# Information Aggregation: ethical and computational issues

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**Abstract**

In this paper we survey on aggregation operators and in particular hierarchies of them. As a case study, we will analyze the problems of the aggregation of truth values of fuzzy predicates and the aggregation of individual opinions into a single group opinion, based upon hierarchical intensity aggregation rules. We will see that hierarchical amalgamations are supported from an ethical and rational point of view. Two different hierarchical procedures will be recalled: cover-based hierarchical aggregations and ordered hierarchical aggregations.

Finally we will see that when we deal with ordered hierarchical aggregations of OWA operators, some interesting computational problems appear quite naturally. Such problem admit polynomial time solutions.

**Key words:** Aggregation rules, fuzzy preferences, group decision making.

## 1 Introduction and preliminaries

The topic of our discussion is intuitively very simple yet at the basis of many applications. Controllers, expert and decision making systems, theorem provers, learning algorithms, to mention a few, are all (computer aided) applications in which it is of primary importance to have mechanisms able to analyze the gathered information and produce from it some aggregated value(s) to be (possibly) used in some next evaluation steps.

Common hypothesis to several applications of such kind is that information is passed to an aggregation operator as an ordered sequence of real numbers, which without loss of generality can be supposed to belong to the unit interval. Formally, an aggregation rule of dimension $n$, will be a mapping

$$\phi^{(n)} : [0,1]^n \to [0,1].$$

To capture the essential meaning of aggregation, $\phi^{(n)}$ must satisfy at least the following two conditions (see [15]):

- $\phi^{(n)}(0,0,\ldots,0) = 0, \quad , \phi^{(n)}(1,1,\ldots,1) = 1$ and
- monotonicity, i.e. $\phi^{(n)}(a_1,a_2,\ldots,a_n) \le \phi^{(n)}(b_1,b_2,\ldots,b_n)$ if $a_i \le b_i$ for all $i$.

If we think of $\phi^{(n)}$ as an algorithm that takes as inputs $n$ numbers and then it outputs the aggregated value of the $n$ numbers, we can obtain aggregation rules that are independent from the number of aggregands by generalizing the above definition as follows. Put

$$\mathcal{I} = \bigcup_{n \geq 2} [0,1]^n.$$

For every $L \in \mathcal{I}$ there exists one and only one $k \geq 2$ such that $L \in [0,1]^k$; such an index $k$ is obviously the "dimension" of $L$, which we will denote by $|L|$.

Then $\phi$ on $\mathcal{I}$ can be defined as

$$\phi(L) = \phi^{(|L|)}(L).$$

For simplicity, we will not specify the dimension of the operator whenever it will be clear from the context.

In what follows we will concentrate on two particular applications

1. the aggregation of truth values of fuzzy predicates in a most general and comprehensive way.
2. the aggregation of preference intensities expressed by individual/groups over finite set of alternatives, so to obtain a *rational* and *democratic* social opinion.

Notice that when we say that we want to aggregate group opinions to obtain a social opinion we are basically saying that we want to aggregate values which are themselves an aggregation of values. Therefore, a further generalization of the above definition of aggregation rules is obtained by introducing the notion of hierarchical aggregations, that is to say aggregations of chunks of information which in turn represent aggregated information.

In general, hierarchical aggregation procedures for individual preferences are defined by means of a basic classification of the individuals. The set of individual is divided into groups, in such a way that each individual is present in at least one of these groups. An immediate example is provided by national Parliaments. Each of the representative of the Parliament has been elected by a particular group of people and therefore he/she represents the aggregated opinion of such people. When the Parliament express its opinion then what we have is an hierarchical aggregation of the people opinions. Since voters are generally divided into electoral districts, we have a fixed "cover" of the set of individuals, and the social opinion is obtained by using such cover. Hierarchical aggregations of this kind will be called "cover-based".

Alternatively, basic classification can be made on the basis of a natural ordering on all possible intensity values, and then we can talk about "ordered" hierarchical aggregation rules. This second type of hierarchical aggregations are obviously more natural when dealing with fuzzy predicates truth aggregations.

## 2 Truth values aggregation

One of the main issues in Fuzzy Logic is the choice of operators to generalize the classical logic operators *and*, *or*. Many proposal have been made and the most general formalizations are represented by T-norms, T-conorms and OWA operators (see [23,24]).

A T-norm is a map $T : [0,1] \times [0,1] \to [0,1]$ such that

**(T1)** $T(a,b) = T(b,a)$

**(T2)** $T(a,T(b,c)) = T(T(a,b),c)$

**(T3)** $T(a,b) \geq T(c,d)$ if $a \geq c$ and $b \geq d$

**(T4)** $T(a,1) = a$

A T-conorm is a map $S : [0,1] \times [0,1] \rightarrow [0,1]$ such that

**(S1)** $S(a,b) = S(b,a)$

**(S2)** $S(a, S(b,c)) = S(S(a,b), c)$

**(S3)** $S(a,b) \geq S(c,d)$ if $a \geq c$ and $b \geq d$

**(S4)** $S(a,0) = a$

T-norms and T-conorms represent aggregation operators that generalize the notion of conjunction and disjunction of classical logic. The min operator is the maximal T-norm and the max operator is the minimal T-conorm, in the sense that for all $a, b$ we have that $T(a,b) \leq \min\{a,b\}$ and $S(a,b) \geq \max\{a,b\}$ hold for any T-norm $T$ and any T-conorm $S$ (see [15]).

*Ordered Weighted Averaging (OWA)* operators fill the gap between min and max, in such a way that by means of these OWA operators we can go from conjunction (intersection) to disjunction (union) in a continuous way. OWA operators were initially introduced by Yager in [24], and they have been used and applied to many fields, such as Neural Networks, Database systems, Learning systems and Fuzzy Logic Controllers (see [25] for a comprehensive review on the subject). In order to get the aggregated values, OWA operators make use of the relative order within intensity values.

Formally, an OWA operator of dimension $n$ is an aggregation operator $\phi$ that has an associated list of weights $W = [w_1, \ldots, w_n]$ such that

1. $w_i \in [0,1]$ for all $1 \leq i \leq n$
2. $\sum_{i=1}^{n} w_i = 1$
3. for any $L = [a_1, a_1, \ldots, a_n] \in [0,1]^n$

$$\phi(L) = \sum_{i=1}^{n} w_i b_i .$$

   where $b_1 \geq \cdots \geq b_n$, is the sequence obtained when we sort in non decreasing order the inputs $a_1, a_2, \ldots, a_n$.

Therefore, OWA operators are commutative, monotone and idempotent, but in general not associative. As a consequence, the semantic problems of using the *same* OWA operator on inputs of different dimensions, i.e. the problem of defining the OWA operator $\phi : \mathcal{I} \rightarrow [0,1]$ given the OWA operators $\phi^{(n)}$ is quite challenging. An answer to such a problem has been given in [10] where it is introduced the notion of recursive families of OWA operators. For the time being, we will ignore such a problem and we will only deal with a particular type of hierarchical aggregation which we will call *ordered* hierarchical aggregation and that will allow us to deal in a very natural way with OWA operators.

# 3 Aggregation of preference intensities

The classical Arrow's paradox in group decision making (cfr. [1]) when translated into a fuzzy context (see [2,13,14,17,20]), can be avoided in several ways, according to axiomatics which are similar to those proposed in the crisp context. As we mentioned above it is very common to obtain aggregated preferences of large groups of people by means of rules which allow the successive aggregation of degrees of preferences. The set of individuals is divided into smaller subsets of individuals -not necessarely disjoint- and the global aggregation will be the aggregation of all partial aggregations within each one of these subsets of individuals. We follow the model proposed in [18,19] and subsequentely characterized in [5], and the formalization given in [7].

At the basis of such a model are non absolutely irrational (in the sense of [18,19]) complete fuzzy preference relations. In particular, it is assumed that each individual is able to express her/his opinion about any possible set of alternatives through some complete fuzzy binary preference relation, as formalized below.

Let $\mu : X \times X \to [0,1]$ be a fuzzy preference relation over an arbitrary finite set of alternatives $X$. $\mu(x,y)$ represents the degree to which the relation $x$ *not worse than* $y$ holds. The completeness hypothesys is expressed by

$$\mu(x,y) + \mu(y,x) \geq 1 \quad \forall x, y \in X \tag{3.1}$$

Following [4], completeness is required in order to assure that all individuals consider the set of alternatives on which they are expressing their opinions, feasible and comprehensive.

The values

$$\begin{aligned} \mu_I(x,y) &= \mu(x,y) + \mu(y,x) - 1 \\ \mu_B(x,y) &= \mu(x,y) - \mu_I(x,y) \\ \mu_W(x,y) &= \mu(y,x) - \mu_I(x,y) \end{aligned} \tag{3.2}$$

can be understood, respectively, as the degree to which the two alternatives are indifferent ($xIy$), the degree of strict preference of $x$ over $y$, ($xBy$, $x$ *is better than* $y$) and the degree of strict preference of $y$ over $x$ ($xWy$, $x$ *is worse than* $y$). We clearly have that

$$\mu_B(x,y) + \mu_I(x,y) + \mu_W(x,y) = 1 \quad \forall x, y . \tag{3.3}$$

We want to define rationality as a fuzzy property. We then consider cycles of preferences over chains $G = (x_1 - x_2 - \cdots - x_k - x_1)$ of $k$ distinct alternatives, defined as

$$x_1 P_1 x_2 P_2 \cdots x_k P_k x_1$$

where $P_h \in \{W, I, B\}$ for all $h = 1, 2, \ldots, k$. A cycle $x_1 P_1 x_2 P_2 \cdots x_k P_k x_1$ is *irrational* if either

- $P_h \in \{B, I\}$ for all $h = 1, 2, \ldots, k$ and $B \in \{P_h : h = 1, 2, \ldots, k\}$; or
- $P_h \in \{W, I\}$ for all $h = 1, 2, \ldots, k$ and $W \in \{P_h : h = 1, 2, \ldots, k\}$.

We say that a cycle is rational if it is not irrational. Then, given any fuzzy preference $\mu$ over a fixed set of alternatives and a chain of alternatives, we can look for all possible rational cycles of preferences, weigh them in some way and assign to the chain a degree of rationality (see [5,18]). Specifically, this is done as follows. Given a a cycle $C \equiv x_1 P_1 \cdots x_k P_k x_1$ where $P_h \in \{B, I, W\}$ for all $h = 1, 2, \ldots, k$, a quite natural weight associated to $C$ and denoted by $\Delta(C)$ is

$$\Delta(C) = \Pi_{h=1}^{k} \mu_{P_h}(x_h, x_{h+1})$$

where $x_{k+1} = x_1$ for convenience.

Therefore, given a chain $G = (x_1 - x_2 - \cdots - x_k - x_1)$ a degree of rationality associated to $G$ and denoted by $A_\mu(G)$ can be defined as

$$A_\mu(G) = \sum_{C \in \text{ rat.cycles}} \Delta(C).$$

As proven in [5,18], $A_\mu(G)$ verifies

$$1 - A_\mu(G) = \Pi_{h=1}^{k} \mu(x_h, x_{h+1}) + \Pi_{h=1}^{k} \mu(x_{h+1}, x_h) - 2\Pi_{h=1}^{k} \mu_I(x_h, x_{h+1}). \tag{3.4}$$

In view of (3.4), once a finite set of alternatives $X$ has been fixed, rationality can be defined as a fuzzy property $A : \mathcal{P}(X) \to [0,1]$ with

$$A(\mu) = \min_G A_\mu(G) \tag{3.5}$$

and where $\mathcal{P}(X)$ is the set of all complete fuzzy preferences. Needless to say, the above is just a particular rationality measure, which may or may not seem "rational". A formal characterization of rationality measures, was given in [8], following the approach described in [6]. The above described rationality measure as well as the most commonly found in literature, are particular instance of such general definition. It would then be desirable to extend the results that we will describe below to "generic" fuzzy rationality measures.

Once a group of $n \geq 2$ individuals is fixed, we should be able to aggregate their opinions about any set of alternatives in a coherent way. Therefore, in [5] were defined aggregation operations that can take into account any extra alternative $x$ so to properly extend any previous aggregated opinion relative to a collection of alternatives not containing $x$. The key properties are the standard conditions

**(IIA)** *Independence of Irrelevant Alternatives*: each aggregated preference relation $\mu(x,y)$ depends solely on the values $\mu^i(x,y)$, i.e. on the individual preference intensities of $x$ over $y$

**(UD)** *Unrestricted Domain*: the aggregation rule is defined over all possible profiles of fuzzy preferences.

Given an aggregation rule $\phi^{(n)}$, if we also assume

**(N)** *Neutrality*: given any permutation of the set of alternatives $\pi$, if $\nu^i(x,y) = \mu^i(\pi(x), \pi(y))$ for all $i = 1, 2, \ldots, n$ and any pair of alternatives $x, y$, then

$$\phi^{(n)}(\nu^1(x,y), \cdots, \nu^n(x,y)) = \phi^{(n)}(\mu^1(x,y), \cdots, \mu^n(x,y))$$

It is clear then that the same aggregation $\phi^{(n)}$ will be associated to any pair of alternatives and therefore each possible aggregation procedure is characterized by one of these aggregation rules.

For the time being, we will suppose that conditions IIA, UD, N hold.

Given $n$ individuals expressing their opinion on the set of alternatives $X$ and $\phi^{(n)}$, the aggregated preference $\mu$ defined on $X \times X$ associated to $\phi^{(n)}$ is defined as

$$\mu(x,y) = \phi^{(n)}(\mu^1(x,y), \cdots, \mu^n(x,y)) \ \ \forall x, y \in X.$$

Standard *ethical* conditions may also be imposed on the intensity aggregation rules, among them:

**(PR)** *Positive Responsiveness*:

$$\phi^{(n)}(a_1, a_2, \ldots, a_n) > \phi^{(n)}(b_1, b_2, \ldots, b_n)$$

if $a_i \geq b_i$ for all $i = 1, 2, \ldots, n$ and there exists $1 \leq j \leq n$ such that $a_j > b_j$.

**(A)** *Anonimity*: given any permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, we have

$$\phi^{(n)}(a_1, a_2, \ldots, a_n) = \phi^{(n)}(a_{\pi(1)}, \ldots, a_{\pi(n)}).$$

**(U)** *Unanimity*: if $a_i = a$ for all $i = 1, 2, \ldots, n$, then

$$\phi^{(n)}(a_1, a_2, \ldots, a_n) = a$$

**(CS)** *Citizen Sovereign*: for any given $a \in [0,1]$ there exists a profile $(a_1, a_2, \ldots, a_n) \in [0,1]^n$ such that $\phi^{(n)}(a_1, a_2, \ldots, a_n) = a$.

**(ND)** *Non Dictatorship*: there is no individual $i$ such that

$$\phi^{(n)}(a_1, a_2, \ldots, a_n) = a_i$$

for any $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in [0,1]^{n-1}$.

We will say that an intensity aggregation rule is *complete* if and only if the associated aggregated fuzzy preference is complete for any profile of complete individual preferences.

A characterization of complete aggregations rules is given by the following lemma proven in [5].

**LEMMA 1** *An intensity aggregation rule $\phi^{(n)} : [0,1]^n \to [0,1]$ is complete if and only if*

$$\phi^{(n)}(a_1, \ldots, a_n) + \phi^{(n)}(b_1, \ldots, b_n) \geq 1$$

*whenever $a_i + b_i \geq 1$ for all $i = 1, 2, \ldots, n$. Moreover, since $\phi^{(n)}$ is monotone non decreasing then it is complete if and only if*

$$\phi^{(n)}(a_1, \ldots, a_n) + \phi^{(n)}(1 - a_1, \ldots, 1 - a_n) \geq 1$$

*for all $(a_1, \ldots, a_n) \in [0,1]^n$.* ■

The above given fuzzy property of rationality is extended to aggregation rules in the following way.

**DEFINITION 1** *Given $n$ individuals, an aggregation rule $\phi^{(n)} : [0,1]^n \to [0,1]$ is* non absolutely irrational *(NAI), or simply* non irrational, *if for any arbitrary finite set of alternatives $X$, the associated aggregated preference $\mu : X \times X \to [0,1]$ is complete and non absolutely irrational, i.e. $A(\mu) > 0$, whenever all individuals are complete and non absolutely irrational themselves, i.e. $A(\mu^i) > 0$ for all $i = 1, 2, \ldots, n$, with $\mu^i : X \times X \to [0,1]$ for all $i$.* □

It is clear that in this way both individual and social opinions are required to belong to the set of Non-Absolutely Irrational (NAI) complete fuzzy preference relations. Therefore, we are in fact modifying the Unrestricted Domain condition.

The main result proven in [5] is the following.

**THEOREM 1** *Let $\phi : [0,1]^n \to [0,1]$ be a complete intensity aggregation rule verifying condition A. Then $\phi$ is NAI if and only if the following conditions hold:*

*(i) if $a_i + b_i > 1$ for all $i = 1, 2, \ldots, n$, then $\phi(a_1, \ldots, a_n) + \phi(b_1, \ldots, b_n) > 1$,*

*(ii) $\phi(a_1, \ldots, a_n) = 1$ implies $a_i = 1$ for all $i = 1, 2, \ldots, n$.* ■

Moreover, from the proof of theorem 1 it can be concluded that in order for a complete intensity aggregation rule to be NAI, conditions (i) and (ii) are sufficient.

Two immediate corollaries of theorem 1 are

**COROLLARY 1** *A complete intensity aggregation rule* $\phi : [0,1]^n \to [0,1]$ *verifying conditions CS and A is NAI if and only if conditions (i)-(ii) of theorem 1 hold.* ■

**COROLLARY 2** *Let* $\phi : [0,1]^n \to [0,1]$ *be a complete intensity aggregation rule verifying condition PR. Then* $\phi$ *is NAI.* ■

**Example 1** By applying Corollary 2 we can easily prove that the following intensity aggregation rule (see [15], pg. 60) is NAI:

- *Weighted Generalized Mean:* given $w_1, \ldots, w_n$, positive real numbers and $r \geq 1$, then

$$WGM(a_1, \ldots, a_n) = \left( \sum_{i=1}^{n} \frac{w_i (a_i)^r}{\sum_{j=1}^{n} w_j} \right)^{\frac{1}{r}}$$

where the assumption $r \geq 1$ has been made in order to assure completeness.

# 4 Cover-based hierarchical aggregation rules

Cover-based hierarchical aggregation rules appear in practice when the whole set of individuals is previously divided into groups (not necessarely disjoint), in such a way that the amalgamation is obtained by means of those subsets of individuals. Sometimes the society under study is naturally classified into those smaller groups, and sometimes such a classification is just claimed because of the large dimension of the group. Hence, final aggregation will be obtained as an amalgamation of all those partial aggregations.

Let us now introduce some notations which will be useful for our future discussions.

By $[i_1, i_1, \ldots, i_n]$ we will denote the ordered list whose first element is $i_1$, second element is $i_2$ and so on. $[]$ will denote the empty list and given a list $L$, $|L|$ will denote the length of the list. Moreover, let $\cdot$ be the classical list composition operator. Finally, given a list $L$ we define the operator $\star$ that produces the set of elements of the list $L$, i.e. $\star L = \{j : j \text{ occurs in } L\}$. Thus, given a list $L$ the notation $j \in L$ has the meaning of $j \in \star L$. $|\star L|$ will denote the cardinality of the set $\star L$ and since sets have no repetion of elements it is clear that $|\star L| \leq |L|$ for any list $L$.

So, if

$$L_1 = [1, 2, 3] \quad L_2 = [3, 2, 4, 3].$$

We have,

$$\star L_1 = \{1, 2, 3\} \ , \ \star L_2 = \{3, 2, 4\}.$$

Given a list $I$ and $m$ lists $I_1, \ldots, I_m$ we say that the list $\mathcal{I} = [I_1, \ldots, I_m]$ is a *cover* of $I$ if the following conditions are verified:

- $I_k \neq []$ for all $k = 1, 2, \ldots, m$;
- for all $k = 1, 2, \ldots, m$, we have $|\star I| = |I|$ i.e. all the elements in $I_k$ are different ;
- $\star I = \bigcup_{k=1}^{m} \star I_k$.

Given a list of indices $I = [i_1, \ldots, i_m]$ and an aggregation rule $\phi$, we introduce the following notation

$$\phi(a_h | h \in I) \equiv \phi(a_{i_1}, \ldots, a_{i_m}).$$

Then we can define the concept of cover-based hierarchical aggregation.

**DEFINITION 2** *Let $I$ be a finite list of indices and let $[I_1, \ldots, I_{c_0}]$ be a fixed cover of $I$. Let $m = |I|$ and $c_k = |I_k|$ for all $k = 1, 2, \ldots, c_0$. A* cover based hierarchical aggregation *$\phi$ is characterized by a collection $\phi_0, \phi_1, \phi_2, \ldots, \phi_{c_0}$ of aggregation rules with $c_0 \geq 2$ and with $m_k > 1$ for some $1 \leq k \leq c_0$, such that $\phi_k : [0,1]^{c_k} \to [0,1]$ for all $k = 0, 1, 2, \ldots, c_0$, in such a way that the composition*

$$\phi = \phi_0(\phi_1, \phi_2, \ldots, \phi_{c_0}) : [0,1]^m \to [0,1]$$

*is defined as*

$$\phi(a_k | k \in I) = \phi_0(\phi_1(a_h | h \in I_1), \ldots, \phi_{c_0}(a_h | h \in I_{c_0}))$$

□

To clarify definition 2 consider the following example. Let $I = [1, 2, 3, 4]$ and for $c_0 = 3$ consider the cover $[I_1, I_2, I_3]$ with $I_1 = [1, 2]$ , $I_2 = [2, 3, 1]$ , $I_3 = [4, 1, 3]$ . So given any input $(a_1, a_2, a_3, a_4)$ we have

$$\phi(a_1, a_2, a_3, a_4) = \phi_0(\phi_1(a_1, a_2), \phi_2(a_2, a_3, a_1), \phi_3(a_4, a_1, a_3)).$$

Assuming an a priori fixed cover $[I_1, \ldots, I_{c_0}]$ of $I$, each hierarchical aggregation is therefore characterized by the composition $\phi_0(\phi_1, \phi_2, \ldots, \phi_{c_0})$.

**DEFINITION 3** *If a property $P$ that holds for $\phi_0, \phi_1, \phi_2, \ldots, \phi_{c_0}$ holds for $\phi_0(\phi_1, \ldots, \phi_{c_0})$ as well, with respect to any fixed cover, we will say that $P$* propagates under cover-based hierarchical aggregations. □

One important property that propagates under cover-based hierarchical aggregation is *stability* as defined and analyzed in [7].

**DEFINITION 4** *An intensity aggregation rule $\phi$ is* stable *if there exists a constant $K$ (called stability constant) such that for all $\epsilon > 0$ and for all $i = 1, \ldots, m$,*

$$|\phi(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_m) - \phi(a_1, \ldots, a_m)| \leq K\epsilon$$

*for all $a_1, \ldots, a_m$ and for $|a_i' - a_i| < \epsilon$.* □

As a simple consequence of functional composition many other properties propagate under cover-based hierarchical aggregations, among them completeness, positive responsiveness (PR), unanimity (U) and conditions (i) and (ii) of theorem 1. Being NAI also propagates, as shown in the following theorem.

**THEOREM 2** *If $\phi_0, \phi_1, \phi_2, \ldots, \phi_{c_0}$ are NAI aggregation rules then $\phi \equiv \phi_0(\phi_1, \phi_2, \ldots, \phi_{c_0})$ is also NAI, i.e. rationality propagates under cover-based hierarchical aggregations.*

**Proof.** Let $X$ be a set of alternatives and let $\mu^1, \ldots, \mu^n$ be $n$ NAI individuals. Let us denote by $\nu$ the aggregation function associated to $\phi$ and by $\nu_k$ the aggregation function associated to $\phi_k$ for all $k = 0, 1, 2, \ldots, c_0$. Since all the individuals $\mu^1, \ldots, \mu^n$ are NAI so are $\nu_1, \ldots, \nu_{c_0}$. On the other hand, since $\phi_0$ is NAI and in view of definition 1, we have that its aggregation function $\nu_0$ defined as

$$\nu_0(x, y) = \phi_0(\nu_1(x, y), \ldots, \nu_{c_0}(x, y))$$

is NAI. Therefore, since

$$\nu(x, y) = \nu_0(x, y)$$

we have that $\phi$ is NAI. ■

It is easy to observe that anonimity (A) and citizen sovereign (CS) do not in general propagate under cover-based hierarchical aggregations. However, it can be proved they propagate in specific but important cases (see [7] for details).

# 5 OWA Operators and Ordered Hierarchies

We will now introduce ordered hierarchical aggregations. Such hierarchical aggregations are related very naturally to the ***Ordered Weighted Averaging (OWA)*** operators.

As previously commented, for any OWA operator $\phi$ we have

$$\min_i(a_1, \ldots, a_n) \leq \phi(a_1, \ldots, a_n) \leq \max_i(a_1, \ldots, a_n) \tag{5.6}$$

Two significative measures are associated with OWA operators (see Yager [24]). Given $\phi$ of dimension $n$ with associated weights $w_1, \ldots, w_n$ these measures are the following.

**(m1)** *degree of orness*, i.e. how close an OWA operator is to the max operator. It is defined as

$$orness(\phi) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i.$$

Dual to the measure of *orness* is the measure of *andness* defined as

$$andness(\phi) = 1 - orness(\phi)$$

which therefore measures how close an OWA operator is to the min operator.

**(m2)** The second measure called ***dispersion*** gives us the degree to which all aggregates are used equally, i.e. the degree to which an OWA operator is close to the simple average operator. The dispersion measure is defined as

$$Disp(\phi) = -\sum_{i=1}^{n} w_i \ln w_i.$$

As shown in [24], $orness(\phi) \leq orness(\phi')$ when their associated weights $(w_1, \ldots, w_n)$ and $(w'_1, \ldots, w'_n)$ verify that

$$\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} w'_j \quad \forall i = 1, \ldots, n \tag{5.7}$$

This property gives in fact the intuitive idea of when an OWA operator must be more an "or" than other OWA operator. An important class of OWA operator is the class of ***buoyancy*** measures defined as follows (see [24]).

**DEFINITION 5** *An OWA operator $\phi$ is a* buoyancy *measure if the associated weights $w_1, w_2, \ldots, w_n$ verify the condition $w_i \geq w_j$ for all $i \leq j$.* □

Buoyancy measures verify the property of being ***orlike*** measures, i.e. if $\phi$ is a buoyancy measure then $orness(\phi) \geq \frac{1}{2}$. Buoyancy measures verify as well an important property for our future discussions (see [11])

**THEOREM 3** *Let $\phi$ be an OWA operator of dimension $n$ with associated weights $w_1, \ldots, w_n$. Then $\phi$ is complete if and only if for all integer $k \leq \frac{n}{2}$ we have*

$$\sum_{i=1}^{k} w_i \geq \sum_{i=1}^{k} w_{n-i+1}. \tag{5.8}$$

■

Therefore, since buoyancy measures verify condition (5.8) they are complete.

A characterization of OWA operators in terms of "fuzzy rationality" is given by the following theorem.

**THEOREM 4** *Let $\phi$ be a complete OWA operator with associated weights $w_1, \dots, w_n$. Then $\phi$ is NAI if and only if $w_n > 0$.*

**Proof.** We remark that OWA operators verify the pre-conditions of theorem 1. We will now show that any complete OWA operator verifies condition (i) of theorem 1.

Suppose then that $a_i + b_i > 1$ for all $i = 1, 2, \dots, n$. Therefore $a_{[n]} > 0$ and $b_{[n]} > 0$ from which we also have that there exists $\epsilon > 0$ such that for all $i = 1, 2, \dots, n$

$$\begin{aligned} a_i + b_i - \epsilon &\geq 1 \\ b_i - \epsilon &\geq 0 \end{aligned}$$

We then have

$$\begin{aligned} \phi(a_1, \dots, a_n) + \phi(b_1, \dots, b_n) &= \sum_{i=1}^{n} w_i a_{[i]} + \sum_{i=1}^{n} w_i b_{[i]} > \\ \sum_{i=1}^{n} w_i a_{[i]} + \sum_{i=1}^{n} w_i (b_{[i]} - \epsilon) &= \phi(a_1, \dots, a_n) + \phi(b_1 - \epsilon, \dots, b_n - \epsilon) \end{aligned}$$

and the latter is not smaller than 1 for the completeness hypothesys.

In order to prove our theorem it is enough to prove that $w_n > 0$ if and only if condition (ii) of theorem 1 holds, i.e. if and only if $\phi(a_1, \dots, a_n) = 1$ implies $a_1 = \dots = a_n = 1$. Suppose that $a_1 \geq a_2 \geq \dots \geq a_n$. We have

$$\phi(a_1, \dots, a_n) = \sum_{i=1}^{n} w_i a_i = 1 \leftrightarrow w_i = 0 \ \forall \, a_i < 1.$$

In turn we have:

- if $\phi(a_1, \dots, a_n) = 1$ and $a_n < 1$ then $w_n = 0$;
- if $w_n = 0$ then $\phi(1, \dots, 1, a_n) = 1$ for all $a_n \in [0, 1]$

and the theorem is proven. ■

As a simple consequence we have

**COROLLARY 3** *A buoyancy measure is NAI if and only if all its associated weights are non null.* ■

## 5.1 Ordered hierarchies of OWA operators

It can easily be observed that cover-based hierarchical aggregations of OWA operators do not in general produce OWA operators since cover-based hierarchical aggregations are characterized by a fixed cover of individuals, independently of their opinions. We will now give a characterization of hierarchical aggregations of OWA operators that produce OWA operators.

Let $\phi_0, \phi_1, \dots, \phi_c$ be $c + 1$ OWA operators such that

- $\phi_0$ has dimension $c$;
- $\phi_i$ has dimension $h_i$ for any $i = 1, 2, \ldots, c$
- $\sum_{i=1}^{c} h_i = n$

Let $w_{0,1}, \ldots, w_{0,c}$ be the weights associated to $\phi_0$, and for all $i = 1, \ldots, c$ let $w_{i,1}, \ldots, w_{i,h_i}$ be the weights associated to $\phi_i$.

**DEFINITION 6** *The* ordered hierachical composition *of $\phi_0, \phi_1, \ldots, \phi_c$ is defined by*

$$\phi_0(\phi_1, \ldots, \phi_c)(a_1, \ldots, a_n) = \phi_0(\phi_1(a_{[1]}, \ldots, a_{[h_1]}), \ldots, \phi_c(a_{[n-h_c+1]}, \ldots, a_{[n]}))$$

*for all n-uples $(a_1, \ldots, a_n)$.* □

**DEFINITION 7** *If a property $P$ that holds for $\phi_0, \phi_1, \phi_2, \ldots, \phi_c$ holds for the ordered hierarchical aggregation $\phi_0(\phi_1, \ldots, \phi_c)$ as well, we will say that $P$* propagates under ordered hierarchical aggregations. □

The result we were looking for follows.

**THEOREM 5** *The property of being an OWA operator propagates under ordered hierarchical aggregations.*

**Proof.** Given any n-uple $a_1, \ldots, a_n$ such that $a_1 \geq a_2 \geq \ldots \geq a_n$, because of property 5.6 for the OWA operators, we have

$$\phi_1(a_1, \ldots, a_{h_1}) \geq \cdots \geq \phi_c(a_{n-h_c+1}, \ldots, a_n)$$

Let $\alpha_j = \sum_{i=1}^{j} h_i$ for any $j = 1, 2, \ldots, c$.
We have

$$\phi_0(\phi_1, \ldots, \phi_c)(a_1, \ldots, a_n) = \phi_0(\sum_{i=1}^{h_1} w_{1,i} a_i, \ldots, \sum_{i=\alpha_{c-1}+1}^{n} w_{c,i-\alpha_{c-1}} a_i) =$$

$$\sum_{j=1}^{c} w_{0,j} \sum_{i=\alpha_{j-1}+1}^{\alpha_j} w_{j,i-\alpha_{j-1}} a_i = \sum_{i=1}^{h_1} w_{0,1} w_{1,i} a_i + \ldots + \sum_{i=\alpha_{c-1}+1}^{n} w_{0,c} w_{c,i-\alpha_{c-1}} a_i$$

Finally, since

$$\sum_{i=1}^{h_1} w_{0,1} w_{1,i} + \ldots + \sum_{i=\alpha_{c-1}+1}^{n} w_{0,c} w_{c,i-\alpha_{c-1}} = 1$$

we can claim that the ordered hierachical aggregation produces an OWA operator ■

Thus, the above defined ordered hierarchical aggregations are such that the property of being an OWA operator propagates. However, as it is easy to see, completeness does not propagate in general for ordered hierarchical aggregations. Moreover -in view of theorem 5- Monotonicity, Idempotency, Commutativity, Stability and Non Absolute Irrationality (whenever completeness propagates) propagate under ordered hierarchical aggregations of OWA operators.

## 5.2 Orness and Dispersion

Given a fixed ordered hierarchical aggregation $\phi \equiv \phi_0(\phi_1, \ldots, \phi_c)$ based upon $c+1$ OWA operators, the degree of orness is given by

$$orness(\phi) = \phi(\frac{n-1}{n-1}, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0) = \phi_0(z_1, \ldots, z_i, \ldots, z_c)$$

where

$$z_i = \phi_i(\frac{n - (\sum_{j=1}^{i-1} h_j + 1)}{n-1}, \ldots, \frac{n - \sum_{j=1}^{i} h_j}{n-1}) = \sum_{m=1}^{h_i} \frac{n - (\sum_{j=1}^{i-1} h_j + m)}{n-1} w_{i,m} =$$

$$\frac{n - \sum_{j=1}^{i} h_j}{n-1} + \frac{\sum_{m=1}^{h_i} (h_i - m) w_{i,m}}{n-1} = \frac{n - \sum_{j=1}^{i} h_j}{n-1} + (h_i - 1)\frac{orness(\phi_i)}{n-1}$$

for all $i = 1, \ldots, c$, with $h_i$ the dimension of each $\phi_i$ and $(w_{i,1}, \ldots, w_{i,h_i})$ its associated weights. Hence,

$$orness(\phi) = \sum_{i=1}^{c} \frac{(h_i - 1) orness(\phi_i) + n - \sum_{j=1}^{i} h_j}{n-1} w_{0,i}$$

Obviously, changes in $\phi_j'$s weights for $j = 1, \ldots, c$ in such a way that $orness(\phi_j)$ does not decrease for any $j = 1, \ldots, c_0$ will never make $orness(\phi)$ decrease. However, increasing $orness(\phi_0)$ does not necessarely increase $orness(\phi)$ as shown by the following example.

**Example 2** Let $\phi_0, \phi_0'$ be two OWA operators of dimension $c$ such that their associated weights are, respectively, $w_{0,2} = 1, w_{0,i} = 0$ for all $i \neq 2$ and $w'_{0,1} = w'_{0,3} = 1/2, w'_{0,i} = 0$ for all $i \neq 1, 3$. Then, $orness(\phi_0) = orness(\phi_0')$. Let now $\phi_i$ be the minimum rule (i.e., $orness(\phi_i) = 0$ for all $i = 1, \ldots, c$). Then the difference between $orness(\phi_0(\phi_1, \ldots, \phi_c))$ and $orness(\phi_0'(\phi_1, \ldots, \phi_c))$ will still depend on the relative sizes $h_2$ and $h_3$ of OWA operators $\phi_2, \phi_3$. □

The dispersion of the hierarchical aggregation can be computed as follows.

$$\begin{aligned}
Disp(\phi) &= -\sum_{j=1}^{c} \sum_{i=1}^{h_j} w_{0,j} w_{j,i} \ln[w_{0,j} w_{j,i}] \\
&= -\sum_{j=1}^{c} \sum_{i=1}^{h_j} (w_{0,j} w_{j,i} \ln w_{0,j} + w_{0,j} w_{j,i} \ln w_{j,i}) \\
&= -\sum_{j=1}^{c} w_{0,j} \ln w_{0,j} - \sum_{j=1}^{c} w_{0,j} Disp(\phi_j) \\
&= Disp(\phi_0) + \sum_{j=1}^{c} w_{0,j} Disp(\phi_j)
\end{aligned}$$

Therefore, the dispersion of the hierarchical aggregations depends directly upon the dispersions of the given OWA operators.

In both cases it is however clear that once the operators $\phi_0$ and $\phi_1, \ldots, \phi_c$ are fixed then the orness and dispersion values of the hierarchical aggregation depend upon the particular ordering of the $c$ OWA operators $\phi_1, \ldots, \phi_c$.

It is then natural to ask the following questions:

(1) How can one *quickly* find an ordering of those $c$ OWA aggregated operators which maximizes [resp. minimizes] the dispersion value of the hierarchical aggregation ?

(2) How can one *quickly* find an ordering of those $c$ OWA aggregated operators which maximizes [resp. minimizes] the orness value of the hierarchical aggregation ?

In the next section we will provide complete solutions for them following the results in [9,12].

# 6 Maximizing and minimizing dispersion and orness

Let $\phi_0$ be an OWA operator of dimension $c$ and let $\Phi = \{\phi_i : 1 \leq i \leq c\}$ be a given set of OWA operators and let $h_1, \ldots, h_c$ their respective dimensions.

## 6.1 Dispersion

We start by providing polynomial algorithms to maximize [resp. minimize] the dispersion of the hierarchical aggregation.

Let $\pi$ be a permutation of the indices $\{1, 2, \ldots, c\}$ such that

$$i \leq j \text{ if and only if } w_{0,\pi(i)} \geq w_{0,\pi(j)}.$$

Intuitively, for $i = 1, 2, \ldots, c$, $w_{0,\pi(i)}$ is the $i$-th weight in non-increasing order.

Let $\phi_1, \ldots, \phi_c$ be a given ordering of the OWA operators in $\Phi$.

We will now prove that the following lemmas hold.

**LEMMA 2** *The value $Disp(\phi) = Disp(\phi_0) + \sum_{j=1}^{c} w_{0,j} Disp(\phi_j)$ is maximum if and only if the ordering $\phi_1, \ldots, \phi_c$ verifies*

$$i \leq j \text{ if and only if } Disp(\phi_{\pi(i)}) \geq Disp(\phi_{\pi(j)}).$$

■

**LEMMA 3** *The value $Disp(\phi) = Disp(\phi_0) + \sum_{j=1}^{c} w_{0,j} Disp(\phi_j)$ is minimum if and only if the ordering $\phi_1, \ldots, \phi_c$ verifies*

$$i \leq j \text{ if and only if } Disp(\phi_{\pi(i)}) \leq Disp(\phi_{\pi(j)})$$

■

In order to prove the two lemmas above, we make use of the elementary result that if $a \geq b \geq 0$ and $c > d \geq 0$ then $ac + bd \geq ad + bc$

**(L1)** Let us then prove Lemma 2.

Suppose first that $Disp(\phi)$ is maximum and suppose by contradiction that there exist $i, j$ with $i < j$ such that $Disp(\phi_{\pi(i)}) < Disp(\phi_{\pi(j)})$. Then since

$$w_{0,\pi(i)} Disp(\phi_{\pi(j)}) + w_{0,\pi(j)} Disp(\phi_{\pi(i)}) > w_{0,\pi(i)} Disp(\phi_{\pi(i)}) + w_{0,\pi(j)} Disp(\phi_{\pi(j)})$$

by switching $\phi_i$ and $\phi_j$ we would obtain an ordering which would cause the dispersion of the hierarchical aggregation to increase, contradicting the initial hypothesys.

Conversely, suppose that the ordering $\phi_1, \ldots, \phi_c$ verifies the condition of the Lemma. Let $\phi'_1, \ldots, \phi'_c$ be an ordering which maximizes $Disp(\phi)$. From what has been proven above, this ordering verifies the condition of the Lemma as well.

Let us prove that

$$\sum_{j=1}^{c} w_{0,j} Disp(\phi_j) = \sum_{j=1}^{c} w_{0,j} Disp(\phi'_j).$$

If $\phi_{\pi(1)} \neq \phi'_{\pi(1)}$ in view of the condition of the Lemma it must be

$$Disp(\phi_{\pi(1)}) = Disp(\phi'_{\pi(1)}).$$

Thus, we are left to prove that

$$\sum_{j \neq \pi(1)} w_{0,j} Disp(\phi_j) = \sum_{j \neq \pi(1)} w_{0,j} Disp(\phi'_j).$$

which can be shown easily to hold, by applying a recursive argument.

**(L2)** Lemma 3 can be proven analogously.

In view of the above Lemmas and from the well-known result (see, e.g., [3]) that any given $n$ numbers can be sorted in $O(n \log n)$ time, we can claim that the following theorem is true.

**THEOREM 6** *Given an OWA operator $\phi_0$ of dimension $c$ and a set of $c$ OWA operators $\Phi$ and given their dispersions, there exists a $O(c \log c)$ algorithm which produces an ordering of $\Phi$ maximizing (or minimizing) the dispersion of the hierarchical aggregation.* □

## 6.2 Orness

The problem of maximizing or minimizing the orness of the hierarchical aggregation at a first sight appears to be quite more difficult from a computational point of view and its (polynomial) solution definitely more subtle. However we can see that we can easily reduce such a problem to the classical combinatorial *assignment problem* for which there exists a $O(c^3)$ algorithm.

Let us go back to the algebraic expression for computing the orness of the hierarchical aggregation once an ordering $\phi_1, \ldots, \phi_c$ of the OWA operators in $\Phi$ has been fixed:

$$orness(\phi) = \sum_{i=1}^{c} \frac{(h_i - 1)orness(\phi_i) + n - \sum_{j=1}^{i} h_j}{n-1} w_{0,i}$$

Thus

$$orness(\phi) = \sum_{i=1}^{c} \frac{(h_i - 1)orness(\phi_i)w_{0,i} + n - \sum_{j=i}^{c} w_{0,j} h_i}{n-1}.$$

We can therefore conclude that the contribution to the orness of $\phi$ of an OWA operator $\bar{\phi} \in \Phi$ with dimension $h$ will be

$$
\begin{array}{ll}
\delta(\bar\phi,1) = \frac{(h-1)orness(\bar\phi)w_{0,1}+n-h}{n-1} & \text{if } \bar\phi \text{ is the first} \\
\delta(\bar\phi,2) = \frac{(h-1)orness(\bar\phi)w_{0,2}+n-(1-w_{0,1})h}{n-1} & \text{if } \bar\phi \text{ is the second} \\
\vdots & \vdots \\
\delta(\bar\phi,i) = \frac{(h-1)orness(\bar\phi)w_{0,i}+n-\sum_{j=i}^{c} w_{0,j}h}{n-1} & \text{if } \bar\phi \text{ is the } i\text{-th} \\
\vdots & \vdots \\
\delta(\bar\phi,c-1) = \frac{(h-1)orness(\bar\phi)w_{0,c-1}+n-(w_{0,c-1}+w_{0,c}h)}{n-1} & \text{if } \bar\phi \text{ is the next to the last} \\
\delta(\bar\phi,c) = \frac{(h-1)orness(\bar\phi)w_{0,c}+n-w_{0,c}h}{n-1} & \text{if } \bar\phi \text{ is the last}
\end{array}
$$

Summing up we can say that:

- denoted with $\phi_1, \phi_2, \ldots, \phi_c$ the OWA operators in $\Phi$, and
- denoted for simplicity with $\delta_{ij}$ the contribution to the orness of $\bar\phi$ of the operator $\phi_i$ once it is chosen as the $j$-th in the hierarchical aggregation, then
- the problem of maximizing [resp. minimizing] the orness of the hierarchical aggregation is equivalent to the problem of finding a permutation $\pi$ of the rows of the matrix

  $$
  M = \begin{pmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1c} \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{c1} & \delta_{c2} & \ldots & \delta_{cc} \end{pmatrix}
  $$

  such that the sum of the elements of the main diagonal of

  $$
  M_\pi = \begin{pmatrix} \delta_{\pi(1)1} & \delta_{\pi(1)2} & \ldots & \delta_{\pi(1)c} \\ \delta_{\pi(2)1} & \delta_{\pi(2)2} & \ldots & \delta_{\pi(2)c} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\pi(c)1} & \delta_{\pi(c)2} & \ldots & \delta_{\pi(c)c} \end{pmatrix}
  $$

  is maximum [resp. minimum], i.e. such that

  $$
  \sum_{i=1}^{c} \delta_{\pi(i)i}
  $$

  is maximum [resp. minimum].

The above is the classical ***assignment problem*** (also known as the ***bipartite weighted matching problem***) (see [21], chapter 11) and can be solved by the so-called ***hungarian method*** (see [16]) in time $\mathcal{O}(c^3)$. More recently in [22] it has been proven that by using special data structures it is possible to improve the above bound to $\mathcal{O}(c^2 \log c)$.

### 6.3 Some particular cases

$\mathcal{O}(c \log c)$ algorithms can instead be given for the most significative cases in practice. Since each one of the given $c$ OWA operators represents partial information to be aggregated into only one index, it is common to define them in such a way that either

- they contain the same amount of information. In this case, all OWA operators in $\Phi$ will have the same dimension $h$ such that $n = hc$; or
- they treat their inputs with the same degree of optimism (pessimism). In this case, all OWA operators will have the same degree of orness.

Let us consider the above two cases in more details.

**(case 1)** In case all OWA operators in $\Phi$ have the same dimension $h = n/c$, we have to find an ordering $\phi_1, \ldots, \phi_c$ such that

$$orness(\phi) = \sum_{i=1}^{c} \frac{(h-1)orness(\phi_i) + n - \sum_{j=1}^{i} h}{n-1} w_{0,i}$$

is maximum. Therefore,

$$orness(\phi) = \frac{(h-1)}{n-1} \sum_{i=1}^{c} orness(\phi_i) w_{0,i} + \frac{\sum_{i=1}^{c}(c-i)h}{n-1} w_{0,i}$$

It follows that the sought ordering can be obtained as in Lemma 2, i.e. by producing any ordering which verifies

$$i \leq j \text{ if and only if } orness(\phi_{\pi(i)}) \leq orness(\phi_{\pi(j)})$$

Analogously, if we want to minimize the orness of the hierarchical aggregation we act as in Lemma 3.

**(case 2)** In case all OWA operators in $\Phi$ have the same orness $r$, we have

$$orness(\phi) = \frac{1}{n-1} \sum_{i=1}^{c} (rh_i + \sum_{j=i+1}^{c} h_j) w_{0,i} - \frac{r}{n-1}.$$

As a consequence, any ordering which sorts the elements of $\Phi$ in increasing [resp. decreasing] order with respect to their dimensions maximizes [resp. minimizes] the orness.

# 7 Final Comments

In this paper, we surveyed on some research work that has been done to obtain a full charaterization of rational aggregation operators and hierarchies of them. In particular we focused on the propagation of key ethical conditions, rationality and stability properties of cover-based hierarchies and ordered hierarchies. In this last case, we have analyzed Yager's OWA operators and seen that the property of being an OWA operator propagates under ordered hierarchical aggregations. As by product, we observed interesting theoretical and computatational results on the propagation of OWA operators significant measures such as *orness* and *dispersion*.

We conclude by again pointing out the practical importance of such hierarchical aggregation rules. Preference intensity of large groups of people are in common life usually aggregated by means of hierarchical procedures. The fact that standard ethical conditions as well as non absolute irrationality, together with stability, propagate under general conditions legitimize hierarchical aggregations from both theoretical and applicative point of view.

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