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A COMPARISON AND EVALUATION OF SOME ALTERNATIVE SOLUTION METHODS TO DYNAMIC STOCHASTIC MODELS

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ABSTRACT

We compare and evaluate the performance of four widely used numerical solution methods to dynamic rational expectations stochastic models, in the context of optimal and nonoptimal Pareto settings using a wide variety of statistical measures and two sample sizes. We find that: (i.) differences between methods do not necessarily increase with the complexity of the solved model. (ii.) For all the example model economies we considered, a log-linear approximation behaves as well as a more complex to implement finite element method. (iii.) Rejection of a particular solution method attending to the fulfilment of the rational expectation hypothesis is compatible with almost no differences between methods attending to other comparison criteria. (iv.) It is proper to consider 'large' sample sizes to check the properties of a particular solution.

Keywords: Approximation, simulation, numerical methods.
JEL Classification: C63, E17

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1 Introduction

The rising importance of computational methods in economics is fairly evident from simple visual inspection of any research review. As pointed out in recent methodological papers by Judd (1997) and Bona and Santos (1997), the interaction between economic theory and computational research is, and it will increasingly be, a central pattern in modern economics. This interaction is particularly important in the research agenda outlined in Lucas (1980). The construction of *fully articulated artificial economies* has led rational–expectations dynamic stochastic modelization to almost all fields of economics (see Marcet (1993), Kydland and Prescott (1994) or Cooley and Prescott (1995) for illustrative reviews).

In this respect, the essential non–linear and stochastic structure embedded in these kind of models has motivated a pararell explosion in numerical solution methods. Although there is a wide variety of numerical approaches at hand[1], there is not many evidence concerning the consequences of using one solution method instead of other to deal with a particular economic problem.

The most complete paper concerning numerical methods comparison is Taylor and Uhlig (1990). In this work they compared fourteen solution methods in the context of the Brock and Mirman (1972) model. The comparison they made was vey rich in terms of discussion of results and comparison measures, and the general conclusion was that differences among methods turned out to be quite substantial for certain aspects of the model. Nonetheless, it suffered certain lack of homogeneity and statistical robustness given the way it was conducted: each researcher sent but one solution time series path and the decision rules, and there was not the same distribution for the technology shock in all the implemented methods.

Another set of papers, in the context of the same model are Christiano (1990) that compared two linear quadratic methods using as comparison criteria a discretization of the state–space solution method, and Christiano and Fisher (1997) that compared a set of finite element methods, using the same comparison criteria and including a binding constraint. İmrohoroğlu (1994) propose a forward solution method and compares it with backsolving and a linear quadratic solution method in the same context, using the Den Haan and Marcet (1994) test as measure of comparison. In that paper Den Haan and Marcet, as illustrations of the ability of the test, compared the Parameterized Expectations Approach with linear quadratic methods by solving the one sector neoclassical growth model and in non–optimal settings using the cash–in–advance model of Cooley and Hansen (1989). Also in a non–optimal environment, Dotsey and Mao (1992) compared different linear and log–linear approximations in a modified version of the growth model with taxes on production, using as comparison criteria a discretization of the Euler equations method.

The picture that emerges from the literature is mixed and scattered: regarding the Brock and Mirman model, linear and log–linear quadratic methods are very similar and not bad, except for the Den Haan and Marcet test. In non–optimal settings things seem to change.

[1] It is not an objective of this paper to describe the state of the art in this branch. For this see: the Winter 1990 number of the *Journal of Business and Economic Statistic*, Cooley and Hansen (1995), Marcet (1993), Danthine and Donaldson (1995) and Judd (1996) for general revisions. See McGrattan (1996a, 1996b) and Christiano and Fisher (1997) for an introduction to finite elements methods and Marcet and Marshall (1994) for the Parameterized Expectations Approach.

Finite element methods seem to behave very similar, although Parameterized Expectations dominates on the basis of speed, accuracy and convenience of implementation[2].

We think there are several interesting questions that arise from this literature and have not been sufficiently considered, or only pointed at:

1. Do differences between different methods increase/appear when increasing complexity in the growth model? In what sense?
2. How many non–linear structure of the original problem is useful to maintain, given the computational costs of more complex methods?
3. Depending on the aim of a research, it is always important if certain solution is rejected by the Den Haan and Marcet (1994) test? That is, how to compare?
4. It is irrrelevant for a business cycle paper to consider a small sample size (say 150 observations) when calculating, for instance, descriptive statistics, or it is necessary to consider larger sample sizes (say 3000 observations) for the sake of reliability on the results?

In this paper we have tried to answer these questions in an unified and complete framework.

Concerning question one, given the importance of the neoclassical growth model for business cycle research, we proceed as usual by computing different methods in the context of the Brock and Mirman (1972) model. Then we increase the complexity of the model considering the real business cycle benchmark model of Hansen (1985) that includes indivisible labor. In a final step we compared in the context of the previous model but including money via a cash–in–advance constraint on consumption, then considering the Cooley and Hansen (1989) model, and so including a non–Pareto–optimal setting.

The answer to the second question is related to the four solution methods we chose to compare. The first is Parameterized Expectations that, in theory at least, can provide us with an approximation as close to the true solution as desired, then maintaining all the non–linear structure of the original problem, but it is very costly in certain respects. On the other extreme we used a standard linear–quadratic method, that directly solves the linear–quadratic version of the original problem as a way to approximate it. Midway we considered the log–linear approximation proposed by Uhlig (1997) and the forward solution proposed by Sims (1989, 1990).

As for the third question, we performed Monte Carlo simulation for a battery of tests to check, on the one hand, the Euler equation residual properties (Den Haan and Marcet test, ARCH test, autocorrelations) and, on the other hand, a set of statistics and its empirical distribution (mean, relative standard deviation, impulse response functions, cross correlations, analysis of the decision rules). Do discrepancies in the first set implies discrepancies in the second one?

We do not pretended to use as a comparison criteria a discretized version of the dynamic programming algorithm for, even if one could obtain arbitrarily accurate approximations, it

[2] In preliminary work Baraño, Iza and Vazquez (1997) find significant differences in an endogenous growth model between Parameterized Expectations and a log–linearization (the same we use in this paper, as we will see).

is very costly and in some cases becomes infeasible. So we prefered to check accuracy in the Euler Equation Residuals (see Santos (1997)).

The fourth question is adressed considering two sample sizes: 150 and 3000. We believe this distinction is relevant because papers on business cycles that use linear–quadratic solution methods often use short sample sizes, and methods like Parameterized Expectations often need large samples to be well defined. Moreover, taking into account that actual data sets are 'small', when one is interested on the estimation of the structural parameters of the model it is not possible to use large samples.

The results point to: (i.) Differences between methods do not necessarily increase with the complexity of the solved model. (ii.) For all the example model economies we considered, the log–linear approximation behaves as well as Parameterized Expectations. (iii.) Rejection of a particular solution attending to the fulfilment of the rational expectation hypothesis is compatible with almost no differences between methods attending to other comparison criteria. (iv.) It is proper to consider 'large' sample sizes to check the properties of a particular solution..

The rest of the paper is organized as follows. Section 2 presents the versions of the neoclassical growth model we used for comparison. Section 3 briefly describes the four methods we chose to compare, while Section 4 set the base for the comparison. In Section 5 we show the results and in Section 6 some concluding remarks. Well feed appendix A provides all necessary information about technical details, and another appendices with additional information and the MATLAB code for all the calculations are available on request.

2 Description of the models

As pointed out in the introduction, we focus on different versions of the neoclassical growth model. See the appendix for more details on the models.

1. *Brock and Mirman (1972).* The social planner maximizes the utility of the representative agent subject to technological and resource constraints.

$$\left\{\begin{array}{l} \max\limits_{k_t, c_t} \quad E_0 \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\eta}-1}{1-\eta}\right] \\ \quad s.t. \\ \qquad\begin{array}{rl} c_t + x_t = & y_t \\ y_t = & z_t k_{t-1}^{\alpha} \\ k_t = & (1-\delta)k_{t-1} + x_t \\ \log(z_t) = & (1-\rho)\log(z_{ss}) + \rho\log(z_{t-1}) + \epsilon_t \\ \epsilon_t \sim & N(0, \sigma_\epsilon^2) \\ \text{given } k_0, z_1 & \end{array} \end{array}\right. \tag{1}$$

Where c_t is consumption at time t, k_{t-1} the capital stock at the beggining of period t, x_t investment, y_t output, and z_t a technology shock to output. $0 < \beta < 1$ is the subjective discount factor, $\eta > 0$ is the coefficient of relative risk aversion, α the capital share in production, $0 < \delta < 1$ the depreciation rate and $0 < \rho < 1$ controls the persistence of the shock. The ss subscript affecting a given variable denotes its deterministic steady state value.

The only expectational first order condition of this problem is

$$c_t^{-\eta} = \beta E_t \left[c_{t+1}^{-\eta}(\alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta)\right] \tag{2}$$

As remarked in the introduction Christiano (1990), Christiano and Fisher (1997), Imrohoroğlu (1994), Taylor and Uhlig (1990) and the first part of Den Haan and Marcet (1994) were made in the context of this model. In any cases the technology shock is considered to follow a discrete Markow chain with two or three states: Christiano (1990), Christiano and Fisher (1997) (due to the fact that they used as comparison criteria a discretization of the state space solution, and the previous assumption simplifies the procedure) and five of the methods in Taylor and Uhlig (1990).

2. *Hansen(1985).* This model is an extension of the previous one that includes indivisible labor to capture better labor market features of the business cycle. See also Hansen (1997). Here the representative household faces the problem,

$$\left\{\begin{array}{l} \max\limits_{k_t, c_t} \quad E_0 \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\eta}-1}{1-\eta} - A_N N_t\right] \\ \quad s.t. \\ \qquad\begin{array}{rl} c_t + x_t = & y_t \\ y_t = & z_t k_{t-1}^{\alpha} N_t^{1-\alpha} \\ k_t = & (1-\delta)k_{t-1} + x_t \\ \log(z_t) = & (1-\rho)\log(z_{ss}) + \rho\log(z_{t-1}) + \epsilon_t \\ \epsilon_t \sim & N(0, \sigma_\epsilon^2) \\ \text{given } k_0, z_1 & \end{array} \end{array}\right. \tag{3}$$

Where N_t is labor and A_N a parameter that measures the weight of labor in the utility function. Other variables and parameters as in the Brock–Mirman model. In this case there is also one expectational first order condition,

$$c_t^{-\eta} = \beta E_t \left[c_{t+1}^{-\eta}(\alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta)\right] \tag{4}$$

3. *Cooley and Hansen(1989).* This economy is a version of the model of Hansen (1985) with money introduced via a cash–in–advance constraint applied to consumption. Then the competitive equilibrium is non–pareto–optimal and the results of the second welfare theorem cannot be used. The firms solve a standard maximization of profits problem, while the households seek to maximize their preferences subject to a certain holdings of nominal money balances and constraints. There are two sources of uncertainty in this economy: a shock to technology and a random money growth rate. The stochastic Euler equations in this case are

$$\lambda_t = \beta E_t \left[\lambda_{t+1}\left(\alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta\right)\right] \tag{5}$$

and

$$\lambda_t c_t = \beta E_t \left[\frac{1}{g_{t+1}} \right] \tag{6}$$

where λ_t denotes the Lagrange multiplier associated with the households budget constraint and g_t is the money growth rate. Den Haan and Marcet (1994) compared the Parameterized Expectation solution to this model with the linear–quadratic used in the Cooley and Hansen paper: the accuracy test suggested important differences between the solutions, although some characteristics of the linear–quadratic solution were very similar to those of the Parameterized Expectations solution, such as descriptive statistics. We confirm and extend these conclusions. For details of the model see the appendix.

3 Methods Used

We have chosen to compare four numerical solution methods. Here we will only give a brief description. We chose these particular methods for their widely use among researchers, and because they present a good framework for comparing how much non–linear structure of the original problem it is important to preserve in the approximation.

1. *Linear Quadratic Approximation (LQA):* The point here is to approximate a non–linear problem by one linear–quadratic for which the solution it is known. For a description see McGrattan (1990), Christiano (1990), Díaz–Giménez (1995) and specially Hansen and Prescott (1995) and Kydland and Prescott (1982). The method has been sucesfully applied to solve representative agent economies and overlapping–generation economies and also to economies that are subject to distorsions where the competitive equilibrium is not a Pareto optimum (see Cooley and Hansen (1989) and Kydland (1989)). Solving the social–planning problem involves solving a dynamic programming problem of the form:

$$\begin{aligned} V^{n+1}(z_t, s_t) &= max_d \left\{ r(z_t, s_t, d_t) + \beta E \left[V^n(z_{t+1}, s_{t+1}|z_t) \right] \right\} \\ st & \\ z_{t+1} &= A(z_t) + \epsilon_{t+1} \\ s_{t+1} &= B(z_t, s_t, d_t) \end{aligned}$$

where $V(z_t, s_t)$ is the optimal value function, z_t a vector of exogenous state variables, s_t a vector of endogenous state variables, d_t a vector of decision variables, $r(z_t, s_t, d_t)$ the return function for the problem, and the two constraints describe the evolution of the state variables. The solution is a function mapping a state space into decisions. In order to solve this problem one may operate directly with the value function. To simplify the previous problem, form a linear quadratic approximation about the steady state equilibrium path of the original economy and look for the solution of this approximate linear quadratic economy. The steps to follow are:

   (a) Find the steady state.
   (b) Substitute the non–linear constraints in the return function.
   (c) Let $W_t = [z_t, s_t, d_t]^T$ and form a second order Taylor expansion of the resulting return function about the steady state. So $r(z_t, s_t, d_t) \approx [1, W_t^T]Q[1, W_t^T]^T$, where Q is a symmetric matrix.
   (d) Then, the approximate problem becomes, using also the certainty equivalence principle,

$$\begin{aligned} V^{n+1}(z_t, s_t) &= max_d \left\{ [1, W_t^T]Q \begin{bmatrix} 1 \\ W_t \end{bmatrix} + \beta V^n(z_{t+1}, s_{t+1}|z_t) \right\} \\ st & \\ z_{t+1} &= A(z_t) \\ s_{t+1} &= B(z_t, s_t, d_t) \end{aligned}$$

   Under suitable conditions, the optimal value function exist, solve this functional equation, and is quadratic. Given this, the associated policy functions are linear.
   (e) Guess an initial conjecture to V^0, say $V^0(z_t, s_t) = F_t^T P^0 F_t$, where $F_t^T = [1, z_t, s_t]$ and P^0 is a symmetric and negative semi–definite matrix.
   (f) Substituting the laws of motion $F_{t+1} = BW_t$ and $V(z_{t+1}, s_{t+1})$ into the problem we get a quadratic expression on (z_t, s_t, d_t). Then the first order conditions of this approximate problem will give us an expression for d_t as a linear function of z_t and s_t (the policy function or decision rule).
   (g) Substituting this into the approximate problem gives us the next approximation, which is a quadratic function on (z_t, s_t).
   (h) Repeat until V^{n+1} is very similar (according to any convergence criteria) to V^n.

   This was the procedure followed to solve the first and the second models with this method. With respect to the third, in which there is a distortion due to the cash–in–advance constraint, important variations are needed. See Cooley and Hansen (1989) for the details.

2. *Forward solution (SIM):* This method was proposed in Sims (1989) and (1990). Blanchard and Khan (1980), İmrohoroğlu (1994) and different versions of backsolving (such as that presented in Vallés (1997) or Novales (1990)) are also related. The idea is to substitute each conditional expectation that might appear in the first order conditions by its realized value plus an expectational error, linearize the resulting problem and add the stability conditions associated with this problem[3] to the original problem. The steps to follow are:

   (a) Obtain the first order conditions of the problem. Find the steady state.
   (b) Substitute the conditional expectation by its realized value plus an expectational error.

[3]Related to the LQA method, SIM tries to make a selective linear approximation in order to preserve the more structure of the original non–linear problem as possible.

(c) Linearize the system of resulting Euler–equations and constraints about the deterministic steady state:

$$\Gamma_0 \check{Y}_{t+1} = \Gamma_1 \check{Y}_t + \Phi \zeta_{t+1}.$$

Where the vector ζ contains the expectational error and the structural perturbations, and $\check{Y}$ contains the state and decision variables in deviations to the steady state.

(d) Locate the unstable root of $\Gamma_0^{-1}\Gamma_1$ and then find the corresponding eigenvector (say μ).

(e) Then add the stability condition $\mu \check{Y}_t = 0$ to the system of first order conditions and constraints. This guarantees a stationary equilibrium of the model. Given an initial condition, the solution starts and remains on the stable subspace of the system, provided we are near the steady state.

In the three models analyzed and with the parameterizations considered, one stability condition was found in each case.

3. *Log–linearization (UHL):* This method consists of a log–linearization of the first order conditions about the steady state. We have used the simplification proposed by Uhlig (1997). The idea is to log–linearize the first order conditions and then solve for the recursive equilibrium law of motion with the method of undetermined coefficients. See also King, Plosser and Rebelo (1987), Campbell (1994) and Binder and Pesaran (1996). For the method of undetermined coefficients see, for example, Chow (1997). The steps to follow are:

(a) Find the first order conditions. Find the steady state.

(b) Log–linearize the necessary equations characterizing the equilibrium to make the system approximately linear in log–deviations from the steady state[4].

(c) Let the recursive equilibrium law of motion be those matrices P, Q, R and S that make stable the system

$$\begin{aligned} x_t &= Px_{t-1} + Qz_t \\ y_t &= Rx_{t-1} + Sz_t \end{aligned}$$

where x_t is an $m \times 1$ endogenous state vector, z_t a vector of exogenous state variables, size $k \times 1$, and y_t a list of other endogenous variables of the system, with size $n \times 1$. The log–linearized system can be written, maintaining the notation of Uhlig (1997),

$$\begin{aligned} 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t \\ 0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right] \\ z_{t+1} &= Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0, \end{aligned}$$

where matrix C is assumed to be of size $l \times n, l \geq n$ and of rank n, F is of size $(m+n-l) \times n$, and N has only stable eigenvalues. Equating coefficients according to the well–known method of undetermined coefficients between the previous two systems let us find P, Q, R and S. See section 6 in Uhlig (1997) for technical details[5].

4. *Parameterized expectations (PEA):* For a detailed explanation see Den Haan and Marcet (1990), Marcet (1993) and Marcet and Marshall (1994) for example. The idea is to parameterize the expectations part of the stochastic Euler equation. The conditional expectation is specified as a function of the state of the system, and the parameters of that function are estimated to solve the model.

The steps to follow are:

(a) Find the first order conditions of the problem. Find the steady state.

(b) Substitute the conditional expectation, say $E_t(\phi_{t+1})$, in each case by a parameterized function $\psi(q; k_{t-1}, z_t)$, a polynomial function, where z_t denotes a vector containing the structural perturbations of the model and q is a vector of parameters. Define the stochastic Euler equation residual as $E_t(\phi_{t+1}) - \psi_t$. In principle ψ can approximate the expectation arbitrarily well by increasing the order of the polynomial.

(c) Choose a value for q. For the searching of the fixed point a drawn of 25000 observations was used in each case. We found useful here to begin with a realization for the variables using another (quicker) method, for the estimation of useful initial conditions of the parameters of ψ.

(d) Use the first order conditions of the problem (with the conditional expectation substituted by $\psi(q; k_{t-1}, z_t)$) and constraints to generate time series paths for the variables of the economy.

(e) Define $S : \Re^m \rightarrow \Re$, where m is the dimension of q. Choose that q that verifies $S(q) = argmin_q E_t\left[\phi_{t+1} - \psi_t(k_{t-1}(q), z_t; q)\right]^2$.

(f) Iterate until $q = S(q)$. This guarantees that if agents use ψ as their expectation function, then q is the best parameter they could use in the sense that it minimize the mean squared error. To find each q^{i+1} starting from a previous q^i, run a non-linear regression of $\phi(q^i)$ on $\psi(q^i)$, as an approximation to $S(q^{i+1})$, and actualize q according to the rule

$$q^{i+1} = q^i + \lambda S(q^i).$$

In the literature nobody has compared this set of methods. For the aim of our comparison, in relation to other papers, Christiano (1990) compared LQA with LQA in logs, Christiano

[4] For this use the following rules:

$$\tilde{x}_t \equiv \log(X_t) - \log(X_{ss}) \iff \tilde{x}_t = \log\left(\frac{X_t}{X_{ss}}\right) \iff X_t = X_{ss}e^{\tilde{x}_t}$$

Then use $e^{\tilde{x}_t + a\tilde{y}_t} \approx 1 + \tilde{x}_t + a\tilde{y}_t$, $\tilde{x}_t\tilde{y}_t \approx 0$ and $E_t\left[ae^{\tilde{x}_{t+1}}\right] \approx E_t\left[a\tilde{x}_{t+1}\right]$ up to a constant.

[5] In Appendix A we show the appropriate matrices A, B, C... for each solved model. MATLAB programs for the implementation of this method are available at Uhlig's home page: http://cwis.kub.nl/ few5/center/STAFF/uhlig/toolkit.dir/toolkit.htm.

and Fisher (1997) evaluates PEA, while in Taylor and Uhlig (1990) LQA, PEA and a different version of SIM is used. In Imrohoroğlu (1994) a very close version of SIM and LQA are compared. All these papers in the context of the Brock and Mirman model.

In different model economies, Den Haan and Marcet (1994) compared LQA and PEA in the first model we present and in the third one, while Dotsey and Mao (1992) evaluated LQA, LQA in logs plus the method proposed in King, Plosser and Rebelo (1987) in logs and in levels of the variables. This last method in logs is very similar to UHL; in this respect our results differ from that of this paper: we find UHL very suitable for the models we have considered, and they found that none of the methods they looked at dominated the others and could have problems for high values of the variance of the shock. Of course this may be context specific: they used a version of the neoclassical growth model in which the only source of perturbation was a process for taxes on production, following a five state Markow chain, and did not try much parametric variation. It could be interesting to extend the comparison we make to a context as theirs and see if our results still hold.

In any case, it is not only that we used a kind of different methods but the framework where we compared and how we compared what we think could be more relevant.

As we can see, we have chosen one dynamic programing based method (LQA) and three Euler equation based methods; this last methods have the adventage that they can be used in Pareto optimal and non-optimal environments without modifications. On the contrary, LQA needs important modifications.

Concerning LQA, SIM and UHL, they have the disadvantage relative to PEA that they cannot be refined depending on the realization of the structural shocks. Nevertheless, the previous adventage cannot be exploited in a comparison of the kind presented here, given that, as remarked in Den Haan and Marcet (1994), for the calculation of the test they proposed, one should use a realization of the stochastic exogenous shocks that is different from the one used in calculating the fixed point.

Another point is that to obtain the decision rules LQA, SIM and UHL are quite fast in relation to PEA.

An advantage of PEA is that it preserves all the non-linear structure of the models at hand, given that the parameterized function selected be near the true expectation. Note that LQA, SIM and UHL preserves different degrees of non-linearity; the first two add policy functions (or stability conditions) to the original system of non-linear first order conditions and constraints, while UHL method gives log-linear decision rules for all variables. In the most simple model LQA and SIM are identical, but in the other two models SIM adds one decision rule while LQA adds two: SIM preserves one of the non-linear decision rules of the original system, and so incurres in the cost of using a non-linear equation solver. For this we used the solver `csolve.m`, available at C. A. Sims home page in the webb.

4 Base for the Comparison

In the first two models, we analize the following parametric cases, changing only the relative risk aversion parameter and the variance of the technology shock, suggested in the literature as the most influential parameters. For sensitivity analysis, we consider three values of σ_ϵ:

from 0.01, close the most usual in the literature (0.00721), to another one six times greater. Concerning risk aversion, we proceed from a low value of 0.5 to a high one of 3.0. The remaining parameter values are standard: $\beta = 0.99, \rho = 0.95, \alpha = 0.36$, and $\delta = 0.025$.

CASE	1	2	3	4	5	6	7	8	9
σ_ϵ^2	0.01	0.01	0.01	0.02	0.02	0.02	0.06	0.06	0.06
η	0.5	1.5	3.0	0.5	1.5	3.0	0.5	1.5	3.0

In the Cooley and Hansen model we focus on the variance of the perturbation, and in the money growth parameter, analysing the cases of the original paper considered by Den Haan and Marcet (1994). Parameter values are $\beta = 0.99, \alpha = 0.36, \delta = 0.025, A_N = 2.86$, and for the two sources of uncertainty the persistence of the postulated autorregressive processes for technology and money growth are $\rho_z = 0.95$ and $\rho_g = 0.48$. We change the money growth rate and the variances of the shocks, according to,

CASE	1	2	3	4	5	6
g_{ss}	1.015	1.15	1.015	1.15	1.015	1.15
$\sigma_{\epsilon z}$	0.01	0.01	0.02	0.02	0.06	0.06
$\sigma_{\epsilon g}$	0.09	0.09	0.09	0.09	0.09	0.09

For the comparison we calculated the mean and standard deviations over 250 simulations of lenght 150 and 3000 of the following set of measures:

- Related to the stochastic Euler equation residual, ξ (to check for the accuracy of the solution):
  - To check for possible correlation with the information set: Den Haan and Marcet (1994) accuracy test. The idea of the test is to check whether there exist any function of variables dated $t-1$ or earlier that should help predict ξ_t. The steps to follow are:
    * First obtain a large number of observations by simulating the model for a realization of the exogenous processes.
    * Run a regression of ξ_t over x_t, matrix of instruments selected from the information set. Then take $\hat{a} = (\sum x_t^T x_t)^{-1}(\sum x_t^T \xi_t)$.
    * To check the null hypothesis that ξ_t is a martingale, form the statistic:

    $$M = \hat{a}^T(\sum x_t^T x_t)(\sum x_t^T x_t \xi_t^2)^{-1}(\sum x_t^T x_t)\hat{a} \sim \chi^2_{qm},$$

    where m is the number of instruments chosen and q is the number of Euler equation errors. It is worth noting that the alternative hypothesis is that the error is not a martingale; so if the value of the statistic belongs to the upper critical region of the χ^2_{qm} distribution, there is evidence against the accuracy of the solution.

The number of observations can be interpreted as a measure of how stringent the criterion is: if the solution passes the test even for a very large number, this is evidence that the solution is very accurate. We choose the set of instruments used by Den Haan and Marcet (1994) for the simple growth model: constant, $k_{t-1}, k_{t-2}, k_{t-3}, \log(z_{t-1}), \log(z_{t-2}), \log(z_{t-3})$.

  - To check also for remaining autocorrelation in mean: an AR(1) process with mean is estimated to ξ. Also the autocorrelation function (ACF) of ξ is calculated to check for remaining dynamic autocorrelation.
  - To check for remaining structure in variance we performed a Lagrange Multiplier test for ARCH structure in ξ. The null hypothesis is constant variance. This test is only indicative because if the acceptance is not very clear, as the sample size increases the statistic goes to the rejection region. Nevertheless, given our aim is to focus on differenciated behaviour of the different methods, the test is useful.

- To check how different are the methods on other dimensions rather than that of the rational expectations fulfilment:
  - Impulse response functions generated using the first order conditions and the decision rules. In the first two models that is to say: generate a unit impulse in the perturbation: $\epsilon_0 = 1, \epsilon_t = 0, \ \forall t > 0$ and compute the response of the approximated systems. In the Cooley and Hansen model we have two sources of perturbation, and so can compute the same exercise for the technology and the monetary shocks.
  - Tabulate the values of the approximate decision rules at alternative points in the state space: give a grid of values for the state variables and then obtain the induced values for the endogenous state (capital) and for the other endogenous variables.
  - Mean, relative–to–output standard deviation, skewness and kurtosis for all variables: point and interval estimation. For the relative volatility also the empirical distribution.
  - Cross correlation functions of output with every other variable: point and interval estimation. For the contemporaneous correlation also the empirical distribution.
  - Estimate autorregressive processes of order one to four for the variables, to approximate the univariate models that the different variables follow. In the case of the UHL method we can derive the exact models, but not for the other methods.

For the Hansen model, in cases 7, 8 and 9 with $T = 3000$, it was almost impossible to found a solution with the LQA method, due to negative values of k for certain draws of the technology shock. For the Cooley and Hansen model the same occured with SIM method in cases 5 and 6 with $T = 3000$; with $T = 150$ it generated negative values for the capital stock for a 30% of the realizations of the technology shock.

Related to PEA, we considered, for each model, a fixed order for the polynomial function used to approximate the conditional expectation (although with different values for the coefficient, of course). This proved useful for all the parametric cases but case 2 in the Cooley and Hansen model, in which it is neccesary to increase the order of the polynomial.

5 Results

We present here selected results, according to their relevance for the aim of the paper. The whole set of results is presented in an Appendix available on request.

5.1 Euler Equation Error properties

5.1.1 *Den Haan and Marcet Test*

Tables 1 to 3 shows the results for the Den Haan and Marcet test for the three models at hand. Concerning the Brock and Mirman model, UHL and PEA pass the test for both sample sizes, and give very similar results, while LQA and SIM (that, remember, are identical for this model) are rejected clearly with $T = 3000$ and in some cases as 2 and 3 may have passed with $T = 150$.

Insert tables 1 to 3

In table 2 it is remarkable the gain in accuracy of SIM relative to LQA when we move from the most simple model to the Hansen model. SIM passes cases 1, 2, 3, 4, 5, 6 and 9 for $T = 150$, but only 2 and 3 for $T = 3000$; LQA is always rejected. As σ_ϵ increases, the accuracy diminishes for SIM and LQA. As η increases, meanwhile, accuracy increases. UHL and PEA again pass for all the cases and both sample sizes, with similar results.

Table 3 presents the results for the Cooley and Hansen model. The first four columns refer to the Euler Equation residual showed in equation (5). The picture is the same as before: UHL and PEA pass all the cases, while LQA and SIM, with similar behaviour, are rejected clearly for T=3000 and for almost all cases even with T=150. The fifth column is refered to the "money" Euler Equation residual of the model, given by equation (6). Note that it is only necessary to check the properties of this error when using the dynamic programming linear–quadratic method, because for the Euler Equation based methods it is, by construction, equal to zero (see appendix for details). The results are very bad for all the parametric cases.

5.1.2 *Autocorrelation and ARCH structure in the residuals*

The results in the next three tables, 4 to 6, reinforce the former results. UHL and PEA always move in the same direction. In the first model for cases 3 and 6 when $T = 3000$, LQA and SIM are better than UHL and PEA, but in the other cases the % of rejections for the constant variance hypothesis is lower for UHL and PEA relative to LQA and SIM. Concerning Hansen's model, from cases 1 to 4 UHL and PEA, and less SIM, give signs of accuracy. But for cases 5 to 9 and $T = 3000$ the test do not let us differenciate between methods. In relation to the Cooley and Hansen model, all the methods give a similar result. In the low variance case exhibit a good behaviour that make worse as σ_{ϵ_t} increases.

Insert tables 4 to 6

In tables 7, 8 and 9 we show direct testing of remaining autocorrelation in the residuals. For the Brock and Mirman model, when we estimate with T=150, nothing is detected. But when we make it with the larger sample size, we cannot reject the existence of a mean different from zero for LQA and SIM in seven cases, so violating the hypothesis of an unbiased error. For UHL we do not detect any case, and for PEA only case 1 for T=3000. In cases 8 and 9, for T=3000 also there is weak evidence of an AR(1) structure in the residuals generated by LQA and SIM.

Insert tables 7 to 9

Table 8 soften this result, and for the Hansen model only in four cases of LQA we find evidence for a mean different from zero. In relation to the Cooley and Hansen model, table 9, for T=3000 present evidence against LQA and SIM; in relation to PEA, if we except case 2, as noted before, there is only weak evidence for a mean different from zero in case 6 for T=3000.

The last piece of evidence in this section is given in table 10, that evaluates a property of the equilibrium of the model that must hold, as shown in the appendix. The correlation mentioned must be almost equal to the unity. Again, the Euler Equation based methods fulfils the property by construction, and LQA fails increasingly with σ_{ϵ_z}.

As a summary, LQA and SIM solutions are not very accurate for these models and parameter cases on the basis of the tests of the previous two subsections, while UHL and PEA are very, and likewise, accurate.

5.2 Other statistical measures

Now we take a look at any other statistics usually reported by business cycle researchers. We selected any cases for each model as benchmark cases.

5.2.1 *Impulse Response Functions*

Figures 1 to 4 plot this statistic for the three models at hand. For the Brock and Mirman model the responses of the system to an unit impulse in technology do not seem to be very different qualitatively speaking: output, capital and the interest rate responses have the same shape and approximate duration. Nevertheless, things change a bit in quantitative terms, with greater capital responses for PEA and UHL. Concerning interest rate, UHL response is lower in the first period relative to LQA, SIM and PEA. These characteristics are maintained across the plotted cases (4, 5 and 6) safe the increase in the response of capital as η increases is proportionally greater for UHL and PEA.

Insert figures 1 to 4

Relative to the Hansen model again there are no significant qualitative differences in the responses, if we except the responses of capital in cases 5 and 6 for SIM, that are a bit out of phase with PEA. SIM tends to present the greater responses for all the variables. LQA reacts little in the cases of output and capital, but in the same way as PEA when dealing with interest rates.

Things seem to change for the Cooley and Hansen model. In this preliminary version we think it is very surprising to see the LQA responses we report in figures 3 and 4, and may be due to the fact that the steady state values for investment and the inverse of real money balances induced by the LQA linear rules are not equal to the values induced by the actual steady state of the nonlinear system, generating somethig like a shift in the first period response. The responses of SIM, UHL and PEA are coherent with the intuition of the model and the other results we report.

5.2.2 *Grids for the decision rules*

In figures 5 to 7 we tabulate the values of the approximate decision rules at alternative points in the state space. All the grids are formed by 21 values for each state variable.

Insert figures 5 to 7

The grid for case 6 in the Brock and Mirman model takes the values 25 to 45 for capital, with $k_{ss} = 37.99$, and 0.7 to 1.3 for z, with $z_{ss} = 1$. It is remarkable that the evolution of consumption as technology increases is non–monotonic for LQA and SIM, a characteristic already reported by Taylor and Uhlig (1990) and Christiano (1990) for the LQA method. The grid for capital is very similar across methods, and that for the interest rate present a more concave grid for UHL as technology increases, while that for SIM, LQA and PEA are identical.

For the Hansen's case 6, capital goes from 11 to 14 ($k_{ss} = 12.66$) and technology from 0.7 to 1.3. LQA maintains the non–monotonic behaviour in consumption and UHL the more concave response in interst rates. Differences are slight in the grids for the Cooley and Hansen model.

5.2.3 *Descriptive statistics, cross correlation functions and autorregressive structure*

In tables 11 to 14 we present descriptive statistics for selected cases of the three models. We do not appreciate any significant differences in sample means or standard deviations for the mean, relative standard deviation, kurtosis and skweness for the three models. Only in Brock and Mirman the distribution of investment for UHL shows a higher skweness and kurtosis. And, of course, some differences when considering the sample size T=150: greater amplitude of the sample standard deviations when comparing with the same case for T=3000, and little differences on relative standard deviations, as one can see looking at the real wage in table 13.

Figure 8 plots the empirical distribution for the relative standard deviation statistics for case 6, T=3000, of the Hansen model. There are certain differences in the shapes of the histograms: for instance, std(R)/std(y) is more symmetric for PEA, while for std(log(z))/std(y) UHL posted a greater number of values on the left tail. But differences are not really important. Concerning histograms of these kind, in the highest variance cases it is possible to find important differences for certain ratios.

The draw for the cross correlation functions (see tables 15 to 17 and figure 9) is the same as that of the two previous paragraphs: we found no differences between methods. Only in the Cooley and Hansen model the correlation of output with interest rates tend to be lower and with a greater standard deviation in the SIM case relative to the other methods. And for the empirical distribution of the contemporaneous correlation of all the variables with output for case 6, T=3000, for the Hansen model we can detect some differences: for example, for the LQA method the correlation of output with consumption is slightly to the right, and for that of output with technology there are more values to the left for SIM. But differences are not very important.

Table 18 presents the last piece of evidence: the AR representation is the same for all the methods.

To sum up:

- We appreciate important differences between methods when checking the rational expectations hypothesis. The Den Haan and Marcet statistic, and other complementary tests, let us differenciate, in the three models considered, the four methods: PEA and UHL passed the tests for all the parametric cases while LQA and SIM presented a bad behaviour.
- Concerning impulse response functions, we only appreciate differences in the Cooley and Hansen model, although qualitatively not very important between SIM, UHL and PEA methods.
- There are almost no differences in the descriptive statistics. Given that the linear approximations are done about the steady state one may expect no differences in the means, but we neither found differences in the relative standard deviations, skweness or kurtosis. And, surprisingly, we neither found differences in the sample standard deviations associated with these statistics.
- The same comment applies to the cross correlation functions and the autocorrelation structure in the series. The time series for the variables inherit the patterns of autocorrelation of the shocks to the system.

6 Concluding remarks

If we consider again the questions possed in the introduction we can answer in the following way:

1. Do differences between different methods increase/appear when increasing complexity in the growth model? The answer is mixed and depends on the adaptability of the method to any proposed model. We have seen how the LQA approximation deteriorates when dealing with the more complex model, but also how SIM improves when passing from the Brock and Mirman model to the Hansen model, and how UHL and PEA, with little exceptions, behave very similar in the three models cosidered. Of course, if one consider more complex extensions of the basic neoclassical growth model things may change.

2. How many non–linear structure of the original problem is useful to maintain, given the computational costs of more complex methods? Our evidence suggest that the log–linearization is good enough for the neoclassical growth model extensions considered. Also, if we had done LQA and SIM approximations considering the logarithms of the series instead of the levels of the series we think the performance of these methods, especially SIM, had improved.

   The complexity of implementing PEA is related with good initial conditions for the parameters of the polynomial function: as already pointed out, we found very useful to generate time series with a linear method (we selected UHL), then estimate initial values for the parameters and iterate until convergence to the fixed point. For the models considered PEA does not present adventages related to UHL.

3. Depending on the aim of a research, it is always important if certain solution is rejected by the Den Haan and Marcet (1994) statistic? We have shown how LQA and SIM solutions are rejected in almost all cases according to this criteria, but we found no important differences in the other set of statistics, if we except LQA in the Cooley and Hansen model. Of course, taking into account these evidence, the answer to this question depends on the loss function of each researcher.

4. It is irrelevant for a business cycle paper to consider a small sample size when calculating, for instance, descriptive statistics, or it is necessary to consider large sample sizes for the sake of reliability on the results? The answer to this question has two slopes. One is statistical, and tell us how both interval and point estimation of the statistics is more reliable with large samples. And also, to achieve convergence with PEA–like methods, for instance, one need large samples. The other slope is related to actual data sets, that use to be small. In this respect we think UHL is preferable.

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Table 4: **LM test for ARCH structure in the residuals.** Brock and Mirman(1972) model. % of acceptance of the constant variance hypothesis.

		LQA	SIM	UHL	PEA
Case 1	T=150	53.2	53.2	77.2	78
	T=3000	0.0	0.0	78.8	78.4
Case 2	T=150	78.8	78.8	80	78.4
	T=3000	50.8	50.8	81.2	81.2
Case 3	T=150	79.2	79.2	78.4	78.4
	T=3000	73.2	73.2	59.2	59.6
Case 4	T=150	39.6	39.6	79.2	78.4
	T=3000	0.0	0.0	84	83.2
Case 5	T=150	68.8	68.8	81.6	80
	T=3000	0.4	0.4	56.8	68.8
Case 6	T=150	75.2	75.2	80	75.2
	T=3000	12.8	12.8	2	0.0
Case 7	T=150	33.6	33.6	81.2	81.6
	T=3000	0.0	0.0	46	26
Case 8	T=150	52	52	77.6	75.2
	T=3000	0.0	0.0	0.0	0.0
Case 9	T=150	53.2	53.2	54.4	48.8
	T=3000	0.0	0.0	0.0	0.0

Table 5: **LM test for ARCH structure in the residuals.** Hansen(1985) model. % of acceptance of the constant variance hypothesis.

		LQA	SIM	UHL	PEA
Case 1	T=150	39.6	78.4	84.4	84
	T=3000	0.0	18.8	89.2	90
Case 2	T=150	7.2	84.4	86.4	85.2
	T=3000	0.0	72.4	81.2	80.8
Case 3	T=150	71.6	79.2	78.8	78.8
	T=3000	0.0	78.4	81.6	76
Case 4	T=150	36.8	63.6	80.8	80.4
	T=3000	0.0	0.0	78.4	75.2
Case 5	T=150	57.2	76	78	79.2
	T=3000	0.0	8.4	58.4	44.4
Case 6	T=150	50.4	76	79.2	78.4
	T=3000	0.0	23.2	52.8	44.4
Case 7	T=150	35.6	20.4	58.8	56.4
	T=3000	-	0.0	0.0	0.0
Case 8	T=150	44	52	71.6	67.6
	T=3000	-	0.0	0.0	0.0
Case 9	T=150	48	57.2	65.2	62.8
	T=3000	-	0.0	0.0	0.0

Table 6: **LM test for ARCH structure in the residuals.** Cooley and Hansen(1989) model. % of acceptance of the constant variance hypothesis.

		LQA	SIM	UHL	PEA
Case 1	T=150	82	82.8	80.8	81.6
	T=3000	81.2	81.6	80.4	81.2
Case 2	T=150	84	84.8	83.2	93.2
	T=3000	83.2	83.2	82.2	84
Case 3	T=150	80	79.6	78.4	76.8
	T=3000	51.6	44.8	40.8	34
Case 4	T=150	79.2	78.4	77.2	78.4
	T=3000	46.4	40.4	42	40.4
Case 5	T=150	52.8	52.4	44.8	44
	T=3000	5.0	-	0.0	0.0
Case 6	T=150	52	54.4	46	43.2
	T=3000	3.0	-	0.0	0.0

Table 7: **AR(1) residual estimation** $\xi_t = \mu + \rho\xi_{t-1} + \epsilon_t$. Brock and Mirman(1972) model. An * denotes non–rejection of the null hypothesis $H_0 : \mu = 0$ or $H_0 : \rho = 0$ at the 95 % level, and ** at the 85% level.

			LQA	SIM	UHL	PEA
Case 1	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	*	*	-	*
		ρ	-	-	-	-
Case 2	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	-	-	-	-
		ρ	-	-	-	-
Case 3	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	-	-	-	-
		ρ	-	-	-	-
Case 4	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	*	*	-	-
		ρ	-	-	-	-
Case 5	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	**	**	-	-
		ρ	-	-	-	-
Case 6	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	**	**	-	-
		ρ	-	-	-	-
Case 7	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	*	*	-	-
		ρ	-	-	-	-
Case 8	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	**	**	-	-
		ρ	**	**	-	-
Case 9	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	-	-	-	-
		ρ	**	**	-	-

Table 8: **AR(1) residual estimation** $\xi_t = \mu + \rho\xi_{t-1} + \epsilon_t$. Hansen(1985) model. An * denotes non-rejection of the null hypothesis $H_0 : \mu = 0$ or $H_0 : \rho = 0$ at the 95 % level, and ** at the 85% level.

			LQA	SIM	UHL	PEA
Case 1	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	*	-	-	-
		ρ	-	-	-	-
Case 2	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	-	-	-	-
		ρ	-	-	-	-
Case 3	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	-	-	-	-
		ρ	-	-	-	-
Case 4	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	*	-	-	-
		ρ	-	-	-	-
Case 5	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	**	-	-	-
		ρ	-	-	-	-
Case 6	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	**	-	-	-
		ρ	-	-	-	-
Case 7	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ		-	-	-
		ρ		-	-	-
Case 8	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ		-	-	-
		ρ		-	-	-
Case 9	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ		-	-	-
		ρ		-	-	-

Table 9: **AR(1) residual estimation** $\xi_t = \mu + \rho\xi_{t-1} + \epsilon_t$. Cooley and Hansen(1989) model. An * denotes non-rejection of the null hypothesis $H_0 : \mu = 0$ or $H_0 : \rho = 0$ at the 95 % level, and ** at the 85% level.

			LQA	SIM	UHL	PEA
Case 1	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	**	*	-	-
		ρ	-	-	-	-
Case 2	T=150	μ	-	-	-	*
		ρ	-	-	-	**
	T=3000	μ	*	**	-	*
		ρ	-	-	-	-
Case 3	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	*	*	-	-
		ρ	-	-	-	-
Case 4	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	*	*	-	-
		ρ	-	-	-	-
Case 5	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	-		-	-
		ρ	**		-	-
Case 6	T=150	μ	-	-	-	-
		ρ	-	-	-	-
	T=3000	μ	-		-	**
		ρ	**		-	-

Table 12: Descriptive Statistics. Hansen model. Case6, T=3000. Numbers are means and standard deviations (in brackets) across simulations.

LQA	mean	rel. Std	skewness	kurtosis	SIM	mean	rel. Std	skewness	kurtosis
y-yss	0.001	1.000	0.198	3.023	y-yss	0.002	1.000	0.250	3.059
	(0.009)	(0.000)	(0.141)	(0.287)		(0.009)	(0.000)	(0.142)	(0.308)
k-kss	0.005	12.319	0.012	2.896	k-kss	0.036	12.372	0.083	2.934
	(0.187)	(0.592)	(0.236)	(0.422)		(0.189)	(0.599)	(0.242)	(0.458)
x-xss	0.000	0.753	0.003	2.962	x-xss	0.001	0.757	0.313	3.128
	(0.005)	(0.013)	(0.118)	(0.200)		(0.005)	(0.012)	(0.124)	(0.288)
c-css	0.001	0.328	0.087	3.339	c-css	0.001	0.321	0.082	2.925
	(0.005)	(0.013)	(0.250)	(0.563)		(0.005)	(0.013)	(0.219)	(0.410)
R-Rss	0.000	0.029	0.237	3.108	R-Rss	0.000	0.029	0.299	3.185
	(0.000)	(0.001)	(0.147)	(0.335)		(0.000)	(0.001)	(0.161)	(0.409)
N-Nss	0.000	0.198	-0.016	2.950	N-Nss	0.000	0.197	0.014	2.920
	(0.003)	(0.005)	(0.181)	(0.326)		(0.003)	(0.005)	(0.178)	(0.315)
$\frac{w}{p}-\frac{w}{p}_{ss}$	0.011	2.501	0.237	3.014	$\frac{w}{p}-\frac{w}{p}_{ss}$	0.014	2.502	0.277	3.047
	(0.036)	(0.112)	(0.228)	(0.502)		(0.037)	(0.108)	(0.230)	(0.541)
UHL	mean	rel. Std	skewness	kurtosis	PEA	mean	rel. Std	skewness	kurtosis
y-yss	0.004	1.000	0.241	3.052	y-yss	0.006	1.000	0.244	3.055
	(0.009)	(0.000)	(0.143)	(0.317)		(0.010)	(0.000)	(0.145)	(0.324)
k-kss	0.065	12.396	0.280	3.032	k-kss	0.179	12.502	0.211	2.973
	(0.190)	(0.632)	(0.250)	(0.576)		(0.193)	(0.614)	(0.245)	(0.518)
x-xss	0.009	0.784	0.710	3.843	x-xss	0.005	0.755	0.264	3.097
	(0.005)	(0.016)	(0.163)	(0.701)		(0.005)	(0.012)	(0.127)	(0.289)
c-css	0.001	0.320	0.109	2.928	c-css	0.002	0.320	0.080	2.919
	(0.005)	(0.013)	(0.218)	(0.412)		(0.005)	(0.013)	(0.217)	(0.404)
R-Rss	0.000	0.029	-0.009	2.994	R-Rss	0.000	0.028	0.191	3.051
	(0.000)	(0.000)	(0.131)	(0.243)		(0.000)	(0.001)	(0.137)	(0.283)
N-Nss	0.001	0.198	0.153	2.981	N-Nss	0.000	0.194	0.109	2.967
	(0.003)	(0.006)	(0.183)	(0.355)		(0.003)	(0.006)	(0.186)	(0.353)
$\frac{w}{p}-\frac{w}{p}_{ss}$	0.014	2.501	0.304	3.067	$\frac{w}{p}-\frac{w}{p}_{ss}$	0.023	2.502	0.274	3.036
	(0.036)	(0.110)	(0.232)	(0.558)		(0.037)	(0.106)	(0.229)	(0.529)

Table 13: Descriptive Statistics. Hansen model. Case6, T=150. Numbers are means and standard deviations (in brackets) across simulations.

LQA	mean	rel. Std	skewness	kurtosis	SIM	mean	rel. Std	skewness	kurtosis
y-yss	-0.001	1.000	0.162	2.672	y-yss	-0.001	1.000	0.200	2.692
	(0.039)	(0.000)	(0.393)	(0.519)		(0.039)	(0.000)	(0.393)	(0.543)
k-kss	-0.039	9.413	0.021	2.265	k-kss	-0.002	9.422	0.086	2.272
	(0.675)	(1.871)	(0.537)	(0.669)		(0.677)	(1.862)	(0.539)	(0.693)
x-xss	-0.001	0.799	0.012	2.699	x-xss	0.000	0.800	0.243	2.779
	(0.022)	(0.042)	(0.368)	(0.487)		(0.022)	(0.034)	(0.375)	(0.577)
c-css	0.000	0.269	0.251	2.664	c-css	0.000	0.261	0.064	2.353
	(0.017)	(0.042)	(0.570)	(0.824)		(0.017)	(0.039)	(0.491)	(0.585)
R-Rss	0.000	0.028	0.157	2.759	R-Rss	0.000	0.028	0.180	2.777
	(0.001)	(0.003)	(0.406)	(0.508)		(0.001)	(0.004)	(0.409)	(0.524)
N-Nss	0.000	0.170	0.006	2.601	N-Nss	0.000	0.170	0.003	2.598
	(0.009)	(0.020)	(0.452)	(0.555)		(0.009)	(0.020)	(0.448)	(0.551)
$\frac{w}{p}-\frac{w}{p}_{ss}$	0.001	2.020	0.114	2.362	$\frac{w}{p}-\frac{w}{p}_{ss}$	0.004	2.020	0.145	2.371
	(0.134)	(0.306)	(0.492)	(0.604)		(0.134)	(0.307)	(0.492)	(0.619)
UHL	mean	rel. Std	skewness	kurtosis	PEA	mean	rel. Std	skewness	kurtosis
y-yss	0.001	1.000	0.168	2.677	y-yss	0.004	1.000	0.161	2.668
	(0.038)	(0.000)	(0.393)	(0.522)		(0.039)	(0.000)	(0.395)	(0.517)
k-kss	0.011	9.402	0.115	2.278	k-kss	0.111	9.476	0.099	2.268
	(0.676)	(1.855)	(0.538)	(0.707)		(0.685)	(1.862)	(0.537)	(0.702)
x-xss	0.007	0.823	0.533	3.104	x-xss	0.003	0.798	0.180	2.738
	(0.022)	(0.052)	(0.409)	(0.919)		(0.022)	(0.033)	(0.374)	(0.534)
c-css	-0.001	0.260	0.057	2.351	c-css	0.000	0.260	0.055	2.352
	(0.017)	(0.039)	(0.491)	(0.582)		(0.017)	(0.039)	(0.490)	(0.580)
R-Rss	0.000	0.028	0.018	2.730	R-Rss	0.000	0.028	0.132	2.748
	(0.001)	(0.003)	(0.400)	(0.488)		(0.001)	(0.003)	(0.403)	(0.502)
N-Nss	0.001	0.171	0.099	2.612	N-Nss	0.001	0.167	0.030	2.588
	(0.009)	(0.024)	(0.453)	(0.549)		(0.009)	(0.023)	(0.454)	(0.546)
$\frac{w}{p}-\frac{w}{p}_{ss}$	0.003	2.018	0.138	2.369	$\frac{w}{p}-\frac{w}{p}_{ss}$	0.011	2.022	0.136	2.369
	(0.133)	(0.308)	(0.492)	(0.614)		(0.134)	(0.302)	(0.490)	(0.611)

Table 14: Descriptive Statistics. Cooley and Hansen model. Case1, T=3000. Numbers are means and standard deviations (in brackets) across simulations.

LQA	mean	rel. Std	skewness	kurtosis	SIM	mean	rel. Std	skewness	kurtosis
y-yss	0.001	1.000	0.095	3.007	y-yss	-0.003	1.000	-0.031	3.000
	(0.009)	(0.000)	(0.166)	(0.304)		(0.009)	(0.000)	(0.176)	(0.295)
k-kss	-0.013	9.807	-0.010	2.935	k-kss	-0.071	10.013	-0.337	3.168
	(0.112)	(0.328)	(0.238)	(0.405)		(0.118)	(0.454)	(0.294)	(0.706)
c-css	0.001	0.521	0.246	3.058	c-css	-0.001	0.518	0.046	2.975
	(0.006)	(0.016)	(0.242)	(0.511)		(0.006)	(0.015)	(0.244)	(0.411)
N-Nss	0.000	0.140	-0.099	3.041	N-Nss	-0.001	0.141	0.029	3.094
	(0.000)	(0.006)	(0.104)	(0.196)		(0.000)	(0.006)	(0.114)	(0.226)
x-xss	0.000	0.604	-0.001	2.997	x-xss	-0.002	0.604	0.039	3.029
	(0.003)	(0.019)	(0.114)	(0.210)		(0.003)	(0.019)	(0.122)	(0.221)
$\hat{p} - \hat{p}_{ss}$	0.001	0.792	0.010	2.944	$\hat{p} - \hat{p}_{ss}$	0.005	0.800	0.215	3.052
	(0.009)	(0.023)	(0.227)	(0.379)		(0.009)	(0.027)	(0.261)	(0.522)
$\lambda - \lambda_{ss}$	0.001	0.769	0.010	2.939	$\lambda - \lambda_{ss}$	0.004	0.777	0.215	3.048
	(0.009)	(0.023)	(0.229)	(0.387)		(0.009)	(0.027)	(0.264)	(0.531)
$\pi - \pi_{ss}$	0.000	0.207	0.035	3.005	$\pi - \pi_{ss}$	0.000	0.205	0.040	3.002
	(0.000)	(0.012)	(0.046)	(0.086)		(0.000)	(0.013)	(0.045)	(0.085)
R-Rss	0.000	0.023	0.133	3.086	R-Rss	0.000	0.024	0.365	3.425
	(0.000)	(0.001)	(0.116)	(0.225)		(0.000)	(0.001)	(0.187)	(0.585)
$\frac{w}{p} - \frac{w}{p}_{ss}$	0.003	1.519	0.244	3.051	$\frac{w}{p} - \frac{w}{p}_{ss}$	-0.004	1.512	0.044	2.971
	(0.017)	(0.047)	(0.244)	(0.517)		(0.017)	(0.045)	(0.246)	(0.420)
UHL	**mean**	**rel. Std**	**skewness**	**kurtosis**	**PEA**	**mean**	**rel. Std**	**skewness**	**kurtosis**
y-yss	0.001	1.000	0.180	3.034	y-yss	0.002	1.000	0.199	3.064
	(0.009)	(0.000)	(0.165)	(0.329)		(0.009)	(0.000)	(0.169)	(0.350)
k-kss	0.010	9.825	0.163	2.978	k-kss	0.036	9.903	0.206	3.020
	(0.112)	(0.321)	(0.245)	(0.470)		(0.114)	(0.329)	(0.253)	(0.514)
c-css	0.000	0.518	0.117	2.966	c-css	0.001	0.517	0.119	2.969
	(0.006)	(0.015)	(0.230)	(0.415)		(0.006)	(0.015)	(0.233)	(0.420)
N-Nss	0.000	0.140	0.102	3.017	N-Nss	0.000	0.139	0.057	3.024
	(0.000)	(0.006)	(0.103)	(0.200)		(0.000)	(0.006)	(0.104)	(0.197)
x-xss	0.003	0.614	0.454	3.369	x-xss	0.001	0.605	0.212	3.118
	(0.003)	(0.019)	(0.135)	(0.423)		(0.003)	(0.018)	(0.126)	(0.296)
$\hat{p} - \hat{p}_{ss}$	0.002	0.794	0.137	2.976	$\hat{p} - \hat{p}_{ss}$	0.001	0.791	0.135	2.979
	(0.009)	(0.027)	(0.228)	(0.398)		(0.009)	(0.028)	(0.229)	(0.400)
$\lambda - \lambda_{ss}$	0.002	0.771	0.136	2.970	$\lambda - \lambda_{ss}$	0.001	0.768	0.134	2.973
	(0.009)	(0.027)	(0.230)	(0.406)		(0.009)	(0.027)	(0.231)	(0.408)
$\pi - \pi_{ss}$	0.000	0.207	0.040	3.002	$\pi - \pi_{ss}$	0.000	0.206	0.039	3.001
	(0.000)	(0.013)	(0.046)	(0.085)		(0.000)	(0.013)	(0.046)	(0.086)
R-Rss	0.000	0.023	0.013	2.996	R-Rss	0.000	0.023	0.087	3.011
	(0.000)	(0.001)	(0.105)	(0.177)		(0.000)	(0.001)	(0.106)	(0.188)
$\frac{w}{p} - \frac{w}{p}_{ss}$	0.001	1.511	0.116	2.961	$\frac{w}{p} - \frac{w}{p}_{ss}$	0.003	1.508	0.118	2.964
	(0.017)	(0.044)	(0.232)	(0.422)		(0.017)	(0.044)	(0.234)	(0.427)

Table 15: Cross Correlation Functions of output with the other variables. Brock and Mirman model. Case6, T=3000. Numbers are means and standard deviations (in brackets) across simulations.

LQA	-4	-3	-2	-1	0	1	2	3	4
y	0.910	0.932	0.954	0.977	1.000	0.977	0.954	0.932	0.910
	(0.016)	(0.012)	(0.008)	(0.004)	(0.000)	(0.004)	(0.008)	(0.012)	(0.016)
k	0.880	0.869	0.856	0.842	0.825	0.806	0.788	0.771	0.753
	(0.017)	(0.019)	(0.021)	(0.024)	(0.026)	(0.030)	(0.033)	(0.036)	(0.039)
x	0.808	0.843	0.880	0.918	0.958	0.936	0.914	0.892	0.872
	(0.021)	(0.016)	(0.012)	(0.006)	(0.003)	(0.005)	(0.009)	(0.012)	(0.016)
c	0.934	0.939	0.944	0.947	0.950	0.928	0.907	0.886	0.865
	(0.010)	(0.009)	(0.008)	(0.007)	(0.007)	(0.010)	(0.014)	(0.017)	(0.021)
R	-0.317	-0.269	-0.218	-0.162	-0.102	-0.101	-0.100	-0.099	-0.098
	(0.081)	(0.087)	(0.093)	(0.100)	(0.107)	(0.107)	(0.106)	(0.106)	(0.105)
SIM	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
y	0.910	0.932	0.954	0.977	1.000	0.977	0.954	0.932	0.910
	(0.016)	(0.012)	(0.008)	(0.004)	(0.000)	(0.004)	(0.008)	(0.012)	(0.016)
k	0.880	0.869	0.856	0.842	0.825	0.806	0.788	0.771	0.753
	(0.017)	(0.019)	(0.021)	(0.024)	(0.026)	(0.030)	(0.033)	(0.036)	(0.039)
x	0.808	0.843	0.880	0.918	0.958	0.936	0.914	0.892	0.872
	(0.021)	(0.016)	(0.012)	(0.006)	(0.003)	(0.005)	(0.009)	(0.012)	(0.016)
c	0.934	0.939	0.944	0.947	0.950	0.928	0.907	0.886	0.865
	(0.010)	(0.009)	(0.008)	(0.007)	(0.007)	(0.010)	(0.014)	(0.017)	(0.021)
R	-0.317	-0.269	-0.218	-0.162	-0.102	-0.101	-0.100	-0.099	-0.098
	(0.081)	(0.087)	(0.093)	(0.100)	(0.107)	(0.107)	(0.106)	(0.106)	(0.105)
UHL	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
y	0.910	0.931	0.954	0.977	1.000	0.977	0.954	0.931	0.910
	(0.016)	(0.012)	(0.008)	(0.004)	(0.000)	(0.004)	(0.008)	(0.012)	(0.016)
k	0.879	0.868	0.855	0.840	0.824	0.805	0.787	0.769	0.751
	(0.017)	(0.019)	(0.021)	(0.023)	(0.026)	(0.029)	(0.032)	(0.035)	(0.038)
x	0.810	0.845	0.882	0.920	0.960	0.937	0.915	0.893	0.872
	(0.021)	(0.016)	(0.011)	(0.006)	(0.001)	(0.005)	(0.009)	(0.013)	(0.016)
c	0.936	0.940	0.945	0.948	0.951	0.929	0.907	0.887	0.866
	(0.010)	(0.009)	(0.008)	(0.006)	(0.006)	(0.010)	(0.013)	(0.017)	(0.020)
R	-0.301	-0.252	-0.199	-0.142	-0.081	-0.081	-0.080	-0.079	-0.079
	(0.072)	(0.078)	(0.084)	(0.090)	(0.097)	(0.097)	(0.097)	(0.097)	(0.096)
PEA	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
y	0.909	0.931	0.953	0.976	1.000	0.976	0.953	0.931	0.909
	(0.016)	(0.012)	(0.008)	(0.004)	(0.000)	(0.004)	(0.008)	(0.012)	(0.016)
k	0.877	0.866	0.853	0.839	0.822	0.803	0.785	0.767	0.749
	(0.017)	(0.019)	(0.021)	(0.023)	(0.026)	(0.030)	(0.033)	(0.036)	(0.039)
x	0.811	0.846	0.882	0.921	0.961	0.938	0.915	0.893	0.872
	(0.021)	(0.016)	(0.011)	(0.006)	(0.001)	(0.005)	(0.009)	(0.012)	(0.016)
c	0.934	0.939	0.944	0.947	0.950	0.928	0.907	0.886	0.865
	(0.010)	(0.009)	(0.007)	(0.006)	(0.006)	(0.009)	(0.013)	(0.017)	(0.020)
R	-0.292	-0.243	-0.190	-0.133	-0.072	-0.071	-0.071	-0.071	-0.070
	(0.073)	(0.078)	(0.084)	(0.091)	(0.098)	(0.097)	(0.097)	(0.097)	(0.097)

Table 16: Cross Correlation Functions of output with the other variables. Hansen model. Case6, T=3000. Numbers are means and standard deviations (in brackets) across simulations.

LQA	-4	-3	-2	-1	0	1	2	3	4
y	0.757	0.812	0.870	0.933	1.000	0.933	0.870	0.812	0.757
	(0.022)	(0.017)	(0.012)	(0.006)	(0.000)	(0.006)	(0.012)	(0.017)	(0.022)
k	0.755	0.734	0.708	0.676	0.638	0.593	0.552	0.513	0.477
	(0.015)	(0.017)	(0.020)	(0.023)	(0.027)	(0.032)	(0.036)	(0.040)	(0.044)
x	0.650	0.720	0.797	0.879	0.969	0.905	0.845	0.788	0.736
	(0.023)	(0.018)	(0.012)	(0.007)	(0.003)	(0.006)	(0.012)	(0.017)	(0.022)
c	0.816	0.821	0.824	0.825	0.824	0.767	0.713	0.664	0.617
	(0.013)	(0.011)	(0.011)	(0.012)	(0.015)	(0.018)	(0.022)	(0.026)	(0.030)
R	-0.136	-0.051	0.044	0.150	0.268	0.252	0.237	0.223	0.209
	(0.053)	(0.057)	(0.062)	(0.068)	(0.074)	(0.073)	(0.073)	(0.072)	(0.072)
N	-0.425	-0.358	-0.281	-0.194	-0.097	-0.088	-0.080	-0.073	-0.066
	(0.045)	(0.050)	(0.057)	(0.064)	(0.071)	(0.072)	(0.073)	(0.074)	(0.075)

SIM	-4	-3	-2	-1	0	1	2	3	4
y	0.756	0.811	0.870	0.933	1.000	0.933	0.870	0.811	0.756
	(0.023)	(0.018)	(0.012)	(0.006)	(0.000)	(0.006)	(0.012)	(0.018)	(0.023)
k	0.757	0.735	0.709	0.677	0.639	0.595	0.553	0.514	0.478
	(0.015)	(0.017)	(0.020)	(0.023)	(0.027)	(0.032)	(0.036)	(0.041)	(0.045)
x	0.649	0.720	0.797	0.881	0.971	0.906	0.845	0.789	0.736
	(0.023)	(0.018)	(0.012)	(0.006)	(0.002)	(0.006)	(0.012)	(0.017)	(0.022)
c	0.825	0.829	0.830	0.829	0.825	0.768	0.716	0.666	0.620
	(0.012)	(0.010)	(0.009)	(0.009)	(0.009)	(0.015)	(0.020)	(0.025)	(0.030)
R	-0.135	-0.051	0.044	0.150	0.268	0.252	0.237	0.222	0.209
	(0.052)	(0.056)	(0.062)	(0.068)	(0.074)	(0.074)	(0.073)	(0.073)	(0.073)
N	-0.426	-0.359	-0.282	-0.195	-0.097	-0.089	-0.081	-0.074	-0.067
	(0.045)	(0.050)	(0.057)	(0.064)	(0.072)	(0.073)	(0.074)	(0.075)	(0.076)

UHL	-4	-3	-2	-1	0	1	2	3	4
y	0.757	0.812	0.871	0.933	1.000	0.933	0.871	0.812	0.757
	(0.023)	(0.018)	(0.012)	(0.006)	(0.000)	(0.006)	(0.012)	(0.018)	(0.023)
k	0.757	0.735	0.709	0.677	0.639	0.594	0.553	0.514	0.477
	(0.015)	(0.017)	(0.020)	(0.023)	(0.027)	(0.032)	(0.036)	(0.041)	(0.045)
x	0.645	0.715	0.792	0.875	0.965	0.901	0.840	0.784	0.731
	(0.023)	(0.018)	(0.012)	(0.006)	(0.003)	(0.006)	(0.012)	(0.017)	(0.023)
c	0.825	0.829	0.831	0.830	0.826	0.769	0.717	0.667	0.621
	(0.012)	(0.010)	(0.009)	(0.008)	(0.009)	(0.014)	(0.020)	(0.025)	(0.030)
R	-0.129	-0.044	0.052	0.159	0.278	0.261	0.245	0.231	0.217
	(0.050)	(0.054)	(0.059)	(0.065)	(0.071)	(0.070)	(0.070)	(0.070)	(0.070)
N	-0.425	-0.358	-0.281	-0.195	-0.097	-0.089	-0.081	-0.073	-0.066
	(0.044)	(0.050)	(0.056)	(0.064)	(0.072)	(0.073)	(0.073)	(0.074)	(0.075)

PEA	-4	-3	-2	-1	0	1	2	3	4
y	0.762	0.815	0.873	0.935	1.000	0.935	0.873	0.815	0.762
	(0.022)	(0.017)	(0.012)	(0.006)	(0.000)	(0.006)	(0.012)	(0.017)	(0.022)
k	0.757	0.735	0.709	0.678	0.640	0.596	0.555	0.517	0.481
	(0.015)	(0.017)	(0.020)	(0.023)	(0.027)	(0.032)	(0.037)	(0.041)	(0.045)
x	0.658	0.727	0.803	0.884	0.972	0.909	0.850	0.794	0.742
	(0.023)	(0.018)	(0.012)	(0.006)	(0.002)	(0.006)	(0.012)	(0.017)	(0.022)
c	0.827	0.832	0.834	0.834	0.832	0.776	0.724	0.675	0.629
	(0.012)	(0.010)	(0.009)	(0.008)	(0.008)	(0.014)	(0.019)	(0.024)	(0.029)
R	-0.130	-0.046	0.048	0.153	0.269	0.254	0.239	0.225	0.212
	(0.050)	(0.055)	(0.060)	(0.065)	(0.072)	(0.071)	(0.071)	(0.071)	(0.071)
N	-0.430	-0.364	-0.289	-0.205	-0.110	-0.101	-0.092	-0.084	-0.076
	(0.045)	(0.050)	(0.057)	(0.064)	(0.072)	(0.073)	(0.074)	(0.075)	(0.076)

Table 17: Cross Correlation Functions of output with the other variables. Cooley and Hansen model. Case1, T=3000. Numbers are means and standard deviations (in brackets) across simulations.

LQA	-4	-3	-2	-1	0	1	2	3	4
y	0.823	0.864	0.907	0.952	1.000	0.952	0.907	0.864	0.823
	(0.021)	(0.016)	(0.011)	(0.006)	(0.000)	(0.006)	(0.011)	(0.016)	(0.021)
k	0.862	0.847	0.827	0.801	0.770	0.732	0.696	0.662	0.630
	(0.013)	(0.015)	(0.017)	(0.020)	(0.024)	(0.029)	(0.033)	(0.038)	(0.042)
N	0.391	0.470	0.557	0.652	0.757	0.721	0.688	0.655	0.625
	(0.020)	(0.017)	(0.017)	(0.018)	(0.021)	(0.020)	(0.019)	(0.020)	(0.022)
$\hat{p}$	-0.888	-0.888	-0.885	-0.879	-0.870	-0.827	-0.787	-0.748	-0.712
	(0.012)	(0.011)	(0.011)	(0.011)	(0.012)	(0.017)	(0.022)	(0.027)	(0.032)
π	0.008	0.001	-0.009	-0.020	-0.032	-0.139	-0.128	-0.120	-0.114
	(0.024)	(0.024)	(0.024)	(0.024)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)
R	0.013	0.097	0.191	0.295	0.411	0.392	0.374	0.357	0.341
	(0.037)	(0.041)	(0.046)	(0.051)	(0.058)	(0.056)	(0.055)	(0.064)	(0.054)

SIM	-4	-3	-2	-1	0	1	2	3	4
y	0.825	0.866	0.908	0.953	1.000	0.953	0.908	0.866	0.825
	(0.022)	(0.017)	(0.012)	(0.006)	(0.000)	(0.006)	(0.012)	(0.017)	(0.022)
k	0.862	0.846	0.826	0.802	0.771	0.734	0.699	0.666	0.634
	(0.013)	(0.015)	(0.017)	(0.020)	(0.025)	(0.030)	(0.034)	(0.039)	(0.043)
N	0.401	0.479	0.564	0.658	0.762	0.727	0.694	0.663	0.633
	(0.024)	(0.020)	(0.017)	(0.017)	(0.019)	(0.018)	(0.019)	(0.021)	(0.023)
$\hat{p}$	-0.889	-0.889	-0.886	-0.880	-0.871	-0.829	-0.790	-0.752	-0.716
	(0.012)	(0.011)	(0.011)	(0.011)	(0.012)	(0.018)	(0.023)	(0.028)	(0.033)
π	0.008	0.000	-0.009	-0.020	-0.032	-0.137	-0.127	-0.119	-0.113
	(0.023)	(0.024)	(0.024)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)
R	-0.012	0.070	0.162	0.264	0.377	0.360	0.344	0.328	0.313
	(0.053)	(0.058)	(0.064)	(0.072)	(0.080)	(0.077)	(0.075)	(0.073)	(0.071)

UHL	-4	-3	-2	-1	0	1	2	3	4
y	0.822	0.863	0.907	0.952	1.000	0.952	0.907	0.863	0.822
	(0.021)	(0.016)	(0.011)	(0.006)	(0.000)	(0.006)	(0.011)	(0.016)	(0.021)
k	0.862	0.846	0.826	0.800	0.769	0.731	0.695	0.661	0.628
	(0.013)	(0.015)	(0.017)	(0.020)	(0.024)	(0.029)	(0.033)	(0.037)	(0.041)
N	0.390	0.470	0.557	0.653	0.759	0.723	0.689	0.657	0.626
	(0.015)	(0.013)	(0.012)	(0.014)	(0.019)	(0.018)	(0.017)	(0.019)	(0.020)
$\hat{p}$	-0.886	-0.886	-0.883	-0.877	-0.867	-0.825	-0.784	-0.746	-0.709
	(0.011)	(0.011)	(0.010)	(0.011)	(0.012)	(0.017)	(0.022)	(0.027)	(0.031)
π	0.008	0.000	-0.009	-0.020	-0.032	-0.138	-0.128	-0.120	-0.114
	(0.024)	(0.024)	(0.024)	(0.024)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)
R	0.014	0.099	0.193	0.298	0.414	0.395	0.377	0.360	0.344
	(0.036)	(0.040)	(0.045)	(0.050)	(0.056)	(0.055)	(0.054)	(0.053)	(0.053)

PEA	-4	-3	-2	-1	0	1	2	3	4
y	0.823	0.865	0.908	0.953	1.000	0.953	0.908	0.865	0.823
	(0.021)	(0.016)	(0.011)	(0.006)	(0.000)	(0.006)	(0.011)	(0.016)	(0.021)
k	0.861	0.846	0.826	0.800	0.769	0.732	0.696	0.662	0.630
	(0.013)	(0.015)	(0.017)	(0.020)	(0.024)	(0.029)	(0.033)	(0.037)	(0.041)
N	0.396	0.475	0.562	0.657	0.761	0.726	0.692	0.660	0.630
	(0.015)	(0.013)	(0.012)	(0.014)	(0.018)	(0.017)	(0.017)	(0.018)	(0.020)
$\hat{p}$	-0.886	-0.886	-0.883	-0.877	-0.868	-0.826	-0.786	-0.748	-0.711
	(0.011)	(0.011)	(0.010)	(0.011)	(0.012)	(0.017)	(0.022)	(0.026)	(0.031)
π	0.008	0.000	-0.009	-0.020	-0.032	-0.138	-0.127	-0.119	-0.113
	(0.024)	(0.024)	(0.024)	(0.024)	(0.024)	(0.023)	(0.023)	(0.023)	(0.023)
R	0.010	0.094	0.188	0.292	0.407	0.389	0.372	0.355	0.339
	(0.037)	(0.041)	(0.046)	(0.051)	(0.057)	(0.056)	(0.055)	(0.054)	(0.054)

Table 18: Estimated autocorrelations for selected variables. Hansen model. Case 6, T=3000. AR(1), AR(2) and AR(3) processes. Means and standard deviations (in brackets) across simulations

LQA	y	c	R	y	c	R	SIM
μ	0.0001	0.0000	0.0000	0.0001	0.0000	0.0000	μ
	(0.0006)	(0.0001)	(0.0000)	(0.0006)	(0.0001)	(0.0000)	
ρ_1	0.9332	0.9846	0.9297	0.9330	0.9891	0.9303	ρ_1
	(0.0063)	(0.0026)	(0.0067)	(0.0064)	(0.0017)	(0.0071)	
μ	0.0001	0.0000	0.0000	0.0001	0.0000	0.0000	μ
	(0.0006)	(0.0001)	(0.0000)	(0.0006)	(0.0001)	(0.0000)	
ρ_1	0.9353	1.0402	0.9321	0.9347	1.0779	0.9322	ρ_1
	(0.0187)	(0.0290)	(0.0186)	(0.0188)	(0.0213)	(0.0188)	
ρ_2	-0.0022	-0.0564	-0.0025	-0.0018	-0.0899	-0.0020	ρ_2
	(0.0186)	(0.0286)	(0.0185)	(0.0188)	(0.0211)	(0.0186)	
μ	0.0001	0.0000	0.0000	0.0001	0.0000	0.0000	μ
	(0.0006)	(0.0001)	(0.0000)	(0.0006)	(0.0001)	(0.0000)	
ρ_1	0.9353	1.0374	0.9321	0.9347	1.0712	0.9322	ρ_1
	(0.0186)	(0.0274)	(0.0186)	(0.0187)	(0.0194)	(0.0187)	
ρ_2	0.0006	-0.0072	0.0007	0.0006	-0.0098	0.0008	ρ_2
	(0.0249)	(0.0364)	(0.0247)	(0.0250)	(0.0255)	(0.0247)	
ρ_3	-0.0029	-0.0472	-0.0035	-0.0026	-0.0743	-0.0031	ρ_3
	(0.0177)	(0.0261)	(0.0177)	(0.0178)	(0.0189)	(0.0177)	
UHL	**y**	**c**	**R**	**y**	**c**	**R**	**PEA**
μ	0.0003	0.0000	0.0000	0.0004	0.0000	0.0000	μ
	(0.0006)	(0.0001)	(0.0000)	(0.0006)	(0.0001)	(0.0000)	
ρ_1	0.9333	0.9890	0.9289	0.9347	0.9887	0.9306	ρ_1
	(0.0064)	(0.0017)	(0.0066)	(0.0063)	(0.0018)	(0.0065)	
μ	0.0003	0.0000	0.0000	0.0004	0.0000	0.0000	μ
	(0.0006)	(0.0001)	(0.0000)	(0.0006)	(0.0001)	(0.0000)	
ρ_1	0.9351	1.0769	0.9317	0.9366	1.0700	0.9328	ρ_1
	(0.0186)	(0.0210)	(0.0185)	(0.0186)	(0.0208)	(0.0186)	
ρ_2	-0.0019	-0.0889	-0.0030	-0.0020	-0.0823	-0.0024	ρ_2
	(0.0186)	(0.0209)	(0.0184)	(0.0186)	(0.0207)	(0.0185)	
μ	0.0003	0.0000	0.0000	0.0004	0.0000	0.0000	μ
	(0.0006)	(0.0001)	(0.0000)	(0.0006)	(0.0001)	(0.0000)	
ρ_1	0.9351	1.0703	0.9318	0.9366	1.0643	0.9328	ρ_1
	(0.0185)	(0.0192)	(0.0184)	(0.0186)	(0.0192)	(0.0185)	
ρ_2	0.0007	-0.0096	0.0004	0.0006	-0.0086	0.0007	ρ_2
	(0.0248)	(0.0256)	(0.0246)	(0.0249)	(0.0255)	(0.0247)	
ρ_3	-0.0028	-0.0737	-0.0037	-0.0028	-0.0689	-0.0034	ρ_3
	(0.0177)	(0.0188)	(0.0177)	(0.0177)	(0.0187)	(0.0177)	

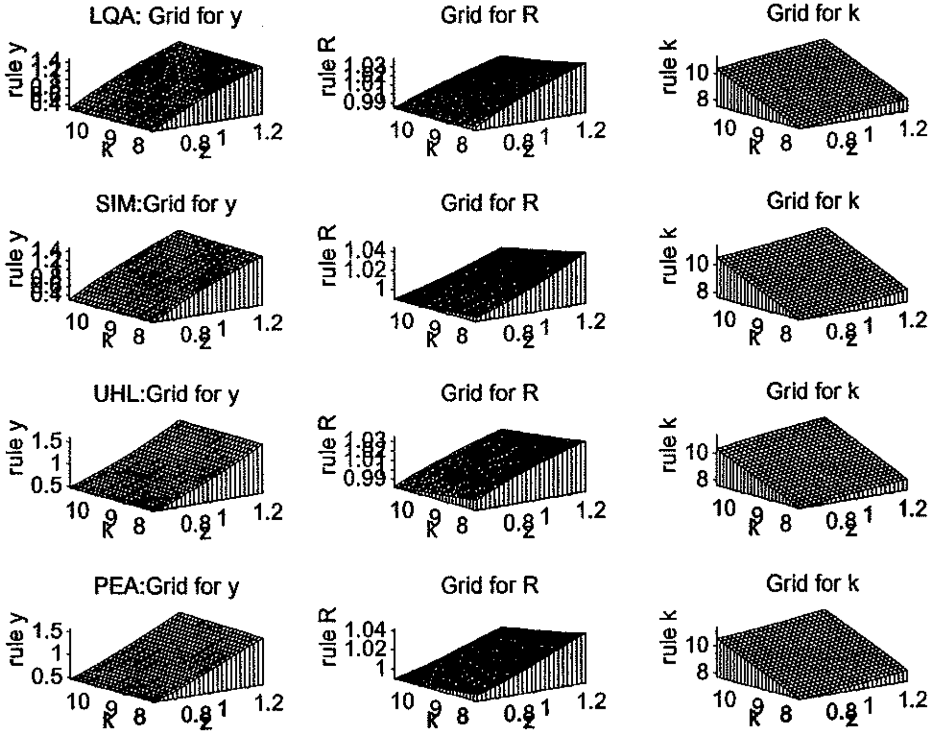

Figure 7: Grid for the decision rules. $g = g_{ss}$. Cooley and Hansen Model. Case 4.

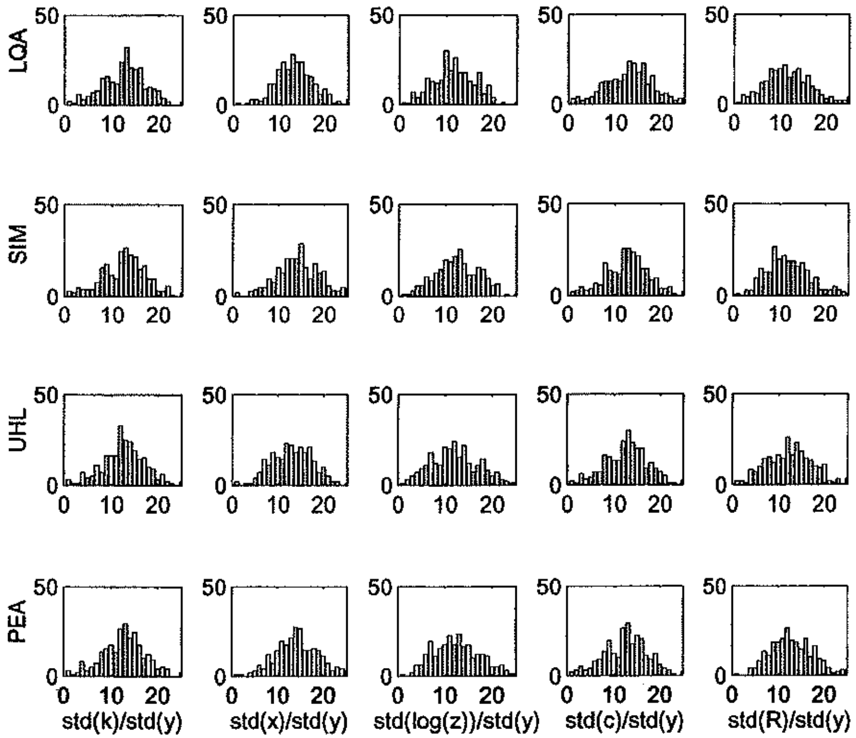

Figure 8: Empirical distribution of the relative standard deviation statistics. Hansen Model. Case 6, T=3000.

Appendix

A Description of the models and keys to solution

A.1 Brock and Mirman (1972)

A.1.1 Description of the model

- **The Problem:** The model is that presented in (1).
- **Lagrangean:**

$$L(k_t, c_t, \lambda_t) = E_0 \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\eta} - 1}{1-\eta} - \lambda_t \left(c_t + k_t - z_t k_{t-1}^{\alpha} - (1-\delta) k_{t-1} \right) \right]$$

- **First Order Conditions:** We need the Euler conditions in order to obtain the deterministic steady state and because the methods SIM, UHL and PE are Euler–equations based methods. The system of optimality conditions that emerge from problem (1) is obtained by deriving the Lagrangean with respect to the endogenous state and decisions,

$$\begin{aligned} \lambda_t:\ 0 &= c_t + k_t - z_t k_{t-1}^{\alpha} - (1-\delta) k_{t-1} \\ c_t:\ 0 &= c_t^{-\eta} - \lambda_t \\ k_t:\ 0 &= -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta \right) \right] \end{aligned}$$

then, eliminating λ_t, we have,

$$c_t = z_t k_{t-1}^{\alpha} + x_t \quad (7)$$

$$R_t = \alpha z_t k_{t-1}^{\alpha-1} + 1 - \delta \quad (8)$$

$$c_t^{-\eta} = \beta E_t \left[c_{t+1}^{-\eta} R_{t+1} \right] \quad (9)$$

and

$$k_t = x_t + (1-\delta) k_{t-1} \quad (10)$$

$$\log(z_t) = (1-\rho) \log(z_{ss}) + \rho \log(z_{t-1}) + \epsilon_t \quad (11)$$

$$\epsilon_t \sim N(0, \sigma_\epsilon^2)$$

- **Stochastic Euler equation residual:** in this model is defined as

$$\xi_{t+1} = c_t^{-\eta} - \beta c_{t+1}^{-\eta} R_{t+1}$$

- **Deterministic Steady State:** We have for interest rate $R_{ss} = \frac{1}{\beta}$, for technology, output and consumption: $z_{ss} = 1$, $y_{ss} = z_{ss} k_{ss}^{\alpha}$ and $c_{ss} = y_{ss} - \delta k_{ss}$, where $k_{ss} = \left(\frac{\alpha z_{ss}}{R_{ss} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$.
- **Parameter Values:** See table in the main text.

A.1.2 Linear–quadratic approximation: value function iteration

- **Return Function:** Substitute the nonlinear constraint $c_t = z_t k_{t-1}^{\alpha} - x_t$ into the utility function, to obtain

$$U(x_t, k_{t-1}, \log(z_t)) = \begin{cases} \log(z_t k_{t-1}^{\alpha} - x_t) & if \quad \eta = 1 \\ \frac{(z_t k_{t-1}^{\alpha} - x_t)^{1-\eta} - 1}{1-\eta} & if \quad \eta \neq 1 \end{cases}$$

  We will consider one exogenous state variable, $\log(z_t)$, one endogenous state variable, k_{t-1}, and one decision variable x_t. Now we will perform a second order Taylor approximation of the return function about the deterministic steady state at the point $(\log(z_{ss}), k_{ss}, x_{ss})$.

  - **First order derivatives**

$$DU_{ss} = \left[\frac{\partial U}{\partial \log(z_t)}, \frac{\partial U}{\partial k_{t-1}}, \frac{\partial U}{\partial x_t}, \right] |_{ss} = \left[c_t^{-\eta} z_t k_{t-1}^{\alpha}, c_t^{-\eta} \alpha z_t k_{t-1}^{\alpha-1}, -c_t^{-\eta}, \right] |_{ss}$$

  - **Second order derivatives:** The hessian

$$D^2 U_{ss} = \begin{bmatrix} \frac{\partial^2 U}{\partial (\log(z_t))^2} & & \\ \frac{\partial^2 U}{\partial \log(z_t) \partial k_{t-1}} & \frac{\partial^2 U}{\partial k_{t-1}^2} & \\ \frac{\partial^2 U}{\partial \log(z_t) \partial x_t} & \frac{\partial^2 U}{\partial k_{t-1} \partial x_t} & \frac{\partial^2 U}{\partial x_t^2} \end{bmatrix} |_{ss}$$

  where,

$$\begin{aligned} \frac{\partial^2 U}{\partial \log(z_t)^2} &= \left[-\eta c_t^{-\eta-1} y_t^2 + c_t^{-\eta} z_t k_{t-1}^{\alpha} \right] \\ \frac{\partial^2 U}{\partial \log(z_t) k_{t-1}} &= \left[-\eta c_t^{-\eta-1} \alpha \frac{y_t^2}{k_{t-1}} + c_t^{-\eta} \alpha \frac{y_t}{k_{t-1}} \right] \\ \frac{\partial^2 U}{\partial \log(z_t) x_t} &= \eta c_t^{-\eta-1} y_t \\ \frac{\partial^2 U}{\partial k_{t-1}^2} &= \left[-\eta \alpha^2 c_t^{-\eta-1} \frac{y_t^2}{k_{t-1}^2} + c_t^{-\eta} \alpha (\alpha - 1) \frac{y_t}{k_{t-1}^2} \right] \\ \frac{\partial^2 U}{\partial x_t^2} &= -\eta c_t^{-\eta-1} \\ \frac{\partial^2 U}{\partial x_t k_{t-1}} &= \eta c_t^{-\eta-1} \alpha \frac{y_t}{k_{t-1}} \end{aligned}$$

  - We can write now the return function. Using the obtained gradient and hessian, and given that $W_t = [\log(z_t), k_{t-1}, x_t]^T$ the return function

$$r(\log(z_t), k_{t-1}, x_t) \approx Q_{11} + 2 \cdot Q_{12} \cdot W_t + W_t^T \cdot Q_{22} \cdot W_t = \left[1 \ W_t^T \right] \cdot Q \cdot \begin{bmatrix} 1 \\ W_t \end{bmatrix}$$

where $Q_{11} = U_{ss} - DU_{ss} \cdot W_{ss} + \frac{1}{2} W_{ss}^T \cdot D^2 U_{ss} \cdot W_{ss}$, $Q_{12} = \frac{1}{2}\left(DU_{ss} - W_{ss}^T \cdot D^2 U_{ss}\right)$ and $Q_{22} = \frac{1}{2} D^2 U_{ss}$.

– Initial value function conjecture. We have, $V^0(\log(z_t), k_{t-1}, x_t) = F_t^T \cdot P^0 \cdot F_t$ $= [1\ W_t]\, B^T P^0 B \begin{bmatrix} 1 \\ W \end{bmatrix}$. In this case $F_t = [1, \log(z_t), k_{t-1}]^T$ and

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 1-\delta & 1 \end{bmatrix}, \quad P^0 = -0.1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

– Rearrange the problem such that

$$V^{n+1}(\log(z_t, k_{t-1}, x_t)) = \underbrace{MAX}_{x_t}\{F^T M_1 F + 2F M_2 x_t + x_t^2 M_3\}$$

with M_1, M_2 and M_3 appropriate matrices. Then, the first order condition $x_t = -M_3^{-1} M_2 F$ is the policy function once convergence is achieved. The actualization in iteration n is $P^n = M_1 - M_2 M_3^{-1} M_2^T$.

- **Policy function:** The output of the algorithm described is, in this case, $x_t = \delta_{x0} + \delta_{xz}\log(z_t) + \delta_{xk} k_{t-1}$, that together with $x_t = k_t - (1-\delta)k_{t-1}$, produces the following decision rule for capital,

$$k_t = \delta_{x0} + \delta_{xz}\log(z_t) + (\delta_{xk} + 1 - \delta)k_{t-1}$$

Estimates of these parameters for the nine parametric cases are shown in the next table. Note that cases with all the same parameter values but standard deviation of the shock have, obviously, the same values.

CASE	δ_{x0}	δ_{xz}	δ_{xk}
1,4,7	1.9190	3.2243	-0.0255
2,5,8	1.0512	2.7668	-0.0027
3,6,9	0.7015	2.7244	0.0065

- **Time series generation:** Once the policy function is calculated, time series are generated using: $\{z_t\}$, $k_0 = k_{ss}$, investment from the decision rule, capital from its law of motion, output from the production function and the interest rate from (8). Using output and investment, consumption arises from the resource constraint.

A.1.3 Forward solution

- **System to be linearized:** From (9), substituting the conditional expectation by its realized value plus an expectational error, we have

$$0 = -c_t^{-\eta} + \beta c_{t+1}^{-\eta}\left(\alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta\right) + \xi_{t+1}, \quad E_t(\xi_{t+1}) = 0 \qquad (12)$$

- Then, the system to be linearized is formed by equation (12), (10) and (11). The states of the problem are k_{t-1} and $\log(z_t)$; the control (decision variable) is x_t. Then, we form the system $\Gamma_0 \tilde{Y}_{t+1} = \Gamma_1 \tilde{Y}_t + \Phi\zeta_{t+1}$ where $\tilde{Y}_{t+1} = [k_t - k_{ss},\ \log(z_{t+1}), x_{t+1} - x_{ss}]^T$ and $\zeta_{t+1} = [\xi_{t+1}, \epsilon_{t+1}]^T$.
- **Linearization about the Steady State**
  – Equation (12): Derivatives with respect to $\tilde{Y}_{t+1}$ and $\tilde{Y}_t$ elements,

$$\frac{\partial(12)}{\partial k_t}|ss = \left[-\beta\eta c_{t+1}^{-\eta-1}\frac{y_{t+1}}{k_t}\alpha R_{t+1} + \beta\alpha(\alpha-1)c_{t+1}^{-\eta}\frac{y_{t+1}}{k_t^2}\right]|ss$$

$$\frac{\partial(12)}{\partial\log(z_{t+1})}|ss = \left[-\beta\eta c_{t+1}^{-\eta-1} y_{t+1} R_{t+1} + \beta\alpha c_{t+1}^{-\eta}\frac{y_{t+1}}{k_t}\right]|ss$$

$$\frac{\partial(12)}{\partial x_{t+1}}|ss = \left[\beta\eta c_{t+1}^{-\eta-1} R_{t+1}\right]|ss$$

$$\frac{\partial(12)}{\partial k_{t-1}}|ss = \left[\eta c_t^{-\eta-1}\alpha\frac{y_t}{k_{t-1}}\right]|ss$$

$$\frac{\partial(12)}{\partial\log(z_t)}|ss = \left[\eta c_t^{-\eta-1} y_t\right]|ss, \ \frac{\partial(12)}{\partial x_t}|ss = \left[-\eta c_t^{-\eta-1}\right]|ss$$

  – Equation (10): Derivatives with respect to $\tilde{y}_{t+1}$ and to $\tilde{y}_t$

$$\frac{\partial(10)}{\partial k_t}|ss = 1, \ \frac{\partial(10)}{\partial\log(z_{t+1})}|ss = 0\ \frac{\partial(10)}{\partial x_{t+1}}|ss = 0,$$

$$\frac{\partial(10)}{\partial k_{t-1}}|ss = -(1-\delta), \ \frac{\partial(10)}{\partial\log(z_t)}|ss = 0, \ \frac{\partial(10)}{\partial x_t}|ss = -1.$$

  – Equation (11): Derivatives with respect to y_{t+1}, and to $\tilde{y}_t$.

$$\frac{\partial(11)}{\partial k_t}|ss = 0, \ \frac{\partial(11)}{\partial\log(z_{t+1})}|ss = -1, \ \frac{\partial(11)}{\partial x_{t+1}}|ss = 0$$

$$\frac{\partial(11)}{\partial k_{t-1}}|ss = 0, \ \frac{\partial(11)}{\partial\log(z_t)}|ss = \rho, \ \frac{\partial(11)}{\partial x_t}|ss = 0$$

- Thus, we have $\Gamma_0 = \begin{pmatrix} \frac{\partial(12)}{\partial\tilde{y}_{t+1}} & \frac{\partial(10)}{\partial\tilde{y}_{t+1}} & \frac{\partial(11)}{\partial\tilde{y}_{t+1}} \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} \frac{\partial(12)}{\partial\tilde{y}_t} & \frac{\partial(10)}{\partial\tilde{y}_t} & \frac{\partial(11)}{\partial\tilde{y}_t} \end{pmatrix}$. Now, we can calculate the stability condition of the linearized system.
- **Policy function:** from the stability condition $\mu_1(k_{t-1} - k_{ss}) + \mu_2\log(z_t) + \mu_3(x_t - x_{ss}) = 0$, and the law of motion for capital in deviations with respect to the steady state, the decision rule for capital becomes,

$$k_t = (x_{ss} + \frac{\mu_1}{\mu_3}k_{ss}) + (\frac{-\mu_2}{\mu_3})\log(z_t) + (1 - \delta - \frac{\mu_1}{\mu_3})k_{t-1}$$

The values of the stability condition weights are shown in the next table. In this simple problem the policy rules obtained from SIM are exactly equal to those induced by LQA.

CASE	μ_1	μ_2	μ_3
1,4,7	0.2240	-28.3064	8.7791
2,5,8	0.0402	-41.6783	15.0639
3,6,9	-0.1375	-57.3516	21.0514

- **Time series generation:** Use the linear stability condition to obtain investment, capital from its law of motion, output from the production function and the interest rate from (8). Using output and investment, consumption arises from the resource constraint. In this case, the decision rule is exactly the same than that obtained by LQA.

A.1.4 Log–linearization

The system to be log–linearized is formed by the first order conditions (7) to (9) and the constraints of problem (1). Let ˜ denote log–deviations from the steady state. The log–linearization produces the following system of equations,

- **Log–linearization**

$$\left.\begin{array}{rcl} 0 &=& -(1-\beta(1-\delta))(1-\alpha)\tilde{k}_{t-1}+(1-\beta(1-\delta))\tilde{z}_t-\tilde{R}_t \\ 0 &=& \frac{-k_{ss}}{c_{ss}}\tilde{k}_t+\frac{k_{ss}}{\beta c_{ss}}\tilde{k}_{t-1}+\frac{y_{ss}}{c_{ss}}\tilde{z}_t-\tilde{c}_t \\ 0 &=& -\tilde{y}_t+\tilde{z}_t+\alpha\tilde{k}_{t-1} \\ 0 &=& E_t\left[-\eta\tilde{c}_{t+1}+\tilde{R}_{t+1}+\eta\tilde{c}_t\right] \\ \tilde{z}_{t+1} &=& \rho\tilde{z}_t+\epsilon_{t+1} \end{array}\right\}$$

- **General form**

  In this particular example, in terms of the general notation we are using, we have: x is capital, y is formed by consumption, output, labor, the interest rate and investment, and z is technology. The matrices for the log–linear approximation are:

$$A = \begin{bmatrix} 0 \\ -k_{ss}/c_{ss} \\ 0 \end{bmatrix}, \; B = \begin{bmatrix} -(1-\beta(1-\delta))(1-\alpha) \\ k_{ss}/(\beta c_{ss}) \\ \alpha \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; D = \begin{bmatrix} 1-\beta(1-\delta) \\ y_{ss}/c_{ss} \\ 1 \end{bmatrix}$$

$$\begin{array}{rclrclrcl} F &=& [0], & G &=& [0], & H &=& [0], \\ J &=& [-\eta,1,0], & K &=& [\eta,0,0], & L &=& [0], \\ M &=& [0], & N &=& [\rho], & Sigma &=& [\sigma_\epsilon^2] \end{array}$$

- **Policy Function and time series generation:** The output of the algorithm are the parameters of the state space form relating endogenous variables to state variables. This form is used to generate the time series. That is,

$$\begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \\ \tilde{R}_t \end{bmatrix} = \begin{bmatrix} \nu_{kk} & \nu_{kz} \\ \nu_{ck} & \nu_{cz} \\ \nu_{Rk} & \nu_{Rz} \\ \nu_{yk} & \nu_{yz} \end{bmatrix} \begin{bmatrix} \tilde{k}_{t-1} \\ \tilde{z}_t \end{bmatrix}$$

  with $k_0 = k_{ss}$ and $x_t = y_t - c_t$. For the analyzed cases, the parameter values are:

CASE	ν_{kk}	ν_{kz}	ν_{ck}	ν_{cz}	ν_{Rk}	ν_{Rz}	ν_{yk}	ν_{yz}
1,4,7	0.9495	0.0849	0.8361	0.1742	-0.0222	0.0348	0.3600	1.000
2,5,8	0.9723	0.0728	0.5210	0.3403	-0.0222	0.0348	0.3600	1.000
3,6,9	0.9815	0.0717	0.3940	0.3557	-0.0222	0.0348	0.3600	1.000

A.1.5 Parameterized Expectations

- **Order of the approximation.** For the PEA method a second order polynomial approximation proved to be useful. (Tolerance=0.0001)

$$\psi(k_{t-1}, z_t) = q_1\, e^{q_2 log(k_{t-1})+q_3 log(z_t)+q_4(log(k_{t-1}))^2+q_5 log(k_{t-1})log(z_t)+q_6(log(z_t))^2}$$

  The gradient takes the form,

$$\nabla\psi = \begin{bmatrix} \partial\psi/\partial q_1 \\ \partial\psi/\partial q_2 \\ \partial\psi/\partial q_3 \\ \partial\psi/\partial q_4 \\ \partial\psi/\partial q_5 \\ \partial\psi/\partial q_6 \end{bmatrix} = \begin{bmatrix} \psi/q_1 \\ \psi\log(k_{t-1}) \\ \psi\log(z_t) \\ \psi(\log(k_{t-1}))^2 \\ \psi\log(k_{t-1})\log(z_t) \\ \psi(\log(z_t))^2 \end{bmatrix}$$

- **Policy function:** The decision rule for capital becomes,

$$k_t = -(\beta\psi(q; k_{t-1}, z_t)))^{-1/\eta} + z_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1}.$$

  The fixed point for q was calculated using a sample size of 25000 observations.

CASE	q_1	q_2	q_3	q_4	q_5	q_6
1	1.8917	-0.2042	-0.2680	-0.0296	0.0500	-0.0454
2	1.8629	-0.3874	-1.0443	-0.0545	0.1466	-0.3106
3	0.8467	-0.3988	-2.0896	-0.1069	0.2981	-0.5457
4	1.8843	-0.2037	-0.2536	-0.0294	0.0462	-0.0348
5	1.5138	-0.2806	-0.4906	-0.0684	0.0015	-0.0492
6	1.6741	-0.7658	-3.5681	-0.0567	0.6828	-0.1533
7	2.4171	-0.3407	-0.2021	-0.0106	0.0315	-0.0207
8	2.6408	-0.5750	-1.0905	-0.0290	0.1612	-0.1231
9	2.8286	-1.0233	-2.3828	-0.0218	0.3569	-0.2553

- **Time series generation:** obtain consumption from condition (9) in which the conditional expectation is substituted by the polynomial function ψ. Then capital from $k_t = -c_t + z_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1}$, investment from the law of motion of capital and the interest rate from (8). Output may then be obtained from the resource constraint (7).

A.2 Hansen (1985)

A.2.1 Description of the model

- **The Problem:** see (3)
- **Lagrangean**

$$L(k_t, c_t, N_t, \lambda_t) = E_0 \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\eta} - 1}{1-\eta} - A_N N_t - \lambda_t \left(c_t + k_t - z_t k_{t-1}^{\alpha} N_t^{1-\alpha} - (1-\delta) k_{t-1} \right) \right]$$

- **First Order Conditions:** to obtain the deterministic steady state and for SIM, UHL and PEA methods.

$$\begin{aligned}
\lambda_t:&\quad 0 = c_t + k_t - z_t k_{t-1}^{\alpha} N_t^{1-\alpha} - (1-\delta) k_{t-1} \\
c_t:&\quad 0 = c_t^{-\eta} - \lambda_t \\
k_t:&\quad 0 = -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta \right) \right] \\
N_t:&\quad 0 = -A_N + \lambda_t z_t k_{t-1}^{\alpha} (1-\alpha) N_t^{-\alpha}
\end{aligned}$$

then, eliminating λ_t, we have,

$$\begin{aligned}
c_t &= z_t k_{t-1}^{\alpha} N_t^{1-\alpha} - k_t + (1-\delta) k_{t-1} && (13)\\
R_t &= \alpha z_t k_{t-1}^{\alpha-1} N_t^{1-\alpha} + 1 - \delta && (14)\\
c_t^{-\eta} &= \beta E_t \left[c_{t+1}^{-\eta} R_{t+1} \right] && (15)\\
A_N &= c_t^{-\eta} z_t k_{t-1}^{\alpha} (1-\alpha) N_t^{-\alpha} = c_t^{-\eta} (1-\alpha) \frac{y_t}{N_t} && (16)\\
& && (17)
\end{aligned}$$

and

$$\begin{aligned}
\log(z_t) &= (1-\rho) \log(z_{ss}) + \rho \log(z_{t-1}) + \epsilon_t && (18)\\
\epsilon_t &\sim N(0, \sigma_\epsilon^2)
\end{aligned}$$

- **Stochastic Euler equation residual:** in this case it is defined as

$$\xi_{t+1} = c_t^{-\eta} - \beta c_{t+1}^{-\eta} R_{t+1}$$

- **Deterministic Steady State:** From (14), $R_{ss} = \alpha \frac{y_{ss}}{k_{ss}} + 1 - \delta$, and from (15), $R_{ss} = \frac{1}{\beta}$, then the ratio of output to capital at the steady state becomes,

$$\frac{y_{ss}}{k_{ss}} = \frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta \right)$$

For employment, the steady state value is fixed to $\frac{1}{3}$. Now, from the production function, $y_{ss} = k_{ss}^{\alpha} N_{ss}^{1-\alpha} \iff \frac{y_{ss}}{k_{ss}} = k_{ss}^{\alpha-1} N_{ss}^{1-\alpha}$, so we have for capital, given that $z_{ss} = 1$,

$$k_{ss} = \left(\frac{y_{ss}}{k_{ss}} \right)^{\frac{1}{\alpha-1}} N_{ss}$$

From the law of motion of capital, $x_{ss} = \delta k_{ss}$, and from the resource constraint, $c_{ss} = y_{ss} - \delta k_{ss}$. Finally, from (16), $A_N = c_{ss}^{-\eta} (1-\alpha) \frac{y_{ss}}{N_{ss}}$.

- **Parameter Values:** See table in the main text.

A.2.2 Linear–quadratic approximation: value function iteration

- **Return Function:** Substitute the nonlinear constraint $c_t = z_t k_{t-1}^{\alpha} N_t^{1-\alpha} - x_t$ into the utility function, to obtain

$$U(x_t, N_t, k_{t-1}, \log(z_t)) = \begin{cases} \log(z_t k_{t-1}^{\alpha} N_t^{1-\alpha} - x_t) - A_N N_t & if \quad \eta = 1 \\ \frac{(z_t k_{t-1}^{\alpha} N_t^{1-\alpha} - x_t)^{1-\eta} - 1}{1-\eta} - A_N N_t & if \quad \eta \neq 1 \end{cases}$$

In this problem we consider one exogenous state variable, $\log(z_t)$, one endogenous state variable, k_{t-1}, and two decision variables, x_t and N_t. Now we will perform a second order Taylor approximation of the return function about the deterministic steady state at the point $(\log(z_{ss}), k_{ss}, x_{ss}, N_{ss})$.

  - **First order derivatives**

$$\begin{aligned}
DU_{ss} &= \left[\frac{\partial U}{\partial \log(z_t)}, \frac{\partial U}{\partial k_{t-1}}, \frac{\partial U}{\partial x_t}, \frac{\partial U}{\partial N_t} \right] \\
&= \left[c_t^{-\eta} y_t,\ c_t^{-\eta} y_t \alpha \frac{1}{k_{t-1}},\ -c_t^{-\eta},\ \left(c_t^{-\eta} y_t (1-\alpha) \frac{1}{N_t} - A_N \right) \right]
\end{aligned}$$

  - **Second order derivatives:** The hessian,

$$D^2 U_{ss} = \begin{bmatrix} \frac{\partial^2 U}{\partial (\log(z_t))^2} & & & \\ \frac{\partial^2 U}{\partial \log(z_t) \partial k_{t-1}} & \frac{\partial^2 U}{\partial k_{t-1}^2} & & \\ \frac{\partial^2 U}{\partial \log(z_t) \partial x_t} & \frac{\partial^2 U}{\partial k_{t-1} \partial x_t} & \frac{\partial^2 U}{\partial x_t^2} & \\ \frac{\partial^2 U}{\partial \log(z_t) \partial N_t} & \frac{\partial^2 U}{\partial k_{t-1} \partial N_t} & \frac{\partial^2 U}{\partial x_t \partial N_t} & \frac{\partial^2 U}{\partial N_t^2} \end{bmatrix} \Bigg|_{ss}$$

where

$$\begin{aligned}
\frac{\partial^2 U}{\partial (\log(z_t))^2} &= \left(-\eta c_t^{-\eta-1} y_t^2 + c_t^{-\eta} y_t \right) \\
\frac{\partial^2 U}{\partial k_{t-1}^2} &= \left[-\eta c_t^{-\eta-1} \left(\frac{\alpha y_t}{k_{t-1}} \right)^2 + c_t^{-\eta} \alpha (\alpha - 1) \frac{y_t}{k_{t-1}^2} \right] \\
\frac{\partial^2 U}{\partial x_t^2} &= -\eta c_t^{-\eta-1} \\
\frac{\partial^2 U}{\partial N_t^2} &= \left[-\eta c_t^{-\eta-1} \left[(1-\alpha) \frac{y_t}{N_t} \right]^2 + c_t^{-\eta} \alpha (\alpha - 1) \frac{y_t}{N_t^2} \right]
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 U}{\partial \log(z_t)\partial k_{t-1}} &= \left[-\eta c_t^{-\eta-1}\alpha\frac{y_t^2}{k_{t-1}} + c_t^{-\eta}\alpha\frac{y_t}{k_{t-1}}\right] \\
\frac{\partial^2 U}{\partial \log(z_t)\partial x_t} &= \eta c_t^{-\eta-1} y_t \\
\frac{\partial^2 U}{\partial \log(z_t)\partial N_t} &= \left[-\eta c_t^{-\eta-1}(1-\alpha)\frac{y_t^2}{N_t} + c_t^{-\eta}(1-\alpha)\frac{y_t}{N_t}\right] \\
\frac{\partial^2 U}{\partial k_{t-1}\partial x_t} &= \eta c_t^{-\eta-1}\alpha\frac{y_t}{k_{t-1}} \\
\frac{\partial^2 U}{\partial k_{t-1}\partial N_t} &= \left[-\eta c_t^{-\eta-1}\alpha(1-\alpha)\frac{y_t^2}{N_t k_{t-1}} + c_t^{-\eta}\alpha(1-\alpha)\frac{y_t}{N_t k_{t-1}}\right] \\
\frac{\partial^2 U}{\partial x_t \partial N_t} &= \eta c_t^{-\eta-1}(1-\alpha)\frac{y_t}{N_t}
\end{aligned}
$$

- We can write now the return function. Using the obtained gradient and the hessian, and given that $W_t = [\log(z_t),\ k_{t-1},\ x_t,\ N_t]^T$ the return function

$$r(\log(z_t), k_{t-1}, x_t, N_t) \approx Q_{11} + 2\cdot Q_{12}\cdot W_t + W_t^T\cdot Q_{22}\cdot W_t = \left[1\ W_t^T\right]\cdot Q\cdot\begin{bmatrix}1\\ W_t\end{bmatrix}$$

  where $Q_{11} = U_{ss} - DU_{ss}\cdot W_{ss} + \frac{1}{2}W_{ss}^T\cdot D^2U_{ss}\cdot W_{ss}$, $Q_{12} = \frac{1}{2}\left(DU_{ss} - W_{ss}^T\cdot D^2U_{ss}\right)$ and $Q_{22} = \frac{1}{2}D^2U_{ss}$.

- Initial value function conjecture. We have, $V^0(\log(z_t), k_{t-1}, x_t) = F_t^T\cdot P^0\cdot F_t = \left[1W_t^T\right]B^TP^0B\begin{bmatrix}1\\ W_t\end{bmatrix}$. In this case $F_t = [1,\ \log(z_t),\ k_{t-1}]^T$ and

$$B = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \rho & 0 & 0\\ 0 & 0 & 1-\delta & 1\end{bmatrix},\quad P^0 = -0.1\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

- Rearrange the problem such that

$$V^{n+1}(\log(z_t), k_{t-1}, x_t, N_t) = MAX_{x_t,N_t}\{F^TM_1F + 2FM_2\begin{bmatrix}x_t\\ N_t\end{bmatrix} + [x_t\ N_t]\,M_3\begin{bmatrix}x_t\\ N_t\end{bmatrix}\}$$

  with M_1, M_2 and M_3 appropriate matrices. Then, the solution to this problem is $\begin{bmatrix}x_t\\ N_t\end{bmatrix} = -M_3^{-1}M_2F$, that is the set of two policy functions once convergence is achieved. The actualization in iteration i is $P^i = M_1 - M_2M_3^{-1}M_2^T$. The stopping criteria was 0.00001.

- **Policy function:** using the decision rule for investment, $x_t = \delta_{x0} + \delta_{xz}\log(z_t) + \delta_{xk}k_{t-1}$, and the law of motion of capital, the decision rule for capital becomes,

$$k_t = \delta_{x0} + \delta_{xz}\log(z_t) + (\delta_{xk} + 1 - \delta)k_{t-1}.$$

Besides, the decision rule for labor,

$$N_t = \delta_{N0} + \delta_{Nz}\log(z_t) + \delta_{Nk}k_{t-1}.$$

For the nine parameter values cases, we have

CASE	δ_{x0}	δ_{xz}	δ_{xk}	δ_{N0}	δ_{Nz}	δ_{Nk}
1,4,7	0.7368	2.6129	-0.0332	0.3801	0.7383	-0.0037
2,5,8	0.7368	1.7499	-0.0332	0.5459	0.3718	-0.0168
3,6,9	0.7368	1.5342	-0.0332	0.6127	0.2242	-0.0221

- **Time series generation:** Once the policy functions are calculated, the time series for investment and labor arise from these policy rules, capital from its law of motion, output from the production function, interest rate from equation (14), and consumption from the resource constraint. Note that it is necessary to add two linear equations to the original system of equations, in order to generate time series.

A.2.3 Sims (1989, 1990)

- **System to be linearized:** From (15), substituting the conditional expectation by its realized value plus an expectational error, we have

$$0 = -c_t^{-\eta} + \beta c_{t+1}^{-\eta}\left(\alpha z_{t+1}k_t^{\alpha-1}N_{t+1}^{1-\alpha} + 1 - \delta\right) + \xi_{t+1},\ E_t(\xi_{t+1}) = 0 \qquad (19)$$

  Then, the system to be linearized is formed by equation (19) and the first order conditions (13), (16) and (18). We have as states k_{t-1} and $\log(z_t)$, and as controls (decision variables) c_t and N_t. We form the system: $\Gamma_0\tilde{Y}_{t+1} = \Gamma_1\tilde{Y}_t + \Phi\zeta_{t+1}$, where $\tilde{Y}_{t+1} = [k_t - k_{ss}, \log(z_{t+1}), c_{t+1} - c_{ss}, N_{t+1} - N_{ss}]^T$ and $\zeta_{t+1} = [\xi_{t+1}, \epsilon_{t+1}]^T$.

- **Linearization about the Steady State:** make a second order Taylor expansion to the previous system.
  - Equation (13),

    $\frac{\partial(13)}{\partial Y_{t+1}}|ss = [-1, 0, 0, 0]$

    $\frac{\partial(13)}{\partial Y_t}|ss = \left[\alpha\frac{y_t}{k_{t-1}} + 1 - \delta,\ y_t,\ -1,\ (1-\alpha)\frac{y_t}{N_t}\right]|ss$

    $\frac{\partial(13)}{\partial \zeta_{t+1}} = [0, 0]$

  - Equation (19),

    $$\frac{\partial(19)}{\partial Y_{t+1}}|ss = \begin{bmatrix}\beta c_{t+1}^{-\eta}\alpha z_{t+1}(\alpha-1)k_t^{\alpha-2}N_{t+1}^{1-\alpha}\\ \beta c_{t+1}^{-\eta}\alpha z_{t+1}k_t^{\alpha-1}N_{t+1}^{1-\alpha}\\ \beta(-\eta)c_{t+1}^{-\eta-1}\left(\alpha z_{t+1}k_t^{\alpha-1}N_{t+1}^{1-\alpha} + 1 - \delta\right)\\ \beta c_{t+1}^{-\eta}\alpha z_{t+1}k_t^{\alpha-1}(1-\alpha)N_{t+1}^{-\alpha}\end{bmatrix}^T |ss$$

    $\frac{\partial(19)}{\partial Y_t}|ss = \left[0, 0, \eta c_t^{-\eta-1}, 0\right]|ss$

$\frac{\partial(19)}{\partial \zeta_{t+1}} = [1, 0]$

– Equation (16),

$$\frac{\partial(16)}{\partial Y_{t+1}}|ss = \begin{bmatrix} c_{t+1}^{-\eta} z_{t+1} \alpha k_t^{\alpha-1}(1-\alpha) N_{t+1}^{-\alpha} \\ c_{t+1}^{-\eta} z_{t+1} k_t^{\alpha}(1-\alpha) N_{t+1}^{-\alpha} \\ -\eta c_{t+1}^{-\eta-1} z_{t+1} k_t^{\alpha}(1-\alpha) N_{t+1}^{-\alpha} \\ c_{t+1}^{-\eta} z_{t+1} k_t^{\alpha}(1-\alpha)(-\alpha) N_{t+1}^{-\alpha-1} \end{bmatrix}^T |ss$$

$\frac{\partial(16)}{\partial Y_t}|ss = [0, 0, 0, 0]$

$\frac{\partial(16)}{\partial \zeta_{t+1}} = [0, 0]$

– Equation (18),
$\frac{\partial(18)}{\partial Y_{t+1}}|ss = [0, -1, 0, 0]$

$\frac{\partial(18)}{\partial Y_t}|ss = [0, \rho, 0, 0]$

$\frac{\partial(18)}{\partial \zeta_{t+1}} = [0, 1]$

- Then we have $\Gamma_0 = \left[\frac{\partial(13)}{\partial \tilde{y}_{t+1}}; \frac{\partial(19)}{\partial \tilde{y}_{t+1}}; \frac{\partial(16)}{\partial \tilde{y}_{t+1}}; \frac{\partial(18)}{\partial \tilde{y}_{t+1}}\right]^T$, $\Gamma_1 = \left[\frac{\partial(13)}{\partial \tilde{y}_t}; \frac{\partial(19)}{\partial \tilde{y}_t}; \frac{\partial(16)}{\partial \tilde{y}_t}; \frac{\partial(18)}{\partial \tilde{y}_t}\right]^T$. Locate now the unstable root of $\Gamma_0^{-1}\Gamma_1$.
- **Policy Function:** add the stability condition $\mu_1(k_{t-1} - k_{ss}) + \mu_2 \log(z_t) + \mu_3(c_t - c_{ss}) + \mu_4(N_t - N_{ss}) = 0$ to the system of first order conditions. For the nine parameter values cases, we have

CASE	μ_1	μ_2	μ_3	μ_4
1,4,7	0.9421	4.2045	-15.6898	2.2111
2,5,8	0.9421	10.8106	-31.7603	2.2111
3,6,9	0.9421	12.4622	-55.8660	2.2111

From the stability condition, (13) and (16) the decision rule for capital becomes,

$$k_t = (1-\delta)k_{t-1} + z_t k_{t-1}^{\alpha} \left[\frac{1}{A_N} c_t^{-\eta}(1-\alpha) z_t k_{t-1}^{\alpha}\right]^{(1-\alpha)/\alpha} - c_t$$

where c_t arises from solving the non–linear equation

$$c_t = c_{ss} + \frac{1}{\mu_3}\left[\mu_1(k_{t-1} - k_{ss}) + \mu_2 \log(z_t) + \mu_4\left(\left[\frac{1}{A_N} c_t^{-\eta}(1-\alpha) z_t k_{t-1}^{\alpha}\right]^{(1-\alpha)/\alpha} - N_{ss}\right)\right].$$

- **Time series generation:** once the stability condition of the linear system is calculated, it is added to the system of first order conditions and constraints. In this case we only add one linear equation. Then solve for labor and consumption using simultaneously (16) and the stability condition. Obtain output from the production function, investment from the resource constraint, capital from its law of motion and interest rate from equation (14).

A.2.4 Uhlig's log–linearization

The system to be log–linearized is formed by the first order conditions (13) to (18) and the constraints of problem (3). Let ˜ denote log–deviations from the steady state. Then, the log–linearization produces the following system of equations,

- **Log–linearization**

$$\left.\begin{array}{rcl} 0 &=& -x_{ss}\tilde{x}_t - c_{ss}\tilde{c}_t + y_{ss}\tilde{y}_t \\ 0 &=& x_{ss}\tilde{x}_t - k_{ss}\tilde{k}_t + (1-\delta)k_{ss}\tilde{k}_{t-1} \\ 0 &=& \alpha\tilde{k}_{t-1} - \tilde{y}_t + (1-\alpha)\tilde{N}_t + \tilde{z}_t \\ 0 &=& -\eta\tilde{c}_t + \tilde{y}_t - \tilde{N}_t \\ 0 &=& -\alpha\frac{y_{ss}}{k_{ss}}\tilde{k}_{t-1} + \alpha\frac{y_{ss}}{k_{ss}}\tilde{y}_t - R_{ss}\tilde{R}_t \\ 0 &=& E_t\left[-\eta\tilde{c}_{t+1} + \tilde{R}_{t+1} + \eta\tilde{c}_t\right] \\ \tilde{z}_{t+1} &=& \rho\tilde{z}_t + \epsilon_{t+1} \end{array}\right\}$$

- **General form:** In this particular example, in terms of the general notation we are using, we have: x is capital, y is formed by consumption, output, labor, the interest rate and investment, and z is technology. The matrices for the log–linear approximation are:

$$A = \begin{bmatrix} 0 \\ -k_{ss} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ (1-\delta)k_{ss} \\ \alpha \\ 0 \\ -\alpha\frac{y_{ss}}{k_{ss}} \end{bmatrix}$$

$$C = \begin{bmatrix} -c_{ss} & y_{ss} & 0 & 0 & -x_{ss} \\ 0 & 0 & 0 & 0 & x_{ss} \\ 0 & -1 & 1-\alpha & 0 & 0 \\ -\eta & 1 & -1 & 0 & 0 \\ 0 & \alpha\frac{y_{ss}}{k_{ss}} & 0 & -R_{ss} & 0 \end{bmatrix}, \ D = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{llll} F = [0], & G = [0], H = [0], & J = [-\eta, 0, 0, 1, 0], & \\ K = [\eta, 0, 0, 0, 0], & L = [0], M = [0], & N = [\rho], Sigma & = [\sigma_\epsilon^2] \end{array}$$

- **Policy Function and time series generation:** The output of the algorithm are the parameters of the state space form relating endogenous variables to state variables. This form is used to generate the time series. That is,

$$\begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \\ \tilde{y}_t \\ \tilde{N}_t \\ \tilde{R}_t \\ \tilde{x}_t \end{bmatrix} = \begin{bmatrix} \nu_{kk} & \nu_{kz} \\ \nu_{ck} & \nu_{cz} \\ \nu_{yk} & \nu_{yz} \\ \nu_{Nk} & \nu_{Nz} \\ \nu_{Rk} & \nu_{Rz} \\ \nu_{xk} & \nu_{xz} \end{bmatrix} \begin{bmatrix} \tilde{k}_{t-1} \\ \tilde{z}_t \end{bmatrix}$$

For the analyzed cases, the parameter values are:

CASE	1,4,7	2,5,8	3,6,9
ν_{kk}	0.9418	0.9418	0.9418
ν_{kz}	0.2063	0.1382	0.1212
ν_{ck}	0.8210	0.3930	0.2206
ν_{cz}	0.4052	0.3989	0.2526
ν_{yk}	0.2702	-0.0481	-0.1763
ν_{yz}	2.4176	1.7139	1.4304
ν_{Nk}	-0.1403	-0.6376	-0.8380
ν_{Nz}	2.2150	1.1155	0.6725
ν_{Rk}	-0.0254	-0.0364	-0.0409
ν_{Rz}	0.0840	0.0596	0.0497
ν_{xk}	-1.3273	-1.3273	-1.3273
ν_{xz}	8.2537	5.5276	4.8461

A.2.5 Parameterized Expectations

- **Order of the approximation.** For the PEA method we tried a third order polynomial approximation.

$$\psi(k_{t-1}, z_t) = q_1\, e^{q_2 log(k_{t-1}) + q_3 log(z_t) + q_4 (log(k_{t-1}))^2 + q_5 log(k_{t-1}) log(z_t) + q_6 (log(z_t))^2}$$

The gradient takes the form,

$$\nabla\psi = \begin{bmatrix} \partial\psi/\partial q_1 \\ \partial\psi/\partial q_2 \\ \partial\psi/\partial q_3 \\ \partial\psi/\partial q_4 \\ \partial\psi/\partial q_5 \\ \partial\psi/\partial q_6 \end{bmatrix} = \begin{bmatrix} \psi/q_1 \\ \psi log(k_{t-1}) \\ \psi log(z_t) \\ \psi(log(k_{t-1}))^2 \\ \psi log(k_{t-1}) log(z_t) \\ \psi(log(z_t))^2 \end{bmatrix}$$

- **Policy function:** The decision rule for capital from (13), (14) and (16) becomes,

$$k_t = (1-\delta)k_{t-1} - [\beta\psi(q; k_{t-1}, z_t)]^{-1/\eta} + z_t k_{t-1}^{\alpha} \left[\frac{1}{A_N}\beta\psi(q; k_{t-1}, z_t) z_t k_{t-1}^{\alpha}(1-\alpha)\right]^{(1-\alpha)/\alpha}.$$

The fixed point for q was calculated using a sample size of 25000 observations.

CASE	q_1	q_2	q_3	q_4	q_5	q_6
1	2.9009	-0.3869	-0.3265	-0.0047	0.049	-0.0611
2	2.1570	0.0846	-0.7523	-0.1310	0.0612	-0.1099
3	1.9385	0.3762	-1.1251	-0.2095	0.1471	0.2196
4	2.8956	-0.3866	-0.3594	-0.0046	0.0620	-0.0978
5	3.9471	-0.3810	-1.1585	-0.0415	0.2274	-0.1638
6	3.9666	-0.2217	-1.1962	-0.0848	0.1683	-0.0769
7	2.7689	-0.3509	-0.3978	-0.0118	0.0765	-0.1213
8	3.8988	-0.3776	-1.0778	-0.0405	0.1882	-0.1368
9	3.5446	-0.1183	-1.2966	-0.1065	0.2031	-0.0884

- **Time series generation:** obtain consumption from condition (15) in which the conditional expectation is substituted by the polynomial function ψ. Then labor from (16), output from the production function, investment from the resource constraint, capital from its law of motion and the interest rate from (14).

A.3 Cooley and Hansen (1989)

A.3.1 Description of the model

- **The Problem:** The economy is a version of the indivisible labor model of Hansen (1985) with money introduced via a cash–in–advance constraint applied to consumption. Consumption is a 'cash good' while leisure and investment are 'credit goods'. Capital letters denote per capita variables that a competitive household takes as parametric; small letters denote specific variables that are chosen by the household. In equilibrium these will be the same.

  Each **household** seeks to maximize her preferences subject to a certain holdings of nominal money balances and constraints,

$$\left\{\begin{array}{rl} \max\limits_{h_t, c_t, k_t, m_t} & E_0 \sum_{t=1}^{\infty} \beta^t [\log(c_t) - A_N n_t] \\ s.t. & \\ M_t = & g_t M_{t-1} \\ \log(g_{t+1}) = & (1-\rho_g)\log(g_{ss}) + \rho_g \log(g_t) + \epsilon_{g_{t+1}} \\ \epsilon_{g_{t+1}} \sim & N(0, \sigma^2_{\epsilon_g}) \\ P_t c_t = & m_{t-1} + (g_t - 1)M_{t-1}, \\ & \left(g_t/E_t[\frac{1}{g_{t+1}}] < \frac{1}{\beta}\right) \\ c_t + x_t + \frac{m_t}{p_t} = & w_t n_t + r_t k_{t-1} + \frac{m_{t-1} + (g_t - 1)M_{t-1}}{P_t} \\ k_t = & (1-\delta)k_{t-1} + x_t \\ given\, k_0, z_1 & \end{array}\right. \tag{20}$$

  Where c_t is consumption at time t, k_{t-1} the capital stock at the beggining of period t, x_t investment, n_t labor, M_t denotes beginning–of–period per capita money balances,

m_t beginning of period money holdings of a particular household, and g_t the gross rate of money, known by all agents at the beginning of period t, with unconditional mean g_{ss}. Moreover, P_t is the price level, w_t the wage rate and r_t the rental rate of capital. Relative to the parameters, $0 < \beta < 1$ is the subjective discount factor, $0 < \delta < 1$ the depreciation rate and $0 < \rho_g < 1$ controls the persistence in the law of motion of money growth. A_N weights labor in utility. The ss subscript affecting a variable denotes its deterministic steady state value. See Cooley and Hansen (1989) for a detailed description of the model.

The Lagrangean for this problem,

$$L(n_t, c_t, m_t, k_t) = E_0 \sum_{t=1}^{\infty} \beta^t \Big[\log(c_t) - A_N n_t - \lambda_t\{c_t + k_t + \frac{m_t}{P_t} - w_t n_t - r_t k_{t-1} - \frac{m_{t-1} + (g_t - 1)M_{t-1}}{P_t} - (1-\delta)k_{t-1}\} - \eta_t\{c_t + \frac{m_{t-1} + (g_t - 1)M_{t-1}}{P_t}\}\Big]$$

And the first order conditions,

$$\begin{aligned}
\eta_t: \quad 0 &= P_t c_t - m_t \\
\lambda_t: \quad 0 &= c_t + x_t + \frac{m_t}{p_t} - w_t n_t - R_t k_{t-1} - \frac{m_{t-1} + (g_t - 1)M_{t-1}}{P_t} \\
c_t: \quad 0 &= \frac{1}{c_t} - \lambda_t - \eta_t \\
n_t: \quad 0 &= -A_N + \lambda_t w_t \\
k_t: \quad 0 &= -\lambda_t + \beta E_t\left[\lambda_{t+1}(r_{t+1} + 1 - \delta)\right] \\
m_t: \quad 0 &= -\frac{\lambda_t}{P_t} + \beta E_t\left[\frac{\lambda_{t+1} + \eta_{t+1}}{P_{t+1}}\right]
\end{aligned}$$

Multiplying equation for m_t with M_t, imposing the equilibrium condition and using the equation for η_t and that for c_t, we obtain,

$$\frac{\lambda_t M_t}{P_t} = \beta E_t \frac{M_t(\lambda_{t+1} + \eta_{t+1})}{P_{t+1}} \iff \lambda_t c_t = \beta E_t \frac{M_t}{P_{t+1} c_{t+1}}$$

then using the equation for c_t and the definition of g_t, we have

$$\lambda_t c_t = \beta E_t \frac{M_t}{M_{t+1}} \iff \lambda_t c_t = \beta E_t \frac{1}{g_{t+1}}$$

If we eliminate $\lambda_t c_t$ using equation for c_t, then we get:

$$1 - \eta_t c_t = \beta E_t\left[\frac{1}{g_{t+1}}\right] \tag{21}$$

that shows that η_t as a fraction of marginal utility is a function of only g_t. An exercise useful for the methods comparison suggested by Den Haan and Marcet (1994) related to this fact is whether each particular solution method verify this fact.

For the **firm**, the problem is to maximize profits subject to technological and productive constraints,

$$\left\{\begin{array}{l}
\underset{N_t, K_{t-1}}{\text{MAX}} \quad \Pi_t = Y_t - w_t N_t - r_t K_{t-1} \\
\quad s.t. \\
\qquad \begin{aligned} Y_t &= z_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \; 0 < \alpha < 1 \\
\log(z_{t+1}) &= \log(z_{ss})(1-\rho_z) + \rho_z \log(z_t) + \epsilon_{z_{t+1}} \\
\epsilon_{z_{t+1}} &\sim N(0, \sigma_{\epsilon_z}^2) \end{aligned}
\end{array}\right. \tag{22}$$

The Lagrangean,

$$L(N_t, K_{t-1}) = z_t K_{t-1}^{\alpha} N_t^{1-\alpha} - w_t N_t - r_t K_{t-1}$$

and the corresponding first order conditions of the problem

$$\begin{aligned}
N_t: \quad w(z_t, K_{t-1}, N_t) &= (1-\alpha) z_t K_{t-1}^{\alpha} N_t^{\alpha} \\
K_{t-1}: \quad r(z_t, K_{t-1}, N_t) &= \alpha z_t K_{t-1}^{\alpha-1} N_t^{1-\alpha}
\end{aligned}$$

Then, imposing equilibrium conditions, the complete system of first order conditions and constraints becomes,

$$\log(z_{t+1}) = \log(z_{ss})(1-\rho_z) + \rho_z \log(z_t) + \epsilon_{z_{t+1}} \tag{23}$$

$$\log(g_{t+1}) = \log(g_{ss})(1-\rho_g) + \rho_g \log(g_t) + \epsilon_{g_{t+1}} \tag{24}$$

$$\lambda_t = \beta E_t\left[\lambda_{t+1}(\alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta)\right] \tag{25}$$

$$\lambda_t c_t = \beta E_t \frac{1}{g_{t+1}} \tag{26}$$

$$A_N = \lambda_t (1-\alpha) \frac{y_t}{N_t} \tag{27}$$

$$c_t + k_t = z_t k_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) k_{t-1} \tag{28}$$

$$P_t c_t = M_t \Rightarrow \hat{p}_t = \frac{1}{c_t} \tag{29}$$

$$M_t = g_t M_{t-1} \tag{30}$$

Which is a system with eight equations and eight unknowns: λ_{t+1}, k_t, N_{t+1}, c_{t+1}, P_{t+1}, M_{t+1}, z_{t+1} and g_{t+1}, with M_0, k_0 and z_1 given and defining $\hat{p}_t = \frac{P_t}{M_t}$.

- **Stochastic Euler equation residual:** in this case it is defined as

$$\xi_{t+1} = -\lambda_t + \beta\left[\lambda_{t+1}(z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta)\right]$$

- **Explicit expression for $E_t\left[1/g_{t+1}\right]$:** We have, from the law of motion of money growth (24),

$$\frac{1}{g_{t+1}} = g_{ss}^{\rho_g - 1} g_t^{-\rho_g} e^{-\epsilon_{g_{t+1}}}$$

then

$$E_t\left[\frac{1}{g_{t+1}}\right] = g_t^{-\rho_g} g_{ss}^{\rho_g - 1} E_t\left[e^{-\epsilon_{g_{t+1}}}\right] = g_t^{-\rho_g} g_{ss}^{\rho_g - 1} \underbrace{E\left[e^{-\epsilon_{g_{t+1}}}\right]}_{X}$$

Given that $\log(X) = -\epsilon_{g_{t+1}} \sim N(0, \sigma^2_{\epsilon_g})$, then X is log–normal, with $E(X) = e^{\frac{\sigma^2_{\epsilon_g}}{2}}$. So, we have,

$$E_t\left[\frac{1}{g_{t+1}}\right] = e^{\frac{\sigma^2_{\epsilon_g}}{2} - (1-\rho_g)\log(g_{ss})} g_t^{-\rho_g} = e^{\frac{\sigma^2_{\epsilon_g}}{2}} g_{ss}^{\rho_g - 1} g_t^{-\rho_g} \tag{31}$$

- **Deterministic Steady State:** By assumption $z_{ss} = 1$. g_{ss} is given exogenously. From (25) we can obtain the ratio

$$\frac{y_{ss}}{k_{ss}} = \frac{1}{\alpha}\left[\frac{1}{\beta} - 1 + \delta\right]$$

From (28),

$$\frac{c_{ss}}{k_{ss}} = \frac{y_{ss}}{k_{ss}} - \delta$$

and from (26) and (27),

$$N_{ss} = \frac{\beta(1-\alpha)}{A_N g_{ss}} \frac{y_{ss}/k_{ss}}{c_{ss}/k_{ss}}$$

Now, using the production function,

$$\frac{y_{ss}}{k_{ss}} = k_{ss}^{\alpha-1} N_{ss}^{1-\alpha} = \left(\frac{k_{ss}}{N_{ss}}\right)^{\alpha-1} \Rightarrow \frac{k_{ss}}{N_{ss}} = \left(\frac{y_{ss}}{k_{ss}}\right)^{\frac{1}{\alpha-1}}$$

Moreover $k_{ss} = \frac{k_{ss}}{N_{ss}} N_{ss}$, $x_{ss} = \delta k_{ss}$, $y_{ss} = \frac{y_{ss}}{k_{ss}} k_{ss}$, $c_{ss} = y_{ss} - x_{ss}$, $\hat{p}_{ss} = \frac{1}{c_{ss}}$ and $\lambda_{ss} = \frac{\beta}{c_{ss} g_{ss}}$

- **Parameter Values:** See table in the main text.

A.3.2 Linear–quadratic approximation: value function iteration

- **Policy function:** As noted before, it is not possible in this problem to compute an equilibrium indirectly by solving for the Pareto Optimal allocation and invoking the second welfare theorem. This is due to the inefficiency introduced by the cash–in–advance. The LQA method for this problem considers the price level as an exogenous process to the agent. The way to do this is by assuming a linear law of motion for the price level, and then interating on this law of motion until it is consistent with the consumption allocations. The method is explained in detail in Cooley and Hansen (1989) and Hansen and Prescott (1995), based on Kydland (1987). The linear rules take the form,

$$\begin{aligned} \hat{p}_t &= \delta_{\hat{p}0} + \delta_{\hat{p}z} log(z_t) + \delta_{\hat{p}g} \log(g_t) + \delta_{\hat{p}k} k_{t-1} \\ x_t &= \delta_{x0} + \delta_{xz} log(z_t) + \delta_{xg} \log(g_t) + \delta_{xk} k_{t-1} \end{aligned}$$

We simply took those parameters reported by Cooley and Hansen (1989):

CASE	$\delta_{\hat{p}0}$	$\delta_{\hat{p}z}$	$\delta_{\hat{p}g}$	$\delta_{\hat{p}k}$	δ_{x0}	δ_{xz}	δ_{xg}	δ_{xk}
1,3,5	1.88633	-0.58175	0.55474	-0.05898	0.64419	1.73073	0.30219	-0.03318
2,4,6	2.07319	-0.66585	0.63537	-0.07726	0.52716	1.51216	0.26423	-0.03318

- **Time series generation:** From the two linear decision rules solve for $\hat{p}_t$ and investment. Then consumption from the cash–in–advance constraint, capital from its law of motion and output from the income identity. Now labor is generated through the production function. Nominal money stock from its law of motion and nominal prices multiplying $\hat{p}_t$ by nominal money.

  As a byproduct we can recover λ from the first order condition (27). From (26) we can recover also the other Euler Equation error,

$$\xi_{g_{t+1}} = \lambda_t c_t \frac{1}{\beta} - e^{\sigma^2_{\epsilon_g}} g_{ss}^{\rho_g - 1} g_t^{-\rho_g}$$

  given that the approximation does not guarantee that this relation be exact, as oposed to the other methods.

A.3.3 Sims (1989, 1990)

- **System to be linearized:** Is formed by five equations. The first three are (23), (24) and (27). Now, eliminating consumption from (26) with (29) and using (31),

$$\frac{1}{\hat{p}_t}\lambda_t = \beta E_t\left[\frac{1}{g_{t+1}}\right] \Rightarrow \hat{p}_t = \left[\frac{1}{\beta} e^{-\frac{\sigma^2_g}{2}} g_{ss}^{1-\rho_g}\right] \lambda_t g_t^{\rho_g} \tag{32}$$

  and that used with (29) in (28) produces

$$0 = -\frac{\beta}{\lambda_t} e^{\frac{\sigma^2_g}{2}} g_{ss}^{\rho_g - 1} g_t^{-\rho_g} - k_t + z_t k_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) k_{t-1}$$

  that we will call (28') . Also, introducing a forecast error in (25) we can obtain

$$\lambda_t = \beta\left[\lambda_{t+1}(\alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta)\right] + \beta \xi_{t+1}$$

  that we will call (25'). The vector of variables for the approximation,

$$\tilde{Y}_{t+1} = [k_t - k_{ss}, \log(z_{t+1}), \log(g_{t+1}) - \log(g_{ss}), \lambda_{t+1} - \lambda_{ss}, N_{t+1} - N_{ss}].$$

  and

$$\zeta_{t+1} = [\epsilon_{z_{t+1}}, \epsilon_{g_{t+1}}, \xi_{t+1}]$$

- **Linearization about the Steady State:** make a second order Taylor expansion to the previous system.

– Equation (23),
$\frac{\partial(23)}{\partial Y_{t+1}}|ss = [0, -1, 0, 0, 0]$

$\frac{\partial(23)}{\partial Y_t}|ss = [0, \rho_z, 0, 0, 0]\,|ss$

$\frac{\partial(23)}{\partial \zeta_{t+1}} = [1, 0, 0]$

– Equation (24),
$\frac{\partial(24)}{\partial Y_{t+1}}|ss = [0, 0, -1, 0, 0]$

$\frac{\partial(24)}{\partial Y_t}|ss = [0, 0, \rho_g, 0, 0]\,|ss$

$\frac{\partial(24)}{\partial \zeta_{t+1}} = [0, 1, 0]$

– Equation (25'),

$$\frac{\partial(25')}{\partial Y_{t+1}}|ss = \begin{bmatrix} \beta\lambda_{t+1}\alpha z_{t+1}(\alpha-1)k_t^{\alpha-2}N_{t+1}^{1-\alpha} \\ \beta\lambda_{t+1}\alpha z_{t+1}k_t^{\alpha-1}N_{t+1}^{1-\alpha} \\ 0 \\ \beta\left(\alpha z_{t+1}k_t^{\alpha-1}N_{t+1}^{1-\alpha}+1-\delta\right) \\ \beta\lambda_{t+1}\alpha z_{t+1}k_t^{\alpha-1}(1-\alpha)N_{t+1}^{-\alpha} \end{bmatrix}^T |ss$$

$\frac{\partial(25')}{\partial Y_t}|ss = [0, 0, 0, -1, 0]\,|ss$

$\frac{\partial(25')}{\partial \zeta_{t+1}} = [0, 0, \beta]$

– Equation (27),

$$\frac{\partial(27)}{\partial Y_{t+1}}|ss = \begin{bmatrix} \lambda_{t+1}z_{t+1}\alpha k_t^{\alpha-1}(1-\alpha)N_{t+1}^{-\alpha} \\ \lambda_{t+1}(1-\alpha)\frac{y_{t+1}}{N_{t+1}} \\ 0 \\ z_{t+1}k_t^{\alpha}(1-\alpha)N_{t+1}^{-\alpha} \\ \lambda_{t+1}z_{t+1}k_t^{\alpha}(1-\alpha)(-\alpha)N_{t+1}^{-\alpha-1} \end{bmatrix}^T |ss$$

$\frac{\partial(27)}{\partial Y_t}|ss = [0, 0, 0, 0, 0]$

$\frac{\partial(27)}{\partial \zeta_{t+1}} = [0, 0, 0]$

– Equation (28'),
$\frac{\partial(28')}{\partial Y_{t+1}}|ss = [-1, 0, 0, 0, 0]$

$$\frac{\partial(28')}{\partial Y_t}|ss = \begin{bmatrix} z_t\alpha k_{t-1}^{\alpha-1}N_t^{1-\alpha}+1-\delta \\ z_t k_{t-1}^{\alpha}N_t^{1-\alpha} \\ \frac{\rho_g\beta}{\lambda_t}e^{\frac{\sigma_g^2}{2}}g_{ss}^{\rho_g-1}g_t^{-\rho_g} \\ \frac{\beta}{\lambda_t^2}e^{\frac{\sigma_g^2}{2}}g_{ss}^{\rho_g-1}g_t^{-\rho_g} \\ z_t k_{t-1}^{\alpha}(1-\alpha)N_t^{-\alpha} \end{bmatrix}^T |ss$$

$\frac{\partial(28')}{\partial \zeta_{t+1}} = [0, 0, 0]$

- Then we have $\Gamma_0 = \left[\frac{\partial(23)}{\partial Y_{t+1}}; \frac{\partial(24)}{\partial Y_{t+1}}; \frac{\partial(25')}{\partial Y_{t+1}}; \frac{\partial(27)}{\partial Y_{t+1}}; \frac{\partial(28')}{\partial Y_{t+1}};\right]^T$, and $\Gamma_1 = \left[\frac{\partial(23)}{\partial Y_t}; \frac{\partial(24)}{\partial Y_t}; \frac{\partial(25')}{\partial Y_t}; \frac{\partial(27)}{\partial Y_t}; \frac{\partial(28')}{\partial Y_t};\right]^T$. Locate now the unstable root of $\Gamma_0^{-1}\Gamma_1$.
- **Policy Function** :add the stability condition $\mu_1(k_{t-1}-k_{ss})+\mu_2\log(z_t)+\mu_3(\log(g_t)-\log(g_{ss}))+\mu_4(\lambda_t-\lambda_{ss})+\mu_5(N_t-N_{ss})=0$ to the system of first order conditions. The calculated weights,

CASE	μ_1	μ_2	μ_3	μ_4	μ_5
1,3,5	0.9421	8.0651	0.6552	15.8988	2.2111
2,4,6	0.9421	7.1183	0.5783	14.0325	2.2111

From the stability condition, (26), (27) and (29), the decision rule for capital is

$$k_t = (1-\delta)k_{t-1} + z_t k_{t-1}^{\alpha}\left[\frac{1-\alpha}{A_N}z_t k_{t-1}^{\alpha}\frac{1}{c_t}\beta e^{\sigma_{\epsilon_g}^2}g_{ss}^{\rho_g-1}g_t^{-\rho_g}\right]^{(1-\alpha)/\alpha} - c_t$$

where c_t arises from solving the non–linear equation

$$\mu_1(k_{t-1}-k_{ss}) + \mu_2\log(z_t) + \mu_3(\log(g_t)-\log(g_{ss})) + \mu_4\left[\frac{1}{c_t}\beta e^{\sigma_{\epsilon_g}^2}g_{ss}^{\rho_g-1}g_t^{-\rho_g} - \lambda_{ss}\right] +$$
$$\mu_5\left[\left[\frac{1-\alpha}{A_N}z_t k_{t-1}^{\alpha}\frac{1}{c_t}\beta e^{\sigma_{\epsilon_g}^2}g_{ss}^{\rho_g-1}g_t^{-\rho_g}\right]^{1/\alpha} - N_{ss}\right] = 0$$

- **Time series generation** :once the stability condition of the linear system is calculated, it is added to the system of first order conditions and constraints. Note that we only add one linear equation. Then solve for λ and labor using simultaneously (27) and the stability condition. Obtain $\hat{p}_t$ from (32), then consumption from (29), capital from (28) and investment from the law of motion of capital. Now, obtain output from the production function, nominal money stock from its law of motion and nominal prices multiplying $\hat{p}_t$ by nominal money.

A.3.4 Uhlig's log–linearization

The system to be log–linearized is formed by equations (23) to (27), (29), (30), the law of motion of capital, the income identity ($y_t = c_t + x_t$), the production function and the expression for the interest rate.

- **Log–linearization:** Let $\tilde{}$ denote log–deviations from the steady state. Then, the system so linearized becomes,

$$
\left.\begin{array}{rcl}
0 & = & -x_{ss}\tilde{x}_t - c_{ss}\tilde{c}_t + y_{ss}\tilde{y}_t \\
0 & = & x_{ss}\tilde{x}_t - k_{ss}\tilde{k}_t + (1-\delta)k_{ss}\tilde{k}_{t-1} \\
0 & = & \alpha\tilde{k}_{t-1} - \tilde{y}_t + (1-\alpha)\tilde{N}_t + \tilde{z}_t \\
0 & = & \tilde{\lambda}_t + \tilde{y}_t - \tilde{N}_t \\
0 & = & \tilde{\hat{p}}_t + \tilde{c}_t \\
0 & = & \tilde{\lambda}_t + \tilde{c}_t + \rho_g\tilde{g}_t \\
0 & = & -R_{ss}\tilde{R}_t + \alpha\frac{y_{ss}}{k_{ss}}\tilde{y}_t - \alpha\frac{y_{ss}}{k_{ss}}\tilde{k}_{t-1} \\
0 & = & E_t\left[-\tilde{\lambda}_t + \tilde{\lambda}_{t+1} + \tilde{R}_{t+1}\right] \\
\tilde{z}_{t+1} & = & \rho_z\tilde{z}_t + \epsilon_{zt+1} \\
\tilde{g}_{t+1} & = & \rho_g\tilde{g}_t + \epsilon_{gt+1}
\end{array}\right\}
$$

- **General form** In this particular example, in terms of the general notation we are using, we have: x is capital, y is formed by consumption, output, labor, investment, $\hat{p}$, λ and the interest rate; z is technology and money growth. The matrices for the log-linear approximation are:

$$
A = \begin{bmatrix} 0 \\ -k_{ss} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ (1-\delta)k_{ss} \\ \alpha \\ 0 \\ 0 \\ 0 \\ -\alpha\frac{y_{ss}}{k_{ss}} \end{bmatrix}
$$

$$
C = \begin{bmatrix}
-c_{ss} & y_{ss} & 0 & -x_{ss} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{ss} & 0 & 0 & 0 \\
0 & -1 & 1-\alpha & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & \alpha\frac{y_{ss}}{k_{ss}} & 0 & 0 & 0 & 0 & -R_{ss}
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \rho_g \\ 0 & 0
\end{bmatrix}
$$

$$
\begin{array}{llll}
F = [0], & G = [0], & H = [0], & J = [0,0,0,0,0,1,1], \\
K = [0,0,0,0,0,-1,0], & L = [0,0], & M = [0,0], & N = \begin{bmatrix} \rho_z & 0 \\ 0 & \rho_g \end{bmatrix},
\end{array}
$$

$$
Sigma = \begin{bmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_g^2 \end{bmatrix}
$$

- **Policy Function and time series generation:** for the log–linearized variables, we use the state space form,

$$
\begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \\ \tilde{y}_t \\ \tilde{N}_t \\ \tilde{x}_t \\ \tilde{\hat{p}}_t \\ \tilde{\lambda}_t \\ \tilde{R}_t \end{pmatrix} = \begin{pmatrix}
\nu_{kk} & \nu_{kz} & \nu_{kg} \\
\nu_{ck} & \nu_{cz} & \nu_{cg} \\
\nu_{yk} & \nu_{yz} & \nu_{yg} \\
\nu_{Nk} & \nu_{Nz} & \nu_{Ng} \\
\nu_{xk} & \nu_{xz} & \nu_{xg} \\
\nu_{\hat{p}k} & \nu_{\hat{p}z} & \nu_{\hat{p}g} \\
\nu_{\lambda k} & \nu_{\lambda z} & \nu_{\lambda g} \\
\nu_{Rk} & \nu_{Rz} & \nu_{Rg}
\end{pmatrix} \begin{pmatrix} \tilde{k}_{t-1} \\ \tilde{z}_t \\ \tilde{g}_t \end{pmatrix}
$$

where all variables are log–deviations with respect to the steady state. We recover the series in levels (no logs) by undoing the original transformation. So, the resulting decision rules for all variables are nonlinear in the levels. For capital, for example, we have,

$$
k_t = k_{ss}\left(\frac{k_{t-1}}{k_{ss}}\right)^{\nu_{kk}}\left(\frac{z_t}{z_{ss}}\right)^{\nu_{kz}}\left(\frac{g_t}{g_{ss}}\right)^{\nu_{kg}}
$$

for consumption,

$$
c_t = c_{ss}\left(\frac{k_{t-1}}{k_{ss}}\right)^{\nu_{ck}}\left(\frac{z_t}{z_{ss}}\right)^{\nu_{cz}}\left(\frac{g_t}{g_{ss}}\right)^{\nu_{cg}}
$$

and henceforth. The parameter values are the same for all the cases, given that no matrix A, B... depends on g_{ss} or $\sigma^2_{\epsilon_g}$.

CASE	All
ν_{kk}	0.9418
ν_{kz}	0.1552
ν_{kg}	0.0271
ν_{ck}	0.5316
ν_{cz}	0.4703
ν_{cg}	-0.4488
ν_{yk}	0.0550
ν_{yz}	1.9417
ν_{yg}	-0.0555
ν_{Nk}	-0.4766
ν_{Nz}	1.4715
ν_{Ng}	-0.0867

CASE	All
ν_{xk}	-1.3273
ν_{xz}	6.2091
ν_{xg}	1.0850
$\nu_{\hat{p}k}$	-0.5316
$\nu_{\hat{p}z}$	-0.4703
$\nu_{\hat{p}g}$	0.4488
$\nu_{\lambda k}$	-0.5316
$\nu_{\lambda z}$	-0.4703
$\nu_{\lambda g}$	-0.0312
ν_{Rk}	-0.0328
ν_{Rz}	0.0675
ν_{Rg}	-0.0019

A.3.5 Parameterized Expectations

- **Order of the approximation:** For the PEA method we tried a third order polynomial approximation as suggested in Den Haan and Marcet (1994).

$$
\begin{array}{rcl}
\psi(k_{t-1}, z_t, g_t) & = & q_1\, e^{q_2 log(k_{t-1}) + q_3 log(z_t) + q_4 log(g_t) + q_5 (log(k_{t-1}))^2} \\
 & \times & e^{q_6 log(k_{t-1}) log(z_t) + q_7 (log(z_t))^2) + q_8 (log(z_t))^3}
\end{array}
$$

that we denote by PEA. The gradient takes the form,

$$
\nabla\psi = \begin{bmatrix} \partial\psi/\partial q_1 \\ \partial\psi/\partial q_2 \\ \partial\psi/\partial q_3 \\ \partial\psi/\partial q_4 \\ \partial\psi/\partial q_5 \\ \partial\psi/\partial q_6 \\ \partial\psi/\partial q_7 \\ \partial\psi/\partial q_8 \end{bmatrix} = \begin{bmatrix} \psi/q_1 \\ \psi\log(k_{t-1}) \\ \psi\log(z_t) \\ \psi\log(g_t) \\ \psi(\log(k_{t-1}))^2 \\ \psi\log(k_{t-1})log(z_t) \\ \psi(\log(z_t))^2 \\ \psi(\log(z_t))^3 \end{bmatrix}
$$

- **Policy function:** The decision rule for capital becomes,

$$
k_t = (1-\delta)k_{t-1} + z_t k_{t-1}^{\alpha}\left[\frac{1}{A_N}(1-\alpha)z_t k_{t-1}^{\alpha}\beta\psi(q;k_{t-1},z_t,g_t)\right]^{(1-\alpha)/\alpha} - \frac{1}{\psi}e^{\sigma_{\epsilon_g}^2} g_{ss}^{\rho_g-1} g_t^{-\rho_g}
$$

The fixed point for q was calculated in each case using a sample size of 25000 observations.

CASE	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
1	3.0710	-0.2498	-0.8460	-0.0362	-0.0569	0.1543	-0.1351	0.2371
2	3.0447	-0.2753	-1.0496	-0.0309	-0.0559	0.2536	-0.0640	-0.1545
3	3.5572	-0.3600	-0.7221	-0.0250	-0.0365	0.1127	-0.1057	-0.4178
4	3.0250	-0.2686	-0.8818	-0.0223	-0.0576	0.1831	-0.1849	-0.6025
5	3.7449	-0.4073	-0.7162	-0.0295	-0.0261	0.1032	-0.0930	0.0467
6	3.5750	-0.4082	-0.7515	-0.0306	-0.0281	0.1247	-0.1235	0.0849

- **Time series generation:** Parameterize the expectation in (25) and obtain λ. Then consumption from equation (26) and labor from (27). From the budget constraint capital, and using (29) and (30) the price level. Interest rate arises from its equation and output from the production function.