

# Invariant Tensors Formulae via Chord Diagrams

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## Abstract

We provide an explicit algorithm to calculate invariant tensors for the adjoint representation of the simple Lie algebra  $sl(n)$ , as well as arbitrary representation in terms of roots. We also obtain explicit formulae for the adjoint representations of the orthogonal and symplectic Lie algebras  $so(n)$  and  $sp(n)$ .

## 1 Introduction

In the last few years, various knot invariants have been discovered. Nowadays, Vassiliev knot invariants are the strongest among all known ones, see e.g. [5]. The essential notion in Vassiliev's theory is the notion of chord diagram.

Chord diagrams have interesting algebraic structure. It turns out [1] that this structure is deeply connected with Lie algebra representations.

In his article [1], D.Bar–Natan shows the way for constructing invariant tensors for the coadjoint action on semisimple Lie algebra tensors by using chord diagrams.

The article [3] describes the case of an arbitrary representation of the algebra  $sl(2)$ , and in [2] the connection between this construction and the adjoint representation of  $sl(2)$  is shown, in terms of which the Four Colour problem can be reformulated in tensorial language; the main features for the case of the adjoint representation for algebras  $sl(n)$  are also described.

The aim of this work is to study the connection between the algebras of Chinese and chord diagrams and the invariant tensors algebra and finding explicit formulae. Note, that chord diagrams describe the Vassiliev knot invariants. This provides a connection between knot and representation theory.

Moreover Chinese character (and chord) diagrams describe the Vassiliev finite-type invariants of knots. Thus, this theory connects knot theory with representation theory.

The method, proposed by Bar–Natan, requires step-by-step contraction of concrete tensors. Each step is a contraction and can be described by a concrete formula.

The main results of the present paper are:

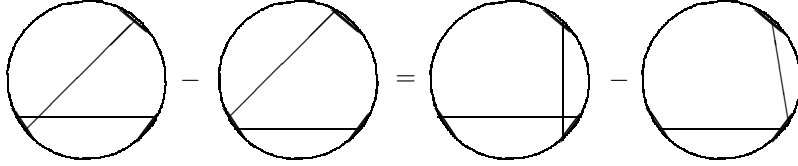


Figure 1: The Four-Term relation

1. To provide an explicit algorithm for calculating invariant tensors for the adjoint representation of  $sl(n)$  (see also [2]). The result of the algorithm action is a scalar function of the variable  $n$ . It turns out that this function is a polynomial of  $n$  whose coefficients are functions on chord diagrams, invariant under the so-called 4T-relation.

One gets some properties of the initial chord diagram from its polynomial. So, the highest possible power of polynomials is obtained only for so-called  $d$ -diagrams, see [4].

2. Formulae for all representations of  $sl(n)$  in the terms of roots.
3. Explicit formulae for the adjoint representation of  $so(n)$  and for the  $k + 1$ -dimensional representation of  $A_1$ .
4. Explicit formulae for the adjoint representation of Lie algebras  $sp(n)$ .

A shortened version of this paper not including results on  $sp(n)$  is published in [6].

**Definition 1.** A chord diagram (CD) [1] is a graph that consists of an oriented cycle (also called circle) and nonoriented edges, connecting points belonging to circle (also called chords). Each graph vertex is incident to just one circle; chord diagrams are considered as combinatorial objects (i.e. up to graph isotopy preserving the circle orientation).

The *degree* of a chord diagram is the number of its chords.

Consider the linear space of all chord diagrams of degree  $n$ , where  $n$  is a non-negative integer over  $\mathbf{Q}$ . Let us define the four-term (4T) relation as follows, see Fig. 1.

The summands represented in Fig. 1. are chord diagrams with  $n$  chords such that  $n - 2$  chords of all diagrams have exactly the same position. The other two chords are shown in the figure. They connect points lying in the marked segments. There are no other vertices on the marked segments except for those shown in the figure.

Now, let us define the chord diagram algebra:

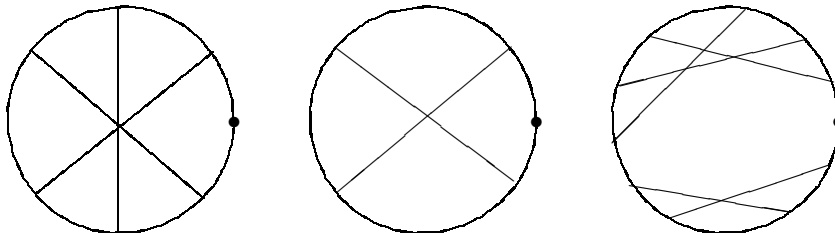


Figure 2: Product of Chord Diagrams

**Definition 2.** The chord diagram algebra  $\mathcal{A}^c$  is a formal algebra, whose elements are equivalence classes of linear combinations of chord diagrams (possibly, having different degrees) modulo the four-term relation.

The unity of this algebra is the equivalence class of the chord diagram without chords.

The multiplication of two diagrams  $A$  and  $B$  is defined as follows: Let us break the circles of these diagrams at arbitrary points which are not vertices, and then connect them together according to the circle orientation. An example of two chord diagrams with marked points (left hand) and their product (right hand) is shown in Figure 2.

Thus we obtain the chord diagram that can be treated as the product  $AB$ . However, this product is not uniquely defined; it depends on the choice of break points. Bar-Natan [1] showed that this choice of fixed point does not change the final result up to the  $4T$ -relation.

Obviously, the product of two linear combinations of chord diagrams is defined according to distributivity rule

Let us denote by  $\mathcal{A}_n^c$  the linear space of chord diagrams of degree  $n$  modulo  $4T$ -relation.

Now, let us define another algebra

**Definition 3.** A Chinese character diagram is a cubic graph, with all vertices either exterior or interior, with indicated oriented cycle, containing all exterior vertices. At each interior vertex one also indicates a cyclic order of outgoing edges counterclockwise. Chinese character diagrams are considered up to graph equivalence preserving the oriented cycle (circle).

The degree of a CCD is half the number of its vertices

**Remark 1.** In the sequel, all circles shown in figures are thought to be oriented counterclockwise; all structures at interior points are taken from the plane.

Although the degree of a CCD is defined by its number of vertices divided by two, it turns out that it is an integer, as can easily be shown.

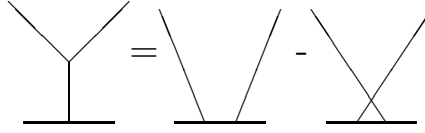


Figure 3:  $STU$ -relation

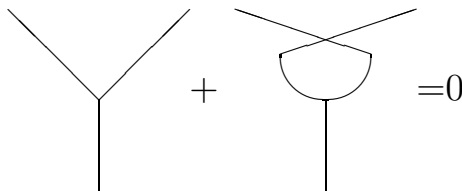


Figure 4: The antisymmetry relation

Note, that each chord diagram is a Chinese character diagram without exterior vertices. In this case, its CD degree coincides with its CCD degree.

Chinese character diagrams admit a similar algebraic structure.

Consider the space of linear combinations of Chinese character diagrams. Let us factorize it by the so-called  $STU$ -relation, see Fig. 3. Denote the obtained space by  $\mathcal{A}^t$ .

As before, we have shown only a small changing part of the CCD. All other vertices remain the same in all three cases shown above.

By using the  $STU$ -relation, each CCD can be transformed into a linear combination of chord diagrams.

Indeed, one can resolve an interior vertex connected with an exterior vertex. Thus, if the number of interior vertices is greater than 0, it can be decreased. The final result (i.e. a linear combination of chord diagrams) is uniquely defined up to  $4T$ -relation, see [1]. Thus, the space  $\mathcal{A}^t$  inherits the algebraic structure of  $\mathcal{A}^c$ .

Let us take into account the two relations that take place in  $\mathcal{A}^t$ , see [1].

**Theorem 1.** *Following identities hold in  $\mathcal{A}^t$ :*

1. *Antisymmetry at interior points, see Fig.4*

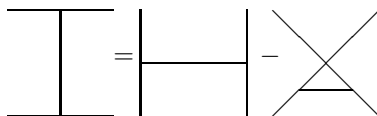


Figure 5: IHX–relation

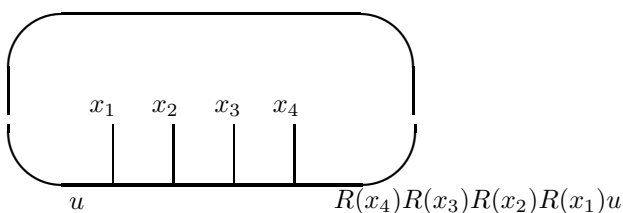


Figure 6: Contraction

2. The IHX–relation, see Fig.5

Both  $\mathcal{A}^c$  and  $\mathcal{A}^t$  are graded according to the half number of the total quantity of vertices (for chord diagrams the degree equals the number of chords); the factorization described above preserves this structures because both the  $4T$ – and  $STU$ – relations are homogeneous according to this graduation.

As it is shown in [1], by using a given representation of a semisimple Lie algebra  $G$  and a  $n$ –Chinese character diagram, one can obtain a tensor of type  $(n, 0)$  invariant under the adjoint action of  $G$ . This can be done as follows: Let  $R$  be a representation of some Lie algebra  $G$ . Consider the  $n$ –Chinese diagram  $C$  without interior vertices (i.e. all trivalent vertices are exterior). One defines the  $n$ –linear form on "tails", corresponding to  $C$  by  $f(x_1, \dots, x_n) = Tr(R(x_1) \dots R(x_n))$ , see figure 6; where  $TR$  is the trace of an operator acting on the Lie algebra.

The computation of the multilinear form for arbitrary  $n$ –chinese diagram can be done by contracting "tails" at interior vertices by using the trivalent structural tensor  $c_{jk}^i$  and the bivalent metrics  $g_{ij}$ . Fix a Lie algebra  $G$  and its representation  $R$ . The map from the set of  $k$ –Chinese character diagrams to the adjoint invariant  $k$ –linear forms on  $G$  is invariant under the  $STU$ –relation. If a linear combination of  $k$ –Chinese character diagrams equals zero, up to the  $STU$ –relation, then the corresponding tensor equals zero by the Jacobi identity and the skew symmetry, see [1]. According to the isomorphism,  $\mathcal{A}^t \cong \mathcal{A}^c$  all  $k$ –linear forms, coming from  $k$ –Chinese character diagrams, can be obtained only

from  $k$ -chord diagrams. Later we shall give an algorithm for calculating these tensors for some representations of Lie algebras and arbitrary  $k$ -chord diagrams.

Note that the case of chord diagrams without tails is also interesting. In this case we get scalars, which are invariant under the 4T-relation. These invariants give some properties of the chord diagram algebra. As shown in [1], this algebra plays a significant role in the Vassiliev invariant theory.

## 2 Adjoint representation of $sl(n)$ and $so(n)$

### 2.1 The case of $sl(n)$

Consider the  $k$ -Chinese character diagram  $L(k)$ , consisting of a circle and  $k$  outgoing tails. In the sequel, we shall denote it simply by  $L$ . As told in the introduction, for such a diagram we get the following invariant tensor:

$$f(x_1, \dots, x_k) = Tr(ad_{x_1} \dots ad_{x_k}) \quad (1)$$

So, we have to find the operator trace  $O = ad_{x_1} \dots ad_{x_k}$  for  $sl(n)$ . First, consider this trace for  $gl(n)$ .

**Lemma 1.** *The final result after a contraction is the same for both  $sl(n)$  and  $gl(n)$ .*

*Proof.* Any contraction means a calculation of a trace for an operator, acting on  $n \times n$  matrices. For these matrices, choose the basis consisting of the unit matrix and of matrices of  $sl(n)$ . The initial operator  $O$  is a composition of commutation operators, so it vanishes on the identity matrix. Thus, all operators obtained from it vanish on the identity as well. Thus, for each of them, the traces in bases of  $sl(n)$  and  $gl(n)$  coincide.  $\square$

Choose the basis  $E_{ij}$ ,  $i, j = 1, \dots, n$  of  $gl(n)$ , where  $E_{ij}$  is the matrix having a 1 in the  $i^{th}$  row and  $j^{th}$  column and zero elsewhere. The dual basis is  $E_{ji}$ ,  $i, j = 1, \dots, n$ . The formula (1) looks like

$$f(x_1, \dots, x_k) = \sum_{i,j=1}^n Tr(E_{ij} ad_{x_1} \dots ad_{x_n} E_{ij}) \quad (2),$$

where  $Tr$  is just the matrix trace.

In the sequel, for the case of the adjoint representation,  $Tr$  is the usual matrix trace.

Now rewrite (2) representing the action  $ad_{x_i}$  on  $u$  as a commutator. So,

$$f(x_1, \dots, x_k) = \sum_{\substack{l=0 \\ i_1 > \dots > i_l, i_{l+1} < \dots < i_n}}^n (-1)^l Tr(x_{i_1} \dots x_{i_k}) Tr(x_{i_{k+1}} \dots x_{i_n}). \quad (3).$$

That is, indices of  $x$  before  $u$  are descending and those after  $u$  are ascending. The sum is taken according to all the decompositions of indices  $1, \dots, k$  into "left ones" and "right ones"

**Remark 2.** For the empty diagram ( $k = 0$ ) the formula (3) does not hold. The corresponding scalar equals  $n^2 - 1$ .

The two following types of members must be determined:

$$\sum_{\alpha} Tr(P\alpha Q\tilde{\alpha}) \quad (4)$$

and

$$\sum_{ij} Tr(P\alpha)Tr(Q\tilde{\alpha}) \quad (5),$$

where  $P, Q \in gl(n)$ , and  $\alpha, \tilde{\alpha}$  run the usual and dual bases of the Lie algebra. Therefore equations (4) and (5) take the form:

$$\sum_{ij} Tr(PE_{ij}QE_{ji}) = \sum_{i,j=1}^n P_{jj}Q_{ii} = Tr(P)Tr(Q) \quad (6)$$

$$\sum_{ij} Tr(PE_{ij})Tr(QE_{ji}) = \sum_{i,j=1}^n P_{ji}Q_{ij} = Tr(PQ) \quad (7).$$

Computing these contractions directly for  $sl(n)$  would result in a much more complicated algorithm.

**Lemma 2.** For  $sl(n)$  the sum (4) equals

$$Tr(P)Tr(Q) - \frac{1}{n}Tr(PQ), \quad (*)$$

and (5) equals

$$Tr(PQ) - \frac{1}{n}Tr(P)Tr(Q). \quad (**)$$

*Proof.* Choose the basis consisting of  $E_{ij}, i \neq j$  and coadjoint elements  $d_l = E_{11} + \dots + E_{l-1,l-1} - (l-1)E_{ll}, l = 1, \dots, n-1$ . of  $sl(n)$ . The dual basis consists of  $E_{ji}$  and  $d_l = E_{11} + \dots + E_{l-1,l-1} - (l-1)E_{ll}, l = 1, \dots, n-1$ .

We have:

$$\begin{aligned} \sum_{i \neq j} Tr(PE_{ij}QE_{ji}) &= \sum_{i,j=1}^n P_{jj}Q_{ii} - \sum_{i=1}^n P_{ii}Q_{ii} = \\ &TrPTrQ - \sum_{i=1}^n P_{ii}Q_{ii} \end{aligned}$$

$$\begin{aligned}
& \sum_{l=1}^{n-1} \text{Tr}(Pd_l Qd_l) \frac{1}{l^2 - l} \\
&= \left( \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n-1)} \right) \sum_{i=1}^n P_{ii} Q_{ii} + \\
& \left( -\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n-1)} \right) \sum_{i \neq j} P_{ij} Q_{ji} = -\frac{1}{n} \text{Tr}(PQ) + \sum_{i=1}^n P_{ii} Q_{ii}
\end{aligned}$$

Collecting these sums together, we get:

$$\text{Tr}(P)\text{Tr}(Q) - \frac{1}{n} \text{Tr}(PQ)$$

$$\begin{aligned}
\sum_{i \neq j} \text{Tr}(PE_{ij})\text{Tr}Q(E_{ji}) &= \sum_{i,j=1}^n P_{ji} Q_{ij} = \text{Tr}(PQ) - \sum_{i=1}^n P_{ii} Q_{ii}. \\
& \sum_{l=1}^{n-1} \text{Tr}(Pd_l)\text{Tr}(Pd_l) \frac{1}{l^2 - l} = \\
& \left( -\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n-1)} \right) \sum_{i \neq j} P_{ii} Q_{jj} + \sum_{i=1}^n P_{ii} Q_{ii}
\end{aligned}$$

Finally, we get:

$$\text{Tr}(P)\text{Tr}(Q) - \frac{1}{n} \text{Tr}(PQ)$$

□

So, in order to calculate the tensor corresponding to a chord diagram of order  $k$  for the case of the adjoint representation of  $so(n)$ , we have to calculate  $2^{2k}$  summands. Since we consider the trace in  $gl(n)$ , the number of summands increases while contracting. We get the following

**Theorem 2.** *Following conditions hold:*

1. *Each  $m$ -linear form on  $x_{i_1}, \dots, x_{i_m}$ , obtained from the coadjoint representation of  $sl(n)$  by contracting the  $k$ -linear form corresponding to  $L$ , is a linear combination of traces of products for  $x_{i_j}$  and  $x_{i_j}^*$  with polynomial (with respect to  $n$ ) coefficients.*
2. *the power of these polynomials does not exceed  $k + 2$ .*

Note, that the statement of the theorem for Chinese character diagrams without univalent vertices was proved by Bar–Natan, [2]. Here we give our own

*Proof.* Consider the desired tensor as a sum of  $2^{2k}$  summands according to (3) and contract it in  $gl(n)$ . For each of  $2^{2k}$  summands we use the induction on the number  $q$  of contracted elements. Let us also observe the number of multipliers (traces) in this product. For  $q = 0$  the statement is evident. Consider the power of a summand and the number of its multipliers. In the case of a contraction like (6), one gets either

1.  $P = Q = E$ . The power increases by two, the number of multipliers decreases by 1.
2. Either  $P$  or  $Q$  equals the identity matrix. The power decreases by 1, the number of multipliers remains the same.
3. None of  $P$  and  $Q$  is the identity matrices. The power stays the same and the number of multipliers increases by 1.

In the case of the contraction (7), one of the following situation occurs:

1.  $P = Q = E$ . The power stays the same, the number of multipliers decreases by 1.
2. At least one of  $P, Q$  is not equal to  $E$ . The power stays the same, the number of multipliers decreases by 1.

Thus we see that contraction implies the induction step for the statement 1). Besides, after  $l$  contractions the sum of coefficient's maximal power for a summand and the number of multipliers of this summand does not exceed  $l + 2$ . Since the number of multipliers is not negative and the number of contractions satisfies  $l \leq k$ , we obtain that the power of a coefficient for a summand does not exceed  $k + 2$ .  $\square$

## 2.2 The case of $so(n)$

As in the case of  $sl(n)$ , we begin with the  $k$ -Chinese diagram  $L$  consisting of a circle with  $k$  outcoming tails. To find (3), we have to calculate (4) and (5).

**Lemma 3.** *For  $so(n)$  the sum (4) looks like*

$$\frac{1}{2}(\text{Tr}(P)\text{Tr}(Q) - \text{Tr}(PQ^*)) \quad (8),$$

and (5) looks like

$$\frac{1}{2}(\text{Tr}(PQ) - \text{Tr}(P)\text{Tr}(Q^*)) \quad (9)$$

*Proof.* Choose the selfadjoint basis  $E_{ij} - E_{ji}, i > j$  of  $so(n)$ , whose elements have length 2. So:

$$\sum_{i>j} \frac{1}{2} \text{Tr}(P(E_{ij} - E_{ji})Q(E_{ji} - E_{ij})) =$$

$$\begin{aligned}
& \frac{1}{2} \sum_{i>j} TrP[P_{jj}E_{ii} + P_{ii}E_{jj} - Q_{ji}E_{ij} - Q_{ij}E_{ji}] = \\
& = \frac{1}{2} \sum_{i>j} (P_{ii}Q_{jj} - P_{ji}Q_{ji} - P_{ij}Q_{ij} = \\
& \frac{1}{2} (Tr(P)Tr(Q) - \sum_{i>j} (P_{ij}Q_{ij} + P_{ji}Q_{ji}) + \sum_i P_{ii}Q_{ii}) = \\
& = \frac{1}{2} (Tr(P)Tr(Q) - Tr(PQ^*)). \\
& \sum_{i>j} Tr(P(E_{ij} - E_{ji}))Tr(Q(E_{ji} - E_{ij})) = \\
& \sum_{i>j} \frac{1}{2} ((P_{ji}Q_{ij} + P_{ij}Q_{ji}) - (P_{ii}Q_{ij} + P_{ji}Q_{ji})) = \\
& \frac{1}{2} (Tr(PQ) - Tr(PQ^*)).
\end{aligned}$$

□

From the formulae above we rewrite equation (3) as:

$$f(x_1, \dots, x_k) = \frac{1}{2} \sum_{\substack{l=0 \\ i_1 > \dots > i_l, i_{l+1} < \dots < i_n}}^n (-1)^l \times (Tr(x_{i_1} \dots x_{i_k})Tr(x_{i_{k+1}} \dots x_{i_n}) + Tr(x_{i_1} \dots x_{i_k} x_{i_n}^* \dots x_{i_{k+1}}^*))$$

**Theorem 3.** *Each  $m$ -linear form on  $x_{i_1}, \dots, x_{i_m}$ , obtained from the adjoint representation for  $so(n)$  is a linear combination of traces of products for  $x_{i_j}$  and  $x_{i_j}^*$  with polynomial coefficients with respect to  $n$ .*

The proof is essentially the same as for  $sl(n)$  with formulae (8), (9) instead of (6) and (7).

### 3 Arbitrary irreducible representations of $sl(n)$

Note that the calculations for arbitrary representations of  $sl(2)$  were done in [3].

In the case of arbitrary representation  $R$  of  $sl(n)$ , it is important to calculate the formulae for  $\sum Tr(P\alpha Q\tilde{\alpha})$  and  $\sum Tr(P\alpha)T(Q\tilde{\alpha})$ , where  $P$  and  $Q$  are products of certain number of matrices, representing  $x_i$ ,  $\alpha = R(\omega)$  and  $\tilde{\alpha} = R(\tilde{\omega})$ , where  $\omega$  runs a basis of  $sl(n)$  and  $\tilde{\omega}$  runs the dual basis.

It is obvious that  $\sum Tr(P\alpha Q\tilde{\alpha}) = ATrPTrQ + BTr(PQ)$ , where  $A$  and  $B$  are numbers depending only on  $n$ . One finds them by using the undefined coefficients method. For  $P = Q = E$ , one gets

$$Dim(R)Dim(G) = ADim(R)^2 + BDim(G) \text{ or } Dim(G) = ADim(R) + B \quad (10)$$

The second equation for calculating  $A, B$  one obtains assuming  $P = Q = R(H)$ , where  $H$  is an element of the Cartan subalgebra of  $G$ . The basis  $\omega$  and the dual basis  $\tilde{\omega}$  consist of vectors of the Cartan subalgebra and root vectors.

So,

$$\sum_{\alpha} Tr(R(H)\alpha R(H)\tilde{\alpha}) = \sum_1 + \sum_2.$$

Here  $\sum_1, \sum_2$  are sums with respect to  $\alpha$  that runs all simple roots and all root vectors.  $\sum_1 = \sum Tr(R(H)^2\alpha\tilde{\alpha})$ ,  $\sum_2 = \sum Tr(R(H)R(e_{\beta})R(H)R(e_{-\beta}))$

Taking into account

$$He_{\beta} = e_{\beta}H + (\beta, H)e_{\beta}$$

and

$$H_{-\beta} = e_{-\beta}H - (\beta, H)e_{-\beta},$$

where  $\beta$  is an arbitrary root of  $G$ , we get:

$$Tr \sum_2 = Tr \sum_2 R(H)e_{\beta}R(H)e_{\beta} = R(H)^2 \sum_2 Tr R(H)R(H)_{\beta} = \sum \beta(H)^2.$$

The last sum is taken with respect to the set of positive roots  $\beta$  of  $G$ .

Thus we obtain

$$\sum \alpha R(H)\tilde{\alpha}R(H) = Dim(R) \sum \alpha\tilde{\alpha} + \sum b(H)^2 = BTrR(H)^2 \quad (11)$$

Equations (10),(11) are sufficient to find  $A, B$ .

In the case of the irreducible  $k+1$ -dimensional representation  $R$  of  $G = sl(2)$ , (10),(11) result in  $A = \frac{2}{k+1}, B = \frac{1-2k}{1+k}$ .

## 4 The case of the adjoint representation of $sl(n)$ and invariants of the chord diagram algebra

Consider a chord diagram  $C$  and construct the corresponding scalar ,depending on  $n$ , by using the adjoint representation of  $sl(n)$ . By theorem 2, this scalar is a polynomial with respect to  $n$ ; its degree does not exceed  $k + 2$ , where  $k$  is the

order of  $C$ . Besides, this polynomial is an invariant under  $4T$ -relation. Denote this function by  $U(C)$ .

Now, let us find those chord diagrams  $C$  for which the power of  $U(C)$  with respect to  $n$  equals  $k + 2$ . Consider the decomposition (3) for a tensor, corresponding to  $C$ , and try to filter those diagrams, generating after  $\pm n^{k+2}$  after complete contraction. Fix a summand  $S$  and observe its contraction. As it is shown in the proof of theorem 2, any contraction of type (6) increases the sum of coefficient power and the number of multipliers (traces) by 1, and any contraction of type (7) decreases this sum. Taking into account that after all contractions the number of multipliers equals zero we see that, in order to obtain the maximal degree  $k + 2$ , we always have to contract by (6).

This means that for each contraction chord ends should belong to the same multiplier.

**Definition 4.** *A  $d$ -diagram is a chord diagram whose set of chords can be split into two families of pairwise nonintersecting chords.*

**Theorem 4.** *The power of  $U(C)$  equals  $k + 2$  on a diagram  $C$  on  $k$  chords iff  $C$  is a  $d$ -diagram. Moreover, the coefficient at  $n^{k+2}$  of  $U(C)$  equals the number of splitting chords of  $C$  into two families (the first and the second) of pairwise nonintersecting chords.*

*Proof.* Consider the diagram  $C$ . As it is shown above, to obtain  $n^{k+2}$  by contracting some summand  $S$ , the ends of contracted chord of the diagram  $C$  should belong to the same multiplier.

This means that for each step any chord must belong to the same multiplier with both ends of it, since otherwise the contraction (7) does not allow to obtain the maximal possible power.

Consider some splitting of our chord diagram vertices into some nonintersecting subsets in such a way that both ends of each chord lie in the same subset. In this case, one can say that each chord belongs to some subset. If we contract along a chord lying in one subset (contraction  $Tr(P\alpha Q\tilde{\alpha}) \rightarrow TrPTrQ$  of type (6)), then the subset containing  $P$  and  $Q$  is decomposed into two subsets:  $P$  and  $Q$ . If we want each chord this splitting to have both ends lying in the same family, each two chords of the same family of the initial diagram must not intersect each other. The inverse statement is also true: if any two chords of the same family for the initial diagram do not intersect each other, then, by contracting a chord by (6), we get a diagram of smaller degree where each two chords of the same family do not intersect each other. In (3) we have  $2^{2k}$  members; each of them corresponds to a splitting of chords of the diagram into two families. Chords in each family must be pairwise nonintersecting, i.e.  $C$  is a  $d$ -diagram. To conclude the proof of the second statement, we only have to consider the coefficient at  $k + 2$ th degree of the polynomial for our  $d$ -diagram  $C$ . Among  $2^{2k}$  members of (3) we have some number of "good" members, giving  $n^{k+2}$  after contraction. Their number equals the number of splittings of chords

into two families in a proper way. After the final contraction, each of them gives  $n^{k+2} \cdot (-1)^l$ , where  $l$  — is the number of chords in one of two families. Since both ends of each chord lie in the same family,  $(-1)^l = 1$ , that concludes the proof.  $\square$

**Theorem 5.** *For each natural  $n$  in each basis of the space  $\mathcal{A}_n^c$  contains at least one  $d$ -diagram.*

*Proof.* Consider a basis of  $\mathcal{A}_k^c$  consisting of some diagrams  $B_1, \dots, B_r$ . Suppose that there is no  $d$ -diagram among them. Then the degree of the polynomial  $\forall i = 1, \dots, r \quad U(B_i)$  is less than  $k + 2$ . Consider any  $d$ -diagram  $C$  on  $k$  chords. Since  $B_1, \dots, B_r$  form a basis of  $\mathcal{A}_k^c$ , then  $C$  must be represented by a linear combination of  $B_1, \dots, B_r$  modulus the  $4T$ -relation. Since  $U(\cdot)$  is invariant under the  $4T$ -relation,  $U(C)$  is a linear combination of  $U(B_i), i = 1, \dots, r$ , that is impossible.  $\square$

**Theorem 6.** *For a chord diagram  $C_k$  on  $k$  chords all monomials of  $U(C_k)$  have the same parity as  $n$ .*

*Proof.* For each contraction of a monomial the sum of its coefficient's power and the number of its multipliers either decrease by 1 or increase by 1. Since this sum was first equal to 2, then, after  $n$  contractions when the number of multipliers equals zero, the parity of the monomial power equals  $n$ .  $\square$

**Theorem 7.** *For an arbitrary chord diagram  $C$  the polynomial  $U(C)$  is divisible by  $n^2 - 1$ , where this number equals the dimension of  $sl(n)$ . Moreover, for arbitrary chord diagrams  $C_1$  and  $C_2$  the following formula holds:*

$$U(C_1 \cdot C_2) = \frac{U(C_1) \cdot U(C_2)}{(n^2 - 1)}$$

*Proof.* Consider the chord diagram  $C$  of degree  $k$  and choose a chord  $a$  of it. Consider the formula (3) for the diagram  $L$ . For each summand of it, let us contract all chords of it except for the chord  $a$ . Thus we obtain a sum of monomials corresponding to the 1-chord diagram obtained from  $C$  by breaking the chord  $a$ .

Now, let us contract the rest in the sense of  $gl(n)$ .

In the case (\*) we get the additional coefficient  $Tr(E)Tr(E) - \frac{1}{n}Tr(E \cdot E) = n^2 - 1$ . In the case (\*\*) we get the coefficient  $Tr(E \cdot E) - \frac{1}{n}Tr(E \cdot E) = 0$ .

Thus, while performing the last step in the sense of  $gl(n)$ , each coefficient either vanishes or is multiplied by  $(n^2 - 1)$ . That completes the proof of the first part of the theorem.

Let us prove now the second part. Let  $C_1 \quad C_2$  be two chord diagrams of degrees  $k$  and  $l$ , respectively. Consider the operators  $O(C_1)$  and  $O(C_2)$ , obtained by contracting the  $Tr(x_k \dots x_1)$  and  $Tr(x_{k+l}, \dots, x_{k+1})$  at pairs of variables among  $x_1, \dots, x_k$  and  $x_{k+1}, \dots, x_{k+l}$  according to diagrams  $C_1$  and

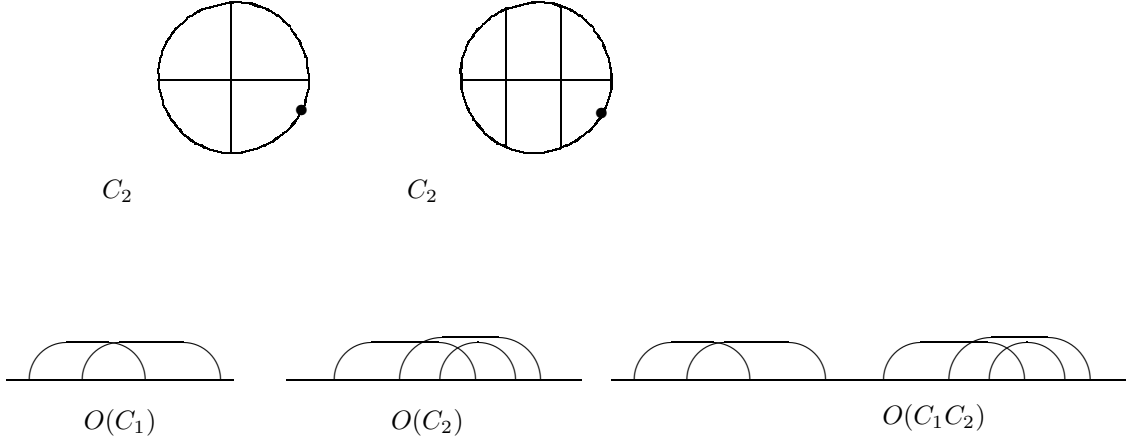


Figure 7: The Product Formula

$C_2$ , respectively, see Fig.4. Taking the traces of these operators, we get just  $U(C_1)$  and  $U(C_2)$ .

Naturally, these operators are adjoint invariant and hence, scalar. Thus we obtain

$$U(C_1) \cdot U(C_2) = Tr(O(C_1)) \cdot Tr(O(C_2)) =$$

$$(n^2 - 1)Tr(O(C_1 \cdot C_2)) = (n^2 - 1)U(C_1 \cdot C_1),$$

That completes the proof of the theorem. □

**Examples 1.** Finally we give the list of chord diagrams of orders  $\leq 4$  with values of  $U$  on them. Taking into account theorem 7, it is sufficient to give polynomials on diagrams which are not products, see fig. 8.

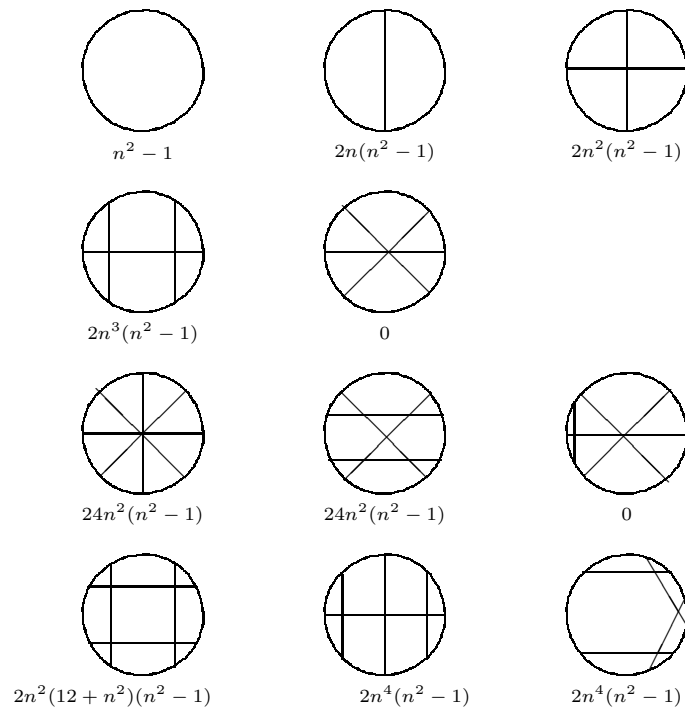


Figure 8: Values  $U$  on chord diagrams

## 5 The case of $sp(n)$

Like in the preceding cases, to determine (3) we have to compute the trace formulae (4) and (5).

**Lemma 4.** *For  $sp(n)$  the sum (4) looks like*

$$\frac{1}{2}(TrPTrQ - Tr(PjQ^*j)) \quad (12),$$

and (5) looks like

$$\frac{1}{2}(Tr(PQ) + Tr(PjQ^*j)), \quad (13)$$

where  $j$  is the block-diagonal  $2n \times 2n$ -matrix:

$$j = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$$

*Proof.* Consider the basis given by the matrices  $E_{ii} - E_{n+i,n+i}$  ( $1 \leq i \leq n$ ),  $E_{ij} - E_{n+j,n+i}$  ( $i \neq j$ ),

$E_{i,n+i}, E_{n+i,i}$  ( $1 \leq i \leq n$ )  $E_{i,n+j} + E_{j,n+i}$ ,  $E_{n+i,j} + E_{n+j,i}$  ( $1 \leq i < j \leq n$ ).

Respect to it, the dual basis is given by  $\frac{1}{2}(E_{ii} - E_{n+i,n+i})$ ,  $\frac{1}{2}(E_{ji} - E_{n+i,n+j})$ ,  $E_{n+i,i}$ ,  $E_{i,n+i}$ ,  $\frac{1}{2}(E_{n+i,j} + E_{n+j,i})$ ,  $\frac{1}{2}(E_{i,n+j} + E_{j,n+i})$ . We now determine the trace formulae by evaluating them on distinct pairs  $(P, Q)$ :

1.

$$(P, Q) = (id_{2n}, id_{2n})$$

$$\sum_{\alpha} Tr(\alpha\alpha') = \dim sp(n) = 2n^2 + n$$

$$\sum_{\alpha} Tr(\alpha)Tr(\alpha') = 0$$

2.

$$(P, Q) = (j, j)$$

$$\sum_{\alpha} Tr(j\alpha j\alpha') = n$$

$$\sum_a Tr(j\alpha)Tr(j\alpha') = 2 \sum_{i=1}^n Tr(-E_{ii})Tr(E_{n+i,n+i}) = -2n$$

3.

$$(P, Q) = (E_{11}, E_{11})$$

$$\sum_a Tr(E_{11}jE_{11}\alpha') = \sum_a Tr(E_{11}\alpha) Tr(E_{11}\alpha') = \frac{1}{2}$$

Taking into account the obvious trace formulae for the identity matrix and the matrix  $j$ , the assertion follows at once.  $\square$

Substituting in equation (3) we obtain:

$$f(x_1, \dots, x_k) = \frac{1}{2} \sum_{\substack{l=0 \\ i_1 > \dots > i_l, i_{l+1} < \dots < i_n}}^n (-1)^l \times (Tr(x_{i_1} \dots x_{i_k}) Tr(x_{i_{k+1}} \dots x_{i_n}) + Tr(x_{i_1} \dots x_{i_k} x_{i_n}^* \dots x_{i_{k+1}}^*))$$

**Theorem 8.** *Each  $m$ -linear form on  $x_{i_1}, \dots, x_{i_m}$ , obtained from the adjoint representation for  $sp(n)$  is a linear combination of traces of products for  $x_{i_j}$  and  $x_{i_j}^*$  with polynomial coefficients with respect to  $n$ .*

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