^{a_s}]$ with r_i and a_i positive integers, the r_i distinct, such that $\sum_i a_i r_i = n$. We say that a representation $\rho : \pi_1(X) \rightarrow GL(k^n)$ is of type τ if the $\pi_1(X)$ -module k^n can be decomposed as a direct sum

$$k^n = \bigoplus_{i=1}^s V_i^{a_i},$$

where the V_i are irreducible representations of dimension $\dim V_i = r_i$. Adapting Proposition 7.3 and Corollary 7.4 of [24] to our situation (see also [27, Section 3]) shows that any representation of $\mathfrak{X}_{SL_n(k)}(X)$ is equivalent, in the GIT quotient, to a representation of type τ , and such τ is unique.

This can be used to decompose the character variety into simpler pieces. Let us denote by $\mathcal{R}^\tau(X)$ the character variety of the representations of type τ . Each representation of $\mathcal{R}^\tau(X)$ is determined by an element of $\prod_i (\mathcal{R}_{GL_{r_i}(k)}^{ir}(X))^{a_i}$ up to permutation of representations of the same dimension. In this way, if S_τ is the subgroup of the symmetric group S_n that preserves τ , we get that $(\prod_i (\mathcal{R}_{GL_{r_i}(k)}^{ir}(X))^{a_i}, S_\tau)$ is a core (see [24, Proposition 4.4]) for the action of $GL_n(k)$ on the representations of type τ . Therefore, we have that

$$\mathcal{R}^\tau(X) = \prod_{i=1}^s \text{Sym}^{a_i}(\mathcal{R}_{GL_{r_i}(k)}^{ir}(X)).$$

Remark 4.1 In particular, for $\tau = [n^1]$ we obtain the stable locus of irreducible representations, $\mathcal{R}^\tau(X) = \mathcal{R}_{GL_n(k)}^{ir}(X)$, and for $\tau = [1^n]$ we get the diagonal representations.

Remark 4.2 This decomposition can be seen as a result of considering the Levi subgroups of $GL_n(k)$ for the given partition τ .

Moreover, the representations of type τ form an open orbitwise-closed set of $\mathcal{R}_{GL_n(k)}(X)$ so by Theorem 4.1 of [24] we get that

$$[\mathcal{R}_{GL_n(k)}(X)] = \sum_{\tau} [\mathcal{R}^\tau(X)] = \sum_{\tau} \prod_{i=1}^s [\text{Sym}^{a_i}(\mathcal{R}_{GL_{r_i}(k)}^{ir}(X))],$$

where the sum runs over the set of all the partitions of n .

In the particular case of rank $n = 2$, we have that the only partitions of 2 are $[2^1]$, corresponding to irreducible representations, and $[1^2]$, corresponding to diagonal representations. Therefore,

$$\begin{aligned} [\mathcal{R}_{GL_2(k)}(X)] &= [\mathcal{R}^{[1^2]}(X)] + [\mathcal{R}^{[2^1]}(X)] = [\text{Sym}^2(\mathcal{R}_{GL_1(k)}^{ir}(X))] + [\mathcal{R}_{GL_2(k)}^{ir}(X)] \\ &= [(k^*)^{2(2g+b-1)} / \mathbb{Z}_2] + \frac{[\mathfrak{X}_{GL_2(k)}(X)] - [\mathfrak{X}_{GL_2(k)}^r(X)]}{[\text{PGL}_2(k)]}, \end{aligned}$$

where we have used that $\mathfrak{X}_{\text{GL}_1(k)}^{ir}(X) = \mathfrak{X}_{\text{GL}_1(k)}(X) = (k^*)^{2g+b-1}$ if X is a closed connected node surface with normalization of genus g and $b > 1$ branches.

In the case of $G = \text{SL}_2(k)$, the argument works verbatim but with the particularity that, now, we have to restrict to representations of unit determinant. Thus, in this case we have

$$[\mathcal{R}_{\text{SL}_2(k)}(X)] = [(k^*)^{2g+b-1}/\mathbb{Z}_2] + \frac{[\mathfrak{X}_{\text{SL}_2(k)}(X)] - [\mathfrak{X}_{\text{SL}_2(k)}^r(X)]}{[\text{PGL}_2(k)]}. \tag{4}$$

At this point, the strategy to compute the former expression is as follows. The virtual class of the total representation variety, $[\mathfrak{X}_{\text{SL}_2(k)}(X)]$ is the hardest part to be computed and is provided by the TQFT. On the other hand, the reducible part $[\mathfrak{X}_{\text{SL}_2(k)}^r(X)]$ can be computed by hand in terms of lower dimensional representations (in this case, 1-dimensional representations which are very easy).

The aim of the following sections is to follow this strategy. From now on, we focus on the case $G = \text{SL}_2(k)$. To shorten the notation, we omit the subscript in the representation and character varieties and denote $\mathfrak{X}(X) = \mathfrak{X}_{\text{SL}_2(k)}(X)$, $\mathcal{R}(X) = \mathcal{R}_{\text{SL}_2(k)}(X)$ and analogous for subsequent strata and parabolic versions.

4.1 Character varieties over orientable node surfaces

Let us fix $G = \text{SL}_2(k)$ and consider the closed connected node surface with normalization of genus g and $b > 1$ branches, $\Sigma_{g,b}$. Recall that, from Eq. (3), we have that $[\mathfrak{X}(\Sigma_{g,b})] = [\mathfrak{X}(\Sigma_g)][\text{SL}_2(k)]^{b-1}$, where Σ_g is the usual compact connected surface of genus g (the normalization of $\Sigma_{g,b}$). Indeed, using the standard presentation of the fundamental group of $\mathfrak{X}(\Sigma_g)$ we have that

$$\mathfrak{X}(\Sigma_{g,b}) = \left\{ (A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_{b-1}) \in \text{SL}_2(k)^{2g+b-1} \mid \prod_{i=1}^g [A_i, B_i] = I \right\}.$$

Here $[A_i, B_i] = A_i B_i A_i^{-1} B_i^{-1}$ denotes the group commutator. Let $A \in \text{SL}_2(k)^{2g+b-1}$ be a tuple of upper-triangular matrices, say

$$A = \left(\begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \beta_1 \\ 0 & \mu_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \lambda_g & \alpha_g \\ 0 & \lambda_g^{-1} \end{pmatrix}, \begin{pmatrix} \mu_g & \beta_g \\ 0 & \mu_g^{-1} \end{pmatrix}, \begin{pmatrix} \eta_1 & \gamma_1 \\ 0 & \eta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_{b-1} & \gamma_{b-1} \\ 0 & \eta_{b-1}^{-1} \end{pmatrix} \right)$$

with $\lambda_i, \mu_i, \eta_i \in k^* = k - \{0\}$ and $\alpha_i, \beta_i, \gamma_i \in k$. A straightforward computation shows that $A \in \mathfrak{X}(\Sigma_{g,b})$ if and only if

$$\sum_{i=1}^g \lambda_i \mu_i \left[(\mu_i - \mu_i^{-1}) \beta_i - (\lambda_i - \lambda_i^{-1}) \alpha_i \right] = 0.$$

We denote by $\pi \subseteq k^{2g+b-1}$ the (α_i, β_i) -plane defined by the previous equation for fixed (λ_i, μ_i) .

Now, let us stratify $\mathfrak{X}^r(\Sigma_{g,b})$ as follows.

- $\mathfrak{X}(\Sigma_{g,b})^t$ is the set of tuples with all the matrices equal to $\pm \text{Id}$ or, equivalently, the set of completely reducible representations with the images of the generators of trace ± 2 . Hence, $\mathfrak{X}(\Sigma_{g,b})^t$ is a set of 2^{2g+b-1} matrices.
- $\mathfrak{X}(\Sigma_{g,b})^v$ is the set of tuples with matrices with trace ± 2 that are not completely reducible. Given $A \in \mathfrak{X}(\Sigma_{g,b})^v$, let

$$\left(\begin{pmatrix} \epsilon_1 & a_1 \\ 0 & \epsilon_1 \end{pmatrix}, \begin{pmatrix} \epsilon_2 & a_2 \\ 0 & \epsilon_2 \end{pmatrix}, \dots, \begin{pmatrix} \epsilon_{2g+b-1} & a_{2g+b-1} \\ 0 & \epsilon_{2g+b-1} \end{pmatrix} \right)$$

be the element of conjugate to A with $\epsilon_i = \pm 1$ and $a_i \in k$ not all zero. Observe that such an element is unique up to simultaneous rescaling of the off-diagonal entries a_i . Thus, the $\mathrm{SL}_2(k)$ -orbit of A , $[A]$, is the set of reducible representations $(B_1, \dots, B_{2g+b-1}) \in X_n$ with a double eigenvalue such that, in their upper triangular form,

$$\left(\begin{pmatrix} \epsilon_1 & b_1 \\ 0 & \epsilon_1 \end{pmatrix}, \begin{pmatrix} \epsilon_2 & b_2 \\ 0 & \epsilon_2 \end{pmatrix}, \dots, \begin{pmatrix} \epsilon_{2g+b-1} & b_{2g+b-1} \\ 0 & \epsilon_{2g+b-1} \end{pmatrix} \right)$$

there exists $\lambda \neq 0$ such that $(a_1, \dots, a_{2g+b-1}) = \lambda(b_1, \dots, b_{2g+b-1})$. Then, taking $\lambda \rightarrow 0$, we find that the closure of the orbit, $\overline{[A]}$, is precisely the set of reducible representations with double eigenvalue such that their off-diagonal entries satisfy $(a_1, \dots, a_{2g+b-1}) = \lambda(b_1, \dots, b_{2g+b-1})$ for some $\lambda \in k$. In particular, for $\lambda = 0$ we get that the special element $(\epsilon_1 \mathrm{Id}, \dots, \epsilon_{2g+b-1} \mathrm{Id}) \in \overline{[A]}$. Therefore, the tuple $(a_1, \dots, a_{2g+b-1}) \in k^{2g+b-1} - \{0\}$ determines the diagonal form up to projectivization. The stabilizer of a Jordan-type matrix under the action of $\mathrm{SL}_2(k)$ by conjugation is k , so the orbit of an element is $\mathrm{SL}_2(k)/k$. Hence, we obtain a regular fibration

$$k^* \longrightarrow \mathrm{SL}_2(k)/k \times \{\pm 1\}^{2g+b-1} \times (k^{2g+b-1} - \{0\}) \longrightarrow \mathfrak{X}(\Sigma_{g,b})^\nu.$$

Observe that this fibration is locally trivial in the Zariski topology so this fibration has trivial monodromy and, thus,

$$\begin{aligned} [\mathfrak{X}(\Sigma_{g,b})^\nu] &= [\{\pm 1\}^{2g+b-1}] [\mathbb{P}^{2g+b-2}] [\mathrm{SL}_2(k)/\mathrm{Stab} J_+] \\ &= 2^{2g+b-1} (q^2 - 1) \frac{q^{2g+b-1} - 1}{q - 1}. \end{aligned}$$

Recall that, as introduced in Sect. 3.1, we denote the Lefschetz motive as $q = [k]$.

- $\mathfrak{X}(\Sigma_{g,b})^\delta$ is the set of completely reducible representations with images not all equal to $\pm \mathrm{Id}$. Given A in this stratum, let

$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \mu_g & 0 \\ 0 & \mu_g^{-1} \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_{b-1} & 0 \\ 0 & \eta_{b-1}^{-1} \end{pmatrix} \right)$$

be the element conjugate to A with $(\lambda_i, \mu_i, \eta_i) \in (k^*)^{2g+b-1}$ and not all equal to ± 1 . The stabilizer of a diagonal matrix is k^* so the orbit of this representation is $\mathrm{SL}_2(k)/k^*$. This canonical diagonal form of an element of $\mathfrak{X}(\Sigma_{g,b})^\delta$ is unique up to simultaneous permutation of the eigenvalues, so we have a double covering

$$\mathrm{SL}_2(k)/k^* \times \left((k^*)^{2g+b-1} - \{(\pm 1, \dots, \pm 1)\} \right) \longrightarrow \mathfrak{X}(\Sigma_{g,b})^\delta.$$

Therefore, we obtain that

$$\mathfrak{X}(\Sigma_{g,b})^\delta = \frac{\mathrm{SL}_2(k)/k^* \times [(k^*)^{2g+b-1} - \{(\pm 1, \dots, \pm 1)\}]}{\mathbb{Z}_2}.$$

Using González-Prieto [24, Remark 5.3], its virtual class is

$$[\mathfrak{X}(\Sigma_{g,b})^\delta] = \frac{q^3 - q}{2} \left((q - 1)^{2g+b-2} + (q + 1)^{2g+b-2} \right) - 2^{2g+b-1} q^2.$$

- $\mathfrak{X}(\Sigma_{g,b})^\rho$ is the set of reducible representations, not completely reducible, so that not all its matrices have double eigenvalue. In this case, any element is conjugated to one

of the form

$$\left(\begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1^{-1} & \beta_1 \\ 0 & \mu_1 \end{pmatrix}, \dots, \begin{pmatrix} \lambda_g & \alpha_g \\ 0 & \lambda_g \end{pmatrix}, \begin{pmatrix} \mu_g & \beta_g \\ 0 & \mu_g \end{pmatrix}, \begin{pmatrix} \eta_1 & \gamma_1 \\ 0 & \eta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_g & \gamma_{b-1} \\ 0 & \eta_{b-1}^{-1} \end{pmatrix} \right)$$

with $(\lambda_i, \mu_i, \eta_i) \in (k^*)^{2g+b-1} - \{(\pm 1, \dots, \pm 1)\}$ and $(\alpha_i, \beta_i, \gamma_i) \in \pi \times k^{b-1} - l$ where l is the line spanned by $(\lambda_1 - \lambda_1^{-1}, \mu_1 - \mu_1^{-1}, \dots, \lambda_g - \lambda_g^{-1}, \mu_g - \mu_g^{-1}, \dots, \eta_{b-1} - \eta_{b-1}^{-1})$. The action of $\text{PGL}_2 = \text{SL}_2/\{\pm \text{Id}\}$ by conjugation on these elements is free, and the canonical form is determined up to action of an upper-triangular matrix of $k \times k^*$. Thus, we have a fibration

$$k \times k^* \longrightarrow \text{PGL}_2 \times \Omega \longrightarrow \mathfrak{X}(\Sigma_{g,b})^e,$$

where Ω is a locally trivial fibration $(\pi \times k^{b-1} - l) \rightarrow \Omega \rightarrow (k^*)^{2g+b-1} - \{(\pm 1, \dots, \pm 1)\}$. Using that $[\pi] = q^{2g-1}$ and $[l] = q$, we get that the virtual class of Ω is $[(\Omega)] = ((q-1)^{2g+b-1} - 2^{2g+b-1})(q^{2g+b-2} - q)$ and the virtual class of this stratum is

$$[\mathfrak{X}(\Sigma_{g,b})^e] = \frac{q^3 - q}{(q-1)q} \left((q-1)^{2g+b-1} - 2^{2g+b-1} \right) (q^{2g+b-2} - q).$$

Therefore, putting all together we have that

$$\begin{aligned} [\mathfrak{X}^r(\Sigma_{g,b})] &= (q+1)(q-1)^{2g+b-1} (q^{2g+b-2} - q) \\ &\quad + \frac{q^3 - q}{2} \left((q-1)^{2g+b-2} + (q+1)^{2g+b-2} \right) - 2^{2g+b-1}(q^2 - 1). \end{aligned}$$

Using the results of Sect. 5.4 of [23] (see also [46, Proposition 11] for the weaker case of E -polynomials) and Eq. (3), we know that virtual class of the total representation variety is

$$\begin{aligned} [\mathfrak{X}(\Sigma_{g,b})] &= [\mathfrak{X}(\Sigma_g)] [\text{SL}_2(k)]^{b-1} = (2^{2g-1}(q-1)^{2g-1}(q+1)q^{2g-1} \\ &\quad + 2^{2g-1}(q+1)^{2g-1}(q-1)q^{2g-1} + (q+q^{2g-1})(q^2-1)^{2g-1} \\ &\quad + \frac{1}{2}(q+1)^{2g-1}(q-1)^2q^{2g-1} + \frac{1}{2}(q-1)^{2g-1}(q+1)(q-3)q^{2g-1}) (q^3 - q)^{b-1}. \end{aligned}$$

Moreover, for the quotient of the reducible locus [24, Corollary 7.5] shows that

$$[(k^*)^{2g+b-1} // \mathbb{Z}_2] = \frac{1}{2} \left((q-1)^{2g+b-1} + (q+1)^{2g+b-1} \right).$$

So, using formula (4), we finally find that

$$\begin{aligned} [\mathcal{R}(\Sigma_{g,b})] &= \frac{1}{2q^3(q^2-1)^3} \left[q^8 \left((q-1)^b + 1 \right) (q-1)^{2g} + q^7 \left((q-1)^b + 1 \right) (q-1)^{2g} \right. \\ &\quad - 2q^6 \left((q-1)^b + 1 \right) (q-1)^{2g} - 2q^5 \left((q-1)^b + 1 \right) (q-1)^{2g} \\ &\quad + q^4 \left((q-1)^b + 1 \right) (q-1)^{2g} + q^3 \left((q-1)^b + 1 \right) (q-1)^{2g} \\ &\quad + (q-1)^3 q^3 \left((q+1)^b - 1 \right) (q+1)^{2g+2} + (q(q^2-1))^b ((q+1)^2(q-1)^{2g} (4^g + q - 3) q^{2g} \\ &\quad + (q-1)^2(q+1)^{2g} (4^g + q - 1) q^{2g} + 2(q^2-1)^{2g} (q^2 + q^2)) \\ &\quad \left. - (q^2-1)q^{2g+1} (2(4^g(q^4 + q^2 + 1) + (q+1)^2(q-1)^{2g+1}) - q^2 \cdot 3 \cdot 2^{2g+1}) \right] \end{aligned} \tag{5}$$

Remark 4.3 Observe that the calculation above is only valid for $g \geq 1$. This is due to the fact that, for the stratum $\mathfrak{X}(\Sigma_{g,b})^e$, the set of feasible eigenvalues and antidiagonal

elements are required to lie in a hyperplane. However, this equation vanishes for $g = 0$ but, in that case, we only need to consider the quotient of a free group, as studied in Sect. 7.1 of [24] or [21].

4.2 The parabolic case with punctures of Jordan type

Now, let us address the computation of the virtual class of the character variety in the parabolic case. In this section, we focus on the case in which the parabolic structure Q only contains punctures of Jordan type

$$J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

that is $Q = \{(p_1, [J_+]), \dots, (p_s, [J_+])\}$ with $p_1, \dots, p_s \in \Sigma_{g,b}$ different points. Observe that the stabilizer of J_+ by conjugation is $\text{Stab } J_+ = k$. In particular, $[\text{SL}_2(k)/\text{Stab } J_+] = q^2 - 1$.

As before, consider a general representation of the form

$$A = \left(\begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \beta_1 \\ 0 & \mu_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_1 & \gamma_1 \\ 0 & \eta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_{b-1} & \gamma_{b-1} \\ 0 & \eta_{b-1}^{-1} \end{pmatrix}, \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & c_s \\ 0 & 1 \end{pmatrix} \right),$$

with $\lambda_i, \mu_i, \eta_i \in k^*, \alpha_i, \beta_i, \gamma_i \in k$ and $c_i \in k^*$. Then, $A \in \mathfrak{X}(\Sigma_{g,b}, Q)$ if and only if

$$\sum_{i=1}^g \lambda_i \mu_i \left[(\mu_i - \mu_i^{-1}) \beta_i - (\lambda_i - \lambda_i^{-1}) \alpha_i \right] + \sum_{i=1}^s c_i = 0. \tag{6}$$

For (λ_i, μ_i) fixed, let us denote by $\pi_s \subseteq k^{2g+s}$ the (α_i, β_i, c_i) -plane defined by the previous equation. In order to compute the virtual class of π_s observe that, solving for c_s , we observe that $\pi_s = k^{2g} \times (k^*)^{s-1} - \pi_{s-1}$. Using as base case that π_1 is k^{2g} minus a hyperplane, we have

$$\begin{aligned} [\pi_s] &= q^{2g}(q-1)^{s-1} - [\pi_{s-1}] \\ &= q^{2g} \sum_{k=1}^s (-1)^{k+1} (q-1)^{s-k} + (-1)^s q^{2g-1} = q^{2g-1} (q-1)^s. \end{aligned}$$

Now, we analyse each stratum of possible reducible representations as above. Using the previous notations for the strata, we have the following.

- $\mathfrak{X}(\Sigma_{g,b}, Q)^t = \emptyset$ since the punctures cannot be $\pm \text{Id}$.
- $\mathfrak{X}(\Sigma_{g,b}, Q)^v$. As in the previous case, an element $A \in \mathfrak{X}(\Sigma_{g,b}, Q)^v$ is conjugated to one of the form

$$\left(\begin{pmatrix} \epsilon_1 & \alpha_1 \\ 0 & \epsilon_1^{-1} \end{pmatrix}, \begin{pmatrix} \epsilon_2 & \beta_1 \\ 0 & \epsilon_2^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \epsilon_{2g+1} & \gamma_1 \\ 0 & \epsilon_{2g+1}^{-1} \end{pmatrix}, \dots, \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & c_s \\ 0 & 1 \end{pmatrix} \right),$$

where $\epsilon_i = \pm 1$ and $c_i \neq 0$. Moreover, this element has to lie in a subset of the plane π_s , which in this case amounts to the hyperplane

$$\tilde{\pi}_s = \left\{ (c_1, \dots, c_s) \in (k^*)^s \mid \sum_{i=1}^s c_i = 0 \right\}.$$

To compute the virtual class of this space, observe that $\tilde{\pi}_s = (k^*)^{s-1} - \tilde{\pi}_{s-1}$. Therefore, using the base case $\tilde{\pi}_1 = \emptyset$, we have

$$[\tilde{\pi}_s] = (q - 1)^{s-1} - [\tilde{\pi}_{s-1}] = \sum_{k=1}^{s-1} (-1)^{k+1} (q - 1)^{s-k} = (-1)^s \left(\frac{(1 - q)^s - 1}{q} + 1 \right).$$

As in the non-parabolic case, such an element is unique up to rescaling of the off-diagonal entries. Therefore, we obtain a regular fibration trivial in the Zariski topology

$$k^* \longrightarrow \mathrm{SL}_2(k)/\mathrm{Stab} J_+ \times \{\pm 1\}^{2g+b-1} \times \left(k^{2g+b-1} \times \tilde{\pi}_s \right) \longrightarrow \mathfrak{X}(\Sigma_{g,b}, Q)^\nu.$$

Hence, by taking virtual classes we obtain

$$\begin{aligned} [\mathfrak{X}(\Sigma_{g,b}, Q)^\nu] &= \left[\{\pm 1\}^{2g+b-1} \right] \frac{[k^{2g+b-1}] [\tilde{\pi}_s]}{[k] - 1} [\mathrm{SL}_2(k)/\mathrm{Stab} J_+] \\ &= 2^{2g+b-1} (q^2 - 1) \frac{q^{2g+b-1}}{q - 1} \left((-1)^s \left(\frac{(1 - q)^s - 1}{q} + 1 \right) \right). \end{aligned}$$

- $\mathfrak{X}(\Sigma_{g,b}, Q)^\delta = \emptyset$ since the holonomies of the punctures are not diagonalizable.
- $\mathfrak{X}(\Sigma_{g,b}, Q)^\ell$. In this case, any element is conjugated to one of the form

$$\left(\begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \mu_g & \beta_n \\ 0 & \mu_g^{-1} \end{pmatrix}, \begin{pmatrix} \eta_1 & \gamma_1 \\ 0 & \eta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_{b-1} & \gamma_{b-1} \\ 0 & \eta_{b-1}^{-1} \end{pmatrix} \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & c_s \\ 0 & 1 \end{pmatrix} \right)$$

with $(\lambda_1, \dots, \mu_g, \eta_1, \dots, \eta_{g-1}) \in (k^*)^{2g+b-1} - \{(\pm 1, \dots, \pm 1)\}$, $\gamma_i \in k$ and the vector of remaining off-diagonal entries $(\alpha_i, \beta_i, c_i) \in \pi_s$. Notice that, now, we no longer have a condition of simultaneously non-vanishing off-diagonal entries, since the punctures are non-diagonalizable. Therefore, we have a fibration

$$k \times k^* \longrightarrow \mathrm{PGL}_2 \times \Omega \longrightarrow \mathfrak{X}(\Sigma_{g,b}, Q)^\ell,$$

where $\pi_s \rightarrow \Omega \rightarrow ((k^*)^{2g+b-1} - \{(\pm 1, \dots, \pm 1)\}) \times k^{b-1}$ is a locally trivial fibration. Thus,

$$[\mathfrak{X}(\Sigma_{g,b}, Q)^\ell] = \frac{q^3 - q}{(q - 1)q} \left((q - 1)^{2g+b-1} - 2^{2g+b-1} \right) q^{b-1} (q^{2g-1} (q - 1)^s).$$

Therefore, putting all together, we obtain that

$$\begin{aligned} [\mathfrak{X}^r(\Sigma_{g,b}, Q)] &= \frac{1}{2q - 2} \left(q^{b+2g-2} \left((q^2 - 1) 2^{b+2g} ((-1)^s ((1 - q)^s + q - 1) - (q - 1)^s) \right. \right. \\ &\quad \left. \left. + 2(q + 1)(q - 1)^{b+2g+s} \right) \right) \end{aligned}$$

In [24, Theorem 5.10], it is proven that the virtual class of the total representation variety of the normalization is

$$\begin{aligned} [\mathfrak{X}(\Sigma_g, Q)] &= (q^2 - 1)^{2g+s-1} q^{2g-1} + \frac{1}{2} (q - 1)^{2g+s-1} q^{2g-1} (q + 1) (2^{2g} + q - 3) \\ &\quad + \frac{(-1)^s}{2} (q + 1)^{2g+s-1} q^{2g-1} (q - 1) (2^{2g} + q - 1). \end{aligned}$$

With this result, we can obtain the irreducible character variety as

$$[\mathcal{R}^{ir}(\Sigma_{g,b}, Q)] = \frac{[\mathfrak{X}(\Sigma_g, Q)] (q^3 - q)^{b-1} - [\mathfrak{X}^r(\Sigma_{g,b}, Q)]}{q^3 - q}$$

Up to this point, the calculation in this parabolic case is analogous to the non-parabolic setting. Nevertheless, for the reducible locus the situation turns completely different from the previous one. Recall that $\mathfrak{X}(\Sigma_{g,b}, Q)^l = \emptyset$ so there are not completely reducible representations (i.e. of type $\tau = [1^2]$). Precisely for this reason, the action of $SL_2(k)$ on $\mathfrak{X}^r(\Sigma_{g,b}, Q)$ is closed (see Sect. 8.1 of [24]). However, this action is not globally free so we have to distinguish between the two strata of $\mathfrak{X}^r(\Sigma_{g,b}, Q)$:

- $\mathfrak{X}(\Sigma_{g,b}, Q)^e$. Here, the action of $PGL_2(k)$ is free so, by [24, Corollary 5.5], we have

$$\begin{aligned} [\mathfrak{X}(\Sigma_{g,b}, Q)^e // SL_2(k)] &= \frac{[\mathfrak{X}(\Sigma_{g,b}, Q)^e]}{q^3 - q} \\ &= \left((q - 1)^{2g+b-1} - 2^{2g+b-1} \right) q^{2g+b-3} (q - 1)^{s-1}. \end{aligned}$$

- $\mathfrak{X}(\Sigma_{g,b}, Q)^v$. Here, the action of $PGL_2(k)$ is not free, but it has a stabilizer isomorphic to $Stab J_+ \cong k$. Hence, the GIT quotient $\mathfrak{X}(\Sigma_{g,b}, Q)^v \rightarrow \mathfrak{X}(\Sigma_{g,b}, Q)^v // SL_2(k)$ is a locally trivial fibration with fibre $SL_2/Stab J_+$ and trivial monodromy. Thus, we have that

$$\begin{aligned} [\mathfrak{X}(\Sigma_{g,b}, Q)^v // SL_2(k)] &= \frac{[\mathfrak{X}(\Sigma_{g,b}, Q)^v]}{q^2 - 1} \\ &= (-1)^s 2^{2g+b-1} \frac{q^{2g+b-1}}{q - 1} \left(\frac{(1 - q)^s - 1}{q} + 1 \right). \end{aligned}$$

Summarizing, the analysis above shows that

$$\begin{aligned} [\mathcal{R}(\Sigma_{g,b}, Q)] &= \frac{q^{2g-3}}{2(q^2 - 1)^3} \left[(4^g - 3) (q(q^2 - 1))^b (q - 1)^{2g+s} \right. \\ &\quad + (q(q^2 - 1))^b \left(q(4^g(q + 2) + q^2 - q - 5) (q - 1)^{2g+s} + 2(q^2 - 1)^{2g+s} \right. \\ &\quad \left. \left. + q(-1)^s (4^g(q - 2) + q^2 - 3q + 3) (q + 1)^{2g+s} + (4^g - 1) (-1)^s (q + 1)^{2g+s} \right) \right] \quad (7) \\ &\quad + (q^2 - 1) (-1)^s 2^{b+2g} q^b \left(q^5 - q^4 + q^3 + (q^2 - 1)^2 (1 - q)^s + 2q^2 + q - 1 \right) \\ &\quad \left. + (q^2 - 1) (-1)^{s+1} q^{b+3} 3 \cdot 2^{b+2g} \right] \end{aligned}$$

4.3 The parabolic case with diagonal punctures

In this section, we study the case of punctures of semi-simple type. That is, punctures whose holonomy is conjugated to

$$D_\xi = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix},$$

for $\xi \in k^* - \{\pm 1\}$. Observe that D_ξ and $D_{\xi^{-1}}$ are conjugated so the conjugacy class of D_ξ , $[D_\lambda]$, is determined by the trace $\text{tr}(D_\xi) = \xi + \xi^{-1}$. Particularly, in this section we focus on a parabolic structure of the form $Q = \{(p_1, [D_{\xi_1}]), \dots, (p_s, [D_{\xi_s}])\}$, for fixed $\xi_1, \dots, \xi_s \in k^* - \{\pm 1\}$. It is useful to consider the set

$$\Lambda_\pm = \{(\epsilon_1, \dots, \epsilon_s) \in \{1, -1\}^s \mid \xi_1^{\epsilon_1} \cdots \xi_s^{\epsilon_s} = \pm 1\},$$

and the integers $\alpha_\pm = \frac{1}{2} |\Lambda_\pm|$.

In order to understand the action of $SL_2(k)$ on the representation variety $\mathfrak{X}(\Sigma_g, Q)$, consider a tuple of matrices $A = (A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_{b-1}, P_1, \dots, P_s)$ of the form

$$\left(\begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \beta_1 \\ 0 & \mu_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_1 & \gamma_1 \\ 0 & \eta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \vartheta_1 & c_1 \\ 0 & \vartheta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \vartheta_s & c_s \\ 0 & \vartheta_s^{-1} \end{pmatrix} \right), \tag{8}$$

with $\lambda_i, \mu_i, \nu_i \in k^*$ and $\alpha_i, \beta_i, \gamma_i, c_i \in k$. Then, we have that

$$\begin{aligned} & \prod_{i=1}^g [A_i, B_i] \prod_{k=1}^s P_k \\ &= \begin{pmatrix} \vartheta_1 \cdots \vartheta_s \sum_{i=1}^g \lambda_i \mu_i \left[(\mu_i - \mu_i^{-1}) \beta_i - (\lambda_i - \lambda_i^{-1}) \alpha_i \right] + \sum_{k=1}^s \left(\prod_{j \neq k} \vartheta_j \right) c_k & \\ 0 & \vartheta_1^{-1} \cdots \vartheta_s^{-1} \end{pmatrix} \end{aligned}$$

Therefore, $A \in \mathfrak{X}(\Sigma_{g,b}, Q)$ if and only if the following system of equations holds

$$\begin{cases} \vartheta_i + \vartheta_i^{-1} = \xi_i + \xi_i^{-1}, & \vartheta_1 \cdots \vartheta_s = 1, \\ \sum_{i=1}^g \lambda_i \mu_i \left[(\mu_i - \mu_i^{-1}) \beta_i - (\lambda_i - \lambda_i^{-1}) \alpha_i \right] + \sum_{k=1}^s \vartheta_k^{-1} c_k = 0. \end{cases} \tag{9}$$

In particular, the first line imposes that $(\vartheta_1, \dots, \vartheta_s) = (\xi_1^{\epsilon_1}, \dots, \xi_s^{\epsilon_s})$ for some $(\epsilon_1, \dots, \epsilon_s) \in \Lambda_+$. Let us denote by π the (α_i, β_i, c_i) -hyperplane of k^{2g+s} given by the second line of equation (9). Observe that we can solve for c_s so $[\pi] = q^{2g+s-1}$.

For the quotient $\mathfrak{X}^r(\Sigma_{g,b}, Q) // SL_2(k)$, as in the non-parabolic case, the diagonal matrices with the action of \mathbb{Z}_2 by interchanging the eigenvalues form a core for the action. In this case, these diagonal matrices are $(k^*)^{2g+b-1} \times \Lambda_+$ and \mathbb{Z}_2 acts on Λ by $(\epsilon_1, \dots, \epsilon_s) \mapsto (-\epsilon_1, \dots, -\epsilon_s)$. Hence, we obtain

$$[\mathfrak{X}^r(\Sigma_{g,b}, Q) // SL_2(k)] = \left[(k^*)^{2g+b-1} \times \Lambda_+ / \mathbb{Z}_2 \right] = \alpha_+(q-1)^{2g+b-1}.$$

The calculation of the virtual class $[\mathfrak{X}^r(\Sigma_{g,b}, Q)]$ can be done by stratifying it as in the non-parabolic case, but taking into account equations (9).

- $\mathfrak{X}(\Sigma_{g,b}, Q)^t = \emptyset$ since the punctures cannot be $\pm Id$.
- $\mathfrak{X}(\Sigma_{g,b}, Q)^v = \emptyset$ since the punctures do not have eigenvalues ± 1 .
- For $\mathfrak{X}(\Sigma_{g,b}, Q)^\delta$ we have that every element is conjugate to one of the form

$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \mu_g & 0 \\ 0 & \mu_g^{-1} \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \eta_{b-1} & 0 \\ 0 & \eta_{b-1}^{-1} \end{pmatrix}, \begin{pmatrix} \xi_1^{\epsilon_1} & 0 \\ 0 & \xi_1^{-\epsilon_1} \end{pmatrix}, \dots, \begin{pmatrix} \xi_s^{\epsilon_s} & 0 \\ 0 & \xi_s^{-\epsilon_s} \end{pmatrix} \right),$$

with $(\lambda_1, \dots, \mu_g, \eta_1, \dots, \eta_{b-1}) \in (k^*)^{2g+b-1}$ and $(\epsilon_1, \dots, \epsilon_s) \in \Lambda_+$. This representation is unique up to permutation of the eigenvalues, so we have a double covering

$$(SL_2(k)/k^*) \times (k^*)^{2g+b-1} \times \Lambda \longrightarrow \mathfrak{X}(\Sigma_{g,b}, Q)^\delta.$$

Therefore, as shown in [24, Remark 5.3], we obtain

$$[\mathfrak{X}(\Sigma_{g,b}, Q)^\delta] = 2\alpha_+(q^2 + q)(q-1)^{2g+b-1}.$$

- For $\mathfrak{X}(\Sigma_{g,b}, Q)^\epsilon \subseteq \mathfrak{X}^r(\Sigma_g, Q)$, we have that any element is conjugated to one of the form (8). According to Eq. (9), this implies that $(\lambda_i, \mu_i, \eta_i) \in (k^*)^{2g+b-1}, (\vartheta_1, \dots, \vartheta_s) =$

$(\xi_1^{\epsilon_1}, \dots, \xi_s^{\epsilon_s})$ for some $(\epsilon_1, \dots, \epsilon_s) \in \Lambda_+$, and the off-diagonal entries $(\alpha_i, \beta_i, c_i) \in \pi$. In order to compute its virtual class, we have a fibration

$$k^* \times k \longrightarrow \mathrm{PGL}_2(k) \times \Omega \longrightarrow \mathfrak{X}(\Sigma_{g,b}, Q)^\ell.$$

Here, $\Omega = ((k^*)^{2g+b-1} \times \Lambda_+) \times (\pi \times k^{b-1} - \ell)$ with ℓ the line spanned by $(\lambda_1 - \lambda_1^{-1}, \dots, \mu_g - \mu_g^{-1}, \eta_1 - \eta_1^{-1}, \dots, \eta_s - \eta_s^{-1}, \lambda_1 - \lambda_1^{-1}, \dots, \lambda_s - \lambda_s^{-1})$. Therefore, the virtual class of Ω is $[\Omega] = \alpha_+(q-1)^{2g+b-1}(q^{2g+s-1}q^{b-1} - q)$ and we have

$$[\mathfrak{X}(\Sigma_{g,b}, Q)^\ell] = 4\alpha_+ \frac{q^3 - q}{(q-1)q} (q-1)^{2g+b-1} (q^{2g+b+s-2} - q).$$

Hence, putting all the computations together we get

$$[\mathfrak{X}^r(\Sigma_{g,b}, Q)] = \frac{1}{q^2} \left(2(q+1)\alpha_+(q-1)^{b+2g-1} (q^3 - 2q^{b+2g+s}) \right)$$

In [25, Theorem 5.6] it is shown that the total representation variety of the normalization is

$$[\mathfrak{X}(\Sigma_g, Q)] = q^{2g+s-1}(q-1)^{2g-1}(q+1)(2^{2g+s-1} - 2^s + (q+1)^{2g+s-2} + q^{2-2g-s}(q+1)^{2g+s-2}) + \mathcal{I}_0(\xi_1, \dots, \xi_s),$$

where the interaction term is given by

$$\mathcal{I}_0(\xi_1, \dots, \xi_s) = q^{s-1}(q-1)^{2g-1}(q+1)(\alpha_+ + \alpha_-) (q(q+1)^{2g-1} + q^{2g}(q+1)^{2g-1} - q^{2g}(q+1)^{2g-1} - q(q+1)^{2g-1}) + q^{2g+s-1}(q-1)^{2g}(q+1)\alpha_+.$$

Hence, plugging all these data into formula (4), we finally get that

$$\begin{aligned} [\mathcal{R}(\Sigma_{g,b}, Q)] &= (q-1)^{2g-2} \left(\frac{4(q-1)^b \alpha_+ (q^3 - q^{b+2g+s})}{q^3} \right. \\ &\quad + (q-1)(q+1) (q(q^2-1))^{b-2} q^{2g+s-1} \left(\left(\frac{q}{q+1} \right)^{-2g-s+2} \right. \\ &\quad + (q+1)^{2g+s-2} + 2^{2g+s-1} - 2^s) - 2(q-1)^b \alpha_+ + (q-1)^{b+1} \alpha_p \Big) \\ &\quad + \mathcal{I}_0(\xi_1, \dots, \xi_s)(q^3 - q)^{b-2} \end{aligned} \tag{10}$$

Compiling all the calculations of Sects. 4.2 and 4.3, we get the main result of this work.

Theorem 4.4 *Let X be the closed node surface with l singular points with a total of $b = r_1 + r_2 + \dots + r_l$ branches, and whose normalization is a closed orientable genus g surface. Fix integers $r, s \geq 0$ and arbitrary values $\xi_1, \dots, \xi_s \in k - \{0, \pm 1\}$. Let α_+ (resp. α_-) be one half of the number of tuples $(\epsilon_1, \dots, \epsilon_s) \in \{\pm 1\}^s$ such that $\xi_1^{\epsilon_1} \dots \xi_s^{\epsilon_s} = 1$ (resp. such that $\xi_1^{\epsilon_1} \dots \xi_s^{\epsilon_s} = -1$). Let Q be a parabolic structure with r punctures with holonomy of Jordan type with trace 2 and s punctures with diagonalizable holonomies with eigenvalues $\lambda_1, \dots, \lambda_s$. Denote $q = [\mathbb{A}^1] \in \mathrm{KVar}_k$ the virtual class of the affine line. The virtual class of the $\mathrm{SL}_2(k)$ -character variety $\mathcal{R}(X, Q) = \mathcal{R}_{\mathrm{SL}_2(k)}(X, Q)$ in the localization of KVar_k by $q, q+1, q-1$ is*

- If $s = r = 0$, then

$$\begin{aligned}
 [\mathcal{R}(X, Q)] &= \frac{1}{2q^3(q^2-1)^3} \left[q^8((q-1)^b+1)(q-1)^{2g} + q^7((q-1)^b+1)(q-1)^{2g} \right. \\
 &\quad - 2q^6((q-1)^b+1)(q-1)^{2g} - 2q^5((q-1)^b+1)(q-1)^{2g} \\
 &\quad + q^4((q-1)^b+1)(q-1)^{2g} + q^3((q-1)^b+1)(q-1)^{2g} \\
 &\quad + (q-1)^3q^3((q+1)^b-1)(q+1)^{2g+2} + (q(q^2-1))^b((q+1)^2(q-1)^{2g}(4g+q-3)q^{2g} \\
 &\quad + (q-1)^2(q+1)^{2g}(4g+q-1)q^{2g} + 2(q^2-1)^{2g}(q^{2g}+q^2)) \\
 &\quad \left. - (q^2-1)q^{2g+1}(2(4g(q^4+q^2+1) + (q+1)^2(q-1)^{2g+1}) - 3q^22^{2g+1}) \right]
 \end{aligned}$$

- If $s = 0$ and $r > 0$, then

$$\begin{aligned}
 [\mathcal{R}(X, Q)] &= \frac{q^{2g-3}}{2(q^2-1)^3} \left[(4g-3)(q(q^2-1))^b(q-1)^{2g+r} \right. \\
 &\quad + (q(q^2-1))^b(q(4g(q+2)+q^2-q-5)(q-1)^{2g+r} + 2(q^2-1)^{2g+r} \\
 &\quad + (-1)^r q(4g(q-2)+q^2-3q+3)(q+1)^{2g+r} + (4g-1)(-1)^r(q+1)^{2g+r}) \\
 &\quad + (-1)^r 2^{b+2g}(q^2-1)q^b(q^5-q^4+q^3+(q^2-1)^2(1-q)^r + 2q^2+q-1) \\
 &\quad \left. + (-1)^{r+1} 2^{b+2g}(q^2-1)3q^{b+3} \right]
 \end{aligned}$$

- If $s > 0$ and $r = 0$, then

$$\begin{aligned}
 [\mathcal{R}(X, Q)] &= (q-1)^{2g-2} \left(\frac{4(q-1)^b \alpha_+ (q^3 - q^{b+2g+s})}{q^3} \right. \\
 &\quad + (q-1)(q+1)(q(q^2-1))^{b-2} q^{2g+s-1} \left(\left(\frac{q}{q+1} \right)^{-2g-s+2} \right. \\
 &\quad + (q+1)^{2g+s-2} + 2^{2g+s-1} - 2^s) - 2(q-1)^b \alpha_+ + (q-1)^{b+1} \alpha_+ \Big) \\
 &\quad + \mathcal{I}_0(\xi_1, \dots, \xi_s)(q^3 - q)^{b-2}
 \end{aligned}$$

where the interaction term is given by

$$\begin{aligned}
 \mathcal{I}_0(\xi_1, \dots, \xi_s) &= q^{s-1}(q-1)^{2g-1}(q+1)(\alpha_+ + \alpha_-)(q(q+1)^{2g-1} + q^{2g}(q+1)^{2g-1} \\
 &\quad - q^{2g}(q+1)^{2g-1} - q(q+1)^{2g-1}) + q^{2g+s-1}(q-1)^{2g}(q+1)\alpha_+.
 \end{aligned}$$

- If $s, r > 0$, then

$$\begin{aligned}
 [\mathcal{R}(X, Q)] &= q^{2g+s-2}(q-1)^{2g+r-2}(2^{2g+s-1} - 2^s + (q+1)^{2g+r+s-2})(q^3 - q)^{b-1} \\
 &\quad + \mathcal{I}_r(\xi_1, \dots, \xi_s)(q^3 - q)^{b-2},
 \end{aligned}$$

where the interaction term is given by

$$\begin{aligned}
 \mathcal{I}_r(\xi_1, \dots, \xi_s) &= q^{2g+s-1}(q-1)^{2g+r-1}(\alpha_+ + \alpha_-) \left(2^{2g} + 2^{2g}q - 2q - 2 \right. \\
 &\quad \left. + (q+1)^{2g+r} + (q+1) \left(1 - 2^{2g-1} - \frac{1}{2}(q+1)^{2g+r-1} \right) \right) \\
 &\quad + q^{2g+s-1}(q-1)^{2g+r}(q+1)\alpha_+.
 \end{aligned}$$

4.4 The parabolic case with arbitrary punctures

In this section, we show that we can compute the character variety of a $SL_2(k)$ -representation variety over the node surface $\Sigma_{g,b}$ with an arbitrary parabolic structure directly from Theorem 4.4. Denote by Q_s^+ the parabolic structure on $\Sigma_{g,b}$ with s punctures with holonomies $[J_+]$, as studied in Sect. 4.2.

Now, consider an arbitrary parabolic structure $Q_{r_+,r_-}^t(\xi_1, \dots, \xi_s)$ with t punctures with holonomy $[-Id]$, r_+ punctures with holonomy $[J_+]$, r_- punctures with holonomy $[J_-]$ and s punctures with semi-simple holonomies $[D_{\xi_1}], \dots, [D_{\xi_s}]$. Set $r = (r_+ + r_-)$ and $\sigma = (-1)^{r_+ + t}$. Observe that $[J_-] = [-J_+]$ and $(-Id)^2 = Id$ so

$$\mathfrak{X}(\Sigma_{g,b}, Q_{r_+,r_-}^t(\xi_1, \dots, \xi_s)) = \begin{cases} \mathfrak{X}(\Sigma_{g,b}, Q_{r,0}^0(\xi_1, \dots, \xi_s)) & \text{if } \sigma = 1, \\ \mathfrak{X}(\Sigma_{g,b}, Q_{r,0}^1(\xi_1, \dots, \xi_s)) & \text{if } \sigma = -1. \end{cases}$$

Then, apart from the cases of Sects. 4.2 and 4.3, the remaining cases reduce to the following scenarios.

- $r, s > 0$ and $\sigma = 1$. Then the action of $PGL_2(k)$ on $\mathfrak{X}(\Sigma_{g,b}, Q)$ is closed and free, since the only matrices that stabilize both a non-diagonalizable and a diagonalizable matrix are the multiples of the identity. Hence, using [25, Theorem 6.1], we get that

$$\begin{aligned} [\mathcal{R}(\Sigma_{g,b}, Q_{r,0}^0(\xi_1, \dots, \xi_s))] &= \frac{[\mathfrak{X}(\Sigma_{g,b}, Q_{r,0}^0(\xi_1, \dots, \xi_s))]}{q^3 - q} \\ &= q^{2g+s-2}(q-1)^{2g+r-2}(2^{2g+s-1} - 2^s + (q+1)^{2g+r+s-2})(q^3 - q)^{b-1} \\ &\quad + \mathcal{I}_r(\xi_1, \dots, \xi_s)(q^3 - q)^{b-2}, \end{aligned}$$

where the interaction term is

$$\begin{aligned} \mathcal{I}_r(\xi_1, \dots, \xi_s) &= q^{2g+s-1}(q-1)^{2g+r-1}(\alpha_+ + \alpha_-) \left(2^{2g} + 2^{2g}q - 2q - 2 \right. \\ &\quad \left. + (q+1)^{2g+r} + (q+1) \left(1 - 2^{2g-1} - \frac{1}{2}(q+1)^{2g+r-1} \right) \right) \\ &\quad + q^{2g+s-1}(q-1)^{2g+r}(q+1)\alpha_+ \end{aligned}$$

- $s > 0$ and $\sigma = -1$. Then we can absorb the $-Id$ puncture into one of the semi-simple punctures so we have $\mathcal{R}(\Sigma_{g,b}, Q_{r,0}^1(\xi_1, \dots, \xi_s)) = \mathcal{R}(\Sigma_{g,b}, Q_{r,0}^0(-\xi_1, \dots, \xi_s))$.
- $s = 0$ and $\sigma = -1$. In this case, all the representations of $\mathfrak{X}(\Sigma_{g,b}, Q_{r,0}^1)$ are irreducible. Indeed, if $(A_1, B_1, \dots, C_1, \dots, C_{b-1}, P_1, \dots, P_r) \in \mathfrak{X}(\Sigma_{g,b}, Q_{r,0}^1(\xi_1, \dots, \xi_s))$ then they satisfy

$$\prod_{i=1}^g [A_i, B_i] \prod_{i=1}^r P_i = -Id.$$

If v is a common eigenvector of these matrices then $\prod_{i=1}^g [A_i, B_i] \prod_{i=1}^r P_i(v) = v$ since the eigenvalues of $[A_i, B_i]$ and P_i are all 1. This is incompatible with $-Id(v) = v$ except if $v = 0$. Therefore, $\mathfrak{X}(\Sigma_{g,b}, Q_{r,0}^1) = \mathfrak{X}^{ir}(\Sigma_{g,b}, Q_{r,0}^1)$ and, thus, using González-Prieto [23, Theorem 5.10]

$$\begin{aligned} [\mathcal{R}(\Sigma_{g,b}, Q_{r,0}^1)] &= \frac{[\mathfrak{X}(\Sigma_{g,b}, Q_{r,0}^1)]}{q^3 - q} = (-1)^{r+1} 2^{2g-1}(q+1)^{2g+r+b-3}(q-1)^{b-2}q^{2g+b-3} \\ &\quad + (q-1)^{2g+r+b-3}(q+1)^{b-2}q^{2g+b-3}((q+1)^{2g+r-2} + 2^{2g-1} - 1). \end{aligned}$$

4.5 Concluding remarks and applications

In this work, we have constructed a lax monoidal TQFT that extends the usual notion of a TQFT to the singular framework, in the sense that it is a functor $Z_G : \mathbf{NBdp}_n(\Lambda) \rightarrow \mathbf{KVar}_k\text{-Mod}_t$ out of the category of nodefold bordisms (i.e. non-smooth bordisms with conic singularities). By looking at this extension, a natural 2-functor structure naturally arises in this singular setting by considering as 2-morphisms in $\mathbf{NBdp}_n(\Lambda)$ the possible degenerations of the conic structure. This also exhibits a novel 2-category structure on $R\text{-Mod}_t$ given by ‘twists’ of morphisms, which must be understood as the algebraic analogue of a singular degeneration.

This singular TQFT Z_G has allowed us to easily compute the virtual classes of G -representation varieties over surfaces, even if they present conic singularities as in a nodal curve. The method is completely algorithmic and provides an effective way of computing virtual classes by decomposing the surface into simpler pieces, including a small ball around the singular points. This strategy, in combination with GIT techniques, has allowed us to explicitly compute virtual classes of $SL_2(k)$ -character varieties, as presented in Sect. 4.1.

These calculations and the associated TQFT can be also extended to the parabolic setting to determine virtual classes over nodefolds with arbitrarily many parabolic points, as conducted in Sects. 4.2–4.4. The result shows that an interaction term $\mathcal{I}_r(\xi_1, \dots, \xi_s)$ appears to account for non-generic situations in the eigenvalues ξ_1, \dots, ξ_s attached to the punctures.

It is a very noticeable fact that these virtual classes of nodal curves turn out to be polynomials in $q = [\mathbb{A}_k^1] \in \mathbf{KVar}_k$, the virtual class of the affine line. This is an unexpected result that cannot be obtained through arithmetic techniques, and whose validity in the general case remains completely unknown. From these explicit expressions, several classical invariants can be readily calculated for these character varieties:

- By setting $q = \mathbb{L}$, the Lefschetz motive, the same expression provides the motive of the algebraic variety in the Chow ring \mathcal{M}_k of pure motives.
- In the case $k = \mathbb{C}$, by setting $q = uv$ as formal variables in $\mathbb{Z}[u, v]$, the virtual class gives rise to the E -polynomial of the character variety.
- Again for $k = \mathbb{C}$, setting $q = 1$ we obtain the Euler characteristic (for compactly supported cohomology) of the underlying complex manifold.

As future work, apart from the utility of the constructed TQFT to compute virtual classes, it would be interesting to study how it varies when, instead of degenerations of nodefolds, we have parametric deformations of complex varieties. From the results of this paper, we can expect that a deformation of complex structure would lead to another 2-category structure in the category of (complex) bordisms, with a natural correlation after the quantization. This would allow us to understand more deeply several limiting cases arising in moduli theory, where two moduli spaces are related after a degeneration, as it happens with the moduli space of Λ -connections and of Higgs bundles [2].

It is also worth mentioning that the TQFT construction for nodefolds we provide deals successfully with both the singularization as well as the resolution operations involved in the geometric transition. Whereas geometric transitions are usually applied on CY 3-folds, and we focus this paper on nodefolds for curves, our construction serves for

more general situations including that of CY 3-folds. In this sense, our work is the first step towards a more general TQFT capturing all the possible degenerations arising in geometric transition.

In general, the extension of the classical smooth context of TQFT to singular and more flexible situations, like the ones arising in moduli theory, is a significant problem to be addressed in the near future.

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Data availability

This work has no associated data.

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