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Part Two

Hybrid Methods and Models of Granular Computing

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A Survey of Interval-Valued Fuzzy Sets

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22.1 Introduction

22 Zadeh presented the theory of fuzzy sets in 1965 [1]. From the beginning it was clear that this theory
23 was an extraordinary tool for representing human knowledge. Nevertheless, Zadeh himself established in
24 1973 (see [2]) that sometimes, in decision-making processes, knowledge is better represented by means
25 of some generalizations of fuzzy sets. The so-called extensions of fuzzy set theory arise in this way.

26 In the applied field, the success of the use of fuzzy set theory depends on the choice of the membership
27 function that we make. However, there are applications in which experts do not have precise knowledge
28 of the function that should be taken. In these cases, it is appropriate to represent the membership degree
29 of each element to the fuzzy set by means of an interval. From these considerations arises the extension of
30 fuzzy sets called *theory of interval-valued fuzzy sets* (IVFSs), that is, fuzzy sets such that the membership
31 degree of each element of the fuzzy set is given by a closed subinterval of the interval $[0, 1]$. Hence, not
32 only vagueness (lack of sharp class boundaries) but also a feature of uncertainty (lack of information)
33 can be addressed intuitively.

34 Therefore (see [3]), membership functions of IVFSs are not as specific as their counterparts of fuzzy
35 sets, but this lack of specificity makes them more realistic in some applications. Their advantage is
36 that they allow us to express our uncertainty in identifying a particular membership function. This
37 uncertainty is involved when IVFSs are processed, making results of the processing less specific but more
38 credible.

39 These sets were born in the 1970s. In May 1975 Sambuc (see [4]) presented in his doctoral thesis the
40 concept of an IVFS named a Φ -fuzzy set. That same year, Jahn [5] wrote about these sets and Zadeh [6]
41 discussed the representation of type 2 fuzzy sets and its potential in approximate reasoning. One year
42 later, Grattan-Guinness [7] established a definition of an interval-valued membership function. In that
43 decade, IVFSs appeared in the literature in various guises and it was not until the 1980s, with the work of
44 Gorzalczany and Turksen [8–16], that the importance of these sets, as well as their name, was definitely
45 established.

46 In this chapter, we present a survey of IVFSs. We describe the most important concepts of this theory
47 and provide a set of references that is intended to represent as best as possible the work carried out on
48 this extension.
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In this introduction, we must indicate the three major problems, in our opinion, of the theory that is the object of our study:

- A large number of contributions are generalized adaptations of the theoretical developments of fuzzy set theory. This prevents us from focusing on the nature of IVFSs themselves and studying the properties possessed exclusively by these sets.
- In the early work on this theory, Gorzalczany presented the concept of the degree of compatibility between two IVFSs as an interval (see [9]). In some of the latest publications that we have read on interval-valued measures of information, these measures are defined as a point on $[0, 1]$. With this modelization, what is achieved is that the relation between different interval-valued measures of information is a copy of the relation that exists between these concepts in fuzzy set theory. However, we consider that in this case we lose the information that would be provided by a modelization of these measures using intervals.
- Currently, there are two names for these sets: some authors call them *interval-valued fuzzy sets* and others *interval type 2 fuzzy sets*. In [17] Mendel writes, ‘it turns out that an interval type 2 fuzzy set is the same as an interval-valued fuzzy set for which there is a very extensive literature. These two seemingly different kinds of fuzzy sets were historically approached from very different starting points, which as we shall explain next has turned out to be a very good thing’. Nonetheless, we consider that this duplicity in the name can cause confusion. We have observed that in some papers results that have already been known for many years for IVFSs are presented for interval type 2 fuzzy sets. For this reason, we believe the name should be set; otherwise, the complete bibliography that exists on IVFSs should be taken into account.

Other objections to these sets can be found in [18].

This chapter is organized in the following way. Starting from the definition of an IVFSs and from the study of two construction methods, we present in Section 22.2 the relation between these sets and other representations of fuzzy set theory. Next, in Sections 22.3 and 22.4, we study the connectives and possible combinations between them. In Section 22.5, we analyze the laws for conjunctions and disjunctions. Then, in Section 22.6, we give an interpretation of interval-valued fuzzy operators together with a construction method. In Section 22.7, we recall information measures, and in Section 22.8, we mention the main fields of application of IVFSs. In the final section (Section 22.9), we analyze the use of IVFSs in granular computing.

22.2 Preliminary Definitions

In fuzzy set theory, a function $n : [0, 1] \rightarrow [0, 1]$ such that $n(0) = 1, n(1) = 0$ that is decreasing is called *negation*. If a negation is strictly decreasing and continuous, it is called a *strict negation*. If $n(n(x)) = x$ for all $x \in [0, 1]$, then n is *involution*. A *strong negation* is a negation that is strict and involutive. In 1979, Trillas [19] characterized strong negations using automorphisms (see also [20]).

In this chapter we shall denote by $\mathcal{FSS}(U)$ the set of all fuzzy sets defined on a finite referential U , where U is a non-empty set. Given a fuzzy set $A = \{(u, \mu_A(u)) | u \in U\} \in \mathcal{FSS}(U)$, the expression $A_n = \{(u, n(\mu_A(u))) | u \in U\}$ will be used here as the complement of the fuzzy set A , where n is a negation. We must point out that two distinct notations are most commonly employed in the literature to denote membership functions. In one of them, the membership function of a fuzzy set A is denoted in the way indicated above by the symbol μ_A ; that is, $\mu_A : U \rightarrow [0, 1]$. In the other one, the function is denoted by A and has, of course, the same form: $A : U \rightarrow [0, 1]$.

We denote by $L([0, 1])$ the set of all *closed intervals* of the closed interval $[0, 1]$; that is,

$$L([0, 1]) = \{\mathbf{x} = [\underline{x}, \bar{x}] | (\underline{x}, \bar{x}) \in [0, 1]^2 \text{ and } \underline{x} \leq \bar{x}\}.$$

$L([0, 1])$ is a partially ordered set with respect to the relation \leq_L defined in the following way: given $\mathbf{x}, \mathbf{y} \in L([0, 1])$,

$$\mathbf{x} \leq_L \mathbf{y} \text{ if and only if } \underline{x} \leq \underline{y} \text{ and } \bar{x} \leq \bar{y}.$$

1 The relation above is transitive and antisymmetric and it expresses the fact that \mathbf{x} links strongly
 2 to \mathbf{y} , so that $(L([0, 1]), \leq_L)$ is a complete lattice (see [21–23]), where the smallest element is $0_L =$
 3 $[0, 0]$ and the largest is $1_L = [1, 1]$. Evidently, it is not a linear lattice, for there exist elements that
 4 are not comparable. A very interesting study on the arithmetic operations in $L([0, 1])$ can be found in
 5 [24–28].

6
 7 **Definition 1.** An interval-valued fuzzy set A on the universe $U \neq \emptyset$ is a mapping $A : U \rightarrow L([0, 1])$.

8
 9 Obviously, $A(u) = [\underline{A}(u), \overline{A}(u)] \in L([0, 1])$ is the membership degree of $u \in U$. We denote by
 10 $IVFSs(U)$ the set of IVFSs on U . We denote by W the length of the interval considered.

11 In [29], two representation theorems and an equivalent classification theorem for IVFSs are presented.
 12 Besides the relation \leq_L above, other relations (see [30]) on IVFSs have been studied, including among
 13 others the following: $\mathbf{x} \leq \mathbf{y}$ if and only if $\underline{x} \leq \underline{y}$ and $\overline{y} \leq \overline{x}$.

14
 15 **22.2.1 Relation to Other Extensions**

16
 17 The concept of a type 2 fuzzy set was introduced in 1975 by Zadeh [6] as a generalization of an ordinary
 18 fuzzy set. Type 2 fuzzy sets are characterized by a fuzzy membership function, that is, the membership
 19 value for each element of the set is itself a fuzzy set in $[0, 1]$.

20 Formally, given the referential set U , a type 2 fuzzy set is defined as an object \overline{A} which has the following
 21 form:

$$\overline{A} = \{(u, x, \mu_u(x)) | u \in U, x \in [0, 1]\},$$

22
 23 where $x \in [0, 1]$ is the primary membership degree of u and $\mu_u(x)$ is the secondary membership level,
 24 specific to a given pair (u, x) .

25
 26 IVFSs can be generalized by assigning to each interval a fuzzy set defined on the referential set $[0, 1]$;
 27 that is, IVFSs can be generalized by means of type 2 fuzzy sets. The following equation shows a way of
 28 constructing a type 2 fuzzy set from an $A \in IVFSs(U)$:

$$\mu_u(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \underline{A}(u) \\ \frac{2}{\overline{A}(u) - \underline{A}(u)}(x - \underline{A}(u)) & \text{if } \underline{A}(u) \leq x \leq \frac{\underline{A}(u) + \overline{A}(u)}{2} \\ \frac{2}{\underline{A}(u) - \overline{A}(u)}(x - \overline{A}(u)) & \text{if } \frac{\underline{A}(u) + \overline{A}(u)}{2} \leq x \leq \overline{A}(u) \\ 0 & \text{if } \overline{A}(u) \leq x \leq 1. \end{cases}$$

29
 30 Evidently, the choice of a triangular shape for the membership function is totally arbitrary; we can
 31 associate another type of membership function, for example, a trapezoidal shape, with the set.

32
 33 A particular case of a type 2 fuzzy set is an interval type 2 fuzzy set (see [17, 31–17]). An interval
 34 type 2 fuzzy set A in U is defined by

$$\overline{A} = \{(u, A(u), \mu_u(x)) | u \in U, A(u) \in L([0, 1])\},$$

35
 36 where $A(u)$ is the membership function presented in Definition 1; that is, it is a closed subinterval of
 37 $[0, 1]$, and the function $\mu_u(x)$ represents the fuzzy set associated with the element $u \in U$ obtained when
 38 x covers the interval $[0, 1]$; $\mu_u(x)$ is given in the following way:

$$\mu_u(x) = \begin{cases} a & \text{if } \underline{A}(u) \leq x \leq \overline{A}(u) \\ 0 & \text{otherwise,} \end{cases}$$

39
 40 where $0 \leq a \leq 1$. In [17, 31–33], it turns out that an interval type 2 fuzzy set is the same as an IVFS if
 41 we take $a = 1$.

1 Another important extension of fuzzy set theory is the theory of Atanassov's intuitionistic fuzzy sets
 2 (A-IFSs) [34, 35]. A-IFSs assign to each element of the universe not only a membership degree, but also
 3 a non-membership degree, which is less than or equal to 1 minus the membership degree.

4 An A-IFS on U is a set

$$5 \quad A = \{(u, \mu_A(u), \nu_A(u)) | u \in U\},$$

6 where $\mu_A(u) \in [0, 1]$ denotes the membership degree and $\nu_A(u) \in [0, 1]$ the non-membership degree of
 7 u in A and where, for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

8 In [34], Atanassov established that every Atanassov's intuitionistic fuzzy set A on U can be represented
 9 by an interval-valued fuzzy set A given by

$$10 \quad A : U \rightarrow L([0, 1])$$

$$11 \quad u \rightarrow [\mu_A(u), 1 - \nu_A(u)], \quad \text{for all } u \in U.$$

12 Using this representation, Atanassov proposed in 1983 that A-IFS theory was equivalent to the theory of
 13 IVFSs. This equivalence was proved in 2003 by Deschrijver and Kerre [23]. Therefore, from a mathe-
 14 matical point of view, the results that we obtain for IVFSs are easily adaptable to A-IFSs and vice versa.
 15 Nevertheless, we need to point out that, conceptually, the two types of sets are totally different. This is
 16 made clear when applications of these sets are constructed (see [36–39]).

17 In 1993, Gau and Buehrer introduced the concept of *vague sets* [40]. Later, in 1996, it was proved that
 18 vague sets are in fact A-IFSs [41].

19 A compilation of the sets that are equivalent (from a mathematical point of view) to IVFSs can be
 20 found in [42]. Two conclusions are drawn from this study:

- 21 1. IVFSs are equivalent to A-IFSs (and therefore vague sets), to grey sets (see [43]), and to L -fuzzy set
 22 in Goguen's sense (see [44, 45]) with respect to a special lattice $L([0, 1])$.
- 23 2. IVFSs are a particular case of probabilistic sets (see [46]), of soft sets (see [47]), of Atanassov's
 24 interval-valued intuitionistic fuzzy sets (see [35]) and evidently of type 2 fuzzy sets.

25 22.2.2 Some Methods of Construction of Interval-Valued Fuzzy Sets

26 In 1975 Sambuc used IVFSs for the construction of a computer system that would help make a diagnosis
 27 of certain thyroidal pathologies. In this paper, the problem of constructing an interval-valued membership
 28 function that is more appropriate for the computer system arises for the first time.

29 Later, Turksen and Yao (see [16]) proposed a method of constructing IVFSs using the fact that in fuzzy
 30 logic it generally occurs that the conjunctive normal form (CNF) gives membership grades greater than
 31 the disjunctive normal form (DNF). Ten years later, Turksen (see [13, 48]) proved that when performing
 32 a fuzzy set-theoretic aggregation of regular membership functions, the simultaneous use of CNF and
 33 DNF forms of the fuzzy connectives yields an IVFSs.

34 The constructions of IVFSs presented in 1996 (see [49]) are divided into two groups: constructions from
 35 a fuzzy set and constructions from two or more fuzzy sets. Next we present an example of each of them.

36 **Construction of an IVFSs from a Fuzzy Set**

37 In 2005 Tizhoosh (see [50]) used IVFSs for determining the threshold of an image Q . The threshold
 38 enables to separate the object contained in the image from its background.

39 In his work, Tizhoosh associates each image with an IVFSs in the following way:

- 40 1. Have an expert assign the image a fuzzy set characterized by the membership function μ_Q .
- 41 2. For each pixel, construct the interval $[(\mu_Q)^\alpha, (\mu_Q)^{\frac{1}{\alpha}}]$ with $\alpha \in (1, \infty)$, which represents its membership
 42 to the IVFS that is going to represent the image (see Figure 22.1).

43 The relevance of this construction has been proved in an experimental way in [50–52]. In these works
 44 one concludes that in images with many pixels for which experts do not agree on whether they belong

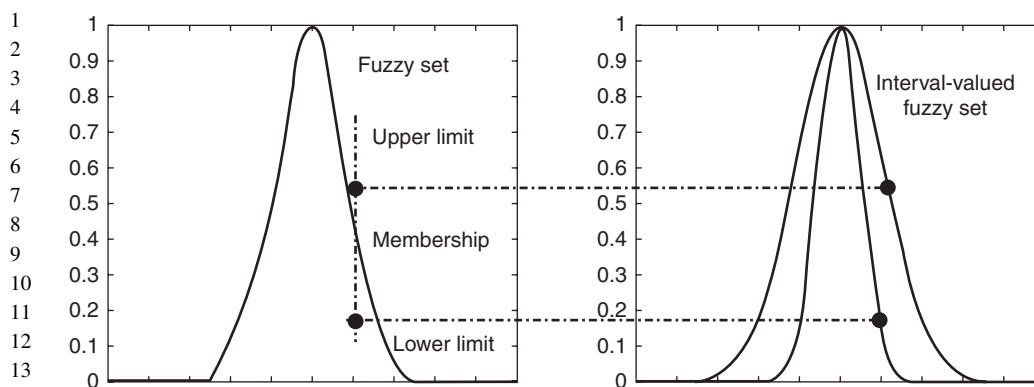


Figure 22.1 Method of construction of an IVFS from a fuzzy set

to the background or to the object, the thresholds obtained with these constructions (together with the algorithm that we will describe in Section 22.8) are much better than those obtained with classical methods or with methods that use fuzzy techniques.

A generalization of this method can be found developed in [53].

Construction of an IVFSs from Two or More Fuzzy Sets

If we ask several experts to construct the membership function that represents the fuzzy set that modelizes a certain action, we discover the fact that in most cases the experts choose different membership functions. Because of this we have uncertainty in choosing the best function. In these conditions it is recommended to work with IVFSs constructed in the following way: each element is assigned an interval whose lower extreme is the lowest value given by the experts for that element and as upper extreme the highest (see [54–57]).

Other construction methods can be found in [38, 58, 59].

22.2.3 A Method of Construction of Fuzzy Sets from Interval-Valued Fuzzy Sets

The operator that we present next will enable us to construct a family of fuzzy sets from an IVFS (see [60]). This concept will be frequently used throughout the whole chapter.

Definition 2. Let $\alpha \in [0, 1]$; we define K_α as a function $K_\alpha : L([0, 1]) \rightarrow [0, 1]$ such that it satisfies the following conditions:

1. If $\underline{x} = \bar{x}$, then $K_\alpha(\mathbf{x}) = \underline{x}$.
2. $K_0(\mathbf{x}) = \underline{x}$, $K_1(\mathbf{x}) = \bar{x}$ for all $\mathbf{x} \in L([0, 1])$.
3. If $\mathbf{x} \leq_L \mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in L([0, 1])$, then $K_\alpha(\mathbf{x}) \leq K_\alpha(\mathbf{y})$ for all $\alpha \in [0, 1]$.
4. $K_\alpha(\mathbf{x}) \leq K_\beta(\mathbf{x})$ if and only if $\alpha \leq \beta$, for all $\mathbf{x} \in L([0, 1])$, with $\beta \in [0, 1]$.

A study of these operators can be found in [35] (and also in [49, 61]). In these papers, the following expressions are proposed for these operators:

$$K_\alpha(\mathbf{x}) = K_\alpha([\underline{x}, \bar{x}]) = K_\alpha([K_0(\mathbf{x}), K_1(\mathbf{x})]) = \underline{x} + \alpha(\bar{x} - \underline{x}) = K_0(\mathbf{x}) + \alpha W_{\mathbf{x}},$$

where $W_{\mathbf{x}}$ represents the length of the interval \mathbf{x} .

1 The operator K_α enables every IVFS to be associated with a fuzzy set in the following way (see [21,
 2 35, 41, 49, 53, 54, 61–66]):

$$K_\alpha : \mathcal{IVFSs}(U) \rightarrow \mathcal{FSs}(U)$$

3
 4
 5 given by

$$K_\alpha(A) = \{(u, \mu_{K_\alpha(A)}(u) = K_\alpha(A(u)) = K_\alpha([\underline{A}(u), \overline{A}(u)]) | u \in U\}.$$

6
 7
 8 Unless otherwise indicated, the operator K_α that we shall use is the general operator presented in
 9 Definition 2. That is, we shall not use any particular expression for it.
 10
 11
 12

13 22.3 Connectives

14 In various papers (e.g., [4, 9, 21, 35, 41, 47, 49, 51–54, 60–66, 68–82]), the union, intersection, and
 15 complementation of $\mathcal{IVFSs}(U)$ are defined in the following way: if $A, B \in \mathcal{IVFSs}(U)$, then
 16
 17

$$A \cap B(u) = [\min(K_0(A(u)), K_0(B(u))), \min(K_1(A(u)), K_1(B(u)))],$$

$$A \cup B(u) = [\max(K_0(A(u)), K_0(B(u))), \max(K_1(A(u)), K_1(B(u)))],$$

$$A_n(u) = [1 - K_1(A(u)), 1 - K_0(A(u))] \text{ for all } u \in U.$$

18
 19
 20
 21
 22 In [21, 64, 65], it is proved that $\{\mathcal{IVFSs}, \cap, \cup\}$ is a distributive, bounded, non-complemented lattice
 23 that satisfies De Morgan’s laws (with respect to the definition of A_n above).

24 Next, in the first subsection we thoroughly study the concept of interval-valued negation and in
 25 subsequent subsections we describe the manner of modelizing the operations of union and intersection
 26 by means t -norms and t -conorms.
 27
 28
 29

30 22.3.1 IV Negations

31 Interval-valued negations, hereinafter referred to as IV negations, are an extension of negations and are
 32 defined as follows:
 33

34 **Definition 3.** An IV negation is a function $N : L([0, 1]) \rightarrow L([0, 1])$ that is decreasing (with respect to
 35 \leq_L) such that $N(1_L) = 0_L$ and $N(0_L) = 1_L$. If for all $\mathbf{x} \in L([0, 1])$, $N(N(\mathbf{x})) = \mathbf{x}$, it is said that N is
 36 involutive.
 37
 38

39 Next, we present a theorem related to the construction of IV negations. Historically, this theorem
 40 provided the first construction method for such negations (see [21, 53]).
 41

42 **Theorem 1.** Let the function $N : L([0, 1]) \rightarrow L([0, 1])$ be given by

$$N(\mathbf{x}) = [n(K_1(\mathbf{x})), n(K_0(\mathbf{x}))],$$

43
 44
 45 where $n : [0, 1] \rightarrow [0, 1]$ is a negation. Under these conditions N is an IV negation.
 46
 47

48 An in-depth study of strict IV negations can be found in [54, 77].
 49

50 Representation Theorem for IV Negations

51 A representation theorem for IV negations was obtained in [54]. Previously Deschrijver *et al.* (see [77])
 52 proved this theorem for A-IFSs. The only small difference between these two studies lies essentially in
 53 the fact that for the negations of the A-IFSs, K_α operators are not used, whereas these operators are used

1 in the characterization of the IV negations developed in [54]. This fact enables us to prove the following
 2 lemma (which is a consequence of Lemma 3.5 proved in [77]):

3 **Lemma 1.** *Let there be a K_α with $\alpha \in [0, 1]$. If N is an involutive IV negation, then for every $\mathbf{x} =$
 4 $[K_0(\mathbf{x}), K_0(\mathbf{x})] \in L([0, 1])$,*

$$5 \quad K_0(N(\mathbf{x})) = K_\alpha(N(\mathbf{x})) = K_1(N(\mathbf{x}))$$

6
 7 holds for all $\alpha \in [0, 1]$.

8
 9 Lemma 1, together with other properties (see [21, 49, 54, 61]) of the operators K_α , has made it possible
 10 to prove the following characterization theorem (which is an adaptation for IVFSs of Theorem 3.6 proved
 11 in [77]).

12 **Theorem 2.** *A function $N : L([0, 1]) \rightarrow L([0, 1])$ is an involutive IV negation if and only if there exists
 13 an involutive negation n such that*

$$14 \quad N(\mathbf{x}) = [n(K_1(\mathbf{x})), n(K_0(\mathbf{x}))].$$

15
 16
 17
 18 **Complement of an Interval-Valued Fuzzy Set**

19 Given an interval-valued fuzzy set $A \in \mathcal{IVFS}(U)$, from the functions N we can define the concept of
 20 the *complement* of A in the following way:

$$21 \quad A_N(u) = N(A(u)) \text{ for all } u \in U.$$

22
 23
 24 **22.3.2 IV t -Norms and IV t -Conorms**

25 In fuzzy set theory, t -norms are used for modeling the intersection (or conjunction) of two fuzzy sets and
 26 t -conorms for modeling the union (or disjunction). Similarly, in the theory of IVFSs, we can model the
 27 intersection and the union using interval-valued t -norms and interval-valued t -conorms in the following
 28 way: for all $u \in U$ and $A, B \in \mathcal{IVFS}(U)$, we have

$$29 \quad A \cap_{\mathbf{T}} B(u) = \mathbf{T}(A(u), B(u)),$$

$$30 \quad A \cup_{\mathbf{S}} B(u) = \mathbf{S}(A(u), B(u)),$$

31
 32 where \mathbf{T} and \mathbf{S} are, according to Definition 4, an IV t -norm and an IV t -conorm, respectively (see [60,
 33 65, 74]).

34
 35 **Definition 4.** *A function $\mathbf{T} : (L([0, 1]))^2 \rightarrow L([0, 1])$ is said to be an interval-valued t -norm (IV t -norm)
 36 if it is commutative, associative, and increasing (in both arguments with respect to the order \leq_L) and
 37 has a neutral element $1_L = [1, 1]$. In the same way, a function $\mathbf{S} : (L([0, 1]))^2 \rightarrow L([0, 1])$ is said to be
 38 an interval-valued t -conorm (IV t -conorm) if it is commutative, associative, and increasing and has a
 39 neutral element $0_L = [0, 0]$.*

40
 41 Evidently, $\mathbf{T}(\mathbf{x}, 0_L) \leq_L \mathbf{T}(1_L, 0_L) = 0_L$, and therefore $\mathbf{T}(\mathbf{x}, 0_L) = 0_L$. In a similar way, we have
 42 $\mathbf{S}(\mathbf{x}, 1_L) = 1_L$. Note that Definition 4 is an extension of the classical definition of the t -norm and
 43 t -conorm in $[0, 1]$. We only need to substitute $L([0, 1])$ for $[0, 1]$.

44 A lot has been written on the way in which IV t -norms and IV t -conorms can be generated from
 45 t -norms and t -conorms in $[0, 1]$. The first idea that appeared was the generation of IV t -norms (and
 46 IV t -conorms) from two t -norms T_a , and T_b (and S_a , and S_b) in $[0, 1]$, from the operator K_α and from
 47 expressions in the following way (see [60, 65]):

$$48 \quad \begin{cases} \mathbf{T}(\mathbf{x}, \mathbf{y}) = [T_a(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y})), T_b(K_\beta(\mathbf{x}), K_\beta(\mathbf{y}))] \\ \mathbf{S}(\mathbf{x}, \mathbf{y}) = [S_a(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y})), S_b(K_\beta(\mathbf{x}), K_\beta(\mathbf{y}))]. \end{cases} \quad (1)$$

49
 50
 51 A justification for the choice of these expressions and a demonstration of the following theorem can
 52 be found in [54].

1 **Theorem 3.** (a) Let $\alpha, \beta \in [0, 1]$ such that $\alpha < \beta$ and let T_a and T_b be two t -norms in $[0, 1]$ such that
 2 $T_a \leq T_b$. Let the function $\mathbf{T} : (L([0, 1]))^2 \rightarrow L([0, 1])$ be given by

$$3 \quad \mathbf{T}(\mathbf{x}, \mathbf{y}) = [T_a(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y})), T_b(K_\beta(\mathbf{x}), K_\beta(\mathbf{y}))] \text{ for all } \mathbf{x}, \mathbf{y} \in L([0, 1]).$$

4
 5 Under these conditions,

$$6 \quad \mathbf{T} \text{ is an IV } t\text{-norm if and only if } \begin{cases} K_\alpha(\mathbf{x}) = K_0(\mathbf{x}) \\ K_\beta(\mathbf{x}) = K_1(\mathbf{x}) \end{cases}$$

7
 8 for all $\mathbf{x} \in L([0, 1])$.

9
 10 (b) Let $\alpha, \beta \in [0, 1]$ such that $\alpha < \beta$ and let S_a and S_b be two t -conorms in $[0, 1]$ such that $S_a \leq S_b$. Let
 11 the function $\mathbf{S} : (L([0, 1]))^2 \rightarrow L([0, 1])$ be given by

$$12 \quad \mathbf{S}(\mathbf{x}, \mathbf{y}) = [S_a(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y})), S_b(K_\beta(\mathbf{x}), K_\beta(\mathbf{y}))] \text{ for all } \mathbf{x}, \mathbf{y} \in L([0, 1]).$$

13
 14 Under these conditions,

$$15 \quad \mathbf{S} \text{ is an IV } t\text{-conorm if and only if } \begin{cases} K_\alpha(\mathbf{x}) = K_0(\mathbf{x}) \\ K_\beta(\mathbf{x}) = K_1(\mathbf{x}) \end{cases}$$

16
 17 for all $\mathbf{x} \in L([0, 1])$.

18
 19 A consequence of Theorem 3 is the following corollary (see [22, 23, 43, 51, 52, 60, 64–66, 68–77]).
 20 This corollary provides a construction method for IV t -norms and IV t -conorms, which was presented
 21 in the first papers on this topic.

22
 23 **Corollary 1.** (a) If T_a and T_b are two t -norms in $[0, 1]$ such that $T_a(x, y) \leq T_b(x, y)$ for all $x, y \in [0, 1]$,
 24 then the function $\mathbf{T} : (L([0, 1]))^2 \rightarrow L([0, 1])$ defined for each $\mathbf{x}, \mathbf{y} \in L([0, 1])$ by

$$25 \quad \mathbf{T}(\mathbf{x}, \mathbf{y}) = [T_a(K_0(\mathbf{x}), K_0(\mathbf{y})), T_b(K_1(\mathbf{x}), K_1(\mathbf{y}))]$$

26
 27 is an IV t -norm.

28
 29 (b) If S_a and S_b are two t -conorms in $[0, 1]$ such that $S_a(x, y) \leq S_b(x, y)$ for all $x, y \in [0, 1]$, then the
 30 function $\mathbf{S} : (L([0, 1]))^2 \rightarrow L([0, 1])$ defined for each $\mathbf{x}, \mathbf{y} \in L([0, 1])$ by

$$31 \quad \mathbf{S}(\mathbf{x}, \mathbf{y}) = [S_a(K_0(\mathbf{x}), K_0(\mathbf{y})), S_b(K_1(\mathbf{x}), K_1(\mathbf{y}))]$$

32
 33 is an IV t -conorm.

34
 35 The following theorem, proved in [54], makes it clear that the converse of Corollary 1 is not true: that
 36 is, there are IV t -norms (and IV t -conorms) that are not generated from expressions of the type (1).

37
 38
 39 **Theorem 4.** (a) Let there be an operator K_α with $\alpha \in [0, 1]$ and let T be any t -norm in $[0, 1]$. Let there
 40 be a function

$$41 \quad \mathbf{T}_\alpha^T(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{y} = 1_L \\ \mathbf{y} & \text{if } \mathbf{x} = 1_L \\ [T(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y})), T(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y}))]. \end{cases}$$

42
 43 Under these conditions,

$$44 \quad \mathbf{T}_\alpha^T \text{ is an IV } t\text{-norm if and only if } K_\alpha(\mathbf{x}) = K_0(\mathbf{x}) \text{ for all } \mathbf{x} \in L([0, 1]).$$

45
 46
 47 (b) Let there be an operator K_α with $\alpha \in [0, 1]$ and let S be any t -conorm in $[0, 1]$. Let there be a function

$$48 \quad \mathbf{S}_\alpha^S(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{y} = 0_L \\ \mathbf{y} & \text{if } \mathbf{x} = 0_L \\ [S(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y})), S(K_\alpha(\mathbf{x}), K_\alpha(\mathbf{y}))]. \end{cases}$$

49
 50
 51
 52
 53

Under these conditions,

$$\mathbf{S}_\alpha^S \text{ is an IV } t\text{-conorm if and only if } K_\alpha(\mathbf{x}) = K_1(\mathbf{x}) \text{ for all } \mathbf{x} \in L([0, 1]).$$

From now on, we shall denote \mathbf{T}_0^T and \mathbf{S}_1^S by \mathbf{T}^T and \mathbf{S}^S , respectively.

t-Representable IV t-norms and s-Representable IV t-Conorms

The objective of constructing characterization theorems for IV *t*-norms (and IV *t*-conorms) led Cornelis *et al.* [74] and Deschrijver *et al.* [77] to introduce the concepts of *t*-representable IV *t*-norm and *s*-representable IV *t*-conorm.

Definition 5. (a) An IV *t*-norm is said to be *t*-representable if there are two *t*-norms T_a and T_b in $[0, 1]$ such that $\mathbf{T}(\mathbf{x}, \mathbf{y}) = [T_a(K_0(\mathbf{x}), K_0(\mathbf{y})), T_b(K_1(\mathbf{x}), K_1(\mathbf{y}))]$ for all $\mathbf{x} \in L([0, 1])$ and for all $\mathbf{y} \in L([0, 1])$, such that $\mathbf{x} = [K_0(\mathbf{x}), K_1(\mathbf{x})]$, $\mathbf{y} = [K_0(\mathbf{y}), K_1(\mathbf{y})]$.

(b) An IV *t*-conorm is said to be *s*-representable if there are two *t*-conorms S_a and S_b in $[0, 1]$ such that $\mathbf{S}(\mathbf{x}, \mathbf{y}) = [S_a(K_0(\mathbf{x}), K_0(\mathbf{y})), S_b(K_1(\mathbf{x}), K_1(\mathbf{y}))]$ for all $\mathbf{x} \in L([0, 1])$ and for all $\mathbf{y} \in L([0, 1])$, such that $\mathbf{x} = [K_0(\mathbf{x}), K_1(\mathbf{x})]$, $\mathbf{y} = [K_0(\mathbf{y}), K_1(\mathbf{y})]$.

We denote by \mathbf{T}_{T_a, T_b} the *t*-representable IV *t*-norms obtained by means of the *t*-norms T_a and T_b in $[0, 1]$. We denote similarly the *s*-representable IV *t*-conorms \mathbf{S}_{S_a, S_b} .

Some examples of *t*-representable IV *t*-norms and *s*-representable IV *t*-conorms are the ones studied in Theorem 3. Some examples of non-*t*-representable IV *t*-norms and non-*s*-representable IV *t*-conorms are the ones studied in Theorem 4 (see [77]).

In [23, 43, 72–77], it is proved that some important representation theorems can be also shown for IV *t*-norms, but not for *t*-representable *t*-norms. This shows that IVFS theory cannot be reduced to an approach in which all operators are *t*-representable. In any case, the theoretical developments described in these publications are similar (although in a certain way they generalize them) to the theoretical developments carried out in [3, 20, 82] in order to characterize *t*-norms (and *t*-conorms) in $[0, 1]$. In [79], the concept of the interval-valued uninorm was analyzed for the first time.

The Archimedean Property of IV t-Norms

Starting from the work on *t*-norms in $[0, 1]$ of Klement *et al.* (see [83–86]), Deschrijver adapted the Archimedean property of *t*-norms in $[0, 1]$ to IV *t*-norms in [78]. In that publication, new and important concepts of weak Archimedean and strong Archimedean properties were presented, and the definition of the *pseudo-t*-representable IV *t*-norm was introduced in the following way:

An IV *t*-norm is called *pseudo-t*-representable if there exists a *t*-norm T on $([0, 1], \leq)$ such that, for all $\mathbf{x}, \mathbf{y} \in L([0, 1])$,

$$\mathbf{T}(\mathbf{x}, \mathbf{y}) = [T(K_0(\mathbf{x}), K_0(\mathbf{y})), \max(T(K_0(\mathbf{x}), K_1(\mathbf{y})), T(K_1(\mathbf{x}), K_0(\mathbf{y})))].$$

In [78], there is a study of the conditions under which *t*-representable IV *t*-norms and pseudo-*t*-representable IV *t*-norms satisfy the Archimedean property, the weak Archimedean property, or the strong Archimedean property. Particularly it is proved that the pseudo-*t*-representable IV *t*-norms satisfy the Archimedean property but the *t*-representable IV *t*-norms do not. This also justifies the comment above regarding the fact that IVFS theory cannot be reduced to an approach in which all operators are *t*-representable.

22.4 Combinations of Operations

Theorem 5. Let N be any involutive IV negation. The following items hold:

- (a) If \mathbf{T} is an IV *t*-norm, then the function defined by $\mathbf{S}^*(\mathbf{x}, \mathbf{y}) = N(\mathbf{T}(N(\mathbf{x}), N(\mathbf{y})))$ for all $\mathbf{x}, \mathbf{y} \in L([0, 1])$ is an IV *t*-conorm.

- 1 (b) If \mathbf{T} is t -representable, then \mathbf{S}^* is s -representable.
 2 (c) If \mathbf{S} is an IV t -conorm, then the function defined by
 3 $\mathbf{T}^*(\mathbf{x}, \mathbf{y}) = N(\mathbf{S}(N(\mathbf{x}), N(\mathbf{y})))$ for all $\mathbf{x}, \mathbf{y} \in L([0, 1])$ is an IV t -norm.
 4 (d) If \mathbf{S} is s -representable, then \mathbf{T}^* is t -representable.

5
 6 Given an IV t -norm \mathbf{T} , we call the expression for item (a) of Theorem 5,

$$7 \quad \mathbf{S}^*(\mathbf{x}, \mathbf{y}) = N(\mathbf{T}(N(\mathbf{x}), N(\mathbf{y}))) \text{ for all } \mathbf{x}, \mathbf{y} \in L([0, 1]),$$

8
 9
 10 the dual IV t -conorm of \mathbf{T} with respect to the IV negation N . Similarly, we call the expression for \mathbf{T}^* in
 11 item (c) the dual IV t -norm of the IV t -conorm \mathbf{S} with respect to the IV negation N [77].

12 Let the triple $(\mathbf{T}, \mathbf{S}, N)$ (where N is an involutive IV negation) denote that \mathbf{T} and \mathbf{S} are dual with respect
 13 to N ; any such triple is called a *dual triple* sometimes also called a De Morgan triple (see [54, 77] and
 14 Section 4 in [84]).

16 22.5 Laws for Conjunctions and Disjunctions

17
 18 In this section, we set out to study properties of set theory such as idempotency, absorption, distribu-
 19 tiveness, the law of contradiction, and the law of the excluded middle. These properties must be defined
 20 for IVFSs. We start from the work of Jenei [30] (see also [54, 76]) and we analyze the conditions under
 21 which IV t -norms and IV t -conorms satisfy or do not satisfy these properties. Some of the theorems that
 22 we present in this section are proved in [54] and others in [30].

24 22.5.1 Law of Contradiction and Law of the Excluded Middle

25
 26 Here, we study the conditions under which the law of contradiction and the law of the excluded middle
 27 are satisfied for IVFSs. For this purpose, we understand the law of contradiction for IVFSs in this
 28 manner, analogously to the cases of classical and fuzzy sets: for all $\mathbf{x} \in L([0, 1])$, it should hold that
 29 $\mathbf{T}(\mathbf{x}, N(\mathbf{x})) = 0_L$. Similarly, the law of the excluded middle says that for all $\mathbf{x} \in L([0, 1])$, it should hold
 30 that $\mathbf{S}(\mathbf{x}, N(\mathbf{x})) = 1_L$.

31
 32 **Theorem 6.** *The following items hold:*

- 33 (a) If \mathbf{T}_{T_a, T_b} is a t -representable IV t -norm, then for any involutive IV negation, the law of contradiction
 34 is not satisfied.
 35 (b) If \mathbf{S}_{S_a, S_b} is an s -representable IV t -conorm, then for any involutive IV negation, the law of the excluded
 36 middle is not satisfied.

37
 38 It is necessary to point out that there exist IV t -norms and IV t -conorms that are not t -representable
 39 or s -representable and satisfy the law of contradiction and the law of the excluded middle, respectively.
 40 This fact is made clear in the following theorem.

41
 42 **Theorem 7.** (a) *Let there be a t -norm T in $[0, 1]$ that satisfies the law of contradiction with respect to
 43 the involutive negation n and let the IV t -norm \mathbf{T}^T be generated by that t -norm. Then \mathbf{T}^T satisfies the
 44 law of contradiction with respect to the involutive IV negation N generated by n .*

45 (b) *Let there be a t -conorm S in $[0, 1]$ that satisfies the law of the excluded middle with respect to the
 46 involutive negation n and let the IV t -conorm \mathbf{S}^S be generated by that t -conorm. Then \mathbf{S}^S satisfies the law
 47 of the excluded middle with respect to the involutive IV negation N generated by n .*

48
 49 From Theorem 7 we deduce that there exist non- t -representable IV t -norms that satisfy the law of con-
 50 tradiction; evidently, we can also deduce that there exist non- s -representable IV t -conorms that satisfy the
 51 law of the excluded middle. Nevertheless, it is important to indicate that there exist non- t -representable
 52 IV t -norms that do not satisfy the law of contradiction, and there also exist non- s -representable IV
 53 IV t -norms that do not satisfy the law of contradiction, and there also exist non- s -representable IV

1 t -conorms that do not satisfy the law of the excluded middle; specific examples of these IV t -norms
 2 and IV t -conorms can be found in [54]. In that work there are also theorems that establish the man-
 3 ner in which one can construct from automorphisms non- t -representable IV t -norms that satisfy the
 4 law of contradiction (and similarly for non- s -representable IV t -conorms and the law of the excluded
 5 middle).
 6
 7

8 22.5.2 Idempotency, Absorption, and Distributivity

9
 10 **Theorem 8.** *The following items hold:*

- 11
 12 (a) $\mathbf{T}(\mathbf{x}, \mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in L([0, 1])$ (i.e., \mathbf{T} is idempotent) if and only if $\mathbf{T} = \mathbf{T}_{\min, \min}$.
 13 (b) $\mathbf{S}(\mathbf{x}, \mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in L([0, 1])$ (i.e., \mathbf{S} is idempotent) if and only if $\mathbf{S} = \mathbf{S}_{\max, \max}$.
 14 (c) $\mathbf{T}(\mathbf{x}, \mathbf{S}(\mathbf{x}, \mathbf{y})) = \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in L([0, 1])$ (property of absorption) if and only if \mathbf{T} is idempotent.
 15 (d) $\mathbf{S}(\mathbf{x}, \mathbf{T}(\mathbf{x}, \mathbf{y})) = \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in L([0, 1])$ (property of absorption) if and only if \mathbf{S} is idempotent.
 16

17 A consequence of Theorem 8 is the following: *the algebraic structure $\{\text{IVFSs}, \mathbf{T}, \mathbf{S}\}$ is only a lattice*
 18 *when we take $\mathbf{T} = \mathbf{T}_{\min, \min}$ and $\mathbf{S} = \mathbf{S}_{\max, \max}$.*
 19

20 **Theorem 9.** *Let \mathbf{T}_{T_a, T_b} be a t -representable IV t -norm and let \mathbf{S}_{S_a, S_b} be an s -representable IV t -conorm.*
 21 *Under these conditions, the following items hold:*
 22

- 23
 24 (a) $\mathbf{S}_{S_a, S_b}(\mathbf{x}, \mathbf{T}_{T_a, T_b}(\mathbf{y}, \mathbf{z})) = \mathbf{T}_{T_a, T_b}(\mathbf{S}_{S_a, S_b}(\mathbf{x}, \mathbf{y}), \mathbf{S}_{S_a, S_b}(\mathbf{x}, \mathbf{z}))$
 25 *for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L([0, 1])$, if and only if $\mathbf{T}_{T_a, T_b} = \mathbf{T}_{\min, \min}$.*
 26 (b) $\mathbf{T}_{T_a, T_b}(\mathbf{x}, \mathbf{S}_{S_a, S_b}(\mathbf{y}, \mathbf{z})) = \mathbf{S}_{S_a, S_b}(\mathbf{T}_{T_a, T_b}(\mathbf{x}, \mathbf{y}), \mathbf{T}_{T_a, T_b}(\mathbf{x}, \mathbf{z}))$
 27 *for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L([0, 1])$, if and only if $\mathbf{S}_{S_a, S_b} = \mathbf{S}_{\max, \max}$.*
 28

29 By Theorems 8 and 9 we have on the one hand the following result: $\{\text{IVFSs}, \mathbf{T}_{\min, \min}, \mathbf{S}_{\max, \max}\}$ is a
 30 distributive lattice, and on the other, the following corollary.
 31

32 **Corollary 2.** *Let \mathbf{T}_{T_a, T_b} be a t -representable IV t -norm and let \mathbf{S}_{S_a, S_b} be an s -representable IV t -conorm.*
 33 *Under these conditions, we have distributiveness if and only if we have absorption, if and only if we have*
 34 *idempotency, and*
 35

$$36 \text{ if and only if } \mathbf{T}_{T_a, T_b} = \mathbf{T}_{\min, \min} \text{ and } \mathbf{S}_{S_a, S_b} = \mathbf{S}_{\max, \max}.$$

37
 38 Regarding the distributive property of non- t -representable IV t -norms and non- s -representable IV
 39 t -conorms, there exist situations in which this property does not hold, as we will see in the following
 40 theorem which is an immediate consequence of this: if $(\mathbf{T}, \mathbf{S}, N)$ satisfies the distribution laws, then
 41 $\mathbf{T} = \mathbf{T}_{\min, \min}$ and $\mathbf{S} = \mathbf{S}_{\max, \max}$, but then \mathbf{T} does not satisfy the law of contradiction, since for any
 42 $\mathbf{x} \in L([0, 1])$, $N(\mathbf{x}) \in L([0, 1])$ (since N is involutive), so $\mathbf{T}(\mathbf{x}, N(\mathbf{x})) \geq_L 0_L$.
 43
 44

45 **Theorem 10.** *Let $(\mathbf{T}, \mathbf{S}, N)$ be a dual triple that satisfies the law of the excluded middle and the law of*
 46 *contradiction. Then $(\mathbf{T}, \mathbf{S}, N)$ does not satisfy the distributive laws.*
 47
 48

49 22.6 Interval-Valued Fuzzy Implication Operators

50
 51 In fuzzy set theory, a fuzzy implication operator I is a function $[0, 1]^2 \rightarrow [0, 1]$ that fulfills a certain set
 52 of properties (see [20]), so that $I(\mu_A(x), \mu_B(y))$ represents the degree of truth of the fuzzy conditional *If*
 53 *x is A then y is B* , where A and B are fuzzy sets.

1 A possible adaptation of the concept of the fuzzy implication operator to the case of IVFS theory has
 2 been given in [62] in the following way.

3 **Definition 6.** An interval-valued fuzzy implication operator is a function
 4

$$5 \quad I_{IV} : (L([0, 1]))^2 \rightarrow L([0, 1])$$

6 that has the following properties:

7
 8 I_{IV0} . If $\mathbf{x}, \mathbf{y} \in L([0, 1])$ are such that $K_0(\mathbf{x}) = K_1(\mathbf{x})$ and $K_0(\mathbf{y}) = K_1(\mathbf{y})$, then $W_{I_{IV}}(\mathbf{x}, \mathbf{y}) = 0$.

9 I_{IV1} . If $\mathbf{x} \leq_L \mathbf{x}'$, then $I_{IV}(\mathbf{x}, \mathbf{y}) \geq_L I_{IV}(\mathbf{x}', \mathbf{y})$ for all $\mathbf{y} \in L([0, 1])$.

10 I_{IV2} . If $\mathbf{y} \leq_L \mathbf{y}'$, then $I_{IV}(\mathbf{x}, \mathbf{y}) \leq_L I_{IV}(\mathbf{x}, \mathbf{y}')$ for all $\mathbf{x} \in L([0, 1])$.

11 I_{IV3} . $I_{IV}(0_L, \mathbf{x}) = 1_L$ for all $\mathbf{x} \in L([0, 1])$.

12 I_{IV4} . $I_{IV}(\mathbf{x}, 1_L) = 1_L$ for all $\mathbf{x} \in L([0, 1])$.

13 I_{IV5} . $I_{IV}(1_L, 0_L) = 0_L$.

14
 15 If $\mathbf{x}, \mathbf{y} \in L([0, 1])$, I_{IV0} establishes that if the sets are fuzzy, that is, the length of the intervals is zero,
 16 then the length of the interval-valued fuzzy implication operator is zero. Moreover, if in Definition 6 we
 17 replace $L([0, 1])$ by $[0, 1]$ and we eliminate the condition I_{IV0} , then we have the definition of a fuzzy
 18 implication operator in the sense of Fodor (see [20]).

19 This definition (Definition 6) extends classical two-valued implication; that is

$$20 \quad I_{IV}(0_L, 0_L) = I_{IV}(0_L, 1_L) = I_{IV}(1_L, 1_L) = 1_L \quad \text{and} \quad I_{IV}(1_L, 0_L) = 0_L.$$

21
 22 In [73], Cornelis *et al.* presented the following definition: An *implicator on* $L([0, 1])$ is
 23 any $(L([0, 1]))^2 \rightarrow L([0, 1])$ mapping \mathcal{I} satisfying $\mathcal{I}(0_L, 0_L) = \mathcal{I}(0_L, 1_L) = \mathcal{I}(1_L, 1_L) = 1_L$ and
 24 $\mathcal{I}(1_L, 0_L) = 0_L$. Moreover, we require \mathcal{I} to be decreasing in its first component and increasing in its
 25 second component.

26 Beginning from this concept, *S*-implicators on $L([0, 1])$ and *R*-implicators on $L([0, 1])$ were defined
 27 in [73]. The axioms of Smets and Magrez (see [87]) were adapted to the case of IVFSs, and various
 28 characterization theorems for *S*- and *R*-implicators on $L([0, 1])$ were analyzed. It was also made clear
 29 that every interval-valued implication operator I_{IV} (Definition 6) is an implicator on $L([0, 1])$.

30 We must point out that the characterization and construction theorems for the implicators on $L([0, 1])$
 31 (see [73]) are a generalization of the construction and characterization theorems for fuzzy *S*-implications
 32 and fuzzy *R*-implications (see [20]). In this connection, we see the following open problem. In the
 33 case of fuzzy sets, various characterizations of fuzzy *S*-implications, fuzzy *R*-implications, fuzzy *QL*-
 34 implications, and fuzzy *D*-implications have been studied, using in some cases the property $I(x, n(x)) =$
 35 $n(x)$ for all $x \in [0, 1]$. It is now necessary to carry out a study parallel to that in [88, 89], adapting the
 36 definitions and properties described in those publications to the implicators on $L([0, 1])$ defined above.

37 In [62], the properties usually required of interval-valued fuzzy implication operators are presented.
 38 These properties can be divided into two groups: the ones that result from adapting the properties of
 39 fuzzy implication operators to the interval-valued case and those that are interval-valued per se. For the
 40 latter group, the following three properties have been proposed, among others:

41
 42 I_{IV6} . $W_{I_{IV}}(\mathbf{x}, \mathbf{y}) \leq \vee(1 - K_0(\mathbf{x}), 1 - K_1(\mathbf{x}), 1 - K_0(\mathbf{y}), 1 - K_1(\mathbf{y}))$.

43 I_{IV7} . If $\mathbf{x} = \mathbf{y}$, then $W_{I_{IV}}(\mathbf{x}, \mathbf{y}) = W_{\mathbf{x}}$.

44 I_{IV8} . If $W_{\mathbf{x}} = W_{\mathbf{y}}$, then $W_{I_{IV}}(\mathbf{x}, \mathbf{y}) = W_{\mathbf{x}}$.

45
 46 Note that if the property I_{IV8} holds, then I_{IV7} holds, but the converse does not hold. Evidently, neither
 47 I_{IV7} nor I_{IV8} is in contradiction with I_{IV6} .

49 22.6.1 A Construction Method

50
 51 In [73], various methods for the construction of implicators on $L([0, 1])$ are presented. These methods are
 52 characterized by the fact that they always use the extremes of the intervals. However, in the construction
 53 method for interval-valued fuzzy implication operators that we shall present next (see [62]), other points

1 not obtained by applying t -norms and t -conorms in $[0, 1]$ to the extremes of the intervals, such as the
 2 average point of the interval, can also be used. This method comes out of the following interpretation of
 3 the condition rule (with IVFSs).

4 In the theory of IVFSs, a general rule for an expert system has the form *If u is A then v is B* , where u
 5 is a variable taking values in U , v is a variable taking values in V , $A \in \mathcal{IVFSs}(U)$, and $B \in \mathcal{IVFSs}(V)$.

6 The interval $A(u)$ is the truth degree of the proposition ' u is A .' Let us take two values, the extremes
 7 of the intervals, $K_0(A(u))$ and $K_1(A(u))$. Since in reality we are interested in assigning a single value to
 8 the degree of truth of the proposition ' u is A ,' we shall say that it is given by an aggregation of $K_0(A(u))$
 9 and $K_1(A(u))$, that is, by $M_1(K_0(A(u)), K_1(A(u)))$. The choice of the aggregation will depend on the
 10 experimental situation we are dealing with.

11 With respect to ' v is B ,' we can say the same. As before, we shall say that the truth degree of the propo-
 12 sition ' v is B ' is given by an aggregation of $K_0(B(v))$ and $K_1(B(v))$, that is, $M_2(K_0(B(v)), K_1(B(v)))$.

13 Therefore, bearing in mind these considerations and the interpretation of the fuzzy implication operator
 14 I , we shall say that a value that represents the truth degree of the interval-valued fuzzy conditional *If u
 15 is A then v is B* (where A, B are interval-valued fuzzy sets) is given by

$$16 \quad I(M_1(K_0(A(u)), K_1(A(u))), M_2(K_0(B(v)), K_1(B(v))))),$$

17
 18
 19 I being any fuzzy implication operator in Fodor's sense.

20 We can perform an analogous reasoning for the degree of non-truth of the proposition: if we take as
 21 the non-truth degree of the proposition ' u is A ' two values that are the negation of the extremes of the
 22 intervals, we obtain the result that the degree of non-truth of the proposition ' u is A ' will be given by any
 23 aggregation of $1 - K_1(A(u))$ and $1 - K_0(A(u))$, that is, by $M_3(1 - K_1(A(u)), 1 - K_0(A(u)))$. Likewise,
 24 the non-truth degree of the proposition ' v is B ' will be given by $M_4(1 - K_1(B(v)), 1 - K_0(B(v)))$, M_4
 25 being an aggregation operator.

26 Following further analogous reasoning, we have the result that the non-truth degree of the interval-
 27 valued fuzzy conditional *If u is A then v is B* is given by

$$28 \quad 1 - I'(1 - M_3(1 - K_1(A(u)), 1 - K_0(A(u))), 1 - M_4(1 - K_1(B(v)), 1 - K_0(B(v))))),$$

29
 30
 31 I' being any fuzzy implication operator in Fodor's sense.

32 Evidently, we can say that another value represents the degree of truth of the interval-valued fuzzy
 33 conditional *If u is A then v is B* (where A and B are interval-valued fuzzy sets). This is given by

$$34 \quad I'(1 - M_3(1 - K_1(A(u)), 1 - K_0(A(u))), 1 - M_4(1 - K_1(B(v)), 1 - K_0(B(v))))).$$

35
 36
 37 Therefore, we can interpret the interval

$$38 \quad [I(M_1(K_0(A(u)), K_1(A(u))), M_2(K_0(B(v)), K_1(B(v))))),$$

$$39 \quad I'(1 - M_3(1 - K_1(A(u)), 1 - K_0(A(u))), 1 - M_4(1 - K_1(B(v)), 1 - K_0(B(v))))]$$

40
 41
 42 as the truth degree of the interval-valued fuzzy conditional *If u is A then v is B* . (Obviously, in the
 43 expression above, I, I', M_1 , etc., are taken such that the expression is an interval.)

44 In the following proposition, we present a construction method for interval-valued fuzzy implication
 45 operators in the sense of Definition 6 (see [62]). The aggregations that we use in this construction method
 46 are the following: an *aggregation operator* is a $[0, 1]^2 \rightarrow [0, 1]$ mapping M that satisfies the following
 47 conditions:
 48

- 49
- 50
- 51 1. $M(0, 0) = 0; M(1, 1) = 1$.
- 52 2. M is increasing in its first and second argument.
- 53 3. $M(x, y) = M(y, x)$ for all $x, y \in [0, 1]$.

1 **Proposition 1.** *Let I be a fuzzy implication operator in Fodor's sense. Let $M_1, M_2, M_3,$ and M_4 be four*
 2 *idempotent aggregation operators such that*

3
 4
$$\begin{cases} M_1(x, y) + M_3(1 - x, 1 - y) \geq 1 \\ M_2(x, y) + M_4(1 - x, 1 - y) \leq 1, \end{cases}$$

5
 6
 7 *for all $x, y \in [0, 1]$. Then*

8
 9
 10
$$I_{IV} : (L([0, 1]))^2 \rightarrow L([0, 1]), \text{ given by}$$

 11
$$I_{IV}(\mathbf{x}, \mathbf{y}) = [I(M_1(K_0(\mathbf{x}), K_1(\mathbf{x})), M_2(K_0(\mathbf{y}), K_1(\mathbf{y}))),$$

 12
$$I(1 - M_3(1 - K_1(\mathbf{x}), 1 - K_0(\mathbf{x})), 1 - M_4(1 - K_1(\mathbf{y}), 1 - K_0(\mathbf{y})))]$$

13
 14 *is an interval-valued fuzzy implication operator in the sense of Definition 6.*

15
 16
 17 In [62], there is a study of the conditions under which the constructions of Proposition 1 fulfill the
 18 properties I_{IV6} – I_{IV8} , among others.

19
 20 **Example 1.** *If $M_1 = M_3 = \max$ and $M_2 = M_4 = \min$, then we obtain the interval-valued fuzzy impli-*
 21 *cation operator introduced by Jenei (see [30]); that is,*

22
 23
$$I_{IV}(\mathbf{x}, \mathbf{y}) = [I(K_1(\mathbf{x}), K_0(\mathbf{y})), I(K_0(\mathbf{x}), K_1(\mathbf{y}))].$$

24
 25 **Example 2.** *If, under the conditions of Example 1, we take the Kleene–Dienes fuzzy implication operator,*
 26 *that is, $I(x, y) = \max(1 - x, y)$, we have the expression for the first interval-valued fuzzy operator*
 27 *introduced by Atanassov (see [35]),*

28
 29
$$I_{IV}(\mathbf{x}, \mathbf{y}) = [\max(1 - K_1(\mathbf{x}), K_0(\mathbf{y})), \max(1 - K_0(\mathbf{x}), K_1(\mathbf{y}))].$$

30
 31
 32 In [62], it is proved that the expression in Example 2 satisfies the properties I_{IV6} , I_{IV7} , and I_{IV8} . Cornelis
 33 *et al.* proved in [73] that this expression is an S -implicator on $L([0, 1])$. Moreover, they also proved that
 34 if, under the conditions of Proposition 1, we take $M_1(x, y) = M_2(x, y) = M_3(x, y) = M_4(x, y) = (x +$
 35 $y)/2$ and the Kleene–Dienes implication operator, the expression that we obtain is not an S -implicator
 36 on $L([0, 1])$ or an R -implicator on $L([0, 1])$.

37 In [73], the definition of an interval-valued fuzzy S -implication (in Fodor's sense) is presented and in
 38 [63] the conditions under which these implications satisfy property I_{IV7} are studied. In this connection,
 39 the following open problems are worth mentioning:

- 40
 41 1. To study the cases for which interval-valued fuzzy S -implications satisfy the property I_{IV8} .
 42 2. To analyze the conditions that must be met in order for interval-valued fuzzy R -implications to satisfy
 43 the property I_{IV8} .
 44 3. To define interval-valued fuzzy D -implications (see [88]) and study the conditions under which these
 45 implications satisfy I_{IV8} and the property that results from generalizing the fuzzy property $I(x, n(x)) =$
 46 $n(x)$ for all $x \in [0, 1]$ to the case of IVFSs.

47
 48
 49 **22.7 IV Information Measures**

50
 51 The purpose of this section is to introduce the main developments and results regarding the best-known
 52 interval-valued information measures. The study that we present is for finite referentials. The non-finite
 53 case has been analyzed in the corresponding references.

Indetermination Index

In 1975, Sambuc [4] presented the following definition: given $A \in IVFSs(U)$, the *indetermination index* of the set A is the following expression:

$$\mathcal{J}(A) = \frac{\sum_{i=1}^{\mathcal{N}} W(A(u_i))}{\mathcal{N}}, \quad \text{where } \mathcal{N} = \text{Card}(U). \tag{2}$$

Sambuc used (2) to determine how far the IVFS considered was from the corresponding fuzzy set that he would have taken if he had used fuzzy set theory in his work. In fact, he reached the expression (2) by applying the concept of the Hamming distance between the extremes of the intervals. He also presented an expression for this index when the Euclidean distance was used for its construction.

IV Distances

A lot has been written on the concept of distance between IVFSs. In the following definition, we present the currently most commonly used expressions (see [90, 91]).

Definition 7. Let U ($\text{Card}(U) = \mathcal{N}$) be the referential set. We define the following distances:

1. The normalized Euclidean distance between A, B belonging to $IVFSs(U)$,

$$D_W^E(A, B) = \sqrt{\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \frac{(K_0(A(u_i)) - K_0(B(u_i)))^2 + (K_1(A(u_i)) - K_1(B(u_i)))^2 + (W(A(u_i)) - W(B(u_i)))^2}{2}}.$$

2. The normalized Hamming distance between $A, B \in IVFSs(U)$,

$$D_W^H(A, B) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \frac{|K_0(A(u_i)) - K_0(B(u_i))| + |K_1(A(u_i)) - K_1(B(u_i))| + |W(A(u_i)) - W(B(u_i))|}{2}.$$

3. The normalized Hausdorff distance between $A, B \in IVFSs(U)$,

$$\mathcal{D}_W(A, B) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \max (|K_0(A(u_i)) - K_0(B(u_i))|, |K_1(A(u_i)) - K_1(B(u_i))|, |W(A(u_i)) - W(B(u_i))|).$$

Historically, the expressions developed in Definition 7 were defined without including the term relative to the length of the intervals (see [4, 34, 60, 61]).

Degree of Compatibility

In 1987, Gorzalczany (see [9]) defined the *degree of compatibility* between two interval-valued fuzzy sets A and B on the same referential U in the following way:

The degree of compatibility $\Gamma(A, B)$ of an interval-valued fuzzy set A (such that there is at least one $u \in U$ with $K_0(u) \neq 0$) with an interval-valued fuzzy set B being an element of $L([0, 1])$ is given by

$$\Gamma(A, B) = \left[\min \left(\frac{\max_{u \in U} (\min(K_0(A(u)), K_0(B(u))))}{\max_{u \in U} K_0(A(u))}, \frac{\max_{u \in U} (\min(K_1(A(u)), K_1(B(u))))}{\max_{u \in U} K_1(A(u))} \right), \max \left(\frac{\max_{u \in U} (\min(K_0(A(u)), K_0(B(u))))}{\max_{u \in U} K_0(A(u))}, \frac{\max_{u \in U} (\min(K_1(A(u)), K_1(B(u))))}{\max_{u \in U} K_1(A(u))} \right) \right].$$

A study of the properties of this concept and a very interesting application of it can be found in [60].

IV Entropies

There are two different definitions in the literature of the concept of an interval-valued fuzzy entropy (IV entropy). The first, \mathcal{E}_F , was presented in 1996 in [61], and the second, \mathcal{E}_c , in 2001 in [91]. The difference between the two definitions lies in the fact that \mathcal{E}_F is a measure of how far an IVFS is from a

1 fuzzy set, whereas \mathcal{E}_c is a measure of how far an IVFS is from a crisp set. Therefore, \mathcal{E}_F is based on the
 2 ideas of Sambuc and \mathcal{E}_c is based on the concept of fuzzy entropy.

3
 4 **Definition 8.** A real function $\mathcal{E}_F : \mathcal{IVFSs}(U) \rightarrow \mathbb{R}^+$ is called an entropy on $\mathcal{IVFSs}(U)$ if \mathcal{E}_F has the
 5 following properties:

- 6
 7 (IF1) $\mathcal{E}_F(A) = 0$ if and only if $A \in \mathcal{FSS}(U)$.
 8 (IF2) $\mathcal{E}_F(A) = \text{Card}(U)$ if and only if $K_0(A(u)) = 0$ and $K_1(A(u)) = 1$ for all $u \in U$.
 9 (IF3) $\mathcal{E}_F(A) = \mathcal{E}_F(A_N)$ for all $A \in \mathcal{IVFSs}(U)$. (N is the involutive and strict IV negation generated by
 10 $n(x) = 1 - x$ for all $x \in [0, 1]$ according to Theorem 2.)
 11 (IF4) If $A \preceq B$, then $\mathcal{E}_F(A) \geq \mathcal{E}_F(B)$.

12
 13 In [61], these entropies \mathcal{E}_F are studied in depth and a theorem for their construction from functions
 14 $\varphi : [0, 1] \rightarrow [0, 1]$ is presented. It is also said there that the most commonly used expression is the
 15 one obtained when we take $\varphi(x) = x$ so that $\mathcal{E}_F(A) = \mathcal{J}(A)$ for all $A \in \mathcal{IVFSs}(U)$. The edge detector
 16 developed in [54] for objects in an image uses the IV entropy \mathcal{E}_F .

17
 18 **Definition 9.** A real function $\mathcal{E}_c : \mathcal{IVFSs}(U) \rightarrow \mathbb{R}^+$ is called an entropy on $\mathcal{IVFSs}(U)$ if \mathcal{E}_c has the
 19 following properties:

- 20
 21 (Ic1) $\mathcal{E}_c(A) = 0$ if and only if A is a crisp set.
 22 (Ic2) $\mathcal{E}_c(A) = 1$ if and only if $K_0(A(u)) = 1 - K_1(A(u))$ for all $u \in U$.
 23 (Ic3) $\mathcal{E}_c(A) = \mathcal{E}_c(A_N)$ for all $A \in \mathcal{IVFSs}(U)$. (N is the involutive and strict IV negation generated by
 24 $n(x) = 1 - x$ for all $x \in [0, 1]$ according to Theorem 2.)
 25 (Ic4) $\mathcal{E}_c(A) \leq \mathcal{E}_c(B)$ if $K_0(A(u)) \leq K_0(B(u))$ and $K_1(A(u)) \leq K_1(B(u))$ for $K_0(B(u)) \leq 1 - K_1(B(u))$
 26 or $K_0(A(u)) \geq K_0(B(u))$ and $K_1(A(u)) \geq K_1(B(u))$ for $K_0(B(u)) \geq 1 - K_1(B(u))$.

27
 28 A study of the main properties of \mathcal{E}_c can be found in [39, 91]. In this latter paper, a first attempt to
 29 relate \mathcal{E}_F to \mathcal{E}_c by means of a novel construction method is presented.

30 The idea of relating \mathcal{E}_c to IV similarities using item (2) of Definition 7 has led Szmidt and Kacprzyk
 31 to modify the definition of \mathcal{E}_c (see [92]).

32 It is necessary to point out that Tizhoosh (see [50]) experimentally proves that in the determination of
 33 the threshold of an image, the entropy \mathcal{E}_F provides very good results. This author at no time considers
 34 the IV entropy: \mathcal{E}_c .

35 In spite of the results obtained by Tizhoosh, we consider that we have the following open problem:
 36 defining interval-valued fuzzy entropy so that it gives as a result an element of $L([0, 1])$ and not an
 37 element of $[0, 1]$. We should also study the conditions under which we recover Definitions 8 and 9 from
 38 the new definition and in this way we can analyze the reasons why the use of \mathcal{E}_F gives good results in
 39 threshold computation in image processing.

40
 41 **IV Similarity**

42 In [60], there is also a section on the similarity of IVFSs. In fact, a normal interval-valued similarity measure
 43 $S(A, B)$ between two IVFSs A and B is defined as one that satisfies the following five properties: (i)
 44 $S(A, B) = S(B, A)$ for all $A, B \in \mathcal{IVFSs}(U)$; (ii) $S(D, D_C) = 0_L$ for all $D \in \mathcal{P}(U)$, where D_C is the
 45 complement of D and $\mathcal{P}(U)$ is the class of all crisp sets of U ; (iii) $S(C, C) = 1_L$ for all $C \in \mathcal{IVFSs}(U)$;
 46 (iv) for all $A, B, C \in \mathcal{IVFSs}(U)$, if $A \preceq B \preceq C$, then $S(A, B) \geq S(A, C)$ and $S(B, C) \geq S(A, C)$; and
 47 (v) if $A, B \in \mathcal{IVFSs}(U)$, then $S(A, B) \in L([0, 1])$. In [60], it is proved that the relation

48
 49
 50
$$S(A, B) = [S_L(A, B), S_U(A, B)], \tag{3}$$

51
 52 where $S_L(A, B)$ is a fuzzy similarity measure (see [93]) between the lower membership functions
 53 of A and B , and $S_U(A, B)$ is a fuzzy similarity measure between the upper membership functions of A

1 and B , satisfies (i)–(v). There are additional results, but these are beyond the scope of this
 2 chapter.

3 Starting from the work of Mitchell (see [94]), Mendel presented in [17] a new expression for IV
 4 similarities. In the same paper, Mendel posed the following open question: finding the possible connection
 5 (if there is one) between the similarity measure (3) and the expression that he proposes.

6 The IV similarities defined in [17, 60] have in common the fact that they give an interval as a result,
 7 that is, an element of $L([0, 1])$. However, the IV similarities proposed in [70, 95–98] are an adaptation
 8 of the well-known measures of similarity between fuzzy sets, so that they give an element of $[0, 1]$ as a
 9 result. Evidently, in this case the expressions that relate IV entropy (\mathcal{E}_c), IV similarity, and IV distances
 10 are identical to those obtained by Liu (see [93]) in 1992 for fuzzy sets. At this point, we would like to
 11 ask the following question:

12
 13 *Should the information measures between IVFSs be such that they give as a result an element of*
 14 *$L([0, 1])$ or an element of $[0, 1]$?*
 15

16 We think that this is the first problem that we should approach. Our particular opinion is that these
 17 measures should give elements of $L([0, 1])$ as a result. Furthermore, we think that if we have an interval-
 18 valued measure Π (e.g., the IV similarity), such that for each pair of elements $A, B \in \mathcal{IVFSs}(U)$ it gives
 19 as the result the interval $\Pi(A, B) \in L([0, 1])$, then it should turn out that for certain values of $\alpha \in [0, 1]$
 20 when a fuzzy measure π is applied (e.g., fuzzy similarity) in association with Π , the following equality
 21 holds:
 22

$$K_\alpha(\Pi(A, B)) = \pi(K_\alpha(A), K_\alpha(B)). \quad (4)$$

23
 24
 25
 26 We consider that in the future we must analyze the conditions under which information measures satisfy
 27 (4). Evidently, these measures should be defined so that they give as a result an element of $L([0, 1])$.

28 We have said before that in the future we must define the entropy of interval-valued fuzzy sets as
 29 an element of $L([0, 1])$. Once this definition has been made we must analyze different methods for the
 30 construction of these entropies and study their generation from IV similarities. We must also relate IV
 31 distances, IV similarities, and the new definition of IV entropy. To do this we must bear in mind each and
 32 every one of the ideas described in the references used up to now on the topic and on the results obtained
 33 in the following works: [75, 100–106].
 34
 35

36 ***IV Inclusion Measures, IV Correlation, and IV Information Energy***

37 In [60], inclusion measures between IVFSs were studied for the first time. These measures give an
 38 element of $L([0, 1])$ as a result. Also, various methods of construction of these measures from fuzzy
 39 implication operators were analyzed. In 2003, Kehagias and Konstantinidou proposed a new version of
 40 these measures [107].

41 In [75], Cornelis and Kerre present a definition of interval-valued inclusion measure in the same sense
 42 as the one presented in [60]; that is, the measure gives as a result an element of $L([0, 1])$. They propose
 43 an axiomatization that enables them to relate their inclusion measure with the expression of IV entropy
 44 \mathcal{E}_c . We consider that any study done to relate IV entropy, IV similarity, and IV distance must take into
 45 account the results obtained by these authors in order to generate \mathcal{E}_c from IV inclusion measures. We
 46 must also highlight the theoretical development made for using IV inclusion measures in approximate
 47 reasoning.

48 In [75], a method for the construction of interval-valued fuzzy entropies from IV inclusion measures
 49 is presented in [96]. However, in this case the method proposed is an adaptation to IVFSs of well-known
 50 results of fuzzy set theory, for the authors believe that the result of any information measure with IVFSs
 51 should always be an element of $[0, 1]$.

52 Gerstenkorn and Manko [108] introduced the concepts of IV *correlation* and IV *information energy*.
 53 Later, in [109], a detailed study of these concepts was carried out.

1 **22.8 Some Applications**

2 Our goal in these pages is not to go into all the details of each and every one of the fields where IVFSs
 3 are being applied; we wish to present only some representative contributions. Nevertheless, we think that
 4 we should make the following remarks:
 5

- 6
- 7 1. In most of the applications that we are going to present it is proved that when we have great imprecision
 8 in the determination of the membership degrees, better results are obtained by modeling with IVFSs
 9 than with FSs.
 - 10 2. The use of IVFSs does not increase the complexity of the algorithms, it only increases the number of
 11 necessary calculations for each algorithm. On the other hand, bearing in mind the latest technological
 12 advances, it results that the expense in time for the execution of the algorithms with IVFSs is practically
 13 the same as that of the algorithms with FSs. (This fact is made especially clear in algorithms that use
 14 IVFSs for image processing.)

15 **Approximate Reasoning**

16 Approximate reasoning is, formally speaking, as Turksen says [110], the process or processes by which
 17 a possible imprecise conclusion is deduced from a collection of imprecise premises. In this section, we
 18 present a short review of the inference methods most commonly used in the literature when the imprecise
 19 premises and the imprecise conclusions are represented using IVFSs.
 20

21 The generalized modus ponens (GMP) inference rule with IVFSs is represented in the following way
 22 (see [9, 65, 73, 111]):

$$\begin{array}{c}
 \text{If } u \text{ is } A \text{ then } v \text{ is } B \\
 u \text{ is } A' \\
 \hline
 v \text{ is } B',
 \end{array}$$

23 where u is a variable taking values in U , v is a variable taking values in V , $A, A' \in \mathcal{IVFSs}(U)$, and
 24 $B, B' \in \mathcal{IVFSs}(V)$.
 25

26 The methods given in order to obtain the conclusion B' can be divided into two groups: the ones that
 27 use an adaptation of Zadeh's compositional rule (see [1, 112]) to the interval-valued fuzzy case and those
 28 that do not.

29 The idea of applying Zadeh's compositional rule to the GMP with IVFSs led to the study of interval-
 30 valued fuzzy relations (IVFRs). In [64, 65, 67, 80, 113], the properties of IVFRs were analyzed and the
 31 composition of such relations was studied. Afterward, these relations were applied to the computation
 32 of the conclusion of the GMP with IVFSs. In [114], interval-valued fuzzy equations were studied for the
 33 first time (see [115]). We must say that the field of IVFRs is the least studied of those presented in this
 34 chapter.
 35

36 Arnould and Tano in 1995 (see [116]) constructed an expert system using rules with IVFSs. In the
 37 inference engine of that expert system, Zadeh's compositional rule is first applied to the lower extremes
 38 of the intervals and then to the upper extremes.
 39

40 With respect to the second group, that is, the applications that do not use Zadeh's composition rule, it is
 41 worth pointing out that all of them use the algorithm proposed by Gorzalczy (see [12]). This algorithm
 42 consists of two steps:
 43

- 44 1. Relate A to A' by means of an information measure.
- 45 2. Build the consequence B' using the result of the comparison above and B .

46 In [12], Gorzalczy used for step 1 the concept of the degree of compatibility, whereas in [60] first
 47 the degree of inclusion was used and then the IV similarities. Note that the measures used always give
 48 an element of $L([0, 1])$ as a result.
 49

50 We must point out that regardless of the method used for calculating $B' \in \mathcal{IVFSs}(V)$, we must always
 51 study properties of the type 'if $A' = A$, then $B' = B$,' etc. In this connection, in [60] there is an in-depth
 52 study of these properties.
 53

1 analysis of the conditions under which the methods developed in that paper satisfy the axioms of Fukami
 2 *et al.* (see [117]) or the axioms of Baldwin and Pilsworth (see [118]). A similar study has been carried
 3 out in [65, 73] for the methods that use Zadeh’s compositional rule adapted to the interval-valued fuzzy
 4 case.

5 In [12, 64, 65, 68, 70, 116, 119], various methods for obtaining the conclusion from a system of
 6 interval-valued rules, in addition to the GMP with IVFSs, can be found.

7
 8 **Image Processing**

9 A very important problem in image processing is the detection of the edges of the objects that make up
 10 an image. A pixel is said to belong to an edge if it has associated with it a big enough change in intensity.
 11 In [54], each image is associated with an IVFS so that the length of the interval associated with each
 12 pixel represents the intensity change between that pixel and its neighbors. Therefore, if the length of the
 13 interval is large enough, then the pixel belongs to an edge. The edge detector developed in [54] surpassed
 14 the classical detectors in the literature in three of the four types in which images were classified. In [54],
 15 an expression for calculating the contrast of an image using interval-valued *S*-implication operators is
 16 given.

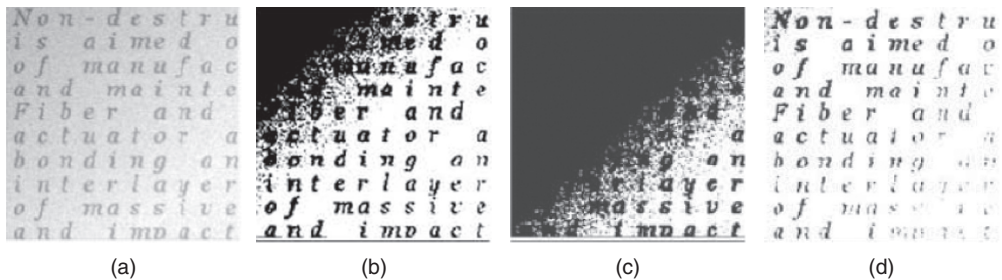
17 In Section 22.2.2 we have said that Tizhoosh developed a method for calculating the threshold of
 18 an image using IVFSs. In [50] and later in [51] and [52] it has been experimentally proved that when
 19 working with images that have a large number of pixels such that the experts are not able to determine
 20 precisely whether they belong to the background or to the object, this method provides better results than
 21 the rest of known methods. In any other type of images, it has also been proved that the results obtained
 22 with this algorithm are similar to those obtained with algorithms that do not use IVFSs.

23 The algorithm proposed by Tizhoosh for images with *L* levels of gray consists of the following steps:

- 24
 25 1. Assign *L* fuzzy sets to each image *Q*.
 26 2. Associate with each fuzzy set its corresponding IVFS constructed with the method described in
 27 Section 22.2.2.
 28 3. For each IVFS, calculate \mathcal{E}_F .
 29 4. Take as threshold the intensity corresponding to IVFS with lowest value of \mathcal{E}_F .
 30

31 We must point out that Tizhoosh calls \mathcal{E}_F (see again [61]) ultrafuzziness and that the algorithm above
 32 has been generalized in [51, 52]. In Figure 22.2 we give an example where it is proved that we obtain
 33 the best result with this algorithm. In [54] it has been proved that the time and memory efficiency of the
 34 algorithms that use IVFSs (for image processing) is practically the same as the efficiency of those that
 35 do not use IVFSs.

36 In [120], Gaussian noise was eliminated from an image using algorithms with IVFSs. In general, each
 37 and every one of the applications constructed with IVFSs for image processing (when it is applied to
 38 images with great imprecision in the membership of the pixels that compose it) gives better results than
 39 those constructed with fuzzy sets. This is due to the fact that they use a characteristic of IVFSs that fuzzy
 40 sets do not have, namely, length (of interval).
 41



50 (a) (b) (c) (d)
 51
 52 **Figure 22.2** (a) Original image; (b) image binarized with the classical Otsu method; (c) image binarized
 53 with the Huang fuzzy method; (d) image binarized with IVFS method

1 **Computing with Words**

2 In [121] Zadeh defines computing with words (CWW) in the following way: *a theory in which the objects*
 3 *of computation are words and propositions drawn from a natural language.* We know that words can
 4 have different meanings depending on the person and the context in which the person is. Therefore there
 5 is uncertainty associated with the words (see [17, 59, 122]). This reasoning has made several authors
 6 consider it necessary to use IVFSs in CWW (see [123]).

7
 8 **Decision Making**

9 In decision making, one works with many types of information, among them interval-valued information.
 10 In [8], various methods of processing information represented in this way are analyzed. Furthermore,
 11 in [124, 125], there is a study of several processes of aggregation of non-homogeneous information
 12 with contexts composed of numerical, interval-valued, and linguistic values. It is also worth point-
 13 ing out the work of Bilgiç (see, e.g., [67]). More applications in decision making can be found in
 14 [126–128].

15
 16 **Other Applications**

17 Obviously there exist a lot of applications of IVFSs; here we present the most representative. For appli-
 18 cations to fuzzy linear programming, see [69, 129]. For economics, see [130, 131]. For medicine, see
 19 [4, 113]. For robotics, see [118, 132–134]. For fuzzy modeling, see [135–137]. For Web intelligence,
 20 see [56]. For the theory of possibility, see [73]. For control, see [10, 57].

21
 22
 23 **22.9 Granular Computing and Interval-Valued Fuzzy Sets**

24 The topic of fuzzy information granulation was first proposed and discussed by Zadeh in 1979 (see [138]).
 25 In this paper, a granule was defined as a collection of indistinguishable objects and granular information
 26 as the grouping of objects into granules. Later, Zadeh established in [139] the following: *the theory of*
 27 *fuzzy information granulation is inspired by the ways in which humans granulate information and reason*
 28 *with it.* On the basis of these considerations Yao (see [140]) says that granular computing may be regarded
 29 as a label of theories, methodologies, techniques, and tools that make of granules, that is, groups, classes,
 30 or clusters of a universe, in the process of problem solving.

31 Therefore, granular computing is based on the principle that the description of the world around us by
 32 means of techniques that use exclusively numeric precision is often unnecessary and has a very high cost.
 33 Furthermore, granular computing also considers the fact that human thinking does not work at a numeric
 34 level of precision but at a much more abstract level. In this sense in [81] the following is established:
 35 *granular computing is a formalism for expressing that abstraction within computational processes,*
 36 *thus endowing computer systems with a more human-centric view of the world. The central notion in*
 37 *granular computing is that there are many ‘levels’ of precision in which information about the real world*
 38 *can be expressed, with numeric precision being the most refined and a binary value the coarsest* (see
 39 [141]).

40 A granule can be interpreted as one of the numerous particles or elements that compose a unit. The
 41 size of the granule is its basic property. Intuitively the size can be interpreted as the degree of abstraction,
 42 detail, or precision.

43 In many applications, when the problem to solve deals with imprecise or vague information, it can be
 44 difficult to identify specific data and then we are forced to use granules. Furthermore, we have said in
 45 the introduction that the use of IVFSs arises in the applications where it is hard to precisely determine
 46 the value of the membership function for the elements. These two arguments allow us to say that IVFSs
 47 can be used in the tasks of representing, operating, and reasoning with granules (see [138]).

48 Depending on the application, a granule can be an element, that is, an interval, or it can be represented
 49 by an IVFS. In Section 22.2.2 we have seen that we can construct IVFSs from fuzzy sets; therefore, we
 50 can construct hierarchies of granules and represent arbitrary groups (clusters) of granules using IVFSs.
 51 This representation has the advantage that it enables us to use all of the results known for IVFSs in
 52 granular computing.
 53

1 **22.10 Conclusion**

2
 3 In this chapter, we have reviewed the basic concepts of IVFSs. We have focused on the elementary
 4 properties and have given a detailed set of references for more advanced properties. We have also
 5 posed several unsolved problems and have demonstrated the importance that the study of properties that
 6 differentiate them from fuzzy sets should have in future research.

7 As with all young theories, there are a great number of open problems in IVFS theory. We have
 8 presented some of them at various points in this chapter.

9 We have detected that currently there is general interest in finding applications of type 2 fuzzy sets
 10 and that the best results are almost always obtained when IVFSs are used. For this reason, we believe
 11 that these sets are going to be thoroughly studied and used in the next few years.

12
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1 **Abstract:** In this chapter we carry out a survey of the theoretical properties and the most important
2 applications of interval-valued fuzzy sets (IVFSs). We present the conditions under which IVFSs should
3 be used and study their relation with other extensions of fuzzy set theory. We also indicate the three major
4 problems of this theory from our point of view. We study structural properties and show the connectives
5 for operating. We review the most commonly used numerical measures of information and pose the
6 problem of the loss of information when they are not defined in an intervalar way. Next we show the
7 importance of these sets in some applications, as, for example, in the thresholding of certain types of
8 images as well as in approximate reasoning. Lastly, we describe their connection with granular computing
9 and present a comprehensive bibliography. Throughout the chapter we list different open problems.

10 **Keywords:** interval-valued fuzzy sets; interval type 2 fuzzy sets; IV negation; IV t -norms; IV t -conorm;
11 interval-valued fuzzy implication operator; indetermination index; IV distance; degree of compatibility;
12 IV entropy; IV similarity; IV inclusion measure
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