UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Combinatorial Structure of Polytopesassociated to Fuzzy Measures

Estructura Combinatoria de Politopos asociados a Medidas Difusas

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Pedro García Segador

Director

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Tesis Doctoral

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Director

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"Es ist wahr, ein Mathematiker, der nicht etwas Poet ist, wird nimmer ein vollkommener Mathematiker sein."

"It is true that a mathematician who is not somewhat of a poet, will never be a perfect mathematician."

Karl Weierstraß

ABSTRACT

Combinatorial Structure of Polytopes associated to Fuzzy Measures

by PEDRO GARCÍA SEGADOR

Abstract in English

This PhD thesis is devoted to the study of geometric and combinatorial aspects of polytopes associated to fuzzy measures. Fuzzy measures are an essential tool, since they generalize the concept of probability. This greater generality allows applications to be developed in various fields, from the Decision Theory to the Game Theory.

The set formed by all fuzzy measures on a referential set is a polytope. In the same way, many of the most relevant subfamilies of fuzzy measures are also polytopes. Studying the combinatorial structure of these polytopes arises as a natural problem that allows us to better understand the properties of the associated fuzzy measures.

Knowing the combinatorial structure of these polytopes helps us to develop algorithms to generate points uniformly at random inside these polytopes. Generating points uniformly inside a polytope is a complex problem from both a theoretical and a computational point of view. Having algorithms that allow us to sample uniformly in polytopes associated to fuzzy measures allows us to solve many problems, among them the identification problem, i.e. estimate the fuzzy measure that underlies an observed data set.

Many of these polytopes associated with subfamilies of fuzzy measures are order polytopes and their combinatorial structure depends on a partially ordered set (poset). In this way we can transform problems of a geometric nature into problems of Combinatorics and Order Theory.

In this thesis, we start by introducing the most important results about posets and order polytopes. Next, we focus on the problem of generating uniformly at random linear extensions in a poset. To this end, we have developed a method, which we have called Bottom-Up, which allows generating linear extensions for several posets quickly and easily. The main disadvantage of this method is that it is not applicable to every poset. The posets for which we can apply Bottom-Up are called BU-feasibles. For this reason we study another method that we call Ideal-based method. This method is more general than Bottom-Up and is applicable to any poset, however its computational cost is much higher.

Among the posets that are not BU-feasible there are important cases that deserve an individualized treatment, as is the case of the Ferrers posets. Ferrers posets are associated to 2-symmetric measures. In this thesis, we study in detail the geometric and combinatorial properties of the 2-symmetric measures. For this purpose we make use of Young diagrams, which are well-known objects in combinatorics. Young diagrams allow us to develop algorithms to generate uniformly at random 2-symmetric measures.

Next, we study the subfamily of 2-additive measures. The polytopes in this subfamily are not order polytopes, and therefore we cannot apply the techniques we have developed related to posets. As we have done with 2-symmetric measures, we study the faces of these polytopes and develop a triangulation that allows us to uniformly sample 2-additive measures.

Monotone games emerge as the non-normalized version of fuzzy measures. As they are non-normalized, the associated polyhedra are not bounded and do not have a polytope structure. In this case these sets form convex cones whose structure also depends on a poset. We have called them order cones. In this thesis, we explain the link between order cones and order polytopes. We also solve the problem of obtaining the extreme rays of the set of monotone games using order cones.

Later in the thesis we study integration techniques to count linear extensions. These techniques can be applied to know the volume of the associated order polytopes. In the same way, it allows us to solve specific combinatorial problems such as counting the number of 2-alternating permutations.

Finally, the thesis ends with a chapter on conclusions and open problems. It is concluded that, in view of the results obtained in the thesis, there is a very close link between Combinatorics, mainly Order Theory, and the geometry of the polytopes associated with fuzzy measures. The more we improve our knowledge about the underlying combinatorics, the better we will understand these polytopes.

Resumen en Español

La presente tesis doctoral está dedicada al estudio de distintas propiedades geométricas y combinatorias de politopos de medidas difusas. Las medidas difusas son una herramienta esencial puesto que generalizan el concepto de probabilidad. Esta mayor generalidad permite desarrollar aplicaciones en diversos campos, desde la Teoría de la Decisión a la Teoría de Juegos.

El conjunto formado por todas las medidas difusas sobre un referencial tiene estructura de politopo. De la misma forma, la mayoría de las subfamilias más relevantes de medidas difusas son también politopos. Estudiar la estructura combinatoria de estos politopos surge como un problema natural que nos permite comprender mejor las propiedades de las medidas difusas asociadas.

Conocer la estructura combinatoria de estos politopos también nos ayuda a desarrollar algoritmos para generar aleatoria y uniformemente puntos dentro de estos politopos. Generar puntos de forma uniforme dentro de un politopo es un problema complejo desde el punto de vista tanto teórico como computacional. Disponer de algoritmos que nos permitan generar uniformemente en politopos asociados a medidas difusas nos permite resolver muchos problemas, entre ellos el problema de identificación que trata de estimar la medida difusa que subyace a un conjunto de datos observado.

Muchos de los politopos asociados a subfamilias de medidas difusas son politopos de orden y su estructura combinatoria depende de un orden parcial (poset). De esta forma podemos transformar problemas de naturaleza geométrica en problemas de Combinatoria y Teoría del Orden.

En esta tesis, empezamos introduciendo los resultados más importantes sobre posets y politopos de orden. A continuación nos centramos en el problema de generar extensiones lineales de forma aleatoria para un poset. A tal fin hemos desarrollado un método, al que hemos llamado Bottom-Up, que permite generar extensiones lineales para diversos posets de forma rápida y sencilla. La principal desventaja de este método es que no es aplicable a cualquier poset. A los posets que se les puede aplicar este método los llamamos BU-factibles. Por este motivo estudiamos otro método al que llamamos método basado en ideales (Ideal-based method). Este método es más general que el Bottom-Up y es aplicable a cualquier poset. Sin embargo, su coste computacional es mucho mayor.

Dentro de los posets que no son BU-factibles hay casos importantes que merecen un estudio individualizado, como es el caso de los posets de Ferrers. Los posets de Ferrers están asociados a las medidas 2-simétricas. En esta tesis estudiamos en detalle las propiedades geométricas y combinatorias de las medidas 2-simétricas. Para ello hacemos uso de unos objetos muy conocidos en Combinatoria, que son los diagramas de Young.

Estos objetos nos permiten desarrollar algoritmos para generar aleatoriamente medidas 2-simétricas de forma uniforme.

A continuación estudiamos la subfamilia de medidas 2-aditivas. Los politopos de esta subfamilia no son politopos de orden, y por tanto no podemos aplicar las técnicas que hemos desarrollado relacionadas con posets. Al igual que hemos hecho con las medidas 2-simétricas, estudiamos las caras de estos politopos y desarrollamos una triangulación que nos permite simular uniformemente medidas 2-aditivas.

Los juegos monótonos surgen como la versión no normalizada de las medidas difusas. Al no estar normalizados, los poliedros asociados no están acotados y no tienen estructura de politopo. En este caso estos conjuntos forman conos convexos cuya estructura también depende de un poset. A estos conos los hemos llamado conos de orden (Order Cones). En la tesis explicamos la relación existente entre conos de orden y politopos de orden. También resolvemos el problema de obtener los rayos extremos de los juegos monótonos usando resultados de conos de orden.

Posteriormente en la tesis estudiamos técnicas de integración para contar extensiones lineales. Estas técnicas se pueden aplicar para conocer el volumen de los politopos de orden asociados. De la misma forma permiten resolver problemas concretos de Combinatoria como el de contar el número de permutaciones 2-alternantes.

Finalmente la tesis acaba con un capítulo sobre conclusiones y problemas abiertos. Se concluye que, a la vista de los resultados obtenidos en la tesis, hay una relación muy estrecha entre la Combinatoria, principalmente aquella asociada a la Teoría del Orden, y la geometría de los politopos asociados a medidas difusas. Según podamos avanzar en nuestros conocimientos teóricos sobre estos objetos combinatorios podremos conocer mejor estos politopos.

Agradecimientos

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en los momentos más duros. Ahora que nuestra familia está creciendo, nos embarcamos juntos en otra etapa de descubrimiento y aprendizaje, seguiremos luchando juntos codo con codo como hemos hecho hasta ahora.

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To Sara

Introduction

Since time began, Humanity has had to deal with situations where a decision must be made between various alternatives. This problem remains central to the lives of all of us. In this way, each day we must decide what clothes to wear, what time to get up, what means of transport to use to get to work, and so on.

In many of these decisions, the consequences depend on a number of factors that cannot be controlled by the decision maker. This makes decision-making a very complicated process governed by uncertainty. In many occasions it is necessary to choose options in competition with other individuals. It goes without saying that a wrong decision can lead to major problems that can extend over a long period of time, making it necessary to reflect on the decision to be made.

In order to try to clarify how this decision-making should be done, the *Decision Theory* and the *Game Theory* have arisen. They propose mathematical tools that allow human behavior to be modeled in decision-making, as well as make the most appropriate decisions for each problem.

Decision-making problems appear first in the works of Pascal and Bernoulli on Probability and in the works on voting systems by Condorcet and Borda in the late eighteenth century. However, despite the evident importance that decision-making has, its study from a formal point of view is quite recent. The first studies emerged in the twentieth century and were done by Neyman and Pearson by using statistical hypothesis contrasts. The extension to other areas of statistic is due to A. Wald. In 1944, J. von Neumann and O. Morgenstern published a work that represents a fundamental advance both in Decision Theory and in Game Theory.

The set of important factors which are not under the control of the decision maker are called *states of nature*. Depending on the information available on the states of nature, the Decision Theory is divided into two: Decision Theory Under Uncertainty and Under Risk. In Decision Theory Under Uncertainty there is no information about the actual state of nature, while in Decision Theory Under Risk we have information that is modeled by a probability distribution on the set of states of nature.

Under these conditions, if we assume that the set of states of nature is finite, each alternative can be identified with a vector. This vector of consequences is made numeric by a utility or loss function. What makes decision-making difficult comes from the fact that an alternative might be better for a state of nature but not for the others. This is because there is not a natural order in the n-dimensional space to compare vectors. To solve this problem, what is done is to replace each vector with a representative value that is a real number; and then the best decision corresponds with the best representative value.

Different ways of choosing the representative value associated with the vectors lead to different decisions, and this gives rise to different decision criteria. If we are in a risk environment, the most usual criterion is the criterion of expected utility, in which the expected value of utilities is taken as representative value.

However, the expected utility criterion cannot explain some behaviors related to risk aversion, such as the Ellsberg and Allais paradoxes, which appear both in an environment of uncertainty and risk. These paradoxes can, however, be modeled through the use of non-additive measures and the Choquet integral instead of mathematical probabilities and expected values.

Another branch of the Decision Theory is the Multicriteria Decision Making. In the case of Multicriteria Decision Making, the problem we are facing does not have a degree of uncertainty as it did in the previous cases. In this case, we have a series of alternatives from which one must be chosen, and we know the evaluations of each of the alternatives with respect to each of the criteria that are important for making the decision. For example, if we want to buy a car we consider several options among the possible cars, such as price, comfort, ... However, the problem takes a similar form to the previous one. Assuming that the evaluation is numerical for each criterion, the different alternatives are identified with their corresponding evaluation vectors, and we again have the problem of comparing vectors in the n-dimensional space. As in the previous case, the solution is to define a representative value of each vector so that the problem is reduced to comparing real values. This is achieved through the so-called aggregation operators. Among the many aggregation operators that can be used, the criterion that consists in assigning weights or amounts to the different criteria and considering a weighting value of the partial evaluations stands out. Formally, this comes down to an expected value. However, the use of this operator does not allow modeling situations that are very common in Multicriteria Decision Making. For example, it does not allow to model interactions between the criteria, nor does it allow veto or favor situations for some criteria. However, these situations can be successfully described using non-additive measures and their corresponding Choquet integral.

Non-additive measures, also known as capacities, monotone measures or fuzzy measures are, from a mathematical point of view, a generalization of probability measures, in which additivity is replaced by a weaker condition of monotonicity. They were defined as a generalization of a measure by Choquet and have been successfully applied in fields as diverse as Fuzzy Logic or Image Recognition. By not requiring additivity in their definition, they allow modeling risk aversion and also interactions between criteria or vetoes and favors. The equivalent of expected value is given in this situation by the Choquet integral, which generalizes the Lebesgue integral.

Another field in which non-additive measures appear is the Cooperative Game Theory. The problem here is a group of players who may or may not cooperate with each other by forming different coalitions. In this case, the game is characterized by the profit that each possible coalition can achieve on its own, even with the opposition of the other players. In this way, each game is characterized by that function. The goal of Cooperative Game Theory is, assuming that all players agree to form the entire coalition, to divide this profit; For this, we must take into account the possible profit of the smaller coalitions. In this context, monotonicity means that a coalition has higher profits than any coalition contained in it. This assumption is reasonable and justifies that the function defining the game is a non-additive measure, although not necessarily normalized.

For all these reasons, non-additive measures have been studied from many points of view. This has led to the development of equivalent representations of the non-additive measures that allow further interpretation. Thus, the Möbius interaction or the Shapley interaction has been defined. Also some axiomatics have been developed that allow to identify when a situation can be modeled by non-additive measures.

However, all this richness in terms of interpretation has its counterpart in the computational complexity of working with non-additive measures. Actually, for a probability measure on a referential of n elements, it is enough to fix $n-1$ values. However, in the case of a non-additive measure $2^n - 2$ values are required. To avoid this problem, two types of solutions have been proposed:

- First, remark that some coalitions might not make sense. These kinds of restrictions are very common in Game Theory, in which some coalitions are impossible.
- On the other hand, additional restrictions may be required on the definition of non-additive measure. In this way, many subfamilies appear preserving much of

the potential of general non-additive measures in terms of interpretation, and considerably reducing the number of values necessary for their definition. This option is the most common in Decision Theory and has given rise to several subfamilies, such as k-additive measures, k-intolerant measures, k-symmetric measures and many more.

One of the problems appearing in the practical application of non-additive measures is determining the measure, generally restricted to a subfamily, that models a specific situation. This is what is known as the identification problem. Obtaining the measure in a specific problem can be solved in different ways. For example, it is possible to assign values to the measure of the different subsets with the help of an expert, by means of a questionnaire. The same can be done to determine the values of other equivalent representations (Möbius, Shapley, ...). Another solution is to use the information provided by a sample. For example, in the case of Multicriteria Decision Making, it is possible that there are several objects for which the score in each of the criteria and the global score of them are known; in this case, an optimization problem must be solved in order to find the corresponding measure or measures.

There are many ways to solve these optimization problems. Quadratic Programming techniques, Genetic Algorithms, and many others have been used. We will mainly use genetic algorithms and we will use the geometric properties of these subfamilies to guarantee the effectiveness of these algorithms.

Related to the problem of testing the effectiveness of these algorithms we have the problem of generating measures in a random way. Also, generating random points within a polytope is a problem with a huge number of applications and much attention has been devoted to it. One of the advantages of knowing ways to sample points randomly and uniformly in a polytope is that it allows us to easily estimate its volume. When the polytope studied coincides with some subfamily of non-additive measures, it is interesting to have techniques that allow us to choose a measure at random in this set, assigning equal probabilities to all the measures.

From a geometric point of view, when we restrict ourselves to subfamilies of non-additive measures, it is very common for the set of non-additive measures of a subfamily to be a convex polytope. Therefore, its points can be easily determined if we know its vertices. Knowing the geometric structure of the different subfamilies that appear in the application of non-additive measures helps us to solve both the problems of random generation and the problems associated with the identification of measures. In this thesis we are going to study some of these subfamilies.

One aspect that many of the most important subfamilies in both Game Theory and Decision Theory have in common is that the corresponding polytopes are a special type of polytopes known as order polytopes. The order polytopes are characterized by the existence of a partially ordered set from which it is possible to determine the entire combinatorial structure of the polytope. This simplifies the problem, since we can reduce our study to the study of the properties of the associated partially ordered set (abbreviated poset), a problem that is usually easier to tackle. Thus, for example, it can be verified that the volume of the polytope can be obtained from the number of linear extensions of the underlying poset. In the same way, it is possible to characterize the vertices in terms of the poset filters, and determine if two vertices are adjacent from the difference of these filters.

In the different chapters of this thesis we will present results for several of these problems. Broadly speaking, the thesis can be divided into the following parts.

- Firstly, a method to generate linear extensions of a poset is studied. Obtaining a linear extension randomly is a complex problem for which only partial results are known. In this thesis we have developed a method, called Bottom-Up, that allows calculating linear extensions quickly and intuitively. It is not applicable to any poset, but for subfamilies that are order polytopes and whose posets allow its application, it allows to find the corresponding vertices and calculate the volume. For example, it allows uniformly generate elements of the set of non-additive measures when there are three elements; it also allows to generate elements in several special cases of the truncated measures; and in specific cases of measures with a restricted set of coalitions.

  Extensions of this method, such as the ideals-based method, allow working with a much wider range of cases. For example it solves the problem of generating elements of the subfamily of 2-truncated measures. The drawback of these generalizations is that they lead to a large increase in the complexity.

- Next, the polytope of 2-symmetric measures is studied. These measures are also order polytopes. We have shown that the associated poset can be identified with Young diagrams, a combinatorial structure that appears when computing the number of partitions of a natural number. This allows us to find a method for randomly generating measures in this polytope. We also obtain some other interesting properties of this polytope.

- The next chapter studies the polytope of 2-additive measures. Contrary to what happened until now, k-additive measures do not constitute an order polytope, and they must be studied differently. The difference between the 2-additive measures and the general k-additive measures is that the vertices of the 2-additive measures are known and are 0/1-valued measures. In this chapter we characterize the adjacency in this polytope, as well as the structure and number of r-dimensional faces. In addition, we give a triangulation of this polytope into simplices of the same volume that allows for a quick random generation.

- The next chapter is devoted to what we have called order cones. The motivation for the problem lies in the fact that, in the field of Game Theory, the condition of normalization is not natural. For this reason, the polyhedra that appear are not upper bounded. In this chapter we see that in a context of restricted cooperation, the set of monotone cooperative games is an order cone. This allows to study its properties again from a poset. Furthermore, we see that many of its geometric properties can be put in terms of the order polytope associated to the underlying poset.

- In the last chapter, we study integral techniques for counting linear extensions. Keep in mind that finding the number of linear extensions of a poset is a complex problem and only partial results are known. These integrals allow us to obtain recursive formulas to get the number of linear extensions of some families of posets. This implies that it is possible to compute the volume of the polytopes whose subfamilies of non-additive measures have an underlying set structure that is identified with one of the families discussed in this chapter.

The thesis ends with a small chapter of conclusions and open problems. While developing the chapters, new problems have arisen, many of which are related to complex problems in Combinatorics or Algebra. However, we hope that this work serves as an example of the advances that the use of algebraic and combinatorial tools can add to the study of non-additive measures.

Introducción

Desde el principio de los tiempos, la Humanidad ha tenido que lidiar con situaciones donde se debe tomar una decisión entre varias alternativas. Este problema sigue siendo nuclear en la vida de todos nosotros. De esta forma cada día debemos decidir qué ropa utilizar, a qué hora levantarnos, qué medio de transporte utilizar para llegar al trabajo, y así un largo etcétera.

En muchas de estas decisiones, las consecuencias dependen de una serie de factores que no pueden ser controlados por el hombre. Esto hace que la toma de decisiones pueda llegar a ser un proceso muy complicado gobernado por la incertidumbre. En muchas ocasiones es necesario elegir opciones en competencia con otros individuos. Ni que decir tiene que una decisión errónea puede acarrear problemas importantes que se pueden extender durante un período prolongado de tiempo, lo que hace que sea necesario una reflexión sobre la decisión a tomar.

Para intentar dilucidar cómo debe hacerse esta toma de decisiones surge la *Teoría de la Decisión* y la *Teoría de Juegos*. En ellas se proponen herramientas matemáticas que permiten modelar el comportamiento humano ante la toma de decisiones, así como establecer las decisiones más adecuadas para cada problema.

Los problemas de toma de decisiones aparecen en primer lugar en los trabajos de Pascal y Bernoulli sobre Probabilidad y en los trabajos sobre sistemas de votación de Condorcet y Borda a finales del siglo XVIII. Sin embargo, a pesar de la evidente importancia que la toma de decisiones tiene, su estudio desde un punto de vista formal es bastante reciente. Los primeros estudios surgen en el siglo XX y son debidos a Neyman y Pearson y sus trabajos sobre contrastes de hipótesis estadísticos. La extensión a otros ámbitos de la estadística es debida a A. Wald. En 1944, J. von Neumann y O. Morgenstern publican un trabajo que supone un avance fundamental tanto en la Teoría de la Decisión como en la Teoría de Juegos.

El conjunto de factores importantes a la hora de decidir y que escapan al control del decisor se llaman *estados de la naturaleza*. Dependiendo de la información disponible sobre los estados de la naturaleza, la Teoría de la Decisión se divide en dos: Teoría de la Decisión Bajo Incertidumbre y Bajo Riesgo. En Teoría de la Decisión Bajo Incertidumbre no se tiene ninguna información sobre cuál puede ser el estado de la naturaleza, mientras que en Teoría de la Decisión Bajo Riesgo se tiene una información que se modela mediante una distribución de probabilidad sobre el conjunto de estados de la naturaleza.

En estas condiciones, si suponemos que el conjunto de estados de la naturaleza es finito, cada alternativa puede ser identificada con un vector. Este vector de consecuencias se hace numérico mediante una función de utilidad o de pérdida. Lo que dificulta la toma de decisiones viene del hecho de que una alternativa puede ser la mejor para un estado de la naturaleza pero no para los otros. Esto se debe a que no existe un orden natural en el espacio n-dimensional para comparar vectores. Para resolver este problema lo que se hace es sustituir cada vector por un representante que ya es un valor real; y entonces se toma como decisión óptima la que tenga un mejor representante.

Diferentes formas de elegir el representante asociado a los vectores conducen a distintas decisiones, y esto da lugar a distintos criterios de decisión. Si estamos en un ambiente de riesgo, el criterio más usual es el criterio de utilidad esperada o derivados del mismo, en el que se toma como representante el valor esperado de las utilidades.

Sin embargo, el criterio de utilidad esperada no puede explicar algunos comportamientos relacionados con la aversión al riesgo, como las paradojas de Ellsberg y Allais, que aparecen tanto en ambiente de incertidumbre como de riesgo. Estas paradojas pueden sin embargo ser modeladas mediante el uso de medidas no aditivas y la integral de Choquet en lugar de probabilidades y esperanzas matemáticas.

Otra rama de la Teoría de la Decisión es la Decisión Multicriterio. En el caso de la Decisión Multicriterio, el problema al que nos enfrentamos no tiene un grado de incertidumbre como ocurría en los casos anteriores. En este caso, tenemos una serie de alternativas de entre las que se debe escoger una, y conocemos las evaluaciones de cada una de las alternativas respecto a cada uno de los criterios que son importantes para tomar la decisión. Por ejemplo, si queremos comprar un coche tenemos en cuenta varias opciones entre los coches posibles, como puede ser el precio, confort, ... Sin embargo, el problema adopta una forma similar al anterior. Asumiendo que la evaluación es numérica para cada criterio, las distintas alternativas se identifican con sus correspondientes vectores de evaluaciones, y volvemos a tener el problema de comparar vectores en el espacio n-dimensional. Como en el caso anterior, la solución pasa por definir un representante de cada vector de forma que el problema se reduzca a comparar valores reales. Esto se consigue mediante los conocidos como operadores de agregación. Entre los muchos operadores de agregación que se pueden utilizar destaca el criterio que consiste en asignar pesos o importancias a los distintos criterios y considerar una ponderación de las evaluaciones parciales. Formalmente, esto se reduce a una esperanza matemática. Sin embargo, el uso de este operador no permite modelar situaciones que son muy habituales en decisión multicriterio. Por ejemplo, no permite modelar interacciones entre los criterios ni tampoco situaciones de veto o favor para algunos criterios. Sin embargo, estas situaciones pueden ser descritas con éxito mediante las medidas no aditivas y su correspondiente integral de Choquet.

Las medidas no aditivas, también conocidas como capacidades, medidas monótonas o medidas difusas son, desde un punto de vista matemático, una generalización de las medidas de probabilidad, en las que la aditividad se sustituye por una condición más débil de monotonía. Fueron definidas como una generalización de medida por Choquet y han sido aplicadas con éxito en campos tan diversos como la Lógica Difusa o el Reconocimiento de Imágenes. Al no exigir aditividad en su definición, permiten modelar la aversión al riesgo y también interacciones entre criterios o vetos y favores. El equivalente a la esperanza matemática viene dado por la integral de Choquet, que generaliza la integral de Lebesgue.

Otro de los campos en los que aparecen las medidas no aditivas es el la Teoría de Juegos Cooperativos. El problema en este caso consiste en un grupo de jugadores que pueden o no cooperar entre sí formando distintas coaliciones. En este caso, el juego queda caracterizado por la ganancia que cada posible coalición puede conseguir por si misma, incluso con la oposición de los demás jugadores. De esta forma, cada juego queda caracterizado por esa función. El objetivo de la teoría de juegos es, suponiendo que todos los jugadores se ponen de acuerdo para formar la coalición total, repartir esta ganancia; para ello se debe tener en cuenta la ganancia que tendría cada una de las coaliciones más pequeñas. En este contexto, la monotonía se traduce en que una coalición tenga mayores ganancias que cualquier coalición contenida en ella. Este supuesto es razonable y justifica el hecho de pensar que la función que define el juego es una medida no aditiva, aunque no necesariamente normalizada.

Por todas estas razones, las medidas no aditivas se han estudiado desde muchos puntos de vista. Esto ha dado lugar a la obtención de representaciones equivalentes de las medidas no aditivas que permiten profundizar en la interpretación. Así, se ha definido la interacción de Möbius o la interacción de Shapley. También se han desarrollado axiomáticas que permiten identificar cuándo una situación puede ser modelada mediante medidas no aditivas.

Sin embargo, toda esta riqueza en términos de interpretación tiene su contrapartida en el gasto computacional de las medidas no aditivas. En efecto, para una medida de probabilidad sobre un referencial de n elementos, basta almacenar $n-1$ valores. Sin embargo, en el caso de una medida no aditiva son necesarios $2^n - 2$ valores. Para evitar este problema se han propuesto dos tipos de soluciones:

- En primer lugar, es posible que los valores sobre algunos conjuntos no tengan sentido. Este tipo de restricciones son muy habituales en Teoría de Juegos, en los que algunas coaliciones son imposibles.
- Por otra parte, se pueden pedir restricciones adicionales sobre la definición de medida no aditiva. De esta forma que aparecen subfamilias que conservan gran parte del potencial de las medidas no aditivas generales en términos de interpretación, mientras que reducen de forma considerable el número de valores necesarios para su definición. Esta opción es la más habitual en Teoría de la Decisión y ha dado lugar a varias subfamilias, como por ejemplo las medidas k-aditivas, las medidas k-intolerantes, las medidas k-simétricas y varias más.

Uno de los problemas que aparece en la aplicación práctica de las medidas no aditivas es el de determinar la medida, generalmente restringida a una subfamilia, que modela una situación concreta; esto lo que se conoce como el problema de identificación. La obtención de la medida en un problema concreto puede resolverse de distintas maneras. Por ejemplo, es posible asignar valores a la medida de los distintos subconjuntos con la ayuda de un experto en el asunto, mediante un cuestionario; esto mismo puede hacerse para determinar los valores de otras representaciones equivalentes (Möbius, Shapley,...). Otra solución es utilizar la información que proporciona una muestra. Por ejemplo, en el caso de Decisión Multicriterio, es posible que se disponga de varios objetos para los que se conoce la puntuación en cada uno de los criterios y la puntuación global de los mismos; en este caso hay que resolver un problema de optimización para hallar la medida o medidas correspondientes.

Hay muchas formas de resolver estos problemas de optimización. Así, se han aplicado técnicas de Programación Cuadrática, Algoritmos Genéticos, y muchas otras. Nosotros usaremos mayoritariamente algoritmos genéticos donde usaremos las diferentes propiedades geométricas de estas subfamilias para garantizar la eficacia de estos algoritmos.

Relacionado con el problema de probar la eficacia de estos algoritmos tenemos el problema de generar medidas de forma aleatoria. Además, generar puntos al azar dentro de un politopo es un problema con una inmensa cantidad de aplicaciones y al que se le ha dedicado mucha atención. Una de las ventajas de conocer maneras de simular puntos aleatoriamente y uniformemente en un politopo es que nos permite estimar de forma sencilla el volumen del mismo. Cuando el politopo estudiado coincide con alguna subfamilia de medidas no aditivas, es interesante disponer de técnicas que nos permitan elegir una medida al azar en este conjunto asignando probabilidades iguales a todas las medidas.

Desde un punto de vista geométrico, cuando nos restringimos a subfamilias de medidas no aditivas, es muy común que el conjunto de medidas no aditivas dentro de una subfamilia sea un politopo convexo. Por ello, sus puntos se pueden determinar de forma sencilla si conocemos sus vértices. Conocer la estructura geométrica de las distintas subfamilias que aparecen en la aplicación de las medidas no aditivas nos ayuda a resolver tanto los problemas de generación aleatoria como los problemas asociados a la identificación de medidas. En esta memoria vamos a estudiar algunas de estas subfamilias.

Un aspecto que tienen en común muchas de las subfamilias más importantes que aparecen tanto en Teoría de Juegos como en Teoría de la Decisión es que los politopos correspondientes son un tipo especial de politopos conocidos como politopos de orden. Los politopos de orden se caracterizan por la existencia de un conjunto parcialmente ordenado a partir del cual es posible determinar toda la estructura combinatoria del politopo. Esto simplifica el problema, ya que podemos reducir nuestro estudio al estudio de las propiedades del conjunto parcialmente ordenado (abreviado poset), un problema que suele ser más sencillo de abordar. Así, por ejemplo puede comprobarse que el volumen del politopo puede ser obtenido a partir del número de extensiones lineales del poset subyacente. De la misma forma, es posible caracterizar los vértices en términos de los filtros del poset, y establecer si dos vértices son adyacentes a partir de la diferencia de estos filtros.

En los distintos capítulos de esta memoria iremos presentando resultados para varios de estos problemas. A grandes rasgos, la memoria se puede dividir en las siguientes partes.

- En primer lugar, se estudia un método para generar extensiones lineales de un poset. Obtener una extensión lineal de forma aleatoria es un problema complejo para el que sólo se conocen resultados parciales. En esta memoria hemos desarrollado un método, llamado Bottom-Up, que permite calcular extensiones lineales de forma rápida e intuitiva. No es aplicable a cualquier poset, pero para subfamilias que sean politopos de orden y su poset permita su aplicación, permite hallar los vértices correspondientes y calcular el volumen. Por ejemplo permite generar uniformemente elementos del conjunto de medidas no aditivas cuando hay tres elementos; también permite generar elementos en varios casos especiales de las medidas truncadas; y en casos concretos de medidas con conjunto de coaliciones restringido.

Extensiones de este método, como el método basado en ideales, permiten trabajar con un rango mucho más amplio de casos, por ejemplo resuelve el problema de generar elementos de la subfamilia de las medidas 2-truncadas. El inconveniente de estas generalizaciones reside en que conllevan un gran aumento en la complejidad de su aplicación.

- A continuación se estudia el politopo de las medidas 2-simétricas. Estas medidas son también politopos de orden. Hemos demostrado que el poset correspondiente puede identificarse con diagramas de Young, una estructura combinatoria que aparece al calcular particiones de números. Esto nos permite hallar un método para generar de forma aleatoria medidas en este politopo. Así como obtener otras propiedades interesantes del mismo.

- El siguiente capítulo estudia el politopo de las medidas 2-aditivas. Al contrario de lo que pasaba hasta este momento, las medidas k-aditivas no constituyen un politopo de orden, y hay que estudiarlas de manera diferente. La diferencia entre las medidas 2-aditivas y las medidas k-aditivas generales es que los vértices de las medidas 2-aditivas son conocidos y son medidas 0/1-valuadas. En este capítulo caracterizamos la adyacencia en este politopo, así como la estructura y número de caras r-dimensionales. Además, damos una triangulación de este politopo en símplices del mismo volumen que permite la generación aleatoria de forma rápida.

- El siguiente capítulo está dedicado a lo que hemos denominado conos de orden. La motivación del problema radica en que en el ámbito de la Teoría de Juegos la condición de normalización no es natural. Por ello, los poliedros que aparecen no están acotados superiormente. En este capítulo vemos que en un contexto de cooperación restringida, el conjunto de juegos cooperativos monótonos es un cono de orden. Esto permite nuevamente estudiar sus propiedades a partir de un poset. Además, vemos que muchas de sus propiedades geométricas se pueden poner en términos del politopo de orden sobre el mismo poset.

- En el último capítulo, se estudian técnicas de integración para contar extensiones lineales . Hay que tener en cuenta que hallar el número de extensiones lineales de un poset es un problema complejo y sólo se conocen resultados parciales. Estas integrales permiten obtener fórmulas recursivas para obtener el número de extensiones lineales de algunas familias de posets. Esto implica que es posible calcular el volumen de los politopos cuyas subfamilias de medidas no aditivas tienen una estructura de conjuntos subyacente que se identifica con alguna de las familias que se analizan en este capítulo.

La memoria acaba con un pequeño capítulo de conclusiones y problemas abiertos. Al desarrollar los distintos capítulos han surgido muchos problemas nuevos, muchos de los cuáles se relacionan con problemas complejos de Combinatoria o Álgebra. Sin embargo, esperamos que este trabajo sirva como ejemplo de los avances que el uso de herramientas algebraicas y combinatorias pueden tener en el estudio de las medidas no aditivas.

Chapter 1

Posets and Polytopes

One of the most important structures along this thesis are the so-called order polytopes. The reason relays in the fact that, as we will see in next chapters, order polytopes can be used to model many subfamilies of fuzzy measures. In order to establish the definition of order polytopes and the properties that will be applied in this thesis, it is first necessary to give some previous results on partially ordered sets (posets for short) and convex polytopes. This is the task we achieve in this chapter. Besides, some results that we have derived about posets and convex polytopes are included.

1.1 Poset theory

For a general introduction on the theory of posets see [1, 2]. Elements of a poset P are denoted x, y and so on, and also $a_1, a_2, ...$ If $|P| = n$, we will also use the notation $P = \{1, ..., n\}$. Subsets of P are denoted by capital letters $A, B, ...$ In order to avoid hard notation, we will often use $i_1 i_2 \cdots i_n$ for denoting the set $\{i_1, i_2, \cdots, i_n\}$, specially for singletons and pairs. We also define $\binom{X}{k}$ as the set of all k-element subsets of X.

Definition 1.1. Let P be a set and $\preceq$ be a binary relation over P. The pair $(P, \preceq)$ is a **partially order set** (or **poset** for short) if $\preceq$ satisfies the following conditions:

i) Reflexivity: $x \preceq x$, $\forall x \in P$,

ii) Antisymmetry: If $x \preceq y$ and $y \preceq x$, then $x = y$, $\forall x, y \in P$,

iii) Transitivity: If $x \preceq y$ and $y \preceq z$, then $x \preceq z$, $\forall x, y, z \in P$.

For a poset P, we can define the **dual poset** $P^{\partial} = (P, \preceq_{\partial})$ such that $x \preceq_{\partial} y \Leftrightarrow y \preceq x$. With some abuse of notation, we will usually omit $\preceq$ and write P instead of $(P, \preceq)$ when referring to posets. We say that y **covers** x, denoted $x \lessdot y$, if $x \preceq y$ and there is no $z \in P \setminus \{x, y\}$ satisfying $x \preceq z \preceq y$. A poset can be represented through *Hasse diagrams*. We draw a Hasse diagram for a poset P representing each element of P by a distinct point so that whenever $x \lessdot y$, there is an upwards segment from x to y.

Example 1.1. *Figure 1.1 shows a Hasse diagram for the poset* $(N, \preceq)$ *where* $N = \{1, 2, 3, 4\}$, *and* $\preceq$ *is given by* $1 \preceq 4, 2 \preceq 4, 2 \preceq 3$ *and satisfying conditions* $i), ii)$ *and* $iii)$.

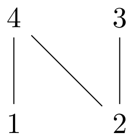


FIGURE 1.1: Poset N and its Hasse diagram.

An element x such that $x \not\lessdot y, \forall y \in P$ is called a **maximal element**. When a poset has just one maximal, this maximal element is called **maximum**. Similarly, if x is such that $y \not\lessdot x, \forall y \in P, x$ is called a **minimal element**. When a poset has just one minimal, this minimal element is called **minimum**. We will denote by $\mathcal{MIN}(P)$ the set of minimal elements of a poset P and $m(P) = |\mathcal{MIN}(P)|$. Similarly, we will denote by $\mathcal{MAX}(P)$ the set of maximal elements of a poset P and $M(P) = |\mathcal{MAX}(P)|$.

For an element x, we define its **level** recursively as follows[1]: maximal elements are in level 0, denoted L_0. Maximal elements of $P \setminus L_0$ are in level L_1; in general, L_i is the set of maximal elements of $P \setminus (L_0 \cup ... \cup L_{i-1})$.

A poset is a **chain** if $x \preceq y$ or $y \preceq x$, $\forall x, y \in P$. We will denote the chain of n elements by $\mathbf{n}$; similarly, an **antichain** is a poset where $\preceq$ is given by $x \preceq y \Leftrightarrow x = y$. We will denote the antichain of n elements by $\bar{\boldsymbol{n}}$. If a pair of different elements (x, y) form an antichain we denote it by $x \parallel y$. A chain $C \subseteq P$ is said to be a **maximal chain** in P if there is not other different chain C' such that $C \subset C'$. Symmetrically, we can define **maximal antichains**. The **height** of P, denoted by $h(P)$, is defined as the cardinality of a longest chain in P. Similarly, the **width** of P, denoted by $w(P)$, is defined as the cardinality of a largest antichain in P. Finite posets can be divided into chains and antichains. Let P be a finite poset and let us denote by $c(P), C(P), a(P)$ and $A(P)$ be the number of chains, the number of maximal chains, the number of antichains and the number of maximal antichains, respectively.

Given an element x, we denote by $\downarrow x$ the subposet of P whose elements are $\{y : y \preceq x\}$ and by $\downarrow \hat{x} := \downarrow x \setminus \{x\}$. Similarly, we denote by $\uparrow x$ the subposet of P whose elements are

[1] Indeed, it may sound rather counterintuitive that in a pair of comparable elements, the greater one has a lower level, i.e. levels are decreasing from the set of maximal elements downwards, while intuitively it should be the opposite. This way of defining the level would be shown useful in the proofs of Chapter 3.

$\{y : x \preceq y\}$ and by $\uparrow \hat{x} := \uparrow x \setminus \{x\}$. These notions can be extended for a general subset A, thus obtaining $\downarrow A, \uparrow A, \downarrow \hat{A}$ and $\uparrow \hat{A}$. Finally, we will denote by $\updownarrow A$ the set of elements related to any element of A and by $\updownarrow \hat{A} = \updownarrow A \setminus A$.

An **ideal** or **downset** I of P is a subset of P such that if $x \in I$, then $\downarrow x \subseteq I$. We will denote the set of all ideals of P by $\mathcal{I}(P)$ and $i(P) := |\mathcal{I}(P)|$. An ideal with exactly one maximum is called **principal ideal**. One of the most important constructions in order theory is the **poset** $\mathcal{I}(P)$ **of down-sets** ordered by inclusion. Symmetrically, a subset F of P is a **filter** or **upset** if for any $x \in F$ and any $y \in P$ such that $x \preceq y$, it follows that $y \in F$. We denote by $\mathfrak{F}(P)$ the set of filters of P and $f(P) := |\mathfrak{F}(P)|$. A filter with exactly one minimum is called **principal filter**. In this thesis, we will assume that P and the empty set are both filters and ideals, therefore $\mathcal{I}(P)$ and $\mathfrak{F}(P)$ have both maximum and minimum.

Two posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ are **isomorphic** if there is a bijection $f : P \to Q$ such that $x \preceq_P y \Leftrightarrow f(x) \preceq_Q f(y)$, and it is denoted by $P \cong Q$ (or $P = Q$). If two posets are isomorphic, then their corresponding Hasse diagrams are the same up to differences in the names of the elements.

A map $f : P \to Q$ is said to be order-preserving (also monotone or a morphism) if $x \preceq_P y$ implies $f(x) \preceq_Q f(y)$.

Definition 1.2. A poset $(Q, \preceq_Q)$ with $Q \subset P$ is a **subposet** of a poset $(P, \preceq_P)$ if the inclusion mapping $\iota : Q \hookrightarrow P$ is order-preserving. A poset $(Q, \preceq_Q)$ is an **induced subposet (or full subposet)** of $(P, \preceq_P)$ if there is an order-preserving injective map $g : Q \to P$ with order-preserving inverse $g^{-1} : g(Q) \to Q$. Observe that an induced subposet Q inherits the order structure of P.

Example 1.2. *Let us consider the* $(N, \preceq)$ *poset from Example 1.1. Note that the antichain formed by elements* 1 *and* 4 *is a subposet of* N*, but it is not an induced subposet of* N*, since* $1 \preceq 4$*. However, the chain formed by the same elements* 1 *and* 4 *is both a subposet and an induced subposet of* N*.*

Two elements $x, y \in P$ are said to be **interchangeable** if there is an automorphism $f : P \to P$ such that $f(x) = y$ and $f(y) = x$.

Now we introduce some important ways of defining new posets from old. Given two posets, $(P, \preceq_P), (Q, \preceq_Q)$, their **direct sum**, denoted $P \oplus Q$, is a poset over the referential $P \cup Q$ (disjoint union) and whose partial order $\preceq_{P \oplus Q}$ is defined as follows: if $x, y \in P$ then $x \preceq_{P \oplus Q} y$ if and only if $x \preceq_P y$; if $x, y \in Q$ then $x \preceq_{P \oplus Q} y$ if and only if $x \preceq_Q y$; and if $x \in P, y \in Q$ then $x \preceq_{P \oplus Q} y$. It is not difficult to show that the direct sum of

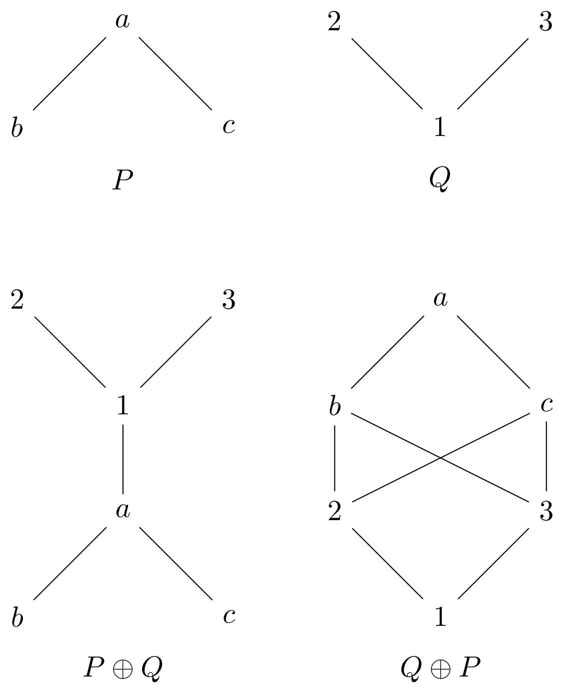


Figure 1.2: Direct sum of posets.

posets is associative but not commutative, see Figure 1.2. A poset is **irreducible** by direct sum if it cannot be written as a direct sum of two posets.

Similarly, the **disjoint union** of two posets $(P, \preceq_P), (Q, \preceq_Q)$, denoted $P \uplus Q$, is a poset $(P \cup Q, \preceq_{P \uplus Q})$ where $x \preceq_{P \uplus Q} y$ whenever $x, y \in P$ and $x \preceq_P y$, or $x, y \in Q$ and $x \preceq_Q y$. It is not difficult to show that the disjoint union is commutative and associative, see Figure 1.3. A poset which cannot be written as disjoint union of two posets is called **connected**. Obviously, the Hasse diagram of a connected poset is also a connected graph.

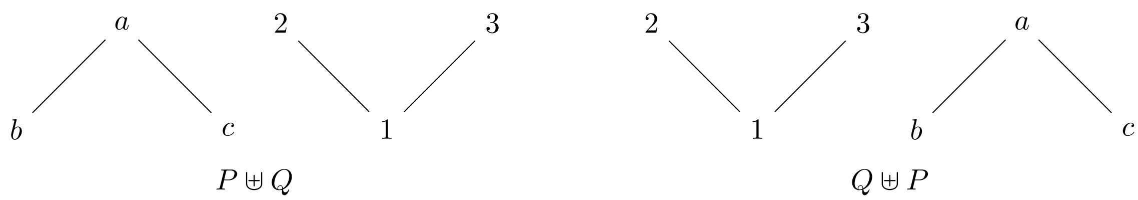


Figure 1.3: Disjoint union of posets.

Given two posets, $(P, \preceq_P), (Q, \preceq_Q)$, their **cartesian product** (or product for short) denoted by $P \times Q$, is a poset over the cartesian product of sets $P \times Q$ and whose partial order $\preceq_{P \times Q}$ is defined as follows: $(x_1, x_2) \preceq_{P \times Q} (y_1, y_2)$ if and only if $x_1 \preceq_P y_1$, and $x_2 \preceq_Q y_2$. It is not difficult to show that the product of posets is commutative and associative. Let $N = \{1, 2, 3, 4\}$ be the poset as in Example 1.1 and $\mathbf{2}$ the chain with two elements. The Hasse diagram of $N \times \mathbf{2}$ is depicted in Figure 1.4.

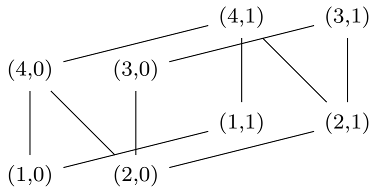


FIGURE 1.4: $N \times \mathbf{2}$.

Another interesting operation is the **lexicographic product**. Given two posets, $(P, \preceq_P)$, $(Q, \preceq_Q)$ their lexicographic product $(P * Q, \preceq_{P*Q})$ is defined over $P \times Q$ as follows: $(x_1, x_2) \preceq_{P*Q} (y_1, y_2)$ if and only if $x_1 \preceq_P y_1$ or $x_1 = y_1$ and $x_2 \preceq_Q y_2$. The lexicographic product is associative but not conmutative in general (see Figure 1.5.)

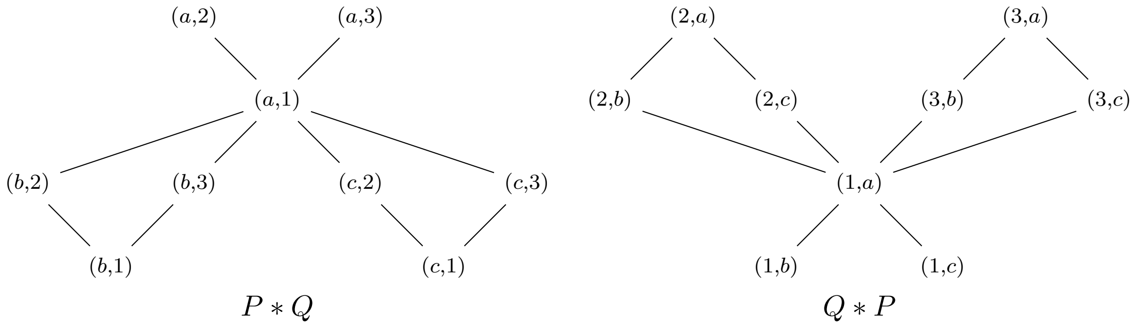


FIGURE 1.5: Lexicographic product of posets.

Let P and Q be two posets, we define the **exponential poset** of Q with respect to P, $Q^{<P>}$, by taking the set of all order-preserving maps from P to Q ordered by pointwise order. Other notation is $< P \to Q >$. It is easy to check that $\mathbf{2}^{<\mathbf{3}>} \cong (\mathcal{P}(\{1, 2, 3\}), \subseteq)$ which is called Boolean poset of order 3, denoted B_3, see Definition 1.4 and Figure 1.6 for its Hasse diagram.

Proposition 1.3. *[1, 2] Some important and known properties of these operations are the following:*

- $(P \uplus Q) \times R = (P \times R) \uplus (Q \times R)$.
- $\mathcal{I}(P)^\partial \cong \mathcal{I}(P^\partial)$.
- $\mathcal{I}(P \oplus \mathbf{1}) \cong \mathcal{I}(P) \oplus \mathbf{1}$ *and* $\mathcal{I}(\mathbf{1} \oplus P) \cong \mathbf{1} \oplus \mathcal{I}(P)$.
- $\mathcal{I}(P \uplus Q) \cong \mathcal{I}(P) \times \mathcal{I}(Q)$.
- $|\mathcal{I}(P)| = |\mathcal{I}(P \setminus \{x\})| + |\mathcal{I}(P \setminus (\downarrow x \cup \uparrow x))|$, $\forall x \in P$ *and* P *finite.*
- *Let* P *and* Q *be finite posets then* $\mathcal{I}(P \oplus Q) \cong \mathcal{I}(P) \overline{\oplus} \mathcal{I}(Q)$, *where* $\mathcal{I}(P) \overline{\oplus} \mathcal{I}(Q)$ *is obtained from* $\mathcal{I}(P) \oplus \mathcal{I}(Q)$ *by identifying the maximum of* $\mathcal{I}(P)$ *with the minimum of* $\mathcal{I}(Q)$.

- $\mathbf{2}^{<P>} \cong \mathcal{I}(P)^{\partial}$.
- $(\mathbf{2}^{<P>})^{\partial} \cong \mathbf{2}^{<P^{\partial}>}$.
- $(R^{<Q>})^{<P>} \cong R^{<P\times Q>}$.

A very important poset is the boolean poset of order n.

Definition 1.4. We define the **boolean poset (or boolean lattice)** of order n, B_n, as the poset of all subsets of a set of n elements $X = \{1, 2, \ldots, n\}$ ordered by inclusion $\subseteq$, i.e. $(\mathcal{P}(X), \subseteq)$.

Figure 1.6 shows the Hasse diagram of B_3.

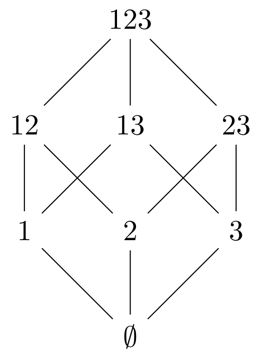


FIGURE 1.6: Hasse diagram of B_3 lattice.

This lattice will play a fundamental role when we study the polytope of fuzzy measures.

Lemma 1.5. *Let $X = \{1, 2, \ldots, n\}$ be a set of n elements and $\mathbf{2}^n = \mathbf{2} \times \mathbf{2} \times \ldots \times \mathbf{2}$. Define $\phi : \mathcal{P}(X) \longrightarrow \mathbf{2}^n$ by $\phi(A) = (\epsilon_1, \ldots, \epsilon_n)$ where*

$$\epsilon_i = \begin{cases} 1, & \textit{if } i \in A \\ 0, & \textit{if } i \notin A \end{cases}.$$

Then, ϕ is an order-isomorphism. Thus, $B_n \cong \mathbf{2}^n \cong \mathbf{2}^{<\overline{n}>}$.

Corollary 1.6. $B_{n+1} \cong B_n \times \mathbf{2}$.

The last result gives us a way of relating the poset B_n with B_{n-1}.

Proposition 1.7. *For the boolean lattice, it holds $\mathcal{I}(B_{n+1}) \cong \mathbf{2}^{<B_{n+1}>} \cong \mathbf{3}^{<B_n>}$.*

Proof. By using the properties of the operations and the fact $B_n = (B_n)^{\partial}$ we have:

$$\mathbf{2}^{<B_{n+1}>} \cong \mathbf{2}^{<(B_{n+1})^{\partial}>} \cong (\mathbf{2}^{<B_{n+1}>})^{\partial} \cong \mathcal{I}(B_{n+1}).$$

On the other hand, note that $\mathbf{2}^{<\mathbf{2}>}$ has three elements, the constant function zero, the constant function one and the identity. Thus, $\mathbf{2}^{<\mathbf{2}>} \cong \mathbf{3}$, and

$$\mathbf{2}^{<B_{n+1}>} \cong \mathbf{2}^{<B_n \times \mathbf{2}>} \cong (\mathbf{2}^{<\mathbf{2}>})^{<B_n>} \cong \mathbf{3}^{<B_n>}.$$

Hence, the result holds. $\square$

Theorem 1.8. *[3] Let P be a finite poset of height $h(P) = k$. Then, there exists a partition of P into k antichains, that is, $P = A_1 \cup \cdots \cup A_k$ where A_i is an antichain $\forall i \in \{1 \ldots k\}$ and $A_i \cap A_j = \emptyset$, $\forall i \neq j$.*

Theorem 1.9 (Dilworth). *[4] Let P be a finite poset of width $w(P) = k$. Then there exists a partition of P into k chains, that is, $P = A_1 \cup \cdots \cup A_k$ where A_i is a chain $\forall i \in \{1 \ldots k\}$ and $A_i \cap A_j = \emptyset$, $\forall i \neq j$.*

Remark 1.10. For this thesis will be decisive to know the cardinalities of the most relevant objects linked to posets (such as ideals or filters) and how this cardinalities relate to each other. An essential relation between these funtions is that the number of non-empty ideals equals the number of antichains. Indeed, let us take the next bijection ϕ, that for any non-empty ideal I, $\phi(I)$ is the set of maximal elements of I which is an antichain. The inverse of this function is given by $\phi^{-1}(a) = \downarrow a$. Moreover, the next map between filters and ideals $\alpha : \mathfrak{I} \to \mathfrak{F}$, $I \mapsto P \setminus I$ is also a bijection. Therefore:

$$i(P) = f(P) = a(P) + 1.$$

Lemma 1.11. *Let P and Q be two non-empty finite posets.*

i) $c(P \oplus Q) = c(P) + c(Q) + c(P)c(Q)$.

ii) $C(P \oplus Q) = C(P)C(Q)$.

iii) $a(P \oplus Q) = a(P) + a(Q)$.

iv) $A(P \oplus Q) = A(P) + A(Q)$.

v) $c(P \uplus Q) = c(P) + c(Q)$.

vi) $C(P \uplus Q) = C(P) + C(Q)$.

vii) $a(P \uplus Q) = a(P) + a(Q) + a(P)a(Q)$.

viii) $A(P \uplus Q) = A(P)A(Q)$.

ix) $i(P \oplus Q) = i(P) + i(Q) - 1$.

x) $i(P \uplus Q) = i(P)i(Q)$.

Proof. *i)* There are three types of chains. Chains with all elements in P; there are $c(P)$ such chains. Chains with all elements in Q, there are $c(Q)$ such chains. And chains combining elements of P and Q. As $x \preceq y$ for all $x \in P$ and $y \in Q$, these chains can be decomposed into a chain in P and a chain in Q, so the number of this type of chains is $c(P)c(Q)$.

ii) It suffices to note that a maximal chain in $P \oplus Q$ is a combination of a maximal chain in P and a maximal chain in Q.

iii) & *iv)* Any (maximal) antichain of P or Q is an (maximal) antichain of $P \oplus Q$. However we cannot join elements of P and Q to get a (maximal) antichain, because Q dominates P.

v) & *vi)* Any (maximal) chain of P or Q is a (maximal) chain of $P \uplus Q$. But we cannot join elements of P and Q to get a (maximal) chain so there are not more (maximal) chains.

vii) An antichain of $P \uplus Q$ is an antichain of P or an antichain of Q or any mixture of antichains of P and Q.

viii) The same as the last one but considering that to be a maximal antichain we need to join maximal antichains of both P and Q.

ix) Since $i(P) = a(P) + 1$ we get

$$\begin{aligned} i(P \oplus Q) &= a(P \oplus Q) + 1 = a(P) + a(Q) + 1 \\ &= i(P) - 1 + i(Q) - 1 + 1 = i(P) + i(Q) - 1. \end{aligned}$$

x) Since $i(P) = a(P) + 1$ we get

$$\begin{aligned} i(P \uplus Q) &= a(P \uplus Q) + 1 = a(P) + a(Q) + a(P)a(Q) + 1 \\ &= i(P) - 1 + i(Q) - 1 + (i(P) - 1)(i(Q) - 1) + 1 = i(P)i(Q). \end{aligned}$$

Hence, the result holds. □

When working with finite posets, it is sometimes convenient to denote elements as natural numbers. A **labeling** is a bijective mapping $L : \{1, 2, \ldots, |P|\} \to P$ (see [2]). There are $n!$ ways to define a labeling. A poset endowed with a labeling is called a **labeled poset**. Observe that a labeled poset is simply a poset with different consecutive natural numbers assigned to its elements. It is important to distinguish between posets and labeled posets. Let us show an example to clarify these issues. In addition, the next example exhibits how labeled posets model preferences.

Example 1.3. *Let A and B be two consumers and consider four consumer goods denoted by numbers $1, 2, 3$ and 4. Suppose that the consumer A and consumer B have preferences as the ones shown in Figure 1.7.*

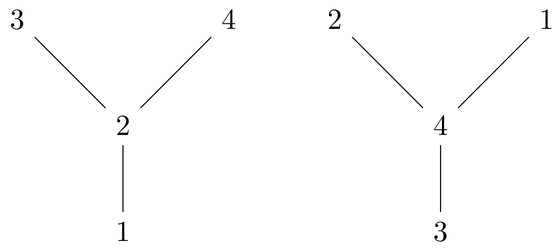


FIGURE 1.7: Labeled posets showing the preferences of consumers A and B.

Note that these two posets are isomorphic but they are not the same as labeled posets. In other words, they have the same shape but not the same numbers. So they are different as labeled posets.

A labeling is **natural** if $x \preceq y$ implies $L^{-1}(x) \leq L^{-1}(y)$ with the natural numbers order. It is well-known that every finite poset admits a natural labeling.

Let P be a poset, we can define $x \vee y$ as the minimum of $\{z \in P \mid z \succeq x,\ z \succeq y\}$ when it exists. Symmetrically, we can define $x \wedge y$ as the maximum of $\{z \in P \mid z \preceq x,\ z \preceq y\}$ when it exists. More generally, for a general subset $S \subseteq P$ we can define $\bigvee_P S$ as the minimum of $\{z \in P \mid z \succeq x, \forall x \in S\}$ when it exists. Dually, we can define $\bigwedge_P S$.

Definition 1.12. Let P be a non-empty poset. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a **lattice**. If $\bigvee_P S$ and $\bigwedge_P S$ exist for all $S \subseteq P$, then P is called a **complete lattice**. A lattice with maximum and minimum elements is called **bounded**.

Usually, the minimum is denoted by $\widehat{0}$ and the maximum by $\widehat{1}$. It easy to see that every finite lattice is bounded.

Remark 1.13. Let L be a non-empty finite lattice and $x, y, z \in L$. The following properties hold:

- If $x \preceq y \Rightarrow x \vee y = y$ and $x \wedge y = x$.
- Associative law (L1): $(x \vee y) \vee z = x \vee (y \vee z)$.
- Associative law $(\mathrm{L1})^\partial$: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

- Conmmutative law (L2): $x \vee y = y \vee x$.
- Conmmutative law $(\mathrm{L}2)^{\partial}$: $x \wedge y = y \wedge x$.
- Idempotency law (L3): $x \vee x = x$.
- Idempotency law $(\mathrm{L}3)^{\partial}$: $x \wedge x = x$.
- Absortion law (L4): $x \vee (x \wedge y) = x$.
- Absortion law $(\mathrm{L}4)^{\partial}$: $x \wedge (x \vee y) = x$.

Remark 1.14. Let L be a non-empty finite lattice and $x, y \in L$. If $x \preceq y \Rightarrow x \vee y = y$ and $x \wedge y = x$.

Let L and K be lattices. A function $f : L \to K$ is a **lattice homomorphism** if $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$, $\forall x, y \in L$. A bijective lattice homomorphism is a **lattice isomorphism**.

An element x of a lattice L is said to be **join-irreducible** if x is not a minimum and $x = a \vee b$ implies $x = a$ or $x = b$, $\forall a, b \in L$. A **meet-irreducible** element is defined dually.

A lattice L is said to be **distributive** if it satisfies the distributive law,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),\ \forall x, y, z \in L.$$

Also, L is **modular** if it satisfies the modular law,

$$x \succeq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z,\ \forall x, y, z \in L.$$

Let $\emptyset \neq M \subseteq L$, then M is a **sublattice (or induced sublattice)** of L if $x, y \in M \Rightarrow x \vee y, x \wedge y \in M$ for all $x, y \in M$ inheriting the lattice structure from L.

The following theorem gives a way of identifying distributive and modular lattices.

Theorem 1.15 (**The M_3-N_5 characterization**). *[1] Let L be a lattice. Then L is modular if and only if L has not the posets N_5 as sublattice. Moreover, L is distributive if and only if L has not the poset N_5 nor M_3 as sublattices. N_5 and M_3 posets are given in the Figure 1.8.*

Let us denote by $\mathcal{J}(L)$ the join-irreducible elements of a lattice L. The join-irreducible elements encode important information about the whole lattice.

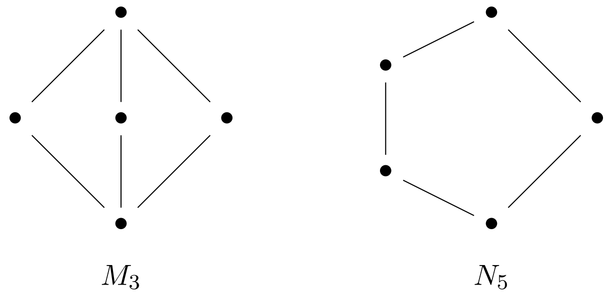


FIGURE 1.8: M_3 and N_5 posets.

Theorem 1.16. *Let P be a finite poset. Then the map $f(x) = \downarrow x$ is an isomorphism between P and $\mathcal{J}(\mathcal{I}(P))$.*

The last theorem has an interesting interpretation. Note that as $\mathcal{I}(P)$ is a lattice of ideals; and ideals are subsets of P, then it is also a lattice of subsets of P. Then, the elements of $\mathcal{J}(\mathcal{I}(P))$ are subsets of P. As $P \cong \mathcal{J}(\mathcal{I}(P))$, we can see any finite poset P as a poset of sets ordered by inclusion. In other words, any finite poset is isomorphic to a subposet of the boolean lattice $(B_n, \subseteq)$.

This is also true for infinite posets. For any general poset, it is enough to consider the poset $\mathcal{S} = \{\downarrow x : x \in P\}$ ordered by inclusion. Obviously, $\mathcal{S}$ is isomorphic to P. Therefore, any poset could be seen as a poset of sets ordered by inclusion.

The last theorem is an inspiration for one of the most important theorems in lattice theory, which is the Birkhoff's representation theorem.

Theorem 1.17 (Birkhoff's representation theorem for finite lattices). *[1]*

Let L be a finite distributive lattice. Then, the map $\eta : L \to \mathcal{I}(\mathcal{J}(L))$, $x \mapsto \mathcal{J}(L) \cap \downarrow x$ is an isomorphism between L and $\mathcal{I}(\mathcal{J}(L))$.

This way, for any distributive lattice L all the information is concentrated in the poset $\mathcal{J}(L)$. Note that the number of elements of $\mathcal{J}(L)$ is in general much lower than the cardinality of L. Indeed, the following proposition can be shown.

Proposition 1.18. *Let L be a finite lattice. Then the following statements are equivalent:*

i) *L is distributive.*

ii) $L \cong \mathcal{I}(\mathcal{J}(L))$.

iii) *L is isomorphisc to a down-set lattice.*

In Figure 1.9 we can see how Birkhoff theorem works.

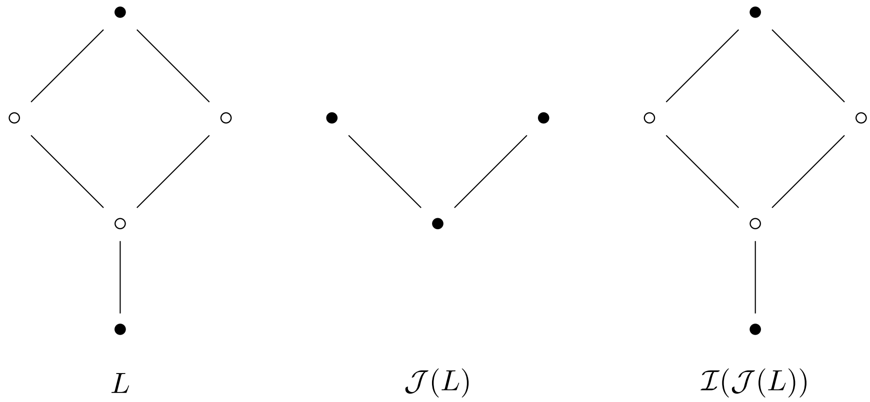


FIGURE 1.9: Birkhoff's representation theorem: the elements $\circ$ are join-irreducibles.

1.1.1 Linear extensions

Linear extensions will play an important role along this thesis because of their connection with order polytopes.

Definition 1.19. [2, 5] A **linear extension** of $(P, \preceq)$ is a sorting of the elements of P that is compatible with $\preceq$, i.e. $x \preceq y$ implies that x is before y in the sorting. In other words, if $|P| = n$, then a linear extension is an order-preserving bijection $\epsilon : P \to \boldsymbol{n}$.

Example 1.4. *The N poset (see Example 1.1) has 5 linear extensions which are displayed in Figure 1.10.*

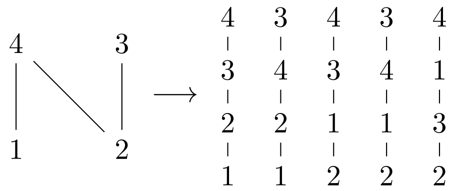


FIGURE 1.10: N poset and its linear extensions.

We normally consider posets endowed with a labeling in order to give a numeric name to its elements. With this labeling, linear extensions are labeled chains.

Linear extensions will be denoted ϵ_1, ϵ_2 and so on, and the i-th element of ϵ is denoted $\epsilon(i)$. We will denote by $\mathcal{L}(P)$ the set of all linear extensions of poset $(P, \preceq)$ and by $e(P) = |\mathcal{L}(P)|$. Two linear extensions are said to be related by a **transposition** if they are identical except for the swapping of two elements. If these elements are consecutive, the linear extensions are said to be related by an **adjacent transposition**.

Definition 1.20. Let P be a finite poset and $\mathcal{L}(P)$ its linear extensions. We define the **transposition graph**, denoted by $(\mathcal{L}(P), \tau)$, as the graph having the elements in $\mathcal{L}(P)$ as vertices and edges between linear extensions that are related by a transposition. The

adjacent transposition graph, denoted $(\mathcal{L}(P), \tau^*)$, is the graph with vertices $\mathcal{L}(P)$ and edges between linear extensions that are related by an adjacent transposition.

Lemma 1.21. *[6] Let P be a finite poset. Both graphs $(\mathcal{L}(P), \tau)$ and $(\mathcal{L}(P), \tau^*)$ are connected.*

Example 1.5. *Consider the N poset, given by four elements $1, 2, 3, 4$ and whose corresponding Hasse diagram is given in Figure 1.10 left. We have seen in Figure 1.10 right that the linear extensions of this poset N are*

$$(1, 2, 3, 4), (1, 2, 4, 3), (2, 1, 3, 4), (2, 1, 4, 3), (2, 3, 1, 4).$$

The corresponding transposition graph and adjacent transposition graph (they are the same for this poset) are given in Figure 1.11 right. Note that we have used a natural labeling for N.

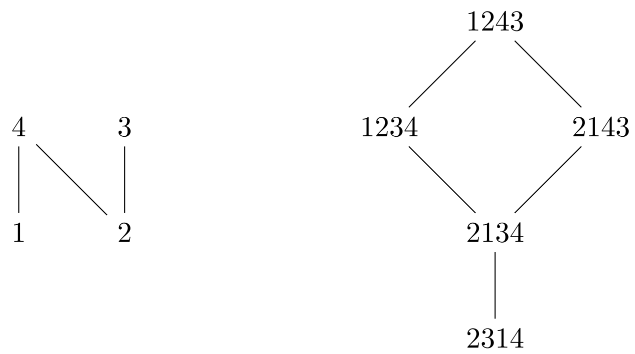


FIGURE 1.11: N poset and its (adjacent) transposition graph.

We start studying some important properties of the function $e(P)$. The first property is completely elemental but it will be used intensively.

Lemma 1.22. *Let P be a poset, then*

$$e(P) = \sum_{x \in \mathcal{MIN}(P)} e(P \setminus \{x\}) = \sum_{x \in \mathcal{MAX}(P)} e(P \setminus \{x\}).$$

Proof. Suppose that ϵ is a linear extension of P. The first element of ϵ must be a minimal element of P. So the set of all linear extensions can be partitioned into so many groups as $|\mathcal{MIN}(P)|$, each one corresponding with the group of linear extensions that start with the i-th minimal element. For maximals the proof is completely symmetric. $\square$

Lemma 1.22 can be extended as follows (see [7]).

Lemma 1.23. *Let P be a poset and let A be an antichain of P intersecting with every maximal chain of P; then,*

$$e(P) = \sum_{x \in A} e(P \setminus \{x\}).$$

Now we are going to study some inequalities involving e function. Given a set P, several partial orders could be defined. We say that $\preceq \subseteq \preceq'$, if $x \preceq y$ implies $x \preceq' y$ for all $x, y \in P$.

Lemma 1.24. *Let $(P, \preceq)$ and $(P, \preceq')$ be two posets such that $\preceq \subseteq \preceq'$. Then $e\left((P, \preceq')\right) \leq e\left((P, \preceq)\right)$. Indeed, $\mathcal{L}\left((P, \preceq')\right) \subseteq \mathcal{L}\left((P, \preceq)\right)$.*

Proof. Let $\epsilon \in \mathcal{L}\left((P, \preceq')\right)$ and let us see that $\epsilon \in \mathcal{L}\left((P, \preceq)\right)$. By definition, $\epsilon \in \mathcal{L}\left((P, \preceq)\right)$ if and only if $\epsilon(i) \prec \epsilon(j)$ implies $i < j$. Suppose that $\epsilon(i) \prec \epsilon(j)$. Since $\preceq \subseteq \preceq'$, $\epsilon(i) \prec' \epsilon(j)$. As $\epsilon \in \mathcal{L}\left((P, \preceq')\right)$, we get $i < j$. □

Example 1.6. *In the figure 1.12, the left-hand poset $(P, \preceq)$ has 2 linear extensions, $(1, 2, 3)$ and $(1, 3, 2)$. The right-hand poset $(P, \preceq')$ satisfies $\preceq \subseteq \preceq'$, therefore it should have at least 2 linear extensions. Indeed it has 3, $(1, 2, 3), (1, 3, 2)$ and $(3, 1, 2)$.*

FIGURE 1.12: $(P, \preceq)$ and $(P, \preceq')$.

Lemma 1.25. *Let $(P, \preceq_P)$ be a finite poset and $(Q, \preceq_Q)$ be a full subposet of P. Then $e(Q) \leq e(P)$.*

Proof. Assume w.l.g. that $Q = \{1, \ldots, q\}$ and let $\epsilon = (\epsilon(1), \epsilon(2), \ldots, \epsilon(q))$ be a linear extension of Q. We will extend this linear extension with the elements of $P \setminus Q$ to a linear extension of P. For this, it suffices to include $z_0 \in P \setminus Q$ and then repeat the process for $z_1 \in P \setminus \{Q \cup z_0\}$, and so on.

Thus, consider z_0 and let us define:

$$S_1 := \{x \in Q : x \prec z_0\} \quad S_2 := \{x \in Q : x \succ z_0\}.$$

We have to consider four different cases:

- If $S_1 = S_2 = \emptyset$, then z_0 is not related to the elements of Q, so it can be added at any position of ϵ.
- If $S_1 = \emptyset$, $S_2 \neq \emptyset$, then z_0 should be located before any element in S_2. For example, it could be placed in the first position.
- If $S_1 \neq \emptyset$, $S_2 = \emptyset$, then z_0 should be located after any element in S_1. For example, it could be placed in the last position.
- If $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, let $h_1 := \max\{j : \epsilon(j) \in S_1\}$ and $h_2 := \min\{j : \epsilon(j) \in S_2\}$. Note that $h_1 < h_2$; otherwise, we have $i_1 > i_2$ such that $\epsilon(i_1) \in S_1$ and $\epsilon(i_2) \in S_2$. Therefore, $\epsilon(i_1) \prec z_0 \prec \epsilon(i_2)$ and since ϵ is a linear extension we get $i_1 < i_2$, a contradiction. Then z_0 can be placed in any position between h_1 and h_2.

Thus, we have included z_0 in a way that we obtain a linear extension of the full subposet $Q \cup \{z_0\}$. Hence, we obtain a linear extension ϵ' of P from ϵ. As ϵ is contained in ϵ', different linear extensions of Q lead to different linear extensions of P. Hence, the result holds. □

Let us know focus on how e function behaves with respect to the operations between posets, see [5].

Lemma 1.26 (**Summation Lemma**). *Let P and Q be finite posets, then*

$$e(P \oplus Q) = e(P)e(Q).$$

Example 1.7. *Let $M_n = \mathbf{1} \oplus \overline{\mathbf{n}} \oplus \mathbf{1}$. By the Summation lemma $e(M_n) = e(\mathbf{1})e(\overline{\mathbf{n}})e(\mathbf{1}) = n!$. Similarly, $e(M_n \oplus M_m) = e(M_n)e(M_m) = n!m!$. If $n = 2$ and $m = 3$, see Figure 1.13, we get $e(M_2 \oplus M_3) = 2! \cdot 3! = 12$.*

Example 1.8. *Consider $P_n = \overline{\mathbf{2}} \oplus \overset{n}{\cdots} \oplus \overline{\mathbf{2}}$, see Figure 1.14. Then, $e(P_n) = e(\overline{\mathbf{2}} \oplus \overset{n}{\cdots} \oplus \overline{\mathbf{2}}) = e(\overline{\mathbf{2}})^n = 2^n$.*

Remark that if $e(P)$ is a prime number, then by the summation lemma P cannot be written as direct sum of posets different from chains.

The disjoint union has also an equivalent result [5].

Lemma 1.27 (**Disjoint Union Lemma**). *Let P_1 and P_2 be finite posets. Then:*

$$e(P_1 \uplus P_2) = \binom{|P_1| + |P_2|}{|P_1|} e(P_1)e(P_2).$$

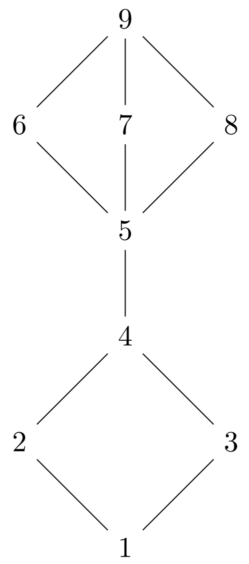


FIGURE 1.13: $P = M_2 \oplus M_3$.

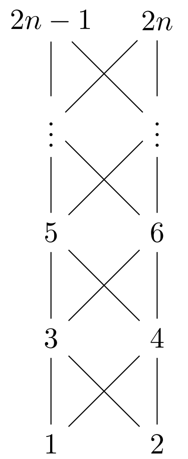


FIGURE 1.14: $P_n = \overline{\mathbf{2}} \oplus \overset{n}{\cdots} \oplus \overline{\mathbf{2}}$.

Corollary 1.28. *Let $P_1, P_2, \ldots, P_m$ be m posets. Then:*

$$e(P_1 \uplus P_2 \uplus \ldots \uplus P_m) = \binom{\sum_{j=1}^{m} |P_j|}{|P_1|, |P_2|, \cdots, |P_m|} \prod_{i=1}^{m} e(P_i),$$

where

$$\binom{\sum_{j=1}^{m} |P_j|}{|P_1|, |P_2|, \cdots, |P_m|} = \frac{\left(\sum_{j=1}^{m} |P_j|\right)!}{\prod_{j=1}^{m} |P_j|!}$$

is the multinomial coefficient.

Definition 1.29. A poset P is **series-parallel** if it can be constructed from singletons using the operations $\oplus$ and $\uplus$.

For example, $P = \mathbf{1} \oplus (\mathbf{1} \uplus \mathbf{1}) \oplus (\mathbf{1} \uplus \mathbf{1}) \oplus \mathbf{1}$ is series-parallel. By using the Summation and the Disjoint Union Lemmas we can compute easily the number of linear extensions of a series-parallel poset. Moreover, series-parallel posets are elegantly characterized.

Proposition 1.30. *[2] A finite poset P is series-parallel if and only if P has not the N poset as induced subposet.*

We move to another basic operation in the poset world, the cartesian product. Unlike direct sum and disjoint union, there are not known fromulas for computing the number of linear extensions of the product $P \times Q$. Instead of thinking about the general product $P \times Q$, we are going to briefly study $\mathbf{2} \times P$.

Definition 1.31. Let $n \in \mathbb{N}$ and $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a permutation of the set $\{1, 2, \ldots, n\}$. We define the torsion poset $\mathbf{T}^{\mathbf{n}}_{\sigma}$ as the resulting poset of adding to $\mathbf{n} \uplus \mathbf{n}$ the relations $i \preceq \sigma(i)$ $\forall i \in \{1, 2, \ldots, n\}$.

To draw the Hasse diagram of $\mathbf{T}^{\mathbf{n}}_{\sigma}$ we normally should remove some relations that hold by transitivity, see Figure 1.15. Note that if $\sigma = (n, n-1, \ldots, 2, 1)$ then $\mathbf{T}^{\mathbf{n}}_{\sigma} = \mathbf{2n}$.

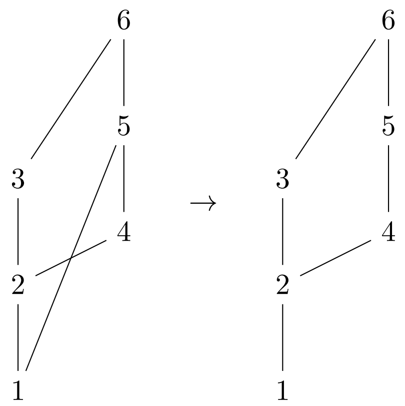


FIGURE 1.15: $\mathbf{T}^{\mathbf{3}}_{\sigma}$ with $\sigma = (2, 1, 3)$.

Now we can state the following result.

Lemma 1.32. *Let P be a finite poset with $|P| = n$. Then:*

$$e(\mathbf{2} \times P) = \sum_{\epsilon_1, \epsilon_2 \in \mathcal{L}(P)} e(\mathbf{T}^{\mathbf{n}}_{\sigma(\epsilon_1, \epsilon_2)}),$$

where $\sigma(\epsilon_1, \epsilon_2)$ is the permutation of $\{1, \ldots, |P|\}$ such that $\sigma(i)$ is the position that the element $\epsilon_1(i)$ takes up in the linear extension ϵ_2.

Proof. Note that every linear extension of $\mathbf{2} \times P$ can be written as a merge between two linear extensions of P that we will name ϵ_1 and ϵ_2. So the key fact here is how many ways of merging these linear extensions are. That is:

$$e(\mathbf{2} \times P) = \sum_{\epsilon_1, \epsilon_2 \in \mathcal{L}(P)} \gamma(\epsilon_1, \epsilon_2)$$

where $\gamma(\epsilon_1, \epsilon_2)$ are the number of ways of merging $\epsilon_1 \in \mathcal{L}(P)$ and $\epsilon_2 \in \mathcal{L}(P)$. The number of ways of merging $\epsilon_1 \in \mathcal{L}(P)$ and $\epsilon_2 \in \mathcal{L}(P)$ is the number of permutations of $2n$ numbers that respect the order of $\mathbf{n} \uplus \mathbf{n}$ and the relations between the elements of ϵ_1 and ϵ_2. This number equals $e(\mathbf{T}^{\mathbf{n}}_{\sigma(\epsilon_1, \epsilon_2)})$ by definition. $\square$

Lemma 1.32 gives us a bound for free.

Corollary 1.33. *Let P be a finite poset. Then,*

$$e(\mathbf{2} \times P) \geq e(P)^2.$$

Proof. Since $e(\mathbf{2} \times P) = \sum_{\epsilon_1, \epsilon_2 \in \mathcal{L}(P)} e(\mathbf{T}^{\mathbf{n}}_{\sigma(\epsilon_1, \epsilon_2)})$ has $e(P)^2$ summands which are all of them greater than 1, the result follows. $\square$

Observe that while the number of liner extensions of the direct sum and the disjoint union of two posets do not depend on the linear extensions of the posets and only depend on the *number* of linear extensions, the product by $\mathbf{2}$ does depend on the linear extensions of P. An interesting open problem is computing a closed formula for $e(\mathbf{T}^{\mathbf{n}}_{\sigma(\epsilon_1, \epsilon_2)})$. In spite of there is not a known closed formula for this value in general, we know the value of some specific cases. That is the case of $e(\mathbf{T}^{\mathbf{n}}_{\mathbf{id}})$ where $id = (1, 2, \ldots, n)$ is the identity function. We know that Catalan numbers count the number of expressions containing n pairs of parentheses which are correctly matched. For example if $n = 3$ some possibilities are ((())), ()()() or (())(). Catalan numbers are one of the most ubiquitous sequence of numbers in combinatorics, since they appear as a the solution of many combinatorial problems [8].

Proposition 1.34. *$e(\mathbf{T}^{\mathbf{n}}_{\mathbf{id}})$ is the n-th Catalan number C_n, i.e.*

$$e(\mathbf{T}^{\mathbf{n}}_{\mathbf{id}}) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Taking into account the shape of $\mathbf{T}^{\mathbf{n}}_{\mathbf{id}}$ (see Figure 1.16) we can consider each x_i as a left parentheses and each y_i as a right one. This way we have a bijection between linear extensions and parentheses matchings. Therefore, the number of linear extensions equals the n-th Catalan number. □

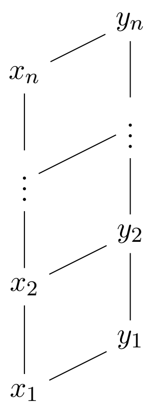


Figure 1.16: $\mathbf{T^n_{id}}$.

Example 1.9. *Let us see now an example with the boolean poset. Then,*

$$e(B_3) = e(\mathbf{2} \times B_2) = \sum_{\epsilon_1,\epsilon_2 \in \mathcal{L}(P)} e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_1,\epsilon_2)}).$$

Remember that B_2 has only 2 linear extensions $\epsilon_1 = (1,2,3,4)$ and $\epsilon_2 = (1,3,2,4)$. Then:

$$e(B_3) = e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_1,\epsilon_1)}) + e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_1,\epsilon_2)}) + e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_2,\epsilon_1)}) + e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_2,\epsilon_2)}).$$

By Proposition 1.34, $e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_1,\epsilon_1)}) = e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_2,\epsilon_2)}) = 14$. Besides, it can be seen that $e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_1,\epsilon_2)}) = e(\mathbf{T}^{\mathbf{4}}_{\sigma(\epsilon_2,\epsilon_1)}) = 10$, so that $e(B_3) = 14 + 14 + 10 + 10 = 48$.

Now we are going to study some links between linear extensions and ideals. Let P be a poset. Then, the number of linear extensions $e(P)$ equals the number of maximal chains of $\mathcal{I}(P)$. To see this, take a linear extension $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and define the function that maps ϵ to the maximal chain of $\mathcal{I}(P)$ given by $(\emptyset, \{\epsilon(1)\}, \{\epsilon(1), \epsilon(2)\}, \ldots, P)$. This function is a bijection and therefore,

$$e(P) = C(\mathcal{I}(P)). \tag{1.1}$$

However, counting maximal chains in $\mathcal{I}(P)$ is a difficult problem [9]. So most of times, this formula is not useful for practical purposes. The next result gives us a connection between linear extensions and filters.

Proposition 1.35. *Let P be a finite poset with width $w(P)$, then $e(P) \leq w(P)^{|P|-1}$.*

Proof. In order to construct a linear extension we can proceed as follows. We start selecting a minimal element. Since the width is $w(P)$, we have at most $w(P)$ choices for this element. Once we have selected this element, we repeat the same argument with the remaining elements. In the last step we have just one remaining element, so we select it. $\square$

The last bound is not an identity because there are a lot of steps in which we have less than $w(P)$ minimal elements to choose. In the next Figure 1.17 we can see the tree of choices for constructing a linear extension from the N poset.

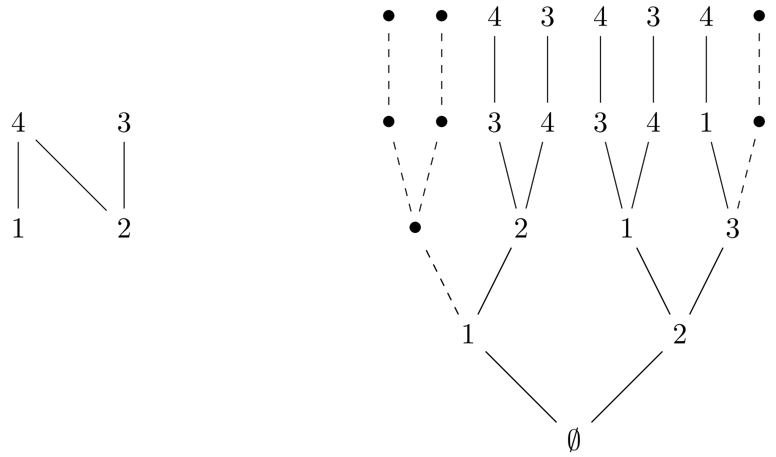


FIGURE 1.17: N poset and choices tree.

Theorem 1.36. *Let P be a finite poset with width $w(P)$, then:*

$$e(P) = w(P)^{|P|-1} - \sum_{i=1}^{w(P)-1} \sum_{F \in \mathcal{F}_i} (w(P) - i) w(P)^{|F|-2} e(P \setminus F),$$

where $\mathcal{F}_i$ is the set of non-singleton filters F such that $|\mathcal{MIN}(F)| = i$ and we consider $e(\emptyset) = 1$.

Proof. To compute $e(P)$ we should remove from $w(P)^{|P|-1}$ the number of non-possible paths (dashed lines). For each filter F different from a singleton (the singleton case correspond to the last step) we count the number of minimal elements of F, suppose that $F \in \mathcal{F}_i$. If this number i is different from $w(P)$ then we find $(w(P) - i)$ non-possible paths associated to this filter. This path branches $|F| - 2$ times so this filter forks into $(w(P) - i)w(P)^{|F|-2}$ non-possible paths. However, we should consider that in general there is not an only way to come to this filter, but $e(P \setminus F)$ ways of doing it. Putting it all together we get the result. $\square$

Corollary 1.37. *Let P be a finite poset with width $w(P) = 2$, then*

$$e(P) = 2^{|P|-1} - \sum_{F \in \mathcal{F}_1} 2^{|F|-2} e(P \setminus F),$$

where $\mathcal{F}_1$ is the set of non-singleton principal filters.

1.2 Convex Polytopes

In this section, we summarize the fundamentals of polyhedral combinatorics. The theory of polytopes is a vast area of study, with deep connections to pure (algebraic geometry, commutative algebra, representation theory) and applied mathematics (optimization, decision theory, cooperative game theory). For a more detailed treatment see [10, 11]. The **convex hull** of a finite set of points $S = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$, denoted by $\mathrm{Conv}(S)$, is the set of all convex combinations of its points, i.e.

$$\mathrm{Conv}(S) = \left\{ \sum_{i=1}^{|S|} \alpha_i x_i \mid \alpha_i \geq 0 \ \wedge \ \sum_{i=1}^{|S|} \alpha_i = 1 \right\}.$$

A **hyperplane** H in $\mathbb{R}^d$ is an affine subspace of dimension $d-1$; it is given by a linear equation $H = \{x \in \mathbb{R}^d \mid \alpha^T \cdot \boldsymbol{x} = c\}$ for some $\boldsymbol{\alpha} \in \mathbb{R}^d \setminus \{0\}$ and $c \in \mathbb{R}$. Similarly, a **halfspace** in $\mathbb{R}^d$ is a set of the form $\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{\alpha}^T \cdot \boldsymbol{x} \leq c\}$ for some vector $\boldsymbol{\alpha} \in \mathbb{R}^d \setminus \{0\}$ and $c \in \mathbb{R}$.

Definition 1.38. A **polyhedron** is the intersection of finitely many halfspaces, $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{A} \cdot \boldsymbol{x} \leq \boldsymbol{b}\}$. An **extreme point** or **vertex** of a polyhedron $\mathcal{P}$ is a point in $\mathcal{P}$ that cannot be expressed as a convex combination of other points in $\mathcal{P}$. A **polytope** is a bounded polyhedron.

We will use calligraphic letters $\mathcal{P}, \mathcal{Q}, \ldots$ for polyhedra and polytopes. There are two equivalent ways of defining convex polytopes: by using halfspaces or vertices.

The V-description: A convex polytope $\mathcal{P}$ is the convex hull of finitely many points $v_1, \dots, v_n$ in $\mathbb{R}^d$, $\mathcal{P} = \mathrm{Conv}(\{v_1, \dots, v_n\})$.
The H-description: A convex polytope $\mathcal{P}$ is a bounded intersection of finitely many halfspaces in $\mathbb{R}^d$.

These two descriptions are equivalent, as next theorem shows.

Theorem 1.39. *[10] A subset $\mathcal{P} \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points if and only if it is a bounded intersection of finitely many halfspaces.*

Remark 1.40. Some examples of simple polytopes are:

- The stardard simplex: $\Delta_{d-1} = \text{Conv}(\{e_1, \ldots, e_d\})$, where $e_1, \ldots, e_d$ is the canonical basis.
- The cube $\mathcal{C}_d = \text{Conv}(\{\pm e_1 \pm \cdots \pm e_d \text{ for any choice of signs}\})$.
- The crosspolytope $\mathcal{CS}_d = \text{Conv}(\{-e_1, e_1, \ldots, -e_d, e_d\})$.

Definition 1.41. Consider $d+1$ affinely independent points in $\mathbb{R}^m, m \geq n$, i.e. $d+1$ points $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ of $\mathbb{R}^m$ where the vectors $\boldsymbol{x}_1 - \boldsymbol{x}_0, \ldots, \boldsymbol{x}_d - \boldsymbol{x}_0$ are linearly independent. The convex hull of these points is called a ***d*-simplex**. When a face of a polytope is a simplex, it is called a **simplicial face**.

This notion is a generalization of the notion of triangle for the m-dimensional space. Note that all vertices of a simplex are adjacent to each other (although there are non-simplicial polytopes in which every two vertices are adjacent, see [10]).

The **dimension** of a polytope $\mathcal{P}$ is defined as the dimension of the smallest affine subspace containing its vertices $v_1, \ldots, v_k$. This affine subspace is denoted by $\text{aff}(\mathcal{P})$. Observe that:

$$\text{aff}(\mathcal{P}) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\ \forall i,\ \alpha_1 + \cdots + \alpha_k = 1\}.$$

For instance, $\dim(\Delta_{d-1}) = d-1$ since Δ_{d-1} is contained in the hyperplane $x_1 + \cdots + x_d = 1$ in $\mathbb{R}^d$.

Definition 1.42. Let $\mathcal{P}$ be a convex polytope and $\boldsymbol{x}$ be a non-collinear point, i.e $\boldsymbol{x} \notin \text{aff}(\mathcal{P})$. Point $\boldsymbol{x}$ is called **apex**. We define a **pyramid** with base $\mathcal{P}$ and apex $\boldsymbol{x}$, denoted by $\text{pyr}(\mathcal{P}, \boldsymbol{x})$, as the polytope whose vertices are the ones of $\mathcal{P}$ and $\boldsymbol{x}$. Observe that $\boldsymbol{x}$ is adjacent to every vertex in $\mathcal{P}$.

Remark that if we consider $\boldsymbol{y} \notin \text{aff}(\text{pyr}(\mathcal{P}, \boldsymbol{x}))$ then $\boldsymbol{y}$ is a possible apex for $\text{pyr}(\mathcal{P}, \boldsymbol{x})$, and we can define a new pyramid $\text{pyr}(\text{pyr}(\mathcal{P}, \boldsymbol{x}), \boldsymbol{y})$, denoted $\text{cpyr}(\mathcal{P}, \{\boldsymbol{x}, \boldsymbol{y}\})$. In general, we can iterate this process to define a **consecutive pyramid** with apexes $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$, denoted by $\text{cpyr}(\mathcal{P}, \mathcal{A})$.

Definition 1.43. A **face** of a polytope $\mathcal{P}$ is defined as a subset $\mathcal{F} \subseteq \mathcal{P}$ satisfying that there exists a vector $\boldsymbol{\alpha}$ and a constant $c \in \mathbb{R}$ such that

$$\boldsymbol{\alpha}^T \cdot \boldsymbol{x} \leq c,\ \forall \boldsymbol{x} \in \mathcal{P} \quad \text{and} \quad \mathcal{F} = \mathcal{P} \cap \{\boldsymbol{\alpha}^T \cdot \boldsymbol{x} = c\}.$$

We will denote the face defined via $\boldsymbol{\alpha}$ and c by $\mathcal{F}_{\boldsymbol{\alpha},c}$. Obviously, a face is also a polytope. The faces of dimension $0, 1, d-2, d-1$ are called **vertices, edges, ridges, and facets**, respectively. The empty set and $\mathcal{P}$ itself are considered faces and they are called improper faces.

Remark 1.44. [10] Some basics facts about faces of polytopes are:

- A polytope is the convex hull of its vertices.
- Note that if $\mathcal{C}$ is the set of vertices of a face $\mathcal{F}$, then $\mathcal{F} = \mathrm{Conv}(\mathcal{C})$. When this is the case, we will denote by $\mathcal{F}_{\mathcal{C}}$ the face defined by vertices in $\mathcal{C}$.
- A polytope is the intersection of the halfspaces determined by its facets.
- The vertices of a face are the vertices of $\mathcal{P}$ in $\mathcal{F}$.
- The intersection of two faces of $\mathcal{P}$ is a face of $\mathcal{P}$.
- If $\mathcal{F}$ is a face of $\mathcal{P}$, then any face of $\mathcal{F}$ is a face of $\mathcal{P}$.

There is a simple way to find faces containing $\boldsymbol{x}$ for a pyramid that we write below.

Proposition 1.45. *For a pyramid of apex $\boldsymbol{x}$ and base $\mathcal{P}$, the k-dimensional faces containing $\boldsymbol{x}$ are given by the $(k-1)$-dimensional faces of $\mathcal{P}$.*

The **interior** of a polytope is equal to its topological interior.

Let $\mathcal{P}$ be a polytope in $\mathbb{R}^d$ and suppose without loss of generality that $\mathbf{0} \in \mathrm{int}(P)$. The **polar polytope** $\mathcal{P}^\Delta$ of $\mathcal{P}$ is defined as $\mathcal{P}^\Delta = \left\{\boldsymbol{a} \in \mathbb{R}^d \mid \boldsymbol{a} \cdot \boldsymbol{x} \leq \mathbf{1},\ \forall \boldsymbol{x} \in \mathcal{P}\right\}$. It holds that $\mathcal{P} = (\mathcal{P}^\Delta)^\Delta$. For example, the cube $\mathcal{C}_d$ and the crosspolytope $\mathcal{CS}_d$ are duals to each other.

The **face lattice** $L(\mathcal{P})$ of a polytope $\mathcal{P}$ is the poset of faces of $\mathcal{P}$, ordered by inclusion. Moreover, this poset is a lattice by considering $\mathcal{F} \wedge \mathcal{G} = \mathcal{F} \cap \mathcal{G}$ and $\mathcal{F} \vee \mathcal{G}$ the intersection of all facets that contains both $\mathcal{F}$ and $\mathcal{G}$.

We say that a poset P is **graded** if it can be equipped with a rank function $\rho : P \to \mathbb{N}$ verifying that $x \lessdot_P y \Rightarrow \rho(y) = \rho(x) + 1$. The face lattice is graded with rank function $\rho(\mathcal{F}) = \dim(\mathcal{F}) + 1$.

Two polytopes $\mathcal{P}$ and $\mathcal{Q}$ are **combinatorially isomorphic (or combinatorially equivalent)** if $L(\mathcal{P}) \cong L(\mathcal{Q})$. We say that two polytopes $\mathcal{P}$ and $\mathcal{Q}$ are **affinely isomorphic** iff there is a square regular matrix A and some vector $\boldsymbol{v}$ such that $\mathcal{P} = A\mathcal{Q} + \boldsymbol{v}$. If two polytopes $\mathcal{P}$ and $\mathcal{Q}$ are affinely isomorphic then they are combinatorially isomorphic.

Remark 1.46. [12, 13] The face lattice of the polar polytope is the dual face lattice

$$L(\mathcal{P}^{\Delta}) \cong L(\mathcal{P})^{\partial}.$$

A d-polytope $\mathcal{P}$ is **simplicial** if every face is a simplex and **simple** if every vertex is on exactly d facets. Indeed, $\mathcal{P}$ is simplicial if and only if $\mathcal{P}^{\Delta}$ is simple.

We say that a polytope is **combinatorial** [14] if its vertices are $\{0,1\}$-valued and for each pair of non-adjacent vertices u, v there exist two other vertices w, h such that $u + v = w + h$.

Let us see now a simple but useful lemma.

Lemma 1.47. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex polytope and $u_1, u_2, v_1, v_2 \in \mathcal{P}$ be vertices of the polytope such that $u_1 + u_2 = v_1 + v_2$. Then, for any face $\mathcal{F} \subseteq \mathcal{P}$, $u_1, u_2 \in \mathcal{F} \Leftrightarrow v_1, v_2 \in \mathcal{F}$.*

Proof. Consider $\mathcal{F}_{\boldsymbol{\alpha},c}$ the face defined by the vector $\boldsymbol{\alpha}$ and $c \in \mathbb{R}$. By definition of face $\boldsymbol{\alpha} \cdot x \leq c$, $\forall x \in \mathcal{P}$ and $\boldsymbol{\alpha} \cdot x = c$, $\forall x \in \mathcal{F}_{\boldsymbol{\alpha},c}$.

$\Rightarrow$) Since $u_1, u_2 \in \mathcal{F}_{\boldsymbol{\alpha},c}$, we get

$$\left.\begin{array}{l} \boldsymbol{\alpha} \cdot u_1 = c \\ \boldsymbol{\alpha} \cdot u_2 = c \end{array}\right\} \Rightarrow \boldsymbol{\alpha} \cdot (v_1 + v_2) = \boldsymbol{\alpha} \cdot (u_1 + u_2) = 2c.$$

If $v_1 \notin \mathcal{F}_{\boldsymbol{\alpha},c}$, then, $\boldsymbol{\alpha} \cdot v_1 < c$, and thus $\boldsymbol{\alpha} \cdot v_2 > c$, implying $v_2 \notin \mathcal{P}$, a contradiction. Thus, $v_1 \in \mathcal{F}_{\boldsymbol{\alpha},c}$, so that $\boldsymbol{\alpha} \cdot v_1 = c$, and hence $\boldsymbol{\alpha} \cdot v_2 = c$, thus concluding $v_2 \in \mathcal{F}_{\boldsymbol{\alpha},c}$.

$\Leftarrow$) Completely symmetric to the previous case. ☐

Definition 1.48. The **f-vector** of a d-polytope $\mathcal{P}$ is

$$f_{\mathcal{P}} = (f_0, f_1, \ldots, f_{d-1}, f_d),$$

where f_k is the number of $(k-1)$-dimensional faces and $f_0 = f_d = 1$.

There are few general results concerning f-vectors, one of the most important ones is the McMullen's Upper Bound Theorem [10] that states that for any polytope $\mathcal{P}$ of dimension d with m vertices, $f_i(\mathcal{P}) \leq f_i(\mathcal{C}_d(m))$ for $i \in \{0, 1, \ldots, d-1\}$ where $\mathcal{C}_d(m)$ is the cyclic polytope, that is defined by $\mathcal{C}_d(m) = \mathrm{Conv}(\{(1, t_i, t_i^2, \ldots, t_i^{d-1}) \mid 1 \leq i \leq m\})$ for some $t_1 < \cdots < t_m$.

1.2.1 Triangulations

Triangulations are one of the most important concepts in combinatorics and discrete geometry, playing also an important role in optimization [12, 15]. Triangulations provide a lot of information about the geometry of a polytope and help us to solve applied problems as sampling points inside a specific polytope. Consider a finite set of labeled points in the real affine space $\mathbb{R}^m$, $\mathcal{A} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.

Definition 1.49. An **abstract simplicial complex** Δ on the vertex set $\mathcal{A}$ is a collection of subsets of $\mathcal{A}$ such that

- $\{v\} \in \Delta$ for all $v \in \mathcal{A}$.
- If $G \in \Delta$ and $F \subseteq G$ then $F \in \Delta$.

The elements of Δ are called **faces**. A face F has dimension d and write $\dim(F) = d$ if $d = |F| - 1$.

Example 1.10. *Consider a triangle ABC. The collection of faces is given by the empty face, vertices, edges, and the triangle itself:*

$$\Delta = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}.$$

Note that Δ is an abstract simplicial complex on the vertex set $\{A, B, C\}$.

Definition 1.50. A **geometric simplicial complex** K in $\mathbb{R}^m$ is a nonempty collection of simplices of different dimensions in $\mathbb{R}^m$ such that

- Every face of a simplex in K is in K.
- The intersection of any two simplices of K is a (possibly empty) common face.

If we assign a simplex to each subset of an abstract simplicial complex we get a geometric simplicial complex. This assignment is done by linking a d-dimensional simplex S_F to each d-dimensional face $F \in \Delta$ in a way that if $F \subseteq G$ then $S_F \subseteq S_G$, $\forall F, G \in \Delta$.

From any geometric simplicial complex K we can get an abstract simplicial complex $\Delta(K)$ by letting the faces of $\Delta(K)$ be the set of vertices of the simplices of K (see again Example 1.10). It can be shown that every abstract simplicial complex Δ can be obtained in this way [15]. In fact, there are many ways of choosing K such that $\Delta(K) = \Delta$. Although an abstract simplicial complex K has associated many geometric simplicial complexes, all of them are unique up to homeomorphism [5, 16]. We refer to this unique topological space as the **geometric realization** of Δ and denote it by $\|\Delta\|$. Now we move to the concept of triangulation. Let us consider a polytope whose vertices lay in a finite set $\mathcal{A}$.

Definition 1.51. A **triangulation** of a set of vertices $\mathcal{A}$ is a collection Δ of simplices, whose vertices lay in $\mathcal{A}$, satisfying the following properties:

i) Clausure Property: All faces of simplices of Δ are in Δ.

ii) Intersection Property: If $\mathcal{F}, \mathcal{F}' \in \Delta \Rightarrow \mathcal{F} \cap \mathcal{F}'$ is a (possibly empty) common face of $\mathcal{F}$ and $\mathcal{F}'$.

iii) Union Property: The union of all these simplices equals $\mathrm{Conv}(\mathcal{A})$.

In other words, a triangulation is just a simplicial complex whose set of vertices is contained in $\mathcal{A}$ and covering $\mathrm{Conv}(\mathcal{A})$. Observe that a triangulation Δ is completely determined by its highest dimensional simplices. Thus, sometimes we just write the highest dimensional simplices to determine the triangulation Δ. A triangulation where we allow the simplices to be general polytopes is called a **subdivision**.

Definition 1.52. A triangulation Δ of $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ is **regular or coherent** if there is a weight vector $\boldsymbol{w} \in \mathcal{R}^n$ for which the following condition holds: a subset $\{i_1, \ldots, i_k\}$ is a face of Δ if and only if there exists a vector $\boldsymbol{c} \in \mathcal{R}^m$ with $\boldsymbol{a}_j \cdot \boldsymbol{c} = w_j$ for $j \in \{i_1, \ldots, i_k\}$, and $\boldsymbol{a}_j \cdot \boldsymbol{c} < w_j$ otherwise. In this case, we denote by $\Delta_{\boldsymbol{w}}$ the triangulation Δ.

The regular triangulation $\Delta_{\boldsymbol{w}}$, can also be constructed geometrically:

i) Using the coordinates of $\boldsymbol{w}$ as "heights", we lift the vertex set $\mathcal{A}$ into the next dimension. The result is the vertex set $\widehat{\mathcal{A}} = \{(a_1, w_1), \ldots, (a_n, w_n)\} \subset \mathbb{R}^{m+1}$.

ii) The "lower faces" of the cone $pos(\widehat{\mathcal{A}}) := \{\lambda_1 \boldsymbol{u}_1 + \cdots \lambda_n \boldsymbol{u}_n : \lambda_1, \ldots, \lambda_n \in \mathbb{R}^+\}$ form a m-dimensional polyhedral complex. A face is "lower" if it has a normal vector with negative last coordinate. The triangulation $\Delta_{\boldsymbol{w}}$ is the image of this complex under projection onto the first m coordinates. Figure 1.18 illustrates the process for a hexagon.

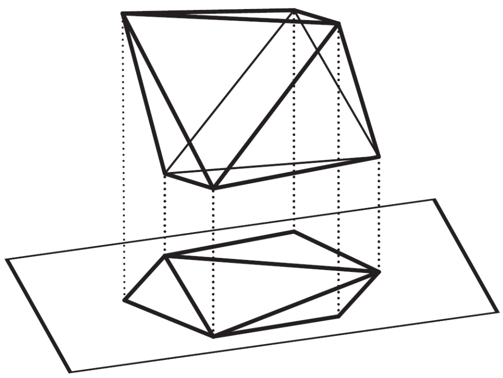


FIGURE 1.18: A regular triangulation of a hexagon.

Remark 1.53.

- Every vertex set has a regular triangulation. Therefore, every polytope has a regular triangulation, see [15] for a proof.
- Obviously, different choices of lifting may produce different triangulations. When we take heights $w(i) = \|\boldsymbol{v}_i\|^2$ for each $\boldsymbol{v}_i \in \mathcal{A}$ the obtained triangulation is called **Delaunay triangulation.**
- Consider the two concentric triangles of Figure 1.19. It can be shown that the triangulation appearing in Figure 1.19 is not produced by any choice of heights and therefore it is not regular [15]. This shows that there are non-regular triangulations.

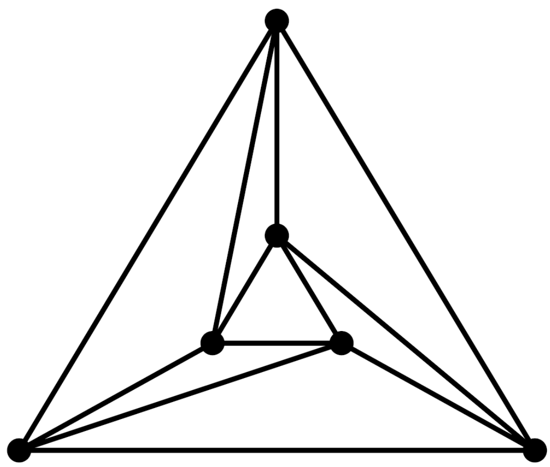


FIGURE 1.19: A non-regular triangulation.

- To every poset P we can associate an abstract simplicial complex $\Delta(P)$ called the **order complex of** P, defined as follows [5]: The vertices of $\Delta(P)$ are the elements of P and the faces are the chains.
- From an abstract simplicial complex Δ we can also construct a poset, called the **face poset** $P(\Delta)$ which is the poset of nonempty faces ordered by inclusion.

- The order complex $\Delta(P(\Delta))$ is known as **barycentric subdivision** of Δ (see [16]). The geometric realizations of these complexes are always homeomorphic, i.e.

$$\|\Delta\| \cong \|\Delta(P(\Delta))\|.$$

Triangulations arise when dealing with the problem of random generating points in a polytope. Generating points in a polytope is a complex problem and several methods, not completely satisfactory, have been presented to cope with this problem [17, 18]. Among them, we have the **triangulation methods** [17]. The triangulation method takes advantage of the fact that random generation in simplices is very simple and fast as we will see below. The triangulation method is based on the decomposition of the polytope into simplices such that any pair of simplices intersects in a (possibly empty) common face. This matchs with the definition of triangulation, but in this case we are only interested in the highest dimensional simplices. Once the triangulation is obtained, we assign to each highest dimensional simplex a probability proportional to its volume; next, these probabilities are used for selecting one of the simplices; finally, a random m-uple in the simplex is generated. The main drawback of this method is that in general it is not easy to split a polytope into simplices. Moreover, even if we are able to decompose the polytope in a suitable way, we have to deal with the problem of determining the volume of each simplex in order to randomly select one of them. Computing the volume of a polytope is a complex problem and only partial results are known. However, in the case of simplices, the volume is given in next result.

Lemma 1.54. *[19] Let Δ be a k-dimensional simplex in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_{k+1}$. Then, the k-dimensional volume of Δ is:*

$$Vol_k(\Delta) = \sqrt{\frac{(-1)^{k+1}}{2^k(k!)^2}\det(CM_\Delta)},$$

where $\det(CM_\Delta)$ is the Cayley-Menger determinant

$$\det(CM_\Delta) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & d_{1,2}^2 & \cdots & d_{1,k}^2 & d_{1,k+1}^2 \\ 1 & d_{2,1}^2 & 0 & \cdots & d_{2,k}^2 & d_{2,k+1}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & d_{k+1,1}^2 & d_{k+1,2}^2 & \cdots & d_{k+1,k}^2 & 0 \end{vmatrix}$$

being $d_{i,j}^2$ the square of the distance between v_i and v_j.

Now we are going to explain how to sample uniformly in a simplex [20]. We start studying the special case of the n-dimensional simplex

$$\mathcal{H}_n := \{\boldsymbol{x} \in \mathbb{R}^n \ : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}.$$

Observe that $\mathcal{H}_n$ has as vertices:

$$(0,0,\ldots,0,0),(0,0,\ldots,0,1),(0,0,\ldots,1,1),\ldots,(0,1,\ldots,1,1) \text{ and } (1,1,\ldots,1,1).$$

For generating a uniformly distributed vector in $\mathcal{H}_n$, we generate an independent and identically distributed sample $\widehat{U}_1 \ldots, \widehat{U}_n$ with uniform distribution $U(0,1)$. Then sort the $\widehat{U}_i$ to give the order statistics with natural order $U_1 \leq U_2 \leq \ldots \leq U_n$. This generates a uniformly distributed vector U in $\mathcal{H}_n$.

Now let us call $\mathcal{S}$ a n-dimensional general simplex. To sample uniformly in $\mathcal{S}$ we consider an affine transformation from $\mathcal{H}_n$ to $\mathcal{S}$, $\boldsymbol{S} = \boldsymbol{A} \cdot \boldsymbol{U} + \boldsymbol{V}_0$ which maps $\mathcal{H}_n$ into the desired simplex $\mathcal{S}$. Suppose that $\mathcal{S} = \mathrm{Conv}(\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. Then, $\boldsymbol{V}_0 = \boldsymbol{v}_0$ and $\boldsymbol{A}$ is the matrix satisfying $S' = \boldsymbol{A} \cdot \boldsymbol{U}'$, where S' is the matrix with column vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \boldsymbol{v}_2 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_n - \boldsymbol{v}_0$ and U' is the matrix with the vertices of $\mathcal{H}_n$ different from the zero vertex. Note that if $h(\boldsymbol{u})$ is the density function of $\boldsymbol{U}$, then the density $g(\boldsymbol{s})$ of $\boldsymbol{S}$ would be

$$g(\boldsymbol{s}) = h(\boldsymbol{u}) |\det(\boldsymbol{A})|^{-1}.$$

Consequently, if $h(\boldsymbol{u})$ is uniform in $\mathcal{H}_n$, then $g(\boldsymbol{s})$ is uniform in $\mathcal{S}$, because $|\det(\boldsymbol{A})|$ is a constant value.

1.2.2 Cones

A **cone** is a non-empty subset $\mathcal{C}$ of $\mathbb{R}^n$ such that if $\boldsymbol{x} \in \mathcal{C}$, then $\alpha \boldsymbol{x} \in \mathcal{C}$ for all $\alpha \geq 0$. Note that $\mathbf{0}$ is in any cone. Additionally, we say that the cone is **convex** if it is a convex set of $\mathbb{R}^n$; equivalently, a cone is convex if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$, it follows

$$\boldsymbol{x} + \boldsymbol{y} \in \mathcal{C}.$$

Given a set $\mathcal{S}$, we define its **conic hull (or conic extension)** as the smallest cone containing $\mathcal{S}$.

A convex cone $\mathcal{C}$ is **polyhedral** if additionally it is a polyhedron. This means that it can be written as

$$\mathcal{C} := \{\boldsymbol{x} : A\boldsymbol{x} \leq \mathbf{0}\}, \tag{1.2}$$

for some matrix $A \in \mathcal{M}_{m\times n}$ of binding conditions. Two polyhedral cones are **affinely isomorphic** if there is a bijective affine map from one cone onto the other. Given a polyhedral cone $\mathcal{C}$ and $\boldsymbol{x} \in \mathcal{C}, \boldsymbol{x} \neq \mathbf{0}$, the set $\{\alpha\boldsymbol{x} : \alpha \geq 0\}$ is called a **ray**. In general we will identify a ray with the point $\boldsymbol{x}$. Notice also that for polyhedral cones, all rays pass through $\mathbf{0}$. Point $\boldsymbol{x}$ defines an **extremal ray** if $\boldsymbol{x} \in \mathcal{C}$ and there are $n-1$ binding conditions for $\boldsymbol{x}$ that are linearly independent. Equivalently, $\boldsymbol{x}$ cannot be written as a convex combination of two linearly independent points of $\mathcal{C}$.

It is well-known that a convex polyhedron only has a finite set of vertices and a finite set of extremal rays. The following result is well-known for convex polyhedra:

Theorem 1.55. *Let $\mathcal{P}$ be a convex polyhedron on $\mathbb{R}^n$. Let us denote by $\boldsymbol{x}_1, ..., \boldsymbol{x}_r$ the vertices of P and by $\boldsymbol{v}_1, ..., \boldsymbol{v}_s$ the vectors defining extremal rays. Then, for any $\boldsymbol{x} \in P$, there exists $\alpha_1, ..., \alpha_r$ such that $\alpha_1 + ... + \alpha_r = 1, \alpha_i \geq 0, i = 1, ..., r,$ and $\beta_1, ..., \beta_s$ such that $\beta_i \geq 0, i = 1, ..., s,$ satisfying that*

$$\boldsymbol{x} = \sum_{i=1}^{r} \alpha_i \boldsymbol{x}_i + \sum_{j=1}^{s} \beta_j \boldsymbol{v}_j.$$

Given a polyhedral cone, if $\boldsymbol{x} \in \mathcal{C}, \boldsymbol{x} \neq \mathbf{0}$, it follows that $\boldsymbol{x}$ cannot be a vertex of $\mathcal{C}$. Thus, for a polyhedral cone, the only possible vertex is $\mathbf{0}$. Thus, for the particular case of polyhedral cones, Theorem 1.55 writes as follows.

Corollary 1.56. *For a polyhedral cone $\mathcal{C}$ whose extremal rays are defined by $\boldsymbol{v}_1, ..., \boldsymbol{v}_s$, any $\boldsymbol{x} \in \mathcal{C}$ can be written as*

$$\boldsymbol{x} = \sum_{j=1}^{s} \beta_j \boldsymbol{v}_j, \quad \beta_j \geq 0, j = 1, ..., s.$$

Consequently, in order to determine the polyhedral cone it suffices to obtain the extremal rays.

We will say that a cone is **pointed** if $\mathbf{0}$ is a vertex. The following result characterizes pointed cones.

Theorem 1.57. *For a polyhedral cone $\mathcal{C}$ the following statements are equivalent:*

- *$\mathcal{C}$ is pointed.*

- *$\mathcal{C}$ contains no line.*
- $\mathcal{C} \cap (-\mathcal{C}) = \mathbf{0}$.

For polyhedral cones, the definitions of face and dimension are practically the same as those given for the case of polytopes. Indeed, given a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, a non-empty subset $\mathcal{F} \subseteq \mathcal{P}$ is a **face** if there exist $\boldsymbol{v} \in \mathbb{R}^n, c \in \mathbb{R}$ such that

$$\boldsymbol{v}^t \boldsymbol{x} \leq c, \forall \boldsymbol{x} \in \mathcal{P}, \quad \boldsymbol{v}^t \boldsymbol{x} = c, \forall \boldsymbol{x} \in \mathcal{F}.$$

We denote this face as $\mathcal{F}_{\boldsymbol{v},c}$. The **dimension** of a face is the dimension of the smallest affine space containing the face. A common way to obtain faces is turning into equalities some of the inequalities of (1.2) defining $\mathcal{P}$.

Theorem 1.58. *[21] Let $A \in \mathcal{M}_{m\times n}$. Then any non-empty face of $\mathcal{P} = \{\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}\}$ corresponds to the set of solutions to*

$$\begin{array}{rcl} \sum_j a_{ij} x_j & = & b_i \textit{ for all } i \in I \\ \sum_j a_{ij} x_j & \leq & b_i \textit{ for all } i \notin I, \end{array}$$

for some set $I \subseteq \{1, \ldots, m\}$.

As in the case of polytopes, the set of faces with the inclusion relation determines a lattice known as the **face lattice** of the polyhedron. In Chapter 6, we will study a specific type of cones, called order cones.

1.3 Order Polytopes

In this section we study a family of polytopes which models many relevant families of fuzzy measures. We recall the most important aspects of the geometry of these polytopes and also enhance a bit this study with some own results.

Definition 1.59. Given a poset $(P, \preceq)$ with n elements, it is possible to associate to P, in a natural way, a polytope, denoted by $\mathcal{O}(P)$, in $\mathbb{R}^n$, called the **order polytope** of P, see [22–24]. The polytope $\mathcal{O}(P)$ is formed by the n-tuples f of real numbers indexed by the elements of P satisfying *i)* $0 \leq f(x) \leq 1$ for every $x \in P$,

ii) $f(x) \leq f(y)$ whenever $x \leq y$ in P.

Thus, the polytope $\mathcal{O}(P)$ consists in the order-preserving functions from P to $[0, 1]$. The following facts are discussed in [22]. It is a well-known fact that $\mathcal{O}(P)$ is a 0/1-polytope, i.e., its extreme points are all in $\{0, 1\}^n$. Indeed, it is easy to prove that the vertices of $\mathcal{O}(P)$ are the characteristic functions v_J of filters J of P.

$$v_J(x) := \begin{cases} 1 & \text{if } x \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\mathcal{O}(P) = \mathrm{Conv}\,(v_J : J \subseteq P \text{ filter})$ and moreover these points are in convex position, i.e., $v_J \notin \mathrm{Conv}\,(v_{J'} : J \neq J' \subseteq P \text{ filter})$.

For example, the order polytope of an antichain with n elements is the n-dimensional unit cube and the order polytope of a chain with n elements is a n-dimensional simplex.

In particular, the number of vertices of $\mathcal{O}(P)$ is the number of filters of P. Any facet is obtained by adding some of the next inequalities which define the polytope:

$f(x) = 0$, for some minimal $x \in P$,

$f(x) = 1$, for some maximal $x \in P$,

$f(x) = f(y)$, for some y covering x in P.

The number of facets of $\mathcal{O}(P)$ is $\mathrm{cov}(P) + m(P) + M(P)$, where $\mathrm{cov}(P)$ is the number of cover relations in P. Also, $\mathcal{O}(P)$ is always a full-dimensional polytope, that is, $\dim(\mathcal{O}(P)) = |P|$.

There are several ways to understand the combinatorial structure of an order polytope. Stanley [22] gave the first combinatorial characterization of the faces of $\mathcal{O}(P)$. Let $\widehat{P} = \widehat{0} \oplus P \oplus \widehat{1}$ be the poset P with a maximum $\widehat{1}$ and a minimum $\widehat{0}$ added. For any face $\mathcal{F}$ of $\mathcal{O}(P)$, we can define the next equivalence relation on $\widehat{P}$:

$$x \sim y \text{ if and only if } f(x) = f(y) \text{ for all } f \in \mathcal{F},$$

where we set $f(\widehat{0}) := 0$ and $f(\widehat{1}) := 1$ $\forall f \in \mathcal{F}$. Denote by $\mathcal{B}(\mathcal{F}) = \{B_1, \cdots, B_m\}$ the set of equivalence classes, which form a partition of $\widehat{P}$. Such a partition is called a **face**

partition. We will denote by $B_{\widehat{0}}$ and $B_{\widehat{1}}$ the blocks containing the elements $\widehat{0}$ and $\widehat{1}$ respectively. Since any face could be obtained by the intersection of facets, the distinct faces yield distint face partitions.

Similarly, for a collection $\mathcal{C}$ of vertices of $\mathcal{O}(P)$, we can define the following equivalence relation on $\widehat{P}: x \sim y \Leftrightarrow f(x) = f(y),\ \forall f \in \text{Conv}(\mathcal{C})$. This determines a partition $\mathcal{B}(\mathcal{C})$ on $\widehat{P}$ which is called the **block partition** of $\widehat{P}$ associated to $\mathcal{C}$. It can be proved that $\mathcal{B}(\mathcal{F}) = \mathcal{B}(\mathcal{V}(\mathcal{F}))$, where $\mathcal{V}(\mathcal{F})$ is the collection of vertices defining $\mathcal{F}$, i.e. $\mathcal{F} = \text{Conv}\left(\mathcal{V}(\mathcal{F})\right)$.

A partition $\mathcal{B} = \{B_1, \cdots, B_m\}$ of $\widehat{P}$ is called **connected** if the blocks B_i, $i \in \{1, 2, \cdots, m\}$ are connected as induced subposets of $\widehat{P}$. A partition $\mathcal{B} = \{B_1, \cdots, B_m\}$ of $\widehat{P}$ is called **compatible** if the binary relation $\preceq_{\mathcal{B}}$ on $\mathcal{B}$ defined by $B_i \preceq B_j$ if $x \preceq y$ for some $x \in B_i$ and $y \in B_j$ is a partial order.

A partition $\mathcal{B} = \{B_1, \cdots, B_m\}$ associated to a collection of vertices $\mathcal{C}$ is said to be **closed** if for any $i \neq j$ there exists $g \in \text{Conv}(\mathcal{C})$ such that $g(B_i) \neq g(B_j)$. Every partition $\mathcal{B}$ has a unique closed coarsening $\mathcal{B}'$ such that $\mathcal{B}'(\mathcal{C}) = \mathcal{B}(\mathcal{C})$.

Theorem 1.60. *[22] Let P be a finite poset. A partition $\mathcal{B}$ of $\widehat{P}$ is a closed face partition if and only if it is connected and compatible.*

However, this is not the only way of characterizing the faces of $\mathcal{O}(P)$. There is another characterization which is less known but is more useful for obtaining faces of small dimension. For a collection $L \subseteq \mathfrak{F}(P)$ of filters we shall write $\mathcal{F}(L) := \text{Conv}(v_J : J \in L)$. An induced subposet $L \subseteq \mathfrak{F}(P)$ is said to be an **embedded sublattice** if for any two filters $J, J' \in \mathfrak{F}(P)$

$$J \cup J', J \cap J' \in L \Leftrightarrow J, J' \in L.$$

Theorem 1.61. *[25] Let P be a finite poset and $L \subseteq \mathfrak{F}(P)$ a collection of filters. Then, $\mathcal{F}(L)$ is a face of $\mathcal{O}(P)$ if and only if L is an embedded sublattice. In this case, $\dim\left(\mathcal{F}(\mathcal{L})\right) = h(L) - 1$, where $h(L)$ is the height of the subposet L.*

These two points of view are equivalent to each other. Indeed, we can provide a more compact version of the last two characterizations. We say that a collection $\mathcal{C}$ is **$\mathcal{B}$-maximal** if there is no collection of vertices $\mathcal{C}' \supsetneq \mathcal{C}$ such that $\mathcal{B}(\mathcal{C}') = \mathcal{B}(\mathcal{C})$.

Theorem 1.62 (Combinatorial structure of Order Polytopes). *[22, 26, 27] Let P be a finite poset with $|P| = n$ and $\mathcal{C}$ a collection of vertices of $\mathcal{O}(P)$ with associated collection of filters $L \subseteq \mathfrak{F}(P)$. Let also $\mathcal{B}(\mathcal{C}) = \{B_{\widehat{0}}, B_{\widehat{1}}, B_1, \cdots, B_r\}$ be the block partition of $\widehat{P}$ associated to $\mathcal{C}$. Then the following are equivalent:*

i) *$Conv(\mathcal{C})$ is a k-dimensional face of $\mathcal{O}(P)$.*

ii) *L is an embedded sublattice and $h(L) - 1 = k$.*

iii) *$\mathcal{B}(\mathcal{C})$ is a $\mathcal{B}$-maximal, connected and compatible partition with $|\mathcal{B}(\mathcal{C})| - 2 = k$.*

It can be proved that faces of order polytopes are again order polytopes (see [22]).

Theorem 1.63. *Let P be a finite poset and $\mathcal{F}$ be a face of $\mathcal{O}(P)$ with associated face partition $\mathcal{B}(\mathcal{F}) = \{B_{\widehat{0}}, B_{\widehat{1}}, B_1, \cdots, B_r\}$. Then $(\mathcal{B}(\mathcal{F}), \preceq_{\mathcal{B}(\mathcal{F})})$ is a poset and $\mathcal{F}$ is affinely equivalent to the order polytope $\mathcal{O}\left((\mathcal{B}(\mathcal{F}) \setminus \{B_{\widehat{0}}, B_{\widehat{1}}\}, \preceq_{\mathcal{B}(\mathcal{F})})\right)$. The dimension of this face is $|\mathcal{B}(\mathcal{F})| - 2$.*

Now we are going to anwser the question of when a collection of vertices are in the same k-dimensional face.

Theorem 1.64. *Let P be a finite poset with $|P| = n$ and $\mathcal{C}$ be a collection of vertices of $\mathcal{O}(P)$ with associated collection of filters $L \subseteq \mathfrak{F}(P)$. Let also be $\mathcal{B}(\mathcal{C}) = \{B_{\widehat{0}}, B_{\widehat{1}}, B_1, \cdots, B_r\}$ the block partition of $\widehat{P}$ associated to $\mathcal{C}$. Then the vertices in $\mathcal{C}$ are in the same k-dimensional face but not in the same $(k-1)$-dimensional face if and only if*

$$\sum_{i=1}^{r} \Phi(B_i) = k,$$

where $\Phi(Q)$ counts the number of connected components of a poset Q and $k \in [n]$.

Proof. Consider $\boldsymbol{x} \in \mathcal{O}(P)$ and $A \subseteq P$. We define:

$$\begin{aligned} X_A &: \mathcal{O}(P) \to \mathbb{R} \\ &\quad \boldsymbol{x} \to X_A(\boldsymbol{x}) = \sum_{i \in A} x_i \\ \overline{\chi}_A &: \mathcal{O}(P) \to \mathbb{R} \\ &\quad \boldsymbol{x} \to \overline{\chi}_A(\boldsymbol{x}) = \sum_{i \preceq_P j} (x_j - x_i) \end{aligned} \tag{1.3}$$

Let us show first that all vertices in $\mathcal{C}$ are in a k-dimensional face with k as defined in the theorem. Consider the halfspace

$$X_{\bigcap_{J \in L} J}(\boldsymbol{x}) \leq |\bigcap_{J \in L} J| + X_{\left(\bigcup_{J \in L} J\right)^c}(\boldsymbol{x}) + \sum_{i=1}^{r} \overline{\chi}_{B_i}(\boldsymbol{x}).$$

Since any vertex is 0/1-valued , for any $\boldsymbol{x} \in \mathcal{O}(P)$, we have

$$X_{\bigcap_{J \in L} J}(\boldsymbol{x}) = \sum_{i \in \bigcap\limits_{J \in L} J} x_i \leq |\bigcap_{J \in L} J|.$$

Moreover, $X_{(\bigcup_{J\in L} J)^c}(\boldsymbol{x}) \geq 0$ and $\overline{\chi}_{B_i}(\boldsymbol{x}) \geq 0$ because $\boldsymbol{x} \in \mathcal{O}(P)$. Consequently, $\mathcal{O}(P)$ is in the halfspace.

Consider now a point $\boldsymbol{x} \in \mathcal{C}$. As $\boldsymbol{x}$ is a vertex in $\mathcal{C}$, it is associated to a filter $J \in L$. Thus, $\boldsymbol{x} = \boldsymbol{v}_J$ and then, $x_i = 1$ if $i \in \bigcap_{J\in L} J$. Hence,

$$X_{\bigcap_{J \in L} J}(\boldsymbol{x}) = |\bigcap_{J \in L} J|.$$

Besides, $x_i = 0$ if $i \notin \bigcup_{J \in L} J$, so that $X_{(\bigcup_{J\in L} J)^c}(\boldsymbol{x}) = 0$.

Finally, for every $B_k \in \mathcal{B}(\mathcal{C})$, it is $x_i = x_j \ \forall i, j \in B_k$. Then,

$$\overline{\chi}_{B_i}(\boldsymbol{x}) = 0, \quad \forall B_i \in \mathcal{B}(\mathcal{C}).$$

We conclude that $\boldsymbol{x} \in \mathcal{C}$ satisfies the equality.

Let us denote by $\mathcal{F}$ the face that is defined via the previous halfspace and let us study its dimension. For this, we cannot apply Theorem 1.63 because it could happen that other vertices outside $\mathcal{C}$ are in $\mathcal{F}$. Such a vertex $\boldsymbol{y}$ would satisfy:

$$X_{\bigcap_{J \in L} J}(\boldsymbol{y}) = |\bigcap_{J \in L} J|; \quad X_{(\bigcup_{J\in L} J)^c}(\boldsymbol{y}) = 0; \quad \overline{\chi}_{B_i}(\boldsymbol{y}) = 0.$$

Take B_i and assume that $B_i = \biguplus_{h=1}^{h(i)} C_h^i$, i.e. B_i can be decomposed into the connected components $C_1^i, \ldots, C_{h(i)}^i$. Since

$$0 = \overline{\chi}_{B_i}(\boldsymbol{y}) = \sum_{h=1}^{h(i)} \overline{\chi}_{C_h^i}(\boldsymbol{y}),$$

we conclude that $\overline{\chi}_{C_h^i}(\boldsymbol{y}) = 0$. Thus, $y_j = y_k$, $\forall j, k \in C_h^i$ and the block partition corresponding to $\mathcal{F}$ is

$$\{B_{\widehat{0}}, B_{\widehat{1}}, C_h^i : i = 1, \ldots, r, \quad h = 1, \ldots, h(i)\}.$$

Applying Theorem 1.63 we conclude that

$$k = \dim(\mathcal{F}) = \sum_{i=1}^{r}\sum_{h=1}^{h(i)} 1 = \sum_{i=1}^{r}\Phi(B_i).$$

It rests to show that it is not possible to find a face $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{C} \subseteq \mathcal{F}'$. Consider such a face and suppose its block partition is

$$\mathcal{B}(\mathcal{F}') = \{\overline{B}_{\widehat{0}}, \overline{B}_{\widehat{1}}, \overline{B}_1, \cdots, \overline{B}_s\}.$$

As $\dim(\mathcal{F}') < \dim(\mathcal{F})$, this implies that $\mathcal{B}(\mathcal{F}')$ has less blocks than $\mathcal{B}(\mathcal{F})$, i.e. $|\mathcal{B}(\mathcal{F}')| < |\mathcal{B}(\mathcal{F})|$. Consequently, there exists $\overline{B}_{s^*}$ such that

$$\overline{B}_{s^*} = C_{h_1}^{i_1} \cup \ldots \cup C_{h_t}^{i_t}, \quad t \geq 2.$$

If $i_1 = i_2 = \cdots = i_t$, then $\overline{B}_{s^*}$ is disconnected, a contradiction. If, say $i_1 \neq i_2$, then $\overline{B}_{s^*}$ contains parts of B_{i_1} and B_{i_2}. As $\mathcal{C}$ is constant on $\overline{B}_{s^*}$, then $\mathcal{C}$ is constant on $B_{i_1} \cup B_{i_2}$, a contradiction with the fact that $\mathcal{B}(\mathcal{C}) = \{B_{\widehat{0}}, B_{\widehat{1}}, B_1, \cdots, B_r\}$. □

Corollary 1.65. *Let P be a finite poset and $\mathcal{C}$ be a collection of vertices of $\mathcal{O}(P)$ with associated collection of filters $L \subseteq \mathfrak{F}(P)$. The face with lower dimension containing $\mathcal{C}$ is the intersection of $\mathcal{O}(P)$ with the hyperplane*

$$X_{\bigcap_{J\in L} J}(\boldsymbol{x}) = |\bigcap_{J\in L} J| + X_{(\bigcup_{J\in L} J)^c}(\boldsymbol{x}) + \sum_{i=1}^{r}\overline{\chi}_{B_i}(\boldsymbol{x}). \tag{1.4}$$

Example 1.11. *Let $P = \overline{\mathbf{3}}$ be the antichain of three elements. Every subset is a filter in this case, therefore $\mathfrak{F}(P) = \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$. Indeed, $\mathcal{O}(P)$ is the unit cube, see Figure 1.20. Let x_i be the variable associated to the element $i \in P$ and take the collection of filters $L_1 = \{1, 123\}$. For this collection its block partition is $B_{\widehat{0}} = \emptyset, B_{\widehat{1}} = \{1\}$ and $B_1 = \{2, 3\}$. Since there is no pair of related elements in B_1 we get $\overline{\chi}(B_1) = 0$. Applying Equation 1.4 the smallest face containing the vertices associated to these filters is*

$$x_1 = 1 + 0 + 0.$$

In this case this is obvious from the Figure 1.20. It is trivial that L_1 is not a face. For example, note that L_1 is not an embedded sublattice because $13 \cup 12 = 123 \in L_1$ and $12 \cap 13 = 1 \in L_1$ but 12 and 13 are not in L_1. Besides, B_1 is not a connected poset.

Consider now the edge $L_2 = \{2, 12\}$*. For this collection its block partition is* $B_{\widehat{0}} = \{3\}, B_{\widehat{1}} = \{2\}$ *and* $B_1 = \{1\}$*. Applying Equation 1.4 the smallest face containing the vertices associated to* L_2 *is*

$$x_2 = 1 + x_3 + 0.$$

Since L_2 *gives us a face, all the blocks are connected.*

Finally take $L_3 = \{\emptyset, 1, 23, 123\}$*. For this collection its block partition is* $B_{\widehat{0}} = \emptyset, B_{\widehat{1}} = \emptyset$ *and* $B_1 = \{1, 2, 3\}$*. Applying Equation 1.4 the smallest face containing the vertices associated to* L_3 *is*

$$0 = 0 + 0 + 0,$$

that is the whole polytope. Note that neither L_3 *is a face nor it is an embedded sublattice.*

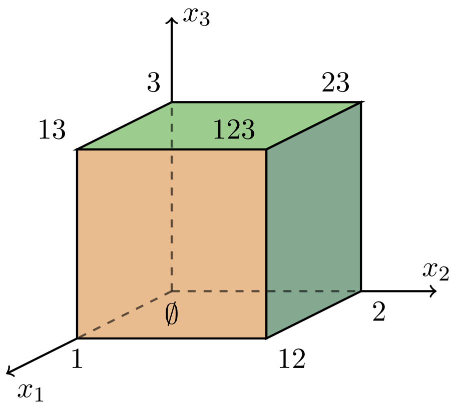


FIGURE 1.20: Polytope $\mathcal{O}(\overline{\mathbf{3}})$.

However, it is not essential to work with block partitions in order to get the last result. Observe that $\biguplus_{i=1}^{r} B_i = \left(\left(\bigcap_{J\in L} J\right) \cup \left(\bigcup_{J\in L} J\right)^c\right)^c$, that is, we could remove the special blocks $B_{\widehat{0}}$ and $B_{\widehat{1}}$ where the vertices of $\mathcal{C}$ take value zero and one respectively. Let us consider $L = \{J_1, \ldots, J_s\} \subseteq \mathfrak{F}(P)$. We define the **generalized symmetric difference** as

$$\Delta(J_1, J_2, \cdots, J_s) := \left(\left(\bigcap_{i=1}^{s} J_i\right) \cup \left(\bigcup_{i=1}^{s} J_i\right)^c\right)^c = \left(\left(\bigcap_{J\in L} J\right) \cup \left(\bigcup_{J\in L} J\right)^c\right)^c.$$

Note that

$$\Delta(J_1, J_2, \cdots, J_s) = \left(\left(\bigcap_{i=1}^{s} J_i\right) \cup \left(\bigcup_{i=1}^{s} J_i\right)^c\right)^c = \left(\bigcap_{i=1}^{s} J_i\right)^c \cap \left(\bigcup_{i=1}^{s} J_i\right) = \left(\bigcup_{i=1}^{s} J_i\right) \cap \left(\bigcup_{i=1}^{s} J_i^c\right).$$

Now we can state the last theorem in the following way:

Corollary 1.66. *Let P be a finite poset with $|P| = n$ and $\mathcal{C}$ a collection of vertices of $\mathcal{O}(P)$ with associated collection of filters $L = \{J_1, \cdots, J_s\} \subseteq \mathfrak{F}(P)$. Then the vertices in $\mathcal{C}$ are in the same k-dimensional face but not in the same $(k-1)$-dimensional face if and only if*

$$\Phi\left(\Delta(J_1, J_2, \cdots, J_s)\right) = k,$$

where $k \in [n]$.

Proof. Let $\mathcal{B}(\mathcal{C}) = \{B_{\widehat{0}}, B_{\widehat{1}}, B_1, \cdots, B_r\}$ be the block partition of $\widehat{P}$ associated to $\mathcal{C}$. Note that $\sum_{i=1}^{r} \Phi(B_i) = \Phi\left(\biguplus_{i=1}^{r} B_i\right)$ and $\biguplus_{i=1}^{r} B_i = \Delta(J_1, J_2, \cdots, J_s)$. □

In particular for $r = 2$:

Corollary 1.67. *Let P be a finite poset with $|P| = n$ and two filters $J_1, J_2 \in \mathfrak{F}(P)$. Then the vertices v_{J_1} and v_{J_2} are in the same k-dimensional face but not in the same $(k-1)$-dimensional face if and only if*

$$\Phi(J_1 \Delta J_2) = k,$$

where Δ is the simmetric difference, i.e. $J_1 \Delta J_2 = (J_1 \setminus J_2) \uplus (J_2 \setminus J_1)$ and $k \in [n]$.

Remark 1.68.

- From Equation 1.4 it follows that a vertex v_J is in the smallest face containing $\mathcal{C}$ if and only if v_J is constant on each block of $\mathcal{B}(\mathcal{C}) = \{B_{\widehat{0}}, B_{\widehat{1}}, B_1, \cdots, B_r\}$ having value zero at $B_{\widehat{0}}$ and one at $B_{\widehat{1}}$.
- The last claim can be stated in another way. Observe that to determine a face partition $\mathcal{B}(\mathcal{F})$ it is of course sufficient to work just with the non-singleton blocks so we define the reduced face partition of $\mathcal{F}$ as $\mathcal{B}^{\circ}(\mathcal{F}) = \{B_i \in \mathcal{B}(\mathcal{F}) : |B_i| > 1\}$. Let us denote $\widehat{J} = J \cup \{\widehat{1}\}$ for any filter J. Let $\mathcal{F} \subseteq \mathcal{O}(P)$ be a face with reduced face partition $\mathcal{B}^{\circ} = \{B_1, \cdots, B_m\}$ and let $J \in \mathfrak{F}(P)$ be a filter. Then $v_J \in \mathcal{F}$ if and only if for all $i = 1, \cdots, m$

$$\widehat{J} \cap B_i = \emptyset \text{ or } \widehat{J} \cap B_i = B_i.$$

  That is, v_J belongs to $\mathcal{F}$ iff for all $i \in [m]$, the filter $\widehat{J}$ does not separate any two elements in B_i.

- From Corollary 1.67 we know that two vertices v_J and $v_{J'}$ are adjacent in $\mathcal{O}(P)$ iff its symmetric difference is connected as induced subposet, see [23]. Therefore, these results are generalizations of the results given in [23].
- We can use these results to know when two collection of filters corresponding with two r-dimensional faces $\mathcal{F}_1$ and $\mathcal{F}_2$ are in the same s-dimensional face. By the last result it happens when $\Phi(\Delta(L_1 \cup L_2)) = s$, where L_i is the set of filters defining the face $\mathcal{F}_i$, for $i \in \{1, 2\}$.

We end this subsection with some important remarks on order polytopes.

Remark 1.69.

- The center of gravity c_P of $\mathcal{O}(P)$ can be computed as follows:

$$c_P = \frac{1}{m+1} H(P),$$

where $H(P)(a) := \frac{1}{e(P)} \sum_{\epsilon \in \mathcal{L}(P)} h_\epsilon(a)$ being $h_\epsilon(a) = |\{x \in P : x \preceq_\epsilon a\}|$. In other words, $h_\epsilon(a)$ is the height of a in the linear extension ϵ [24].

- Let P be a finite poset, the **chain polytope** $\mathfrak{C}(P)$ [22] associated to P is defined as

$$0 \leq g(x) \text{ for every } x \in P,$$
$$g(x_1) + g(x_2) + \cdots + g(x_k) \leq 1 \text{ for every chain } x_1 < x_2 < \cdots < x_k \text{ of } P.$$

This polytope has as vertices the characteristic functions of the antichains of P. It can be shown that the two polytopes $\mathcal{O}(P)$ and $\mathfrak{C}(P)$ are affinely equivalent and hence combinatorially equivalent.

1.3.1 Triangulating $\mathcal{O}(P)$ via linear extensions

We are ready to describe Stanley's triangulation of the order polytope $\mathcal{O}(P)$, see [22]. Let P be a finite poset and consider a maximal chain C of filters in $\mathfrak{F}(P)$, $J_0 \subset J_1 \subset \cdots \subset J_n$. Observe that $K(C) := \mathrm{Conv}(v_{J_i} : J_i \in C)$ is a simplex of dimension n. If we put together all these simplices we get a triangulation of $\mathcal{O}(P)$. Since the number of maximal chains of $\mathfrak{F}(P)$ equals $e(P)$ (see Eq. 1.1), we can relate each highest dimensional simplex of the last triangulation with a linear extension of P. Using Cayley-Menger determinant (see Lemma 1.54) is easy to show that all of these simplices share the same volume.

Theorem 1.70. *[22, 24] Let P be a poset of n elements.*

i) *Let C be a maximal chain of filters in $\mathfrak{F}(P)$. The convex hull of the vertices associated to these filters $K(C) := Conv(v_{J_i} : J_i \in C)$ is a simplex of dimension n and volume $\frac{1}{n!}$.*

ii) *The collection $\Delta = \{K(C) : C$ is a maximal chain of $\mathfrak{F}(P)\}$ is a triangulation of $\mathcal{O}(P)$. Therefore,*

$$Vol(\mathcal{O}(P)) = \frac{e(P)}{n!}. \tag{1.5}$$

Remark 1.71.

- From the last result we draw an important conclusion: computing the volume of $\mathcal{O}(P)$ is equivalent to compute $e(P)$. We will exploit this idea along this thesis. However, the problem of counting the number of linear extensions of a general poset is a $\sharp P$-complete problem [28]. Therefore, obtaining procedures with low complexity for counting linear extensions in a family of posets is a relevant and interesting problem.

- With respect to the problem of generating points uniformly in $\mathcal{O}(P)$, it suffices to generate randomly a linear extension of P and then generate a point in the corresponding simplex.

- The abstract simplicial complex underlying this triangulation Δ is the order complex of $\mathfrak{F}(P)$, $\Delta(\mathfrak{F}(P))$.

Example 1.12. *Consider the poset $P = \mathbf{1} \uplus \mathbf{2}$. Its order polytope $\mathcal{O}(P)$ is a half-cube as it can be seen in Figure 1.21 left. Note that $e(P) = 3$ and therefore we can split this polyhedron into three simplices, see Figure 1.21 right. Moreover, in this case $Vol(\mathcal{O}(P)) = e(P)/3! = 3/6 = 1/2$.*

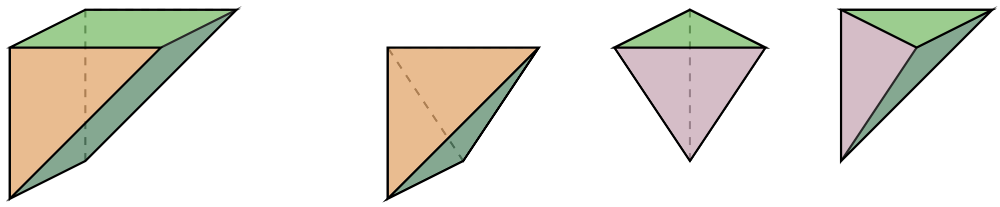


FIGURE 1.21: Polytope $\mathcal{O}(\mathbf{1}\uplus\mathbf{2})$ (left) and its triangulation via linear extensions (right)

Chapter 2

Fuzzy measures and Games

2.1 Introduction

In this chapter we give a brief introduction to fuzzy measures (also called non-additive measures or capacities). We study many important subfamilies of fuzzy measures. The main goal of this work is to be able to provide as much information as possible about the geometry of these subfamilies. In concrete terms, it is very important to know which of these subfamilies correspond to order polytopes, since we have many results about the geometry of these objects (see Chapter 1). In this chapter we will also study some aspects of cooperative game theory and we will see the basic properties of the Choquet Integral. Choquet Integral appears as an essential part of the Identification problem.

In what follows, let $X = \{1, \ldots, n\}$ be a finite referential set for n elements. Elements of X are criteria in the field of Multicriteria Decision Making, players in Cooperative Game Theory, and so on. We will denote subsets of X by $A, B, \ldots$. The set of subsets of X is denoted by $\mathcal{P}(X)$.

2.2 Fuzzy measures

In this section we introduce the basic definitions related to fuzzy measures, see [29].

Definition 2.1. A **fuzzy measure** is a map $\mu : \mathcal{P}(X) \to \mathbb{R}$ that is:

i) **Grounded**: $\mu(\emptyset) = 0$,

ii) **Monotone**: $\mu(A) \leq \mu(B)$, $\forall A \subseteq B$.

A **normalized fuzzy measure** is a fuzzy measure which also is:

iii) **Normalized**: $\mu(X) = 1$.

Sometimes we just call them fuzzy measures when the property of $\mu(X) = 1$ is clear from the context. This way we can see normalized fuzzy measures as a generalization of probability measures where the additivity axiom has been removed and monotonicity is imposed instead. The term "fuzzy measure" was coined by Sugeno [30]. The notion of fuzzy measure was also proposed independently by Denneberg under the name of non-additive measure [31] and by Choquet under the name of capacity [32].

Capacities are mainly used in decision theory and game theory. There are two main interpretations of these functions. The first one is to consider capacities as a means to represent the importance/power/worth of a group. Here the subsets $A \subseteq X$ represent a set of persons, usually called players, agents, voters, experts, criteria, etc., depending on the specific issues addressed. In game theory, a group of individuals A is usually called coalition. The second interpretation is about the representation of uncertainty. In this case, X is the set of possible outcomes of some experiment, and it is supposed that X is exhaustive (i.e., any experiment produces an outcome that belongs to X), and that each experiment produces a single outcome. Any subset $A \subseteq X$ is called an event, and $\mu(A)$ quantifies the uncertainty that the event A contains the outcome of an experiment.

The set of normalized fuzzy measures on the referential X is denoted by $\mathcal{FM}(X)$. It can be seen that $\mathcal{FM}(X)$ is a polytope. Indeed, monotonicity ($\mu(A) \leq \mu(B)$ if $A \subseteq B$) gives the hyperplanes defining the polyhedron and the grounded ($\mu(\emptyset) = 0$) and normalized ($\mu(X) = 1$) conditions ensure the boundedness. Moreover, $\mathcal{FM}(X)$ is an order polytope. Indeed, it is bounded between 0 and 1 and the associated poset is the Boolean lattice with the empty and total sets removed, since they take constant values 0 and 1 respectively. Therefore if $X = [n]$,

$$\mathcal{FM}(X) = \mathcal{O}(B_n \setminus \{\emptyset, X\}).$$

The Hasse diagram of the Boolean poset $B_4 \setminus \{\emptyset, X\}$ is given in Figure 2.1.

Fuzzy measures are a specific type of set functions.

Definition 2.2. A **set function on X** is a map $\xi : \mathcal{P}(X) \to \mathbb{R}$, associating to each subset of X a real number. A set function is

i) **Additive** if $\xi(A \cup B) = \xi(A) + \xi(B)$, $\forall A, B \subseteq X$ such that $A \cap B = \emptyset$;

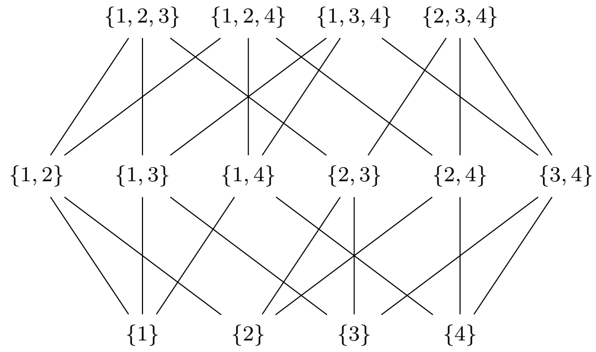


FIGURE 2.1: Hasse diagrams of the Boolean poset for $|X| = 4$.

ii) **Monotone** if $\xi(A) \leq \xi(B)$, $\forall A \subseteq B$;

iii) **Grounded** if $\xi(\emptyset) = 0$;

iv) **Normalized** if $\xi(X) = 1$;

v) a **Measure** if ξ is nonnegative and additive;

vi) **Symmetric** if $\xi(A) = \xi(B)$ whenever $|A| = |B|$;

Remark 2.3.

- A fuzzy measure is just a grounded and monotone set function. Also a normalized measure is a probability measure.
- The value of any additive set function on $A \subseteq X$ is $\xi(A) = \sum_{x \in A} \xi(\{x\})$, therefore if we know the value of ξ on every singleton $\{x\}$, we know the value of ξ on any subset. Even more, as $\xi(X) = 1$, just $n - 1$ values suffice to define ξ.
- The normalized and additive fuzzy measures are the well known probability measures on X, denoted by $\mathcal{PM}(X)$. They are not order polytopes because of the condition $\sum_{i \in X} P(i) = 1$.

Let us introduce a few ways of getting new set functions from old.

Definition 2.4. Let ξ be a set function on X. The **monotone cover** of ξ, denoted $mc(\xi)$, is the smallest fuzzy measure μ such that $\mu \geq \xi$. It is defined by

$$mc(\xi)(A) = \max_{B \subseteq A} \xi(B).$$

Now let us define the conjugate of a set function.

Definition 2.5. Let ξ be a set function on X. We define its **conjugate or dual** as:

$$\overline{\xi}(A) = \xi(X) - \xi(A^c),$$

for all $A \subseteq X$.

The following properties concerning conjugation can be easily proved .

Lemma 2.6. *[29] Let $\xi : \mathcal{P}(X) \to \mathbb{R}$ be a set function. Then:*

- *If $\xi(\emptyset) = 0$, then $\overline{\xi}(X) = \xi(X)$ and $\overline{\overline{\xi}} = \xi$.*
- *If ξ is monotone, then $\overline{\xi}$ is monotone.*
- *If ξ is additive, then $\overline{\xi} = \xi$.*

Let us now define some important types of fuzzy measures.

Definition 2.7. A fuzzy measure μ on X is

i) a 0/1-**capacity** if $\mu(A) \in \{0, 1\}$, $\forall A \subseteq X$;

ii) a **unanimity game** centered on $\emptyset \subset A \subseteq X$, u_A, if

$$u_A(B) = \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{otherwise.} \end{cases}$$

When $A = \{x\}$, u_A is called the **Dirac measure** centered at x. The set $\{u_A : \emptyset \subset A \subseteq X\}$ determines a base of $\mathcal{FM}(X)$, i.e. it is possible to write $\mu = \sum_{\emptyset \subset A \subseteq X} \alpha_A u_A$ for some α_A, $\forall \mu \in \mathcal{FM}(X)$.

iii) k-**monotone** (for $k \geq 2$) if $\forall A_1, A_2, \ldots, A_k \subseteq X$,

$$\mu\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} A_i\right).$$

If a fuzzy measure μ is k-monotone for all $k \geq 2$ we say that it is **totally monotone (or ∞-monotone)**.

iv) k-**alternating** (for $k \geq 2$) if $\forall A_1, A_2, \ldots, A_k \subseteq X$,

$$\mu\left(\bigcap_{i=1}^{k} A_i\right) \leq \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mu\left(\bigcup_{i \in I} A_i\right).$$

If a fuzzy measure μ is k-alternating for all $k \geq 2$ we say that it is **totally alternating (or ∞-alternating)**.

v) a **belief (plausibility)** measure on X is normalized and totally monotone (alternating) fuzzy measure.

vi) a **λ-measure** for $\lambda > -1$ if it is normalized and satisfies

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B),$$

for every $A, B \subseteq X$ such that $A \cap B = \emptyset$. The conjugate of a λ-measure is a λ'-measure with $\lambda' = -\dfrac{\lambda}{\lambda+1}$. A λ-measure is a belief measure if and only if $\lambda > 0$, and is a plausibility measure otherwise ($\lambda \in (-1, 0)$).

Remark 2.8.

- The last definitions could be generalized to grounded set functions (also called games, see Section 2.7).
- A 0/1-capacity is uniquely determined by the antichain of its minimal winning coalitions. A set A is a **winning coalition** for μ if $\mu(A) = 1$. It is minimal if in addition $\mu(B) = 0$ for all $B \subset A$. Note that two minimal winning coalitions are incomparable by inclusion. Hence the collection of minimal winning coalitions of μ is an antichain in $\mathcal{P}(X)$ and any antichain in $\mathcal{P}(X)$ can be identified to the set of minimal winning coalitions of a 0/1-capacity. The number of antichains in the Boolean lattice B_n is known as the Dedekind number $M(n)$, see Table 2.1. For $M(n)$ the empty set is considered as an antichain. Since the number of filters equals the number of antichains (counting the empty set), the Dedekind numbers also count the filters of B_n. Therefore, the number of 0/1-capacities is $M(n) - 2$. In other words, the number of vertices of $\mathcal{FM}(X)$ is $M(n) - 2$.

n	$M(n)$
0	2
1	3
2	6
3	20
4	168
5	7 581
6	7 828 354
7	2 414 682 040 998
8	56 130 437 228 687 557 907 788

Table 2.1: First Dedekind numbers.

  Remark the exponential growth of $M(n)$. There is not known closed-form formula for an easy computation of these numbers.
- k-monotonicity implies k'-monotonicity for all $2 \leq k' \leq k$.

- k-monotonicity and the k-alternating properties generalize the Inclusion-Exclusion Principle, which holds for any additive measure:

$$\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1}\mu\left(\bigcap_{i\in I} A_i\right).$$

This equality comes from the classic identity:

$$\Big|\bigcup_{i=1}^{k} A_i\Big| = \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1}\Big|\bigcap_{i\in I} A_i\Big|.$$

- $\forall k \geq 2$, μ is a k-monotone game (respectively, k-alternating) if and only if $\overline{\mu}$ is a k-alternating game (respectively, k-monotone).

2.3 Choquet Integral

Consider a given function $f : X \to [0,1]$, whose corresponding scores on each criterium are $f(x_1), \ldots, f(x_n)$. To compare different functions, we need to obtain an overall score from $f(x_1), \ldots, f(x_n)$. This is done through an aggregation operator. The Choquet integral [32] is one of the most popular.

Definition 2.9. Let $(X, \mathcal{X})$ be a measurable space and $f : X \to \mathbb{R}$. We say that f is a measurable function if $\{x|f(x) \geq \alpha\}$ is in the σ-algebra $\mathcal{X}$ for any $\alpha \geq 0$.

The Choquet integral can be defined as follows.

Definition 2.10. Let μ be a fuzzy measure on $(X, \mathcal{X})$ and $f : X \to \mathbb{R}^+$ a measurable function. The **Choquet integral** of f with respect to μ is defined by

$$\mathcal{C}_\mu(f) := (\mathcal{C})\int_X f d\mu = \int_0^\infty \mu\left(\{x|f(x) > \alpha\}\right) d\alpha.$$

In the case of finite referential sets $X = \{x_1, x_2, \ldots, x_n\}$, the last expression simplifies to

$$\mathcal{C}_\mu(f) = \sum_{i=1}^{n} \left(f(x_{(i)}) - f(x_{(i-1)})\right)\mu(B_i),$$

where $\{x_{(1)}, \ldots, x_{(n)}\}$ is a permutation of the set X satisfying

$$0 = f(x_{(0)}) \leq f(x_{(1)}) \leq f(x_{(2)}) \leq \cdots \leq f(x_{(n)}),$$

and $B_i = \{x_{(i)}, \ldots, x_{(n)}\}$.

Remark 2.11.

- The Choquet integral is a generalization of the Lebesgue integral, i.e. it can be shown that for any additive measure the Choquet and the Lebesque integral give the same result. Thus, we can see that the Choquet integral is a generalization of the concept of expected value for non-additive measures. This allows to generalize the Expected Utility Theory (see Section 2.4).
- For finite referential sets X, the chosen σ-algebra $\mathcal{X}$ is $\mathcal{P}(X)$, this way every measure is measurable.

In the case of general functions (not necessarily positive) we can also define the Choquet integral; however in this case a couple of possible definitions are available. Let us denote the positive and negative parts of a function f by $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

Definition 2.12. Let μ be a fuzzy measure on $(X, \mathcal{X})$ and $f : X \to \mathbb{R}$ a measurable function. The **(asymmetric) Choquet integral** of f with respect to μ is defined by

$$\mathcal{C}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_{\overline{\mu}}(f^-).$$

Definition 2.13. Let μ be a fuzzy measure on $(X, \mathcal{X})$ and $f : X \to \mathbb{R}$ a measurable function. The **(symmetric) Choquet integral or Šipoš integral** of f with respect to μ is defined by

$$\check{\mathcal{C}}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_\mu(f^-).$$

Without further indication, by "Choquet integral" we always mean its asymmetric extension. We will now examine some properties of the Choquet integral.

Definition 2.14. Two functions $f, g : X \to \mathbb{R}$ are **comonotonic** if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \quad \forall x, y \in X.$$

Proposition 2.15. *Let μ be a normalized fuzzy measure on $(X, \mathcal{X})$ and $f, g : X \to \mathbb{R}$ two measurable functions. Then,*

i) $\mathcal{C}_\mu(1_A) = \mu(A)$ *for all* $A \in \mathcal{X}$, *where* $1_A(x) = 1$ *if* $x \in A$, *and* $1_A(x) = 0$ *otherwise.*

ii) $\mathcal{C}_\mu(af) = a\mathcal{C}_\mu(f)$ *for every* $a \geq 0$.

iii) $\mathcal{C}_\mu(-f) = -\mathcal{C}_{\overline{\mu}}(f)$.

iv) *If* $f \geq g$, *then* $\mathcal{C}_\mu(f) \geq \mathcal{C}_\mu(g)$.

v) $\mathcal{C}_\mu(f + a) = \mathcal{C}_\mu(f) + a$, *for every* $a \in \mathbb{R}$.

vi) *If* f, g *are comonotonic functions, then* $\mathcal{C}_\mu(f + g) = \mathcal{C}_\mu(f) + \mathcal{C}_\mu(g)$.

2.4 Applications in Decision Theory

In this section we introduce some applications of fuzzy measures, and more exactly, of Choquet integral, in the framework of Decision Theory. Decision Theory is the branch of Mathematics treating decision problems in which a person or a group of people (called *the decision maker*) must make a choice among some options or alternatives.

Fuzzy integrals have been applied to subjective evaluation of objects whose characteristics are depending on multiple criteria. This first application was given by Sugeno in [30].

The section is divided into two parts:

- The first part is devoted to Decision under Uncertainty and Risk. It is also subdivided into two different parts. For decision under risk and uncertainty, pioneering works using Choquet integral are due to Schmeidler and Quiggin [33].
- The second part of the Chapter deals with Multicriteria Decision Making (MCDM). In MCDM we look for an aggregation operator for compounding a group of evaluations. We give a survey of the usual properties required to such function. Then, we see that Choquet integral is a suitable operator.

These are the two fundamental situations treated by Decision Theory. Anyway, this is not an exhaustive description; for example, we are not concerned with the Theory of Social Choice developed by Arrow [34].

2.4.1 Decision under Uncertainty

For Decision under Uncertainty and under Risk, it is supposed that the consequences of the choice depend on some incertitude factors that cannot be controlled by the decision maker. This section is divided in two parts. In the first one, we describe some of the usual criteria for solving a problem of Decision under Uncertainty. In the second part, we deal with Expected Utility Theory.

The fundamental elements of a decision problem in Decision under Uncertainty and Decision under Risk are the *set of states of the world*, denoted by S; over S we have a σ-algebra of subsets, denoted by $\mathcal{S}$; hence, $(S, \mathcal{S})$ is a measurable space. The set of states of the world represents the possible situations that can happen and we assume that only one of them happens but we do not know which one (it will be known after making our decision). The *set of possible consequences* is denoted by X. The *set of possible acts* is $\mathcal{F}$, where an act f is a mapping from S to X. For any state of the world $s \in S$ and for any act $f \in \mathcal{F}$, there is a consequence $f(s) \in X$ (not necessarily a numerical value) representing the gain or loss if we choose act f when the state of the world is s. When these consequences are numerical they are usually called **utilities**. Hence $\mathcal{F}$ is X^S or is limited to a subset of it. In decision problems it is usually required that there exist a mapping $u : X \mapsto \mathbb{R}$, that allows us to work with utilities.

Consequently, acts can be compared in terms of its corresponding consequences (utilities). Let us explain these concepts through an example:

Example 2.1. *An economist must choose among some investments* v_1, v_2, v_3, v_4. *The benefits of each investment depend on the future economical situation of the country. The possible situations are inflation, recession and depression.*

In this problem, we have five possible acts $\mathcal{F} = \{f_1, ..., f_5\}$ *where* f_i *denotes that the economist makes the investment* v_i, $i = 1, ..., 4$ *and* f_5 *denotes the fact that the economist decides to keep the money at the bank. The set of states of the world is the set of possible economical situations; thus, we have three possible states of the world:* s_1 *(inflation),* s_2 *(recession) and* s_3 *(depression). Now suppose that we have the following table of utilities, each value representing the expected benefit in million euros.*

	x_1	x_2	x_3
f_1	*1*	*0.5*	*-0.1*
f_2	*0.7*	*2*	*-0.6*
f_3	*0.2*	*-0.5*	*-1*
f_4	*-0.6*	*0*	*1.5*
f_5	*0*	*0*	*0*

In the next section, we will give some methods to solve this problem.

In order to make a decision, we need to establish a rationality criterion; this criterion leads to a "representative value" for each act and then the act with the best representative value is chosen.

For Decision under Uncertainty, we assume that we do not have any prior objective information about which is the real state of the world. For Decision under Risk, we suppose that we have a probability distribution on S.

2.4.1.1 Some criteria for making decision under Uncertainty

As we have seen, the solution of this problem is given by a rationality criterion. Let us recall the most commonly used criteria in the literature; for each of them we will solve the problem given in Example 2.1. For the rest of this section, we will assume that the consequences are measured in terms of utilities.

The MaxiMin criterium

This is a pessimist criterium where the decision maker considers that the worst situation will happen, i.e. the representative value for act f is $\inf_{s\in S} f(s)$. Then, the optimal act should be the one maximizing the worst value; mathematically,

$$\sup_{f\in\mathcal{F}} \inf_{s\in S} f(s).$$

A decision maker acts following this criterion when he is very conservative or cautious.

Example 2.2. *In our example*

	x_1	x_2	x_3	*Min*
f_1	*1*	*0.5*	*-0.1*	*-0.1*
f_2	*0.7*	*2*	*-0.6*	*-0.6*
f_3	*0.2*	*-0.5*	*-1*	*-1*
f_4	*-0.6*	*0*	*1.5*	*-0.6*
f_5	*0*	*0*	*0*	*0*

Then, the investor should keep the money at the bank (as expected).

The MaxiMax criterium

The MaxiMin criterium has its counterpart in the MaxiMax criterium. For this criterium, the decision maker is optimist and considers that the true state of world will be the best situation for this act, i.e. the representative value for act f is $\sup_{s\in S} f(s)$. Then, the optimal act should be the one maximizing the best value; formally,

$$\sup_{f\in\mathcal{F}} \sup_{s\in S} f(s).$$

Example 2.3. *In our example*

	x_1	x_2	x_3	*Max*
f_1	*1*	*0.5*	*-0.1*	*1*
f_2	*0.7*	*2*	*-0.6*	*2*
f_3	*0.2*	*-0.5*	*-1*	*0.2*
f_4	*-0.6*	*0*	*1.5*	*1.5*
f_5	*0*	*0*	*0*	*0*

Then, the investor should choose the second investment.

The Hurwicz criterium

This criterium is a middle term between the MaxiMin and the MaxiMax. For this criterium, the decision maker has a degree of optimism given by a value $\alpha \in [0, 1]$ and then the value considered for an act f is $\alpha \sup_{s\in S} f(s) + (1-\alpha) \inf_{s\in S} f(s)$. Then, the optimal act should be the one maximizing the representative value; mathematically,

$$\sup_{f\in\mathcal{F}} \left[\alpha \sup_{s\in S} f(s) + (1-\alpha) \inf_{s\in S} f(s) \right].$$

Example 2.4. *In our example for* $\alpha = 0.2$*,*

	x_1	x_2	x_3	$H_{0.4}$
f_1	*1*	*0.5*	*-0.1*	*0.12*
f_2	*0.7*	*2*	*-0.6*	*-0.08*
f_3	*0.2*	*-0.5*	*-1*	*-0.76*
f_4	*-0.6*	*0*	*1.5*	*-0.18*
f_5	*0*	*0*	*0*	*0*

Then, the investor should choose the first investment.

It is interesting to remark that this criterion only uses the best and the worst possiblities for each act.

2.4.1.2 The Expected Utility Model

Suppose that we have a preference relation $\succeq$ on $\mathcal{F}$. Assume that this relation is a complete preorder (i.e. it is reflexive, transitive, and two acts can always be compared) so that it can be represented by some functional $V : \mathcal{F} \mapsto \mathbb{R}$; we will denote by $\succ$ the strict preference and by $\sim$ the indifference. We want to extract some properties about the behaviour of the decision maker that leads to this functional.

In this section we address the Bayesian model of Anscombe and Aumann [35] leading to an additive model, i.e. we assume that the preference relation on $\mathcal{F}$ can be modelled by the expected utility

$$f \preceq g \Leftrightarrow \int_S u(f(s))dP \leq \int_S u(g(s))dP,$$

where P is a probability distribution over $(S, \mathcal{S})$, and $u : X \mapsto \mathbb{R}$ is an utility function.

For introducing the axiomatic structure proposed by Anscombe and Aumann we need to introduce some previous notations and definitions:

- We denote by Y the set of all *finite gambles* on X, the set of consequences, where it is assumed that X has at least two elements. From a formal point of view, $y \in Y \Leftrightarrow y : X \mapsto [0,1]$, $y(x) \neq 0$ only for a finite number of consequences and $\sum_{x \in X} y(x) = 1$.
- We denote by $\mathcal{L}_0$ the set of *simple acts*, i.e. the set of acts f defined by

$$f(s) = \sum_{i=1}^{n} y_i(1_{A_i}(s)),\ y_i \in Y,\ A_i \in \mathcal{S},\ \bigcup_{i=1}^{n} A_i = S.$$

Now, the axioms are:

- **A1.** Ranking: $(\mathcal{F}, \succeq)$ is a complete preorder.

Here, it must be noted that the set of constant acts, $\mathcal{L}_0^C = \{f \in \mathcal{L}_0, f(s) = y, \forall s \in S\}$ induces a complete preorder on Y, that is denoted $\succeq$, too.

- **A2.** Independence: $\forall f, g, h \in \mathcal{L}_0$, $\forall \alpha \in (0,1]$,

$$f \succeq g \Leftrightarrow \alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h.$$

- **A3.** Continuity: $\forall f, g, h \in \mathcal{L}_0$, $f \succ g \succ h$, $\exists \alpha, \beta \in (0,1)$ such that

$$\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h.$$

Given two acts f, g, let us now denote by (f_A, g_{A^c}) the act that coincides with f in A and with g in A^c.

- **A4.** Strict monotonicity: $\forall f, g \in \mathcal{L}_0$, $\forall y, z \in Y$, $A \in \mathcal{S}$, if $f \succ g$ with $f = (y_A, f_{A^c})$, $g = (z_A, g_{A^c})$, then $y \succ z$.
- **A5.** Non-triviality: $\exists f, g \in \mathcal{L}_0$ such that $f \succ g$.

Lemma 2.16. *If* **A1** *and* **A4** *hold, then the following axiom holds:*

- **A4'.** *Monotonicity:* $\forall f, g \in \mathcal{L}_0$, $f(s) \succeq g(s)$, $\forall s \in S$, *then* $f \succeq g$.

Then, the following can be proved:

Theorem 2.17. *[35] If $\succeq$ satisfies* **A1**, **A2**, **A3**, **A4** *and* **A5**, *then there exists a unique probability measure on $(S, \mathcal{S})$ and an affine* [1] *utility function $u : Y \mapsto \mathbb{R}$ such that for all $f, g \in \mathcal{L}_0$*

$$f \succeq g \Leftrightarrow \int u(f)\, dP \geq \int u(g)\, dP. \tag{2.1}$$

u is unique up to positive affine transformation[2].

Moreover, if $\succeq$ can be represented by Equation (2.1), then it satisfies **A1** *to* **A5**.

It must be remarked that this is not the only the only axiomatic structure of Expected Utility Theory. Another model has been proposed by Savage in [36].

2.4.1.3 Some drawbacks of Expected Utility Theory

In this section we give two examples where problems arise if the Expected Utility Model is used.

Ellsberg's paradox

This example is in the framework of Decision under Uncertainty. It was proposed in [37].

Consider a box with 90 balls. There are 30 red balls and other 60 balls that are either black or white, but we do not know exactly how many black or white balls there are.

A ball is extracted from the box and the decision maker must choose between betting for a red ball (f_1) or betting for a white ball (g_1). In a second step, he is asked to choose between betting for a red or black ball (f_2) against betting for a white or black ball (g_2).

[1] $u(\alpha y + (1-\alpha)z) = \alpha u(y) + (1-\alpha)u(z)$, $\forall \alpha \in (0, 1)$, $\forall y, z \in Y$
[2] $v = \alpha u + \beta, \alpha \geq 0$, $\beta \in \mathbb{R}$

Typical preferences are $f_1 \succ g_1$ and $f_2 \prec g_2$. In the first case he knows that there are 30 red balls and he does not know the number of black balls; then he feels more sure to bet for red ball, even if maybe there are 60 black balls, because the possibility of 0 black balls makes him conservative. In the second case, the situation reverses because now he knows that there are 60 balls that are either black or white, but he does not know how many balls either red or white are; then, as in the first situation, he prefers security.

This example violates the Independence Principle (**A2**) and then no probability distribution over the states of the world is able to represent the behaviour of the decision maker.

Allais' paradox

This example is in the framework of Decision under Risk. It was proposed in [38].

Suppose that a decision maker is faced to the following acts:

- f_1: he wins 300 euros.
- f_2: he wins 400 euros with probability 0.8 and nothing with probability 0.2.
- f_3: he wins 300 euros with probability 0.25 and nothing with probability 0.75.
- f_4: he wins 400 euros with probability 0.2 and nothing with probability 0.8.

Usual preferences are $f_1 \succ f_2$ and $f_3 \prec f_4$. However, this contradicts the independence axiom **A2** as $f_3 = 0.25f_1 + 0.75f_0$, $f_4 = 0.25f_2 + 0.75f_0$, with f_0 the act that gives a gain of 0 euros with probability 1. Indeed, for the first choice the decision maker prefers having a sure gain instead of risking; in the second choice he has not a sure gain and then he decides to risk a little bit more to obtain a higher gain. It can be said that in the second situation he thinks that he may not win and then he prefers risking.

The reason for both paradoxes is the fact that the decision maker has a *risk aversion* and this cannot be modelled with a probability distribution. We will see that these paradoxes can be modelled with the Choquet Expected Utility Model.

2.4.1.4 Choquet Expected Utility Model

We have seen in last section that the Expected Utility fails to model some behaviors which exhibit risk aversion. Schmeidler [39] proposed to replace probability measures by a general non additive measure and the classical expected value by Choquet integral.

The central part of Choquet Expected Utility Model is to change the independence axiom **A2** by a weaker axiom of independence restricted to comonotone acts:

- **Sch2.** Comonotonic independence. For all pairwise comonotonic acts f, g, h and for all $\alpha \in (0,1)$, then $f \succ g$ implies $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

The result of Schmeidler is:

Theorem 2.18. *[39] Suppose that $\succeq$ satisfies* **A1, Sch2, A3, A4', A5**. *Then, there exists a unique capacity μ on $(S, \mathcal{S})$ and an affine function $u : Y \mapsto \mathbb{R}$, unique up to affine transformation, such that*

$$f \succeq g \Leftrightarrow (\mathcal{C}) \int u(f)\, d\mu \geq (\mathcal{C}) \int u(g)\, d\mu.$$

Reciprocally, if there exist μ and u in these conditions, then $\succeq$ satisfies **A1, Sch2, A3, A4', A5**.

We have seen that Choquet integral is a generalization of Lebesgue integral. Then, Schmeidler model is a generalization of the Expected Utility model.

Let us now see another axiomatic, called the *Schmeidler simplified model*, proposed in [40] by Chateauneuf. Let us denote now the set $\mathcal{F}$ of all acts such that $\mathcal{F} = \{f : S \mapsto \mathbb{R}, f \text{ measurable } \}$. Consider the following axioms:

- **Cha1.** Ranking: $\succeq$ is a non-trivial complete preorder.
- **Cha2.** Continuity: $[f_n, f, g \in \mathcal{F},\ f_n \succeq g,\ f_n \downarrow f] \Rightarrow f \succeq g$.

  $[f_n, f, g \in \mathcal{F},\ f_n \preceq g,\ f_n \uparrow f] \Rightarrow g \succeq f$.
- **Cha3.** Monotonicity: $f(s) \geq g(s) + \epsilon,\ \forall s \in S, \epsilon > 0, \Rightarrow f \succ g$.
- **Cha4.** Comonotonic independence: For all pairwise comonotonic acts $f, g, h \in \mathcal{F}$, we have $f \sim g$ implies $f + h \sim g + h$.

Then, the following can be proved:

Theorem 2.19. *[40] Let $\succeq$ be a preference relation over the set of acts. The following statements are equivalent:*

- $\succeq$ *satisfies* **Cha 1, Cha 2, Cha 3, Cha4**.

- *The preference relation can be represented by a Choquet integral with respect to a fuzzy measure, i.e.*
$$f \succ g \Leftrightarrow (\mathcal{C}) \int f \, d\mu \geq (\mathcal{C}) \int g \, d\mu.$$
Moreover, μ is unique.

Consider the following axiom:

- **Cha4'.** For all $f, g, h \in \mathcal{F}$ with h, g comonotone,
$$f \sim g \Rightarrow f + h \succeq g + h.$$

The following holds:

Theorem 2.20. *[40] $\succeq$ satisfies* **Cha1, Cha2, Cha3, Cha4'** *if and only if the preference relation can be represented by a Choquet integral with respect to a* **convex** *fuzzy measure (Definition 2.33).*

Let us now return to the Ellsberg's paradox. In this case, $S = \{R, B, W\}$. The available information in terms of probability is $P(R) = \frac{1}{3}$, $P(W, B) = \frac{2}{3}$. Let us denote by $\mathcal{P}$ the set of all probability distributions over $(S, \mathcal{P}(S))$ that are compatible with this information. Let us define

$$\mu(A) = \inf_{P \in \mathcal{P}} P(A), \forall A \subset S.$$

Then, μ is given by

A	$\emptyset$	R	B	W	R,B	R,W	W,B	R,W,B
μ	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1

Then, representing our preferences by the Choquet integral with respect to μ we obtain:

$$(\mathcal{C}) \int f_1 \, d\mu = 100\mu(R) = \frac{100}{3}.$$

$$(\mathcal{C}) \int g_1 \, d\mu = 100\mu(B) = 0.$$

Thus, $f_1 \succ g_1$. On the other hand

$$(\mathcal{C}) \int f_2 \, d\mu = 100\mu(R, W) = \frac{100}{3}.$$

$$(\mathcal{C}) \int g_2 \, d\mu = 100\mu(B, W) = \frac{200}{3}.$$

Thus, $f_2 \prec g_2$.

In respect of Allais paradox, we can also use the Choquet Expected Utility model to find a solution. In this case, to represent risk aversion, we apply a *distorsion function* φ to the probability distribution of each lottery being φ an increasing bijection on $[0,1]$. This way we use Choquet integral with respect to $\mu = \varphi \circ P$, where P is the probability distribution of the lottery,

$$(\mathcal{C}) \int f \, d\mu = \sum_i f_i \varphi(p_i).$$

This way, by taking Choquet integrals, the preference $f_1 \succ f_2$ translates into

$$300 > 400\varphi(0.8)$$

hence we should have $0 < \varphi(0.8) < 0.75$. Next, the preference $f_3 \prec f_4$ imposes

$$300\varphi(0.25) < 400\varphi(0.2)$$

i.e., $\varphi(0.2)/\varphi(0.25) \in (0.75, 1)$. For instance, this is achieved by taking $\varphi(0.2) = 0.2$ and $\varphi(0.25) = 0.25$.

2.4.2 Multicriteria Decision Making

For Multicriteria Decision Making we start from a *set of criteria*, $N = \{1, ..., n\}$, representing the different factors that can influence the decision maker in making a decision. The set of potential alternatives is $X = X_1 \times ... \times X_n$, where each factor is related to a descriptor or attribute, whose possible values belong to a set X_i. It must be remarked that in many practical situations we do not have all potential alternatives, but only a subset of X. Fixing an alternative $x = (x_1, ..., x_n) \in X$ and a criterion i, x_i is the value (possibly non-numerical) of attribute i for alternative x. The next step is to map the spaces X_i to a common scale in $\mathbb{R}$, so as to reflect preferences. This is done through n functions $u_i : X_i \mapsto \mathbb{R}$, called utility functions. $u_i(x_i)$ represents the "score" of alternative x with respect to criterion i. In order that the construction makes sense, all u_i must be commensurable, i.e. if $u_i(x_i) = u_j(x_j)$, then the decision maker should feels an equal satisfaction for x_i and x_j. Finally, scores $u_i(x_i)$, $i = 1, ..., n$ are aggregated into an overall score representing the satisfaction degree of x.

Example 2.5. *A customer wants to buy a car and he must decide among five different models $f_1, ..., f_5$. His choice is based on some characteristics of the car, namely price, consumption and security system. Let us suppose that for any criterion we have six possibilities, namely that, for this criterion, the car is very bad, bad, average, good, very good or excellent, denoted by VB, B, A, G, VG and E, respectively. Then, he has the following table of scores:*

	Price	*Consumption*	*Security*
f_1	*G*	*B*	*A*
f_2	*VG*	*E*	*VB*
f_3	*B*	*G*	*VG*
f_4	*E*	*A*	*VB*
f_5	*B*	*A*	*G*

In this example, the set of alternatives is the set of cars and the set of criteria is the set of characteristics (price, comsumption, security) that will play a role in the decision. For each criterion, the set X_i takes six different values (VB, B, A, G, VG, E). Now, in order to have numerical values, we can define $u_i : X_i \mapsto \mathbb{R}$ by

$$u_i(VB) = 0,\ u_i(B) = 1,\ u_i(A) = 2,\ u_i(G) = 3,\ u_i(VG) = 4,\ u_i(E) = 5,\ i = 1, 2, 3.$$

Finally, if the aggregation function is defined by the sum of partial numerical scores, we have that the overall evaluations are

$$V(f_1) = 6,\ V(f_2) = 9,\ V(f_3) = 8,\ V(f_4) = 7,\ V(f_5) = 6.$$

Then, the best car is the second one.

Assume that there is more than one criterion; then, the decision should depend on the evaluations of each alternative over each criterion. It must be remarked that in a Multicriteria Decision problem we have no uncertainty. The difficulty of the problem comes from the definition of the aggregation function.

The idea consists in assigning to any object $x \in X$, a representative numerical value aggregating the values of x over each criterium. Thus, we would obtain a total order over the set of alternatives, and the best object should be the one with maximal representative value. In other words, we want to build an aggregation function $V : X \mapsto \mathbb{R}$ such that

$$x \succeq y \Leftrightarrow V(x) \geq V(y),$$

where $x \succeq y$ represents the fact that alternative x is preferred to alternative y.

There are a lot of functions aggregating scores. A simple way to solve this problem is to consider V defined by the aggregation of unidimensional utility functions. As we are in the finite case, this reduces to

$$V(x_1, ..., x_n) = \mathcal{H}(u_1(x_1), ..., u_n(x_n)),$$

where $\mathcal{H}$ is called an *aggregation operator*. Usually, the values of $u_i(x_i)$ are supposed to be normalized, i.e. in [0,1]. Then, aggregation operators are defined by:

Definition 2.21. [41, 42] An **aggregation operator (or aggregation operation)** is a function $\mathcal{H} : [0,1]^n \mapsto [0,1]$ satisfying:

- $\mathcal{H}(0, ..., 0) = 0$, $\mathcal{H}(1, ..., 1) = 1$.
- $\mathcal{H}$ is non decreasing for any argument.

Of course, an aggregation operator can be required to satisfy some additional properties. Usual properties required for aggregation operators are [43]:

- **Idempotence**:
$$\mathcal{H}(c, ..., c) = c,\ \forall c \in [0,1].$$
- **Conmutativity (or symmetry)**: For any permutation π on the set of indices, it is
$$\mathcal{H}(f_1, ..., f_n) = \mathcal{H}(f_{\pi(1)}, ...f_{\pi(n)}).$$
- **Decomposition**:
$$\mathcal{H}^n(f_1, ..., f_k, f_{k+1}, ..., f_n) = \mathcal{H}^n(a, ..., a, f_{k+1}, ..., f_n),$$
with $a = \mathcal{H}^k(f_1, ..., f_k)$.

Other *behavioural properties* can be asked, namely:

- $\mathcal{H}$ must be able to assign weights to the different criteria if necessary.
- $\mathcal{H}$ is able to model the decision maker way of acting (tolerance, intolerance).
- $\mathcal{H}$ is easy to understand, i.e. we can derive some properties about the way the decision maker acts.
- $\mathcal{H}$ is able to model compensatory effects, as interaction among criteria.

Let us see some examples of aggregation operators.

- Quasi-arithmetic means:

$$\mathcal{H}_h(f) = h^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} h(f_i)\right],$$

with $h : \mathbb{R} \mapsto \mathbb{R}$ being a continuous strictly increasing function. This family includes usual means as arithmetic means, geometric means, ...

- Weighted minimum and maximum:

$$wmax_{w_1,\dots,w_n}(f_1, ..., f_n) = \bigvee_{i=1}^{n} [w_i \wedge f_i]$$

$$wmin_{w_1,\dots,w_n}(f_1, ..., f_n) = \bigwedge_{i=1}^{n} [(1 - w_i) \vee f_i],$$

for $w_1, ..., w_n \in [0, 1]$ and $\bigvee_{i=1}^{n} w_i = 1$.

- Ordered weighted averaging (OWA) operators operators :

$$\mathrm{OWA}_w(f) := \sum_{i=1}^{n} w_i f_{(i)},$$

where w is a weight vector, $w = (w_1, \dots, w_n) \in [0, 1]^n$, such that $\sum_{i=1}^{n} w_i = 1$ and the $f_{(i)}$ are defined as in the Choquet Integral.

2.4.2.1 Choquet integral as aggregation operator

We turn now to Choquet integral. The Choquet integral is an aggregation operator. Moreover, this integral satisfies the following properties (see [29, 42, 44] for the proofs):

Proposition 2.22. *Let $\mathcal{C}_\mu$ be the Choquet integral with respect to a normalized fuzzy measure μ, then:*

i) $\min(f_1, ..., f_n) \leq \mathcal{C}_\mu(f_1, ..., f_n) \leq \max(f_1, ..., f_n)$, $\forall \mu$.

ii) Choquet integral is stable for linear transformations.

iii) If μ is additive, $\mathcal{C}_\mu$ is the weighted arithmetic mean where the weights w_i are given by $\mu(\{i\})$.

iv) $\mathcal{C}_\mu = OWA_w$ *if and only if* μ *is symmetric, with* $w_i = \mu(C_{n-i+1}) - \mu(C_{n-i})$, $i = 2, \ldots, n$, *and* $w_1 = 1 - \sum_{i=2}^{n}$, *where* C_i *is any subset of* X *with* $|C_i| = i$.

v) *Choquet integral contains all statistical orderings (in particular the maximum and minimum).*

Based on the last proposition, in Section 2.6.3, we give a generalization of OWA operator to k-symmetric OWA operators. To achieve that, we need to define k-symmetric measures which generalize the concept of symmetry for fuzzy measures.

As seen above, Choquet integral can be considered and used as aggregation operator. In addition, Choquet integral is able to represent interaction among different criteria (see [29, 44]).

Definition 2.23. Let μ be a fuzzy measure on X. The **interaction index** between the elements i and j is defined by

$$I_{ij} = \sum_{A \subset X \setminus \{i,j\}} \frac{(n - |A| - 2)!|A|!}{(n-1)!} \left[\mu(A \cup \{i,j\}) - \mu(A \cup \{i\}) - \mu(A \cup \{j\}) + \mu(A)\right].$$

When $I_{ij} > 0$ (resp. < 0), it is said that i and j are *complementary* (resp. *substitutive*) or that there exists a *cooperation* (resp. a *conflict*). When $I_{ij} = 0$, criteria i and j are said to be *independent*.

This definition can be generalized to general subsets (see [45]).

Definition 2.24. Let μ be a fuzzy measure on X. The **Shapley interaction index** of $A \subset X$, is defined by:

$$I_\mu(A) = \sum_{B \subset X \setminus A} \frac{(n-b-a)!b!}{(n-a+1)!} \sum_{C \subset A} (-1)^{a-c} \mu(B \cup C),$$

with $a = |A|$, $b = |B|$ and $c = |C|$.

The Shapley interaction index is an alternative representation of fuzzy measures. For some subfamilies of fuzzy measures, the Choquet integral with respect to a fuzzy measure μ only depends on the Shapley interaction indexes of 2-elements subsets, which correspond to interaction indexes I_{ij}. This is the case of 2-additive measures (see Section 2.6.4).

We finish studying the role of fuzzy measures in modelling vetos, favors and interactions among criteria.

Definition 2.25. [46] Suppose $\mathcal{H}$ is an aggregation operator. A criterion i is a **veto** for $\mathcal{H}$ if for any n-uple $(x_1, ..., x_n) \in \mathbb{R}^n$ of scores

$$\mathcal{H}(x_1, ..., x_n) \leq x_i.$$

Similarly, criterion i is a **favor** if for any n-uple $(x_1, ..., x_n)$ of scores,

$$\mathcal{H}(x_1, ..., x_n) \geq x_i.$$

The concepts of veto and favor can be generalized to subsets as follows:

Definition 2.26. [47] Suppose $\mathcal{H}$ is an aggregation operator. A subset A is a **veto** for $\mathcal{H}$ if for any n-uple $(x_1, ..., x_n) \in \mathbb{R}^n$ of scores

$$\mathcal{H}(x_1, ..., x_n) \leq \bigwedge_{i \in A} x_i.$$

Similarly, subset A is a **favor** if for any n-uple $(x_1, ..., x_n)$ of scores,

$$\mathcal{H}(x_1, ..., x_n) \geq \bigvee_{i \in A} x_i.$$

If criterion i is a veto, it means that if the evaluation over i is high, it has no effect on the evaluation, but if it is low, then the global score will be low too; similarly we can obtain a dual interpretation for favor. These two concepts have been already proposed in the context of social choice functions [48].

A veto and a favor can be represented by a fuzzy measure as follows:

Proposition 2.27. *[47] If we consider as aggregation operator the Choquet integral, i is a veto if and only if the fuzzy measure satisfies $\mu(A) = 0$ whenever $i \notin A$. Similarly, i is a favor if and only if the fuzzy measure satisfies $\mu(A) = 1$ whenever $i \in A$, or equivalently, when $\mu(i) = 1$.*

In terms of Shapley interaction, vetos and favors are characterized for 2-additive measures (see Section 2.6.4) by:

Proposition 2.28. *[47] Let μ be a 2-additive measure. Criterion i is a veto for the Choquet integral if and only if the following conditions are satisfied:*

1. $I_{ik} \geq 0, \forall k \neq i.$
2. $I_{kl} = 0, \forall k, l \neq i.$

3. *$\varphi_k = \frac{1}{2}I_{ik}$, $\forall k \neq i$.*

Similarly, i is a favor if and only if

1. *$I_{ik} \leq 0$, $\forall k \neq i$.*

2. *$I_{kl} = 0$, $\forall k, l \neq i$.*

3. *$\varphi_k = -\frac{1}{2}Iik$, $\forall k \neq i$.*

where φ_i is the Shapley value of the measure (see Section 2.7.1.2).

Note that it is not possible to have a veto on i and a favor on j since $\mathcal{H}(x_1, ..., x_n) \leq x_i$ and $\mathcal{H}(x_1, ..., x_n) \geq x_j$ are not compatible in general.

2.5 The Möbius Transform

Definition 2.29. [49] Let μ be a set function (not necessarily a fuzzy measure) on X. The **Möbius transform (or inverse)** of μ is another set function on X defined by

$$m^\mu(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \; \forall A \subseteq X.$$

We write m instead of m^μ when the measure μ is clear from the context.

The Möbius transform is an alternative representation of fuzzy measures, in the sense that given the Möbius transform m of a fuzzy measure μ, this measure can be recovered through the *Zeta transform* [50]:

$$\mu(A) = \sum_{B \subseteq A} m^\mu(B).$$

The Möbius transform corresponds to the *basic probability mass assignment* in Dempster-Shafer theory of evidence [51] and the Harsanyi *dividends* [52] in Cooperative Game Theory. From the point of view of Game Theory, the Möbius transform gives a measure of the importance of a coalition by itself, without taking account of its different parts. As explained before, the Möbius transform can be applied to any set function; if related to a fuzzy measure, the Möbius transform can be characterized as follows.

Proposition 2.30. *[50] A set of 2^n coefficients $m(A)$, $A \subseteq N$, corresponds to the Möbius transform of a normalized fuzzy measure if and only if*

1. $m(\emptyset) = 0, \sum_{A \subseteq N} m(A) = 1,$

2. $\sum_{i \in B \subseteq A} m(B) \geq 0$, *for all* $A \subseteq N$, $\forall i \in A$.

Let us see some other important properties of the Möbius transform [29].

Proposition 2.31. *Let v be a grounded set function (also called game, see Section 2.7):*

i) *v is additive if and only if $m^v(A) = 0$ for all $A \subseteq X, |A| > 1$.*

ii) *v is monotone if and only if for all $K \subseteq X$, $i \in K$ holds*

$$\sum_{i \in L \subseteq K} m^v(L) \geq 0.$$

iii) *Let $k \geq 2$ be fixed. v is k-monotone if and only if for all $A, B \subseteq X$, $A \subseteq B$ and $2 \leq |A| \leq k$*

$$\sum_{L \in [A,B]} m^v(L) \geq 0.$$

iv) *If v is k-monotone for some $k \geq 2$, then $m^v \geq 0$ for all $A \subseteq X$ such that $2 \leq |A| \leq k$.*

v) *v is a nonnegative totally monotone game if and only if $m^v \geq 0$.*

vi) *The Möbius transform of a unanimity game u_A is*

$$m^{u_A}(B) = \begin{cases} 1, & \text{if } B = A \\ 0, & \text{otherwise.} \end{cases}$$

vii) *The coefficients of expressing a measure μ using the basis of unanimity games are given by the Möbius transform m^μ, i.e. $\mu = \sum_{A \subseteq X} m^\mu(A) u_A$.*

viii) *The polytope of belief measures $\mathcal{B}(X)$ expressed using the basis of unanimity games read as follows*

$$\mathcal{B}(X) = \{m \in \mathbb{R}^{2^{|X|}-1} \mid m(A) \geq 0,\ \emptyset \neq A \subseteq X\ \wedge \sum_{A \subseteq X, A \neq \emptyset} m(A) = 1\}.$$

ix) *The Möbius transform of λ-measure μ is given by*

$$m^\mu(A) = \lambda^{|A|-1} \prod_{i \in A} \mu(i),\ A \neq \emptyset.$$

Normalized capacities take their value in $[0, 1]$. One may think that the Möbius transform of normalized capacities takes values in the symmetrized interval $[-1, 1]$. The following result shows that this is not true [29].

Theorem 2.32. *For any normalized capacity μ, its Möbius transform satisfies for any $A \subseteq N$, $|A| > 1$:*

$$-\binom{|A|-1}{l'_{|A|}} \leqslant m^{\mu}(A) \leqslant \binom{|A|-1}{l_{|A|}},$$

with

$$l_{|A|} = 2\left\lfloor \frac{|A|}{4} \right\rfloor, \quad l'_{|A|} = 2\left\lfloor \frac{|A|-1}{4} \right\rfloor + 1 \tag{2.2}$$

and for $|A| = 1 < n$:

$$0 \leqslant m^{\mu}(A) \leqslant 1,$$

and $m^{\mu}(A) = 1$ if $|A| = n = 1$. These upper and lower bounds are attained by the normalized capacities μ^*_A, μ_{A*}, respectively:

$$\mu^*_A(B) = \begin{cases} 1, & \text{if } |A| - l_{|A|} \leqslant |B \cap A| \leqslant |A| \\ 0, & \text{otherwise} \end{cases}, \quad \mu_{A*}(B) = \begin{cases} 1, & \text{if } |A| - l'_{|A|} \leqslant |B \cap A| \leqslant |A| \\ 0, & \text{otherwise} \end{cases}$$

for any $B \subseteq N$.

We give in Table 2.2 the first values of the bounds. Using the well-known Stirling's

$\lvert A\rvert$	1	2	3	4	5	6	7	8	9	10	11	12
u.b. of $m^{\mu}(A)$	1	1	1	3	6	10	15	35	70	126	210	462
l.b. of $m^{\mu}(A)$	1(0)	-1	-2	-3	-4	-10	-20	-35	-56	-126	-252	-462

TABLE 2.2: Lower and upper bounds for the Möbius transform of a normalized capacity

approximation $\binom{2n}{n} \simeq \frac{4^n}{\sqrt{\pi n}}$ for $n \to \infty$, we deduce that

$$-\frac{4^{\frac{n}{2}}}{\sqrt{\frac{\pi n}{2}}} \leqslant m^{\mu}(N) \leqslant \frac{4^{\frac{n}{2}}}{\sqrt{\frac{\pi n}{2}}}$$

when n tends to infinity.

2.6 Subfamilies of fuzzy measures

As we have explained before, fuzzy measures can be applied in many different fields. However, this flexibility has to be paid via an increment in the complexity; in this sense, for a referential set of n elements, $2^n - 2$ values are needed to completely define the fuzzy measure. In order to reduce this complexity, several attempts have been proposed. For example, we could just reduce the number of coalitions allowed to happen; this situation is perfectly justified in many cases in Game Theory (see e.g., [53]). Another way to reduce the complexity is to consider some subfamilies of fuzzy measures, aiming to combine flexibility with a reduction in the number of coefficients; following this line, many subfamilies have been proposed, as for example k-additive measures [45, 54], k-symmetric measures [55], k-intolerant measures [56], and many others. In this section we study some of the most important subfamilies of fuzzy measures. Let us start with some of them.

Definition 2.33. A fuzzy measure μ on X is

i) **superadditive** if $\forall A, B \subseteq X$, such that $A \cap B = \emptyset$,

$$v(A \cap B) \geq \mu(A) + \mu(B).$$

The fuzzy measures is **subadditive** if $\mu(A \cap B) \leq \mu(A) + \mu(B)$.

ii) **supermodular** if $\forall A, B \subseteq X$,

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

The fuzzy measures is **submodular** if the reverse inequality holds. Supermodular measures are also called **convex measures.** A fuzzy measure that is both supermodular and submodular is said to be modular.

iii) **maxitive** if $\forall A, B \subseteq X$,

$$\mu(A \cup B) = \max\{\mu(A), \mu(B)\}.$$

iv) **minitive** if $\forall A, B \subseteq X$,

$$\mu(A \cap B) = \min\{\mu(A), \mu(B)\}.$$

v) a **possibility (necessity)** measure if it is normalized and maxitive (minitive).

Remark 2.34.

- The last definitions can also be applied to general set functions.
- Super(sub)modularity implies super(sub)additivity but not the converse [29]. Also, it is easy to check that additivity and modularity are equivalent properties [29]. This is no longer true when fuzzy measures are defined on subcollections of $\mathcal{P}(X)$ [29].

- 2-monotonicity corresponds to supermodularity, while the 2-alternating property corresponds to submodularity.
- If μ is a superadditive grounded set function, then $\overline{\mu} \geq \mu$.
- A λ-measure is superadditive if $\lambda > 0$, and subadditive otherwise.
- The set of supermodular grounded set functions (as well as the set of submodular grounded set functions) is a non-pointed cone [29].
- If $\mu \geq 0$ is a supermodular grounded set function, then μ is monotone.
- μ is maxitive if and only if $\overline{\mu}$ is minitive.
- The set of possibility (or necessity) measures is a polytope affinely isomorphic to the order polytope $\mathcal{O}(\overline{\mathbf{n}}) = \mathcal{C}_n$. Indeed, it is sufficient to specify the value of each singleton $\Pi(A) = \bigvee_{x \in A} \Pi(\{x\})$ and $Nec(A) = \bigwedge_{x \in A} Nec(\{x\})$.

Figure 2.2 shows a diagram with the relations between the different subfamilies of normalized fuzzy measures.

As the set of normalized fuzzy measures is a polytope, whichever property from above may be mixed with normalized fuzzy measures to get new polytopes. For example, the polytope of normalized supperadditive games. Removing $\mu(X) = 1$, leads to convex cones, as the set of non-normalized fuzzy measures that we will study later in Chapter 6.

2.6.1 Fuzzy Measures with restricted cooperation

Let $\xi : \mathcal{P}(X) \to \mathbb{R}$, be a set function. Sometimes there are situations in which the players of some coalition $A \subseteq X$ do not cooperate together, thus A is not in the domain of ξ. In these cases we say that the problem has restricted cooperation and the domain of ξ is changed for a general collection of subsets Ω. Limiting the number of coalitions is also a good way to reduce the complexity of the problem.

Definition 2.35. Let X be a finite set and $\Omega \subseteq \mathcal{P}(X)$. We define a **$\Omega$-restricted set function** as a set function on Ω, $\xi : \Omega \to \mathbb{R}$.

All the definitions on fuzzy measures seen previously could be generalized to the Ω-restricted case. One can think of Ω as the collection of feasible coalitions. If $\emptyset, X \in \Omega$ then Ω is said to be a **set system** [53]. Observe that $(\Omega, \subseteq)$ is a poset and the set of **Ω-restricted normalized fuzzy measures** $\mathcal{RFM}(\Omega)$ is the order polytope

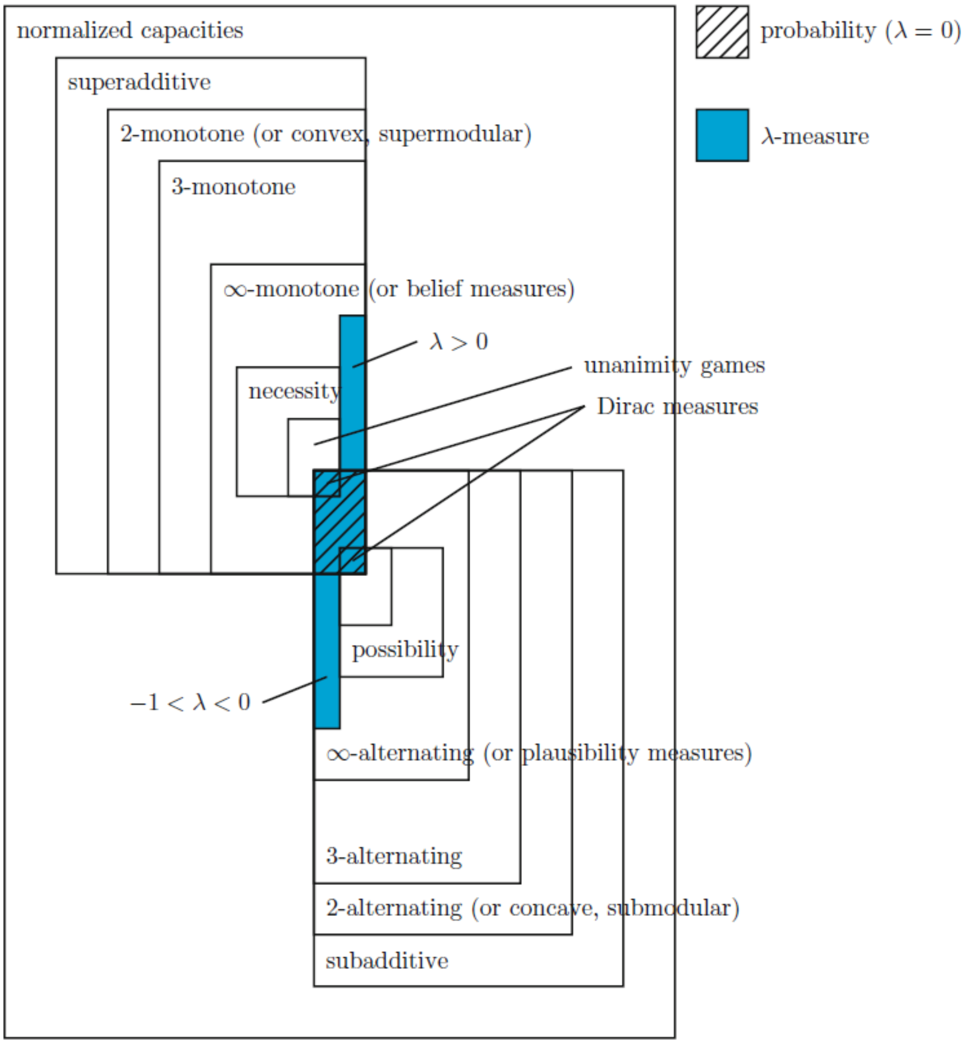


Figure 2.2: Several families of normalized fuzzy measures on a finite set.

$\mathcal{O}(\Omega \setminus \{\emptyset, X\})$. Since any poset is isomorphic to a subposet of the Boolean lattice, for every poset P there is some collection of feasible coalitions $\Omega(P)$ such that $\mathcal{RFM}(\Omega(P)) = \mathcal{O}(P \setminus \{\emptyset, X\})$. In other words, every poset P models some problem with possibly restricted cooperation. Obviously, when $\Omega = \mathcal{P}(X) \Rightarrow \mathcal{RFM}(\Omega) = \mathcal{FM}(X)$.

In the rest of the work we will be devoted to the study of the geometry of particular choices of the poset of feasible coalitions Ω.

2.6.2 k-intolerant fuzzy measures

Here we introduce the k-intolerant and k-tolerant fuzzy measures [56].

Definition 2.36. Let $|X| = n$ and $k \in [n]$. A fuzzy measure μ is **k-intolerant** if $\mu(A) = 0$ for all $A \subseteq X$ such that $|A| \leq n-k$ and there is $A^* \subseteq X$, with $|A^*| = n-k+1$, such that $\mu(A^*) \neq 0$. Symmetrically, μ is **k-tolerant** if $\mu(A) = 1$ for all $A \subseteq X$ such that $|A| \geq k$ and there is $A^* \subseteq X$, with $|A^*| = k-1$, such that $\mu(A^*) \neq 1$.

Observe that assigning a specific value to a set A is equivalent to remove this set from the set of feasible coalitions. Therefore the set of normalized k-intolerant measures can be identified with the Ω-restricted normalized measures set $\mathcal{RFM}(\Omega)$ such that $\Omega = \{A \in \mathcal{P}(X) \mid |A| > n-k\}$. Hence, the geometric properties of k-intolerant measures are the same as the properties of the order polytope $\mathcal{O}(\Omega)$.

Similarly, for the k-tolerant fuzzy measures we get $\mathcal{RFM}(\Omega)$ such that $\Omega = \{A \in \mathcal{P}(X) \mid |A| < k\}$. Again we get an order polytope. All the theory about order polytopes from Chapter 1 (see Section 1.3) can be applied to know the combinatorial structure (vertices, faces, etc.) of the order polytopes associated to k-intolerant and k-tolerant measures. In particular, the linear extensions of poset $(\Omega, \subseteq)$ gives us a triangulation (see Section 1.3.1) of these order polytopes and therefore a way to sample points uniformly inside it. We will study more closely how to sample points inside these polytopes in Chapter 3.

2.6.3 k-symmetric measures

Let us now introduce the concept of k-symmetry. For this concept, the notion of subset of indifference is the crux of the issue.

Definition 2.37. [55] Let $\mu : \mathcal{P}(X) \to [0,1]$ be a fuzzy measure. Given a subset $A \subseteq X$, we say that A is a **subset of indifference** for μ if for any $B_1, B_2 \subset A$, such that $|B_1| = |B_2|$ and for any $C \subseteq X \setminus A$, then

$$\mu(B_1 \cup C) = \mu(B_2 \cup C).$$

From the point of view of Game Theory, this definition translates the idea that we do not care about which elements inside A are included in a coalition, and we just need to know how many of them take part in it. Remark that if A is a subset of indifference, so is any $B \subseteq A$. Moreover, for a given fuzzy measure μ, X can always be partitioned in $\{A_1, ..., A_k\}$ subsets of indifference for a suitable value of k; there may be several partitions of X in subsets of indifference, but it can be proved that there is a partition that is the coarsest one [55], being any other refinement of this partition another partition in subsets of indifference for μ.

Definition 2.38. [55] Let $\mu \in \mathcal{FM}(X)$. We say that μ is a $k-$**symmetric measure** if and only if the (unique) coarsest partition of the referential set in subsets of indifference has k non-empty subsets.

In particular, if we consider the partition $\{X\}$ we recover the set of symmetric fuzzy measures, i.e., measures μ satisfying $\mu(A) = \mu(B)$ whenever $|A| = |B|$. These measures and their corresponding Choquet integral are very important in Fuzzy Logic, where they are related to OWA operators [57]. Therefore, k-symmetric measures define a gradation between symmetric measures (where we just care about the number of elements in the subset) and general fuzzy measures.

The Proposition 2.22 shows the equivalence of the OWA operators with symmetric Choquet integrals, themselves being bijectively related to symmetric capacities. Following this line, we can generalize the concept of OWA operator by considering Choquet integrals with respect to k-symmetric measures.

Definition 2.39. An operator $\mathcal{H}$, is said to be a k-symmetric OWA if there is a k-symmetric normalized capacity such that $\mathcal{H} = \mathcal{C}_\mu$.

The expression of a k-symmetric OWA is given in the next proposition [42].

Proposition 2.40. *Let μ be a k-symmetric fuzzy measure on X with indifference partition $\{A_1, \dots, A_k\}$. Then, for all $\boldsymbol{x} = (x_1, \dots, x_n) \in [0,1]^n$, the Choquet integral w.r.t μ is given by*

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{k} \mathcal{C}_{\mu_{|A_i}}(\boldsymbol{x}_{|A_i}) + \sum_{A \nsubseteq A_j, \forall j} m^\mu(A) \bigwedge_{i \in A} x_i,$$

where $\mu_{|A_i}$ is the restriction of μ to A_i.

If $\{A_1, A_2, \cdots, A_k\}$ is a partition of X, the set of all fuzzy measures μ such that $\{A_1, A_2, \cdots, A_k\}$ is a partition of indifference for μ is denoted by $\mathcal{FM}(A_1, A_2, \cdots, A_k)$. Note that $\mu \in \mathcal{FM}(A_1, A_2, \cdots, A_k)$ does not imply that $\{A_1, A_2, \cdots, A_k\}$ is the coarsest partition in subsets of indifference for μ; indeed, all symmetric measures belong to $\mathcal{FM}(A_1, A_2, \cdots, A_k)$, no matter the partition.

As all elements in a subset of indifference have the same behavior, when dealing with a fuzzy measure in $\mathcal{FM}(A_1, ..., A_k)$, it suffices to know the number of elements of each A_i that belong to a given subset C of the referential set X. Therefore, the following result holds:

Lemma 2.41. *[55] If $\{A_1, ..., A_k\}$ is a partition of X, then any $C \subseteq X$ can be identified with a k-dimensional vector $(c_1, ..., c_k)$ with $c_i := |C \cap A_i|$.*

Then, $c_i \in \{0, ..., |A_i|\}$ and in order to build a $k-$symmetric measure we just need

$$(|A_1| + 1) \times (|A_2| + 1) \times \cdots \times (|A_k| + 1) - 2$$

coefficients, a number far away from $2^n - 2$ needed for a general fuzzy measure.

Observe that the set $\mathcal{FM}(A_1, ..., A_k)$ is the order polytope associated to the poset $(P(A_1, ..., A_k), \preceq)$, where

$$P(A_1, ..., A_k) := \{(i_1, ..., i_k) : i_j \in \{0, ..., |A_j|\}\} \backslash \{(0, ..., 0), (|A_1|, ..., |A_k|)\}, \tag{2.3}$$

and $\preceq$ is given by $(c_1, ..., c_k) \preceq (b_1, ..., b_k) \Leftrightarrow c_i \leq b_i, i = 1, ..., k$ (see [23]). The Hasse diagrams of $P(\{1, 2\}, \{3, 4\})$ is given in Figure 2.3.

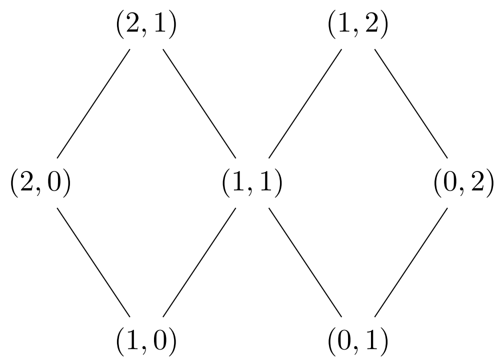


FIGURE 2.3: Hasse diagrams of the poset $P(\{1, 2\}, \{3, 4\})$.

Thus, the problem of random generation of points in $\mathcal{FM}(X)$ and $\mathcal{FM}(A_1, ..., A_k)$ reduces to obtain a random procedure for generating points in an order polytope (see Section 1.3.1). In Chapter 4 we will study more about k-symmetric measures, in particular about 2-symmetric measures.

2.6.4 k-additive games

As explained above, the Möbius transform gives a measure of the importance of a coalition by itself, without taking account of its different parts. Sometimes it could be difficult for an expert to assess values to the interactions of say, 4 criteria, and interpret what these interactions mean. Then, it makes sense to restrict the range of allowed interactions to coalitions of a reduced number of criteria, i.e. no interactions among more than k criteria are permitted. This translates into the condition $m(A) = 0$ if $|A| > k$. Based on this fact, we arrive to the concept of k-additivity in a natural way.

Definition 2.42. [54] A fuzzy measure μ is said to be k**-additive** if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A of exactly k elements such that $m(A) \neq 0$.

From this definition, it follows that a probability measure is just a 1-additive measure; therefore, k-additive measures generalize probability measures and constitute a gradation between probability measures and general fuzzy measures (n-additive measures). For a k-additive measure, the number of coefficients is reduced to

$$\sum_{i=1}^{k} \binom{n}{i} - 1,$$

a middle term between $n-1$ (probabilities) and 2^n-2 (general fuzzy measures). We will denote by $\mathcal{FM}^k(X)$ the set of all normalized fuzzy measures being *at most* k-additive. Specially appealing is the 2-additive case, that allows to model interactions between two criteria, that are the most important ones, while keeping a reduced (indeed quadratic) complexity. The same considerations can be done for non-normalized measures denoting the set of at most k-additive measures as $\mathcal{MG}^k(X)$. Moreover, simple expressions for μ in terms of the Möbius transform, and the corresponding Choquet integral can be obtained (see [54]). For example, for 2-additive measures the following holds:

Proposition 2.43. *Let μ a 2-additive fuzzy measure and $f = (f_1, \ldots, f_n)$ takes positive values. Then,*

$$\mathcal{C}_\mu(f) = \sum_{i \in X} \varphi_i(\mu) f_i - \frac{1}{2} \sum_{\{i,j\} \subseteq X} I_{ij} |f_i - f_j|,$$

where $\varphi_i(\mu)$ is the Shapley value of μ (see Section 2.7.1.2) and the interaction indexes are the ones of Definition 2.23.

Unlike other subfamilies of fuzzy measures, k-additive measures are not order polytopes. Hence, we cannot use the theory of order polytopes to study the geometry of k-additive measures. We will study in more depth the k-additive measures in Chapter 5, in particular we will see the combinatorial structure of 2-additive measures and a method to triangulate the 2-additive measures polytope.

2.7 Games

A grounded set function is also called a **game**. We denote the set of games on X by $\mathcal{G}(X)$. The set of games $\mathcal{G}(X)$ has a natural structure of vectorial space of dimension $2^{|X|} - 1$. The collection of Dirac games $\{\delta_A\}_{A \in \mathcal{P}(X) \setminus \{\emptyset\}}$ and the collection of unanimity measures $\{u_A\}_{A \in \mathcal{P}(X) \setminus \{\emptyset\}}$ are bases.

Observe that a fuzzy measure is just a monotone game. In other words, a monotone game is a non-normalized fuzzy measure. Let us denote the set of fuzzy measures, that is the set of monotone games, as $\mathcal{MG}(X)$. In what follows, we will use ξ for general set functions, v for games and μ for fuzzy measures.

The value $v(A)$ represents the minimum gain that can be achieved by cooperation of the players in the coalition A, even if the other players do everything possible against coalition A.

All properties and subfamilies defined in the previous sections can be straightforwardly extended in the framework of cooperative games. Some of these sets are convex cones. We will study the properties of some of these cones in Chapter 6.

In the case of restricted cooperation, $\Omega \subset \mathcal{P}(X)$, the set of **$\Omega$-restricted monotone games** $\mathcal{MG}_{\Omega}(X)$ is a cone whose geometric properties depend on the poset $(\Omega, \subseteq)$, as we will see in Chapter 6.

2.7.1 Cooperative Game Theory

In cooperative game theory each element i in the referential set X represents a player and each subset $A \subseteq X$ represents a coalition [29]. The main assumption in cooperative game theory is that the total coalition X will form. The challenge is then to allocate the total payoff $v(X)$ of the game v among the players in some fair way satisfying $\sum_{i=1}^{|X|} x_i = v(X)$. There are several ways of allocating this total payoff. Normally, these methods are classified into two big groups of solutions: one-point solutions and set solutions. The most famous concepts of solutions are the core (set solution) and the Shapley value (one-point value).

2.7.1.1 Core of games

A set solution to the payoff allocation problem is given by a set of "reasonable" payoffs according to some criterion. Let us denote by $\mathcal{G}(X, \Omega)$ the set of Ω-restricted games being Ω a set system. The **core** of a game $v \in \mathcal{G}(X, \Omega)$ is defined by

$$\mathbf{core}(v) = \{x \in \mathbb{R}^{|X|} \ : \ x(S) \geq v(S), \ \forall S \in \Omega, \ x(X) = v(X)\},$$

where $x(S)$ is a shorthand for $\sum_{i \in S} x_i$. By convention, $x(\emptyset) = 0$. Note that the $\mathbf{core}(v)$ is a convex polyhedron which may be empty. We say that a collection $\mathcal{B}$ of non-empty sets is a **balanced collection** if given $A \in \mathcal{B}$, there exists $\lambda_A > 0$ satisfying $\sum_{A \in \mathcal{B}, \ i \in A} \lambda_A = 1$,

$\forall i \in X$. In shorter form $\sum_{A\in\mathcal{B}} \lambda_A \mathbf{1}_A = \mathbf{1}_X$. We say that the quantities λ_A form a **system of balancing weights.** Here we use the notation $y(A) = \sum_{i\in A} y_i$.

Theorem 2.44. *[29] A collection $\mathcal{B}$ of nonempty sets is balanced if and only if for every vector $y \in \mathbb{R}^{|X|}$ such that $y(X) = 0$, either $y(S) = 0$ for every $S \in \mathcal{B}$ or there exist $S, T \in \mathcal{B}$ such that $y(S) > 0$ and $y(T) < 0$.*

A game v on X is balanced if for any balanced collection $\mathcal{B}$ it holds

$$v(X) \geq \sum_{A\in\mathcal{B}} \lambda_A v(A).$$

Theorem 2.45 (Bondareva-Shapley Theorem, weak form). *[29] Let $v \in \mathcal{G}(X)$. Then **core**(v) is nonempty if and only if v is balanced.*

We say that a balanced collection is minimal if it does not contain a proper subcollection that is balanced.

Theorem 2.46 (Bondareva-Shapley Theorem, sharp form). *[29] Let $v \in \mathcal{G}(X)$. Then **core**(v) is nonempty if and only if for any minimal balanced collection $\mathcal{B}$ with system of balancing weights $\{\lambda_A\}_{A\in\mathcal{B}}$, we have*

$$v(X) \geq \sum_{A\in\mathcal{B}} \lambda_A v(A).$$

Let $\sigma \in \mathcal{S}_n$ and $v \in \mathcal{G}(X)$ we define its **marginal vector** $x^{\sigma,v} \in \mathbb{R}^{|X|}$ as $x^{\sigma,v}_{\sigma(i)} = v(\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(i-1)\})$. The convex hull of the marginal vectors is called the Weber set, $\mathbf{Web}(v) = \mathrm{Conv}(x^{\sigma,v} : \sigma \in \mathcal{S}_n)$. The Weber set is another important set solution highly related to the core.

Proposition 2.47. *i) For any game $v \in \mathcal{G}(X)$, **core**$(v) \subseteq$ **Web**(v).*

*ii) If v, v' are supermodular games, then **core**(v) + **core**(v') = **core**$(v + v')$.*

*iii) **core**(v) + **core**$(v') \subseteq$ **core**$(v + v')$, where v, v' are balanced games.*

*iv) **Web**(v) + **Web**$(v') \supseteq$ **Web**$(v + v')$, where v, v' are balanced games.*

Theorem 2.48 (Structure of the core of supermodular games). *Let v be a game in $\mathcal{G}(X)$. The following are equivalents.*

i) v is supermodular.

*ii) $x^{\sigma,v} \in$ **core**(v) for all $\sigma \in \mathcal{S}_n$.*

iii) ***core***$(v) =$ ***Web***(v).

iv) The vertives of ***core***(v) *are the marginal vectors* $x^{\sigma,v}$ *with* $\sigma \in \mathcal{S}_n$.

This way the core equals the Weber set for supermodular games. In this case its vertices are the marginal vectors. For references on other famous concepts of set solutions as the selectope see [29].

2.7.1.2 Shapley value

Shapley value is one of the most famous solution concepts in game theory. This is a one-point solution since the value gives just one possible payoff allocation. Let $v \in \mathcal{G}(X)$ then its **Shapley value** [29] is given by:

$$\varphi_i(v) = \sum_{S \subseteq X \setminus \{i\}} \frac{|S|!\,(|X| - |S| - 1)!}{|X|!}(v(S \cup \{i\}) - v(S)).$$

Theorem 2.49. *[29] Let* $v \in \mathcal{G}(X)$. *The Shapley value* $\varphi_i(v)$ *is the only payment rule satisfying:*

i) **Efficiency** *:* $\sum_{i \in X} \varphi_i(v) = v(X)$.

ii) **Symmetry** *: If* $i, j \in X$ *are equivalent in the sense that* $v(S \cup \{i\}) = v(S \cup \{j\})$, $\forall S \subseteq X \setminus \{i, j\}$ *then* $\varphi_i(v) = \varphi_j(v)$.

iii) **Linearity** *:* $\varphi_i(v + w) = \varphi_i(v) + \varphi_i(w)$ *for all* $i \in X$.

iv) **Null player** *: If* i *is a player such that* $v(S \cup \{i\}) = v(S)$, $\forall S \subseteq X \setminus \{i\}$ *then* $\varphi_i(v) = 0$.

The Shapley value can also be seen as the solution of some optimization problem.

Theorem 2.50. *Let* $v \in \mathcal{G}(X)$. *Consider the optimization problem:*

$$\min_{\varphi \in \mathbb{R}^X} \sum_{S \subseteq X} \alpha_S \left(v(S) - \varphi(S)\right)^2$$

subject to $\varphi(X) = v(X)$. *Then, the Shapley value is the unique solution when the coefficients* α_S *are given by*

$$\alpha_S = \frac{1}{\binom{|X|-2}{|S|-1}}, \quad \emptyset \neq S \subseteq X.$$

Other alternative definition of the Shapley value can be expressed in terms of all possible orders of the players. Let $\sigma \in \mathcal{S}_n$ be a permutation that assigns to each position k the player $\sigma(k)$. Given a permutation σ, let us denote by $\mathrm{Pre}^i(\sigma)$ the set of predecessors of the player i in the order σ, that is $\mathrm{Pre}^i(\sigma) = \{\sigma(1), \sigma(2), \ldots, \sigma(k-1)\}$ where $i = \sigma(k)$. In this case, the Shapley value can be expressed in the following way:

$$\varphi_i(v) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \left(v\left(\mathrm{Pre}^i(\sigma) \cup i\right) - v\left(\mathrm{Pre}^i(\sigma)\right)\right), \ \forall i \in [n].$$

The last expression is the population mean value of the variable $\chi_i(\sigma) = v\left(\mathrm{Pre}^i(\sigma) \cup i\right) - v\left(\mathrm{Pre}^i(\sigma)\right)$ with respect to the uniform distribution on the set of permutations S_n, that is

$$\varphi_i(v) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi_i(\sigma).$$

Sometimes computing the Shapley value involves too many computations, then we can use statistical sampling theory to estimate the Shapley value, see [58, 59]. In a sampling with replacement scheme and equal probabilities if we choose a sample $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ of m permutations we can estimate the Shapley value as:

$$\widehat{\varphi}_i(v) = \frac{1}{m} \sum_{j=1}^{m} \chi_i(\sigma_j).$$

This estimator is unbiased and consistent with variance $Var(\widehat{\varphi}_i(v)) = \dfrac{\sigma_i^2}{m}$, where $\sigma_i^2 = \dfrac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \left(\chi_i(\sigma) - \varphi_i(v)\right)^2$.

The last construction can be generalized to Ω-restricted games $v \in \mathcal{G}(X, \Omega)$ having the additional structure of augmenting systems [60].

Definition 2.51. A set system $\Omega \subseteq \mathcal{P}(X)$ is an **augmenting system** if

i) $\emptyset \in \Omega$,

ii) for $S, T \in \Omega$ with $S \cap T \neq \emptyset$, we have $S \cup T \in \Omega$,

iii) for $S, T \in \Omega$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup i \in \Omega$.

Let $\Omega \subseteq \mathcal{P}(X)$ be an augmenting system and $v \in \mathcal{G}(X, \Omega)$, the Shapley value for a player $i \in X$ is defined as [60]

$$\varphi_i(v,\Omega) = \frac{1}{C(\Omega)} \sum_{C\in \mathrm{Ch}(\Omega)} \left(v\left(\mathrm{Pre}^i(C)\cup i\right) - v\left(\mathrm{Pre}^i(C)\right)\right),$$

where $\mathrm{Ch}(\Omega)$ is the set of all maximal chains of Ω and $\mathrm{Pre}^i(C)$ is the set in the chain C being inmediately before the smallest set in C containing i. By definition, both $\mathrm{Pre}^i(C)\cup i$ and $\mathrm{Pre}^i(C)$ belong to Ω. This generalization of the Shapley value preserves good axiomatization properties [60]. Let us denote $\chi_i(C) = v\left(\mathrm{Pre}^i(C)\cup i\right) - v\left(\mathrm{Pre}^i(C)\right)$. Recall that maximal chains of distributive lattices are in bijection with the linear extensions of the poset of join-irreducible elements of the lattice (see Equation 1.1). Therefore, if Ω is in addition a distributive lattice with $\mathcal{J}(\Omega)\cong P$ we get

$$\varphi_i(v,\Omega) = \frac{1}{e(P)} \sum_{\epsilon\in \mathcal{L}(P)} \left(v\left(\mathrm{Pre}^i(\epsilon)\cup i\right) - v\left(\mathrm{Pre}^i(\epsilon)\right)\right),$$

where $\mathrm{Pre}^i(\epsilon)$ is the set of element in the linear extension ϵ being inmediately before i. Let us denote $\chi_i(\epsilon) = v\left(\mathrm{Pre}^i(\epsilon)\cup i\right) - v\left(\mathrm{Pre}^i(\epsilon)\right)$. Following the same reasoning as above, sometimes this Shapley value is really difficult to compute since the number of linear extensions could be huge. In this case we can use statistical sampling to get a good estimation of this value. In a sampling with replacement scheme and equal probabilities if we choose a sample $\{\epsilon_1,\epsilon_2,\ldots,\epsilon_m\}$ of m linear extensions of $P\cong\mathcal{J}(\Omega)$ we can estimate the Shapley value as:

$$\widehat{\varphi}_i(v,\Omega) = \frac{1}{m}\sum_{j=1}^{m}\chi_i(\epsilon_j).$$

This estimator is unbiased and consistent with variance $Var(\widehat{\varphi}_i(v,\Omega)) = \dfrac{\sigma_i^2}{m}$, where

$$\sigma_i^2 = \frac{1}{e(P)}\sum_{\epsilon\in\mathcal{L}(P)} \left(\chi_i(\epsilon) - \varphi_i(v,\Omega)\right)^2.$$

This constitutes an application of linear extensions to cooperative game theory.

2.8 Identification of fuzzy measures

Consider a situation that can be modeled by the Choquet integral with respect to an unknown fuzzy measure μ. How can we get μ? This is known as the identification problem. The identification problem when sample data are available can be stated as follows: Consider m objects represented by the functions $f_1, \dots, f_m$. Assume the aggregation operator applied to obtain an overall score is the Choquet integral and that the corresponding value of function f_i is y_i , $i = 1, \dots, m$. Consider a subfamily of fuzzy measures $\mathfrak{F}$. We look for $\mu \in \mathfrak{F}$ minimizing

$$\sum_{i=1}^{m} (\mathcal{C}_\mu(f_i) - y_i)^2$$

That is, we look for the fuzzy measure μ in $\mathfrak{F}$ that best fits our data, with the squared error as a criterion of fitness. Note, however, that other criteria might be applied instead of the squared error. The problem of identifying fuzzy measures from sample data can be written as a quadratic problem if $\mathfrak{F}$ is convex (see [61]) and then solved with the usual methods. This approach will always lead to the exact solution, but it can be very time-consuming and has a strong tendency to overfit, thus leading to bad approximations when there exists some noise in the data. We will study how to overcome these drawbacks using genetic algorithms in Chapter 4.

Chapter 3

BU-feasible posets and random linear extension sampling

3.1 Introduction

In the last chapters we have seen that the problem of sampling uniformly inside a polytope associated to certain subfamilies of fuzzy measures is linked to the problem of sampling linear extensions from posets. This way the problem of generating a point in an order polytope can be turned into the problem of generating a linear extension of the subjacent poset in a random way. In this chapter we will study several methods for sampling and counting linear extensions. As stated in Chapter 1, this problem has drawn the attention of many researchers and many algorithms have been proposed to cope with it (see e.g. [6, 62–65] for generating all linear extensions of a poset and [7, 66, 67] for counting the number of linear extensions).

The classic methods for sampling linear extensions could be divided into three types. The first type is the simplest one. This method relies on the generation of all linear extensions to randomly choose one. One of the most classical algorithms to do that is Kahn algorithm (see [68], [69]). Kahn Algorithm works by choosing vertices in the same order as the eventual topological sort. This algorithm has complexity $O(n{\cdot}e(P))$. In 1994 Pruesse and Ruskey [65] introduce the first constant amortized time algorithm reaching complexity $O(e(P))$. Note that even in the best case with amortized time $O(e(P))$ the computational cost of these algorithms depends on the size of the set of linear extensions which is huge most of times. Therefore, for big posets these techniques cannot be applied due to its high computational cost. Indeed, Brightwell and Winkler have shown in [28] that, for general posets the problem of counting the number of linear extensions is a

$\sharp P$-complete problem, and consequently, it is not possible to derive an easy procedure for sampling linear extensions for general posets using these methods.

The second kind of solutions is based on a geometrical approach. These methods have received little attention. The best known approach of this kind was made by Peter Matthews in 1991 [70]. This technique uses a coupling for a random walk on a polygonal subset of the unit sphere in $\mathbb{R}^n$. This method is not exact but we can choose the precision of the algorithm. If the precision is required to be higher, the computational cost increases but our sampled linear extension will be closer to uniform distribution. The convergence time of this method is about $O(n^8 \cdot log(n))$, which is a polynomial but very high cost.

The third and last type of method is based on Markov chains and follows the work of Karzanov and Khachiyan [71] (see also [72] for an introduction on mixing Markov chains). The idea is to generate a sequence of linear extensions and consider the n-th term of the sequence. It has been proved [73] that it is possible to obtain a bound n such that for the n-th element in the sequence, all linear extensions of the poset have the same probability of appearence, no matter the initial linear extension considered. Thus, a way to generate a linear extension is to follow the sequence until the n-th term. Some work in this line can be seen e.g. in [74–77]. The number of steps required to reach this target is called the *mixing time* of the chain. Note that most of these methods analyse just the mixing time cost, which can be reduced to $O(n^3 \cdot log(n))$. However, this is not the computational cost of the algorithm because we should add some cost referred to the above computational complexities.

The main difficulty when trying to generate a random linear extension relies in the combinatorial nature of the problem. Usually, the quantities appearing in the problem are very large and grow very fast when the cardinality of the poset grows, and then are considered intractable (see unsolvable) for generating a linear extension in a random way.

In this chapter we present a new procedure for generating linear extensions, that we have called Bottom-Up method (Sections 3.2 to 3.6) [78]. We will see how to use the Bottom-Up method to sample linear extensions in a simple way. However, this method just applies to some posets. For this reason, we will look for some generalizations of the Bottom-Up method (see Section 3.7). We will discuss some generalizations relying on the number of ideals of P. The most important generalization is the Ideal-based method. Finally, we introduce this method and use it to solve some interesting problems. Among these problems we highlight the problem of sampling uniformly inside the 2-truncated capacities polytope.

3.2 Bottom-Up method

Briefly speaking, in the Bottom-Up method we look for a vector of weights w^* on the elements of P, so that an element has a probability to be selected as the next element of the linear extension given by the quotient between its weight and the sum of all elements that can be selected for this position. The problem we solve in this section is how to obtain a weight vector so that any linear extension has the same probability of being obtained. We will see that this can be done by solving a linear system of equations. Moreover, this system does not depend on the number of linear extensions, but on the number of what we have called *positioned antichains*, whose number is usually very small compared with the number of linear extensions. As it will become apparent below, the algorithm is simple and fast; besides, it allows one to compute the number of linear extensions of the poset very easily. Moreover, once the weight vector w^* is obtained, the computational cost of deriving a random extension is very reduced. This is because it just suffices to consider the possible next elements and select one of them with probability proportional to the corresponding weight. On the other hand, this method cannot be applied to every poset and indeed, we show that the existence of a suitable weight w^* depends on whether a linear system has an infinite number of solutions. We show some examples of families of posets where it can be applied, illustrating the procedure. We also provide some sufficient conditions on the poset to admit such a weight vector and study how this weight vector can be obtained.

Let us consider a poset $(P, \preceq)$ and let us treat the problem of building a linear extension. The first element of the linear extension is a minimal element of P, say x_1, next an element in $\mathcal{MIN}(P\backslash\{x_1\})$ is selected, say x_2, then a minimal element of $P\backslash\{x_1, x_2\}$ and so on. In order to generate a random linear extension, the problem we have to face is the way each minimal element is selected. In what follows, we develop a procedure that assigns to each element of the poset a weight value, so that the probability of selecting this element in a step is proportional to the quotient between its weight and the sum of the weights of all minimal elements of the corresponding subposet. The basic steps of our algorithm are given in Algorithm 1.

Lemma 3.1. *Let $w = (w_1, ..., w_n)$ be a weight vector for poset P such that $w_i > 0, \forall i$, and suppose that we derive a linear extension via the previous algorithm. Then, we obtain a probability distribution on $\mathcal{L}(P)$.*

Algorithm 1 BOTTOM-UP

1. **ASSIGNING WEIGHTS**
   Step 1.1: For any $x \in P$, assign a weight value $w_x > 0$.
2. **SAMPLING STEP**
   $P' \leftarrow P$ ▷ Initialization
   $S \leftarrow \emptyset$

while $|P'| > 0$ **do**
  Step 2.1: Select a minimal element x of P' with probability the quotient between w_x and the sum of the weights of $\mathcal{MIN}(P')$ and $S \leftarrow S \cup x$.
  Step 2.2: $P' \leftarrow P' \backslash \{x\}$.
end
  return S

Proof. By construction, we know that $P(\epsilon) > 0$ for any $\epsilon \in \mathcal{L}(P)$. Thus, it suffices to show $\sum_{\epsilon \in \mathcal{L}(P)} P(\epsilon) = 1$. We will prove it by induction on $|P|$.

For $|P| = 2$ we have a chain or an antichain, and the result trivially holds in both cases.

Assume the result holds for $|P| \leq n$ and consider the case $|P| = n+1$. Let $m_1, ..., m_r$ be the minimal elements of P. Then, one of them is the first element in the linear extension and thus,

$$\begin{aligned} \sum_{\epsilon \in \mathcal{L}(P)} P(\epsilon) &= \sum_{i=1}^{r} P(\epsilon(1) = m_i) \sum_{\epsilon' \in \mathcal{L}(P \backslash \{m_i\})} P(\epsilon') = \sum_{i=1}^{r} \frac{w_{m_i}}{\sum_{j=1}^{r} w_{m_j}} \sum_{\epsilon' \in \mathcal{L}(P \backslash \{m_i\})} P(\epsilon') \\ &= \sum_{i=1}^{r} \frac{w_{m_i}}{\sum_{j=1}^{r} w_{m_j}} = \frac{\sum_{i=1}^{r} w_{m_i}}{\sum_{j=1}^{r} w_{m_j}} = 1. \end{aligned}$$

Thus, the result holds.

□

With some abuse of notation, we will denote $w_A = \sum_{x \in A} w_x$. Note that for a given linear extension ϵ and a weight function w, the probability of the appearance of ϵ is given by

$$P(\epsilon) = \frac{w_{\epsilon(1)}}{w_{\mathcal{MIN}(P)}} \times \frac{w_{\epsilon(2)}}{w_{\mathcal{MIN}(P \backslash \{\epsilon(1)\})}} \times \cdots = \prod_{i=1}^{n} \frac{w_{\epsilon(i)}}{w_{\mathcal{MIN}(P \backslash \{\epsilon(1), \ldots, \epsilon(i-1)\})}}. \tag{3.1}$$

We look for a weight vector satisfying that all linear extensions share the same probability, that we will denote by w^*. The critical point in this procedure is the way we assign weights to elements in P so that w^* serves this purpose. Two questions arise:

1. Is it possible to derive such a weight for any poset?

2. If a poset admits a weight vector in these conditions, how can it be computed?

The answer to the first question is negative, as our next example shows.

Example 3.1. *Consider the poset given in Figure 3.1, some of its linear extensions are:* $\epsilon_1 = (1,2,4,3,5), \epsilon_2 = (1,2,4,5,3)$*,* $\epsilon_3 = (5,2,1,3,4), \epsilon_4 = (5,2,3,1,4), \epsilon_5 = (1,5,2,3,4)$ *and* $\epsilon_6 = (5,1,2,3,4)$.

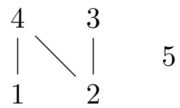


FIGURE 3.1: Example of a poset where w^* does not exist.

Then, if w^* *exists,* $P(\epsilon_1) = P(\epsilon_2)$ *leads to* $w_3^* = w_5^*$*; next,* $P(\epsilon_3) = P(\epsilon_4)$ *leads to* $w_1^* = w_3^* + w_4^*$*; finally,* $P(\epsilon_5) = P(\epsilon_6)$ *leads to* $w_1^* = w_5^*$*. As a conclusion, no* w^* *satisfying* $w_i^* > 0, \forall i \in P$ *exists.*

If a poset P admits a weight vector $w^* > 0$, we say that P is **BU-feasible**.

Assume P is BU-feasible; we treat the problem of obtaining w^* below. Let us start with an example.

Example 3.2. *Consider the poset* N *(see Figure 1.1) and let* $w = (w_1, w_2, w_3, w_4)$ *be a (possible) vector of weights. Then, for each linear extension of poset* N*, we obtain the probabilities:*

$$\epsilon_1 = (1,2,3,4) \Rightarrow p(\epsilon_1) = \frac{w_1}{w_1+w_2} \times \frac{w_2}{w_2} \times \frac{w_3}{w_3+w_4} \times \frac{w_4}{w_4}$$
$$\epsilon_2 = (1,2,4,3) \Rightarrow p(\epsilon_2) = \frac{w_1}{w_1+w_2} \times \frac{w_2}{w_2} \times \frac{w_4}{w_3+w_4} \times \frac{w_3}{w_3}$$
$$\epsilon_3 = (2,1,3,4) \Rightarrow p(\epsilon_3) = \frac{w_2}{w_1+w_2} \times \frac{w_1}{w_1+w_3} \times \frac{w_3}{w_3+w_4} \times \frac{w_4}{w_4}$$
$$\epsilon_4 = (2,1,4,3) \Rightarrow p(\epsilon_4) = \frac{w_2}{w_1+w_2} \times \frac{w_1}{w_1+w_3} \times \frac{w_4}{w_3+w_4} \times \frac{w_3}{w_3}$$
$$\epsilon_5 = (2,3,1,4) \Rightarrow p(\epsilon_5) = \frac{w_2}{w_1+w_2} \times \frac{w_3}{w_1+w_3} \times \frac{w_1}{w_1} \times \frac{w_4}{w_4}$$

In order to sample uniformly, we should get a vector $w = (w_1, w_2, w_3, w_4)$ *such that* $p(\epsilon_1) = p(\epsilon_2) = p(\epsilon_3) = p(\epsilon_4) = p(\epsilon_5)$*. It is easy to see that* $w^* = (2,3,1,1)$ *satisfies these conditions.*

Note that we have two problems to face: First, as it can be seen in the previous example, we have to solve a non-linear system of equations. This is the case for general posets.

Second, the system involves $e(P)-1$ equations and relies on the knowledge of the whole set of linear extensions. Therefore, the problem is intractable at this stage. Below we will see that it is possible to derive an equivalent linear system involving a reduced number of equations and such that it does not depend on the knowledge of $\mathcal{L}(P)$.

We start transforming the system into an equivalent one just involving linear equations. To achieve this, we will use our knowledge about the adjacent transposition graph $(\mathcal{L}(P), \tau^*)$, see Definition 1.20.

Lemma 3.2. *Let us consider two adjacent linear extensions ϵ_1, ϵ_2 in $(\mathcal{L}(P), \tau^*)$. Then, the equation $p(\epsilon_1) = p(\epsilon_2)$ is linear.*

Proof. Let ϵ_1 and ϵ_2 be two adjacent linear extensions in $(\mathcal{L}(P), \tau^*)$. They can be written as $\epsilon_1 = (a_1, a_2, \ldots, a_k, x, y, b_1, b_2, \ldots, b_s)$ and $\epsilon_2 = (a_1, a_2, \ldots, a_k, y, x, b_1, b_2, \ldots, b_s)$. Now, denoting $P_i = P\backslash\{\epsilon_1(1), ..., \epsilon_1(i-1)\}$ and $P'_i = P \setminus \{\epsilon_2(1), ..., \epsilon_2(i-1)\}$,

$$P(\epsilon_1) = \frac{w_{a_1}}{w_{\mathcal{MIN}(P_1)}} \times \cdots \times \frac{w_{a_k}}{w_{\mathcal{MIN}(P_k)}} \times \frac{w_x}{w_{\mathcal{MIN}(P_{k+1})}} \times \frac{w_y}{w_{\mathcal{MIN}(P_{k+2})}} \times \frac{w_{b_1}}{w_{\mathcal{MIN}(P_{k+3})}} \times \cdots \times \frac{w_{b_s}}{w_{P_n}},$$

$$P(\epsilon_2) = \frac{w_{a_1}}{w_{\mathcal{MIN}(P'_1)}} \times \cdots \times \frac{w_{a_k}}{w_{\mathcal{MIN}(P'_k)}} \times \frac{w_y}{w_{\mathcal{MIN}(P'_{k+1})}} \times \frac{w_x}{w_{\mathcal{MIN}(P'_{k+2})}} \times \frac{w_{b_1}}{w_{\mathcal{MIN}(P'_{k+3})}} \times \cdots \times \frac{w_{b_s}}{w'_{P_n}}.$$

Notice that $w_{\mathcal{MIN}(P_i)} = w_{\mathcal{MIN}(P'_i)}$, if $i \neq k+2$. Consequently,

$$P(\epsilon_1) = P(\epsilon_2) \Leftrightarrow w_{\mathcal{MIN}(P'_{k+2})} = w_{\mathcal{MIN}(P_{k+2})}.$$

Hence, we have obtained a linear equation. □

From this result we can transform the system of equations into a linear system.

Theorem 3.3. *Let P be a finite poset. The system of $e(P)-1$ non-linear equations $p(\epsilon_1) = p(\epsilon_2) = \ldots = p(\epsilon_{e(P)})$ can be transformed into a linear system of the same number of equations.*

Proof. Consider a linear extension ϵ_1. As $(\mathcal{L}(P), \tau^*)$ is connected by Lemma 1.21, there exists a linear extension ϵ_2 being adjacent to ϵ_1. Applying the previous Lemma 3.2, we conclude that $p(\epsilon_1) = p(\epsilon_2)$ is a linear equation. Now, there exists $\epsilon_3 \in \mathcal{L}(P)\backslash\{\epsilon_1, \epsilon_2\}$ adjacent to one of them, say ϵ_2. Therefore, $p(\epsilon_2) = p(\epsilon_3)$ is linear. Acting like this, we obtain a system of $e(P)-1$ linear equations. □

Example 3.3. *In Table 3.1, we show the way to obtain the weight vector for poset N. In the first column, we consider the set of adjacent linear extensions. Although there are five pairs of adjacent linear extensions (see Fig. 1.11), note that we just need four of them in order to define the system involving all linear extensions. Moreover, note that there are just three different equations, as some of them coincide.*

Linear extensions	Equation	Incomparable pair	V
$p(\epsilon_1) = p(\epsilon_2)$	$w_3 = w_4$	$\{3, 4\}$	$\emptyset$
$p(\epsilon_1) = p(\epsilon_3)$	$w_2 = w_1 + w_3$	$\{1, 2\}$	$\emptyset$
$p(\epsilon_2) = p(\epsilon_4)$	$w_2 = w_1 + w_3$	$\{1, 2\}$	$\emptyset$
$p(\epsilon_3) = p(\epsilon_4)$	$w_3 = w_4$	$\{3, 4\}$	$\emptyset$
$p(\epsilon_3) = p(\epsilon_5)$	$w_1 = w_3 + w_4$	$\{1, 3\}$	$\emptyset$

Table 3.1: Linear system for poset N.

Therefore, we obtain the following linear system:

$$\begin{cases} w_3 = w_4 \\ w_2 = w_1 + w_3 \\ w_1 = w_3 + w_4 \end{cases}$$

whose solution is $w^* = (2\lambda, 3\lambda, \lambda, \lambda), \lambda > 0.$

Now, a problem may be considered: How can the adjacent linear extensions be chosen? Of course, the natural answer is to consider a Hamiltonian path in the graph $(\mathcal{L}(P), \tau^*)$. However, such a path does not exist in general and it is indeed a problem that has attracted the attention of many researchers (see [6] and references therein). We will show below that the problem can be solved in a more suitable way without the need of considering linear extensions and connections between them, so we can avoid this problem.

Let us now deal with the problem of reducing the number of equations of the linear system. To shed light on what follows, let us have a look to the system obtained in Example 3.3. As we have seen in this example, it could be the case that some equations coming from different pairs of adjacent linear extensions in $(\mathcal{L}(P), \tau^*)$ coincide. This is the case for $p(\epsilon_1) = p(\epsilon_2)$ and $p(\epsilon_3) = p(\epsilon_4)$, or $p(\epsilon_1) = p(\epsilon_3)$ and $p(\epsilon_2) = p(\epsilon_4)$. Let us take a deeper look at these equations. As they come from adjacent linear extensions, they differ in two consecutive uncomparable elements that have swapped positions, the position of the other elements remaining unaltered. Thus, as we have shown in Lemma 3.2, the equation depends on these pairs of elements. Moreover, as the equations rely on the minimal elements of the corresponding subposets when these elements are selected, we have to take into account the elements that have been selected before (or equivalently, that will be selected after). The elements in $\downarrow\hat{x}\cup\downarrow\hat{y}$ are of course selected before x, y, and elements in $\uparrow\hat{x}\cup\uparrow\hat{y}$ are selected after. Then, it just suffices to know which are the elements outside these subsets that have been selected before x, y. This is the set V in the fourth column and it is an ideal of $P\setminus\updownarrow\{x, y\}$. This leads us to the following definition.

Definition 3.4. Let P be a finite poset. We say that $(a = \{x, y\}, V)$ is a **positioned antichain** if $a = \{x, y\}$ is an antichain in P and V is an ideal of $P\setminus\updownarrow\{x, y\}$. Notice that V can be the empty set.

Let us denote by $\mathcal{PA}(P)$ the set of all positioned antichains for P and by $\mathrm{pa}(P)$ its cardinality. Note that for any pair of adjacent linear extensions in $(\mathcal{L}(P), \tau^*)$, a positioned antichain is associated with the pair, but it could be the case that several pairs of adjacent linear extensions share the same positioned antichain, as we have seen in the previous example. Therefore, the linear system based on adjacent linear extensions can be transformed into another one based on positioned antichains. This is formally shown below.

Definition 3.5. Let P be a finite poset and (a, V) a positioned antichain of P with $a = \{x, y\}$. We define the **set of linear extensions generated** by (a, V) as

$$\mathcal{G}(a, V) := \{\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \mathcal{L}(P) : \epsilon_1 \in \mathcal{L}(\downarrow\widehat{x}\cup\downarrow\widehat{y}\cup V), \epsilon_2 \in \mathcal{L}(a), \epsilon_3 \in \mathcal{L}(\uparrow\widehat{x}\cup\uparrow\widehat{y}\cup V_a^{\mathrm{c}})\}$$

where $V_a^{\mathrm{c}} := P\setminus(\updownarrow\{x, y\} \cup V)$. Notice that $\epsilon_2 = (x, y)$ or $\epsilon_2 = (y, x)$.

Lemma 3.6. *Let us consider a pair of adjacent linear extensions (ϵ_1, ϵ_2) in $(\mathcal{L}(P), \tau^*)$. Then, there is a unique positioned antichain (a, V) such that $\epsilon_1, \epsilon_2 \in \mathcal{G}(a, V)$ and ϵ_2 is obtained by interchanging the elements of a in ϵ_1. We call it the* **positioned antichain associated** *with the pair (ϵ_1, ϵ_2).*

Proof. Let $\epsilon_1 = (a_1, a_2, \ldots, a_k, x, y, b_1, b_2, \ldots, b_s)$ and $\epsilon_2 = (a_1, a_2, \ldots, a_k, y, x, b_1, b_2, \ldots, b_s)$. Now we necessarily have to choose $a = \{x, y\}$ and $V = (P\setminus\updownarrow\{x, y\}) \cap \{a_1, a_2, \ldots, a_k\}$. Since ϵ_1 and ϵ_2 are linear extensions, then V is an ideal of $P\setminus\updownarrow\{x, y\}$. Finally $\epsilon_1, \epsilon_2 \in \mathcal{G}(a, V)$. □

Note that for any positioned antichain $(a = \{x, y\}, V)$, it is always possible to obtain a pair of linear extensions (ϵ_1, ϵ_2) such that ϵ_1 and ϵ_2 are adjacent in $(\mathcal{L}(P), \tau^*)$ and whose associated positioned antichain is (a, V). To see this, it just suffices to consider $\epsilon_1 = (\epsilon_1^1, x, y, \epsilon_1^3) \in \mathcal{G}(a, V)$ and $\epsilon_2 = (\epsilon_1^1, y, x, \epsilon_1^3)$.

Theorem 3.7. *Let us consider two pairs of adjacent linear extensions (ϵ_1, ϵ_2) and (ϵ_3, ϵ_4) in $(\mathcal{L}(P), \tau^*)$ and suppose they share the same positioned antichain. Then, the linear* *equations for $p(\epsilon_1) = p(\epsilon_2)$ and $p(\epsilon_3) = p(\epsilon_4)$ are the same. Consequently, it just suffices to consider the linear equations corresponding to different positioned antichains.*

Proof. Let (a, V), where $a = \{x, y\}$, denotes the common positioned antichain associated with both pairs. From the proof of Lemma 3.2, we know that the linear equations for $p(\epsilon_1) = p(\epsilon_2)$ and $p(\epsilon_3) = p(\epsilon_4)$ only depend on the terms corresponding to the (consecutive) positions for x, y in the linear extension. On the other hand, these terms depend only on the elements appearing before in the linear extension. As these elements are in both cases $\downarrow\widehat{x} \cup \downarrow\widehat{y} \cup V$, the result holds. $\square$

Definition 3.8. We define the **linear equation associated** with the positioned antichain $(a = \{x, y\}, V)$ as the equation given by

$$w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a} \cup V \cup \{x\}))) = w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a} \cup V \cup \{y\}))). \tag{3.2}$$

Note that this equation arises for $p(\epsilon_1) = p(\epsilon_2)$ where (ϵ_1, ϵ_2) is a pair of two adjacent linear extensions whose associated positioned antichain is (a, V).

3.3 $\mathrm{pa}(P) \leq e(P)$

Now, the following question is of relevance: Does the system based on positioned antichains involve a reduced number of equations? For this, we have to compare $\mathrm{pa}(P)$ and $e(P)$; the following holds:

Theorem 3.9. *Let P be a finite poset. Then, $\mathrm{pa}(P) \leq e(P)$.*

This section is devoted to prove this theorem. In an attempt to clarify the proof, we have considered some previous results before the main part of the proof. In what follows, we will denote $P_x := P \setminus \{x\}$.

Lemma 3.10. *Let P be a finite poset. The following holds:*

i) *For every $x \in \mathcal{MIN}(P)$, $i(P_x) \leq i(P) \leq 2i(P_x)$.*

ii) $\mathrm{pa}(P) = \sum_{(x,y)\in\mathcal{A}_2(P)} i(P \setminus \updownarrow\{x, y\})$*, where $\mathcal{A}_2(P)$ denotes the set of antichains of two elements of P.*

iii) $\mathrm{pa}(P_1 \oplus P_2) = \mathrm{pa}(P_1) + \mathrm{pa}(P_2)$.

iv) $\mathrm{pa}(P_1 \uplus P_2) = i(P_2)\mathrm{pa}(P_1) + i(P_1)\mathrm{pa}(P_2) + \sum_{(x,y)\in P_1\times P_2} i((P_1 \setminus \updownarrow\{x\}) \uplus (P_2 \setminus \updownarrow\{y\}))$.

v) Let P be a poset with $m(P) = 2$. Suppose that P has a minimal element x_1 less than every non-minimal element in P; then, $\mathrm{pa}(P) = \mathrm{pa}(P_{x_1}) + 1$. *Therefore, if P^* is a poset with an only minimum, then* $\mathrm{pa}(P^* \uplus \mathbf{1}) = \mathrm{pa}(P^*_{x_1} \uplus \mathbf{1}) + 1$.

vi) Let P be a poset with $m(P) = 3$. Suppose that P has a minimal element x_1 less than every non-minimal element in P; then, $\mathrm{pa}(P) = \mathrm{pa}(P_{x_1}) + 5$. *Therefore, if P^* is a poset with an only minimum, then* $\mathrm{pa}(P^* \uplus \mathbf{1} \uplus \mathbf{1}) = \mathrm{pa}(P^*_{x_1} \uplus \mathbf{1} \uplus \mathbf{1}) + 5$.

vii) $i(P) = i(P^\partial)$ and $\mathrm{pa}(P) = \mathrm{pa}(P^\partial)$.

Proof. i) Consider $x \in \mathcal{MIN}(P)$ fixed and let us define

$$\begin{array}{rccc} F: & \mathcal{I}(P_x) & \to & \mathcal{I}(P) \\ & I & \mapsto & I \cup \{x\} \end{array}$$

As $x \in \mathcal{MIN}(P)$, if $I \in \mathcal{I}(P_x)$, it follows that $I \cup \{x\} \in \mathcal{I}(P)$, and F is injective, and hence the first inequality holds.

Let us now consider the function $G : \mathcal{I}(P) \to \mathcal{I}(P_x) \times 2$ given by

$$G(I) = \begin{cases} (I, 0), & x \notin I \\ (I \setminus \{x\}, 1), & x \in I \neq \{x\} \\ (\emptyset, 1), & I = \{x\} \end{cases}$$

Note that if $x \notin I$ then $I \in \mathcal{I}(P_x)$; on the other hand, if $x \in I$, then $I \setminus \{x\} \in \mathcal{I}(P_x)$; we conclude that G is well-defined. As G is injective, the second inequality holds.

ii) For a fixed antichain $\{x, y\}$ of two elements, the number of positioned antichains associated with $\{x, y\}$ is given by $i(P \setminus \updownarrow \{x, y\})$. Then,

$$\mathrm{pa}(P) = \sum_{(x,y) \in \mathcal{A}_2(P)} i(P \setminus \updownarrow \{x, y\}).$$

iii) As $P = P_1 \oplus P_2 \oplus ... \oplus P_n$, if $x \in P_i, y \in P_j, i < j$, it always holds that $x \prec y$. Therefore, positioned antichain in P is always a positioned antichain in P_i for some i.

By definition of $\oplus$, given $x \in P_i$ and $y \in P_j$ with $i < j$, $x \preceq_P y$. Consequently, any antichain in P contains elements from a single P_i. Therefore, $P \setminus \updownarrow \{x, y\} = P_i \setminus \updownarrow \{x, y\}$ and we conclude that V is an ideal of $P_i \setminus \updownarrow \{x, y\}$.

Finally,

$$\mathcal{PA}(P) = \bigcup_{i=1}^{n} \mathcal{PA}(P_i),$$

and therefore $\mathrm{pa}\,(P_1 \oplus P_2 \oplus \ldots \oplus P_n) = \sum_{i=1}^{n} \mathrm{pa}(P_i)$. In particular, $\mathrm{pa}(P_1 \oplus P_2) = \mathrm{pa}(P_1) + \mathrm{pa}(P_2)$.

iv) The set $\mathcal{PA}(P_1 \uplus P_2)$ can be partitioned into three parts in terms of the antichain a: $a \subseteq P_1, a \subseteq P_2$ and $a = \{x, y\}, x \in P_1, y \in P_2$. Then, if $\mathcal{A}_2(P)$ denotes the set of antichains of two elements of P, iii) implies that $\mathrm{pa}(P_1 \uplus P_2)$ is given by

$$\sum_{(x,y)\in\mathcal{A}_2(P_1)} i((P_1\backslash \updownarrow a)\uplus P_2) + \sum_{(x,y)\in\mathcal{A}_2(P_2)} i(P_1\uplus(P_2\backslash \updownarrow a)) + \sum_{(x,y)\in P_1\times P_2} i((P_1\backslash \updownarrow\{x\})\uplus(P_2\backslash \updownarrow\{y\})).$$

On the other hand, note that $i((P_1\backslash \updownarrow a) \uplus P_2) = i(P_1\backslash \updownarrow a)i(P_2)$. Therefore,

$$\sum_{(x,y)\in\mathcal{A}_2(P_1)} i((P_1\backslash \updownarrow a) \uplus P_2) = \mathrm{pa}(P_1)i(P_2).$$

Similarly,

$$\sum_{(x,y)\in\mathcal{A}_2(P_2)} i(P_1 \uplus (P_2\backslash \updownarrow a)) = i(P_1)\mathrm{pa}(P_2).$$

v) Let x_1, x_2 be the minimal elements of P. Observe that x_2 is the only element in P that is not related to x_1. Then, every antichain of two elements a in P is related to or contains x_1. Therefore, $P\backslash \updownarrow a = P_{x_1}\backslash \updownarrow a$ and $\mathcal{I}(P\backslash \updownarrow a) = \mathcal{I}(P_{x_1}\backslash \updownarrow a)$. Then, if (a, V) is a positioned antichain of P such that $x_1 \notin a$ then (a, V) is a positioned antichain of P_{x_1}. Also, if (a, V) is a positioned antichain of P_{x_1}, then (a, V) is a positioned antichain of P. Then, the positioned antichains in P are the same as P_{x_1} plus the positioned antichains with $x_1 \in a$. Since x_1 is less than every element apart from x_2, then the only position antichain with $x_1 \in a$ is $(a = \{x_1, x_2\}, V = \emptyset)$. Therefore, $\mathrm{pa}(P) = \mathrm{pa}(P_{x_1}) + 1$.

vi) Let x_1, x_2, x_3 be the minimal elements of P and let us study the different kinds of positioned antichains (a, V) in P and P_{x_1}.

- Suppose a contains some element in $\uparrow\widehat{x_1}$. Then, $x_1 \notin V$ and $P\backslash \updownarrow a = P_{x_1}\backslash \updownarrow a$; thus, V is an ideal of P_{x_1}, and then (a, V) is a positioned antichain for P_{x_1}. Reciprocally, any $(a, V) \in \mathcal{PA}(P_{x_1})$ satisfying $a \cap \uparrow\widehat{x_1} \neq \emptyset$ can be associated with $(a, V) \in \mathcal{PA}(P)$. Then,

$$\begin{array}{rccc} f: & \mathcal{I}(P\backslash \updownarrow a) & \rightarrow & \mathcal{I}(P_{x_1}\backslash \updownarrow a) \\ & V & \rightarrow & V \end{array}$$

  is a bijective function.

- Suppose on the other hand that $a \cap \uparrow \widehat{x_1} = \emptyset$. Observe that this case only arises if $a \subset \{x_1, x_2, x_3\}$, since x_1 is related to every non-minimal element. The possible positioned antichains are $(a = \{x_1, x_2\}, V = \emptyset), (a = \{x_1, x_3\}, V = \emptyset), (a = \{x_1, x_2\}, V = \{x_3\}), (a = \{x_1, x_3\}, V = \{x_2\})$ and the positioned antichains for $a = \{x_2, x_3\}$. Note that the first four positioned antichains are not positioned antichains of P_{x_1}.

  Let us then turn to positioned antichains with $a = \{x_2, x_3\}$. We have two possible choices for V in this case: either $x_1 \in V$ or $V = \emptyset$. In the first case, note that

$$\begin{array}{rccc} f: & \mathcal{I}(P \setminus \updownarrow \{x_2, x_3\}) & \to & \mathcal{I}(P_{x_1} \setminus \updownarrow \{x_2, x_3\}) \\ & V & \to & V \setminus \{x_1\} \end{array}$$

  is a bijective function. Then, the number of ideals associated with the positioned antichain $a = \{x_2, x_3\}$ in P is the number of ideals associated with the positioned antichain $a = \{x_2, x_3\}$ in P_{x_1} plus one (the remaining case $V = \emptyset$). This positioned antichain plus the four ones above which were not in P_{x_1} give us the result $\mathrm{pa}(P) = \mathrm{pa}(P_{x_1}) + 5$.

vii) Note that $i(P) = i(P^{\partial})$ because $A \in \mathcal{I}(P) \Leftrightarrow A^{c} \in \mathcal{F}(P) = \mathcal{I}(P^{\partial})$.

Let us now prove that $\mathrm{pa}(P) = \mathrm{pa}(P^{\partial})$. Let a be an antichain in P; then, a is an antichain in P^{∂}. On the other hand, the number of positioned antichains associated with this antichain in P is $i(P \setminus \updownarrow a)$, and the number of positioned antichains associated with a in P^{∂} is $i(P^{\partial} \setminus \updownarrow a)$. Since $P^{\partial} \setminus \updownarrow a = (P \setminus \updownarrow a)^{\partial}$, then $i(P^{\partial} \setminus \updownarrow a) = i((P \setminus \updownarrow a)^{\partial}) = i(P \setminus \updownarrow a)$. By iii) we get $\mathrm{pa}(P) = \mathrm{pa}(P^{\partial})$.

Then, the result holds. □

Lemma 3.11. *Let P be a finite poset with exactly one minimal element x; then*

$$\mathrm{pa}(P) = \mathrm{pa}(P_x).$$

Proof. Obviously, $\mathrm{pa}(P) \geq \mathrm{pa}(P_x)$. Now, as $x \preceq y$, $\forall y \in P_x$, for any positioned antichain (a, V) in P, it follows that $x \notin a, x \notin V$. Consequently, any positioned antichain in P is also a positioned antichain in P_x, and hence $\mathrm{pa}(P) \leq \mathrm{pa}(P_x)$ and thus, $\mathrm{pa}(P) = \mathrm{pa}(P_x)$. □

Lemma 3.12. *Let P be a finite poset such that $m(P) \geq 4$. Then,*

$$\mathrm{pa}(P) \leq \sum_{x \in \mathcal{MIN}(P)} \mathrm{pa}(P_x).$$

Proof. The set $\mathcal{PA}(P)$ (resp. $\mathcal{PA}(P_x)$) can be partitioned into three sets attending the number of minimal elements of P in the antichain, $\mathcal{PA}_0(P), \mathcal{PA}_1(P)$, $\mathcal{PA}_2(P)$ (resp. $\mathcal{PA}_0(P_x), \mathcal{PA}_1(P_x), \mathcal{PA}_2(P_x)$). We will show that for each of these three cases the result holds.

- **Case 1:** $a = \{a_1, a_2\}, a_1, a_2 \notin \mathcal{MIN}(P)$**.** Consider an element $x(a_1, a_2) \in \mathcal{MIN}(P)$ such that $x(a_1, a_2) \in \downarrow \hat{a}$. Then, $x(a_1, a_2) \notin a, x(a_1, a_2) \notin V$, and hence $(a, V) \in \mathcal{PA}(P_{x(a_1,a_2)})$. Consequently,

$$\mathrm{pa}_0(P) \leq \sum_{x \in \mathcal{MIN}(P)} \mathrm{pa}_0(P_x).$$

- **Case 2:** $a = \{x, a_0\}, a_0 \notin \mathcal{MIN}(P), x \in \mathcal{MIN}(P)$**.** As in the previous case, there exists an element $x(a_0) \in \mathcal{MIN}(P), x(a_0) \neq x$ such that $x(a_0) \in \downarrow a$. Then, $x(a_0) \notin a, x(a_0) \notin V$, so that $(a, V) \in \mathcal{PA}(P_{x(a_0)})$. Consequently,

$$\mathrm{pa}_1(P) \leq \sum_{x \in \mathcal{MIN}(P)} \mathrm{pa}_1(P_x).$$

- **Case 3:** $a = \{x_i, x_j\}, x_i, x_j \in \mathcal{MIN}(P)$**.** Note that for fixed $\{x_i, x_j\}$ the number of possible positioned antichains in P (resp. P_x, for $x \in \mathcal{MIN}(P) \backslash \{x_i, x_j\}$) is given by $i(P \backslash \uparrow \{x_i, x_j\})$ (resp. $i(P_x \backslash \uparrow \{x_i, x_j\})$). Thus, by Lemma 3.10 iii),

$$\mathrm{pa}_2(P) = \sum_{(x_i,x_j) \in \mathcal{MIN}(P)^2} i(P \backslash \uparrow \{x_i, x_j\}),$$

$$\sum_{x \in \mathcal{MIN}(P)} \mathrm{pa}_2(P_x) = \sum_{x \in \mathcal{MIN}(P)} \sum_{(x_i,x_j) \in (\mathcal{MIN}(P) \backslash \{x\})^2} i(P_x \backslash \uparrow \{x_i, x_j\}).$$

Now, defining $P^{ij} = P \backslash \uparrow \{x_i, x_j\}$, we have

$$\begin{aligned}
\sum_{x \in \mathcal{MIN}(P)} \sum_{(x_i,x_j) \in (\mathcal{MIN}(P) \backslash \{x\})^2} i(P_x \backslash \uparrow \{x_i, x_j\}) &= \sum_{x \in \mathcal{MIN}(P)} \sum_{(x_i,x_j) \in (\mathcal{MIN}(P) \backslash \{x\})^2} i(P^{ij} \setminus \{x\}) \\
&= \sum_{(x_i,x_j) \in \mathcal{MIN}(P)^2} \sum_{x \in \mathcal{MIN}(P) \backslash \{x_i,x_j\}} i(P^{ij} \setminus \{x\})
\end{aligned}$$

Since there are at least four minimal elements in P, the last expression has at least two addends. If $x^*_{i,j}$ is the minimal element with least $i(P^{ij} \setminus \{x^*_{i,j}\})$, it follows from Lemma 3.10 ii)

$$\begin{aligned}
\sum_{(x_i,x_j)\in\mathcal{MIN}(P)^2}\ \sum_{x\in\mathcal{MIN}(P)\setminus\{x_i,x_j\}} i(P^{ij}\setminus\{x\}) &\geq \sum_{(x_i,x_j)\in\mathcal{MIN}(P)^2} 2i(P^{ij}\setminus\{x^*_{i,j}\}) \\
&\geq \sum_{(x_i,x_j)\in\mathcal{MIN}(P)^2} i(P^{ij}) \\
&= \sum_{(x_i,x_j)\in\mathcal{MIN}(P)^2} i(P\setminus\uparrow\{x_i,x_j\}) \\
&= \mathrm{pa}_2(P)
\end{aligned}$$

Adding up these three cases, the result holds. □

It is not difficult to find examples showing that there are not similar results to Lemmas 3.11 and 3.12 when $m(P) = 2$ and $m(P) = 3$. However, there are special cases where a similar result holds.

Lemma 3.13. *Let P be a finite poset such that $m(P) = 2$ and P has at least two maximal elements which are both non-minimal elements. Then:*

$$\mathrm{pa}(P) \leq \mathrm{pa}(P_{x_1}) + \mathrm{pa}(P_{x_2}).$$

Proof. The proof is quite similar to the previous one. The set $\mathcal{PA}(P)$ (and $\mathcal{PA}(P_{x_1})$, $\mathcal{PA}(P_{x_2})$) can be partitioned into three sets, $\mathcal{PA}_0(P), \mathcal{PA}_1(P), \mathcal{PA}_2(P)$ (resp. $\mathcal{PA}_0(P_{x_i})$, $\mathcal{PA}_1(P_{x_i})$, $\mathcal{PA}_2(P_{x_i})$) attending the number of minimal elements of P in the antichain. The first two cases can be treated as in the previous lemma. In particular, if we consider M_1 and M_2 maximal and non-minimal elements then $(\{M_1, M_2\}, V = \emptyset)$ is linked to $(\{M_1, M_2\}, \emptyset)$ in, say, $\mathcal{PA}(P_{x_1})$, with $x_1 \in\downarrow\{M_1, M_2\}$.

Let us then deal with the case of $\mathcal{PA}_2(P)$. In this case, the only positioned antichain is given by $(a = \{x_1, x_2\}, V = \emptyset)$. Then, we can link this positioned antichain to $(a = \{M_1, M_2\}, V = \emptyset) \in \mathcal{PA}(P_{x_2})$, which has not been linked to any other positioned antichain. □

Lemma 3.14. *Let P be a finite poset such that $\mathcal{MIN}(P) = \{x_1, x_2, x_3\}$ and P has at least three maximal elements which are non-minimals. Then,*

$$\mathrm{pa}(P) \leq \mathrm{pa}(P_{x_1}) + \mathrm{pa}(P_{x_2}) + \mathrm{pa}(P_{x_3}).$$

Proof. The proof is very similar to the previous proofs. The set $\mathcal{PA}(P)$ (and $\mathcal{PA}(P_{x_i}), x_i \in \mathcal{MIN}(P)$) can be partitioned into three sets, $\mathcal{PA}_0(P), \mathcal{PA}_1(P), \mathcal{PA}_2(P)$ (resp. $\mathcal{PA}_0(P_{x_i})$, $\mathcal{PA}_1(P_{x_i})$, $\mathcal{PA}_2(P_{x_i})$) attending the number of minimal elements of P in the antichain

a. The first two cases can be treated as in Lemma 3.12. In particular, if we denote by M_1, M_2, M_3 three maximal non-minimal elements, the positioned antichain $(a = \{M_i, M_j\}, V = \emptyset)$ is linked to $(a, V) \in \mathcal{PA}(P_{x(M_i,M_j)})$ for a $x(M_i, M_j) \in\downarrow \{M_i, M_j\}$.

Let us then deal with the situation where $a = \{x_i, x_j\}$ and consider the positioned antichain (a, V). In this case, either $V = \emptyset$ or $x_k \in V$, where x_k is the minimal outside a. If $x_k \in V$, we link (a, V) to $(a, V \setminus \{x_k\}) \in \mathcal{PA}(P_{x_k})$. If $V = \emptyset$, we link $(a = \{x_i, x_j\}, V = \emptyset)$ to $(a = \{M_i, M_j\}, V = \emptyset) \in \mathcal{PA}(P_{x^*(M_i,M_j)})$ for $x^*(M_i, M_j) \neq x(M_i, M_j)$. □

Lemma 3.15. *Let P be a finite poset with 2 minimals x_1 and x_2 and such that $P \setminus \{x_1, x_2\}$ is neither the empty set nor a chain. Then,*

$$\sum_{x \in P} i(P \setminus \updownarrow x) \leq (|P| - 1)e(P)$$

Proof. We are going to build an injective function F from $\bigcup_{x \in P} \mathcal{I}(P \setminus \updownarrow x) \times \{x\}$ to $\mathcal{L}(P) \times \{1, 2, \ldots, |P|-1\}$. Define $f : P \to \{1, 2, \ldots, |P|-1\}$ such that $f(x_1) = f(x_2) = 1$, $f(P \setminus \{x_1, x_2\}) = \{2, ..., |P| - 1\}$ and f is a natural labeling between $P \setminus \{x_1, x_2\}$ and $\{2, ..., |P| - 1\}$, thus $f(i) < f(j)$ if $i \preceq j$.

Take $x \in P \setminus \{x_1, x_2\}$ and consider $(I, x) \in \mathcal{I}(P \setminus \updownarrow x) \times \{x\}$. We define $F(I, x) = (\epsilon, f(x))$, where ϵ is a linear extension given by $\epsilon := (\downarrow \hat{x}, I, x, R)$ where $R := P \setminus (I \cup \downarrow x)$ and such that the order in the elements of each part are given according to f and x_1 is placed before x_2 when they are in the same part. F is well-defined: note that $I \cap \downarrow x = \emptyset$ and there is no contradiction with the order if we place elements of I after elements of $\downarrow \hat{x}$. Indeed, if $y \in I, z \in\downarrow \hat{x}, y \preceq z$, then $y \in\downarrow x$, and hence $y \in I \cap \downarrow x = \emptyset$, that is not possible. Also, if $x \preceq y$ then x would be related to some element of I which is in $P \setminus \updownarrow x$, what is impossible, so we can place x after I. Similarly, there is no contradiction placing elements of R after I or $\downarrow x$. Note that F is injective so far. Indeed, as f is bijective on $P \setminus \{x_1, x_2\}$, the value $f(x)$ provides element x. And then, I can be found as the elements placed before x and outside $\downarrow \hat{x}$.

Consider now $(I, x_k) \in \mathcal{I}(P \setminus \updownarrow x_k) \times \{x_k\}$. We define $F(I, x_k) = (\epsilon, 1)$. Let us then define ϵ:

- If $I \neq \emptyset$ (then I contains a minimal element) then we define $\epsilon = (I, x_k, R)$ where elements in I and in R are placed in increasing order according to f.

- Finally, $F(\emptyset, x_1) = (x_1, x_2, R^*)$ and $F(\emptyset, x_2) = (x_2, x_1, R^*)$, where R^* is a linear extension of $P \setminus \{x_1, x_2\}$ such that it is not in increasing order according to f. Note that this is always possible as $P \setminus \{x_1, x_2\} \neq \emptyset$ and it is not a chain.

Note that this function is injective. □

Now we can prove the principal theorem of this section.

Proof of Theorem 3.9: We are going to prove it by induction on $n = |P|$.

For $n = 1$ the only poset is 1 with $\mathrm{pa}(1) = 0 < 1 = e(1)$.

Assume $\mathrm{pa}(P) \le e(P)$ holds if $|P| \le n$ and let us prove the result for a poset with $n+1$ elements. We have several cases:

- **Case 1:** $m(P) = 1$. Applying the induction hypothesis and Lemma 3.11, if x is the minimum,

$$\mathrm{pa}(P) = \mathrm{pa}(P_x) \le e(P_x) = e(P),$$

hence the result.

- **Case 2:** $m(P) \ge 4$. Applying the induction hypothesis and Lemma 3.12

$$\mathrm{pa}(P) \le \sum_{x \in \mathcal{MIN}(P)} \mathrm{pa}(P_x) \le \sum_{x \in \mathcal{MIN}(P)} e(P_x) = e(P),$$

hence the result.

- **Case 3:** $m(P) = 2$. We consider several cases in terms of the number of maximal elements that are not minimal in P. Let us denote this set by S.

  – If $|S| = 0$, then $P = \overline{2}$ and $\mathrm{pa}(\overline{2}) = 1 < 2 = e(\overline{2})$.

  – If $|S| = 1$, then either P has a maximum or $P = P^* \uplus \mathbf{1}$, where P^* has an only minimum and an only maximum. If P has a maximum, applying $\mathrm{pa}(P) = \mathrm{pa}(P^\partial)$ (Lemma 3.10 viii)), $e(P) = e(P^\partial)$ and the first case,

$$\mathrm{pa}(P) = \mathrm{pa}(P^\partial) \le e(P^\partial) = e(P).$$

  Otherwise, if $P = P^* \uplus \mathbf{1}$, where P^* has an only minimum, we can apply Lemma 3.10 i), vi) and the induction hypothesis, so that

$$\mathrm{pa}(P^* \uplus 1) = \mathrm{pa}(P^*_{x_1} \uplus \mathbf{1}) + 1 \le e(P^*_{x_1} \uplus \mathbf{1}) + 1 = (|P^*_{x_1}| + 1)e(P^*_{x_1}) + 1$$

$$\le (|P^*_{x_1}|+1)e(P^*_{x_1})+e(P^*_{x_1}) = (|P^*_{x_1}|+2)e(P^*_{x_1}) = (|P^*|+1)e(P^*) = e(P^* \uplus \mathbf{1}).$$

– If $|S| \geq 2$, then we can use Lemma 3.13 and the induction hypothesis to conclude

$$\mathrm{pa}(P) \leq \mathrm{pa}(P_{x_1}) + \mathrm{pa}(P_{x_2}) \leq e(P_{x_1}) + e(P_{x_2}) = e(P).$$

- **Case 4:** $m(P) = 3$**.** In this last case we can suppose that P has exactly 3 maximal elements because otherwise we can apply Lemma 3.10 viii), $e(P) = e(P^{\partial})$ and the corresponding case studied before to conclude:

$$\mathrm{pa}(P) = \mathrm{pa}(P^{\partial}) \leq e(P^{\partial}) = e(P).$$

Then, let us suppose that P has three maximal elements; let us denote by S the set of maximal elements of P that are not minimal. We consider different cases in terms of $|S|$.

– If $|S| = 0$, then $P = \overline{3}$ and $\mathrm{pa}(\overline{3}) = 6 = e(\overline{3})$.

– If $|S| = 1$, then $P = P^* \uplus \mathbf{1} \uplus \mathbf{1}$, where P^* has just one minimal x_1. Then, applying Lemma 3.10 vii) and the induction hypothesis,

$$\mathrm{pa}(P) = \mathrm{pa}(P_{x_1}) + 5 \leq e(P_{x_1}) + 5.$$

We will prove that in this case $e(P_{x_1}) + 5 \leq e(P)$. To prove this, observe that if we take any linear extension of P_{x_1}, it can be obtained from a linear extension of P removing x_1. Then, the map

$$\begin{array}{cccc} G: & \mathcal{L}(P) & \rightarrow & \mathcal{L}(P_{x_1}) \\ & (\epsilon_1, ..., \epsilon_i, x_1, \epsilon_{i+2}, ..., \epsilon_n) & \rightarrow & (\epsilon_1, ..., \epsilon_i, \epsilon_{i+2}, ..., \epsilon_n) \end{array}$$

is a surjective function. Note that:

∗ $G(x_1, x_2, x_3, z, \ldots) = G(x_2, x_1, x_3, z, \ldots) = G(x_2, x_3, x_1, z, \ldots) = (x_2, x_3, z, \ldots)$.

∗ $G(x_1, x_3, x_2, z, \ldots) = G(x_3, x_1, x_2, z, \ldots) = G(x_3, x_2, x_1, z, \ldots) = (x_3, x_2, z, \ldots)$.

∗ $G(x_1, x_3, z, x_2, \ldots) = G(x_3, x_1, z, x_2, \ldots) = (x_3, z, x_2, \ldots)$.

Thus, $e(P) \geq e(P_{x_1}) + 5$ and the result holds.

– If $|S| = 2$, then $P = P^* \uplus \mathbf{1}$, where P^* is a poset with 2 minimal elements and two (different) maximal elements. Then, we can apply Lemma 3.10 v), Lemma 3.15 and the induction hypothesis to conclude

$$\mathrm{pa}(P^* \uplus \mathbf{1}) = 2\mathrm{pa}(P^*) + \sum_{x \in P^*} i(P^* \setminus \updownarrow x) \leq 2e(P^*) + \sum_{x \in P^*} i(P^* \setminus \updownarrow x)$$

$$\leq 2e(P^*) + (|P^*| - 1)e(P^*) = (|P^*| + 1)e(P^*) = e(P^* \uplus \mathbf{1}).$$

– If $|S| = 3$, we can use Lemma 3.14 and the induction hypothesis to conclude

$$\mathrm{pa}(P) \leq \mathrm{pa}(P_{x_1}) + \mathrm{pa}(P_{x_2}) + \mathrm{pa}(P_{x_3}) \leq e(P_{x_1}) + e(P_{x_2}) + e(P_{x_3}) = e(P).$$

Therefore we have proved the inductive step for any P with $n+1$ elements, and then by induction the result holds. □

In general, $\mathrm{pa}(P) < e(P)$ and $\mathrm{pa}(P)$ is very small compared to $e(P)$. Note however that it could be the case that the number of linear extensions and the number of positioned antichains could be the same.

Example 3.4. *Consider the antichain of three elements, $\bar{3}$. In this case, there are six linear extensions. On the other hand, each pair of elements are uncomparable, and V can be either the element outside the antichain or the empty set. Then, there are six positioned antichains.*

Finally, it is important to note that the number of equations could be further reduced, as redundances may appear. Next example illustrates this situation and shows how positioned antichains can reduce the complexity of the problem.

Example 3.5. *Let us consider the case of the Boolean poset of order three B_3 (Figure 3.2), that has 48 linear extensions.*

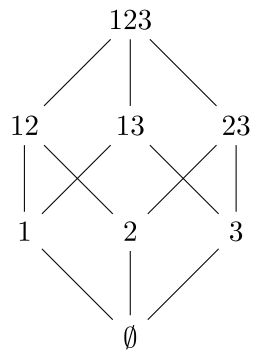


FIGURE 3.2: B_3 lattice.

Note that the empty set and the total set have fixed positions and any weight is valid for these elements. If we remove these two elements we obtain an irreducible poset. We have the following positioned antichains and equations:

Positioned Antichain	*Equation*	*Positioned Antichain*	*Equation*
$(12, 13)\ V = \emptyset$	$w_{13} = w_{12}$	$(12, 13)\ V = 23$	$w_{13} = w_{12}$
$(12, 23)\ V = \emptyset$	$w_{12} = w_{23}$	$(12, 23)\ V = 13$	$w_{12} = w_{23}$
$(13, 23)\ V = \emptyset$	$w_{13} = w_{23}$	$(13, 23)\ V = 12$	$w_{13} = w_{23}$
$(1, 23)\ V = \emptyset$	$w_1 = w_{12} + w_{13} + w_{23}$	$(2, 13)\ V = \emptyset$	$w_2 = w_{12} + w_{13} + w_{23}$
$(3, 12)\ V = \emptyset$	$w_3 = w_{12} + w_{13} + w_{23}$	$(1, 2)\ V = \emptyset$	$w_1 = w_2$
$(1, 2)\ V = 3$	$w_1 + w_{23} = w_2 + w_{13}$	$(1, 3)\ V = \emptyset$	$w_1 = w_3$
$(1, 3)\ V = 2$	$w_1 + w_{23} = w_3 + w_{12}$	$(2, 3)\ V = \emptyset$	$w_2 = w_3$
$(2, 3)\ V = 1$	$w_2 + w_{13} = w_3 + w_{12}$		

Note that we have reduced the number of equations from 47 to 15. In fact, the number of equations can be further reduced to:

$$\begin{cases} w_{12} = w_{13} = w_{23} \\ w_1 = w_{12} + w_{13} + w_{23} \\ w_2 = w_{12} + w_{13} + w_{23} \\ w_3 = w_{12} + w_{13} + w_{23} \end{cases}$$

Then, a solution is $w^* = (w_1^*, w_2^*, w_3^*, w_{12}^*, w_{13}^*, w_{23}^*) = (3, 3, 3, 1, 1, 1)$ *or, with* $w_{\emptyset}^* = 1$ *and* $w_{123}^* = 1$, $w^* = (1, 3, 3, 3, 1, 1, 1, 1)$.

Remark 3.16. Let P be BU-feasible. Then, $\forall \epsilon \in \mathcal{L}(P)$, it is possible to obtain $P(\epsilon)$ applying Eq. (3.1). On the other hand, $P(\epsilon) = \frac{1}{e(P)}$, and hence $e(P) = \frac{1}{P(\epsilon)}$.

For example, considering B_3, it can be checked that:

$P(\epsilon = (\emptyset, 1, 2, 3, 12, 13, 23, 123)) = 1 \cdot \frac{3}{9} \cdot \frac{3}{6} \cdot \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{48} \Rightarrow e(B_3) = 48.$

Thus, our procedure also provides an easy way to obtain the number of linear extensions. Note that, as already pointed out in the introduction, the problem of counting all linear extensions of a poset is a hard problem.

3.4 Properties of BU-feasible posets

Let us now show some properties regarding BU-feasibility.

Proposition 3.17. *Let* $P_1, ..., P_n$ *be finite posets and consider* $P := P_1 \oplus P_2 \oplus ... \oplus P_n$. *Then, there exists a weight vector* w_P^* *if and only if there exists* $w_{P_i}^*, i = 1, ..., n$. *In* *other words, P is BU-feasible if and only if P_i is BU-feasible $\forall i = 1, ..., n$. Moreover, $w^*_P = (w^*_{P_1}, ..., w^*_{P_n})$.*

Proof. Let $P = P_1 \oplus P_2 \oplus ... \oplus P_n$, as we saw in proof of Lemma 3.10 iii),

$$\mathcal{PA}(P) = \bigcup_{i=1}^{n} \mathcal{PA}(P_i).$$

For a positioned antichain $(a = \{x, y\}, V)$ in P_i, notice that the minimal elements of $P \setminus (\downarrow \widehat{a} \cup V \cup \{x\})$ and $P \setminus (\downarrow \widehat{a} \cup V \cup \{y\})$ belong to P_i. Therefore, we obtain a linear equation that just involves elements in P_i, and it is the same equation arising when dealing with P_i instead of P.

Therefore, the equations for P can be divided in n groups of equations, each of them just involving elements of a P_i. Thus, it is possible to obtain a weight vector w^*_P if and only if it is possible to obtain a group of n weight vectors $w^*_{P_1}, ..., w^*_{P_n}$ and in this case $w^*_P = (w^*_{P_1}, ..., w^*_{P_n})$. $\square$

Now, we prove an important property about feasibility of subposets.

Theorem 3.18. *Let P be a BU-feasible finite poset with weight vector w^*_P. If F is a filter of P, then F is BU-feasible and $w^*_F := w^*_{P|F}$ is a possible weight vector for the subposet F.*

Proof. We will prove that every linear equation associated with the linear system generated by F is in the linear system generated by P. Consider a positioned antichain $(a_F = \{a_1, a_2\}, V_F)$ in F. The associated equation for this positioned antichain is given by (see Eq. (3.2))

$$w(\mathcal{MIN}(F \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_1\}))) = w(\mathcal{MIN}(F \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_2\}))).$$

Consider (a, V), where $a = a_F$ and $V = V_F \cup F^c_a$ where $F^c_a := F^c \setminus \updownarrow a$. First, note that V is an ideal of $P \setminus \updownarrow a$. As F^c is an ideal of P, then $F^c \setminus \updownarrow a = F^c_a$ is an ideal of $P \setminus \updownarrow a$. Take $x \in V$ and $z \preceq x, z \in P \setminus \updownarrow a$. Then, if $x \in F^c_a$ so is z. Suppose on the other hand that $x \in V_F$. If $z \in F$, then $z \in V_F$ because V_F is an ideal of $F \setminus \updownarrow a$. Otherwise, $z \in F^c$ and $z \in P \setminus \updownarrow a$, so that $z \in F^c_a \subseteq V$.

Then (a, V) is a positioned antichain of P and the associated equation is

$$
\begin{aligned}
w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a} \cup V \cup \{a_1\}))) &= w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a} \cup V \cup \{a_2\}))) \Leftrightarrow \\
w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a_F} \cup (V_F \cup F_a^c) \cup \{a_1\}))) &= w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a_F} \cup (V_F \cup F_a^c) \cup \{a_2\}))) \Leftrightarrow \\
w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a_F} \cup V_F \cup \{a_1\} \cup (F^c \setminus \uparrow a_F)))) &= w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a_F} \cup V_F \cup \{a_2\} \cup (F^c \setminus \uparrow a_F)))) \Leftrightarrow \\
w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a_F} \cup V_F \cup \{a_1\} \cup F^c))) &= w(\mathcal{MIN}(P \setminus (\downarrow\widehat{a_F} \cup V_F \cup \{a_2\} \cup F^c))) \Leftrightarrow \\
w(\mathcal{MIN}(F \setminus (\downarrow\widehat{a_F} \cup V_F \cup \{a_1\}))) &= w(\mathcal{MIN}(F \setminus (\downarrow\widehat{a_F} \cup V_F \cup \{a_2\}))).
\end{aligned}
$$

Observe that we have used above that $F^c \setminus \uparrow a_F = F^c$ since $\uparrow a_F \subseteq F$. Finally, as the system for P has a positive solution, then so has the one for F, and a possible solution is the restriction of w_P^* to F. $\square$

Note however that this does not mean that BU-feasibility for every proper filter F implies BU-feasibility for P (see the poset of Example 3.1).

Corollary 3.19. *Let $P_1, P_2, \ldots, P_n$ be a collection of finite connected posets. If $P = P_1 \uplus P_2 \uplus \ldots \uplus P_n$ is BU-feasible, then each P_i is BU-feasible and $w_{P_i}^* = w_{P|P_i}^*$ is a possible weight vector.*

Proof. As P_i is a filter of P, it just suffices to apply Theorem 3.18. $\square$

Lemma 3.20. *Let P be a finite poset and let $\mathbf{M} = L_0 = \{m_1, m_2, \ldots, m_s\}$ be the set of maximal elements of P. If P is BU-feasible, then $w_{m_i}^* = w_{m_j}^*, \forall m_i, m_j \in \mathbf{M}$.*

Proof. For every pair of maximal elements m_i, m_j, consider the positioned antichain $(\{m_i, m_j\}, V = P \setminus (\downarrow m_i \cup \downarrow m_j))$. Then, $P \setminus (\downarrow\widehat{m_i} \cup \downarrow\widehat{m_j} \cup V) = \{m_i, m_j\}$, and hence $\mathcal{MIN}(P \setminus (\downarrow\widehat{m_i} \cup \downarrow\widehat{m_j} \cup V \cup m_j) = \{m_i\}$ and $\mathcal{MIN}(P \setminus (\downarrow\widehat{m_i} \cup \downarrow\widehat{m_j} \cup V \cup m_i)) = \{m_j\}$ and thus by Eq. (3.2), the corresponding equation is $w_{m_i} = w_{m_j}$. $\square$

Let us now deal with the problem of BU-feasibility of a general poset P. From Proposition 3.17, we just need to focus on the case of posets P that are irreducible by direct sum $\bigoplus$.

Theorem 3.21. *Let P be a finite poset irreducible by direct sum $\oplus$. Then, P is BU-feasible if and only if the system (3.2) has infinitely many solutions.*

Proof. $\Rightarrow$) Note that the system based on positioned antichains has the null vector as trivial solution, but this vector is not a valid weight vector as weights should be positive. Thus, BU-feasibility implies that the linear system has infinitely many solutions.

$\Leftarrow$) Let us prove that if the system generated by the positioned antichains has infinitely many solutions, then it is possible to obtain a weight vector; i.e. we shall prove that a vector with positive coordinates can be always obtained.

We are going to show it by induction on the levels of P. Let us start proving the induction step. Suppose that there is a solution w which is positive on the first k levels of P. Now we are going to show it for the elements in L_{k+1}. As P is irreducible, there exist at least two elements in $\cup_{i=0}^{k} L_i$ and there exists at least one $x \in L_{k+1}$ such that it is not covered by every element in $\cup_{i=0}^{k} L_i$. If there are several elements in $\cup_{i=0}^{k} L_i$ non-related to x we choose an element y in the L_t with lowest t, therefore in L_{t-1} every element is related to x. Consider the positioned antichain $(\{x, y\}, V)$ where $V = P \setminus \updownarrow \{x, y\}$. Then, from Def. 3.8, the corresponding equation is given by

$$w_x = w_y + \sum_{h \in \cup_{i=0}^{k} L_i, y \not< h, x < h} w_h. \tag{3.3}$$

By the induction hypothesis, the right hand side of the last equation is positive so $w_x > 0$. Consider now an element $z \in L_{k+1}$ such that it is covered by any element in $\cup_{i=0}^{k} L_i$. Since there exists $x \in L_{k+1}$ that is not covered by all elements in $\cup_{i=0}^{k} L_i$, let us consider the positioned antichain $(\{z, x\}, V)$ where $V = P \setminus \updownarrow \{x, z\}$. Again, this leads to the equation

$$w_z = w_x + \sum_{h \in \cup_{i=0}^{k} L_i, x \not< h, z < h} w_h. \tag{3.4}$$

The right hand side of the last equation is also positive. Hence, the induction step is finished. It just remains to show the basis step. So we have to prove that there is a solution having positive value for the maximals of P, that is for L_0. First, let us show that if the system has infinitely many solutions, then there exists a solution such that $w_{m_i} \neq 0$ for a maximal element. If any solution has $w_{m_i} = 0$ for all maximal elements, we can apply Eqs. (3.3) and (3.4) to conclude that $w_x = 0, \forall x \in L_1$. But then, we can repeat the procedure to conclude $w_x = 0, \forall x \in L_2$; and so on. Then, $w_x = 0, \forall x \in P$, and hence the system has just an only solution, which is a contradiction.

Assume then that we have a solution for the system satisfying $w_{m_i} \neq 0$, for a maximal element m_i. We can fix this value to be $w_{m_i} = 1$. Now, all maximal elements have the same positive weight by Lemma 4, so the induction is finished. $\square$

The proof of the last theorem has many interesting consequences. First, observe that in the case of an irreducible BU-feasible poset we can compute all the values of w^* by applying Eqs. (3.3) and (3.4). In this sense, just these equations are needed. However, the rest of equations derived from positioned antichains play an important role in the BU-feasibility of the poset, as they determine whether the previous solution holds. Moreover, the following corollaries follow as a direct consequence of this result.

Corollary 3.22. *Let P be a BU-feasible finite poset irreducible by direct sum $\oplus$. If w_1^* and w_2^* are two possible weights, then $\exists \lambda \in \mathbb{R}^+$ such that $w_1^* = \lambda \cdot w_2^*$.*

Proof. Applying equations (3.3) and (3.4), it can be concluded that all the coordinates in a suitable weight vector w^* depend on the (common) value of the maximal elements. If we multiply the value corresponding to the maximal elements by a coefficient λ, all other coordinates of the weight vector are also multiplied by λ just by linearity. □

Corollary 3.23. *Let P be a BU-feasible finite poset irreducible by direct sum $\oplus$. If $x \prec y$ then $w_x > w_y$.*

Proof. If we apply Equations 3.3 and 3.4 on each level and we use induction on $k \geq 1$, where the induction step supposes that the statement is true for the first k levels, $\cup_{i=0}^{k} L_i$, we get the result. □

Corollary 3.24. *Let P be a finite poset irreducible by direct sum $\oplus$. If $\mathrm{pa}(P) < |P|$, then there is a positive solution w^*.*

Joining Theorem 3.21, Proposition 3.17 and Corollary 3.24, the following results hold.

Corollary 3.25. *Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Then, there exists w_P^* if and only if the linear systems associated with $P_i, i = 1, ..., n$ have infinitely many solutions.*

Corollary 3.26. *Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. If $\mathrm{pa}(P_i) < |P_i|, i = 1, ..., n$, then there exists a weight vector w_P^*.*

Note that this condition is sufficient but not necessary (see for example the boolean lattice B_3 developed in Example 3.5).

Next, we are going to study a useful tool to compute w^*.

Proposition 3.27. *Let P be a BU-feasible finite poset and $x, y \in P$. If x and y are interchangeable, then $w_x^* = w_y^*$.*

Proof. Note that if x and y are interchangeable, then there is an automorphism $f : P \to P$ such that $f(x) = y$ and $f(y) = x$. If x and y are maximals, then $w_x^* = w_y^*$ by Lemma 3.20 and we are done, so we can suppose that they are not maximals. Since f is an isomorphism, then $\uparrow\{x,y\}\cong_f\uparrow\{f(x),f(y)\}=\uparrow\{x,y\}$. Therefore, f is an automorphism on $\uparrow\{x,y\}$. Note that $\uparrow\{x,y\}=\uparrow x\cup\uparrow y$ and define $F_1:=\uparrow\widehat{x}\cup\uparrow y$ and $F_2:=\uparrow x\cup\uparrow\widehat{y}$; then, $F_1\cong_f F_2$. Now, since P is BU-feasible and F_1, F_2 are filters of P then, by Theorem 3.18, we know that $w^*_{F_1}=w^*_{P|F_1}$ and $w^*_{F_2}=w^*_{P|F_2}$. Next, let us write F_1 and F_2 as sum of irreducible subposets, so that $F_1=F_1^1\oplus\ldots\oplus F_1^s$ and $F_2=F_2^1\oplus\ldots\oplus F_2^s$, where obviously $F_1^i\cong_f F_2^i$. If x and y are related to every non-minimal element of $\uparrow x\cup\uparrow y$, then the associated equation to $(a=\{x,y\},V=P\setminus\updownarrow\{x,y\})$ is $w^*_x=w^*_y$ and we are done, so we can suppose that x and y are not related to every non-minimal of $\uparrow x\cup\uparrow y$. Then, F_1^1 and F_2^1 have at least 2 elements and there exists some common element $z\in F_1^1\cap F_2^1$. Now, since F_1^1 is isomorphic to F_2^1, then $w^*_{F_1^1}$ and $w^*_{F_2^1}$ are solutions for the same subposet, and hence by Corollary 3.22 $w^*_{F_1^1}$ and $w^*_{F_2^1}$ are equal or proportional. Since both vectors have a common element z with a common weight w^*_z, we conclude that they are equal and thus, $w^*_x=w^*_{f(x)}=w^*_y$. □

The final version of the Bottom-Up algorithm is given in Algorithm 2.

Algorithm 2 BOTTOM-UP

1. ASSIGNING WEIGHTS

Step 1.1: Compute all possible positioned antichains.

Step 1.2: For any positioned antichain, build the corresponding linear equation.

Step 1.3: Solve the system of linear equations and choose a weight vector (if possible).

return w^*

2. SAMPLING STEP

$P'\leftarrow P$ ▷ Initialization

$S\leftarrow\emptyset$

while $|P'|>0$ **do**

Step 2.1: Select a minimal element x of P' with probability the quotient between w_x and the sum of the weights of $\mathcal{MIN}(P')$ and $S\leftarrow S\cup x$.

Step 2.2: $P'\leftarrow P'\setminus\{x\}$.

end

return S

Finally, note that sometimes we find that P is not BU-feasible but P^∂ is. Since there is an easy bijection between $\mathcal{L}(P)$ and $\mathcal{L}(P^\partial)$, we can use Bottom-Up with P^∂ and then change $\epsilon^*\in\mathcal{L}(P^\partial)$ for the corresponding $\epsilon\in\mathcal{L}(P)$.

3.5 Some relevant BU-feasible posets

In this section, we apply the Bottom-Up method to some families of posets in order to illustrate the performance of this algorithm.

3.5.1 Rooted Trees

In this case we are going to show that every rooted tree is BU-feasible. A rooted tree T [5] is a poset satisfying that its Hasse diagram (considered as an undirected graph) is connected and has no cycles, and it has just one minimal element (called root). An example of a rooted tree can be seen in Figure 3.3.

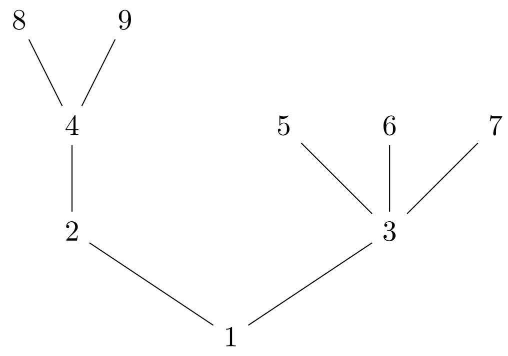


FIGURE 3.3: Rooted tree.

Consider a positioned antichain $(a = \{x, y\}, V)$ for a rooted tree. Note that, as T has no cycles and for any $z \in \mathcal{MIN}(P \setminus (\downarrow \hat{a} \cup V)), z \neq x, y$, it follows

$$z \in \mathcal{MIN}(P \setminus (\downarrow \hat{a} \cup V \cup \{x\})) \cap \mathcal{MIN}(P \setminus (\downarrow \hat{a} \cup V \cup \{y\})),$$

we can conclude that for two positioned antichains (a, V) and (a, V') with common antichain a, they share the same associated equation. In other words, the choice of the ideal V is not relevant.

Now, for $x \in T$, let us consider $\lambda_x := |\{y \in T | y \succeq x\}|$. As T has no cycles, it is easy to see by induction on the level that $\lambda_x = 1 + \sum_{x \lessdot h} \lambda_h$. Then, for $a = \{x, y\}$, the associated equation is

$$w_x + \sum_{y \lessdot z} w_z = w_y + \sum_{x \lessdot h} w_h.$$

Taking $w_x = \lambda_x$, we have

$$w_x+\sum_{y\lessdot z} w_z = \lambda_x+\sum_{y\lessdot z}\lambda_z = 1+\sum_{x\lessdot h}\lambda_h+\sum_{y\lessdot z}\lambda_z = \sum_{x\lessdot h}\lambda_h+1+\sum_{y\lessdot z}\lambda_z = \sum_{x\lessdot h}\lambda_h+\lambda_y = w_y+\sum_{x\lessdot h} w_h.$$

Consequently, any rooted tree is BU-feasible and $w_x^* = \lambda_x$ is a possible solution. Observe that the last argument remains valid for disjoint union of rooted trees by the same reasoning.

Finally, let us compute $e(T_n)$ for any rooted tree with n elements. Again, we have to keep in mind that $\lambda_x = 1 + \sum_{x\lessdot h}\lambda_h$. Then, it can be easily seen by induction that for a disjoint union of rooted trees P, $\sum_{x\in\mathcal{MIN}(P)}\lambda_x = n$ holds with $n = |P|$. Therefore, for any linear extension ϵ of a rooted tree

$$P(\epsilon) = \frac{\lambda_{\text{root}}}{n}\cdot\frac{\lambda_{x_1}}{n-1}\cdot\frac{\lambda_{x_2}}{n-2}\cdots,$$

hence

$$e(T_n) = \frac{n!}{\prod_{x\in T_n}\lambda_x}. \tag{3.5}$$

This formula was already shown by Stanley [5].

3.5.2 P_n family

Let us consider the following family of posets P_n, where P_n has $n+1$ levels, starting on level 0 for minimals[1]. In each level, we have an antichain of two elements, and they are related to the previous and the next level in the following way: one of the elements is related to any element of these levels, while the other one is related only to one element in each level; to fix notation, we assume that level i consists in $\{2i+1, 2i+2\}$ and odd numbers are related to all the elements of the previous and next level, while even numbers are related to odd numbers in the previous and next level. Figure 3.4 shows the Hasse diagrams of P_1, P_2 and P_3.

Thus labeled, the possible positioned antichains for P_n are: $(\{2i-1, 2i\}, V = \emptyset)$ and $(\{2i, 2i+2\}, V = \emptyset)$. Consequently, there are $2n+1$ positioned antichains and $2n+2$ elements. By Corollary 3.24, we conclude that P_n is BU-feasible. Let us then compute a weight vector w^*.

[1] Here we are reversing the definition of level of Chapter 1. However, the results for this example are more easy to follow with this change.

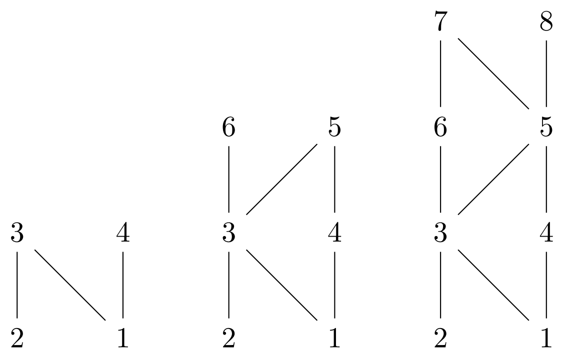


FIGURE 3.4: P_1, P_2 and P_3.

Applying Lemma 3.20, we can assign $w^*_{2n+1} = 1, w^*_{2n+2} = 1$ to the two top elements. Now, considering the positioned antichain $(\{2n, 2n+2\}, V = \emptyset)$, we obtain $w^*_{2n} = w^*_{2n+1} + w^*_{2n+2} = 2$. For $(\{2n-1, 2n\}, V = \emptyset)$, we obtain $w^*_{2n-1} = w^*_{2n} + w^*_{2n+2} = 3$. And we can continue this process until we reach the first level. For example, for P_3 we obtain the vector $w^* = (17, 12, 7, 5, 3, 2, 1, 1)$.

In order to derive a general formula, it is convenient to reverse the order, so that element i is assigned to w^*_{2n+3-i}; for example, for P_3 we would obtain the vector $w^* = (1, 1, 2, 3, 5, 7, 12, 17)$. Let us denote by o_n the n-th odd number of this sequence and e_n the n-th even number; then, it is easy to show by induction that

$$\begin{cases} o_n = o_{n-1} + e_{n-1} \\ e_n = o_n + o_{n-1} \\ o_1 = e_1 = 1 \end{cases}$$

Merging the second equation into the first one we have $o_n = 2o_{n-1} + o_{n-2}$ with $o_0 = 0$ and $o_1 = 1$. Let us solve this recursive equation through generating functions [79]. Let $F(x) := \sum_{k=0}^{\infty} x^k o_k$. Observe that:

$$F(x) = o_0 + o_1 x + \sum_{k=2}^{\infty} x^k o_k = x + 2\sum_{k=2}^{\infty} x^k o_{k-1} + \sum_{k=2}^{\infty} x^k o_{k-2} = x + 2xF(x) + x^2 F(x).$$

Thus, $F(x) = \dfrac{x}{1-2x-x^2}$ and it can be written as

$$F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \frac{A + B - (\beta A + \alpha B)x}{(1-\alpha x)(1-\beta x)}.$$

As $(1-\alpha x)(1-\beta x) = 1-2x-x^2$, we obtain that α and β satisfy

$$\begin{cases} \alpha\beta = -1 \\ \alpha+\beta = 2 \end{cases}$$

and hence $\alpha = \frac{2+\sqrt{4+4}}{2} = 1+\sqrt{2}$ and $\beta = \frac{2-\sqrt{4+4}}{2} = 1-\sqrt{2}$. On the other hand,

$$\left.\begin{array}{c} A+B=0 \\ \beta A+\alpha B=-1 \end{array}\right\} \Rightarrow A = \frac{1}{2\sqrt{2}},\ B = -\frac{1}{2\sqrt{2}}.$$

$$F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A\sum_{k=0}^{\infty}\alpha^k x^k + B\sum_{k=0}^{\infty}\beta^k x^k,$$

and hence

$$o_n = A\alpha^n + B\beta^n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$$

and

$$\begin{aligned} e_n = o_n + o_{n-1} &= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n + (1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right] \\ &= \frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]. \end{aligned}$$

Finally, let us obtain the number of linear extensions for P_n. Note that:

$$\begin{aligned} P(\epsilon = (2,1,4,3,...,2n+2,2n+1)) &= \frac{o_{n+1}}{o_{n+1}+e_{n+1}} \cdot 1 \cdot \frac{o_n}{o_n+e_n} \cdot 1 \cdot \frac{o_{n-1}}{o_{n-1}+e_{n-1}} \cdot \ldots \frac{o_2}{o_2+e_2} \cdot 1 \cdot \frac{1}{o_1+e_1} \cdot 1 \\ &= \frac{o_{n+1}}{o_{n+2}} \cdot 1 \cdot \frac{o_n}{o_{n+1}} \cdot 1 \cdot \frac{o_{n-1}}{o_n} \cdot \ldots \frac{o_2}{o_3} \cdot 1 \cdot \frac{1}{o_2} \cdot 1 \\ &= \frac{1}{o_{n+2}}. \end{aligned}$$

Hence, $e(P_n) = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+2} - (1-\sqrt{2})^{n+2}\right]$. The first values of $e(P_n)$ are given in Table 3.2.

n	2	3	4	5
$e(P_n)$	12	29	70	169

TABLE 3.2: Number of linear extensions of P_n.

3.5.3 Family H_n^k

In this example we are going to study a more complex family of posets.

Definition 3.28. We define the poset H_n^k as the poset having levels from 0 to n with k elements on each level. Every element of level $k > 0$ is lower than $k-1$ elements of level $k-1$ in such a way that two elements of level k cannot be lower than the same $k-1$ elements of level $k-1$.

See Figure 3.5 for a figure showing H_1^3, H_2^3 and H_3^3. Observe that H_n^k is well-defined, because if we take two posets A_n^k, B_n^k satisfying the last definition for some k and n we can get an isomorphism $f : A_n^k \to B_n^k$. Indeed, let us send minimals of A_n^k to minimals of B_n^k (level n) in some order, for every i in level $n-1$ we make $f(i) = k$, where i is the unique element in A_n^k covering $i_1, \ldots, i_{k-1}$ and k is the unique element in B_n^k covering $f(i_1), \ldots, f(i_{k-1})$. For levels $n-2, \ldots, 1, 0$ we continue inductively to get an isomorphism f. The last argument proves that minimals elements of H_n^k are interchangeable to each other. With a similar argument we can see that elements of the same level are interchangeable for any level. This way, in case of existing some solution w^*, it should be constant on each level.

We are not going to compute all positioned antichains but we are going to use the properties we know about it. As an example, consider H_3^3. The 3 maximals should have equal weight, by Lemma 3.20. Every positioned antichain with an antichain involving only maximals give us this condition. Now consider the next level and take an antichain, for example, $(a = (7, 12), V = \emptyset)$. For this antichain we have $w_7^* = w_{10}^* + w_{11}^* + w_{12}^* = 3$. Note that this is the only choice for V and by symmetry we have $w_7^* = w_8^* = w_9^* = 3$.

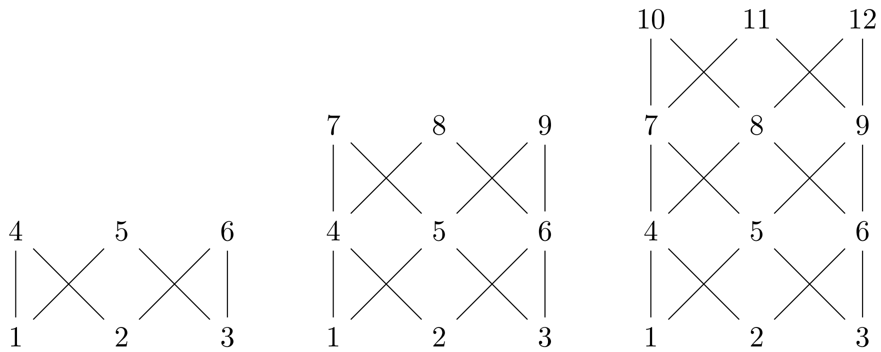


FIGURE 3.5: H_1^3, H_2^3 and H_3^3 .

Every row is just related with the one above and below. Now let do the same with the next level. To $(a = (4, 9), V = \emptyset)$ we obtain $w_4^* = w_7^* + w_8^* + w_9^* = 9$. Therefore $w_4^* = w_5^* = w_6^* = 9$ and $w_1^* = w_2^* = w_3^* = 27$. Note that this solution satisfies every equation because every positioned antichain has been considered. We can generalise this result and we obtain for H_n^3:

$$w_3^* = (3^n, 3^n, 3^n, \cdots, 3, 3, 3, 1, 1, 1)$$

Finally, for $\epsilon = (1, 2, 3, 4, \cdots, 3n-2, 3n-1, 3n)$ we get:

$$P(\epsilon) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3^n}{3^{n-1} + 3^n} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3^{n-1}}{3^{n-2} + 3^{n-1}} \cdots \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{3^{n+1}} \cdot \frac{1}{2^{n+1}} \cdot \left(\frac{3}{4}\right)^n = \frac{1}{6 \cdot 8^n}$$

Then,

$$e(H_n^3) = 6 \cdot 8^n.$$

Note that H_1^3 is B_3 without maximum and minimum (see Chapter 1) and has $6 \cdot 8 = 48$ linear extensions.

Now we can consider $k = 4$, i.e. we take 4 elements on each level. See Figure 3.6.

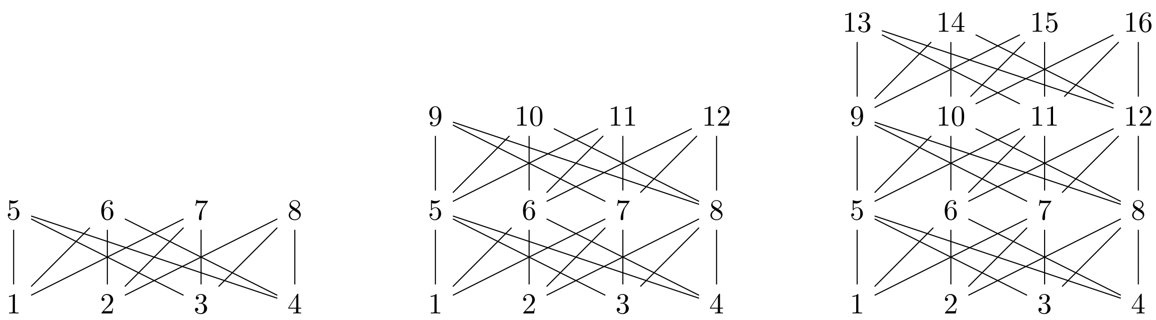


FIGURE 3.6: H_1^4, H_2^4 and H_3^4 .

Note that every element is connected with the above element and the next 2 elements from the above level, where it comes to the end it starts again. By a similar argument as the last one we get:

$$w_4^* = (4^n, 4^n, 4^n, 4^n, \cdots, 4, 4, 4, 4, 1, 1, 1, 1)$$

Moreover, for $\epsilon = (1, 2, 3, 4, \cdots, 4n-2, 4n-1, 4n)$ we get:

$$P(\epsilon) = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{4^n}{4^{n-1} + 4^n} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{4^{n-1}}{4^{n-2} + 3^{n-1}} \cdots \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4^{n+1}} \cdot \frac{1}{3^{n+1}} \cdot \frac{1}{2^{n+1}} \cdot \left(\frac{4}{5}\right)^n = \frac{1}{24 \cdot 30^n}$$

Then

$$e(H_n^4) = 24 \cdot 30^n$$

Now in the general case for H_n^k, by symmetry, we just have to take into account one positioned antichain between elements of different levels. This way we always obtain $w^* = 1$ for maximals and for level $k > 0$ we get

$$w_k^* = w_{k-1}^* + \overset{k}{\cdots} + w_{k-1}^* = k \cdot w_{k-1}^*,$$

where w_s^* is the value of the solution on the level s. This way we obtain that H_n^k is BU-feasible for every $n \geq 1$ and $k \geq 3$, with solution:

$$w_k^* = (k^n, k^n, \cdots, k^n, k^n, \cdots, k, k, \cdots, k, k, 1, 1, \cdots, 1, 1)$$

And

$$e(H_n^k) = k^{n+1} \cdot (k-1)^{n+1} \cdots 2^{n+1} \cdot \left(\frac{k+1}{k}\right)^n = k![(k+1) \cdot (k-1)!]^n.$$

3.5.4 $TB_n(2,2)$ twisted band

In the following examples we are going to study a family of posets consisting in two chains and different connections between them. We call this kind of posets **bands**. Specifically, when each element of the first chain is connected with the one which is k_1 levels above in the other chain and the elements of the second chain are connected with the elements which are in the first chain k_2 levels above we call this poset $(\mathbf{k_1}, \mathbf{k_2}) -$ **twisted band** or for short $TB(k_1, k_2)$. Observe that $(1,1)-$twisted bands are just $2 \oplus 2 \oplus 2 \cdots$. In this example we are going to study the $(2,2)$- twisted bands. See Figure 3.7.

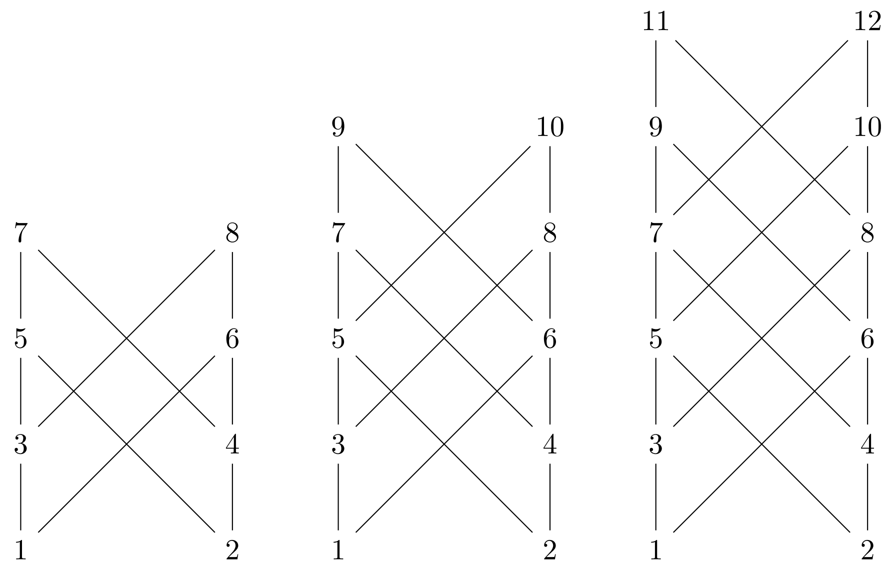


FIGURE 3.7: $TB_4(2,2), TB_5(2,2)$ and $TB_6(2,2)$.

Observe that every antichain must have one element of the first chain and one of the second chain. Then, the only choice for V is the emptyset. Also by symmetry, the elements in the same level have the same weight. Working with $TB_4(2,2)$ we have that

$w_7^* = w_8^* = 1$. The antichain $a = (5, 8)$ give us $w_5^* = w_7^* + w_8^* = 2$. And by symmetry $w_6^* = 2$. For the following level, we have $a = (3, 6)$ with $w_3^* = w_5^* + w_6^* = 4$. We can use this argument for every level and we will be considering every positioned antichain. Indeed, for level k we get

$$w_k^* = 2w_{k-1}^*,$$

where w_s^* is the value of the solution on the level s.

Thus, in general $TB_n(2,2)$ is BU-feasible and

$$w^* = (2^n, 2^n, 2^{n-1}, 2^{n-1}, \cdots, 4, 4, 2, 2, 1, 1)$$

Moreover,

$$P(\epsilon = id) = \frac{1}{2} \cdot \frac{2^n}{2^n + 2^{n-1}} \cdot \frac{1}{2} \cdot \frac{2^{n-1}}{2^{n-1} + 2^{n-2}} \cdots \frac{1}{2} \cdot 1 = \frac{1}{2^n} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{2 \cdot 3^{n-1}}.$$

Thus:

$$e(TB_n(2,2)) = 2 \cdot 3^{n-1}.$$

3.5.5 $TB_n(1,2)$ twisted band

In this example we are going to study the $(1,2)$-twisted bands. See Figure 3.8.

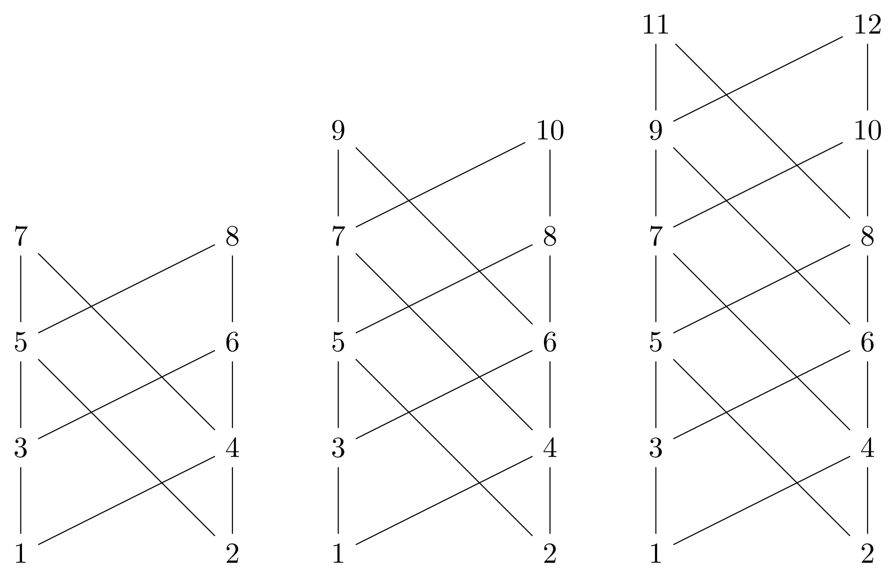


FIGURE 3.8: $TB_4(1,2), TB_5(1,2)$ and $TB_6(1,2)$.

As before, observe that every antichain must have one element of the first chain and one of the second chain. Then the only choice for V is the emptyset. Working with $TB_4(1,2)$ we have that $w_7^* = w_8^* = 1$. Now for the next level we can consider the antichain $a = (6,7)$ which link the elements between the first and the second level. This antichain give us $w_6^* = w_7^* + w_8^* = 2$. Now considering the antichain $a = (5,6)$ which link 2 elements in the same level we have $w_5^* = w_6^* + w_7^* = 1 + 2 = 3$. Now for the next level we have $a = (4,5)$ with $w_4^* = w_5^* + w_6^* = 3 + 2 = 5$ and $a = (3,4)$ with $w_3^* = w_4^* + w_5^* = 5 + 3 = 8$.

We can follow this procedure to obtain a solution w^* for $TB_n(1,2)$. If we consider an antichain with two elements of the same level k we get the equation on w^*

$$w_{k,l}^* = w_{k,r}^* + w_{k-1,l}^*,$$

where $w_{s,t}^*$ is the weight associated to element of level s on the right if $t = r$ or on the left if $t = l$. For an antichain with two elements of different levels k and $k-1$ we get

$$w_{k,r}^* = w_{k-1,r}^* + w_{k-1,l}^*.$$

Observe that if we label the elements of $TB_n(1,2)$ as in Figure 3.8 we get that the value of w_i^* is the sum of the last 2 values, $w_i^* = w_{i-1}^* + w_{i-2}^*$. Then we obtain a BU-feasible poset with weights:

$$w_n^* = f_n$$

where f_n is the n-th Fibonacci number. Remember that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{(1+\sqrt{5})}{2}\right)^n - \left(\frac{(1-\sqrt{5})}{2}\right)^n\right].$$

For example for $TB_4(1,2)$ we get $w^* = (21, 13, 8, 5, 3, 2, 1, 1)$. Finally we have, if $\epsilon = (2, 1, 4, 3, 6, 5, \cdots, 2n, 2n-1)$:

$$\begin{aligned} P(\epsilon = id) &= \frac{f_{2n-1}}{f_{2n} + f_{2n-1}} \cdot 1 \cdot \frac{f_{2n-3}}{f_{2n-2} + f_{2n-3}} \cdot 1 \cdots \frac{f_1}{f_2 + f_1} \cdot 1 = \\ &= \frac{f_{2n-1}}{f_{2n+1}} \cdot 1 \cdot \frac{f_{2n-3}}{f_{2n-1}} \cdot 1 \cdots \frac{f_1}{f_3} \cdot 1 = \frac{f_1}{f_{2n+1}}. \end{aligned}$$

Thus:

$$e(TB_n(1,2)) = f_{2n+1}.$$

3.6 Application to fuzzy measures

As we saw in Section 2.6.1, fuzzy measures can be generalized to the case of a restricted collection of coalitions $\Omega \subseteq \mathcal{P}(X)$. In this context, $(\Omega, \subseteq)$ is a poset and the set of Ω-restricted normalized fuzzy measures, $\mathcal{RFM}(\Omega)$, is the order polytope $\mathcal{O}(\Omega)$. Moreover, for every poset P there is some collection of feasible coalitions $\Omega(P)$ such that $\mathcal{RFM}(\Omega(P)) = \mathcal{O}(P)$. In other words, for some finite poset P, its order polytope $\mathcal{O}(P)$ models some set of restricted normalized fuzzy measures. To sample points uniformly inside $\mathcal{O}(P)$ we need to know how to sample linear extensions (see Section 1.3.1). In order to achieve this, when the poset P is BU-feasible, we can use the BU-method. In this section, we study the performance of BU-method to some relevant families of fuzzy measures.

Unfortunately, the Boolean lattice B_n is not BU-feasible if and only if $n \geq 3$. This way the polytope of unrestricted normalized fuzzy measures on a referential X, $\mathcal{FM}(X)$, cannot be studied via BU-method when $|X| > 3$.

Some other interesting subfamilies of $\mathcal{FM}(X)$ are associated to BU-feasible posets. As we saw in Section 2.6.2, a k-tolerant capacity can be identified with a fuzzy measure restricted to sets of size lower than k. We can generalize this concept by considering sets of k different cardinalities, not necessarily the first k cardinalities.

3.6.1 Boolean lattice B_n, $n > 3$

For $n > 3$, B_n is not BU-feasible. Consider B_4, see Figure 3.9.

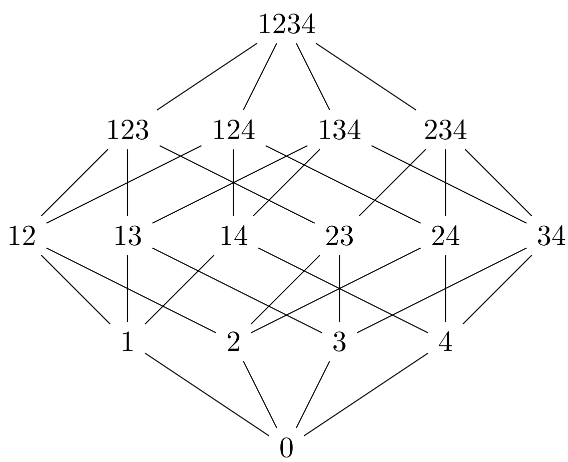


FIGURE 3.9: B_4 lattice.

We are going to see that the associated linear system of equations does not have any positive solution. The maximal and the minimal elements are not important and we can remove them to work. The maximal elements 123,124,134 and 234 have the same value of w^* by Lemma 3.20. Now we take the positioned antichain with $a = (12, 134)$ and V the set of all the elements not related to a. Then we get $w^*_{12} = w^*_{123} + w^*_{124} + w^*_{134} = 6$. By simmetry, every element in the same level has the same value w^*. Now for the first level we can choose $a = (1, 23)$ and V all the elements not related to a. Then, we get $w^*_{12} + w^*_{13} + w^*_{14} + w^*_{23} = w^*_1 + w^*_{234} \Leftrightarrow w^*_1 = 22$. Now, taking $a = (1, 23)$ and $V = \emptyset$ we get $w^*_1 = w^*_{12} + w^*_{13} + w^*_{14} + w^*_{23} = 24$, and then there is no solution to this system, so B_4 is not BU-feasible. Now since $B_n = 2 \times B_{n-1}$ we know that B_{n-1} is a filter of B_n. Thus, by Theorem 3.18, B_n is not BU-feasible for $n \geq 4$.

On the number of linear extensions of B_n, observe that for $n \geq 3$, $pa(B_n) \geq 2\binom{n}{2} i(B_{n-2})$. To see this, it is enough to count the positioned antichains where the antichain a consists of 2 singletons or 2 subsets with $n-1$ elements. Observe that $i(B_n)$ are the Dedekind numbers $M(n)$ (see Section 2.2). As $pa(B_n) \leq e(B_n)$ we get the following inequality:

$$e(B_n) \geq 2\binom{n}{2} M(n-2).$$

This statement allows to know that $e(B_n)$ grows at least as quickly as the Dedekind numbers, that is surprisingly fast.

3.6.2 Boolean lattice B_n, $n \leq 3$

For $n \leq 3$, B_n is BU-feasible. For $n = 1, 2$ is obvious. If $n = 3$ we have seen in Example 3.5 that B_3 is BU-feasible with $w^* = (1, 6, 6, 6, 2, 2, 2, 1)$ where coordinates $2, 3, 4$ stand for singletons and coordinates $5, 6, 7$ for subsets of two elements (we have multiplied by 2 the solution obtained in the Example 3.5, since the only important things are the proportions).

3.6.3 k-truncated fuzzy measures.

Let us define now the concept of k-truncated fuzzy measures.

Definition 3.29. We say that a fuzzy measure over a referential set of size n is **k-truncated** with respect to some cardinalities $r_1, \ldots, r_k$ if the set of feasible subsets is the set of all subsets whose cardinalities are $r_1, \ldots, r_k$. We will denote the set of k-truncated fuzzy measures by $\mathcal{TFM}^n(r_1, \ldots, r_k)$.

Note that k-truncated measures are the Ω-restricted capacities where Ω is the collection of all sets with k different cardinalities $r_1, \ldots, r_k$.

In a similar way, we denote by $B_k^n(r_1, \ldots, r_k)$ the Boolean poset (without maximum and minimum) restricted to the sets of size $r_1, \ldots, r_k$. In the next section we will study the case $\mathcal{TFM}^n(1, n-1)$.

3.6.4 Truncated fuzzy measures $\mathcal{TFM}^n(1, n-1)$.

In this section we study the set of truncated fuzzy measures $\mathcal{TFM}^n(1, n-1)$. By definition, $\mathcal{TFM}^n(1, n-1)$ is the order polytope associated to $B_2^n(1, n-1)$. Let $n \in \mathbb{N}, X = \{1, ..., n\}$ and consider the poset $B_2^n(1, n-1)$ consisting of all the subsets of X that are either singleton or whose complementary is a singleton, and consider the order relation given by $x \prec y \Leftrightarrow x \subset y$. Figure 3.10 shows $B_2^3(1, 2), B_2^4(1, 3)$ and $B_2^5(1, 4)$.

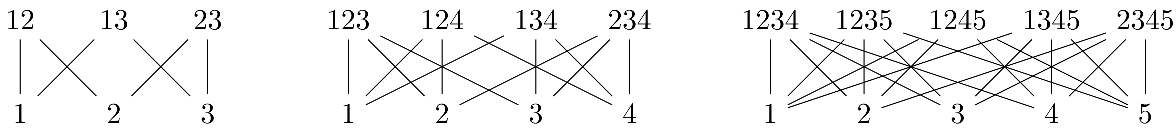


FIGURE 3.10: $B_2^3(1, 2), B_2^4(1, 3)$ and $B_2^5(1, 4)$.

Let us first find out the possible positioned antichains. We have three different cases:

- Case 1: Derived from $\{i, j\}$. We have $\binom{n}{2}$ possibilities. For each of them, we have the $n-2$ singletons that are incomparable to both of them, so that we have 2^{n-2} possible choices of V.
- Case 2: Derived from $\{X \backslash \{i\}, X \backslash \{j\}\}$. As in the previous case, there are $\binom{n}{2}$ possibilities. For each of them, we have the $n-2$ subsets of cardinality $n-1$ that are incomparable to both of them, so that we have 2^{n-2} possible choices of V.
- Case 3: Derived from $\{i, X \backslash \{i\}\}$. In this case, there are n possibilities and any other element in the poset compares to one of them, so that $V = \emptyset$.

Then, we have $n(n-1)2^{n-2} + n$ positioned antichains, a number much larger than $|B_2^n(1, n-1)| = 2n$ and consequently, Corollary 3.24 cannot be applied. However, we will show that in this case it is possible to find a weight vector w^*.

Note that by Lemma 3.20, we know that if w^* exists, then all maximal elements have the same weight, say 1; thus $w^*_{X \backslash \{i\}} = 1, \forall i \in X$. Now, take $i \in X$ and consider the positioned antichain $(\{i, X \backslash \{i\}\}, \emptyset)$; then, it follows that $w_i^* = n$. It suffices to show that no contradiction arises for any other condition derived from other positioned antichain.

- For $(\{i, j\}, V)$ we derive $w_i^* = w_j^*$.

- For $(\{X\backslash\{i\}, X\backslash\{j\}\}, V)$ we derive $w^*_{X\backslash\{i\}} = w^*_{X\backslash\{j\}}$.

Thus, our solution fits all the equations. For example, for $B_2^4(1,3)$, $B_2^5(1,4)$ and $B_2^5(1,5)$ the corresponding weight vectors are $(3,3,3,1,1,1)$, $(4,4,4,4,1,1,1,1)$ and $(5,5,5,5,5,1,1,1,1,1)$, respectively, and for $B_2^n(1,n-1)$ we obtain $w^* = (n,...,n,1,...,1)$.

Finally, let us use this vector to compute $e(B_2^n(1,n-1))$. If we choose an extension ϵ whose first n elements are the singletons, we obtain:

$$p(\epsilon) = \frac{n}{n^2} \cdot \frac{n}{(n-1)n} \cdot \frac{n}{(n-2)n} \cdots \frac{n}{2n} \cdot \frac{n}{n+1} \cdot \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{n}{n!(n+1)!}.$$

Therefore,

$$e(B_2^n(1,n-1)) = (n+1)!(n-1)!.$$

3.7 Extensions of BU method: Ideal-based method

In this section we provide a generalization of the Bottom-Up method which can be applied to any poset. However, the complexity of the associated algorithm will increase significantly.

Definition 3.30. Let P be a finite poset, $I \subseteq P$ be an ideal of P and $x \in \mathcal{MIN}(P \setminus I)$. Then, we define the **conditional probability** of x given I as

$$P(x|I) = \frac{e\left(P \setminus (I \cup \{x\})\right)}{e(P \setminus I)}.$$

Remark 3.31.

- Note that this probability represents the proportion of linear extensions of $P \setminus I$ which start by x. Obviously, if we sum $P(x|I)$ over all the minimal elements $x \in \mathcal{MIN}(P \setminus I)$ we get $\sum_{x \in \mathcal{MIN}(P\setminus I)} P(x|I) = 1$. Thus, the conditional probability is a probability distribution on $\mathcal{MIN}(P \setminus I)$ for a fixed ideal I.
- If I is an ideal such that $P \setminus I \cong \overline{n}$, being $\overline{n}$ the antichain of n elements then for any $x \in P \setminus I$, $P(x|I) = \dfrac{1}{n}$ because $e\left(P \setminus (I \cup \{x\})\right) = (n-1)!$ and $e\left(P \setminus I\right) = n!$.
- We can extend the last definition to any $x \in P$ by doing $P(x|I) = 0$ for any $x \notin \mathcal{MIN}(P \setminus I)$, so that we obtain a probability distribution on $P \setminus I$.

In what follows, let us denote $P_x = P \setminus \{x\}$ and $P_{xy} = P \setminus \{x, y\}$.

Lemma 3.32. *Let P be a finite poset, $I \subseteq P$ be an ideal of P, and $x, y \in \mathcal{MIN}(P \setminus I)$. Then*

$$P(x|I) \cdot P(y|I \cup \{x\}) = P(y|I) \cdot P(x|I \cup \{y\}). \tag{3.6}$$

Proof. It follows almost trivially:

$$\begin{aligned} P(x|I) \cdot P(y|I \cup \{x\}) &= \frac{e\left(P_x \setminus I\right)}{e(P \setminus I)} \cdot \frac{e\left(P_{xy} \setminus I\right)}{e(P_x \setminus I)} = \\ &= \frac{e\left(P_y \setminus I\right)}{e(P \setminus I)} \cdot \frac{e\left(P_{xy} \setminus I\right)}{e(P_y \setminus I)} = P(y|I) \cdot P(x|I \cup \{y\}). \end{aligned}$$

Therefore, the result holds. □

By the last lemma, we can compute the probabilities $P(x|I)$ where $|I| = m$ if we know the probabilities associated to the ideals with one element more $P(x|I')$, with $|I| = m+1$. On the other hand, if $I' = P_x$ then $P(x|I') = 1$ and we can obtain inductively the value of $P(x|I)$ for smaller ideals I. Indeed, the way of doing this is as simple as solving the next linear system. Fix $m \in \mathbb{N}$ and consider that we know the value of $P(x|I')$ for any ideal I' with $m+1$ elements and $x \in \mathcal{MIN}(P \backslash I')$. Suppose $\mathcal{MIN}(P \backslash I) = \{x_1, x_2, \ldots, x_k\}$. The goal is to compute $P(x_i|I)$, by using $P(x_i|I) \cdot P(x_j|I \cup \{x_i\}) = P(x_j|I) \cdot P(x_i|I \cup \{x_j\})$. Now,

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ p_{1,2} & -p_{2,1} & 0 & \cdots & 0 \\ p_{1,3} & 0 & -p_{3,1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{1,k} & 0 & 0 & \cdots & -p_{k,1} \end{pmatrix} \cdot \begin{pmatrix} P(x_1|I) \\ P(x_2|I) \\ \cdots \\ P(x_k|I) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \tag{3.7}$$

where $p_{i,j} = P(x_j|I \cup \{x_i\})$. The first equation comes from Remark 3.31 and the next ones form Equation 3.6. If we call A to the last matrix of probabilities, by making the numbers under the first element of the matrix into zero using Gauss-Jordan elimination it is straightforward to show that

$$det(A) = (-1)^{k-1} \prod_{i=2}^{k} p_{i,1} \left(1 + \sum_{i=2}^{k} \frac{p_{1,i}}{p_{i,1}} \right) \neq 0.$$

Then, A is invertible with inverse

$$A^{-1} = \begin{pmatrix} \dfrac{1}{(1+\Sigma)} & \dfrac{1}{p_{2,1}(1+\Sigma)} & \dfrac{1}{p_{3,1}(1+\Sigma)} & \dots & \dfrac{1}{p_{k,1}(1+\Sigma)} \\ \dfrac{p_{1,2}}{p_{2,1}(1+\Sigma)} & \dfrac{p_{1,2}-p_{2,1}(1+\Sigma)}{p_{2,1}^2(1+\Sigma)} & \dfrac{p_{1,2}}{p_{2,1}p_{3,1}(1+\Sigma)} & \dots & \dfrac{p_{1,2}}{p_{2,1}p_{k,1}(1+\Sigma)} \\ \dfrac{p_{1,3}}{p_{3,1}(1+\Sigma)} & \dfrac{p_{1,3}}{p_{2,1}p_{3,1}(1+\Sigma)} & \dfrac{p_{1,3}-p_{3,1}(1+\Sigma)}{p_{3,1}^2(1+\Sigma)} & \dots & \dfrac{p_{1,3}}{p_{3,1}p_{k,1}(1+\Sigma)} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{p_{1,k}}{p_{k,1}(1+\Sigma)} & \dfrac{p_{1,k}}{p_{2,1}p_{k,1}(1+\Sigma)} & \dfrac{p_{1,k}}{p_{3,1}p_{k,1}(1+\Sigma)} & \dots & \dfrac{p_{1,k}-p_{k,1}(1+\Sigma)}{p_{k,1}^2(1+\Sigma)} \end{pmatrix}$$

where $\Sigma = \sum_{i=2}^{k} \frac{p_{1,i}}{p_{i,1}}$. Therefore,

$$P(x_1|I) = \frac{1}{(1+\Sigma)}, \qquad P(x_i|I) = \frac{p_{1,i}}{p_{i,1}(1+\Sigma)} \quad i \neq 1.$$

Thus, if we know the value of $P(x|I')$ for any ideal I' with $m+1$ elements we can easily compute the value of $P(x|I)$ for every ideal I with m elements. Applying the last idea a finite number of times we get the value of $P(x|I)$ for every ideal I and $x \in P \setminus I$.

Remark 3.33.

- We can use these ideas to sample linear extensions uniformly in a similar way as we do with the Bottom-Up method. Indeed, the Ideal-based method is a generalization of the Bottom-Up method. As we do with the BU algorithm we choose a minimal element in each step with a given probability, then we save the chosen element x and continue with the minimal elements after removing x. In the Ideal-based method we can do the same, we draw a minimal element x with probability proportional to $P(x|I)$, where I is the set of elements that have already been drawn. This way we get a linear extension with uniform probability,

$$P(\epsilon) = P(x_1|\emptyset) \cdot P(x_2|x_1) \cdots P(x_n|P \setminus x_{n-1}) = \frac{e(P_{x_1})}{e(P)} \cdot \frac{e(P_{x_1 x_2})}{e(P_{x_1})} \cdots 1 = \frac{1}{e(P)}.$$

  Thus, we can also use this method to compute the number of linear extensions.

- The main advantage of the Ideal-based method is that it can be applied to every poset. Indeed, we just need to compute $P(x|I)$ for any combination of $x \in \mathcal{MIN}(P \setminus I)$ and $I \in \mathcal{I}(P)$.

- The main disadvantage of the Ideal-based method is that we need to know every ideal of P in advance. Sometimes, there are too many ideals to work with and this method is unfeasible to apply. Even in the case of knowing the structure of the different ideals, we need to do a lot of computations to get the value of every probability. Note however that the number of these probabilities is bounded by $w(P) \cdot i(P)$.

- If P is BU-feasible, then we know that there are some $w_x > 0$, $\forall x \in P$, such that

$$P(x|I) = \frac{w_x}{\sum_{z \in \mathcal{MIN}(P \setminus I)} w_z}.$$

  In this case the BU method allows us to compute all the probabilities $P(x|I)$, for every ideal I and $x \in P \setminus I$, by solving an easier system of linear equations involving just $|P|$ variables.

- For any series-parallel poset P we can compute all the conditional probabilities associated to each ideal recursively by following the next two rules. If $P \setminus I$ is a disjoint union of filters $C_1 \uplus C_2$ such that $x \in \mathcal{MIN}\,(C_1)$, we get

$$P(x|I) = \frac{e(P_x \setminus I)}{e(P \setminus I)} = \frac{\binom{|C_1|+|C_2|-1}{|C_1|-1, |C_2|} e(C_1 \setminus \{x\}) e(C_2)}{\binom{|C_1|+|C_2|}{|C_1|, |C_2|} e(C_1) e(C_2)} = \frac{|C_1|}{|C_1| + |C_2|} \cdot \frac{e(C_1 \setminus \{x\})}{e(C_1)},$$

  where $\dfrac{e(C_1 \setminus \{x\})}{e(C_1)}$ is a probability associated to a smaller filter. This way we can focus on the connected component where x belongs. Similarly if $P \setminus I$ is a direct sum $C_1 \oplus C_2$ and $x \in \mathcal{MIN}\,(C_1)$,

$$P(x|I) = \frac{e(P_x \setminus I)}{e(P \setminus I)} = \frac{e(C_1 \setminus \{x\}) e(C_2)}{e(C_1) e(C_2)} = \frac{e(C_1 \setminus \{x\})}{e(C_1)},$$

  this way the direct sum factors can be removed to compute conditional probabilities.

The Ideal-based algorithm is given in Algorithm 3. Similar algorithms can be found in the literature (see [9]):

Example 3.6. *Let us see a simple example of the last method. For the N poset (see Figure 1.1) we have the following ideals $\mathcal{I}(N) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}, N\}$, and the probabilities can be computed recursively obtaining the ones in Table 3.3.*

Algorithm 3 IDEAL-BASED ALGORITHM

1. COMPUTING IDEALS

Step 1.1: Compute all possible ideals (see [9] for an algorithm doing this).

2. COMPUTING PROBABILITIES

Step 2.1: Compute recursively all possible conditional probabilities by starting with the biggest ideals.

3. SAMPLING STEP

$P' \leftarrow P$ ▷ Initialization

$S \leftarrow \emptyset$

while $|P'| > 0$ **do**

Step 3.1: Select a minimal element x of P' with probability $P(x|S)$.

Step 3.2: $P' \leftarrow P' \setminus \{x\}$ and $S \leftarrow S \cup \{x\}$.

end

return S

Elements / Ideals	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$	$\{1,2,4\}$
1	2/5	0	2/3	0	1	0	0
2	3/5	1	0	0	0	0	0
3	0	0	1/3	1/2	0	0	1
4	0	0	0	1/2	0	1	0

TABLE 3.3: Ideal-Based method for N, each cell contains $P(x|I)$.

These values have been obtained as follows:

$$\begin{aligned} P(1|\{2\})P(3|\{1,2\}) = P(3|\{2\})P(1|\{2,3\}) \quad &\Leftrightarrow \quad P(1|\{2\}) \cdot \frac{1}{2} = P(3|\{2\}) \cdot 1 \Rightarrow \\ P(1|\{2\}) = \frac{2}{3}, \quad &\textit{and} \quad P(3|\{2\}) = \frac{1}{3}. \\ P(1|\emptyset)P(2|\{1\}) = P(2|\emptyset)P(1|\{2\}) \quad &\Leftrightarrow \quad P(1|\emptyset) \cdot 1 = P(2|\emptyset) \cdot \frac{2}{3} \Rightarrow \\ P(1|\emptyset) = \frac{2}{5}, \quad &\textit{and} \quad P(2|\emptyset) = \frac{3}{5}. \end{aligned}$$

As we already know, this poset is BU-feasible indeed, for examples regarding the performance of Ideal-based method on non BU-feasible posets see Section 3.7.1.

Observe that most of the probabilities $P(x|I)$ take trivial values 0 or 1, sometimes because $P \setminus I$ contains just one element, or in other cases because x is not a minimal of $P \setminus I$. As we will see below, all the non-trivial probabilities are associated to positioned antichains of P. We say that Equation 3.6 is *not trivial* for some ideal I and $x, y \in P \setminus I$ if this equation is different from $0 = 0$ and $1 = 1$.

Lemma 3.34. *Let P be a finite poset, I some ideal of P and $x, y \in P \setminus I$. Then:*

i) The Equation 3.6 is not trivial if and only if $x, y \in \mathcal{MIN}(P \setminus I)$.

ii) *If Equation 3.6 is not trivial, then* $(a = (x, y), I \setminus \downarrow \widehat{a})$ *is a positioned antichain of* P.

Proof. *i)* $\Rightarrow$) Since $P(x|I) \cdot P(y|I \cup \{x\}) \neq 0$ we get $P(x|I) \neq 0$ and $P(y|I \cup \{x\}) \neq 0$. Similarly, $P(y|I) \neq 0$ and $P(x|I \cup \{y\}) \neq 0$. Since $P(x|I) \neq 0$ and $P(y|I) \neq 0$ then $x, y \in \mathcal{MIN}(P \setminus I)$.

$\Leftarrow$) In this case, $P(x|I) = \dfrac{e(P_x \setminus I)}{e(P \setminus I)} \notin \{0, 1\}$ because we have at least two minimal elements and $P(y|I \cup \{x\}) = \dfrac{e(P_{xy} \setminus I)}{e(P_x \setminus I)} \neq 0$. The same is obtained for $P(y|I)$ and $P(x|I \cup \{y\})$, therefore the equation

$$P(x|I) \cdot P(y|I \cup \{x\}) = P(y|I) \cdot P(x|I \cup \{y\}),$$

is not trivial.

ii) Now, let us suppose that the Equation 3.6 is not trivial. By *i*), we know that $x, y \in \mathcal{MIN}(P \setminus I)$. Clearly, $a = (x, y)$ is an antichain and $I \setminus \updownarrow a$ is an ideal of $P \setminus \updownarrow a$. As $x, y \in \mathcal{MIN}(P \setminus I)$ thus $I \setminus \updownarrow a = I \setminus \downarrow \widehat{a}$. Obviously $I \setminus \updownarrow a \subseteq I \setminus \downarrow \widehat{a}$. For proving $I \setminus \downarrow \widehat{a} \subseteq I \setminus \updownarrow a$, let $z \in I \setminus \downarrow \widehat{a}$ and let us check that $z \notin \uparrow a$. As $z \in I$ and $a \cap I = \emptyset$, thus $z \neq x$ and $z \neq y$. Assume that $z \in \uparrow \widehat{a}$ and suppose without loss of generality that $z \succ x$. Since $z \in I$, which is an ideal, we get $x \in I$, which is a contradiction. Consequently $I \setminus \updownarrow a = I \setminus \downarrow \widehat{a}$ and $(a, I \setminus \downarrow \widehat{a})$ is a positioned antichain of P.

Then, the result holds. $\square$

This way it is enough to compute all the positioned antichains $(a = (x, y), V)$ and use recursively from big to small ideals the equation

$$P(x|V \cup \downarrow \widehat{a}) \cdot P(y|V \cup \downarrow \widehat{a} \cup \{x\}) = P(y|V \cup \downarrow \widehat{a}) \cdot P(x|V \cup \downarrow \widehat{a} \cup \{y\}).$$

This technique leads to a system where the probabilities which are out of the system are trivial, i.e. which value 0 or 1. Observe that in the Ideal-based method the positioned antichains plays a similar role that in BU method. However, here we do not work with any weight w_x, but we work directly with the conditional probabilities $P(x|I)$.

These equations are not linear but can be transformed into linear equations by taking logarithms.

$$\log\left(P(x|V \cup \downarrow \widehat{a})\right) + \log\left(P(y|V \cup \downarrow \widehat{a} \cup \{x\})\right) = \log\left(P(y|V \cup \downarrow \widehat{a})\right) + \log\left(P(x|V \cup \downarrow \widehat{a} \cup \{y\})\right).$$

This way, by Lemma 3.34, we obtain a system of $pa(P)$ linear equations with unique solution. Using the last idea we can state Algorithm 3 using positioned antichains, see Algorithm 4. Computing all the positioned antichains is an easy combinatorial problem for posets with small width. For big width posets, computing all the positioned antichains is a problem as difficult as the problem of computing all the ideals (see [9]).

Algorithm 4 IDEAL-BASED ALGORITHM 2.0

1. COMPUTING POSITIONED ANTICHAINS

Step 1.1: Compute all possible positioned antichains.

2. COMPUTING PROBABILITIES

Step 2.1: Solve the system of linear equation associated to the non-trivial probabilities.

3. SAMPLING STEP

$P' \leftarrow P$ ▷ Initialization

$S \leftarrow \emptyset$

while $|P'| > 0$ **do**

Step 3.1: Select a minimal element x of P' with probability $P(x|S)$.

Step 3.2: $P' \leftarrow P' \backslash \{x\}$ and $S \leftarrow S \cup \{x\}$.

end

return S

3.7.1 Application of Ideal-based method

In this section we are going to study some applications of the Ideal-based method. In general we could compute all the neccesary probabilities with a computer, however in some cases we are able to use the last theory to derive closed or recursive formulas.

3.7.1.1 Poset $2 \times \mathbf{n}$

Ideal-based method give us an easy recursive formula for computing the conditional probabilities of $2 \times \mathbf{n}$, see Figure 3.11.

Note that the positioned antichains of $2 \times \mathbf{n}$ are the ones of the form $(a = (x_i, y_j), V = \emptyset)$ where $i > j$. Let us denote by $\gamma_{i,j}$ to the ideal associated to $(a = (x_i, y_j), V = \emptyset)$, i.e. $\gamma_{i,j} = \downarrow \{x_i, y_j\}$. By Equation 3.6 follows:

$$P(x_{i+1}|\gamma_{i,j}) \cdot P(y_{j+1}|\gamma_{i+1,j}) = P(y_{j+1}|\gamma_{i,j}) \cdot P(x_{i+1}|\gamma_{i,j+1}).$$

Therefore,

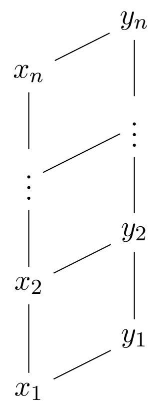


FIGURE 3.11: Poset $2 \times \mathbf{n}$.

$$P(x_{i+1}|\gamma_{i,j}) = \frac{P(x_{i+1}|\gamma_{i,j+1})}{P(x_{i+1}|\gamma_{i,j+1}) + P(y_{j+1}|\gamma_{i+1,j})},$$

$$P(y_{j+1}|\gamma_{i,j}) = \frac{P(y_{j+1}|\gamma_{i+1,j})}{P(x_{i+1}|\gamma_{i,j+1}) + P(y_{j+1}|\gamma_{i+1,j})}.$$

Obviously $P(x_n|\gamma_{n-1,n-2}) = P(y_{n-1}|\gamma_{n-1,n-2}) = 1$ are the initial values of the last recursion.

3.7.1.2 Family GS_n

Consider the poset GS_n with $4n + 2$ elements consisting in two antichains of $2n + 1$ points, $\{x_1, \ldots, x_{2n+1}\}$ and $\{y_1, \ldots, y_{2n+1}\}$, and such that any point in the antichain $\{x_1, \ldots, x_{2n+1}\}$ is dominated by any point in the antichain $\{y_1, \ldots, y_{2n+1}\}$, except an element in the first antichain, say x_{n+1}, that is not dominated by other element of the second antichain, say y_{n+1}. Figure 3.12 shows the Hasse diagrams of GS_1 and GS_2.

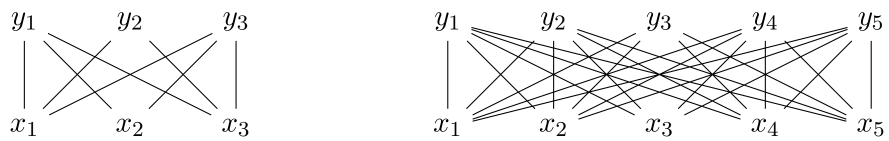


FIGURE 3.12: GS_1 and GS_2 .

In this example, it is not possible to find a vector w^*. In other words, these posets are not BU-feasible. To see this, consider the positioned antichain $(x_1, x_{n+1}), V = \emptyset$; then, we obtain the equation $w^*_{x_1} = w^*_{x_{n+1}}$. On the other hand, consider the positioned antichain $(x_1, x_{n+1}), V = \{x_2, ..., x_n, x_{n+2}, ..., x_{2n+1}\}$; in this case, we obtain the equation $w^*_{x_1} = w^*_{x_{n+1}} + w^*_{y_{n+1}}$. As $w^*_{y_{n+1}} > 0$, no solution can be obtained.

However, we can use the Ideal-based method to derive a way of sampling linear extensions uniformly. For this, we should compute all the non trivial probabilities $P(x|I)$. Start observing the easy cases:

- If $\{x_1, x_2, \ldots, x_{2n+1}\} \subseteq I$, then $GS_n \setminus I$ is an antichain and $P(y_k|I) = \dfrac{1}{|GS_n \setminus I|}$, $\forall y_k \in GS_n \setminus I$.
- If $x_{n+1} \in I$ but $\{x_1, x_2, \ldots, x_{2n+1}\} \nsubseteq I$, then all the minimal elements are interchangeable to each other so $P(x_k|I) = \dfrac{1}{|\mathcal{MIN}(GS_n \setminus I)|}$, $\forall x_k \in GS_n \setminus I$.

So it suffices to study the case $x_{n+1} \notin I$. If $I = I_0 = \{x_1, x_2, \ldots, x_n, x_{n+2}, \ldots, x_{2n+1}\}$. In this case $\mathcal{MIN}(GS_n \setminus I_0) = \{x_{n+1}, y_{n+1}\}$, or in other words, we are working with the positioned antichain $(a = (x_{n+1}, y_{n+1}), V = \emptyset)$. Via Equation 3.6,

$$P(x_{n+1}|I_0) \cdot P(y_{n+1}|I_0 \cup \{x_{n+1}\}) = P(y_{n+1}|I_0) \cdot P(x_{n+1}|I_0 \cup \{y_{n+1}\}).$$

Hence,

$$P(x_{n+1}|I_0) \cdot \frac{1}{2n+1} = P(y_{n+1}|I_0) \cdot 1,$$

and thus $P(x_{n+1}|I_0) = \dfrac{2n+1}{2n+2}$ and $P(y_{n+1}|I_0) = \dfrac{1}{2n+2}$.

The remaining cases are the ones where $x_{n+1} \notin I$ and $I \subset \{x_1, x_2, \ldots, x_n, x_{n+2}, \ldots, x_{2n+1}\}$. Let us call I_k to some of these ideals satisfying $|I \cap \{x_1, x_2, \ldots, x_n, x_{n+2}, \ldots, x_{2n+1}\}| = 2n - k$. Since the non-central elements $x_i \in I_k$ are interchangeable, then all the ideals with $2n - k$ elements in $\{x_1, x_2, \ldots, x_n, x_{n+2}, \ldots, x_{2n+1}\}$ are isomorphic to each other. Let us see the case $k = 1$,

$$P(x_{n+1}|I_1) \cdot P(x_i|I_1 \cup \{x_{n+1}\}) = P(x_i|I_1) \cdot P(x_{n+1}|I_1 \cup \{x_i\})$$

Hence,

$$P(x_{n+1}|I_1) \cdot 1 = P(x_i|I_1) \cdot \frac{2n+1}{2n+2},$$

and thus $P(x_{n+1}|I_1) = \dfrac{2n+1}{4n+3}$ and $P(x_i|I_1) = \dfrac{2n+2}{4n+3}$. For $k > 1$, there is a strong link between I_{k+1} and I_k. Indeed,

$$P(x_{n+1}|I_{k+1}) \cdot P(x_i|I_{k+1} \cup \{x_{n+1}\}) = P(x_i|I_{k+1}) \cdot P(x_{n+1}|I_k)$$

Hence,

$$P(x_{n+1}|I_{k+1}) \cdot \frac{1}{k+1} = P(x_i|I_{k+1}) \cdot P(x_{n+1}|I_k),$$

and also $(k+1) \cdot P(x_i|I_{k+1}) + P(x_{n+1}|I_{k+1}) = 1$. Therefore,

$$P(x_i|I_{k+1}) = \frac{P(x_{n+1}|I_{k+1})}{(k+1)P(x_{n+1}|I_k)},$$

and

$$\frac{P(x_{n+1}|I_{k+1})}{P(x_{n+1}|I_k)} + P(x_{n+1}|I_{k+1}) = 1 \Leftrightarrow \frac{1}{P(x_{n+1}|I_{k+1})} = 1 + \frac{1}{P(x_{n+1}|I_k)}.$$

As the initial condition, $k=1$, for the last recursive formula is $\frac{1}{P(x_{n+1}|I_1)} = 2 + \frac{1}{2n+1}$, we get

$$\frac{1}{P(x_{n+1}|I_k)} = (k+1) + \frac{1}{2n+1}.$$

Therefore, we obtain for $1 \leq k \leq 2n$:

$$P(x_{n+1}|I_k) = \frac{2n+1}{(k+1)(2n+1)+1},$$

and since $P(x_{n+1}|I_k) + kP(x_i|I_k) = 1$,

$$P(x_i|I_k) = \frac{k(2n+1)+1}{k(k+1)(2n+1)+k}.$$

In particular, $P(x_{n+1}|I_{2n}) = \frac{2n+1}{(2n+1)^2+1}$ and we can use this to compute $e(GS_n)$. As we did in the BU method take $\epsilon = (x_{n+1}, x_1, x_2, \ldots, x_{2n+1}, y_1, y_2, \ldots, y_{2n+1})$ then:

$$e(GS_n) = \frac{1}{P(\epsilon)} = \frac{(2n+1)^2+1}{2n+1} \cdot (2n) \cdot (2n-1) \cdots 1 \cdot (2n+1) \cdot (2n) \cdots 1 = \left[(2n)!\right]^2 (4n^2+4n+2).$$

3.7.1.3 2-truncated fuzzy measures

As we saw in Section 3.6, we can generalize the concept of k-intolerant measures by considering measures restricted to all sets of k different cardinalities, $\mathcal{TFM}^n(r_1, \ldots, r_k)$. This section is devoted to the case of 2-truncated fuzzy measures over sets of cardinalities

r_1 and r_2, $\mathcal{TFM}^n(r_1, r_2)$ over a referential set of size n, especially we will study the case $r_1 = 1$. We also denote by $B_2^n(r_1, r_2)$ the Boolean poset (without maximum and minimum) restricted to the sets of size r_1 and r_2 (see Figure 3.13). It is obvious that $\mathcal{TFM}^n(r_1, r_2)$ is an order polytope with associated poset $B_2^n(r_1, r_2)$.

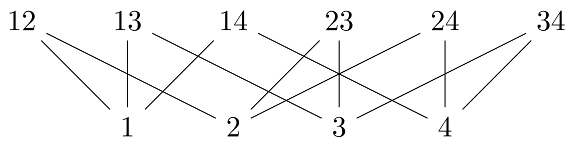


FIGURE 3.13: Hasse diagram of $B_2^4(1, 2)$.

Remark that $\mathcal{TFM}^n(1, 2)$ is the set of 3-tolerant fuzzy measures and $\mathcal{TFM}^n(n-2, n-1)$ is the set of 3-intolerant fuzzy measures. Here we are going to solve the problem of sampling uniformly capacities of $\mathcal{TFM}^n(1, r_2)$ by sampling linear extensions of $B_2^n(1, r_2)$. The case $r_1 \neq 1$ involves more complex computations and it is an open problem. However, if $r_2 = n-1$, then $B_2^n(r_1, n-1)^\partial \cong B_2^n(1, n-r_1)$ and hences if we solve the case $B_2^n(1, r_2)$, the case $B_2^n(r_1, n-1)$ follows by duality. Note that I is an ideal of P if and only if $F = P \setminus I$ is a filter. For convenience reasons, we will work with filters instead of ideals. In this case, the Ideal-based method could be used to draw a minimal element of F on each step with the proper probability. Let us compute this probability. Before this, let us show that the filters of $B_2^n(1, r_2)$, just depend on the number of singletons and r_2-subsets in it.

Lemma 3.35. *Let F be a filter of $B_2^n(1, r_2)$, with 2 or more singletons. If $\{x\}, \{y\} \in F$ then $F \setminus \{x\} \cong F \setminus \{y\}$.*

Proof. Let $\phi : F \setminus \{x\} \to F \setminus \{y\}$ be a map such that interchanges x and y, i.e.

$$\phi(A) = \begin{cases} A, & \text{if } x \notin A \\ (A \setminus \{x\}) \cup \{y\}, & \text{otherwise} \end{cases}$$

It is clear that $A \subseteq B \Leftrightarrow \phi(A) \subseteq \phi(B)$ so $F \setminus \{x\} \cong F \setminus \{y\}$. □

This way, all singletons are interchangeable (see Section 1.1).

Lemma 3.36. *Let F_1 and F_2 be two filters of $B_2^n(1, r_2)$. Then $F_1 \cong F_2 \Leftrightarrow F_1$ and F_2 have the same number of singletons and r_2-subsets.*

Proof. Let us denote

$$F_1 = \{x_1, \ldots, x_s, A_1, \ldots, A_l, B_1, \ldots, B_p\},$$

where $x_1, \ldots, x_s$ are the singletons in F_1, $A_1, \ldots, A_l$ are the r_2-subsets in F_1 such that $A_i \cap \{x_1, \ldots, x_s\} \neq \emptyset$, and $B_1, \ldots, B_p$ are the r_2-subsets in F_1 such that $B_i \cap \{x_1, \ldots, x_s\} = \emptyset$.

Thus defined, filter F_1 can be written as $F_1 = G_1 \uplus H_1$, where

$$G_1 := \uparrow x_1 \cup \uparrow x_2 \cup \cdots \cup \uparrow x_s = \{x_1, \ldots, x_s, A_1, \ldots, A_l\}$$

and $H_1 := \{B_1, \ldots, B_p\}$. Similarly, $F_2 = G_2 \uplus H_2$.

Now, $F_1 \cong F_2 \Leftrightarrow G_1 \cong G_2$ and $H_1 \cong H_2$. As H_1 and H_2 are antichains, $H_1 \cong H_2 \Leftrightarrow |H_1| = |H_2|$.

Let us now show that $G_1 \cong G_2$ iff they have the same number of singletons.

$\Rightarrow$) If $G_1 \cong G_2$, then they have exactly the same number of minimal elements, and minimal elements are exactly the singletons.

$\Leftarrow$) Let us denote by $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_s\}$ the singletons in G_1 and G_2 respectively. Now, for $A \in G_1$, $A = \{x_{i_1}, \ldots, x_{i_q}, z_1, \ldots, z_h\}$, let us define a function $\phi(A) \in G_2$ given by $\phi(A) = \{y_{i_1}, \ldots, y_{i_q}, z_1, \ldots, z_h\}$. Thus, $A \subseteq B \Leftrightarrow \phi(A) \subseteq \phi(B)$, so $G_1 \cong G_2$. $\square$

From now on we will denote by $F(t_1, t_2)$ to some filter having t_1 singletons and t_2 r_2-subsets. We can ask about how many r_2-subsets cover some of the t_1 singletons. If we consider all the posible $\binom{n}{r_2}$ r_2-subsets and take away the $\binom{n-t_1}{r_2}$ r_2-subsets without any of the t_1 singletons we get that the number of r_2-subsets covering some of the t_1 singletons is $\binom{n}{r_2} - \binom{n-t_1}{r_2}$. Then, $0 \leq t_1 \leq n$ and $\binom{n}{r_2} - \binom{n-t_1}{r_2} \leq t_2 \leq \binom{n}{r_2}$.

Corollary 3.37. *The number of non-isomorphic filters of $B_2^n(1, r_2)$ is $\binom{n+1}{r_2+1} + n + 1$.*

Proof. Since $F(t_1, t_2)$ are non-isomorphic filters for different values of $0 \leq t_1 \leq n$ and $\binom{n}{r_2} - \binom{n-t_1}{r_2} \leq t_2 \leq \binom{n}{r_2}$, it holds

$$\sum_{t_1=0}^{n} \left[\binom{n}{r_2} - \left(\binom{n}{r_2} - \binom{n-t_1}{r_2} \right) + 1 \right] = \sum_{t_1=0}^{n} \binom{n-t_1}{r_2} + n + 1$$

$$= \sum_{h=r_2}^{n} \binom{h}{r_2} + n + 1 = \binom{n+1}{r_2+1} + n + 1.$$

Hence, the result holds. $\square$

Theorem 3.38. *Let $F(t_1,t_2)$ be a filter of $B_2^n(1,r_2)$ and let us denote $I(t_1,t_2) = B_2^n(1,r_2) \setminus F(t_1,t_2)$ to its associated ideal. Let x be a singleton of $\mathcal{MIN}(F(t_1,t_2))$ and y a r_2-subset of $\mathcal{MIN}(F(t_1,t_2))$. Then:*

$$P(x|I(t_1,t_2)) = \frac{t_1 + \binom{n}{r_2} - \binom{n-t_1}{r_2}}{t_1(t_1+t_2)}, \qquad P(y|I(t_1,t_2)) = \frac{1}{t_1+t_2}.$$

Proof. As we know by the last lemmas, $F(t_1,t_2)$ has t_1 interchangeable singletons, all of them minimals and $T = t_2 + \binom{n-t_1}{r_2} - \binom{n}{r_2}$ minimal interchangeable r_2-subsets. Let $\alpha = P(x|I(t_1,t_2))$ and $\beta = P(y|I(t_1,t_2))$. Since these minimals can be interchanged if they have the same size then $t_1\alpha + T\beta = 1$. The key fact here is to apply the Disjoint union Lemma 1.27,

$$\beta = \frac{e(F(t_1,t_2)\setminus\{y\})}{e(F(t_1,t_2))} = \frac{e(F(t_1,T-t_2)\uplus \overline{T-1})}{e(F(t_1,T-t_2)\uplus \overline{T})}$$
$$= \frac{\binom{|F(t_1,T-t_2)|+T-1}{T-1} e(F(t_1,T-t_2))(T-1)!}{\binom{|F(t_1,T-t_2)|+T}{T} e(F(t_1,T-t_2))T!} = \frac{1}{|F(t_1,T-t_2)|+T}.$$

Since $T = t_2 + \binom{n-t_1}{r_2} - \binom{n}{r_2}$ and $|F(t_1,T-t_2)| = t_1 + \binom{n}{r_2} - \binom{n-t_1}{r_2}$, then

$$\beta = \frac{1}{t_1+t_2} = \frac{1}{|F(t_1,t_2)|}.$$

From $t_1\alpha + T\beta = 1$ we obtain the value of α,

$$\alpha = \frac{1-\beta T}{t_1} = \frac{t_1 + \binom{n}{r_2} - \binom{n-t_1}{r_2}}{t_1(t_1+t_2)} = \frac{|F(t_1,T-t_2)|}{t_1|F(t_1,t_2)|}.$$

Therefore, the result holds. □

With the last probabilities we can sample linear extensions of $B_2^n(1,r_2)$ and therefore we can sample 2-truncated fuzzy measures of the form $\mathcal{TFM}^n(1,r_2)$. Finally, we can use the last result to get the number of linear extensions of $B_2^n(1,r_2)$.

Theorem 3.39. *The number of linear extensions of the 2-truncated boolean poset $B_2^n(1,r_2)$ is given by:*

$$e\left(B_2^n(1,r_2)\right) = n! \prod_{t=r_2}^{n} \prod_{j=1}^{\binom{t-1}{r_2-1}} \left[n + \binom{n}{r_2} - (j-1) - t - \binom{t-1}{r_2} \right].$$

Proof. As we know, the number of linear extensions is the inverse of the probability of any linear extension of being uniformly drawn. The extension which we are going to choose is the one starting by singletons until some r_2-subset becomes minimal. If some r_2-subset becomes minimal, then we draw it, else we continue selecting singletons. In the first r_2-1 drawn elements we do not get any r_2-subset as a new minimal. From then on we get on each step $\binom{t}{r_2} - \binom{t-1}{r_2} = \binom{t-1}{r_2-1}$ new minimal r_2-subsets. The size of the filter when we have removed t singletons and its associated $\binom{t-1}{r_2}$ r_2-subset involving the first $t-1$ singletons is given by $n+\binom{n}{r_2}-t-\binom{t-1}{r_2}$. Therefore, inverting probabilities and using the last theorem $P(y|I(t_1,t_2)) = \dfrac{1}{t_1+t_2}$ we get the result. Observe that the factor $n!$ is because when we select singletons the probability is $\dfrac{1}{k}$ for each one. When r_2-subset are released (for $r_2 \leq t \leq n$) the inverse of the probability is just the cardinality of the remaining filter which is $n+\binom{n}{r_2}-t-\binom{t-1}{r_2}$ minus the number of r_2-subsets already selected $j-1$. □

The first values of $e(B_2^n(1,r_2))$ are given in Table 3.4. Observe that the number of linear extensions of $B_2^n(1,n-1)$ were computed in Subsection 3.6.4.

$n \setminus r_2$	2	3	4	5
2	2			
3	48	6		
4	34 560	720	24	
5	1 383 782 400	746 496 000	17 280	120

TABLE 3.4: First values of $e(B_2^n(1,r_2))$.

3.7.1.4 Treefication

We finish this section with a construction that allows to generalize the results about rooted trees seen in Section 3.5.1 in an easy way. Let us start with the definition of treefication.

Definition 3.40. Let T be a rooted tree and P a finite poset. We define the **treefication of P based on T** as the lexicographic product $T * P$.

Example 3.7. *Let N be the poset from Example 1.1 and T be the rooted tree of Figure 3.3, see Figure 3.14 for the Hasse diagram of the treefication of poset N based on T,*

$T * P$. In general the lexicographic product of BU-feasible posets is not BU-feasible. For proving this, we take the filter $F = N \uplus \mathbf{1}$. We know by Example 3.1 that F is not BU-feasible and then by Theorem 3.18 the treefication $T * P$ is not BU-feasible. Indeed, $T * P$ is not BU-feasible in this example.

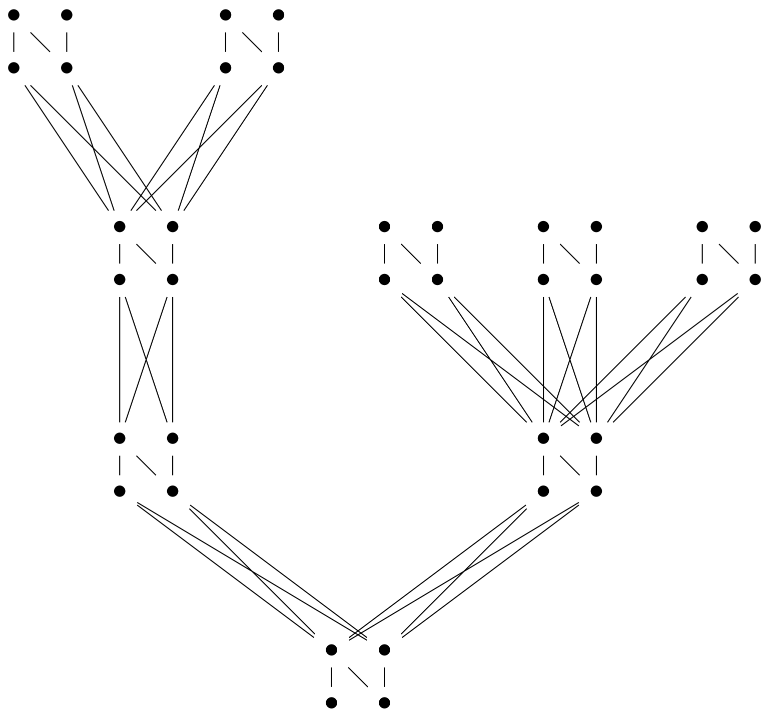


FIGURE 3.14: Treefication $T * N$.

Let us denote $|T| = n$ and $|P| = p$.

Theorem 3.41. *Consider a rooted tree T and a poset P. Let I be an ideal of the treefication of P based on T, $T * P$, and $x \in \mathcal{MIN}\left((T * P) \setminus I\right)$, then*

$$P(x|I) = \frac{|\uparrow P(x) \setminus I|}{|T * P \setminus I|} \cdot P_P(x|I_P),$$

where $P(x)$ is the copy of P in $T * P$ where x lives, $I_P = I \cap P(x)$, and P_P is the probability inside poset $P(x)$.

Proof. Let $C_1, \ldots, C_k$ the connected components of $(T * P) \setminus I$, and suppose, without loss of generality, that $x \in C_1$. Then by using the Disjoint Union Lemma 1.27,

$$\frac{e((T * P_x) \setminus I)}{e((T * P) \setminus I)} = \frac{\binom{|C_1|+|C_2|+\cdots+|C_k|-1}{|C_1|-1,|C_2|,\cdots,|C_k|} e(C_1 \setminus \{x\}) \prod_{i=2}^{k} e(C_i)}{\binom{|C_1|+|C_2|+\cdots+|C_k|}{|C_1|,|C_2|,\cdots,|C_k|} e(C_1) \prod_{i=2}^{k} e(C_i)}$$

$$= \frac{|C_1|}{\sum_{i=1}^{k}|C_i|}\frac{e(C_1\setminus\{x\})}{e(C_1)} = \frac{|\uparrow P(x)\setminus I|}{|(T*P)\setminus I|}\cdot P_P(x|I_P).$$

Hence, the result holds. □

This theorem is an extension of the case seen in Subsection 3.5.1.

When P is BU-feasible we can compute all the conditional probabilities from the weights w_i,

$$P(x|I) = \frac{|\uparrow P(x)\setminus I|}{|T*P\setminus I|}\cdot\frac{w_x}{\sum_{z\in\mathcal{MIN}(P\setminus I_P)} w_z}.$$

Finally, we can give a generalization of the formula counting the number of linear extension of rooted trees, see Equation 3.5.

Theorem 3.42. *Consider a rooted tree T and a poset P. Then the number of linear extensions of the treefication of P based on T, $T*P$, is*

$$e(T*P) = e(P)^n\frac{\prod_{i=1}^{n}\binom{ip}{p}}{\prod_{x\in T}\binom{\lambda_x p}{p}},$$

where $\forall x\in T,\ \lambda_x := |\{y\in T | y\succeq x\}|$.

Proof. Since any linear extension has the same probability we are going to choose one linear extension passing through all the elements of each copy of P consecutively. Multiplying the inverse of the probabilities we get that the factors $P_P(x|I_P)$ team up n times to get $e(P)^n$. The other factors are $\frac{|T*P\setminus I|}{|\uparrow P(x)\setminus I|}$ for each ideal. Since the ideal has one element less on each step, $\prod_I |T*P\setminus I| = (np)!$. Let $P(x)$ be the copy of P associated to some element x in $T*P$, the product of the ideals inside $P(x)$ is $\prod_I |\uparrow P(x)\setminus I| = (\lambda_x p)(\lambda_x p-1)\cdots(\lambda_x p - p+1) = \frac{(\lambda_x p)!}{(\lambda_x(p-1))!}$. Therefore,

$$e(T*P) = e(P)^n\frac{(np)!}{\prod_{x\in T}\frac{(\lambda_x p)!}{(\lambda_x(p-1))!}} = e(P)^n\frac{\frac{(np)!}{(p!)^n}}{\prod_{x\in T}\frac{(\lambda_x p)!}{(\lambda_x(p-1))!p!}} = e(P)^n\frac{\prod_{i=1}^{n}\binom{ip}{p}}{\prod_{x\in T}\binom{\lambda_x p}{p}}.$$

This finishes the proof. □

Chapter 4

Combinatorial structure of the polytope of 2-symmetric fuzzy measures

In this chapter we apply Young diagrams, a well-known object appearing in Combinatorics and Group Representation Theory, to study some properties of the polytope of 2-symmetric fuzzy measures with respect to a given partition. The main result in this chapter allows to build a simple and fast algorithm for generating points on this polytope in a random fashion. Besides, we also study some other properties of this polytope, as for example its volume. In the last section, we give an application of this result to the problem of identification of general fuzzy measures [80].

4.1 Young diagrams and Ferrers posets

In this section, we are going to introduce the basic properties of Young diagrams. Young diagrams are not only a useful combinatorial structure but also play an important role in other branches of mathematics such as Representation Theory or Schubert Calculus [81].

Definition 4.1. Consider a natural number n. A **partition** of n (do not confuse this concept with the partition of a set) is a sequence of natural numbers $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 1$ and $\sum_{i=1}^{k} \lambda_i = n$.

Each partition represents a way to obtain n as a sum of natural numbers. Partitions of n are represented through Young diagrams (see e.g., [82, 83] for the basic properties of this object).

Definition 4.2. Let $\boldsymbol{\lambda}$ be a partition of a natural number $n \in \mathbb{N}$. Then, we define the **Young diagram** (or **Ferrers diagram**) of shape $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$, $\mathbf{X}(\boldsymbol{\lambda})$, as an array of cells, $x_{i,j}$, arranged in left-justified rows[1], $i = 0, \cdots, k-1$ and $j = 0, \ldots, \lambda_i - 1$.

Example 4.1. *For instance, if $\boldsymbol{\lambda} = (5, 4, 4, 1)$ we obtain the Young diagram for $n = 14$ shown in Figure 4.1 left.*

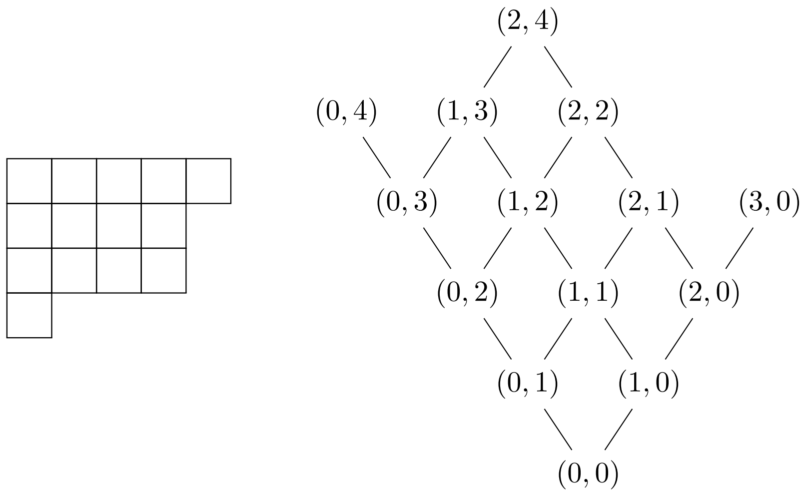


FIGURE 4.1: Young diagram associated to $\boldsymbol{\lambda} = (5, 4, 4, 1)$ and its corresponding Ferrers poset.

Definition 4.3. Let $\boldsymbol{\lambda}$ be a partition of n. A **Young tableau** (plural **Young tableaux**) of shape $\boldsymbol{\lambda}$ is an assignment of the integers $1, \ldots, n$ to the cells in the diagram of shape $\boldsymbol{\lambda}$. A Young tableau is said to be **standard** if all rows and columns form increasing sequences.

Example 4.2. *A (standard) Young tableau for the Young diagram of shape $\boldsymbol{\lambda} = (5, 4, 4, 1)$ is presented in Figure 4.2.*

1	4	6	7	10
2	8	11	12	
3	9	13	14	
5				

FIGURE 4.2: (Standard) Young tableau of shape (5,4,4,1).

Let us now introduce Hook-length formula. For this, some previous concepts are needed.

[1] Although it is usual to name the cells in a Young diagram indexing i from 1 to k and j from 1 to λ_i, we have decided in this chapter to start at value 0. This provides a deeper insight into the relationship between 2-symmetric measures and Young diagrams, in the sense that subset (i, j) will be associated to cell (i, j) (see Lemma 4.8 below).

Definition 4.4. Let $\mathbf{X}(\boldsymbol{\lambda})$ be a Young diagram of shape $\boldsymbol{\lambda}$. For a cell (i,j) in the diagram, the **Hook** of cell (i,j), denoted $H_{\boldsymbol{\lambda}}(i,j)$, consists of the cells that are either below (i,j) in column j or to the right of (i,j) in row i, along with (i,j) itself. That is:

$$H_{\boldsymbol{\lambda}}(i,j) := \{(a,b) \in \mathbf{X}(\boldsymbol{\lambda}) | (a = i, b \geq j) \vee (a \geq i, b = j)\}.$$

The **Hook length** $h_{\boldsymbol{\lambda}}(i,j)$ is the number of cells in the hook $H_{\boldsymbol{\lambda}}(i,j)$.

Example 4.3. *For the example with $\boldsymbol{\lambda} = (5,4,4,1)$, the set $H(0,0)$ is drawn in Figure 4.3.*

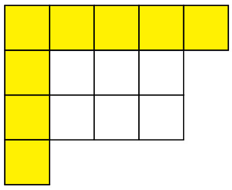


Figure 4.3: Hook $H_{\boldsymbol{\lambda}}(0,0)$ with $\boldsymbol{\lambda} = (5,4,4,1)$.

In this case, $h_{\boldsymbol{\lambda}}(0,0) = 8$.

The number of standard Young tableaux of shape $\boldsymbol{\lambda}$ can be written in terms of the Hook lengths of the different cells. This is the famous Hook-length formula.

Theorem 4.5. *[82, 83] Let $\boldsymbol{\lambda}$ be a partition of n and $d_{\boldsymbol{\lambda}}$ be the number of standard Young tableaux of shape $\boldsymbol{\lambda}$. Then,*

$$d_{\boldsymbol{\lambda}} = \frac{n!}{\prod_{(i,j)\in\mathbf{X}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(i,j)}.$$

Finally, let us introduce Ferrers posets. For a given Young diagram, we can build a poset $(P, \preceq)$ in the following way. Elements of P are the cells (written as the pair of coordinates) and given two cells $(i,j),(k,l)$, we define

$$(i,j) \preceq (k,l) \Leftrightarrow i \leq k, j \leq l.$$

This poset is known as the **Ferrers poset** associated to shape $\boldsymbol{\lambda}$, denoted $\mathbf{P}(\boldsymbol{\lambda})$. For example, the Ferrers poset associated to shape $(5,4,4,1)$ can be seen in Figure 4.1 right.

Note that if we consider a Young diagram $\mathbf{X}(\boldsymbol{\lambda}), \boldsymbol{\lambda} = (\lambda_1, ..., \lambda_r)$ such that $\lambda_i = \lambda_j, \forall i,j$, i.e., when the Young diagram adopts the form of a rectangle, the corresponding Ferrers poset has a minimum (corresponding to cell $(0,0)$) and a maximum (associated to cell

$(r-1, \lambda_r - 1)$). We call **normalized Ferrers poset**, $\mathbf{P}^*(\boldsymbol{\lambda})$ to the poset resulting from removing maximum and minimum from $\mathbf{P}(\boldsymbol{\lambda})$ for a Young diagram in these conditions.

Remark 4.6.

- Note that the same Ferrers poset arises if we swap rows and columns and build the corresponding Young diagram, i.e., the Ferrers poset is the same for shapes $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$ and $\boldsymbol{\eta} = (\eta_1, ..., \eta_l)$ where $\eta_i := \sharp\{\lambda_j : \lambda_j \geq i\}$.

- Remark also that (i, j) covers $(i-1, j)$ and $(i, j-1)$. In terms of the Young diagram, this means that a cell covers just the cell to the left and the cell to the top. Following the terminology of graphs, we will say that two cells are **adjacent** if one of them covers the other in the corresponding Ferrers poset, i.e., when they are consecutive cells in a column or a row.

  We define a **path** in a Young diagram as a sequence of adjacent cells. Finally, we will say that a subset F of cells in the Young diagram is **connected** if there is a path inside F connecting any pair of cells in F. This will play an important role in the following section.

Now, for a fixed shape, observe that any standard Young tableau can be related to a possible linear extension of the corresponding Ferrers poset, where the labels give the position of each element in the linear extension (label one corresponds to the first element, label two to the second one and so on); reciprocally, given a linear extension of the Ferrers poset, we can build a standard Young tableau just assigning to each cell the position of the cell in the linear extension. Therefore, we have as many standard Young tableaux as linear extensions of the Ferrers poset and applying Theorem 4.5, the following holds for the number of linear extensions.

Corollary 4.7. *Let $\boldsymbol{\lambda}$ be a partition of n, then:*

$$e(\mathbf{P}(\boldsymbol{\lambda})) = \frac{n!}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}.$$

4.2 $\mathcal{FM}(A_1, A_2)$ as an order polytope

We start with a fundamental property relating $\mathcal{FM}(A_1, A_2)$ to the normalized Ferrers poset $\mathbf{P}^*(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda}$ the $(|A_2|+1)$-vector whose coordinates are all of them $|A_1|+1$. From now on, we will write this vector as $\boldsymbol{\lambda} = (|A_1|+1, |A_1|+1, \overset{|A_2|+1}{\ldots}, |A_1|+1)$.

Lemma 4.8. *Let $\{A_1, A_2\}$ be a partition of the referential set X and let us denote $|A_1| = a_1$ and $|A_2| = a_2$. The set $\mathcal{FM}(A_1, A_2)$ can be seen as an order polytope associated to the normalized Ferrers poset $\mathbf{P}^*(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (a_1+1, a_1+1, \overset{a_2+1}{\ldots}, a_1+1)$.*

Proof. As explained in Section 2.6.3, the set $\mathcal{FM}(A_1, A_2)$ is an order polytope whose subjacent poset is $P(A_1, A_2)$. Let us consider $P^+(A_1, A_2)$, where we have added $\emptyset$ and X, given by $(0,0)$ and (a_1, a_2), resp. to $P(A_1, A_2)$; then, $P^+(A_1, A_2)$ can be identified to a $(a_1+1) \times (a_2+1)$ table, where entry $(i,j), i = 0, ..., a_1, j = 0, ..., a_2$, corresponds to element (i,j). But this table is the Young diagram of shape $\boldsymbol{\lambda} = (a_1+1, a_1+1, \overset{a_2+1}{\cdots}, a_1+1)$. Thus, $P^+(A_1, A_2)$ can be associated to $\mathbf{P}(\boldsymbol{\lambda})$ and consequently, $P(A_1, A_2)$ can be associated to $\mathbf{P}^*(\boldsymbol{\lambda})$. □

Swapping positions of A_1 and A_2, we conclude that $\mathcal{FM}(A_1, A_2)$ can be associated to the normalized Ferrers poset $\mathbf{P}^*(\boldsymbol{\eta})$ with $\boldsymbol{\eta} = (a_2+1, a_2+1, \overset{a_1+1}{\cdots}, a_2+1)$. Compare this with Remark 4.6.

An interesting consequence which follows from the last theorem gives us the exact volume of the polytope $\mathcal{FM}(A_1, A_2)$.

Theorem 4.9. *Let $\{A_1, A_2\}$ be a partition of X in two subsets of indifference and let $|A_1| = a_1$ and $|A_2| = a_2$. Then,*

$$Vol\,(\mathcal{FM}(A_1, A_2)) = [(a_1+1)(a_2+1)]\,[(a_1+1)(a_2+1)-1]\prod_{k=0}^{a_1}\frac{k!}{(a_2+1+k)!}$$

Proof. By Theorem 1.70, we know that for an order polytope,

$$Vol(O(P, \preceq)) = \frac{1}{|P|!}e(P),$$

where $e(P)$ denotes the number of linear extensions of poset $(P, \preceq)$. Thus, it suffices to find the number of linear extensions of the subjacent poset. For $\mathcal{FM}(A_1, A_2)$, we have to sort out the number of linear extensions of $P(A_1, A_2)$. Note that the number of linear extensions of this poset is the same as the polytope $P^+(A_1, A_2)$ defined in Lemma 4.8 or equivalently, to poset $\mathbf{P}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (a_1+1, a_1+1, \overset{a_2+1}{\cdots}, a_1+1)$. Now, by Corollary 4.7, for shape $\boldsymbol{\lambda}$,

$$e(\mathbf{P}(\boldsymbol{\lambda})) = \frac{[(a_1+1)(a_2+1)]!}{\prod_{(i,j)\in\mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(i,j)}.$$

Let us then find the values of $h_{\boldsymbol{\lambda}}(i,j)$ in this case. Note that for row i fixed, the values of $h_{\boldsymbol{\lambda}}(i,j)$, where $j = 0, ..., a_2$ are $a_1 + a_2 + 1 - i, ..., a_1 + 1 - i$ resp. (see Figure 4.4).

Thus,

7	6	5	4	3
6	5	4	3	2
5	4	3	2	1

FIGURE 4.4: Hook lengths for a rectangular Young tableau.

$$\prod_{(i,j)\in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(i,j) = \prod_{i=0}^{a_1} \frac{(a_1+a_2+1-i)!}{(a_1-i)!} = \prod_{k=0}^{a_1} \frac{(a_2+1+k)!}{k!}.$$

Consequently,

$$\begin{aligned} Vol\,(\mathcal{FM}(A_1,A_2)) &= \frac{1}{[(a_1+1)\times(a_2+1)-2]!} e(P(A_1,A_2)) \\ &= \frac{1}{[(a_1+1)\times(a_2+1)-2]!} e(P^+(A_1,A_2)) \\ &= \frac{1}{[(a_1+1)\times(a_2+1)-2]!}\,[(a_1+1)(a_2+1)]! \prod_{k=0}^{a_1} \frac{k!}{(a_2+1+k)!} \\ &= [(a_1+1)(a_2+1)]\,[(a_1+1)(a_2+1)-1] \prod_{k=0}^{a_1} \frac{k!}{(a_2+1+k)!}. \end{aligned}$$

Therefore, the result holds. $\square$

Some values of this volume for different values of a_1 and a_2 can be seen in the next table.

$a_1\backslash a_2$	1	2	3	4	5
1	1	$\frac{6\cdot 5}{3!4!}$	$\frac{8\cdot 7}{4!5!}$	$\frac{10\cdot 9}{5!6!}$	$\frac{12\cdot 11}{6!7!}$
2		$\frac{9\cdot 8\cdot 2!}{3!4!5!}$	$\frac{12\cdot 11\cdot 2!}{4!5!6!}$	$\frac{15\cdot 14\cdot 2!}{5!6!7!}$	$\frac{18\cdot 17\cdot 2!}{6!7!8!}$
3			$\frac{16\cdot 15\cdot 2!3!}{4!5!6!7!}$	$\frac{20\cdot 19\cdot 2!3!}{5!6!7!8!}$	$\frac{24\cdot 23\cdot 2!3!}{6!7!8!9!}$
4				$\frac{25\cdot 24\cdot 2!3!4!}{5!6!7!8!9!}$	$\frac{30\cdot 29\cdot 2!3!4!}{6!7!8!9!10!}$
5					$\frac{36\cdot 35\cdot 2!3!4!5!}{6!7!8!9!10!11!}$

It must be remarked at this point that obtaining the volume of an order polytope is a complex problem depending on the number of linear extensions (and counting linear extensions is another complex problem), so that usually bounds are considered. In this case, some bounds for the volume of $\mathcal{FM}(A_1,A_2)$ were given in [23].

Young tableaux also gives an interesting combinatorial approach to the adjacency in $\mathcal{FM}(A_1, A_2)$. Consider a $n \times m$ grid on quadrant I, i.e., the left-bottom corner corresponds to $(0,0)$ and right-top corner corresponds to (n,m).

Definition 4.10. Given a $n \times m$ grid, a **staircase walk** is a path from $(0,0)$ to (n,m) which uses just up and right steps.

An example of a staircase walk in a 4×4 grid can be seen in Figure 4.5. Staircase walks play an important role in Combinatorics, since they are intimately linked to Catalan numbers [8].

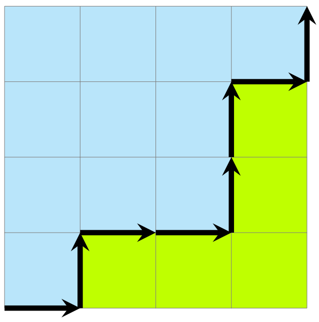


FIGURE 4.5: Staircase walk in a 4×4 grid.

Let us now show the relationship between vertices and staircase walks, and apply it to characterize adjacency in $\mathcal{FM}(A_1, A_2)$.

Proposition 4.11. *Let $\{A_1, A_2\}$ be a partition of X in two subsets of indifference and let $|A_1| = a_1$ and $|A_2| = a_2$. Then,*

i) There is a bijection between the set of vertices of $\mathcal{FM}(A_1, A_2)$ and the staircase walks on a $(a_1+1) \times (a_2+1)$ grid which do not cross $(0, a_2+1)$ nor $(a_1+1, 0)$.

ii) $\mathcal{FM}(A_1, A_2)$ has $\binom{a_1+a_2+2}{a_1+1} - 2$ vertices.

iii) Let F_1 and F_2 be two vertices of $\mathcal{FM}(A_1, A_2)$. F_1 and F_2 are adjacent if and only if the set of cells between the two associated staircase paths is connected, as defined in Remark 4.6.

Proof.

i) By Lemma 4.8, we know that $\mathcal{FM}(A_1, A_2)$ is an order polytope associated to the normalized Ferrers poset $\mathbf{P}^*(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (a_1+1, a_1+1, \overset{a_2+1}{\ldots}, a_1+1)$. Since $\mathcal{FM}(A_1, A_2)$ is an order polytope, its vertices are the characteristic functions of the filters of $\mathbf{P}^*(\boldsymbol{\lambda})$. Let us then characterize filters in this poset. For this, note that $(i,j) \leq (k,l)$ in $\mathbf{P}^*(\boldsymbol{\lambda})$ means $i \leq k, j \leq l$. Consequently, a filter of $\mathbf{P}(\boldsymbol{\lambda})$

translates in the Young diagram as a set of cells satisfying that if $(i, j) \in F$ then $H_{\boldsymbol{\lambda}}(i, j) \subseteq F$. Therefore, a filter defines a set of cells whose border is given by a staircase path from the cell $(a_1, 0)$ to the cell $(0, a_2)$, or equivalently, a path in a $(a_1 + 1) \times (a_2 + 1)$ grid from $(0, 0)$ to $(a_1 + 1, a_2 + 1)$. For example, the cells appearing to the right of the staircase in Figure 4.5 define a filter of $\mathbf{P}^*(\boldsymbol{\lambda})$. Finally, we have to take into account that the underlying poset is $\mathbf{P}^*(\boldsymbol{\lambda})$ instead of $\mathbf{P}(\boldsymbol{\lambda})$, whence we have to remove two possible paths, namely the path crossing $(0, a_2 + 1)$ and the path crossing $(a_1 + 1, 0)$.

ii) It suffices to remark that the number of staircase walks in a $(a_1 + 1) \times (a_2 + 1)$ grid is given by

$$\binom{(a_1+1)+(a_2+1)}{a_1+1} = \binom{a_1+a_2+2}{a_1+1}.$$

By *i*), we just should remove 2 staircase walks, whence the result.

iii) To prove this, we will use the fact that in an order polytope two vertices are adjacent to each other if and only if their associated filters satisfy either $F_1 \subset F_2$ or $F_2 \subset F_1$ and the difference $F_1 \setminus F_2$ or $F_2 \setminus F_1$ is connected [23]. Consider two vertices in these conditions and let us assume $F_2 \subset F_1$. Suppose $F_1 \setminus F_2$ is connected as poset. Note that by *i*) the elements in $F_1 \setminus F_2$ correspond to the cells between the two staircase paths associated to both filters. Now, given two cells $(i_1, j_1), (i_2, j_2)$ in $F_1 \setminus F_2$, there exists a path

$$(i_1, j_1) = a_1 - a_2 - \ldots - a_r = (i_2, j_2)$$

in $F_1 \setminus F_2$ such that either a_i covers a_{i+1} or the other way round. Indeed, (i, j) covers (k, l) if and only if either $i = k + 1$ or $j = l + 1$. But this means that cells (i, j) and (k, l) are adjacent in $\mathbf{P}(\boldsymbol{\lambda})$. Thus, connection of $F_1 \setminus F_2$ is equivalent to connection of the corresponding set of cells $F_1 \setminus F_2$ in the Young diagram, in the sense of Remark 4.6.

This finishes the proof. □

4.3 A procedure for random generation in $\mathcal{FM}(A_1, A_2)$

In this section, we tackle the problem of generating measures in $\mathcal{FM}(A_1, A_2)$ in a random way. As explained in Section 4.2, this is equivalent to generate a linear extension in a random way for the underlying poset $P(A_1, A_2)$. Now, as explained in Section 4.2, generating a linear extension of $P(A_1, A_2)$ is equivalent to generate a linear extension of $\mathbf{P}^*(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (|A_1| + 1, |A_1| + 1, \overset{|A_2|+1}{\ldots}, |A_1| + 1)$; and finally, as stated in Section 4.1, this is equivalent to generate a standard Young tableau for a rectangular Young diagram of shape $\boldsymbol{\lambda}$. The main difficulty here is the following: once an element i is selected as the last one in the linear extension of poset $\mathbf{P}^*$, this element is removed and we have to select another one from the poset $\mathbf{P}^* \backslash \{i\}$ to be the element before i in the linear extension; and this has to be done in a way such that all linear extensions have the same probability.

Definition 4.12. Given a Young diagram of shape $\boldsymbol{\lambda}$, we say that a position (i, j) is **maximal** if there is no $(k, l) \neq (i, j)$ in the diagram satisfying $i \leq k, j \leq l$.

Example 4.4. *For the Young diagram in Figure 4.1, it can be seen that we have three maximal cells, namely (0,4), (2,3) and (3,0).*

Note that at each step, we aim to select a maximal element of the diagram. For a $(|A_1|+1)\times(|A_2|+1)$ rectangular Young diagram, the only maximal element in the linear extension is the one in position $(|A_1|, |A_2|)$. Next step is to select the previous element in the linear extension and two candidates (maximal elements) arise: $(|A_1| - 1, |A_2|)$ and $(|A_1|, |A_2| - 1)$; the question now is: which probability should be assigned to select $(|A_1| - 1, |A_2|)$? This translates to step i: once several elements have been selected, we have some other positions that are candidates to be chosen as the previous element in the linear extension; which are the corresponding probabilities for each of them?

At this point, it should be noted that when an element is selected to be the last one, we obtain a new Young diagram (not necessarily a rectangular one), as next lemma shows:

Lemma 4.13. *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition of n. If a maximal cell is removed from the $i-th$ row in $\mathbf{X}(\boldsymbol{\lambda})$, then we obtain another Young diagram $\mathbf{X}(\boldsymbol{\lambda}')$ with $\lambda'_i = \lambda_i - 1, \lambda'_j = \lambda_j, j \neq i$.*

Proof. Since $\mathbf{X}(\boldsymbol{\lambda})$ is a Young diagram and the removed element is a maximal element, it follows that below and on the right side of the the cell associated to this maximal element there is no cell. Then, $\lambda_i > \lambda_{i+1}$, whence $\lambda'_i = \lambda_i - 1 \geq \lambda_{i+1} = \lambda'_{i+1}$. We conclude that $\boldsymbol{\lambda}'$ is a partition of $n-1$ and $\mathbf{X}(\boldsymbol{\lambda}')$ is its corresponding Young diagram. □

Consequently, the problem reduces to obtain a procedure to select a maximal element in a general Young diagram so that we obtain a random standard Young tableau. The main result of the section solves this question.

Theorem 4.14. *Let $\mathbf{P}(\boldsymbol{\lambda})$ be a Ferrers poset with n elements and $x \in \mathbf{P}(\boldsymbol{\lambda})$ be a maximal element. Then, the probability of x being the last element in a linear extension of $\mathbf{P}(\boldsymbol{\lambda})$ is given by:*

$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{1}{n} \prod_{z \in H_{\boldsymbol{\lambda}}^{-1}(x)} \frac{h_{\boldsymbol{\lambda}}(z)}{h_{\boldsymbol{\lambda}}(z) - 1},$$

where $H_{\boldsymbol{\lambda}}^{-1}(x) = \{z \in \mathbf{P}(\boldsymbol{\lambda}) \mid x \in H_{\boldsymbol{\lambda}}(z) \text{ and } z \neq x\}$.

Proof. We start noting that

$$e\left(\mathbf{P}(\boldsymbol{\lambda})\right) = \sum_{x \text{ maximal of } \mathbf{P}(\boldsymbol{\lambda})} e\left(\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}\right)$$

and

$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{e\left(\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}\right)}{e\left(\mathbf{P}(\boldsymbol{\lambda})\right)}.$$

From Lemma 4.13, $\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}$ is a Ferrers poset with partition $\boldsymbol{\lambda}'$, with $\lambda'_i = \lambda_i - 1, \lambda'_k = \lambda_k, k \neq i$ if $x = (i-1, j)$. Then, both $e\left(\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}\right)$ and $e\left(\mathbf{P}(\boldsymbol{\lambda})\right)$ can be computed from Corollary 4.7, whence

$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{\dfrac{(n-1)!}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda}')} h_{\boldsymbol{\lambda}'}(z)}}{\dfrac{n!}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}} = \frac{1}{n} \frac{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda}')} h_{\boldsymbol{\lambda}'}(z)}.$$

Observe that if we remove a maximal element from a Young diagram, then the only elements with different hook lengths are the elements in the same row or in the same column as the removed maximal element (see Figure 4.6). Therefore,

9	7	5	3	1
7	5	3	1	
5	3	1		
3	1			
1				

9	7	4	3	1
7	5	2	1	
4	2			
3	1			
1				

FIGURE 4.6: Hook lengths before (left) and after (right) removing a maximal element.

$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{1}{n} \frac{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda}')} h_{\boldsymbol{\lambda}'}(z)} = \frac{1}{n} \frac{\prod_{z \in H_{\boldsymbol{\lambda}}^{-1}(x)} h_{\boldsymbol{\lambda}}(z)}{\prod_{z \in H_{\boldsymbol{\lambda}}^{-1}(x)} h_{\boldsymbol{\lambda}'}(z)} = \frac{1}{n} \prod_{z \in H_{\boldsymbol{\lambda}}^{-1}(x)} \frac{h_{\boldsymbol{\lambda}}(z)}{h_{\boldsymbol{\lambda}'}(z)},$$

where in the last step we have used that $h_{\boldsymbol{\lambda}}(x) = 1$ because it is a maximal element.

Finally, since we have removed x we obtain $h_{\boldsymbol{\lambda}'}(z) = h_{\boldsymbol{\lambda}}(z) - 1$ for every element z in the same row or column as x. Therefore, we get the desired formula. $\square$

Theorem 4.14 provides the probability of selecting a maximal element as the previous element in a linear extension at each step. Therefore, we can state the following procedure for deriving a random standard Young tableau and thus, a random 2-symmetric measure (see Algorithm 5).

Algorithm 5 SAMPLING ALGORITHM FOR 2-SYMMETRIC MEASURES

1. Consider a Young diagram of shape $\boldsymbol{\lambda} = (|A_1|+1, |A_1|+1, \overset{|A_2|+1}{\cdots}, |A_1|+1)$.

2. Sampling algorithm for standard Young tableaux.

Step 2.1: Select a maximal cell (i,j) of the Young diagram with probability

$$P\left((i,j) \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{1}{n} \prod_{(k,l)\in H_{\boldsymbol{\lambda}}^{-1}(i,j)} \frac{h_{\boldsymbol{\lambda}}(k,l)}{h_{\boldsymbol{\lambda}}(k,l)-1}.$$

Step 2.2: Remove this cell and repeat the previous step for the new Young diagram.

3. Given the standard Young tableau obtained in the previous step, build the corresponding linear extension of the Ferrers poset, thus obtaining a linear extension ϵ of the poset $(P, \preceq)$.

4. Remove the first $(0,0)$ and last element $(|A_1|, |A_2|)$ of the linear extension. Thus, we obtain a linear extension ϵ^* of the poset $\mathbf{P}^*(\boldsymbol{\lambda})$.

5. Sampling the values of the 2-symmetric measure.

Step 5.1: Generate a $[(a_1+1)(a_2+1)-2]$ vector $\vec{u}$ of random variables $U(0,1)$.

Step 5.2: Sort $\vec{u}$, to get a $[(a_1+1)(a_2+1)-2]$ vector $\vec{v}$ with the values generated in the previous step in increasing order.

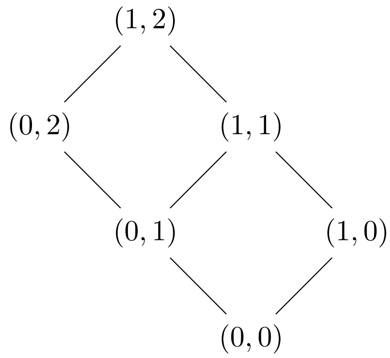
Step 5.3: Assign the value $v[k]$ to $\mu(i,j)$ if the element associated to the cell (i,j) is placed at position k in ϵ^*. We denote this by $\epsilon^*[(i,j)] = k$.
return μ.

Example 4.5. *Let* $X = \{1,2,3\}$, $A_1 = \{1,2\}$ *and* $A_2 = \{3\}$. *Therefore* $n = 6, a_1 = 2$ *and* $a_2 = 1$. *Then by Lemma 4.8, this polytope is associated to the Young diagram of shape* $\boldsymbol{\lambda} = (3,3)$. *The initial Young diagram and its Ferrers poset are given in Figure 4.7.*

- *At the beginning we have just one maximal cell, namely* $(1,2)$, *so we remove it and* $\epsilon[(1,2)] = 6$. *That is, the element associated to subset* $(1,2)$ *is at position 6 in the linear extension* ϵ. *Update* $\boldsymbol{\lambda}' = (3,2)$.

FIGURE 4.7: Standard Young tableau associated to $\boldsymbol{\lambda} = (3,3)$ (left) and its Ferrers poset (right).

$(0,0)$	$(0,1)$	$(0,2)$
$(1,0)$	$(1,1)$	$(1,2)$



- *Now* $(1,1)$ *and* $(0,2)$ *are maximal cells. We compute their probabilities through Theorem 4.14.*

$$P\left((1,1) \mid \mathbf{P}(\boldsymbol{\lambda}')\right) = \frac{1}{5} \cdot \frac{3}{2} \cdot \frac{2}{1} = \frac{3}{5}, \quad P\left((0,2) \mid \mathbf{P}(\boldsymbol{\lambda}')\right) = \frac{1}{5} \cdot \frac{3}{2} \cdot \frac{4}{3} = \frac{2}{5}.$$

Suppose $(1,1)$ *is selected,* $\epsilon[(1,1)] = 5$*. Update* $\boldsymbol{\lambda}'$ *to* $\boldsymbol{\lambda}'' = (3,1)$*.*

- *Now* $(1,0)$ *and* $(0,2)$ *are maximal cells. We compute their probabilities through Theorem 4.14.*

$$P\left((1,0) \mid \mathbf{P}(\boldsymbol{\lambda}'')\right) = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}, \quad P\left((0,2) \mid \mathbf{P}(\boldsymbol{\lambda}'')\right) = \frac{1}{4} \cdot \frac{2}{1} \cdot \frac{4}{3} = \frac{2}{3}.$$

Suppose $(1,0)$ *is selected,* $\epsilon[(1,0)] = 4$*. Update* $\boldsymbol{\lambda}''$ *to* $\boldsymbol{\lambda}''' = (3,0) \equiv 3$*.*

- *The remaining poset is a chain, so we have just one maximal in each step and* $\epsilon[(0,2)] = 3$*,* $\epsilon[(0,1)] = 2$ *and* $\epsilon[(0,0)] = 1$*. Therefore,* $\epsilon = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$*.*

Now we obtain ϵ^* *from* ϵ *by removing the maximum and minimum,* $\epsilon^* = \{(0,1),(0,2),(1,0),(1,1)\}$*.*

Finally, we have to generate 4 variables $U(0,1)$ *in order to sample inside the simplex* ϵ^**. Using any mathematical software (for example R), we obtain say*

$$(0.77, 0.65, 0.73, 0.09).$$

Now, $v = (0.09, 0.65, 0.73, 0.77)$*, and* μ *is given by:*

$$\mu(0,0) = 0.00,\ \mu(0,1) = 0.09\ \mu(0,2) = 0.65,\ \mu(1,0) = 0.73,\ \mu(1,1) = 0.77\ \mu(1,2) = 1.00.$$

Next, let us study the computational complexity.

Proposition 4.15. *The computational complexity of the previous algorithm is $O(n^2)$, where $n = (a_1 + 1)(a_2 + 1)$.*

Proof. We will count the number of operations that the algorithm performs to obtain a random 2-symmetric measure.

- **Initializing**: Build the initial hook matrix. The complexity is n, since we just have to save $n = (a_1 + 1) \cdot (a_2 + 1)$ numbers in memory, corresponding to the hook length of the n cells.
- **Updating position vector**: When an element is removed, we should identify the new maximal cells. The number of maximal elements is at most $min\,\{a_1 + 1, a_2 + 1\}$ and we repeat this search n times, one for each iteration, we need say $min\,\{a_1 + 1, a_2 + 1\} \cdot n \leq n^2$ computations at most.
- **Updating hook matrix**: When removing an element, we have to subtract 1 to the elements above and to the left of it. The number of such cells is limited by $(a_1 + 1) + (a_2 + 1) \leq n$. And taking it for the n iterations, we obtain the upper bound n^2.
- **Computing probability vectors for maximals**: For each combination of maximals we should compute the probability of each maximal.

  First, let us assume w.l.g. that $a_1 \leq a_2$, and that this holds for the Young diagram arising at each iteration. Note that the number of maximals is bounded by 1 in the first iteration, by 2 for iterations 2, 3, by 3 for iterations 4, 5, 6 and so on. Next, after iteration $1 + 2 + 3 + 4 + ... + a_1$ the number of maximals is bounded by $a_1 + 1$ (the number of lines of the initial Young diagram) and this remains for $(a_1+1)(a_2-a_1+1)$ iterations, i.e., until the last $1+2+3+4+...+a_1$ iterations, where the number of maximals is again bounded by $a_1, a_1-1, ...$ Now, for maximal (i, j), the number of products to compute the probability of this maximum is $2 \cdot (i + j)$. Therefore, we have to compute

$$\sum_{t=1}^{n} \sum_{\{(i,j)\ maximal\ in\ iteration\ t\}} [2 \cdot (i + j)] \,.$$

  We are going to split this sum into the two parts mentioned above.

  For the first $1 + 2 + 3 + 4 + ... + a_1$ iterations, we know that there are a_1 kinds of iterations depending on the bound for the number of maximals. Moreover, for the kind of iteration p there are p of these iterations (as many as cells in the diagonal), and in each of these iterations there are at most p maximals; for each maximal we should do $2 \cdot (i+j) \leq 2(a_1+a_2)$ computations at most. This also applies for the last $1+2+3+4+...+a_1$ iterations, so we have to double the bound.

$$2\sum_{p=1}^{a_1} 2(a_1+a_2)p^2 = 4(a_1+a_2)\frac{a_1(a_1+1)(2a_1+1)}{6}$$

$$\leq \frac{16}{6} \cdot (a_1+1)^3(a_2+1) \leq 3 \cdot (a_1+1)^2(a_2+1)^2 = 3n^2.$$

For the second part, we work with the $(a_1+1)(a_2-a_1+1)$ iterations related to the cells in the middle of the Young tableau. In each of these iterations, there are at most (a_1+1) maximals and for each maximal we should do $2 \cdot (i+j) \leq 2(a_1+a_2)$ computations at most. Therefore we have:

$$2(a_1+a_2) \cdot (a_1+1)^2 \cdot (a_2-a_1+1) \leq 4 \cdot (a_1+1)^2(a_2+1)^2 = 4n^2,$$

whence the quadratic complexity is obtained.

- **Sampling**: We sample n uniform random variables. Obviously, it takes a complexity of n.

By adding these steps we observe that the important part is the computation of the probability vector which reaches a quadratic complexity. □

4.4 Application: the problem of identification

In this section, let us present an application of the previous results. Consider a problem in the framework of Multicriteria Decision Making; then, we have a set of $|X| = n$ criteria and we have to choose between a set of objects. Each object has to be given an overall value y from the partial evaluations on each criterion, denoted $x_1, ..., x_n$; we assume that the problem can be modeled via Choquet integral, i.e., we assume that y is the Choquet integral of a function f defined by $f(i) = x_i, i = 1, ..., n$, with respect to an unknown fuzzy measure μ (possibly restricted to a subfamily of fuzzy measures). The goal is to identify the fuzzy measure μ modeling this situation and for this, we have a sample of m objects for which we know both the partial scores $(x_1^i, ..., x_n^i), i = 1, ...m$, and the overall score $y_i, i = 1, ..., m$ (possibly affected by some random noise); it could be the case that y, $(x_1, ..., x_n)$ or both of them are ordinal values, not necessarily numerical. We look for a fuzzy measure (not necessarily unique) that best fits these data. If the quadratic error is considered, this amounts to looking for a fuzzy measure μ_0 minimizing

$$F(\mu) := \sum_{i=1}^{m} \left(\mathcal{C}_\mu(x_1^i, ..., x_n^i) - y_i\right)^2.$$

Several techniques have been proposed to solve this problem [61, 84, 85]. One of them is based on genetic algorithms [86]. Genetic algorithms are general optimization methods based on the theory of natural evolution; starting from an initial population, at each iteration some individuals are selected with probability proportional to their fitness (measured according to the function that we want to optimize) and new individuals are generated from them using a cross-over operator. These new individuals replace the old ones (their parents) and the process continues until an optimum is found or the maximum number of generations is reached; besides, some other individuals may mutate via a mutation operator. Finally, the best individual in the last population is returned as a possible solution to the problem.

In [87], a procedure based on genetic algorithms has been proposed. In this procedure, the cross-over operator is given by the convex combination between parents and the mutation operator is given by a convex combination between the selected measure and another one. The algorithm seems to be fast and work properly because it is not needed to check at each iteration if the offsprings are in the search region. However, it has the drawback that this reduces the search region at each iteration; and this forbids to go back if the solution of the problem is left outside this region in an iteration. To cope with this problem, there are several options; the most natural is to consider the set of vertices as the initial population of the algorithm and use as mutation operator the convex combination of the measure with one of the vertices; unfortunately, for the general case of fuzzy measures and also for many subfamilies, the number of vertices increases too fast [88]. Thus, this solution is most of times unfeasible from a computational point of view. Then, the most practical way of proceeding is to consider as initial population a set of fuzzy measures selected randomly with respect to the uniform distribution on the set of all fuzzy measures in $\mathcal{FM}(X)$. In this section, we are going to develop a procedure based on the previous results on 2-symmetric measures.

To apply the previous results on 2-symmetric measures to this problem, the following result is the cornerstone. It tells that the whole set of fuzzy measures $\mathcal{FM}(X)$ can be recovered from 2-symmetric measures, just allowing different partitions. First, let us define

$$\mathcal{FMS}_2 := \bigcup_{A \subseteq X} \mathcal{FM}(A, A^c),$$

the set of all fuzzy measures being at most 2-symmetric for a partition.

Proposition 4.16. *Let X be a referential set of n elements and consider $\mu \in \mathcal{FM}(X)$. Then, there exist $k, k' \in \mathbb{N}$ such that μ can be written as*

$$\mu = \mu_1 \vee \mu_2 \vee \cdots \vee \mu_k,$$

and also

$$\mu = \mu_1^* \wedge \mu_2^* \wedge \cdots \wedge \mu_{k'}^*,$$

where $\mu_i \in \mathcal{FMS}_2$, $\forall i \in 1, \cdots, k$ and $\mu_i^* \in \mathcal{FMS}_2$, $\forall i \in 1, \cdots, k'$.

Proof. Observe that

$$\mu = \bigvee_{\emptyset \subset A \subset X} \mu_A,$$

where μ_A is defined by

$$\mu_A(C) := \begin{cases} \mu(A) & \text{if} \quad C \supseteq \text{A}, \\ 0 & \text{otherwise} \end{cases}$$

As $\mu_A \in \mathcal{FM}(A, A^c)$, the result holds.

For the second statement, note that

$$\mu = \bigwedge_{\emptyset \subset A \subset X} \mu_A^*,$$

where μ_A^* is given by

$$\mu_A^*(C) := \begin{cases} \mu(A) & \text{if} \quad C \subseteq \text{A}, \\ 1 & \text{otherwise} \end{cases}$$

As $\mu_A^* \in \mathcal{FM}(A, A^c)$, the result holds. □

Based on the previous proposition, we could modify the procedure in [87], and consider as cross-over operator

$$\mu_1 \oplus_\lambda \mu_2 := \lambda \left(\mu_1 \vee \mu_2\right) + (1 - \lambda) \left(\mu_1 \wedge \mu_2\right), \; \lambda \in [0, 1], \tag{4.1}$$

instead of the convex combination. This cross-over operator shares similar properties to the convex combination, in the sense that it is not necessary to check at each iteration whether the children are inside the set $\mathcal{FM}(X)$. In other words, we can achieve any element of $\mathcal{FM}(X)$ from a proper random initial population of elements in $\mathcal{FMS}_2$.

To study the behavior of this algorithm, we have conducted a simulation study. We have studied two different situations.

- **Case 1: Identification of 2-symmetric measures for a fixed partition.** We try to identify a measure $\mu \in \mathcal{FM}(A, A^c)$ chosen at random. The data to identify the measure are the 3^n points consisting in all possible n-vectors $(x_1, ..., x_n)$, where $x_i \in \{0, 0.5, 1\}$ (this is the same data as considered in [87]). For any possible vector, the overall value y is the corresponding Choquet integral with respect to μ. The initial population consists in 30 measures in $\mathcal{FM}(A, A^c)$ selected at random via the algorithm presented in the previous section. At each step, the population splits into three groups. The first group consists in the measures with the highest scores (lowest mean Choquet errors $F(\mu)$) computed with the first quartile. The second group includes the elements with the lowest scores, using the third quartile. Finally, the third group are the rest of the elements. The cross-over operator is the convex combination. In each step, we apply the cross-over operator to groups 2 and 3. The mutation operator is as follows: the elements of group 2 are replaced by new ones generated at random from $\mathcal{FM}(A, A^c)$. In addition, groups 1 and 3 are mutated by adding gaussian noise on some coordinates chosen at random. As the algorithm progresses, the three types of population change and better scored elements arrive to group 1. This method stops when the mean Choquet error is below a threshold ($\epsilon = 10^{-8}$) or a maximum number of iterations is achieved.

- **Case 2: Identification of general fuzzy measures.** In this case, we try to identify a measure $\mu \in \mathcal{FM}(X)$ chosen at random. For this generation we can use some high complexity algorithm such as an acceptance-rejection method. As in the previous case, the data are the 3^n points consisting in all possible n-vectors $(x_1, ..., x_n)$, where $x_i \in \{0, 0.5, 1\}$ and the overall value y is the corresponding Choquet integral with respect to μ. The initial population is a set of $n = 30$ 2-symmetric fuzzy measures in $\mathcal{FMS}_2$ selected at random; for this, we have first to choose a partition $\{A, A^c\}$ at random, and this can be done using the volume of each $\mathcal{FM}(A, A^c)$ (that has been obtained in Corollary 4.7) to know the probability of each partition; next, once A is fixed, we generate a measure in $\mathcal{F}M(A, A^c)$ via the algorithm proposed in the previous section. The population again splits into three groups as defined above. The rest of the algorithm is also similar as above but selecting the cross-over operator given by (4.1). At each step, we apply the cross-over operator for a value of λ selected at random in [0,1] to groups 2 and 3. The mutation operator is also similar as above: The elements of group 2 are replaced by the new ones generated at random from $\mathcal{FMS}_2$. In addition, groups 1 and 3 are mutated by adding gaussian noise on some coordinates chosen at random.

This method stops when the mean Choquet error is below 10^{-8} or a maximum number of iterations is achieved.

We have repeated both models 50 times for each size n. The mean identification errors for the different referential sizes are given in Table 4.1. From these values, it can be seen that the algorithm seems to perform very well in both cases.

Identification Algorithm Performance		
Referencial Size	2-Symmetric	Fuzzy measures
$n = 3$	1.388e-05	2.653e-05
$n = 4$	3.918e-05	4.128e-05
$n = 5$	6.365e-05	2.413e-04
$n = 6$	7.831e-05	4.480e-04
$n = 7$	8.134e-05	7.279e-04
$n = 8$	1.915e-04	9.011e-04
$n = 9$	2.353e-04	9.545e-04
$n = 10$	2.561e-04	1.025e-03

TABLE 4.1: Identification errors.

Obviously, when we identify a general fuzzy measure by using an initial random population of 2-symmetric measures we get a higher identification error. However, this error is acceptable even for big values of n. Moreover, this algorithm gets lower errors than other methods in the literature (see [87]). Thus, we have managed to achieve a way of avoiding the computational problems related to the huge number of vertices in $\mathcal{FM}(X)$ and identifying fuzzy measures by using the combinatorial properties of $\mathcal{FMS}_2$.

Chapter 5

Combinatorial structure of the polytope of 2-additive fuzzy measures

In this chapter we study the polytope of 2-additive measures, an important subpolytope of the polytope of fuzzy measures. For this polytope, we obtain its combinatorial structure, namely the adjacency structure and the structure of 2-dimensional faces, 3-dimensional faces, and so on. Basing on this information, we build a triangulation of this polytope satisfying that all simplices in the triangulation have the same volume. As a consequence, this allows a very simple and appealing way to generate points in a random way in this polytope, an interesting problem arising in the practical identification of 2-additive measures. Finally, we also derive the volume, the centroid, and some properties concerning the adjacency graph of this polytope (see [89]).

5.1 Combinatorial structure of $\mathcal{FM}^2(X)$

In this section we tackle the problem of obtaining the combinatorial structure of $\mathcal{FM}^2(X)$, that is, its k-dimensional faces. For the different concepts relating polytopes appearing in this section, see Chapter 1. It can be easily seen that $\mathcal{FM}^2(X)$ is a convex polyhedron in $\mathbb{R}^{2^n-2}$, i.e., a polytope, as it is the intersection of the polytope $\mathcal{FM}(X)$ and the hyperplanes $m(A) = 0$, $|A| > 2$. Then, it can be characterized in terms of its vertices. The vertices of $\mathcal{FM}^2(X)$ have been obtained in [90] and are given in next proposition.

Proposition 5.1. *The set of vertices of $\mathcal{FM}^2(X)$ are given by the $\{0,1\}$-valued fuzzy measures in $\mathcal{FM}^2(X)$, i.e., u_i, u_{ij}, μ_{ij}, that are defined by*

$$u_A(B) := \begin{cases} 1 & \text{if } A \subseteq \mathrm{B} \\ 0 & \text{otherwise} \end{cases},$$

$$\mu_{ij}(B) := \begin{cases} 1 & \text{if } i \in \mathrm{B} \text{ or } j \in \mathrm{B} \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathcal{FM}^2(X)$ has n^2 vertices.

This result shows an important difference between 2-additive measures and general k-additive measures, $k > 2$, as it has been proved in [90] that there are vertices in $\mathcal{FM}^k(X), k > 2$ that are not $\{0,1\}$-valued.

For $\mathcal{FM}^k(X)$, it is convenient in many situations to use the equivalent Möbius transform. For this reason, let us define a map that will be of aid in the following:

$$\begin{array}{rccc} \mathsf{m}: & \mathcal{FM}(X) & \rightarrow & \mathcal{M}(X) \\ & \mu & \hookrightarrow & m_\mu \end{array}$$

We will denote $\mathcal{M}^2(X) := \mathsf{m}(\mathcal{FM}^2(X))$. As m is a nonsingular linear application, we conclude that $\mathcal{M}^2(X)$ is a convex polytope that is combinatorially equivalent to $\mathcal{FM}^2(X)$ (see [10]). Thus, this function maps k-dimensional faces into k-dimensional faces and keeps adjacency and any other result concerning the combinatorial structure. Note however that it does not keep the volume nor distances.

This way, we can study k-dimensional faces of $\mathcal{M}^2(X)$ (a simpler problem, as it will be seen below) and apply m^{-1} to get the same conclusions about the k-dimensional faces of $\mathcal{FM}^2(X)$. In particular, m maps vertices into vertices; hence, the vertices of $\mathcal{M}^2(X)$ are:

For u_i : we get m_i defined as $m_i(i) = 1$ and $m_i(A) = 0, \forall A \neq i$.

For u_{ij} : we get m_{ij} defined as $m_{ij}(ij) = 1$ and $m_{ij}(A) = 0, \forall A \neq ij$.

For μ_{ij} : we get $\overline{m}_{ij}$ defined as $\overline{m}_{ij}(ij) = -1, \overline{m}_{ij}(i) = 1, \overline{m}_{ij}(j) = 1$ and $\overline{m}_{ij}(A) = 0, \forall A \notin \{i, j, ij\}$.

We start studying the dimension of $\mathcal{FM}^2(X)$.

Lemma 5.2. *$dim(\mathcal{FM}^2(X)) = \binom{n}{2} + n - 1$.*

Proof. It is clear that $dim(\mathcal{FM}(X)) = 2^n - 2$. Hence, $dim(\mathcal{M}(X)) = 2^n - 2$. Now, remark that

$$\mathcal{M}^2(X) = \mathcal{M}(X) \bigcap_{|A|\geq 3} \{m : m(A) = 0\}.$$

Therefore,

$$dim(\mathcal{M}^2(X)) = 2^n - 2 - \sum_{i=3}^{n} \binom{n}{i} = \binom{n}{2} + n - 1.$$

Hence, the result holds. □

In what follows, we will study the properties of $\mathcal{FM}^2(X)$ as a polytope in $\mathbb{R}^{\binom{n}{2}+n-1}$. We start showing that, although the vertices of $\mathcal{FM}^2(X)$ are $\{0,1\}$-valued, it is not an order polytope.

Proposition 5.3. *If $|X| > 2$, the polytope $\mathcal{FM}^2(X)$ is not an order polytope.*

Proof. Assume there is a poset P such that $\mathcal{O}(P) = \mathcal{FM}^2(X)$. Then, P is a filter for P and thus there is a maximum vertex with all coordinates equal to 1, i.e. there is a vertex $\mu \in \mathcal{FM}^2(X)$ such that $\mu(A) \geq \mu'(A), \forall A \subseteq X, \forall \mu' \in \mathcal{FM}^2(X)$. However, if $|X| > 2$, there is not $\mu \in \mathcal{FM}^2(X)$ dominating say μ_{12}, and this measure does not dominate all 2-additive measures. Therefore, we get a contradiction and the result holds. If $|X| = 2$, then $\mathcal{FM}^2(X) = \mathcal{FM}(X)$ which is an order polytope. □

As $\mathcal{FM}^2(X)$ is not an order polytope, we cannot apply the results of [22, 23] to sort out the combinatorial structure of this polytope and we have to look for another way to solve the problem.

The problem of determining all the faces of a polytope consists in characterizing the conditions for a subset of vertices $\mathcal{C}$ to determine a face. Next lemma shows the basic result for characterizing the faces of $\mathcal{FM}^2(X)$.

Lemma 5.4. *Let $\mathcal{F}$ be a face of $\mathcal{FM}^2(X)$. Then, $u_{ij}, \mu_{ij} \in \mathcal{F}$ if and only if $u_i, u_j \in \mathcal{F}$.*

Proof. It is enough to point out that $u_i + u_j = u_{ij} + \mu_{ij}$, $\forall i \neq j$ in X, and use Lemma 1.47. □

We are now in a position to present the main result of this section, in which we give a complete characterization of the faces of $\mathcal{FM}^2(X)$, see Definition 1.43. In it, we show that the necessary condition of Lemma 5.4 is also sufficient.

Theorem 5.5. ***Combinatorial structure of*** $\mathcal{FM}^2(X)$. *Let $\mathcal{C}$ be a collection of vertices of $\mathcal{FM}^2(X)$. Then the following are equivalent:*

i) $\mathrm{Conv}(\mathcal{C})$ *is a face of* $\mathcal{FM}^2(X)$.

ii) $u_i, u_j \in \mathcal{C} \Leftrightarrow u_{ij}, \mu_{ij} \in \mathcal{C}$.

Proof. *i)* $\Rightarrow$ *ii)* This is Lemma 5.4.

ii) $\Rightarrow$ *i)* Consider a set of vertices $\mathcal{C}$ satisfying ii) and consider the corresponding vertices in $\mathcal{M}^2(X)$. Let us define

$$H := \{i \in X : u_i \in \mathcal{C}\}, \quad I := \{ij \in \binom{X}{2} : u_{ij}, \mu_{ij} \in \mathcal{C}\},$$

$$A := \{ij \in \binom{X}{2} : u_{ij} \in \mathcal{C}, \mu_{ij} \notin \mathcal{C}\}, \quad B := \{ij \in \binom{X}{2} : \mu_{ij} \in \mathcal{C}, u_{ij} \notin \mathcal{C}\}.$$

Now, we define the following halfspace

$$2\sum_{i\in H} m(i) + 2\sum_{ij\in A\cup I} m(ij) + \sum_{ij\notin A\cup B\cup I,\ i\in H, j\notin H} m(ij) - 2\sum_{ij\in B, i\notin H, j\notin H} m(ij) \leq 2.$$

Let us show that this halfspace defines a face in $\mathcal{M}^2(X)$ such that the corresponding face in $\mathcal{FM}^2(X)$ has $\mathcal{C}$ as vertices. First, let us see that $\mathcal{M}^2(X)$ is in the halfspace. To show this, it suffices to check that all vertices in $\mathcal{M}^2(X)$ are in the halfspace. By ii), we have $ij \in I \Leftrightarrow i, j \in H$. Thus, we avoid the cases $ij \in B, i, j \in H$ and $ij \in A, i, j \in H$.

For $m_i,\ i \in H$: we get $2 \leq 2$.

For $m_i,\ i \notin H$: we get $0 \leq 2$.

For $m_{ij},\ ij \in I$: we get $2 \leq 2$.

For $m_{ij},\ ij \in A$: we get $2 \leq 2$.

For $m_{ij},\ ij \in B$: we get $0 \leq 2$ if $i \in H, j \notin H$ and $-2 \leq 2$ if $i \notin H, j \notin H$.

For $m_{ij},\ ij \notin A \cup B \cup I$: we get $1 \leq 2$ if $i \in H, j \notin H$ and $0 \leq 2$ if $i \notin H, j \notin H$.

For $\overline{m}_{ij},\ ij \in I$: we get $2 \leq 2$.

For $\overline{m}_{ij},\ ij \in A$: we get $0 \leq 2$ if $i \in H, j \notin H$ and $-2 \leq 2$ if $i \notin H, j \notin H$.

For $\overline{m}_{ij},\ ij \in B$: we get $2 \leq 2$ if $i \in H, j \notin H$ and $2 \leq 2$ if $i \notin H, j \notin H$.

For $\overline{m}_{ij},\ ij \notin A \cup B \cup I$: we get $1 \leq 2$ if $i \in H, j \notin H$ and $0 \leq 2$ if $i \notin H, j \notin H$.

Note further that equality holds exactly for the vertices in $\mathcal{M}^2(X)$ whose image by m^{-1} is in $\mathcal{C}$. Therefore, $\mathcal{C}$ defines a face in $\mathcal{FM}^2(X)$ and the result holds. $\square$

From this theorem, we can derive in an easy and fast way the adjacency structure of $\mathcal{FM}^2(X)$.

Corollary 5.6. *Let μ_1 and μ_2 be two different vertices of $\mathcal{FM}^2(X)$. Then, μ_1 and μ_2 are adjacent vertices in $\mathcal{FM}^2(X)$ except if $\mu_1 = u_i$, $\mu_2 = u_j$ or $\mu_1 = u_{ij}, \mu_2 = \mu_{ij}$.*

Proof. It suffices to remark that two vertices are adjacent if and only if they form a 1-dimensional face, i.e. an edge. Let $\mathcal{C}$ be a collection of two vertices of $\mathcal{FM}^2(X)$. Applying Theorem 5.5, it follows that $\mathcal{C}$ defines an edge if and only if $\mathcal{C} \neq \{u_i, u_j\}$ or $\mathcal{C} \neq \{u_{ij}, \mu_{ij}\}$. □

Next, we study the structure of k-dimensional faces. From Corollary 5.6 and Theorem 5.5 it is also possible to determine whether a face is a simplex. Next theorem describe the geometry of each face.

Theorem 5.7. *Let $\mathcal{F}_{\mathcal{C}}$ be a face of $\mathcal{FM}^2(X)$ and let us consider the following sets*

$$\mathcal{U} := \{i \in X : u_i \in \mathcal{C}\}, \quad \mathcal{V} := \left\{ij \in \binom{X}{2} : |\{u_{ij}, \mu_{ij}\} \cap \mathcal{C}| = 1\right\}.$$

Then, the following holds:

i) If $|\mathcal{U}| \leq 1$, then $\mathcal{F}_{\mathcal{C}}$ is a simplicial face of dimension $|\mathcal{C}| - 1$, (see Def. 1.41).

ii) If $|\mathcal{U}| > 1$, then $\mathcal{F}_{\mathcal{C}}$ is a non-simplicial face of dimension $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| + |\mathcal{V}| - 1$. Moreover, if $\mathcal{V} = \emptyset$, then $\mathcal{F}_{\mathcal{C}} = \mathcal{FM}^2(\mathcal{U})$. Otherwise, $\mathcal{F}_{\mathcal{C}} = cpyr(\mathcal{FM}^2(\mathcal{U}), \mathcal{V})$, where cpyr denotes the consecutive pyramid (see Def. 1.42).

Proof. *i)* Since $|\mathcal{U}| \leq 1$ and $\mathcal{F}_{\mathcal{C}}$ is a face, we conclude from Theorem 5.5 that $\{u_{ij}, \mu_{ij}\} \nsubseteq \mathcal{C}, \forall ij \in \binom{X}{2}$. We are going to show that vertices in $\mathcal{C}$ are affinely independent and therefore they form a simplex. To show this, we are going to work in $\mathcal{M}^2(X)$ with Möbius coordinates. Let us write $\mathsf{m}(\mathcal{C}) = \{v_0, \ldots, v_s\}$, with $s \geq 1$, as otherwise the result trivially holds. We will show that no $v_i - v_0$ can be written as a linear combination of $v_j - v_0$, $j \neq i$. We consider two cases:

- Consider $m_{ij} \in \mathsf{m}(\mathcal{C})$ and suppose we can write $m_{ij} - v_0$ as a linear combination of the other $v_j - v_0$ (the case for $\overline{m}_{ij}$ follows exactly the same reasoning). Without loss of generality, let us denote $v_1 = m_{ij}$. Remark that $m_{ij}(ij) \neq 0 = v_k(ij)$, $\forall k \neq 1$ (as $\overline{m}_{ij} \notin \mathsf{m}(\mathcal{C})$). Thus,

$$(v_1 - v_0)(ij) \neq 0 = (v_k - v_0)(ij), \ \forall k > 1,$$

and hence we conclude that $v_1 - v_0$ cannot be written as a linear combination of $v_k - v_0,\ k > 1$.

- Let us now suppose that we can write the (possibly) only $m_i \in \mathsf{m}(\mathcal{C})$ in the way that $m_i - v_0$ is a linear combination of the other $v_j - v_0$. Let us assume without loss of generality $v_1 = m_i$. Then

$$v_1 - v_0 = \sum_{i=2}^{s} \alpha_i (v_i - v_0),$$

for some α_i not all of them null, say $\alpha_2 \neq 0$. Then, we can rewrite the previous expression, thus obtaining

$$v_2 - v_0 = \sum_{k=3}^{s} \frac{-\alpha_k}{\alpha_2}(v_k - v_0) + \frac{1}{\alpha_2}(v_1 - v_0).$$

As v_2 is either m_{kr} or $\overline{m}_{kr}$, this is a contradiction with the previous case.

Then, the vertices of $\mathcal{C}$ are affinely independent and $\mathcal{F}_{\mathcal{C}}$ is a simplicial face of dimension $|\mathcal{C}| - 1$.

ii) Since $|\mathcal{U}| > 1$, there are two vertices u_i and u_j that are not adjacent to each other by Corollary 5.6. Therefore, $\mathcal{F}_{\mathcal{C}}$ is not a simplex.

If $\mathcal{V} = \emptyset$, we conclude from Lemma 5.4 that

$$\mathcal{C} = \{u_i : i \in \mathcal{U}\} \cup \left\{ u_{ij}, \mu_{ij} : ij \in \binom{\mathcal{U}}{2} \right\}.$$

Hence, $\mathcal{F}_{\mathcal{C}} = \mathcal{FM}^2(\mathcal{U})$ and the dimension is $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| - 1$.

Let us suppose now that $\mathcal{V} \neq \emptyset$. By Lemma 5.4, we know that

$$\{u_i : i \in \mathcal{U}\} \cup \left\{ u_{ij}, \mu_{ij} : ij \in \binom{\mathcal{U}}{2} \right\} \subset \mathcal{C}.$$

Therefore, $\mathcal{FM}^2(\mathcal{U}) \subset \mathrm{Conv}(\mathcal{C})$. Take u_{ij} such that $ij \in \mathcal{V}$ (the case for μ_{ij} is completely similar) or, in Möbius coordinates, m_{ij} with $ij \in \mathcal{V}$. Note that for all vertices $\mu \in \mathcal{FM}^2(\mathcal{U})$, it is $m_\mu(ij) = 0$ and thus, they are non-collinear with m_{ij}. Then, when we add u_{ij} we get $\mathrm{pyr}(\mathcal{FM}^2(\mathcal{U}), u_{ij})$. The dimension is the dimension of the base plus 1. If we repeat this procedure with the other elements of $\mathcal{V}$ and we get the consecutive pyramid $\mathrm{cpyr}(\mathcal{FM}^2(\mathcal{U}), \mathcal{V})$ with dimension $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| + |\mathcal{V}| - 1$.

Therefore, the result holds. □

Finally, we study the number of k-dimensional faces of $\mathcal{FM}^2(X)$. In next result, we compute the f-vector of $\mathcal{FM}^2(X)$, (see Def. 1.48).

Theorem 5.8. *Let fs_k and fns_k be the number of simplicial and non-simplicial k-dimensional faces of $\mathcal{FM}^2(X)$, respectively. Then:*

i) The number of simplicial k-dimensional faces is given by:

$$fs_k = \begin{cases} 2^{k+1}\binom{\binom{n}{2}}{k+1} + n2^k\binom{\binom{n}{2}}{k} & \text{if } k \leq \binom{n}{2} - 1 \\ n2^{\binom{n}{2}} & \text{if } k = \binom{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

ii) If we denote $p(j) := \binom{j}{2} + j$, and by $s(k)$ the maximum value of j such that $k + 1 - p(j) \geq 0$, then the number of non-simplicial k-dimensional faces is given by:

$$\sum_{j=2}^{s(k)} 2^{k+1-p(j)} \binom{n}{j} \binom{\binom{n}{2} - \binom{j}{2}}{k+1-p(j)}.$$

Finally, $f_k = fs_k + fns_k$.

Proof. *i)* Applying Theorem 5.7 and following the same notation, we have two kinds of simplicial faces, namely the ones with $|\mathcal{U}| = 0$ and the ones where $|\mathcal{U}| = 1$.

For the first case, as the dimension is the number of vertices minus 1, we need to select $k+1$ vertices derived from $\mathcal{V}$. Note that for a chosen pair $ij \in \mathcal{V}$, either u_{ij} or either μ_{ij} are in the face. Therefore, the number of possible faces is given by

$$2^{k+1}\binom{\binom{n}{2}}{k+1}.$$

If $|\mathcal{U}| = 1$ we need to select k vertices derived from $\mathcal{V}$. As we have n possible choices for the vertex in $\mathcal{U}$, we conclude that the number of such faces is given by

$$n2^k\binom{\binom{n}{2}}{k}.$$

Thus,

$$fs_k = \begin{cases} 2^{k+1}\binom{\binom{n}{2}}{k+1} + n2^k\binom{\binom{n}{2}}{k} & \text{if } k \leq \binom{n}{2} - 1 \\ n2^{\binom{n}{2}} & \text{if } k = \binom{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

ii) In order to get a non-simplicial k-dimensional face, by Th. 5.7, we need $|\mathcal{U}| \geq 2$; now, we look for the number of possibilities of $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| + |\mathcal{V}| = k + 1$. Remark that the number of pairs suitable for $\mathcal{V}$ is at most

$$\binom{n}{2} - \binom{|\mathcal{U}|}{2}.$$

Then, the number of possible faces is the number of combinations of $\mathcal{U}$ and $\mathcal{V}$ in these conditions. In addition, consider that we must choose between u_{ij} and μ_{ij} for each pair in $\mathcal{V}$. If we denote $j := |\mathcal{U}|$, $p(j) = \binom{j}{2} + j$, and by $s(k)$ the maximum value of j such that $k + 1 - p(j) \geq 0$, then the number of non-simplicial faces is

$$\sum_{j=2}^{s(k)} \binom{n}{j} 2^{k+1-p(j)} \binom{\binom{n}{2} - \binom{j}{2}}{k+1-p(j)},$$

where we are assuming that $\binom{a}{b} = 0$ if $a < b$.

Hence, the result holds. $\square$

5.2 A random procedure for generating points in $\mathcal{FM}^2(X)$

Inspired by the adjacency structure of $\mathcal{FM}^2(X)$ obtained previously, in this section we are going to develop a procedure for generating random points uniformly distributed in $\mathcal{FM}^2(X)$. As explained in Chapter 1, generating points in a polytope is a complex problem and several methods, not completely satisfactory, have been presented to cope with this problem [17, 18]. Among them, we have the triangulation method [17]. The triangulation method takes advantage of the fact that random generation in simplices is very simple and fast [20].

The triangulation method is based on the decomposition of the polytope into simplices such that any pair of simplices intersects in a (possibly empty) common face. Once the decomposition is obtained, we assign to each simplex a probability proportional to its volume; next, these probabilities are used for selecting one of the simplices; finally, a random m-uple in the simplex is generated.

The main drawback of this method is that in general it is not easy to split a polytope into simplices. Moreover, even if we are able to decompose the polytope in a suitable way, we have to deal with the problem of determining the volume of each simplex in order to randomly select one of them. Computing the volume of a polytope is a complex problem and only partial results are known. However, in the case of simplices, the volume is given in Lemma 1.54.

The triangulation method is specially appealing for order polytopes, as it is easy to decompose the polytope in simplices having the same volume (see Theorem 1.70). However, as we have seen in Proposition 5.3, $\mathcal{FM}^2(X)$ is not an order polytope and thus, we have to look for another way to split the polytope into simplices. This is the task we achieve in this section.

To develop an algorithm to generate random points in $\mathcal{FM}^2(X)$, we will profit its combinatorial structure, and more concretely the adjacency structure developed in the previous section. We will use the fact that

$$u_i + u_j = u_{ij} + \mu_{ij}.$$

Lemma 5.9. *Given $\mu \in \mathcal{FM}^2(X)$, it is possible to write μ as a unique convex combination of vertices of $\mathcal{FM}^2(X)$ in a way such that either u_{ij} or either μ_{ij} has null coefficient, for all pairs $ij \in \binom{X}{2}$.*

Proof. Given a measure $\mu \in \mathcal{FM}^2(X)$, let us write μ as

$$\mu = \sum_{i=1}^{n} \alpha_i u_i + \sum_{ij} \alpha_{ij} u_{ij} + \sum_{ij} \beta_{ij} \mu_{ij}$$

such that

$$\sum_{i=1}^{n} \alpha_i + \sum_{ij} \alpha_{ij} + \sum_{ij} \beta_{ij} = 1.$$

Notice that this convex combination might not be unique to represent μ. Now, if say $\alpha_{ij} > \beta_{ij}$, we apply

$$\alpha_{ij} u_{ij} + \beta_{ij} \mu_{ij} = \beta_{ij} u_i + \beta_{ij} u_j + (\alpha_{ij} - \beta_{ij}) u_{ij}.$$

Similarly, if $\alpha_{ij} < \beta_{ij}$, we apply

$$\alpha_{ij} u_{ij} + \beta_{ij} \mu_{ij} = \alpha_{ij} u_i + \alpha_{ij} u_j + (\beta_{ij} - \alpha_{ij}) \mu_{ij}.$$

Hence, we have proved the first part. Assume now that two such representations are possible; then,

$$\begin{aligned}
\mu &= \sum_{i=1}^{n} \alpha_i u_i + \sum_{ij} \alpha_{ij} u_{ij} + \sum_{ij} \beta_{ij} \mu_{ij} \\
&= \sum_{i=1}^{n} \alpha'_i u_i + \sum_{ij} \alpha'_{ij} u_{ij} + \sum_{ij} \beta'_{ij} \mu_{ij}.
\end{aligned}$$

Now, for a pair ij, its Möbius transform is given by

$$m(ij) = \alpha_{ij} - \beta_{ij} = \alpha'_{ij} - \beta'_{ij}.$$

But as one α_{ij}, β_{ij} (resp. $\alpha'_{ij}, \beta'_{ij}$) vanishes by hypothesis and they are all non-negative, this implies $\alpha_{ij} = \alpha'_{ij}$, $\beta_{ij} = \beta'_{ij}$, whence the result. $\square$

The previous lemma leads to the key idea for triangulating $\mathcal{FM}^2(X)$. Let us consider $\binom{X}{2}$ and for each pair $ij \in \binom{X}{2}$, either u_{ij} or μ_{ij} is assigned. We define $\mathcal{A}^-$ as the subset of pairs of $\binom{X}{2}$ where u_{ij} is selected and $\mathcal{A}^+$ the set of pairs for which μ_{ij} is selected. There are $2^{\binom{n}{2}}$ different $\mathcal{A}^-$, so that there are $2^{\binom{n}{2}}$ different $\mathcal{A}^-, \mathcal{A}^+$. For fixed $\mathcal{A}^-, \mathcal{A}^+$, we define $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ as the convex hull of $\{u_i : i \in X\} \cup \{u_{ij} : ij \in \mathcal{A}^-\} \cup \{\mu_{ij} : ij \in \mathcal{A}^+\}$. In other words, $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ consists of all fuzzy measures μ in $\mathcal{FM}^2(X)$ such that the unique representation of μ in terms of Lemma 5.9 is such that $ij \in \mathcal{A}^-$ if u_{ij} appears in the representation and $ij \in \mathcal{A}^+$ if it is μ_{ij} who appears in the representation. Remark that $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is a polytope whose vertices are

$$\{u_i : i \in X\} \cup \{u_{ij} : ij \in \mathcal{A}^-\} \cup \{\mu_{ij} : ij \in \mathcal{A}^+\}. \tag{5.1}$$

We have $2^{\binom{n}{2}}$ different subsets $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, one for each possible $\mathcal{A}^-, \mathcal{A}^+$.

In next results we will show that it is possible to derive an appealing algorithm for random generation in $\mathcal{FM}^2(X)$ from these subsets applying triangulation methods. For this, we will prove in Theorem 5.10 that $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ for different choices of $\mathcal{A}^-, \mathcal{A}^+$ provide a triangulation of $\mathcal{FM}^2(X)$. Next, we will show in Proposition 5.11 that all of them share the same volume. Thus, in order to generate a fuzzy measure in $\mathcal{FM}^2(X)$ in a random fashion, it just suffices to select randomly one of the possible $\mathcal{A}^-, \mathcal{A}^+$ and then generate a point in the corresponding simplex $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$.

Theorem 5.10. *Let Δ be the collection of all the polytopes $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ where $\{\mathcal{A}^-, \mathcal{A}^+\}$ is any possible decomposition of $\binom{X}{2}$. Then, Δ are the top elements of a triangulation of $\mathcal{FM}^2(X)$ (see Def. 1.51).*

Proof. By Lemmma 5.9,

$$\mathcal{FM}^2(X) = \bigcup_{\mathcal{A}^-,\mathcal{A}^+} \mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X).$$

Let us show that each $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is a simplex. For this, we have to prove that the vertices of every $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ are affinely independent and this is equivalent to prove that the Möbius transform of these vertices form an affinely independent set.

As the vertices of $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ are given in Eq. (5.1), it follows that the vertices of $\mathsf{m}(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X))$ are

$$\{m_i : i = 1, ..., n\} \cup \{m_{ij} : ij \in \mathcal{A}^-\} \cup \{\overline{m}_{ij} : ij \in \mathcal{A}^+\}.$$

Let us rename these vertices as $\{v_0, ..., v_s\}$ and let us assume that $v_0 = m_1$. We will show that no $v_i - v_0$ can be written as a linear combination of the other $v_j - v_0,\ j \neq i$. We consider two cases:

- Consider a pair $ij \in \mathcal{A}^-$ and suppose we can write $m_{ij} - v_0$ as a linear combination of the other $v_j - v_0$ (the case $ij \in \mathcal{A}^+$ is completely symmetric). Without loss of generality, let us denote $v_1 = m_{ij}$. Remark that by construction $m_{ij}(ij) \neq 0 = v_k(ij),\ \forall k \neq 1$ as $\overline{m}_{ij}$ is not a vertex of $\mathsf{m}(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X))$. Thus, $(v_1 - v_0)(ij) \neq 0 = (v_k - v_0)(ij),\ \forall k > 1$, and hence we conclude that $v_1 - v_0$ cannot be written as a linear combination of the vectors $v_k - v_0,\ k > 1$.

- Let us now consider $m_i, i \neq 1$, and let us assume without loss of generality $v_1 = m_i$. Suppose
$$v_1 - v_0 = \sum_{i=2}^{s} \alpha_i (v_i - v_0),$$
for some α_i not all of them null. Besides, as $m_j(i) = 0,\ \forall j \neq i$, it follows that there exists, say v_2, corresponding to $\overline{m}_{ij}$ and such that $\alpha_2 \neq 0$. Then, we can rewrite the previous expression, thus obtaining
$$v_2 - v_0 = \sum_{k=3}^{s} \frac{-\alpha_k}{\alpha_2}(v_k - v_0) + \frac{1}{\alpha_2}(v_1 - v_0).$$
But this a contradiction with the previous case.

Thus, we have shown that the vertices of $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ are affinely independent and then they form a simplex.

Besides, as the number of vertices in $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is $\binom{n}{2}+n$, it follows that its dimension is $\binom{n}{2} + n - 1$. As this is the dimension of $\mathcal{FM}^2(X)$, we conclude that $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is a full-dimensional simplex in $\mathcal{FM}^2(X)$.

It just rests to show that the intersection of two of these simplices is a (possibly empty) common face. Consider $\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)$ and $\mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$ and suppose that they have non-empty intersection. Let us denote by $\mathcal{CV}$ the common vertices of these two simplices, i.e.,

$$\{u_i : i \in X\} \cup \{u_{ij} : ij \in \mathcal{A}_1^- \cap \mathcal{A}_2^-\} \cup \{\mu_{ij} : ij \in \mathcal{A}_1^+ \cap \mathcal{A}_2^+\}.$$

By Lemma 5.9, any fuzzy measure in $\mathcal{FM}^2(X)$ can be written as a unique convex combination such that either the coefficient of u_{ij} or the coefficient of μ_{ij} vanishes. Then,

$$\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X) \cap \mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X) = \mathrm{Conv}(\mathcal{CV}).$$

It follows that $\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X) \cap \mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$ is a simplicial face of $\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)$ and $\mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$.

$\square$

Next step is to prove that all the simplices obtained with the triangulation developed above share the same volume.

Proposition 5.11. *All simplices $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ have the same $\left(\binom{n}{2}+n-1\right)$-dimensional volume.*

Proof. By Lemma 1.54, the volume of a simplex only depends on the distances between each pair of vertices. Now consider two decompositions $\{\mathcal{A}_1^-,\mathcal{A}_1^+\}, \{\mathcal{A}_2^-,\mathcal{A}_2^+\}$ that differ just in one vertex. Without loss of generality, we can suppose that this vertex is $u_{ij} \in \mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)$, and $\mu_{ij} \in \mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$. We are going to show that the simplices associated to these decompositions share the same volume. Observe that since $u_i + u_j = u_{ij} + \mu_{ij}$ this implies that

$$d^2(u_i, v) + d^2(u_j, v) = d^2(u_{ij}, v) + d^2(\mu_{ij}, v)$$

for any vertex v because we are dealing with $0,1$-valued vectors.

With this result, we can derive the Cayley-Menger matrix (see Lemma 1.54) associated to $\mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$ from the one associated to $\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)$. These matrices differ in row with distances $d^2(u_{ij}, v)$ and row $d^2(\mu_{ij}, v) = d^2(u_i, v) + d^2(u_j, v) - d^2(u_{ij}, v)$ and also in columns $d^2(v, u_{ij})$ and $d^2(v, \mu_{ij}) = d^2(v, u_i) + d^2(v, u_j) - d^2(v, u_{ij})$. Since the determinant is invariant under this operation, both matrices have the same determinant and hence, the simplices share the same volume.

Finally, we can repeat this argument changing the desired vertices to prove that all simplices have the same volume. $\square$

As a consequence, we can apply the triangulation method as follows: As all subsets $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ have the same volume, we need to select one of these subsets at random. For this, it suffices to choose at random for each pair ij if it is included in $\mathcal{A}^-$ or $\mathcal{A}^+$.

Next step is to generate a point in the selected $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$. Generation in simplices is easy as we saw in Subsection 1.2.1. For the sake of completeness, we explain the procedure in the following.

Let us reuse the notation $p(n) = \binom{n}{2} + n$. Since the dimension of each simplex is $p(n) - 1$ we are going to use just the first $p(n) - 1$ coordinates. In other words, we work with the projection of $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ into the subspace consisting in intersecting the hyperplanes $X_A = 0$ for $|A| > 2$ and $X_{(n-1)n} = 0$. We call this projection $\pi : \mathbb{R}^{2^n} \to \mathbb{R}^{p(n)-1}$. Let us see each step in more detail.

1) Sample a uniformly distributed random point in

$$\mathcal{H}_n = \{\boldsymbol{U} \in [0,1]^{p(n)-1} \ : U_1 \geq U_2 \geq \cdots \geq U_{p(n)-1}\}.$$

For this, generate an independent and identically distributed sample $\widehat{U}_1 \ldots, \widehat{U}_{p(n)-1}$ with distribution $U(0,1)$. Then sort the $\widehat{U}_i$ to give the order statistics with the reverse order $U_1 \geq U_2 \geq \ldots \geq U_{p(n)-1}$. This generates a uniformly distributed vector U in $\mathcal{H}_n$. Note that the vertices of $\mathcal{H}_n$ are $(0,0,\ldots,0,0), (1,0,\ldots,0,0), (1,1,\ldots,0,0)$, $\ldots, (1,1,\ldots,1,0)$ and $(1,1,\ldots,1,1)$.

2) Apply the affine transformation $\boldsymbol{X} = \boldsymbol{A} \cdot \boldsymbol{U} + \boldsymbol{V}_0$ which maps $\mathcal{H}_n$ into the desired simplex $\pi\left(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)\right)$ associated to the partition $\mathcal{A}^-, \mathcal{A}^+$. Note that if $h(\boldsymbol{u})$ is the density function of $\boldsymbol{U}$, then the density $g(\boldsymbol{x})$ of $\boldsymbol{X}$ would be

$$g(\boldsymbol{x}) = h(\boldsymbol{u}) |\det(\boldsymbol{A})|^{-1}.$$

Consequently, if $h(\boldsymbol{u})$ is uniform in $\mathcal{H}_n$, then $g(\boldsymbol{x})$ is uniform in $\pi\left(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)\right)$, because $|\det(\boldsymbol{A})|$ is a constant value.

3) Finally, observe that our simplex $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is a $(p(n)-1)$-dimensional simplex in $\mathbb{R}^{2^n}$. We should recover the rest of coordinates to get the final vector $\boldsymbol{X}^* \in \mathbb{R}^{2^n}$. Obviously $\boldsymbol{X}^*(A) = \boldsymbol{X}(A)$ for the first $p(n) - 1$ coordinates. Applying the Zeta transform (see Section 2.5) and replacing the values $m(i), m(ij)$ by the corresponding expressions in terms of $\mu(i), \mu(ij)$ according to Definition 2.29 we recover the rest of coordinates, it follows

$$\mu(A) = \sum_{ij \in A} \mu(ij) - (|A| - 2) \sum_{i \in A} \mu(i), \ \forall |A| > 2, \tag{5.2}$$

Finally, as the sum of Möbius coefficients is 1 (see Proposition 2.30),

$$\mu(n-1,n) = 1 - \sum_{\substack{ij \in \binom{X}{2} \\ ij \neq (n-1,n)}} \mu(ij) + (n-2)\sum_{i=1}^{n} \mu(i). \tag{5.3}$$

By using Eqs. (5.2) and (5.3), we recover the rest of coordinates. The last map is again an affine transformation respecting the uniformity.

It just remains to give a full description of the affine transformation $\boldsymbol{X} = \boldsymbol{A} \cdot \boldsymbol{U} + \boldsymbol{V}_0$. This affine transformation maps the vertices of $\mathcal{H}_n$ into the vertices of $\pi\left(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)\right)$. Recall that the vertices of $\pi\left(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)\right)$ are the first $p(n)-1$ coordinates of u_i and v_{ij} where $v_{ij} = u_{ij}$ if $ij \in \mathcal{A}^-$ and $v_{ij} = \mu_{ij}$ if $ij \in \mathcal{A}^+$. We denote the restriction of a vector v to the first $p(n)-1$ coordinates with an overline, $\overline{v}$. Indeed, we are going to denote the vertices by $v_1, v_2, \ldots, v_{p(n)}$ in the natural order, that is $u_1, u_2, \ldots, v_{12}, v_{13}, \ldots, v_{(n-1)n}$. We also identify $\boldsymbol{V}_0 = \overline{u}_1$. Now consider the matrix having as i-th column the vector $v_{i+1} - v_i$, that is, we define matrix $\boldsymbol{A}$ by

$$\left[\overline{u}_2 - \overline{u}_1, \overline{u}_3 - \overline{u}_2, \cdots, \overline{v}_{12} - \overline{u}_n, \cdots, \overline{v}_{(n-1)n} - \overline{v}_{(n-2)n}\right].$$

It is easy to see that with these choices $\boldsymbol{X} = \boldsymbol{A} \cdot \boldsymbol{U} + \boldsymbol{V}_0$. Therefore,

$$\boldsymbol{X}_k = \overline{u}_{1k} + \sum_{j=2}^{p(n)} (\overline{v}_{jk} - \overline{v}_{(j-1)k}) \boldsymbol{U}_{j-1}$$

where $\overline{v}_{jk}$ is the k-th position of the vertex $\overline{v}_j$. Thus, we are in conditions to present our algorithm for generating random points in $\mathcal{FM}^2(X)$, see Algorithm 6.

Algorithm 6 SAMPLING IN $\mathcal{FM}^2(X)$

Step 1: Choose randomly between u_{ij} and μ_{ij} for any pair of elements $ij \in \binom{X}{2}$ to get the partition $\mathcal{A}^-, \mathcal{A}^+$.
Step 2: Generate an iid sample $\widehat{U}_1 \ldots, \widehat{U}_{p(n)-1}$ with distribution $U(0,1)$. Then sort the $\widehat{U}_i$ with the reverse order to get U, s.t. $U_1 \geq U_2 \geq \ldots \geq U_{p(n)-1}$.
Step 3: Apply the linear map

$$\boldsymbol{X}_k = \overline{u}_{1k} + \sum_{j=2}^{p(n)} (\overline{v}_{jk} - \overline{v}_{(j-1)k}) \boldsymbol{U}_{j-1}, \forall k \in \{1, \ldots, p(n)-1\}.$$

Step 4: Find the other coordinates using Eqs. (5.2) and (5.3).

If we work just with the first $p(n)-1$ coordinates the last algorithm has quartic complexity as we show in the next result.

Proposition 5.12. *The computational complexity of the sampling algorithm for 2-additive measures is $O(n^4)$.*

Proof. We compute the complexity of each part:

1. We should choose if each pair ij is associated to u_{ij} or μ_{ij}. Then, we compute a vector of $\binom{n}{2}$ values 0 and 1. The complexity is $O(\binom{n}{2}) = O(n^2)$.
2. We generate an iid sample $\widehat{U}_1 \dots, \widehat{U}_{p(n)-1}$ with distribution $U(0,1)$. Then sort the U_i with the reverse order $U_1 \geq U_2 \geq \dots \geq U_{p(n)-1}$. Since the complexity is linear for sampling and for sorting this step needs $O(\binom{n}{2} + n) = O(n^2)$ computations.
3. Apply the transformation
$$\boldsymbol{X}_k = \overline{u}_{1k} + \sum_{j=2}^{p(n)} (\overline{v}_{jk} - \overline{v}_{(j-1)k})\boldsymbol{U}_{j-1}.$$
In this step, we multiply a $(p(n)-1) \times (p(n)-1)$ matrix by a vector. Then, the complexity is $O\left(\left(\binom{n}{2} + n\right)^2\right) = O(n^4)$.
4. The last step is not be necessary because we are working just with the first $p(n)-1$ coordinates.

Therefore, the complexity is $O(n^4)$. □

Obviously, if we work with all the $2^n - 2$ coordinates, the complexity increases to $O(2^n)$, because we need to recover the value for each subset of X.

We finish this section with two results that can be derived from the proposed triangulation. The first one refers to the volume of $\mathcal{FM}^2(X)$. For this, we can apply Lemma 1.54 to obtain the volume of one of the simplices, and multiply this value by $2^{\binom{n}{2}}$ to obtain $Vol(\mathcal{FM}^2(X))$.

Corollary 5.13.
$$Vol(\mathcal{FM}^2(X)) = 2^{\frac{(n-1)(n-2)}{4}} \frac{\sqrt{|\det(CM_\Delta)|}}{[\binom{n}{2} + n - 1]!},$$
where Δ is any simplex of the triangulation.

The volumes for the first values of n are given in next table.

$\lvert X \rvert$	2	3	4	5
$Vol(\mathcal{FM}^2(X))$	1	0.1632	0.0298	0.0001

Another consequence of the triangulation proposed in this section is that it allows a simple way of computing the **center of gravity** or **centroid** of $\mathcal{FM}^2(X)$, i.e. the mean position of all the points in all of the coordinate directions. From a mathematical point of view, the centroid of $\mathcal{P}$ is given by

$$C = \frac{\int x I_{\mathcal{P}}(x)dx}{\int I_{\mathcal{P}}(x)dx},$$

where $I_{\mathcal{P}}(x)$ is the characteristic function of $\mathcal{P}$. At this point, note that computing the center of gravity of a polytope is a difficult problem and usually, complicated methods and formulas are given. Only for special cases, the center of gravity has been obtained. One of this cases is the case of simplices, for which the following can be shown.

Lemma 5.14. *[91] Consider an n-dimensional simplex whose vertices are $v_0, \dots, v_n$; then, considering the vertices as vectors, the centroid of the simplex is*

$$C = \frac{1}{n+1}\sum_{i=0}^{n} v_i.$$

Now, for a given polytope and a decomposition, the following can be shown.

Lemma 5.15. *Let $\mathcal{P}$ be a polytope and $\{\mathcal{P}_1, ..., \mathcal{P}_r\}$ a partition of $\mathcal{P}$. Suppose that the centroid of $\mathcal{P}_i$ is C_i, $i = 1, .., r$ and let us denote $Vol(\mathcal{P}_i) = V_i$. Then, the centroid of $\mathcal{P}$ is given by*

$$C = \frac{\sum_{i=1}^r C_i V_i}{\sum_{i=1}^r V_i}.$$

Proof. We have

$$C = \frac{\int x I_{\mathcal{P}}(x)dx}{\int I_{\mathcal{P}}(x)dx} = \frac{\sum_{i=1}^r \int x I_{\mathcal{P}_i}(x)dx}{\sum_{i=1}^r \int I_{\mathcal{P}_i}(x)dx} = \frac{\sum_{i=1}^r V_i \frac{\int x I_{\mathcal{P}_i}(x)dx}{V_i}}{\sum_{i=1}^r V_i} = \frac{\sum_{i=1}^r C_i V_i}{\sum_{i=1}^r V_i}.$$

Thus, the result holds. □

Applying the triangulation proposed in this section, the following can be shown.

Proposition 5.16. *The centroid of $\mathcal{FM}^2(X)$ is given by $\overline{\mu}$*

$$\overline{\mu}(B) = \frac{|B|}{n}.$$

Proof. Consider one of the simplices of the triangulation proposed in this section; this simplex is defined via the sets $\mathcal{A}^-, \mathcal{A}^+$; besides, we have shown that all these simplices have the same volume. As we have $2^{\binom{n}{2}}$ subsets $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, by Lemma 5.15,

$$\overline{\mu} = \frac{1}{2^{\binom{n}{2}}} \sum_{\mathcal{A}^-,\mathcal{A}^+} \overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+},$$

where $\overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+}$ is the centroid of $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$. On the other hand, for a given simplex $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, we know by Lemma 5.14 that

$$\overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+} = \frac{1}{\binom{n}{2}+n} \left[\sum_{i=1}^{n} u_i + \sum_{ij \in \mathcal{A}^-} u_{ij} + \sum_{ij \in \mathcal{A}^+} \mu_{ij} \right].$$

Next, for a given $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, let us consider $\mathcal{FM}^2_{\mathcal{A}^+,\mathcal{A}^-}(X)$, i.e. the simplex such that $u_{ij} \in \mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X) \Leftrightarrow \mu_{ij} \in \mathcal{FM}^2_{\mathcal{A}^+,\mathcal{A}^-}(X)$ and $\mu_{ij} \in \mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X) \Leftrightarrow u_{ij} \in \mathcal{FM}^2_{\mathcal{A}^+,\mathcal{A}^-}(X)$. Then, the sum of the center of gravity of these two simplices is

$$\mu_{aux} = \frac{1}{\binom{n}{2}+n} \left[2\sum_{i=1}^{n} u_i + \sum_{ij} u_{ij} + \sum_{ij} \mu_{ij} \right] = \frac{1}{\binom{n}{2}+n} \left[(n+1)\sum_{i=1}^{n} u_i \right] = 2\frac{\sum_{i=1}^{n} u_i}{n}.$$

Then,

$$\overline{\mu} = \frac{1}{2^{\binom{n}{2}}} \sum_{\mathcal{A}^-,\mathcal{A}^+} \overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+} = \frac{2^{\binom{n}{2}-1}}{2^{\binom{n}{2}}} \mu_{aux} = \frac{\sum_{i=1}^{n} u_i}{n},$$

and thus, $\overline{\mu}(B) = \frac{|B|}{n}$. □

5.3 The adjacency graph of $\mathcal{FM}(X)$

We finish this chapter presenting some properties of the adjacency graph of this polytope. Given a polytope $\mathcal{P}$, we define its **associated graph** $G(\mathcal{P})$ (also called adjacency graph or 1-skeleton) as the graph whose vertices are the vertices of $\mathcal{P}$ and two nodes are adjacent if the corresponding vertices are adjacent in $\mathcal{P}$.

For example, any n-dimensional simplex has the complete graph as associated graph, because all vertices are adjacent to each other. In Figure 5.1 we can compare the graph of $\mathcal{FM}^2(X)$ with the graph of a (n^2-1)-dimensional simplex. We observe that when the size of X grows, $G(\mathcal{FM}^2(X))$ tends to be very similar to the complete graph.

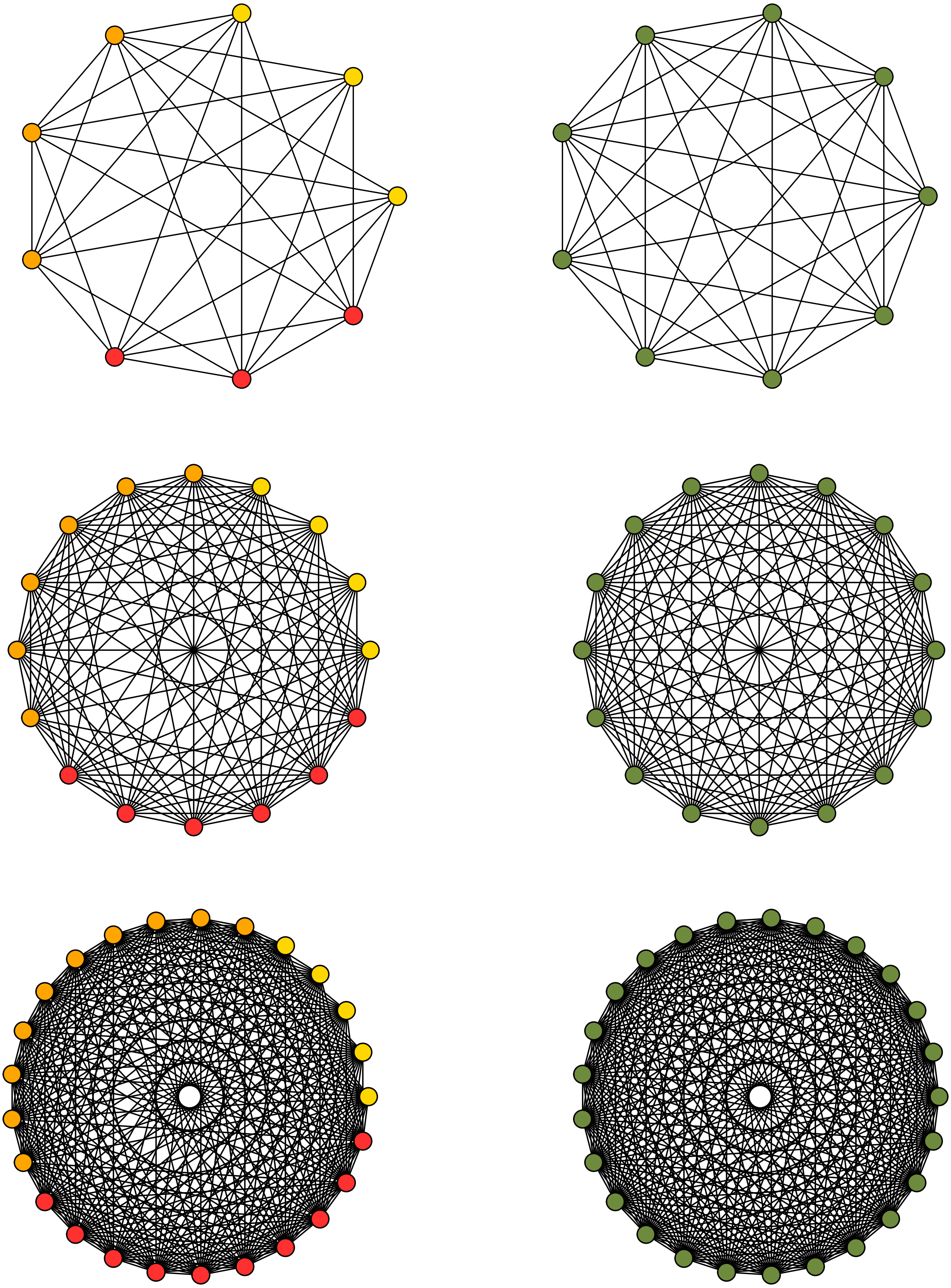


FIGURE 5.1: Adjacency graphs of $\mathcal{FM}^2(X)$, with $|X| = 3, 4, 5$ (left) and simplices (right). In these figures, the yellow vertices are u_i, the orange ones u_{ij}, and the red ones μ_{ij}.

The **distance** between two vertices of a polytope $\mathcal{P}$ is defined as the shortest path connecting the corresponding nodes in $G(\mathcal{P})$. The **diameter** of a polytope, $\mathrm{diam}(\mathcal{P})$, is defined as the longest distance between any pair of vertices.

A straight consequence of Theorem 5.5 is the following.

Corollary 5.17. *The diameter of $\mathcal{FM}^2(X)$ is 2.*

Proof. By Corollary 5.6, the distance between two vertices is 1 except for u_i, u_j and u_{ij}, μ_{ij}. But in these cases, we have the paths $u_i - \mu_{ij} - u_j$ and $u_{ij} - u_i - \mu_{ij}$. $\square$

One important feature of a graph is the chromatic number [92]. The **chromatic number**, $\chi(G)$, of a graph G is the smallest number of colors needed to color the vertices of G so that no adjacent vertices share the same color. Figure 5.2 shows a graph coloring for $|X| = 4$.

Theorem 5.18. *Let $|X| = n$. Then $\chi\left(G(\mathcal{FM}^2(X))\right) = \binom{n}{2} + 1$.*

Proof. Start observing that all the vertices u_{ij} are adjacent to each other, so that we need $\binom{n}{2}$ colors. Since μ_{ij} is not adjacent to u_{ij} we can use for μ_{ij} the same color as u_{ij}. This way we can color all the vertices μ_{ij} using the same $\binom{n}{2}$ colors. Finally, as the u_i vertices are not related to each other but are related with the rest of vertices we need one last color. $\square$

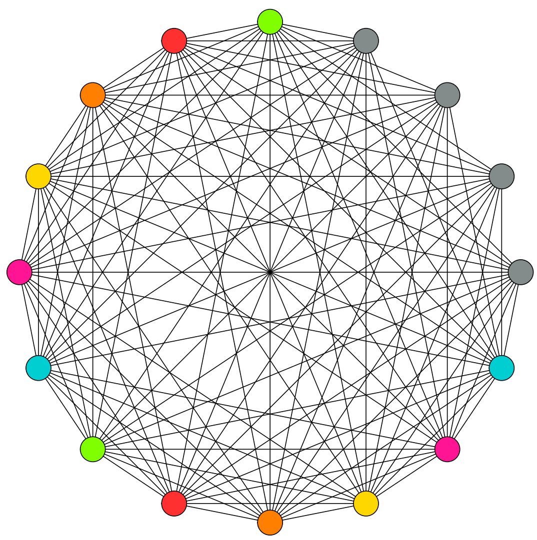


FIGURE 5.2: $\mathcal{FM}^2(X), |X| = 4$ graph coloring.

Let us analyze the Hamiltonicity of these graphs. Recall that a **Hamiltonian path** is a path that visits each vertex exactly once. Also, a graph is **Hamilton connected** if there exists a Hamiltonian path between each pair of vertices.

To see that $G(\mathcal{FM}^2(X))$ is Hamilton connected we need some previous results.

Lemma 5.19. *$\mathcal{FM}^2(X)$ is a combinatorial polytope.*

Proof. By Proposition 5.1 all the vertices are $\{0,1\}$-valued. Moreover, by Corollary 5.6, the only pairs that are not adjacent are (u_i, u_j) and (u_{ij}, μ_{ij}), and they satisfy $u_i + u_j = u_{ij} + \mu_{ij}$. $\square$

For combinatorial polytopes we can use the following result.

Proposition 5.20. *[14] Let $\mathcal{P}$ be a combinatorial polytope. Then, $G(\mathcal{P})$ is either Hamilton connected or the graph of a hypercube.*

Theorem 5.21. *Let $|X| > 2$. Then $G(\mathcal{FM}^2(X))$ is Hamilton connected.*

Proof. It suffices to note that a hypercube has no complete subgraphs, i.e. it has no simplicial faces of dimension greater than 1. As $G(\mathcal{FM}^2(X))$ has such faces if $|X| > 2$ by Theorem 5.7, the result follows. $\square$

Finally, let us study the planarity of this graph. A **planar graph** is a graph that can be drawn on the plane in such a way that its edges do not cross each other. By the four color theorem, every planar graph should have a chromatic number lower or equal than 4, i.e. $\chi(G) \leq 4$.

The **complete bipartite graph** $K_{n,m}$ has $n+m$ vertices and edges joining every vertex of the first n vertices to every vertex of the last m vertices.

A **minor** of a graph is a subgraph which can be obtained by deleting edges and vertices and by contracting edges. Edge contraction removes an edge from the graph while simultaneously merging the endpoints.

Theorem 5.22. *[92]* ***Wagner's theorem.*** *Let G be a finite graph. Then G is planar if and only if its minors include neither the complete graph of five elements K_5 nor the complete bipartite graph $K_{3,3}$.*

Theorem 5.23. *Let $|X| > 2$. Then $G(\mathcal{FM}^2(X))$ is not planar.*

Proof. If $|X| > 2$, we consider the minor formed by deleting every vertex but u_1, u_2, u_3, u_{12}, u_{13}, u_{23} and deleting the edges between u_{ij} and u_{ik}. Hence, we obtain the complete bipartite graph $K_{3,3}$ and therefore $G(\mathcal{FM}^2(X))$ is not planar. $\square$

Chapter 6

Order Cones

In this chapter we introduce the concept of order cone. This concept is inspired by the concept of order polytopes. Similarly to order polytopes, order cones are a special type of polyhedral cones whose geometrical structure depends on the properties of a poset. This allows to study these properties in terms of the subjacent poset, a problem that is usually simpler to solve. From the point of view of applicability, it can be seen that many cones appearing in the literature of monotone TU-games are order cones. Especially, it can be seen that the cones of monotone games with restricted cooperation are order cones, no matter the structure of the set of feasible coalitions.

As we saw in Section 2.6.1, it could be the case that some coalitions fail to form. Thus, v cannot be defined on some of the elements of $\mathcal{P}(X)$ and we have a subset Ω of $\mathcal{P}(X)$ containing all *feasible coalitions*. By a similar argument, coalitions with a fixed value may be left outside Ω. In this chapter, we will not include $\emptyset$ in Ω. Usually, Ω has a concrete structure.

As it will become clear below, order cones are a class of cones including the cones of monotone games with restricted cooperation, no matter which the set Ω is. Thus, order cones allow to study this set of cones in a general way. For example, we will characterize the set of extremal rays of the cone $\mathcal{MG}(X)$, a problem that to our knowledge has not been solved yet [29].

Interestingly enough, order cones can be applied to other situations different to monotone games with restricted cooperation. As an example dealing with such a case, we study the cone of monotone k-symmetric games. This also adds more insight about the relationship between order cones and order polytopes.

The rest of the chapter goes as follows: In next section we define order cones and study some of its geometrical properties. We then apply these results for some special cases of monotone games with restricted cooperation.

6.1 Order Cones

Let us now turn to the concept of order cones. The idea is to remove the condition $f(a) \leq 1$ from Definition 1.59. Thus, the resulting set is no longer bounded. This is what we will call an order cone. Formally,

Definition 6.1. Let P be a finite poset with n elements. The **order cone** $\mathcal{C}(P)$ is formed by the n-tuples f of real numbers indexed by the elements of P satisfying

i) $0 \leq f(x)$ for every $x \in P$,

ii) $f(x) \leq f(y)$ whenever $x \preceq y$ in P.

For example, we will see in Section 6.2.1 that the set of monotone games $\mathcal{MG}(X)$ as a subset of $\mathbb{R}^{2^n-1}$ is an order cone with respect to the poset $P = \mathcal{P}(X)\backslash\{\emptyset\}$ with the partial order given by $A \prec B \Leftrightarrow A \subset B$. Another example is given at the end of the section.

The name order cone is consistent, as next lemma shows.

Lemma 6.2. *Given a finite poset P, then $\mathcal{C}(P)$ is a pointed polyhedral cone.*

Proof. It is a straightforward consequence of the definition that $\mathcal{C}(P)$ is a polyhedron. Let us then show that it is indeed a cone. For this, take $f \in \mathcal{C}(P)$ and consider $\alpha f, \alpha \geq 0$. For $x \preceq y$ in P, we have $f(x) \leq f(y)$ and thus, $\alpha f(x) \leq \alpha f(y)$. Hence $\alpha f \in \mathcal{C}(P)$ and the result holds.

Moreover, as $f(x) \geq 0, \forall x \in P, f \in \mathcal{C}(P)$, it follows that $\mathcal{C}(P) \cap -(\mathcal{C}(P)) = \{\mathbf{0}\}$, and by Theorem 1.57, $\mathcal{C}(P)$ is a pointed cone. □

Consequently, $\mathcal{C}(P)$ has just one vertex, $\mathbf{0}$.

Definition 6.1 suggests a strong relationship between order polytopes and order cones. The following results study some straightforward aspects of this relation.

Lemma 6.3. *Let P be a finite poset. Then, $\mathcal{C}(P)$ is the conical extension of $\mathcal{O}(P)$.*

Proof. If $f \in \mathcal{O}(P)$, it follows that for $x, y \in P, x \prec y$, it is $0 \leq f(x) \leq f(y)$. Thus, $f \in \mathcal{C}(P)$.

On the other hand, consider a cone $\mathcal{C}$ such that $\mathcal{O}(P) \subset \mathcal{C}$. For $f \in \mathcal{C}(P)$, and $\alpha > 0$ small enough, we have $\alpha f \in \mathcal{O}(P) \subset \mathcal{C}$. Then, $\frac{1}{\alpha}\alpha f = f \in \mathcal{C}$, and hence $\mathcal{C}(P) \subseteq \mathcal{C}$. $\square$

Indeed, the following holds:

Lemma 6.4. *Consider a finite poset P. Then,*

$$\mathcal{C}(P) \cap \{\boldsymbol{x} : \boldsymbol{x} \leq \mathbf{1}\} = \mathcal{O}(P).$$

Proof. $\subseteq$) Consider $f \in \mathcal{C}(P) \cap \{\boldsymbol{x} : \boldsymbol{x} \leq \mathbf{1}\}$. Hence, $f(x) \leq 1, \forall x \in P$, and if $x \preceq y$, then $0 \leq f(x) \leq f(y) \leq 1$. Thereofore, $f \in \mathcal{O}(P)$.

$\supseteq$) For $f \in \mathcal{O}(P)$, we have $f \in \mathcal{C}(P)$ by Lemma 6.3 and $f(x) \leq 1, \forall x \in P$. $\square$

As $\mathcal{C}(P)$ is a polyhedral cone by Lemma 6.2 and according to Corollary 1.56, this cone can be given in terms of its corresponding extremal rays. Next theorem characterizes the set of extremal rays of $\mathcal{C}(P)$ in terms of filters of P.

Theorem 6.5. *Let P be a finite poset and $\mathcal{C}(P)$ its associated order cone. Then, its extremal rays are given by*

$$\{\alpha \cdot \boldsymbol{v}_F : \alpha \in \mathbb{R}^+\},$$

where $\boldsymbol{v}_F$ is the characteristic function of a non-empty connected filter F of P.

Proof. We know that extremal rays of a pointed cone are rays passing through $\mathbf{0}$. Let us show that extremal rays of $\mathcal{C}(P)$ are related to vertices of $\mathcal{O}(P)$ adjacent to $\mathbf{0}$. Consider an extremal ray, that is given by a vector $\boldsymbol{v}$. We can assume that $\boldsymbol{v}$ is such that $\boldsymbol{v} \leq \mathbf{1}$ and there exists a coordinate i such that $v_i = 1$. Hence, by Lemma 6.4, $\boldsymbol{v} \in \mathcal{O}(P)$. Let us show that $\boldsymbol{v}$ is indeed a vertex of $\mathcal{O}(P)$. If not, there exist two different points $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{O}(P)$ such that

$$\boldsymbol{v} = \alpha \boldsymbol{w}_1 + (1-\alpha)\boldsymbol{w}_2, \quad \alpha \in (0,1).$$

Besides, $\alpha \boldsymbol{w}_1, (1-\alpha)\boldsymbol{w}_2 \in \mathcal{C}(P)$. Remark that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are linearly independent because there exists a coordinate i such that $v_i = 1$. Consequently, $\boldsymbol{v}$ does not define an extremal ray, a contradiction.

Next, let us now show that $\boldsymbol{v}$ is adjacent to $\mathbf{0}$. Otherwise, the segment $[\mathbf{0}, \boldsymbol{v}]$ is not an edge of $\mathcal{O}(P)$. Consequently, $\frac{1}{2}\boldsymbol{v}$ can be written as

$$\frac{1}{2}\boldsymbol{v} = \alpha \boldsymbol{y}_1 + (1-\alpha)\boldsymbol{y}_2,$$

where $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{O}(P)$ such that they are outside $[\mathbf{0}, \boldsymbol{v}]$. Thus,

$$\boldsymbol{v} = 2\alpha \boldsymbol{y}_1 + 2(1-\alpha)\boldsymbol{y}_2.$$

Finally, $2\alpha\boldsymbol{y}_1, 2(1-\alpha)\boldsymbol{y}_2 \in \mathcal{C}(P)$, so we conclude that $\boldsymbol{v}$ does not define an extremal ray, which is a contradiction.

Now, $\boldsymbol{v}$ is related to a filter $F \subseteq P$. On the other hand, $\mathbf{0}$ is related to the empty filter. As $\boldsymbol{v}$ is adjacent to $\mathbf{0}$, we can apply Corollary 1.67 to conclude that $F = F\Delta\emptyset$ is a connected filter of P.

Let us now prove the reverse. Consider $\boldsymbol{v}$ an adjacent vertex to $\mathbf{0}$ in $\mathcal{O}(P)$ and assume that $\boldsymbol{v}$ does not define an extremal ray. Then, there exists $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{C}(P)$ and not proportional to $\boldsymbol{v}$ such that

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2 = \frac{1}{2}2\boldsymbol{w}_1 + \frac{1}{2}2\boldsymbol{w}_2 = \frac{1}{2}\boldsymbol{w}_1' + \frac{1}{2}\boldsymbol{w}_2'.$$

Now, for $\epsilon > 0$ small enough, we have

$$\epsilon\boldsymbol{v} = \frac{1}{2}\epsilon\boldsymbol{w}_1' + \frac{1}{2}\epsilon\boldsymbol{w}_2',$$

and $\epsilon\boldsymbol{w}_1' \leq \mathbf{1}, \epsilon\boldsymbol{w}_2' \leq \mathbf{1}$. Hence, $\epsilon\boldsymbol{w}_1', \epsilon\boldsymbol{w}_2' \in \mathcal{O}(P)$ by Lemma 6.4, and hence $[\mathbf{0}, \boldsymbol{v}]$ is not an edge of $\mathcal{O}(P)$, in contradiction with $\boldsymbol{v}$ adjacent to $\mathbf{0}$. $\square$

Let us now turn to the problem of obtaining the faces of $\mathcal{C}(P)$. As explained in Theorem 1.58, faces arise when inequalities turn into equalities. Let us consider the inequality $f(x) \leq f(y)$ for $x \lessdot y$ and assume this inequality is turned into an equality. This means that x and y identified to each other; let us call z this new element. In terms of posets, this translate into transforming P into another poset $(P', \preceq')$ defined as $P' := P\backslash\{x, y\} \cup \{z\}$ and $\preceq'$ given by:

$$\begin{cases} a \preceq' b \Leftrightarrow a \preceq b & \text{if } a, b \neq z \\ z \preceq' b \Leftrightarrow x \preceq b \\ a \preceq' z \Leftrightarrow a \preceq y \end{cases}$$

Similar conclusions arise when $0 \leq f(x)$ turns into an equality. Moreover, if $\mathcal{F}$ is the face obtained by turning inequalities into equalities, the projection

$$\begin{array}{cccc} \pi : & \mathcal{F} & \to & \mathcal{C}(P') \\ & (f(a), ..., f(x), f(y), ..., f(b)) & \hookrightarrow & (f(a), ..., f(x), f(y), ..., f(b)) \end{array}$$

is a bijective affine map. Consequently, the following holds.

Lemma 6.6. *The faces of an order cone are affinely isomorphic to order cones.*

Compare this result with the corresponding result for order polytopes [22].

Lemma 6.7. *For an order cone $\mathcal{C}(P)$, the vertex $\mathbf{0}$ is in all non-empty faces. Consequently, all faces can be written as $\mathcal{F}_{\boldsymbol{v},0}$.*

Proof. It suffices to show that for a non-empty face $\mathcal{F}_{\boldsymbol{v},c}$, it is $c = 0$. First, $\boldsymbol{v}^t\mathbf{0} \leq c$, so that $c \geq 0$.

Suppose $c > 0$. As $\mathcal{F}_{\boldsymbol{v},c}$ is non-empty, there exist $\boldsymbol{x} \in \mathcal{C}(P)$ such that $\boldsymbol{v}^t\boldsymbol{x} = c$. But then, $\boldsymbol{v}^t 2\boldsymbol{x} = 2c > c$, a contradiction. Thus, $c = 0$ and $\mathbf{0} \in \mathcal{F}_{\boldsymbol{v},0}$. □

With this in mind, Theorem 6.5 can be extended to characterize all the faces of the order cone, not only the extremal rays.

Theorem 6.8. *Let P be a finite poset and $\mathcal{C}(P)$ and $\mathcal{O}(P)$ the corresponding order cone and order polytope, respectively. For a pair $(\boldsymbol{v}, 0)$, the set $\mathcal{F}'_{\boldsymbol{v},0} = \mathcal{C}(P) \cap \{\boldsymbol{x} : \boldsymbol{v}^t\boldsymbol{x} = 0\}$ is a face of $\mathcal{C}(P)$ if and only if $\mathcal{F}_{\boldsymbol{v},0} = \mathcal{O}(P) \cap \{\boldsymbol{x} : \boldsymbol{v}^t\boldsymbol{x} = 0\}$ is a face of $\mathcal{O}(P)$. Moreover, $dim(\mathcal{F}'_{\boldsymbol{v},0}) = dim(\mathcal{F}_{\boldsymbol{v},0})$.*

Proof. Let $\mathcal{F}_{\boldsymbol{v},0}$ be a face of $\mathcal{O}(P)$ containing $\mathbf{0}$ and let us show that it determines a face on $\mathcal{C}(P)$. First, let us show that $\boldsymbol{v}^t\boldsymbol{x} \leq 0, \forall \boldsymbol{x} \in \mathcal{C}(P)$. Otherwise, there exists $\boldsymbol{x}_0 \in \mathcal{C}(P)$ such that $\boldsymbol{v}^t\boldsymbol{x}_0 > 0$. But then $\boldsymbol{v}^t\epsilon\boldsymbol{x}_0 > 0, \forall \epsilon > 0$. As ϵ can be taken small enough so that $\epsilon\boldsymbol{x}_0 \leq \mathbf{1}$, it follows by Lemma 6.4 that $\epsilon\boldsymbol{x}_0 \in \mathcal{O}(P)$ and as $\boldsymbol{v}^t\epsilon\boldsymbol{x}_0 > 0$, and we get a contradiction. Hence, the pair $(\boldsymbol{v}, 0)$ determines a face $\mathcal{F}'_{\boldsymbol{v},0}$ of $\mathcal{C}(P)$.

Consider now a face $\mathcal{F}'_{\boldsymbol{v},0}$ of $\mathcal{C}(P)$. Hence, $\boldsymbol{v}^t\boldsymbol{x} \leq 0, \forall \boldsymbol{x} \in \mathcal{C}(P)$. But then, $\boldsymbol{v}^t\boldsymbol{x} \leq 0, \forall \boldsymbol{x} \in \mathcal{O}(P)$ and as $\mathbf{0} \in \mathcal{F}'_{\boldsymbol{v},0}$, this determines a face of $\mathcal{O}(P)$.

Let us now see that for each pair $(\boldsymbol{v}, 0)$, $dim(\mathcal{F}'_{\boldsymbol{v},0}) = dim(\mathcal{F}_{\boldsymbol{v},0})$. First, as $\mathcal{F}_{\boldsymbol{v},0} \subseteq \mathcal{F}'_{\boldsymbol{v},0}$, we have $dim(\mathcal{F}_{\boldsymbol{v},0}) \leq dim(\mathcal{F}'_{\boldsymbol{v},0})$.

On the other hand, let k be the dimension of $\mathcal{F}'_{\boldsymbol{v},0}$. This implies that there are k vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ linearly independent in $\mathcal{F}'_{\boldsymbol{v},0}$. But now, we can find $\epsilon > 0$ small enough such that $\epsilon\boldsymbol{v}_1 \leq \mathbf{1}, ..., \epsilon\boldsymbol{v}_k \leq 1$. Thus, $\epsilon\boldsymbol{v}_1, ..., \epsilon\boldsymbol{v}_k \in \mathcal{F}_{\boldsymbol{v},0}$ and hence, $dim(\mathcal{F}_{\boldsymbol{v},0}) \geq dim(\mathcal{F}'_{\boldsymbol{v},0})$. □

As a consequence, we can adapt Theorem 1.61 for order cones as follows.

Theorem 6.9. *[25] Let $L \subseteq \mathcal{F}(P)$. Then, L determines a face of $\mathcal{C}(P)$ if and only if L is an embedded lattice of $\mathcal{F}(P)$ containing the empty filter.*

Remark 6.10. From Theorem 6.8, in order to find faces of an order cone, we need to look for faces of the corresponding order polytope containing $\mathbf{0}$. As previously explained in Theorem 1.58, if we consider the expression of $\mathcal{O}(P)$ as a polyhedron, faces arise turning inequalities into equalities. Vertices in the face are the vertices of the polyhedron satisfying these equalities. If we consider $\hat{P}$, vertex $\mathbf{0}$ corresponds to function

$$f(x) = \begin{cases} 0 & x \neq \widehat{1} \\ 1 & x = \widehat{1} \end{cases}$$

Consequently, $\mathbf{0}$ satisfies $f(x) = f(y)$ when $y \neq \widehat{1}$. Thus, we look for the faces where the inequalities turned into equalities do not depend on $\widehat{1}$.

In terms of Theorem 1.60, we have to look for partitions defining faces containing $\mathbf{0}$. Note that each block B_i defines a subset of P such that all elements in B_i attain the same value for all points in the face. Therefore, faces containing $\mathbf{0}$ mean that there is a block containing only $\widehat{1}$.

Example 6.1. *Consider the polytope given in Figure 6.1 left.*

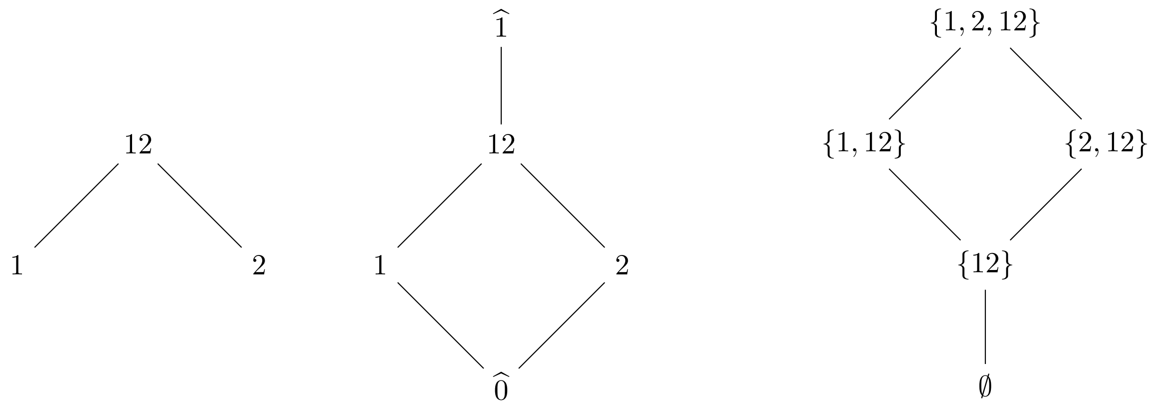


FIGURE 6.1: Example of poset P (left), his extension $\hat{P}$ (center) and his filter lattice (right).

In this case, we have three elements and both the order polytope and the cone order cone can be depicted in $\mathbb{R}^3$, with the first coordinate corresponding to 1, the second one to 2 and the third to 12, see Figure 6.2. The cone $\mathcal{C}(P)$ is given by 3-dimensional vectors f satisfying

$$0 \leq f(1),\ 0 \leq f(2),\ f(1) \leq f(12),\ f(2) \leq f(12).$$

Let us then explain the previous results for this poset. First, let us start obtaining the vectors defining extremal rays. According to Theorem 6.5, it suffices to obtain the non-empty filters that are connected subposets of P. Non-empty filters of P are:

$$\{\{12\}, \{1, 12\}, \{2, 12\}, \{1, 2, 12\}\}.$$

All of them are connected subposets of P. Hence, we have 4 extremal rays, whose respective vectors are

$$(0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1).$$

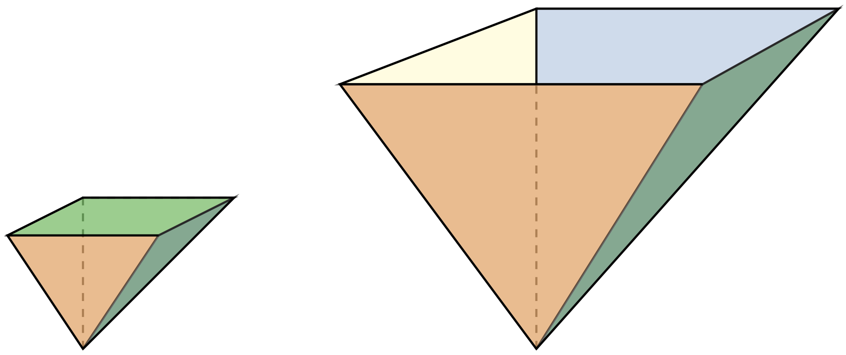


FIGURE 6.2: Order polytope $\mathcal{O}(P)$ (left) and order cone $\mathcal{C}(P)$ (right).

Let us now deal with the facets. For this, consider the poset $\hat{P} = \widehat{0} \oplus P \oplus \widehat{1}$ (see Figure 6.1 center). According to Theorems 1.60 and 6.8, the facets are given by considering one of the following equalities:

$$f(\widehat{0}) = f(1),\ f(\widehat{0}) = f(2),\ f(1) = f(1, 2),\ f(2) = f(1, 2),\ f(1, 2) = f(\widehat{1}).$$

This translates into transforming poset $\hat{P}$ in a new poset where the elements in the equality identify to each other (see Lemma 6.6). The posets for the previous equalities are given in Figure 6.3.

Note that the facets containing $\mathbf{0}$ *are those whose defining equality does not involve $\widehat{1}$, as* $\mathbf{0}$ *satisfies any other equality. In our case, they correspond to the first four cases. Thus, we have four facets containing* $\mathbf{0}$ *and all of them are simplices (indeed triangles) because the corresponding polytope is a chain.*

For the 1-dimensional faces, we have to consider two equalities. However, we have to be careful with the selected equalities because they might imply other equalities. For example, if we consider $f(\widehat{0}) = f(1), f(1) = f(1, 2)$, this also implies $f(\widehat{0}) = f(2)$, and

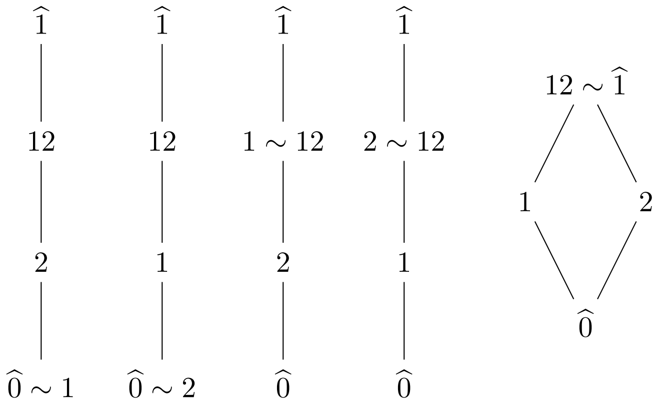


FIGURE 6.3: Subposets when turning an inequaity into an equality.

hence we obtain a point instead of an edge. In our case, the edges containing $\mathbf{0}$ *are given by the pairs of equalities defining an edge and not involving* $\widehat{1}$*. There are four pairs in these conditions that are*

$$\{f(\widehat{0}) = f(1), f(\widehat{0}) = f(2)\},\ \{f(\widehat{0}) = f(1), f(2) = f(1,2)\},$$

$$\{f(\widehat{0}) = f(2), f(1) = f(1,2)\},\ \{f(1) = f(1,2), f(2) = f(1,2)\}.$$

Alternatively, we could use the characterization given in Theorem 6.9. In this case, we have to consider the filter lattice (see Figure 6.1 right).

Hence, edges are given by pairs of filters defining a sublattice and involving the empty filter. Thus, the possible choices are the following pairs:

$$\{\{\emptyset\}, \{12\}\},\ \{\{\emptyset\}, \{1, 12\}\},\ \{\{\emptyset\}, \{2, 12\}\},\ \{\{\emptyset\}, \{1, 2, 12\}\}.$$

Thus, the extremal rays of $\mathcal{C}(P)$ *are given by vectors* $(0,0,1), (1,0,1), (0,1,1), (1,1,1)$.

For 2-dimensional faces, we have to consider all possible sublattices of height 2 and involving the $\emptyset$ *filter. These sublattices are:*

$$\{\{\emptyset\}, \{1, 12\}, \{1, 2, 12\}\},\ \{\{\emptyset\}, \{2, 12\}, \{1, 2, 12\}\},\ \{\{\emptyset\}, \{12\}, \{1, 12\}\},\ \{\{\emptyset\}, \{12\}, \{2, 12\}\}.$$

Hence, the 2-dimensional faces for $\mathcal{C}(P)$ *are defined by vectors*

$$\{(1,0,1),(1,1,1)\},\{(0,1,1),(1,1,1)\},\{(0,0,1),(1,0,1)\},\{(0,0,1),(0,1,1)\}.$$

Notice that we cannot consider

$$\{\{\emptyset\},\{1,12\},\{2,12\},\{1,2,12\}\},\ \{\{\emptyset\},\{12\},\{1,12\},\{2,12\}\},$$

because they are not embedded sublattices.

6.2 Application to Game Theory

In this section, we show that some well-known cones appearing in the field of monotone games can be seen as order cones. Hence, all the results developed in the previous section can be applied to these cones. The first example deals with the general case of monotone games when all coalitions are feasible. We next extend this to the case where $\Omega \subset \mathcal{P}(X)\backslash\{\emptyset\}$. As an example of applicability for subfamilies of monotone games satisfying a property on v but not on the set of feasible coalitions, we also treat the case of k-symmetric monotone games.

6.2.1 The cone of general monotone games

Consider monotone games when all coalitions are feasible, i.e. the set $\mathcal{MG}(X)$. We consider $\mathcal{MG}(X)$ as a subset of $\mathbb{R}^{2^n-1}$ (we have removed the coordinate for $\emptyset$ because its value is fixed). This set is given by all games satisfying $v(A) \leq v(B)$ whenever $A \subset B$. Thus, a game $v \in \mathcal{MG}(X)$ is characterized by the following conditions:

- $0 \leq v(A)$.
- $v(A) \leq v(B)$ if $A \subseteq B$.

Then, $\mathcal{MG}(X) = \mathcal{C}(\mathcal{P}(X)\backslash\{\emptyset\})$, where the order relation $\prec$ on $\mathcal{P}(X)\backslash\{\emptyset\}$ is given by $A \prec B$ if and only if $A \subset B$. For example, for $|X| = 3$, this poset is given in Figure 6.4. However, little else is known about $\mathcal{MG}(X)$; for instance, the set of extremal rays is not known and this question appears in [29] as an open problem. We will study this set at the light of the results of the previous section. Let us first deal with the extremal rays.

Corollary 6.11. *The vectors defining an extremal ray of $\mathcal{MG}(X)$ are defined by non-empty filters of $\mathcal{P}(X)\backslash\{\emptyset\}$.*

Proof. Following Theorem 6.5, we need to find the non-empty filters of $\mathcal{P}(X)\backslash\{\emptyset\}$ that are connected. But in this case, all filters except the empty filter, corresponding to vertex $\mathbf{0}$, contain X. Hence, all of them are connected. $\square$

For obtaining the number of extremal rays, note that any filter in a poset is characterized in terms of its minimal elements and that these minimal elements are an antichain of the poset. For the boolean poset $\mathcal{P}(X)$, the number of antichains (counting the empty set) are the Dedekind numbers, $M(n)$.

For $\mathcal{MG}(X)$, we have to remove the antichain $\{\emptyset\}$ because the poset defining the order cone is $\mathcal{P}(X)\backslash\{\emptyset\}$. Besides, the empty antichain corresponds to $\mathbf{0}$ and thus, it should be removed, too. Hence, the number of extremal rays of $\mathcal{MG}(X)$ is $M(n) - 2$.

Example 6.2. *Let us compute the extremal rays of the order cone $\mathcal{MG}(X)$ where $X = \{1, 2, 3\}$. Note that $\mathcal{C}(P)$ is a cone in $\mathbb{R}^7$. Then, considering $P = B_3 \setminus \{\emptyset\}$, it suffices to compute the filters of P.*

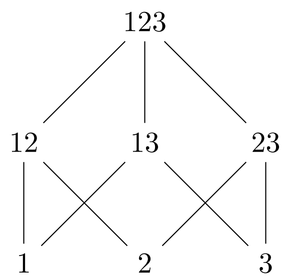


FIGURE 6.4: Boolean poset $P = B_3 \setminus \{\emptyset\}$.

A list with these filters is:

$$\mathfrak{CF}(P) = \{\emptyset, \{123\}, \{12, 123\}, \{13, 123\}, \{23, 123\}, \{12, 13, 123\}, \{12, 23, 123\}, \{13, 23, 123\},$$

$$\{12, 13, 23, 123\}, \{1, 12, 13, 123\}, \{1, 12, 13, 23, 123\}, \{2, 12, 23, 123\}, \{2, 12, 13, 23, 123\},$$

$$\{3, 13, 23, 123\}, \{3, 12, 13, 23, 123\}, \{1, 2, 12, 13, 23, 123\}, \{1, 3, 12, 13, 23, 123\},$$

$$\{2, 3, 12, 13, 23, 123\}, \{1, 2, 3, 12, 13, 23, 123\}\}.$$

Removing $\emptyset$, we have a total of 18 extremal rays. Note that $M(3) = 20$.

Similarly, we can apply Theorem 6.8 to obtain all k-dimensional faces of the cone $\mathcal{MG}(X)$.

Corollary 6.12. *The non-empty faces of $\mathcal{MG}(X)$ are given by the non-empty faces of $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset\})$ containing vertex* $\mathbf{0}$.

However, we will see that in this case we can do better by using pyramids.

From now on, in order to simplify the notation, we will assume that the last coordinate in vector $\boldsymbol{v}$ corresponds to the value $v(X)$.

Proposition 6.13. *Consider the poset $\mathcal{P}(X)\backslash\{\emptyset\}$ with the relation order $A \prec B \Leftrightarrow A \subset B$. Then, the order polytope $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset\})$ is a pyramid with apex $\mathbf{0}$ and base* $\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})\}$.

Proof. Note that for any non-empty filter F, it follows that $N \in F$. Then, the characteristic function of any non-empty filter v_F satisfies $v_F(X) = 1$. Hence, any vertex of $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset\})$ except $\mathbf{0}$ is in the hyperplane $v(X) = 1$. Consequently, $\mathcal{O}(\mathcal{P}\backslash\{\emptyset\})$ is a pyramid with apex $\mathbf{0}$. Finally, the points $\boldsymbol{v}$ of $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset\})$ in the hyperplane $v(X) = 1$ satisfy $v(A) \leq v(B)$ if $A \subseteq B$. Thus, these points can be associated to the order polytope $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$, where the order relation $\preceq$ is given by $A \preceq B \Leftrightarrow A \subseteq B$. $\square$

This allows us to study the k-dimensional faces of $\mathcal{MG}(X)$ from a different point of view that the one of Theorems 6.9 and 6.8. In particular, as apex $\boldsymbol{x}$ is adjacent to every vertex in the base $\mathcal{P}$, edges are given by segments $[\boldsymbol{x}, \boldsymbol{y}]$ with $\boldsymbol{y}$ a vertex of $\mathcal{P}$, thus recovering the result of Corollary 6.11. In general, applying Proposition 1.45, the following holds.

Corollary 6.14. *The k-dimensional faces of $\mathcal{MG}(X)$ are given by the $(k-1)$-dimensional faces of $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$.*

Recall that the order polytope $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$ is the polytope corresponding to the set of capacities or fuzzy measures $\mathcal{FM}(X)$.

It is worth-noting that the geometrical structure (apart the dimension) of $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$ is quite different from the geometrical structure of $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset\})$. For example, in $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset\})$ all vertices are adjacent to $\mathbf{0}$, while this is not the case for $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$ (see [23]).

For this order polytope, many results are known, as for example wether two vertices are adjacent or the centroid [23, 88], see Chapter 1. Applying Corollary 6.14, we conclude that 2-dimensional faces of $\mathcal{MG}(X)$ are given by an edge of $\mathcal{FM}(X) = \mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$. On the other hand, an edge in $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$ is given by two adjacent vertices $\boldsymbol{v}_{F_1}, \boldsymbol{v}_{F_2}$. Another characterization specific for $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$ is given in [88]. Moreover, as both F_1, F_2 are adjacent to $\mathbf{0}$, the following holds.

Corollary 6.15. *Any 2-dimensional face of $\mathcal{MG}(X)$ are defined in terms of 2-dimensional simplices given by $\{\mathbf{0}, \boldsymbol{v}_{F_1}, \boldsymbol{v}_{F_2}\}$ where $F_2 \backslash F_1$ is a connected subposet of $\mathcal{P}(X)\backslash\{\emptyset, X\}$.*

Example 6.3. *Continuing with the previous example, the previous discussion allows to derive the 2-dimensional faces of $\mathcal{MG}(X)$, as by Corollary 6.14 they can be given in terms of edges of $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$. The filters of $\mathcal{P}(X)\backslash\{\emptyset, X\}$ are:*

$$\mathcal{F}(P) = \{\emptyset, \{12\}, \{13\}, \{23\}, \{12, 13\}, \{12, 23\}, \{13, 23\},$$

$$\{12, 13, 23\}, \{1, 12, 13\}, \{1, 12, 13, 23\}, \{2, 12, 23\}, \{2, 12, 13, 23\}, \{3, 13, 23\},$$

$$\{3, 12, 13, 23\}, \{1, 2, 12, 13, 23\}, \{1, 3, 12, 13, 23\}, \{2, 3, 12, 13, 23\}, \{1, 2, 3, 12, 13, 23\}\}.$$

Now, we have to search for pairs of adjacent vertices in $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, N\})$, for example using Corollary 1.67. It is easy but tedious to show that there are 76 pairs in these conditions.

6.2.2 The cone of games with restricted cooperation

Let us now treat the problem when we face a situation of restricted cooperation. Then, several coalitions are not allowed and we have a set $\Omega \subset \mathcal{P}(X)\backslash\{\emptyset\}$ of feasible coalitions. Many papers have been devoted to this subject, usually imposing an algebraic structure on Ω (see e.g. [53, 93–95]). From the point of view of polyhedra, if a coalition is not feasible, this implies that this subset is removed from Ω. We will denote by $\mathcal{MG}_\Omega(X)$ the set of all monotone games whose feasible coalitions are Ω. Thus, a game $v \in \mathcal{MG}_\Omega(X)$ is characterized by the following conditions:

- $0 \leq v(A), A \in \Omega$.
- $v(A) \leq v(B)$ if $A \subseteq B, A, B \in \Omega$.

Then, $\mathcal{MG}_\Omega(X) = \mathcal{C}(\Omega)$, where the order relation $\prec$ on Ω is given by $A \prec B$ if and only if $A \subset B$.

Assume first that $X \in \Omega$. This is the usual situation, as most of the solution concepts on Game Theory assume that all players agree to form the grand coalition (see e.g. [96]). In this case, the following holds.

Corollary 6.16. *If $X \in \Omega$, then the set of extremal rays of $\mathcal{MG}_\Omega(X)$ are given by*

$$\{\boldsymbol{v}_F : \emptyset \neq F, F \textit{ filter of } \Omega\}.$$

Proof. Applying Theorem 6.5, the set of extremal rays is given by the set of vertices $\boldsymbol{v}_F$ of $\mathcal{O}(\Omega)$ such that F is a connected filter in Ω. As $X \in \Omega$, it follows that all filters are connected subposets of Ω, so that we have as many extremal rays as vertices in $\mathcal{O}(\Omega)$ different from $\mathbf{0}$. And this value is given by the number of filters minus one (for the empty filter corresponding to vertex $\mathbf{0}$). □

Indeed, we can translate in this case the results obtained for $\mathcal{MG}(X)$. Assuming the last coordinate corresponds to subset X, the following holds.

Proposition 6.17. *Assume $X \in \Omega$ and consider the poset Ω with the relation order $A \prec B \Leftrightarrow A \subset B$. Then, the order polytope $\mathcal{O}(\Omega)$ is a pyramid with apex $\mathbf{0}$ and base $\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(\Omega \backslash \{X\})\}$.*

Proof. It is a straightforward translation of the proof of Proposition 6.13. □

This implies that have two possibilities for studying $\mathcal{MG}_\Omega(X)$. First, we can apply the general results for any order cones developed in Section 6.1. Alternatively, we can apply Proposition 1.45 and derive the results from the structure of the order polytope $\mathcal{O}(\Omega \backslash \{X\})$ just as it has been done for $\mathcal{MG}(X)$. In this last case, the following holds.

Corollary 6.18. *The k-dimensional faces of $\mathcal{MG}_\Omega(X)$ are given by the $(k-1)$-dimensional faces of $\mathcal{O}(\Omega \backslash \{X\})$.*

Example 6.4. *Suppose a situation with four players, and assume that the only feasible coalitions are $\Omega = \{12, 23, 34, 1234\}$. The corresponding Hasse diagram is given in Figure 6.5.*

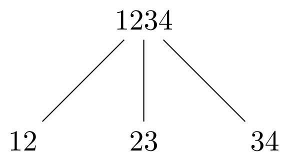


FIGURE 6.5: Hasse diagram of the poset of a game with restricted cooperation.

For this example, the non-empty filters of Ω are:

$$F_1 = \{1234\},\ F_2 = \{12, 1234\},\ F_3 = \{23, 1234\},\ F_4 = \{34, 1234\},\ F_5 = \{12, 23, 1234\},$$

$$F_6 = \{12, 34, 1234\},\ F_7 = \{23, 34, 1234\},\ F_8 = \{12, 23, 34, 1234\}.$$

Thus, we have 8 extremal rays. For example, the extremal ray corresponding to F_5 is given by vector $\boldsymbol{v} = (1, 1, 0, 1)$, where the third coordinate corresponds to subset $\{34\}$.

For k-dimensional faces, it just suffice to note that $\Omega\backslash\{1234\}$ is an antichain. Then, $\mathcal{O}(\Omega\backslash\{X\})$ is a cube. For example, for finding 2-dimensional faces, we have to consider pairs of adjacent vertices of the cube $\mathcal{O}(\Omega\backslash\{X\})$ (there are 12 pairs). Similarly, for 3-dimensional faces we have to consider 2-dimensional faces of the cube (six cases), and there is just one 4-dimensional face.

Now, assume $X \notin \Omega$. This situation is more tricky and needs to study each case applying Theorems 6.5 and 6.8. For example, in this situation it could happen that some vertices are not adjacent to $\mathbf{0}$ and thus, they do not define an extremal ray. Moreover, the 2-dimensional faces are not defined necessarily via 2-dimensional simplices.

As examples for this case, we study two situations. Assume $\Omega \cup \{\emptyset\}$ is a poset with a top element $\widehat{1}$ and thus, we can extend all the results that we have obtained when $X \in \Omega$.

Proposition 6.19. *Consider the poset with top element $\Omega \cup \{\emptyset\}$ with the relation order $A \prec B \Leftrightarrow A \subset B$ and top element $\widehat{1}$. Then, the order polytope $\mathcal{O}(\Omega\backslash\{\emptyset\})$ is a pyramid with apex $\mathbf{0}$ and base $\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(\Omega\backslash\{\widehat{1}\})\}$.*

Corollary 6.20. *The k-dimensional faces of $\mathcal{MG}(\Omega)$ are given by the $(k-1)$-dimensional faces of $\mathcal{O}(\Omega\backslash\{\widehat{1}\})$.*

Suppose as a second example that Ω is a union of connected posets

$$\Omega = P_1 \cup ... \cup P_r, \quad P_i \text{ connected.}$$

In this case, the only connected filters are the connected filters $F_i \subseteq P_i$. Then, we have:

Proposition 6.21. *If $\Omega = P_1 \cup ... \cup P_r$, where P_i is a connected poset, $i = 1, ..., r$, then the extremal rays of $\mathcal{MG}_\Omega(X)$ are given by $\boldsymbol{v}_{F_i}$ where F_i is a non-empty connected filter of P_i.*

For example, if $|P_i| = 1 \ \forall i$, then Ω is an antichain and the only connected filters are the singletons. Thus, there are just r extremal rays for $\mathcal{MG}_\Omega(X)$. Indeed, note that the corresponding order polytope is the r-dimensional cube and thus the vertices adjacent to $\mathbf{0}$ are $\boldsymbol{e}_i$, $i = 1, ..., r$.

In general, we have to study the properties of the corresponding poset.

Example 6.5. *Assume again a 4-players game and let us consider the coalitions given in Figure 6.6 left. We have in this case a 4-dimensional cone order.*

Fixing the order for coordinates $12, 13, 34, 123$, the vertices of the corresponding order polytope are given in Table 6.1.

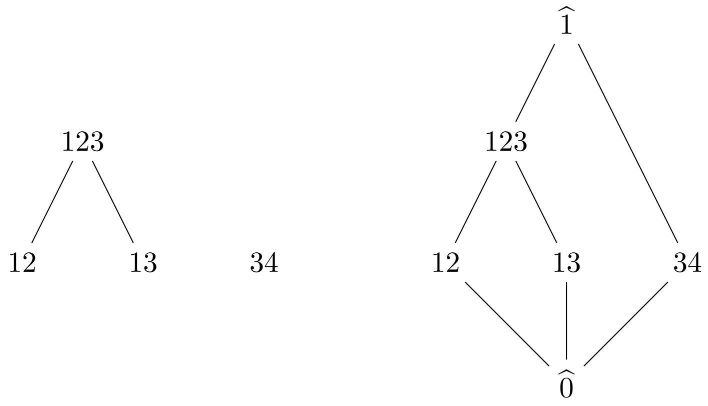


FIGURE 6.6: Hasse diagram of the poset P of a game with restricted cooperation (left) and his extension $\hat{P}$.

Filter	$\emptyset$	**123**	**34**	34, 123	**12, 123**
Vertex	(0,0,0,0)	**(0,0,0,1)**	**(0,0,1,0)**	(0,0,1,1)	**(1,0,0,1)**
Filter	**13, 123**	**12, 12, 123**	12, 34, 123	13, 34, 123	12, 13, 34, 123
Vertex	**(0,1,0,1)**	**(1,1,0,1)**	(1,0,1,1)	(0,1,1,1)	(1,1,1,1)

TABLE 6.1: Filters and vertices of poset of Figure 6.6.

Vertices defining an extremal ray are those whose corresponding filter is connected. The five vertices in these conditions are written in boldface.

In order to obtain the facets of this order cone, we look for facets of the corresponding order polytope containing **0** *(Theorem 6.8). For this, we consider* $\widehat{0} \oplus P \oplus \widehat{1}$ *(see Figure 6.6 right). As we are looking for facets, we just turn an inequality not involving* $\widehat{1}$ *into an equality. Then, the facets are given in Table 6.2.*

Restriction	$f(\widehat{0}) = f(12)$	$f(\widehat{0}) = f(13)$	$f(\widehat{0}) = f(34)$	$f(12) = f(123)$	$f(13) = f(123)$
Vertices	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
	(0,0,0,1)	(0,0,0,1)	(0,0,0,1)	(1,0,0,1)	(0,1,0,0)
	(0,0,1,0)	(0,0,1,0)	(1,0,0,1)	(1,0,1,1)	(0,1,1,1)
	(0,0,1,1)	(0,0,1,1)	(0,1,0,1)	(1,1,0,1)	(1,1,0,1)
	(0,1,0,1)	(0,1,0,1)	(1,1,0,1)	(1,1,1,1)	(1,1,1,1)
	(0,1,1,1)	(1,0,1,1)			

TABLE 6.2: Facets of the order cone of poset of Figure 6.6.

Another way to look for extremal rays is Theorem 1.61. For this, we need to build the lattice of filters, that is given in Figure 6.7.

Then, the extremal rays are given by filters that together with $\emptyset$ *form an embedded sublattice. These filters are*

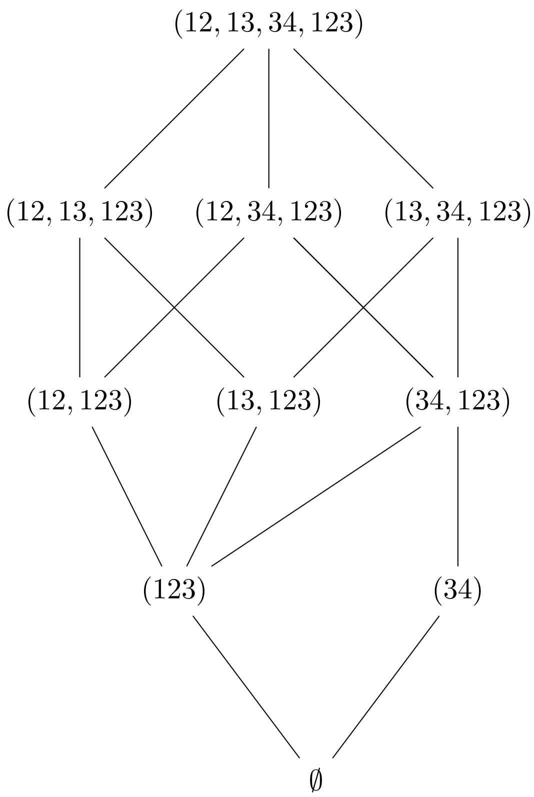


FIGURE 6.7: Lattice of filters.

$$\{123\}, \{34\}, \{12, 123\}, \{13, 123\}, \{12, 13, 123\}.$$

6.2.3 The cone of k-symmetric measures

As explained before, order cones can be applied to more general situations than games with restricted cooperation. In this subsection we will apply it to k-symmetric monotone games. We have chosen this case because the set of k-symmetric capacities with respect to a fixed partition is an order polytope [23], see Section 2.6.3 and Chapter 4.

We denote by $\mathcal{MG}^k(A_1, ..., A_k)$ the set of monotone games v such that $A_1, ..., A_k$ are subsets of indifference for v (but not necessarily k-symmetric; for example, any symmetric monotone game, in which all players are indifferent, belongs to $\mathcal{MG}^k(A_1, ..., A_k)$. Then, $v \in \mathcal{MG}^k(A_1, ..., A_k)$ is characterized as follows:

- $v(0, ..., 0) = 0$.
- $v(a_1, ..., a_k) \leq v(b_1, ..., b_k)$ if $a_i \leq b_i, i = 1, ..., k$.

Consider then the poset

$$P = \{(c_1, ..., c_k) : c_i = 0, ..., |A_i|, i = 1, ..., k\}$$

with the order relation $(c_1, ..., c_k) \preceq (b_1, ..., b_k)$ if and only if $c_i \leq b_i, i = 1, ..., k$.

Then, it follows that $\mathcal{MG}^k(A_1, ..., A_k) = \mathcal{C}(P \backslash \{(0, ..., 0)\})$ and the results of Section 3 can be applied to obtain the geometrical aspects of this cone. Moreover, as $(|A_1|, ..., |A_k|)$ is a top element in the poset, we can apply the results obtained for $\mathcal{MG}(X)$.

Corollary 6.22. *The vectors defining an extremal ray of $\mathcal{MG}^k(A_1, ..., A_k)$ are defined by non-empty filters of $P \backslash \{(0, ..., 0)\}$.*

Proposition 6.23. *Consider the poset $P = \{(c_1, ..., c_k) : c_i = 0, ..., |A_i|, i = 1, ..., k\}$. Then, the order polytope $\mathcal{O}(P \backslash \{(0, ..., 0)\})$ is a pyramid with base*

$$\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(P \backslash \{(0, ..., 0), (|A_1|, ..., |A_k|)\})\}$$

and apex $\mathbf{0}$.

Corollary 6.24. *The k-dimensional faces of $\mathcal{MG}^k(A_1, ..., A_k)$ are given by the $(k-1)$-dimensional faces of $\mathcal{O}(P \backslash \{(0, ..., 0), (|A_1|, ..., |A_k|)\}$.*

Let us study two particular cases.

Example 6.6. *For $\mathcal{MG}^1(X)$, the set of monotone symmetric games, the corresponding order polytope is a chain of n elements. Thus, we have n non-empty filters $F_1, ..., F_n$, given by $F_i := \{i, ..., n\}$ and $\boldsymbol{v}_{F_i} = (0, ..., 1, ..., 1)$. Therefore, we have n extremal rays.*

Besides, by Corollary 1.67, we conclude that all vertices are adjacent to each other. Hence, we have $\binom{n}{2}$ 2-dimensional faces and in general, the number of k-dimensional faces is $\binom{n}{k}$, for $k \geq 2$.

Example 6.7. *For the 2-symmetric case $\mathcal{MG}^2(A_1, A_2)$, it has been proved in Chapter 4 [80], that the order polytope $\mathcal{FM}^2(A_1, A_2)$ can be associated to a Young diagram [83] of shape $\boldsymbol{\lambda} = (|A_2|, ..., |A_2|)$.*

Chapter 7

Integral techniques for counting Linear Extensions

Counting linear extensions becomes essential to understand the geometry of order polytopes. From Equation 1.5 we can use the number of linear extensions $e(P)$ to compute the volume of $\mathcal{O}(P)$. In this chapter we follow the opposite path, that is, we are going to deal with the volume of $\mathcal{O}(P)$ to get recursive formulas for computing the number of linear extensions of some important families of posets.

The technique described in the chapter can be used to count the number of linear extensions of certain posets. However, a deeper research about how to extend this technique to generate linear extensions randomly is still pending. Despite this, we find it an interesting tool for studying subfamilies of non-additive measures.

As a theoretical application we develop a procedure for counting the number of 2-alternating permutations. Next we show that the volume can be given in terms of the coefficients of these polynomials, thus obtaining a linear recurrence for the volume and the number of linear extensions. We also apply these integral techniques to solve a specific problem related to fuzzy measures. Finally, we argue that this procedure can also be applied to other important families of fence-type posets to derive formulas for the number of linear extensions.

7.1 2-alternating permutations and the generalized fence

A permutation $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \mathcal{S}_n$ is called **alternating** if

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots .$$

The number of alternating permutations for a referential of n elements is denoted E_n and is called the n-th Euler number (see [5]). Euler numbers have been deeply studied and there are good linear recurrences to generate them [97].

In this section we deal with 2-alternating permutations. We say that a permutation $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \mathcal{S}_n$ is **2-alternating** if

$$\sigma_1 > \sigma_2 > \sigma_3 < \sigma_4 < \sigma_5 > \cdots.$$

Let us denote by A_n^2 the number of 2-alternating permutations in $\mathcal{S}_n$. As it will become clear below, this problem and other similar can be easily translated in terms of computing the number of linear extensions of suitable posets. For example, for 2-alternating permutations, we will deal with a poset that we have named generalized fence.

As explained above, the problem of counting alternating permutations can be seen as the problem of counting linear extensions of some fence problem. To see this clearly, let us first define the family of **generalized fences** or generalized zig-zag posets. The n-th member of the family is given by $(GZ_n, \preceq)$ where

$$GZ_n := \{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}, z_1, ..., z_{2n+1}\}$$

and $\preceq$ is given by

$$x_1 \prec z_1 \prec y_1 \succ z_2 \succ x_2 \prec z_3 \prec y_2 \succ ... \succ x_n \prec z_{2n-1} \prec y_n \succ z_{2n} \succ x_{n+1} \prec z_{2n+3} \prec y_{n+1}.$$

The Hasse diagram of GZ_n is given in Figure 7.1.

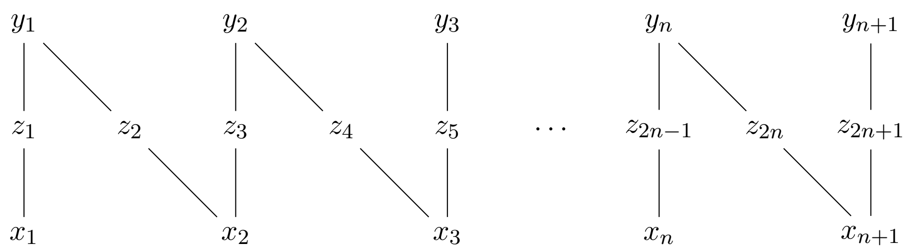


FIGURE 7.1: Generalized Fence or generalized zigzag poset GZ_n.

Then, GZ_n is a poset of $4n+3$ elements. Obviously, the number of linear extensions of GZ_n corresponds to A_{4n+3}^2. Let us fix our attention on the elements z_i. Note that they are not related to each other. Besides, given $x_i, i > 1$, it follows that $x_i \prec z_{2i-2}$ and $x_i \prec z_{2i-1}$. Similarly, for $y_i, i < n+1$, it follows that $y_i \succ z_{2i}$ and $y_i \succ z_{2i-1}$. Finally, $x_1 \prec z_1$

and $y_{n+1} \succ z_{2n+1}$. Thus, the corresponding order polytope is the polytope in $\mathbb{R}^{4n+3}$ given by points $(f(x_1), ..., f(x_{n+1}), f(z_1), ..., f(z_{2n+1}), f(y_1), ..., f(y_{n+1}))$ satisfying

$$\begin{cases} f(z_i) \in [0,1] \\ f(x_i) \in [0, \min(f(z_{2i-2}), f(z_{2i-1}))], i > 1, \quad f(x_1) \in [0, f(z_1)] \\ f(y_i) \in [\max(f(z_{2i}), f(z_{2i-1})), 1], i \leq n, \quad f(y_{n+1}) \in [f(z_{2n+1}), 1] \end{cases}.$$

With these facts in mind, the volume of the corresponding order polytope is given by

$$\int_0^1 ... \int_0^1 \int_0^{\min\{z_{2n+1}, z_{2n}\}} ... \int_0^{z_1} \int_{z_{2n+1}}^1 \int_{\max\{z_{2n}, z_{2n-1}\}}^1 ... \int_{\max\{z_1, z_2\}}^1 1 dy_1 ... dy_{n+1} dx_1 ... dx_{n+1} dz_1 ... dz_{2n+1}$$

$$= \int_0^1 ... \int_0^1 z_1(1 - z_{2n+1}) \prod_{i=1}^{n} \min(z_{2i+1}, z_{2i})(1 - \max(z_{2i-1}, z_{2i})) dz_1 ... dz_{2n+1}.$$

Then, let us focus on the integral

$$\int_0^1 \int_0^1 \min(y, z)(1 - \max(x, y)) p(x) dx dy,$$

where $p(x)$ is a polynomial. This integral is solved in the next lemmas.

Lemma 7.1. *Let $p(x)$ be a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, ..., a_n \in \mathbb{R}.$$

Then, for fixed $y \in [0,1]$ it follows

$$\int_0^1 \min(x, y) p(x) dx = -\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2} + \left(\sum_{k=0}^{n} \frac{a_k}{k+1} \right) y.$$

Proof. Let us denote

$$P(y) := \int_0^y p(x) dx = \sum_{k=0}^{n} \frac{a_k}{k+1} y^{k+1},$$

$$\hat{P}(y) := \int_0^y x p(x) dx = \int_0^y \left(\sum_{k=0}^{n} a_k x^{k+1} \right) dx = \left[\sum_{k=0}^{n} \frac{a_k}{k+2} x^{k+2} \right]_0^y = \sum_{k=0}^{n} \frac{a_k}{k+2} y^{k+2}.$$

Therefore, for fixed $y \in [0,1]$ it follows

$$\begin{aligned}
\int_0^1 \min(x,y)p(x)dx &= \int_0^1 [yI(y \leq x) + xI(x < y)]\, p(x)dx \\
&= \int_y^1 yp(x)dx + \int_0^y xp(x)dx \\
&= y\,(P(1) - P(y)) + \hat{P}(y) \\
&= y\left(\sum_{k=0}^{n} \frac{a_k}{k+1} - \sum_{k=0}^{n} \frac{a_k}{k+1} y^{k+1}\right) + \sum_{k=0}^{n} \frac{a_k}{k+2} y^{k+2} \\
&= y\sum_{k=0}^{n} \frac{a_k}{k+1} + \sum_{k=0}^{n} \left[-\frac{a_k}{k+1} + \frac{a_k}{k+2}\right] y^{k+2} \\
&= \left(\sum_{k=0}^{n} \frac{a_k}{k+1}\right) y - \sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2},
\end{aligned}$$

and the result holds. □

Lemma 7.2. *Let $p(x)$ be a polynomial*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, ..., a_n \in \mathbb{R}.$$

Then, for fixed $y \in [0,1]$, it follows

$$\int_0^1 [1 - \max(x,y)]\, p(x)dx = -\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2} + \left(\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)}\right).$$

Proof. Consider $P(y)$ and $\hat{P}(y)$ as defined in the proof of the previous lemma. Then,

$$\begin{aligned}
\int_0^1 [1 - \max(x,y)]\, p(x)dx &= \int_0^1 p(x)dx - \int_0^1 \max(x,y)p(x)dx \\
&= \int_0^1 p(x)dx - \int_0^1 y\,(I(x \leq y))\, p(x)dx - \int_0^1 x\,(I(y \leq x))\, p(x)dx \\
&= \int_0^1 p(x)dx - \int_0^y yp(x)dx - \int_y^1 xp(x)dx \\
&= P(1) - yP(y) - \hat{P}(1) + \hat{P}(y).
\end{aligned}$$

Now,

$$P(1) - \hat{P}(1) = \sum_{k=0}^{n} \frac{a_k}{k+1} - \sum_{k=0}^{n} \frac{a_k}{k+2} = \sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)},$$

and

$$\hat{P}(y) - yP(y) = \sum_{k=0}^{n} \frac{a_k}{k+2} y^{k+2} - \sum_{k=0}^{n} \frac{a_k}{k+1} y^{k+2} = -\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2}.$$

Therefore,

$$\int_0^1 [1-\max(x,y)]\, p(x)dx = -\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2} + \left(\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} \right)$$

and the result holds. □

Lemma 7.3. *Let $p(x)$ be a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, ..., a_n \in \mathbb{R}.$$

Then,

$$\int_0^1 \int_0^1 \min(y,z)\, [1-\max(x,y)]\, p(x)dxdy =$$

$$= \sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)(k+3)(k+4)} z^{k+4} - \frac{1}{2} \left(\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} \right) z^2 + \left(\sum_{k=0}^{n} \frac{(k+2)a_k}{(k+1)(k+2)(k+3)} \right) z.$$

Proof. Note that

$$\int_0^1 \int_0^1 \min(y,z)\, [1-\max(x,y)]\, p(x)dxdy = \int_0^1 \min(y,z) \left(\int_0^1 [1-\max(x,y)]\, p(x)dx \right) dy.$$

Now, we apply Lemma 7.2 and we get

$$\int_0^1 [1-\max(x,y)]\, p(x)dx = -\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2} + \left(\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} \right).$$

Next, if we apply Lemma 7.1 for $p(y) = -\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2} + \left(\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)}\right)$, we obtain a polynomial in z. The term for z is given by

$$\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} - \sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)(k+3)} = \sum_{k=0}^{n} \frac{(k+2)a_k}{(k+1)(k+2)(k+3)}.$$

The term for z^2 is given by

$$-\frac{1}{2}\left(\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)}\right),$$

and the term for $z^{k+4}, k \geq 0$ (note that there is no term for z^3 as there is no term for y in $p(y)$) is given by

$$\frac{a_k}{(k+1)(k+2)(k+3)(k+4)}.$$

Therefore, the result holds. $\square$

Briefly speaking, note that the result of this integral is a polynomial in z. Thus, considering again the expression for the volume and integrating with respect to z_1, z_2 we obtain that $\mathrm{Vol}(\mathcal{O}(GZ_n))$ is given by

$$\int_0^1 \cdots \int_0^1 (1-z_{2n+1}) \min(z_{2n}, z_{2n+1}) (1 - \max(z_{2n-1}, z_{2n})) \cdots [\min(z_2, z_3) (1 - \max(z_1, z_2)) z_1] \, d\overrightarrow{z}$$

$$= \int_0^1 \cdots \int_0^1 (1-z_{2n+1}) \min(z_{2n}, z_{2n+1}) (1 - \max(z_{2n-1}, z_{2n})) \cdots \min(z_4, z_5) (1 - \max(z_3, z_4)) \, p(z_3) d\overrightarrow{z \backslash \{z_1, z_2\}},$$

where $p(z_3)$ is the polynomial

$$p(z_3) = \int_0^1 \int_0^1 \min(z_2, z_3) (1 - \max(z_1, z_2)) \, z_1 dz_2 dz_1.$$

Next step is to solve again the integral with respect to z_3, z_4, and so on. This defines a sequence $a^{(0)}, a^{(1)}, \ldots$ of polynomials with $a^{(0)}(z) = z$ that can be obtained recursively. The first members of the sequence are:

$$\begin{aligned} a^{(1)}(z) &= \frac{z}{8} - \frac{z^2}{12} + \frac{z^5}{120}. \\ a^{(2)}(z) &= \frac{59}{5\,760}z - \frac{71}{10\,080}z^2 + \frac{1}{960}z^5 - \frac{1}{4\,320}z^6 + \frac{1}{362\,880}z^9. \end{aligned}$$

Next Lemma shows several facts about the polynomials in this sequence.

Lemma 7.4. *Let $a^{(0)}(z) := z$ and $a^{(k+1)}(z)$ be defined recursively by*

$$a^{(k+1)}(z) := \int_0^1 \int_0^1 [1 - \max(x, y)] \min(y, z) a^{(k)}(x) dx dy.$$

Then, we have:

i) $a^{(n)}(z)$ is a polynomial of degree $4n + 1$.

ii) If $n \geq 1$, then $a_3^{(n)} = 0$, and if $n \geq 0$, then $a_0^{(n)} = 0$.

iii) $a_k^{(n)} = \dfrac{1}{k(k-1)(k-2)(k-3)} a_{k-4}^{(n-1)}$, $\forall n \geq 1$ and $k \in \{4, 5, \cdots, 4n + 1\}$.

iv) $a_{4n+1}^{(n)} = \dfrac{1}{(4n+1)!}$, $\quad a_{4k}^{(n)} = a_{4k+3}^{(n)} = 0$, for $0 \leq k \leq n$.

v) Let us define $\Gamma_k := a_1^{(k)}$ and $\beta_k := a_2^{(k)}$, then, $a_{4k+1}^{(n)} = \dfrac{1}{(4k+1)!}\Gamma_{n-k}$, and $a_{4k+2}^{(n)} = \dfrac{2}{(4k+2)!}\beta_{n-k}$, where $0 \leq k \leq n - 1$.

Proof. i) We will show the result by induction on n. For $n = 1$, the result holds. Now, assuming the result holds for $1, \ldots, n-1$, we can apply Lemma 7.3 to conclude that $a^{(n)}(x)$ has degree four units greater than $a^{(n-1)}$. Hence, the result holds.

ii) Trivial by Lemma 7.3.

iii) Trivial by Lemma 7.3.

iv) For $a_{4n+1}^{(n)}$, applying iii) n times, we have

$$\begin{aligned} a_{4n+1}^{(n)} &= \frac{1}{(4n+1)(4n)(4n-1)(4n-2)} a_{4n-3}^{(n-1)} \\ &= \frac{1}{(4n+1)(4n)(4n-1)(4n-2)} \cdot \frac{1}{(4n-3)(4n-4)(4n-5)(4n-6)} a_{4n-7}^{(n-2)} \\ &= \ldots \\ &= \frac{1}{(4n+1)(4n)(4n-1)(4n-2)} \cdot \frac{1}{(4n-3)(4n-4)(4n-5)(4n-6)} \cdots \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} a_1^{(0)} \\ &= \frac{1}{(4n+1)!}. \end{aligned}$$

Similarly, applying ii), for $a_{4k}^{(n)}$ we obtain

$$\begin{aligned}
a_{4k}^{(n)} &= \frac{1}{(4k)(4k-1)(4k-2)(4n-3)} a_{4k-4}^{(n-1)} \\
&= \frac{1}{(4k)(4k-1)(4k-2)(4k-3)} \cdot \frac{1}{(4k-4)(4k-5)(4k-6)(4k-7)} a_{4k-8}^{(n-2)} \\
&= ... \\
&= \frac{1}{(4k)(4k-1)(4k-2)(4k-3)} \cdot \frac{1}{(4k-4)(4k-5)(4k-6)(4k-7)} \cdots a_{(0)}^{(n-k)} \\
&= 0.
\end{aligned}$$

For $a_{4k+3}^{(n)}$ we obtain

$$\begin{aligned}
a_{4k+3}^{(n)} &= \frac{1}{(4k+3)(4k+2)(4k+1)(4k)} a_{4k-1}^{(n-1)} \\
&= \frac{1}{(4k+3)(4k+2)(4k+1)(4k)} \cdot \frac{1}{(4k-1)(4k-2)(4k-3)(4k-4)} a_{4k-5}^{(n-2)} \\
&= ... \\
&= \frac{1}{(4k+3)(4k+2)(4k+1)(4k)} \cdot \frac{1}{(4k-1)(4k-2)(4k-3)(4k-4)} \cdots a_{(3)}^{(n-k)} \\
&= 0.
\end{aligned}$$

v) For $a_{4k+1}^{(n)}$, applying iii), we have

$$\begin{aligned}
a_{4k+1}^{(n)} &= \frac{1}{(4k+1)(4k)(4k-1)(4k-2)} a_{4k-3}^{(n-1)} \\
&= \frac{1}{(4k+1)(4k)(4k-1)(4k-2)} \cdot \frac{1}{(4k-3)(4k-4)(4k-5)(4k-6)} a_{4k-7}^{(n-2)} \\
&= ... \\
&= \frac{1}{(4k+1)(4k)(4k-1)(4k-2)} \cdot \frac{1}{(4k-3)(4k-4)(4k-5)(4k-6)} \cdots \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} a_1^{(n-k)} \\
&= \frac{1}{(4k+1)!} \Gamma_{n-k}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
a_{4k+2}^{(n)} &= \frac{1}{(4k+2)(4k+1)(4k)(4k-1)} a_{4k-2}^{(n-1)} \\
&= \frac{1}{(4k+2)(4k+1)(4k)(4k-1)} \cdot \frac{1}{(4k-2)(4k-3)(4k-4)(4k-5)} a_{4k-6}^{(n-2)} \\
&= \dots \\
&= \frac{1}{(4k+2)(4k+1)(4k)(4k-1)} \cdot \frac{1}{(4k-2)(4k-3)(4k-4)(4k-5)} \cdots \frac{1}{6 \cdot 5 \cdot 4 \cdot 3} a_2^{(n-k)} \\
&= \frac{2}{(4k+2)!} \beta_{n-k}.
\end{aligned}
$$

□

The values of Γ_n and β_n can be computed recursively as follows:

Lemma 7.5. *The sequences of* Γ_n *and* β_n *are given by:*

$$
\begin{cases}
\Gamma_1 = \frac{1}{8}, \;\; \beta_1 = -\frac{1}{12}, \\
\Gamma_{n+1} = \frac{4n+3}{(4n+4)!} + \sum_{k=0}^{n-1} \frac{4k+3}{(4k+4)!} \Gamma_{n-k} + 2 \sum_{k=0}^{n-1} \frac{4k+4}{(4k+5)!} \beta_{n-k}, \\
\beta_{n+1} = -\frac{1}{2(4n+3)!} - \sum_{k=0}^{n-1} \frac{1}{2(4k+3)!} \Gamma_{n-k} - \sum_{k=0}^{n-1} \frac{1}{(4k+4)!} \beta_{n-k}.
\end{cases}
$$

Proof. The first values Γ_1 and β_1 arise from Lemma 7.3.

Now, by definition of Γ_n and β_n and applying again Lemma 7.3, we obtain

$$
\Gamma_{n+1} = a_1^{(n+1)} = \sum_{k=0}^{4n+1} \frac{(k+2) a_k^{(n)}}{(k+1)(k+2)(k+3)}.
$$

Now, applying Lemma 7.4, we obtain that all values congruent with 0 and 3 vanish, and by definition of Γ_n and β_n, we obtain that

$$
\Gamma_{n+1} = \frac{4n+3}{(4n+4)!} + \sum_{k=0}^{n-1} \frac{4k+3}{(4k+4)!} \Gamma_{n-k} + 2 \sum_{k=0}^{n-1} \frac{4k+4}{(4k+5)!} \beta_{n-k}.
$$

Similarly,

$$
\beta_{n+1} = a_2^{(n+1)} = -\frac{1}{2} \sum_{k=0}^{4n+1} \frac{a_k^{(n)}}{(k+1)(k+2)}.
$$

Then,

$$\beta_{n+1} = -\frac{1}{2(4n+3)!} - \sum_{k=0}^{n-1} \frac{1}{2(4k+3)!}\Gamma_{n-k} - \sum_{k=0}^{n-1} \frac{1}{(4k+4)!}\beta_{n-k}.$$

Thus, the result holds. □

Theorem 7.6. *Consider GZ_n. Then:*

$$e(GZ_n) = -2(4n+3)!\beta_{n+1}.$$

Proof. Let us obtain the volume of the corresponding order polytope. For this, if we apply Lemma 7.3 several times, we get that $\mathrm{Vol}(\mathcal{O}(GZ_n))$ is given by:

$$\int_0^1 \cdots \int_0^1 (1-z_{2n+1}) \min(z_{2n}, z_{2n+1}) \, (1 - \max(z_{2n-1}, z_{2n})) \cdots \min(z_2, z_3) \, (1 - \max(z_1, z_2)) \, z_1 d\overrightarrow{z} =$$

$$= \int_0^1 \cdots \int_0^1 (1-z_{2n+1}) \min(z_{2n}, z_{2n+1}) \, (1 - \max(z_{2n-1}, z_{2n})) \cdots \min(z_2, z_3) \, (1 - \max(z_1, z_2)) \, a^{(0)}(z_1) d\overrightarrow{z} =$$

$$= \int_0^1 \cdots \int_0^1 (1-z_{2n+1}) \min(z_{2n}, z_{2n+1}) \, (1 - \max(z_{2n-1}, z_{2n})) \cdots \min(z_4, z_5) \, (1 - \max(z_3, z_4)) \, a^{(1)}(z_3) d\overrightarrow{z \setminus \{z_1, z_2\}} =$$

$$= \cdots = \int_0^1 (1 - z_{2n+1}) a^{(n)}(z_{2n+1}) dz_{2n+1} = \sum_{k=0}^{4n+1} \frac{a_k^{(n)}}{k+1} - \sum_{k=0}^{4n+1} \frac{a_k^{(n)}}{k+2} = \sum_{k=0}^{4n+1} \frac{a_k^{(n)}}{(k+1)(k+2)} = -2\beta_{n+1},$$

where the last equality arises from Lemma 7.3. Since $e(GZ_n) = (4n+3)!\mathrm{Vol}(\mathcal{O}(GZ_n))$, the result holds. □

The first values of $e(GZ_n)$ are given in Table 7.1.

n	1	2	3	4	5
Γ_n	$\frac{1}{8}$	$\frac{59}{5\,760}$	$\frac{12\,031}{14\,515\,200}$	$\frac{1\,402\,815\,833}{20\,922\,789\,888\,000}$	$\frac{8\,573\,648\,137}{1\,580\,833\,013\,760\,000}$
β_n	$-\frac{1}{12}$	$-\frac{71}{10\,080}$	$-\frac{45\,541}{79\,833\,600}$	$-\frac{120\,686\,411}{2\,615\,348\,736\,000}$	$-\frac{908\,138\,776\,681}{243\,290\,200\,817\,664\,000}$
$e(GZ_n)$	71	45 541	120 686 411	908 138 776 681	$\cdots$

TABLE 7.1: First values of Γ_n, β_n and $e(GZ_n)$.

It just remains to compute the rest of values for A_n^2 as last formula only gives us A_{4n+3}^2. Now if we remove 1, 2 or 3 elements of the corresponding generalized zigzag poset, we get the equivalent version with $4n+2, 4n+1$ and $4n$ elements.

Theorem 7.7. *Let A_n^2 be the number of 2-alternating permutations. Then:*

i) $A_1^2 = A_2^2 = A_3^2 = 1$ *and* $A_5^2 = 6$.

ii) $A_{4n+3}^2 = -2(4n+3)!\beta_{n+1},\ n \geq 1$.

iii) $A_{4n+2}^2 = 1 + \sum_{k=0}^{n-1} \frac{(4n+2)!}{(4k+2)!}\Gamma_{n-k} + 2\sum_{k=0}^{n-1} \frac{(4n+2)!}{(4k+3)!}\beta_{n-k},\ n \geq 1$.

iv) $A_{4n+1}^2 = \frac{(4n-1)(4n)}{2} + \sum_{k=0}^{n-2} \frac{(4n+1)!}{2(4k+2)!(4k+5)}\Gamma_{n-k-1} + \sum_{k=0}^{n-2} \frac{(4n+1)!}{(4k+3)!(4k+6)}\beta_{n-k-1}$, $n \geq 2$.

v) $A_{4n}^2 = (4n)!\Gamma_n,\ n \geq 1$.

Proof. *i)* Trivial.

ii) This is Theorem 7.6.

iii) In this case we remove y_{n+1} from GZ_n and the factor $1 - z_{2n+1}$ dissapears. Then, the volume of the corresponding order polytope is

$$\int_0^1 \cdots \int_0^1 \min(z_{2n}, z_{2n+1})\,(1 - \max(z_{2n-1}, z_{2n})) \cdots \min(z_2, z_3)\,(1 - \max(z_1, z_2))\, z_1 d\overrightarrow{z} =$$

$$= \cdots = \int_0^1 a^{(n)}(z_{2n+1}) dz_{2n+1} = \sum_{k=0}^{4n+1} \frac{a_k^{(n)}}{k+1}.$$

Now by Lemma 7.4 iv) and v) we get

$$\sum_{k=0}^{4n+1} \frac{a_k^{(n)}}{k+1} = \frac{1}{(4n+2)!} + \sum_{k=0}^{n-1} \frac{1}{(4k+2)!}\Gamma_{n-k} + 2\sum_{k=0}^{n-1} \frac{1}{(4k+3)!}\beta_{n-k}.$$

iv) In this case we remove y_{n+1}, z_{2n+1} from GZ_n and the factors $1 - z_{2n+1}$ and $\min(z_{2n}, z_{2n+1})$ dissapear. But it appears z_{2n}. Consequently, the volume is

$$\int_0^1 \cdots \int_0^1 z_{2n}\,(1 - \max(z_{2n-1}, z_{2n})) \cdots \min(z_2, z_3)\,(1 - \max(z_1, z_2))\, z_1 d\overrightarrow{z} =$$

$$= \cdots = \int_0^1 \int_0^1 z_{2n}\,(1 - \max(z_{2n-1}, z_{2n}))\, a^{(n-1)}(z_{2n-1}) dz_{2n-1} dz_{2n} =$$

$$= \int_0^1 z_{2n} \left[-\sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)} z_{2n}^{k+2} + \left(\sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)} \right) \right] dz_{2n} =$$

$$= -\sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)(k+4)} + \frac{1}{2} \left(\sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)} \right) =$$

$$= \sum_{k=0}^{4(n-1)+1} \frac{(k+2)(k+3)a_k^{(n-1)}}{2(k+1)(k+2)(k+3)(k+4)}.$$

Now by Lemma 7.4 iv) and v) we get that the last expression is

$$\frac{(4n-1)(4n)}{2(4n+1)!} + \sum_{k=0}^{n-2} \frac{(4k+3)(4k+4)}{2(4k+5)!} \Gamma_{n-k-1} + \sum_{k=0}^{n-2} \frac{(4k+4)(4k+5)}{(4k+6)!} \beta_{n-k-1}.$$

$v)$ In this case we remove $y_{n+1}, z_{2n+1}, x_{n+1}$ from GZ_n and the factors $1 - z_{2n+1}$ and $\min(z_{2n}, z_{2n+1})$ dissapear. Consequently, the volume is

$$\int_0^1 \cdots \int_0^1 (1 - \max(z_{2n-1}, z_{2n})) \cdots \min(z_2, z_3) \,(1 - \max(z_1, z_2))\, z_1 d\overrightarrow{z} =$$

$$= \cdots = \int_0^1 \int_0^1 (1 - \max(z_{2n-1}, z_{2n}))\, a^{(n-1)}(z_{2n-1}) dz_{2n-1} dz_{2n} =$$

$$= \int_0^1 \left[-\sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)} z_{2n}^{k+2} + \left(\sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)} \right) \right] dz_{2n} =$$

$$= -\sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)(k+3)} + \sum_{k=0}^{4(n-1)+1} \frac{a_k^{(n-1)}}{(k+1)(k+2)} = \sum_{k=0}^{4(n-1)+1} \frac{(k+2)a_k^{(n-1)}}{(k+1)(k+2)(k+3)}.$$

But by Lemma 7.3, this is Γ_n, and hence the result holds.

This way we get a recursive formula for A_n^2. □

Finally, Table 7.2 shows the first number of 2-alternating permutations, A_n^2.

n	1	2	3	4	5	6	7	8	9	10	11	12
A_n^2	1	1	1	3	6	26	71	413	1 456	10 576	45 541	397 023

TABLE 7.2: First A_n^2 numbers.

7.2 Applying the procedure for other families of posets

The procedure applied for the generalized fence can be successfully applied to other families of posets. We give in this section some other examples.

7.2.1 The fence or zig-zag poset. Alternating permutations

Let us introduce the family of **fences** or "zigzag posets". We will consider two cases. Let Z_{2n} denote the $2n-$element "zigzag poset" or (even) fence, with elements $y_1, y_2, \cdots, y_n, x_1, ..., x_n$ and cover relations $y_i \succ x_i, y_i \succ x_{i+1},\ i < n$ and $y_n > x_n$. Similarly, let Z_{2n+1} denote the $(2n+1)-$element "zigzag poset" or (odd) fence, with elements $y_1, y_2, \cdots, y_n, x_1, ..., x_n, x_{n+1}$ and cover relations $y_i \succ x_i, y_i \succ x_{i+1}, i = 1, ..., n$. The corresponding Hasse diagrams are displayed in Figure 7.2. Note that the number of linear extensions of these posets gives us the number of alternating permutations.

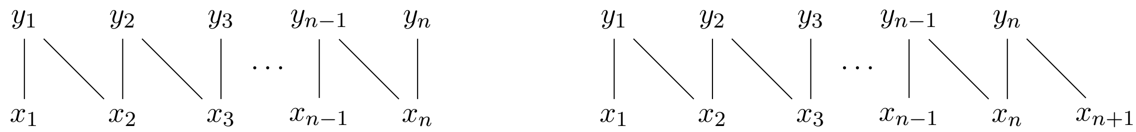


FIGURE 7.2: Fence or zigzag posets Z_{2n} (left) and Z_{2n+1} (right).

The number of linear extensions $e(Z_n)$ is known (see [5]) and is given by $e(Z_n) = E_n$, where E_n is the $n-$th Euler number. The first Euler numbers are given in Table 7.3.

n	1	2	3	4	5	6	7	8	9	10
E_n	1	1	2	5	16	61	272	1 385	7 936	50 521

TABLE 7.3: First Euler numbers.

Besides, E_{2n+1} are the Taylor coefficients of the tangent function. For this reason, they are also called tangent numbers [5]. Numbers E_{2n} are the Taylor coefficients of the secant function. For this reason, they are also called secant numbers [5]. Indeed,

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec(x) + \tan(x).$$

Let us apply the previous procedure to this family of posets. We start with the odd fence. For this, let us focus our attention on elements x_i. Note that they are not related to each other. Now, the volume of $\mathcal{O}(Z_{2n+1})$ is given by

$$\begin{aligned} Vol(\mathcal{O}(Z_{2n+1})) &= \int_0^1 ... \int_0^1 \int_0^{\min(x_1,x_2)} ... \int_0^{\min(x_n,x_{n+1})} 1 dy_1 ... dy_n dx_{n+1} ... dx_1 \\ &= \int_0^1 ... \int_0^1 \prod_{i=1}^{n} \min(x_i, x_{i+1}) dx_1 ... dx_{n+1}. \end{aligned}$$

Thus, we focus on

$$\int_0^1 \min(x,y) p(x) dx,$$

where $p(x)$ is a polynomial. This integral has been already solved in Lemma 7.1. Then, if we integrate with respect to x_{n+1}, we obtain

$$\begin{aligned} \mathrm{Vol}(\mathcal{O}(Z_{2n+1})) &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} \min(x_i, x_{i+1}) dx_{n+1} ... dx_1 \\ &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n-1} \min(x_i, x_{i+1}) p(x_n) dx_n ... dx_1, \end{aligned}$$

where $p(x_n)$ is the polynomial

$$p(x_n) = \int_0^1 \min(x_n, x_{n+1}) dx_{n+1}.$$

Next step is to solve again the integral with respect to x_n, then x_{n-1}, and so on. We are going to study this sequence of polynomials when $a^{(0)}(x) = 1$. Then, simple calculi show that the first polynomials of the sequence are:

$$\begin{aligned} a^{(1)}(x) &= x - \frac{1}{2}x^2, \\ a^{(2)}(x) &= \frac{1}{3}x - \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4. \end{aligned}$$

This defines a sequence $a^{(0)}, a^{(1)}, ...$ of polynomials that can be obtained recursively. Next lemma shows several facts about the polynomials in this sequence.

Lemma 7.8. *Let $a^{(0)}(x) = 1$ and $a^{(k+1)}(x)$ be defined recursively as*

$$a^{(k+1)}(x) := \int_0^1 \min(x,y) a^{(k)}(y) dy.$$

Then, the following facts hold:

i) $a^{(n)}(x)$ *is a polynomial of degree* $2n$.

ii) *If* $n \geq 1$, *then* $a_0^{(n)} = 0$.

iii) $a_{k+1}^{(n)} = -\dfrac{1}{k(k+1)} a_{k-1}^{(n-1)}$, $\forall n \geq 1$ *and* $k \in \{1, 2, \cdots, 2n-1\}$.

iv) $a_{2n}^{(n)} = (-1)^n \dfrac{1}{(2n)!}$ *and* $a_{2k}^{(n)} = 0$, *for* $0 \leq k \leq n-1$.

v) $a_{2k+1}^{(n)} = (-1)^k \dfrac{1}{(2k+1)!} \mathcal{T}_{n-k}$, *where* $0 \leq k \leq n-1$ *and* $\mathcal{T}_k := a_1^{(k)}$.

Proof. i) By Lemma 7.1 we conclude that $a^{(n)}(x)$ has degree two units greater than $a^{(n-1)}$. Hence, the result holds.

ii) Trivial by Lemma 7.1.

iii) It follows again from Lemma 7.1.

iv) By iii) $a_{2k}^{(n)} = -\dfrac{1}{2k(2k-1)} a_{2(k-1)}^{(n-1)}$. Hence,

$$a_{2n}^{(n)} = (-1)\frac{1}{2n(2n-1)} a_{2(n-1)}^{(n-1)} = (-1)^2 \frac{1}{2n(2n-1)} \cdot \frac{1}{(2n-2)(2n-3)} a_{2(n-2)}^{(n-2)} = \ldots$$

$$= (-1)^n \frac{1}{2n(2n-1)} \cdot \frac{1}{(2n-2)(2n-3)} \cdot \frac{1}{(2)(1)} \cdots a_0^{(0)} = (-1)^n \frac{1}{(2n)!}.$$

Now, we use ii) for the case $0 \leq k \leq n-1$:

$$a_{2k}^{(n)} = (-1)\frac{1}{2k(2k-1)} \cdot (-1)\frac{1}{(2k-2)(2k-3)} \cdots a_0^{(n-k)} = 0.$$

v) Applying iii) and induction, we obtain

$$a_{2k+1}^{(n)} = (-1)\frac{1}{(2k+1)(2k)} a_{2k-1}^{(n-1)} = (-1)^2 \frac{1}{(2k+1)(2k)} \cdot \frac{1}{(2k-1)(2k-2)} a_{2k-3}^{(n-2)} = \ldots$$

$$= (-1)^k \frac{1}{(2k+1)(2k)} \cdot \frac{1}{(2k-1)(2k-2)} \cdots \frac{1}{(2)(1)} a_1^{(n-k)} = (-1)^k \frac{1}{(2k+1)!} a_1^{(n-k)}.$$

Hence, the lemma holds. □

Theorem 7.9. *Given the odd fence* Z_{2n+1}, *it follows*

$$e(Z_{2n+1}) = (2n+1)! \mathcal{T}_{n+1}.$$

Proof. Note that $e(Z_{2n+1}) = (2n+1)!\mathrm{Vol}(\mathcal{O}(Z_{2n+1}))$. Now,

$$\begin{aligned}
\mathrm{Vol}(\mathcal{O}(Z_{2n+1})) &= \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^{\min(x_1,x_2)} \int_0^{\min(x_2,x_3)} \cdots \int_0^{\min(x_n,x_{n+1})} 1 dx_{n+1} dx_n \cdots dx_1 \\
&= \int_0^1 \int_0^1 \cdots \int_0^1 \min(x_1,x_2) \min(x_2,x_3) \cdots \min(x_n,x_{n+1}) a^{(0)}(x_{n+1}) dx_{n+1} dx_n \cdots dx_1.
\end{aligned}$$

Now, we use $\int_0^1 \min(x_i, x_{i+1}) a^{(n-i)}(x_{i+1}) dx_{i+1} = a^{(n-i+1)}(x_i)$, so we obtain,

$$\begin{aligned}
\mathrm{Vol}(\mathcal{O}(Z_{2n+1})) &= \int_0^1 \int_0^1 \cdots \int_0^1 \min(x_1,x_2) \min(x_2,x_3) \cdots \min(x_{n-1},x_n) a^{(1)}(x_n) dx_n \cdots dx_1 \\
&= \int_0^1 \int_0^1 \cdots \int_0^1 \min(x_1,x_2) \min(x_2,x_3) \cdots \min(x_{n-2},x_{n-1}) a^{(2)}(x_{n-1}) dx_{n-1} \cdots dx_1 \\
&= \ldots \\
&= \int_0^1 a^{(n)}(x_1) dx_1
\end{aligned}$$

Finally,

$$\mathrm{Vol}(\mathcal{O}(Z_{2n+1})) = \int_0^1 a^{(n)}(x_1) dx_1 = \left[\sum_{k=0}^{2n} \frac{a_k^{(n)}}{k+1} x_1^{k+1} \right]_0^1 = \sum_{k=0}^{2n} \frac{a_k^{(n)}}{k+1} = \mathcal{T}_{n+1},$$

where the last equality arises from Lemma 7.1. Hence, the result holds. □

Now, let us obtain the sequence of coefficients $\mathcal{T}_0, \mathcal{T}_1, \ldots$ For this sequence, the following can be shown.

Lemma 7.10. *The coefficients $\mathcal{T}_n$ can be obtained recursively by:*

$$\begin{cases} \mathcal{T}_0 = 0, \ \mathcal{T}_1 = 1 \\ \mathcal{T}_{n+1} = \dfrac{(-1)^n}{(2n+1)!} + \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+2)!} \mathcal{T}_{n-k} \end{cases}$$

Proof. By Lemma 7.1 we know that $\mathcal{T}_{n+1} = \sum_{k=0}^{2n} \frac{a_k^{(n)}}{k+1}$. Now, applying Lemma 7.8 iv), we get

$$\begin{aligned}\mathcal{T}_{n+1} &= 0+\frac{a_1^{(n)}}{2}+0+\frac{a_3^{(n)}}{4}+\cdots+\frac{a_{2n-1}^{(n)}}{2n}+\frac{a_{2n}^{(n)}}{2n+1}\\ &= 0+\frac{a_1^{(n)}}{2}+0+\frac{a_3^{(n)}}{4}+\cdots+\frac{a_{2n-1}^{(n)}}{2n}+(-1)^n\frac{1}{(2n+1)(2n)!}\end{aligned}$$

Finally, by Lemma 7.8 v),

$$\mathcal{T}_{n+1}=0+\frac{\mathcal{T}_n}{2}+0-\frac{\mathcal{T}_{n-1}}{4!}+\cdots+(-1)^{n-1}\frac{\mathcal{T}_1}{(2n)!}+(-1)^n\frac{1}{(2n+1)!}.$$

This finishes the proof. □

Table 7.4 shows the first values of $\mathcal{T}_n$.

n	1	2	3	4	5	6	7	8
$\mathcal{T}_n$	1	$\frac{1}{3}$	$\frac{2}{15}$	$\frac{17}{315}$	$\frac{62}{2\,835}$	$\frac{1\,382}{155\,925}$	$\frac{21\,844}{6\,081\,075}$	$\frac{929\,569}{638\,512\,875}$

TABLE 7.4: First $\mathcal{T}_n$ values.

From the last linear recurrence, it is possible to show that $E_{2n+1}=(2n+1)!\mathcal{T}_{n+1}$.

Let us work now with the even part. In this case,

$$\begin{aligned}Vol(\mathcal{O}(Z_{2n})) &= \int_0^1 ... \int_0^1 \int_0^{y_1} \int_0^{\min(y_1,y_2)} ... \int_0^{\min(y_{n-1},y_n)} 1 dx_n...dx_1dy_1...dy_n\\ &= \int_0^1 ... \int_0^1 y_1 \prod_{i=1}^{n-1}\min(y_i,y_{i+1})dy_1...dy_n.\end{aligned}$$

Then, if we integrate with respect to y_1, we obtain

$$\begin{aligned}\mathrm{Vol}(\mathcal{O}(Z_{2n})) &= \int_0^1\cdots\int_0^1 y_1\prod_{i=1}^{n-1}\min(y_i,y_{i+1})dy_n...dy_1\\ &= \int_0^1\cdots\int_0^1 \prod_{i=1}^{n-1}\min(y_i,y_{i+1})p(y_2)dy_{n-1}...dy_1,\end{aligned}$$

where $p(y_2)$ is the polynomial

$$p(y_2)=\int_0^1 y_1\min(y_1,y_2)dy_1.$$

We are going to work with the same polynomial recurrence as above, but in this case we start from $b^{(0)}(x) = x$. Then, the first terms in the sequence are:

$$\begin{aligned} b^{(1)}(x) &= \frac{x}{2} - \frac{x^3}{6}, \\ b^{(2)}(x) &= \frac{5x}{6 \cdot 4} - \frac{x^3}{4 \cdot 3} + \frac{x^5}{6 \cdot 5 \cdot 4}. \end{aligned}$$

Next lemma provides some basic properties of this particular sequence. The proof is very similar to that of Lemma 7.8 and we omit it.

Lemma 7.11. *Let $b^{(0)}(x) = x$ and $b^{(k+1)}(x)$ be defined recursively as*

$$b^{(k+1)}(x) := \int_0^1 \min(x, y) b^{(k)}(y) dy.$$

Then, the following holds:

i) $b^{(n)}(x)$ *is a polynomial of degree* $2n + 1$.

ii) If $n \geq 0$, *then* $b_0^{(n)} = 0$.

iii) $b_{k+1}^{(n)} = -\dfrac{1}{k(k+1)} b_{k-1}^{(n-1)}$, $\forall n \geq 1$ *and* $k \in \{1, 2, \cdots, 2n+1\}$.

iv) $b_{2k}^{(n)} = 0$, *for* $0 \leq k \leq n$.

v) $b_{2k+1}^{(n)} = (-1)^k \dfrac{1}{(2k+1)!} \mathcal{S}_{n-k}$, *where* $0 \leq k \leq n$ *and* $\mathcal{S}_k := b_1^{(k)}$.

Theorem 7.12. *Consider* Z_{2n}. *Then,*

$$e(Z_{2n}) = (2n)! \mathcal{S}_n.$$

Proof. Proceeding as in Theorem 7.9, we have:

$$\begin{aligned} \mathrm{Vol}(\mathcal{O}(Z_{2n})) &= \int_0^1 \cdots \int_0^1 \int_0^{\min(y_1, y_2)} \cdots \int_0^{\min(y_{n-1}, y_n)} \int_0^{y_1} 1 dy_n \cdots dy_1 dx_n \cdots dx_1 \\ &= \int_0^1 \cdots \int_0^1 \min(y_1, y_2) \min(y_2, y_3) \cdots \min(y_{n-1}, y_n) y_1 dy_1 \cdots dy_n \end{aligned}$$

Now, we use $\int_0^1 \min(x_i, x_{i+1}) b^{(n-i)}(x_{i+1}) dx_{i+1} = b^{(n-i+1)}(x_i)$, so we obtain,

$$
\begin{aligned}
\mathrm{Vol}(\mathcal{O}(Z_{2n})) &= \int_0^1 \cdots \int_0^1 \min(y_2, y_3) \cdots \min(y_{n-1}, y_n) b^{(1)}(y_2) dy_2 \cdots dy_n \\
&= \int_0^1 \cdots \int_0^1 \min(y_3, y_4) \min(x_4, x_5) \cdots \min(y_{n-1}, y_n) b^{(2)}(y_3) dy_3 \cdots dy_n \\
&= \ldots \\
&= \int_0^1 b^{(n-1)}(y_n) dy_n.
\end{aligned}
$$

Finally,

$$
\mathrm{Vol}(\mathcal{O}(Z_{2n})) = \left[\sum_{k=0}^{2n-1} \frac{b_k^{(n-1)}}{k+1} y_n^{k+1} \right]_0^1 = \sum_{k=0}^{2n-1} \frac{b_k^{(n-1)}}{k+1} = \mathcal{S}_n.
$$

Hence, the result holds. □

Let us consider the sequence $\mathcal{S}_0, \mathcal{S}_1, \ldots$ For this sequence, the following holds.

Lemma 7.13. *The sequence $\mathcal{S}_0, \mathcal{S}_1, \ldots$ can be defined via the following recurrence:*

$$
\begin{cases}
\mathcal{S}_0 = 1 \\
\mathcal{S}_{n+1} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+2)!} \mathcal{S}_{n-k}
\end{cases}
$$

Proof. By Lemma 7.1 we know

$$
\mathcal{S}_{n+1} = \sum_{k=0}^{2n+1} \frac{b_k^{(n)}}{k+1} = \frac{b_0^{(n)}}{1} + \frac{b_1^{(n)}}{2} + \frac{b_2^{(n)}}{3} + \ldots
$$

Now, applying Lemma 7.11 ii), we conclude that $b_k^{(n)} = 0$ if k is even. Thus,

$$
\mathcal{S}_{n+1} = \frac{b_1^{(n)}}{2} + \frac{b_3^{(n)}}{4} + \cdots + \frac{b_{2n+1}^{(n)}}{2n+2}.
$$

Finally, applying Lemma 7.11 v),

$$
\mathcal{S}_{n+1} = \frac{\mathcal{S}_n}{2} - \frac{\mathcal{S}_{n-1}}{4!} + \cdots + (-1)^n \frac{\mathcal{S}_0}{(2n+2)!}.
$$

Hence, the result holds. □

n	1	2	3	4	5	6	7	8
$\mathcal{S}_n$	$\frac{1}{2}$	$\frac{5}{24}$	$\frac{61}{720}$	$\frac{277}{8\,064}$	$\frac{50\,521}{3\,628\,800}$	$\frac{540\,553}{95\,800\,320}$	$\frac{199\,360\,981}{87\,178\,291\,200}$	$\frac{3\,878\,302\,429}{4\,184\,557\,977\,600}$

TABLE 7.5: First $\mathcal{S}_n$ values.

Table 7.5 shows the first values of $\mathcal{S}_n$.

From the last linear recurrence, it is possible to show that $E_{2n} = (2n)!\mathcal{S}_n$.

7.2.2 The butterfly poset

Let us consider the **butterfly poset**, a poset of four elements x_1, x_2, y_1, y_2 such that $x_i \prec y_j, \forall i, j$. We will denote this poset by BF_1, and we will consider a sequence of n butterflies posets such that for butterfly j, $1 < j < n$ one of the elements y is common with butterfly $j-1$ and the other element y is common with butterfly $j+1$.

Thus defined, poset BF_n has $n+1$ elements type y and $2n$ elements type x and the relation $\prec$ is given by

$$y_i \succ x_{2i-3},\ y_i \succ x_{2i-2},\ y_i \succ x_{2i-1},\ y_i \succ x_{2i}, \quad i \neq 1, n{+}1, \quad y_1 \succ x_1,\ y_1 \succ x_2,\ y_{n+1} \succ x_{2n-1},\ y_{n+1} \succ x_{2n}.$$

Figure 7.3 shows the Hasse diagram of BF_1 and BF_n.

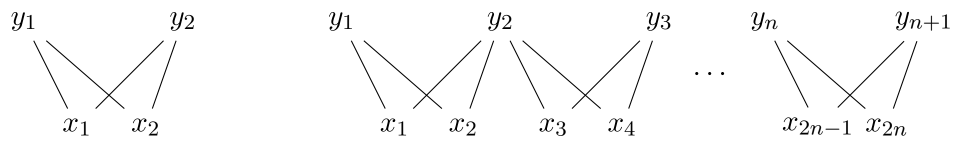


FIGURE 7.3: Butterfly BF_1 (left) and Butterfly BF_n (right).

We will apply our procedure to compute the number of linear extensions of this family of posets. In this case,

$$\begin{aligned}
\mathrm{Vol}(\mathcal{O}(BF_n)) &= \int_0^1 \dots \int_0^1 \int_0^{\min(y_1,y_2)} \int_0^{\min(y_1,y_2)} \dots \int_0^{\min(y_{n-1},y_n)} \int_0^{\min(y_{n-1},y_n)} 1 dx_{2n} \dots dx_1 dy_1 \dots dy_{n+1} \\
&= \int_0^1 \dots \int_0^1 \prod_{i=1}^{n} \min^2(y_i, y_{i+1}) dy_1 \dots dy_{n+1}.
\end{aligned}$$

Then, if we integrate with respect to y_1, we obtain

$$\begin{aligned}\mathrm{Vol}(\mathcal{O}(BF_n)) &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} \min{}^2(y_i, y_{i+1}) dy_1 ... dy_{n+1} \\ &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n-1} \min{}^2(y_i, y_{i+1}) p(y_2) dy_2 ... dy_{n+1},\end{aligned}$$

where $p(y_2)$ is the polynomial

$$p(y_2) = \int_0^1 \min{}^2(y_1, y_2) dy_1.$$

Lemma 7.14. *Let $p(x)$ be a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, ..., a_n \in \mathbb{R}.$$

Then,

$$\int_0^1 min^2(x, y) p(x) dx = -\sum_{k=0}^{n} \frac{2a_k}{(k+1)(k+3)} y^{k+3} + \left(\sum_{k=0}^{n} \frac{a_k}{k+1} \right) y^2.$$

Proof. It is very similar to the proof of Lemma 7.1, so we omit it. □

Now suppose that we consider an initial polynomial $c^{(0)}(x)$ and we define recursively the polynomial $c^{(k+1)}(x)$ as

$$c^{(k+1)}(x) := \int_0^1 \min{}^2(x, y) c^{(k)}(y) dy.$$

We are going to study this sequence of polynomials when $c^{(0)}(x) = 1$. Then, simple calculi show that the first terms of this sequence are:

$$\begin{aligned}c^{(1)}(x) &= x^2 - \frac{2}{3} x^3, \\ c^{(2)}(x) &= \frac{1}{6} x^2 - \frac{2}{15} x^5 + \frac{1}{18} x^6.\end{aligned}$$

Let us denote

$$a!!! := a(a-3)(a-6)...$$

By convention, $1!!! = 1, 2!!! = 2$ and $3!!! = 3$. The basic properties of sequence $c^0(x), c^1(x), ...$ are given in next lemma, whose proof is completely similar to the proofs seen previously.

Lemma 7.15. *Let $c^{(0)}(x) = 1$ and $c^{(k+1)}(x)$ be defined recursively as*

$$c^{(k+1)}(x) := \int_0^1 min^2(x,y)c^{(k)}(y)dy.$$

Then, the following facts follow:

i) $c^{(n)}(x)$ *is a polynomial of degree* $3n$.

ii) *If* $n \geq 1$, *then* $c_0^{(n)} = c_1^{(n)} = 0$.

iii) $c_k^{(n)} = -\frac{2}{k(k-2)}c_{k-3}^{(n-1)}$, $\forall n \geq 1$ *and* $k \in \{3, 4, \cdots, 3n-1\}$.

iv) $c_{3n}^{(n)} = (-1)^n \frac{2^n(3n-1)!!!}{(3n)!}$.

v) $c_{3k}^{(n)} = c_{3k+1}^{(n)} = 0$ *and* $c_{3k+2}^{(n)} = (-1)^k \frac{2^{k+1}(3k+1)!!!}{(3k+2)!}\mathcal{R}_{n-k}$, *where* $1 \leq k \leq n-1$ *and* $\mathcal{R}_k := c_2^{(k)}$

Theorem 7.16. *The number of linear extensions of BF_n is given by*

$$e(BF_n) = (3n+1)!\mathcal{R}_{n+1}.$$

Proof. We have

$$\begin{aligned}
\text{Vol}(\mathcal{O}(BF_n)) &= \int_0^1\int_0^1 \cdots \int_0^1 \min^2(y_1,y_2)\min^2(y_2,y_3)\cdots\min^2(y_{n-1},y_n)c^{(0)}(y_{n+1})dy_1...dy_{n+1} \\
&= \int_0^1\int_0^1 \cdots \int_0^1 \min^2(y_1,y_2)\min^2(y_2,y_3)\cdots\min^2(y_{n-2},y_{n-1})c^{(1)}(y_n)dy_1...dy_n \\
&= \int_0^1 c^{(n)}(y_1)dy_1 = \sum_{k=0}^{3n}\frac{c_k^{(n)}}{k+1} = \mathcal{R}_{n+1},
\end{aligned}$$

where the last equality arises from Lemma 7.14. □

The solution is given by a linear recurrence in $\mathcal{R}_n$. Thus defined, we have a sequence $\mathcal{R}_0, \mathcal{R}_1, ...$ Next lemma gives some insight about this sequence.

Lemma 7.17. *The sequence $\mathcal{R}_n$ is given by:*

$$\begin{cases}\mathcal{R}_0 = 0,\ \mathcal{R}_1 = 1 \\ \mathcal{R}_{n+1} = (-2)^n\frac{(3n-1)!!!}{(3n+1)!} + \frac{\mathcal{R}_n}{3} + \sum_{k=1}^{n-1}(-1)^k\frac{2^{k+1}(3k+1)!!!}{(3(k+1))!}\mathcal{R}_{n-k}\end{cases}$$

Proof. By Lemma 7.14 we know that

$$\mathcal{R}_{n+1} = \sum_{k=0}^{3n} \frac{c_k^{(n)}}{k+1} = \frac{c_2^{(n)}}{3} + \frac{c_3^{(n)}}{4} + \frac{c_4^{(n)}}{5} + \frac{c_5^{(n)}}{6} + \cdots + \frac{c_{3n-1}^{(n)}}{3n} + \frac{c_{3n}^{(n)}}{3n+1}.$$

Applying now Lemma 7.15 v), we see that terms $c_{3k}^{(n)} = 0, 0 \le k \le n-1$, and $c_{3k-1}^{(n)} = 0, 0 \le k \le n$. Moreover, by Lemma 7.15 iv), $c_{3n}^{(n)} = (-1)^n \frac{2^n(3n-1)!!!}{(3n)!}$, so that

$$\mathcal{R}_{n+1} = \sum_{k=0}^{3n} \frac{c_k^{(n)}}{k+1} = \frac{c_2^{(n)}}{3} + \frac{c_5^{(n)}}{6} + \cdots + \frac{c_{3n-1}^{(n)}}{3n} + (-1)^n \frac{2^n(3n-1)!!!}{(3n+1)!}.$$

Finally, by Lemma 7.15 v), we obtain that $c_{3k+2}^{(n)} = (-1)^k \frac{2^{k+1}(3k+1)!!!}{(3k+2)!} \mathcal{R}_{n-k}$ and hence the result holds. □

Table 7.6 provides the first values of $\mathcal{R}_n$ and $e(BF_n)$.

n	1	2	3	4	5
$\mathcal{R}_n$	1	$\frac{1}{6}$	$\frac{13}{315}$	$\frac{17}{1\,620}$	$\frac{9\,301}{3\,474\,900}$
$e(BF_n)$	4	208	38 080	16 667 392	$\cdots$

TABLE 7.6: First values of $\mathcal{R}_n$ and $e(BF_n)$.

7.2.3 The Y_n fence

Let us now consider the next family of Y **fences**. The poset Y_n has $3n+1$ elements, $x_1, ..., x_n, z_1, ..., z_n, y_1, ..., y_{n+1}$ and the order relation is given by $x_i \prec z_i, z_i \prec y_i, z_i \prec y_{i+1}, i = 1, ..., n$. Figure 7.4 shows the Hasse diagram of this poset.

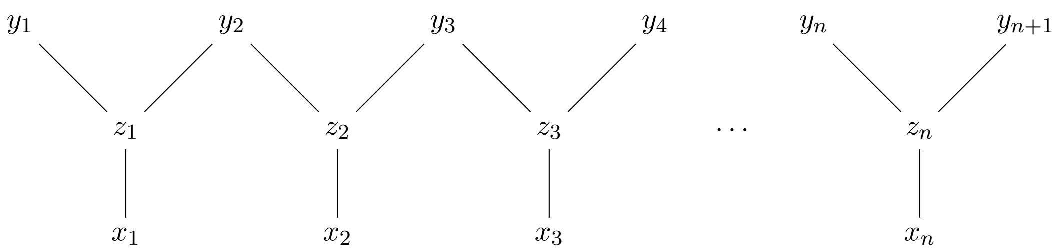


FIGURE 7.4: Fence Y_n.

We proceed as usual. First, let us note the shape of this integral,

$$\mathrm{Vol}(\mathcal{O}(Y_n)) = \int_0^1 \cdots \int_0^1 \int_0^{z_1} \cdots \int_0^{z_n} \int_{z_1}^1 \int_{\max(z_1,z_2)}^1 \int_{\max(z_1,z_2)}^1 \cdots \int_{z_n}^1 1 d\overrightarrow{y}\, d\overrightarrow{x}\, d\overrightarrow{z} =$$

$$\int_0^1 \cdots \int_0^1 (1-z_1) z_1 \left(1 - \max(z_1, z_2)\right) z_2 \left(1 - \max(z_2, z_3)\right) \cdots z_{n-1} \left(1 - \max(z_{n-1}, z_n)\right) z_n (1-z_n) d\overrightarrow{z}.$$

Thus, we focus on

$$\int_0^1 x \left[1 - \max(x, y)\right] p(x) dx,$$

where $p(x)$ is a polynomial.

Lemma 7.18. *Let $p(x)$ be a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, ..., a_n \in \mathbb{R}.$$

Then,

$$\int_0^1 x \left[1 - \max(x, y)\right] p(x) dx = -\sum_{k=0}^{n} \frac{a_k}{(k+2)(k+3)} y^{k+3} + \sum_{k=0}^{n} \frac{a_k}{(k+2)(k+3)}.$$

Proof. Similar to Lemma 7.3. □

Now suppose that we consider an initial polynomial $d^{(0)}(x)$ and we define recursively the polynomial $d^{(k+1)}(x)$ as

$$d^{(k+1)}(x) := \int_0^1 y \left[1 - \max(x, y)\right] d^{(k)}(y) dy.$$

We obtain then a sequence of polynomials $d^{(0)}(x), d^{(1)}(x), d^{(2)}(x), ...$ We are going to study this sequence of polynomials when $d^{(0)}(x) = 1 - x$. Then, simple calculi show that the first terms of this sequence are:

$$\begin{aligned} d^{(1)}(x) &= \frac{1}{12} - \frac{1}{6} x^3 + \frac{1}{12} x^4, \\ d^{(2)}(x) &= \frac{13}{1\,260} - \frac{1}{72} x^3 + \frac{1}{180} x^6 - \frac{1}{504} x^7. \end{aligned}$$

Next lemma gives more insight about this sequence. The proof of this lemma and the next ones are omitted.

Lemma 7.19. *Let $d^{(0)}(x) = 1 - x$ and $d^{(k+1)}(x)$ be defined recursively by*

$$d^{(k+1)}(x) := \int_0^1 y\,[1 - \max(x,y)]\, d^{(k)}(y)dy.$$

Then we have:

i) $d^{(n)}(x)$ *is a polynomial of degree* $3n+1$.

ii) *If* $n \geq 1$, *then* $d_1^{(n)} = d_2^{(n)} = 0$.

iii) $d_k^{(n)} = -\dfrac{1}{k(k-1)} d_{k-3}^{(n-1)}$, $\forall n \geq 1$ *and* $k \in \{3, \cdots, 3n+1\}$.

iv) $d_{3n+1}^{(n)} = (-1)^{n+1}\dfrac{(3n-1)!!!}{(3n+1)!}$, $d_{3n}^{(n)} = (-1)^n \dfrac{(3n-2)!!!}{(3n)!}$ *and* $d_{3k+2}^{(n)} = d_{3k+1}^{(n)} = 0$, *for* $0 \leq k \leq n-1$.

v) $d_{3k}^{(n)} = (-1)^k \dfrac{(3k-2)!!!}{(3k)!}\Upsilon_{n-k}$, *where* $0 \leq k \leq n-1$ *and* $\Upsilon_k := d_0^{(k)}$.

Lemma 7.20. *The sequence* $\Upsilon_0, \Upsilon_1, ..., \Upsilon_n, ...$ *is given by:*

$$\begin{cases} \Upsilon_0 = 1, \;\; \Upsilon_1 = 1/12 \\ \Upsilon_{n+1} = (-1)^{n+1}\dfrac{(3n+2)!!!}{(3n+4)!} + (-1)^n \dfrac{(3n+1)!!!}{(3n+3)!} + \displaystyle\sum_{k=0}^{n-1} (-1)^k \dfrac{(3k+1)!!!}{(3k+3)!}\Upsilon_{n-k} \end{cases}$$

Theorem 7.21.

$$e(Y_n) = (3n+1)!\Upsilon_n.$$

Proof. Since $\mathrm{Vol}(\mathcal{O}(P)) = \dfrac{e(P)}{|P|!}$, then $e(Y_n) = (3n+1)!\mathrm{Vol}(\mathcal{O}(Y_n))$. Now, the volume is given by

$$\int_0^1 \cdots \int_0^1 (1-z_1)z_1\,(1-\max(z_1,z_2))\,z_2\,(1-\max(z_2,z_3)) \cdots z_{n-1}\,(1-\max(z_{n-1},z_n))\,z_n(1-z_n)d\vec{z} =$$

$$= \int_0^1 (1-z_1)z_1 d^{n-1}(z_1)dz_1 = \sum_{k=0}^{3(n-1)+1} \frac{d_k^{(n-1)}}{k+2} - \sum_{k=0}^{3(n-1)+1} \frac{d_k^{(n-1)}}{k+3} = \sum_{k=0}^{3(n-1)+1} \frac{d_k^{(n-1)}}{(k+2)(k+3)} = \Upsilon_n,$$

where the last equality arises from Lemma 7.18. □

The first values of $e(Y_n)$ are given in Table 7.7.

n	1	2	3	4	5
Υ_n	$\frac{1}{12}$	$\frac{13}{1\,260}$	$\frac{17}{12\,960}$	$\frac{9\,301}{55\,598\,400}$	$\frac{398\,641}{18\,681\,062\,400}$
$e(Y_n)$	2	52	4 760	1 041 712	446 477 920

Table 7.7: First values of Υ_n and $e(Y_n)$.

7.3 An application to Fuzzy Measures: the rhombus poset

Suppose the following situation. There are to be elections, and n political parties participate. We will denote them by $1, \ldots, n$. Suppose that they are ordered according to their ideology, such that the most left-wing party is the party 1 and the most right-wing party is the party n. In order to form a goverment we suppose that different coalitions must adhere to the following rules:

- A party can govern alone. This means, we allow the coalitions:

$$\{1\}, \{2\}, \ldots, \{n\}.$$

- A party can form a coalition with the party immediately to the left or right. This means, we allow as possible coalitions:

$$\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}.$$

- A coalition of three parties can be formed, if these parties are consecutive according to their order. This means, we allow coalitions

$$\{1, 2, 3\}, \{2, 3, 4\}, \ldots, \{n-2, n-1, n\}.$$

- A coalition of more than three parties is considered to be too unstable to be formed.

Besides, we consider that the importance of a coalition is greater the more parties are in it, i.e. we consider that monotoncity holds, so we can use fuzzy measures. In the rest of this section we are goint to study the poset associted to this collection of feasible coalitions, that we will call the rhombus posets. In what follows, we are going to see that integral techniques can be applied for the rhombus posets.

Let us now consider the next family of **rhombic fences** RF_n. These posets have $3n+1$ elements $\{x_1, ..., x_n, z_1, ..., z_{n+1}, y_1, ..., y_n\}$ and the order relation is given by $x_i \prec z_i, x_i \prec$

$z_{i+1}, y_i \succ z_i, y_i \succ z_{i+1}, i = 1, ..., n$. Figure 7.5 shows the Hasse diagram of RF_n. Note that this poset fits the requirements done in the situation described above about political parties. Here, we have changed the notation a little, being x_i the elements associated to one party coalitions, z_i for the coalitions of two parties and y_i for the ones with three parties.

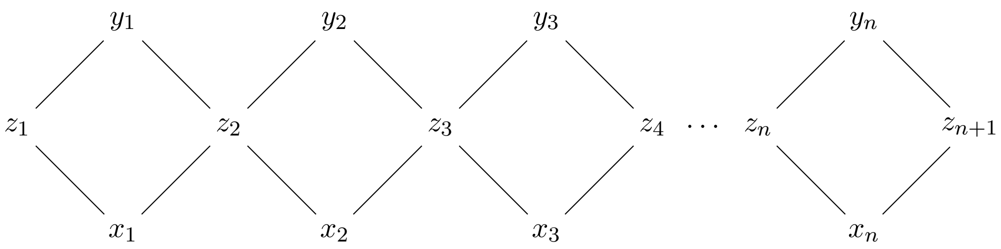


FIGURE 7.5: Rhombic Fence RF_n

We proceed as usual. First, the volume is given by

$$\mathrm{Vol}(\mathcal{O}(R_n)) = \int_0^1 \cdots \int_0^1 (1 - \max(z_1, z_2)) \min(z_1, z_2) \cdots (1 - \max(z_n, z_{n+1})) \min(z_n, z_{n+1}) d\overrightarrow{z}.$$

Thus, we focus on

$$\int_0^1 \min(x, y) \left[1 - \max(x, y)\right] p(x) dx,$$

where $p(x)$ is a polynomial.

Lemma 7.22. *Let $p(x)$ be a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, ..., a_n \in \mathbb{R}.$$

Then,

$$\int_0^1 \min(x, y) \left[1 - \max(x, y)\right] p(x) dx = -\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} y^{k+2} + \left(\sum_{k=0}^{n} \frac{a_k}{(k+1)(k+2)} \right) y.$$

Proof. Similar to Lemma 7.3. □

Now suppose that we consider an initial polynomial $e^{(0)}(x)$ and we define recursively the polynomial $e^{(k+1)}(x)$ as

$$e^{(k+1)}(x) := \int_0^1 \min(x,y)\,[1-\max(x,y)]\,e^{(k)}(y)dy.$$

We obtain then a sequence of polynomials $e^{(0)}(x), e^{(1)}(x), e^{(2)}(x), ...$ In our case, we are interested in this sequence when $e^{(0)}(x) = 1$. Then, simple calculi show that the first terms of this sequence are:

$$\begin{array}{rcl} e^{(1)}(x) & = & \frac{1}{2}x - \frac{1}{2}x^2, \\ e^{(2)}(x) & = & \frac{1}{24}x - \frac{1}{12}x^3 + \frac{1}{24}x^4. \end{array}$$

Next lemma gives more insight about this sequence. Again, for the sake of dynamicity, the proof is omitted.

Lemma 7.23. *Let $e^{(0)}(x) = 1$ and $e^{(k+1)}(x)$ be defined recursively by*

$$e^{(k+1)}(x) := \int_0^1 \min(x,y)\,[1-\max(x,y)]\,e^{(k)}(y)dy.$$

Then, we have:

i) $e^{(n)}(x)$ *is a polynomial of degree* $2n$.

ii) If $n \geq 1$, *then* $e_0^{(n)} = 0$.

iii) $e_{k+1}^{(n)} = -\dfrac{1}{k(k+1)} e_{k-1}^{(n-1)}$, $\forall n \geq 1$ *and* $k \in \{1, 2, \cdots, 2n-1\}$.

iv) $e_{2n}^{(n)} = (-1)^n \dfrac{1}{(2n)!}$ *and* $e_{2k}^{(n)} = 0$, *for* $0 \leq k \leq n-1$.

v) $e_{2k+1}^{(n)} = (-1)^k \dfrac{1}{(2k+1)!} \Theta_{n-k}$, *where* $0 \leq k \leq n-1$ *and* $\Theta_k := e_1^{(k)}$.

Lemma 7.24. *The value Θ_n can be derived recursively by:*

$$\begin{cases} \Theta_0 = 0, \;\; \Theta_1 = 1/2 \\ \Theta_{n+1} = (-1)^n \dfrac{1}{(2n+2)!} + \sum_{k=0}^{n-1} \dfrac{(-1)^k}{(2k+3)!} \Theta_{n-k} \end{cases}$$

Proof. The values Θ_0 and Θ_1 are trivial. By Lemmas 7.22 and 7.23 we know that

$$\begin{aligned}\Theta_{n+1} &= \sum_{k=0}^{2n} \frac{e_k^{(n)}}{(k+1)(k+2)} \\ &= 0 + \frac{e_1^{(n)}}{2\cdot 3} + 0 + \frac{e_3^{(n)}}{4\cdot 5} + \cdots + \frac{e_{2n-1}^{(n)}}{(2n)\cdot(2n+1)} + \frac{e_{2n}^{(n)}}{(2n+1)(2n+2)} \\ &= 0 + \frac{\Theta_n}{3!} + 0 - \frac{\Theta_{n-1}}{5!} + \cdots + (-1)^{n-1}\frac{\Theta_1}{(2n+1)!} + (-1)^n \frac{1}{(2n+2)!}.\end{aligned}$$

Hence, the result holds. □

As it will be explained below, for this poset, we will need another sequence $\Pi_0, \Pi_1, ...$ defined by

$$\Pi_{n+1} := \sum_{k=0}^{2n} \frac{e_k^{(n)}}{k+1}.$$

This sequence can be obtained from the sequence $\Theta_0, \Theta_1, ...$ as shown in next lemma.

Lemma 7.25. *The sequence $\Pi_0, \Pi_1, ...$ can be obtained recursively by:*

$$\begin{cases} \Pi_0 = 1, \ \Pi_1 = 1 \\ \Pi_{n+1} = (-1)^n \dfrac{1}{(2n+1)!} + \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+2)!}\Theta_{n-k} \end{cases}$$

Proof. The values Π_0 and Π_1 are trivial. By Lemma 7.23 we know that

$$\Pi_{n+1} = \sum_{k=0}^{2n} \frac{e_k^{(n)}}{k+1} = 0 + \frac{e_1^{(n)}}{2} + 0 + \frac{e_3^{(n)}}{4} + \cdots + \frac{e_{2n-1}^{(n)}}{2n} + \frac{e_{2n}^{(n)}}{2n+1} =$$

$$= 0 + \frac{\Theta_n}{2!} + 0 - \frac{\Theta_{n-1}}{4!} + \cdots + (-1)^{n-1}\frac{\Theta_1}{(2n)!} + (-1)^n \frac{1}{(2n+1)!}.$$

Thus, the result holds. □

Theorem 7.26. *The number of linear extensions of RF_n is given by*

$$e(RF_n) = (3n+1)!\Pi_{n+1}.$$

Proof. Since $\mathrm{Vol}(\mathcal{O}(P)) = \dfrac{e(P)}{|P|!}$, then $e(\mathcal{RF}_n) = (3n+1)!\mathrm{Vol}(\mathcal{O}(\mathcal{RF}_n))$.

$$\mathrm{Vol}(\mathcal{O}(\mathcal{RF}_n)) = \int_0^1 \cdots \int_0^1 (1 - \max(z_1, z_2)) \min(z_1, z_2) \cdots (1 - \max(z_n, z_{n+1})) \min(z_n, z_{n+1}) d\overrightarrow{z} =$$

$$= \int_0^1 \cdots \int_0^1 (1-\max(z_1,z_2))\min(z_1,z_2)\cdots(1-\max(z_{n-1},z_n))\min(z_{n-1},z_n)e^{(1)}(z_n)d\overrightarrow{z} =$$

$$= \int_0^1 e^{(n)}(z_1)dz_1 = \left[\sum_{k=0}^{2n} \frac{e_k^{(n)}}{k+1}x_1^{k+1}\right]_0^1 = \sum_{k=0}^{2n} \frac{e_k^{(n)}}{k+1} = \Pi_{n+1}.$$

This finishes the proof. □

The first values of $e(RF_n)$ are given in Table 7.8.

n	1	2	3	4	5
Θ_n	$\frac{1}{2}$	$\frac{1}{24}$	$\frac{1}{240}$	$\frac{17}{40\,320}$	$\frac{31}{725\,760}$
Π_n	1	$\frac{1}{12}$	$\frac{1}{120}$	$\frac{17}{20\,160}$	$\frac{31}{362\,880}$
$e(RF_n)$	2	42	3 060	531 960	$\cdots$

TABLE 7.8: First values of Θ_n, Π_n and $e(RF_n)$.

7.4 Towards a general framework

In the previous sections we have obtained expressions for the number of linear extensions of several families of posets. As explained, the basic idea is to find a formula computing the volume of the corresponding order polytope, and then use the relationship between the volume of an order polytope and number of linear extensions of the subjacent poset. This way of acting is possible for any poset. However, we have to face the problem of obtaining the volume. For this, it is necessary to compute an integral, and this could be tedious (see unfeasible) depending on the polytope. Besides, when dealing with families of posets, it is necessary to derive a recursive formula allowing to descend to the previous member of the family. This translated in the results developed before in a sequence of polynomials. Finally, in the previous results, the volume is given in terms of a recursive sequence of values. Needless to say, any of these steps may fail for a given family of polytopes. The question we address in this section is: Is it possible to find a framework where this procedure can be successfully applied?

To get a clue about how this framework could be, we focus on the properties of the families solved in this paper. Hence, some of the rules that could be suitable for this procedure could be summarized as follows:

First, the families considered in the paper arise from gluing several times a "small" poset P in some way. This poset is always connected. It also has at least two interchangeable elements. Let us call poset P the **initial tile**. Then, for the families considered in the paper, the initial tile corresponds to the first member of the family.

From the point of view of the integral, it seems advisable to have a poset P with low height, because the integrals become harder when the poset height increases. Note that for the posets considered in this paper, the height is 2 or 3.

The target family F_n is usually built by doing $F_1 = P$ and F_n arises by gluing one interchangeable element of the last P tile and one of a new tile. So F_n has $|P| - 1$ elements more than F_{n-1}. For example, for generalized fences, we are gluing y_2 with the corresponding y_1 of the next tile and the same happens for the other families in the paper.

The repetition of the same pattern tends to make easier to get solvable polynomial recurrences. Note that one initial poset P could lead to different families F_n if P has more than one pair of interchangeable elements. In other words, sometimes there are several ways of gluing posets this way.

Note also that the posets studied in this work fit the last properties. In Figure 7.6, we can see the poset tiles of the families studied along this papers in order of appearance.

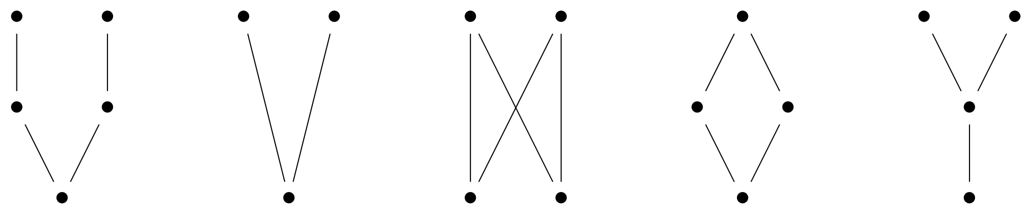


FIGURE 7.6: P tiles of studied F_n families.

We do not know if this method works for any structure of this type. To deal with this problem we should define properly the type of solution that we consider valid. Should any kind of recurrence be considered a good solution? How do we know a priori if the integrals involved lead to good recurrences?

Finally, in Figure 7.7 we provide a complete list of the non-isomorphic connected posets with at most 5 elements and having at least two interchangeable elements. These posets tiles should be the most reasonable scope of this methodology. Of course, it could be the case that gluing two elements instead of one could lead to a family of posets where this procedure works properly. Or it could be the case that an initial tile with large height leads to a simple integral. The rules established before are just clues that in our opinion seem to be the most reasonable for the applicability of the procedure.

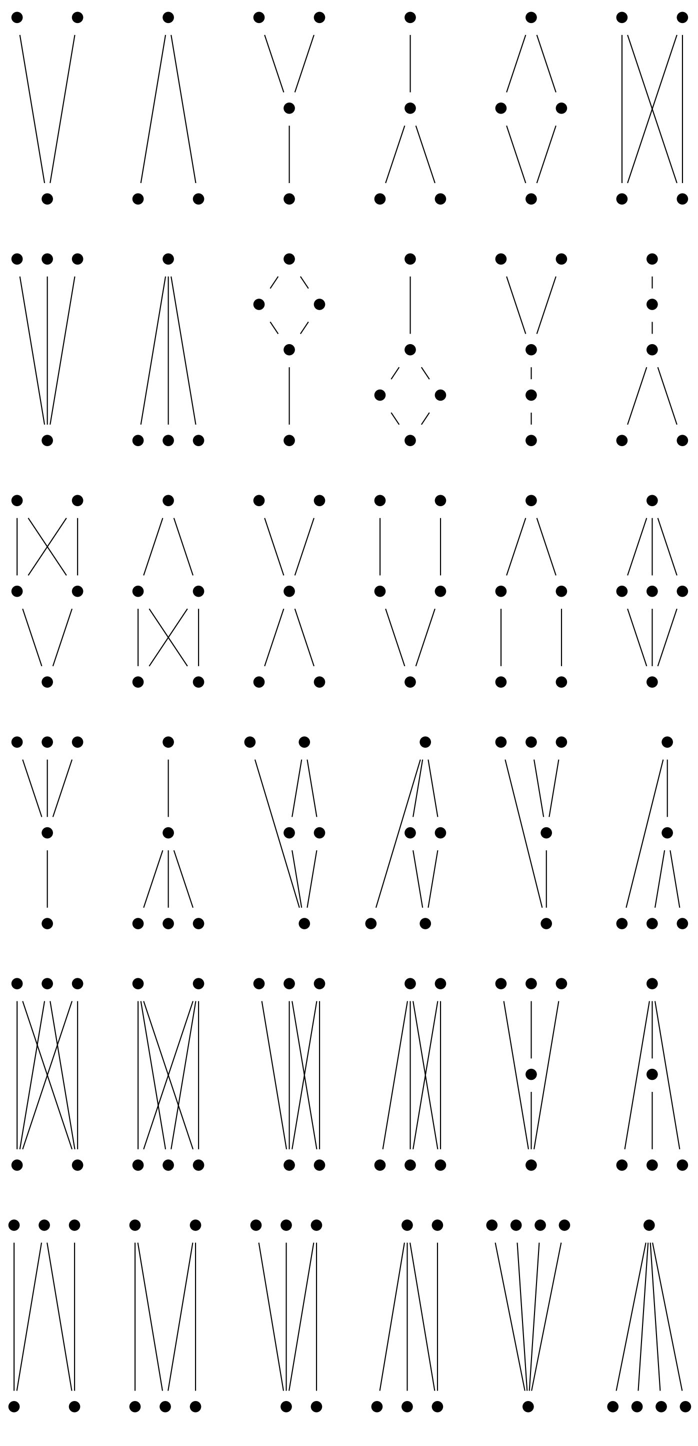


Figure 7.7: Non-isomorphic connected posets with at most 5 elements and having at least two interchangeable elements.

Chapter 8

Conclusions and open problems

Throughout this thesis we have studied different aspects of polytopes associated to fuzzy measures. In Chapter 1, we start introducing the basic results on Poset theory. Then we define the concept of linear extension of a poset and analyse its properties. Here we highlight that computing linear extension is not an easy task in general, indeed the problem of counting linear extensions of a general poset is a $\sharp P$-complete problem. Therefore, obtaining procedures with low complexity valid for counting linear extensions in a family of posets is a relevant and interesting problem. In Chapter 1, we also introduce the basic theory regarding convex polytopes and cones. Order polytopes link posets with convex polytopes and most importantly they model many important subfamilies of fuzzy measures. Most of the geometrical properties of these polytopes can be expressed in terms of algebra associated to its associated posets. Following this line we can characterize its faces, see Theorem 1.62 and also the smallest face containing a collection of vertices, see Theorem 1.64. Finally, we study how to triangulate order polytopes. Theorem 1.70 shows that one of the possible regular triangulations of order polytopes connects simplices to linear extensions of posets. Triangulations provide a lot of information about the geometry of a polytope and help us to solve applied problems as sampling points inside a specific polytope.

In Chapter 2, we give an introduction to the theory of fuzzy measures. We study the most important features of some of their subfamilies. Most of these subfamilies can be seen as convex polytopes that we can study using the theory of Chapter 1. We also examine some tools in order to work with fuzzy measures in Decision Theory. One of these tools is the Choquet Integral, see Section 2.3. The Choquet integral is a generalization of Lebesque integral which acts as an aggregation operator for fuzzy measures. The Choquet integral has many applications in Decision Theory. These applications can be divided into applications in Decision under Uncertainty and Risk and Multicriteria

Decision Making. It is worth noting that the Choquet integral solves the Ellberg's and Allais' paradoxes. Other important tool is the Möbius transform which gives a measure of the importance of a coalition by itself, without taking account of its different parts. The Möbius transform is necessary to define the k-additive measures. In Section 2.6, we explain the most important families of fuzzy measures and study some of their geometric aspects. Some of these subfamilies are studied more closely in other Chapters. In this Chapter, we also provide some basics on Cooperative Game Theory. We conclude by explaining how Choquet integral can be used to solve the Identification problem of finding the fuzzy measure that best fits some specific data.

In Chapter 3, we deal with the problem of generating linear extensions of a finite poset. This problem is again a $\sharp P$-complete problem. In this chapter, we derive an algorithm for generating linear extensions, that we have called Bottom-Up method. This procedure looks for a vector of weights on the elements of the poset so that any linear extension has the same probability of being obtained. We prove that this can be done by solving a linear system of equations. Moreover, this system does not depend on the number of linear extensions, but on the number of what we have called positioned antichains, whose number is usually very small compared with the number of linear extensions. In Section 3.3 we prove that $\mathrm{pa}(P) \leq e(P)$. This algorithm allows to compute the number of linear extensions of the a poset very easily. However, this algorithm cannot be applied to every poset but only to BU-feasible posets. In Theorem 3.21 we give an algebraic characterization of BU-feasible posets. Then, we examine some examples of families of posets where it can be applied, illustrating the procedure. Since most of subfamilies of fuzzy measures are order polytopes we study some application of Bottom-Up method to fuzzy measures (see Section 3.6). This is the case of the 2-truncated measures $\mathcal{TFM}^n(1, n-1)$. Finally, we explain how this methodology can be extended to general posets, not necessarily BU-feasible. Section 3.7 develops the Ideal-based method generalizing the Bottom-Up. This procedure can be applied to any poset, however, the complexity of the associated algorithm increases significantly. We give some applications where the ideals and the conditional probabilities can be computed directly.

In Chapter 4 we study more closely the polytope of k-symmetric measures, in particular the polytope of 2-symmetric measures. The novelty in this chapter is the use of combinatorial tools, in this case Young diagrams, to sort out the problem. We use Young diagrams to study some combinatorial properties of the polytope of 2-symmetric measures. For that purpose, we use the Hook length formula, see Theorem 4.5, to compute the number of linear extensions associated to Ferrers posets. Following this line, in Theorem 4.9 we compute the exact volume of these family of polytopes. We also give a combinatorial characterization of the adjacency on $\mathcal{FM}(A_1, A_2)$. In Section 4.3 we derive an algorithm to sample random points inside the polytope of 2-symmetric measures $\mathcal{FM}(A_1, A_2)$ for each fixed partition $\{A_1, A_2\}$. This procedure is appealing from a visual point of view, easy to implement and has a reduced computational cost. Finally, in Section 4.4, we deal with a problem of identification of fuzzy measures. Two types of algorithms have been proposed. In the first case we identify 2-symmetric random measures by using genetic algorithms with the convex combination as cross-over operator. In the second case, we use Proposition 4.16 to construct a new cross-over operator which let us identify general fuzzy measures with a low mean Choquet error. Although these results do not provide us with a random sampler in $\mathcal{FM}(X)$ because a fuzzy measure can be put as a maximum or minimum of 2-symmetric measures in different ways, this could be a startpoint for a fast pseudo-random procedure.

In Chapter 5 we study the combinatorial structure of the polytope of 2-additive measures, $\mathcal{FM}^2(X)$, an important subpolytope of $\mathcal{FM}(X)$. This polytope has the disadvantage of not being an order polytope, so we cannot use results from Chapter 1. For this polytope, we obtain its combinatorial structure in Theorem 5.5. We also describe the geometric structure of its faces in Theorem 5.7. Using these results, we build a triangulation of this polytope satisfying that all simplices in the triangulation have the same volume, see Theorem 5.10. This result allows us to compute the volume of this polytope. Moreover, this allows a very simple and appealing way to generate random points in this polytope. We finsig this chapter studying the adjacency graph of $\mathcal{FM}^2(X)$.

In Chapter 6, we introduce the concept of order cone. This concept is inspired by the concept of order polytopes. Similarly to order polytopes, order cones are a special type of polyhedral cones whose geometrical structure depends on the properties of a poset. In Theorem 6.5 we characterize all the extremal rays defining an order cone. Moreover, we provide in Theorem 6.8 a characterization of the faces of an order cone using the geometry of the associated order polytope. In Section 6.2, we use the last results to study some important order cones appearing in Game Theory. We start by studying the cone of monotone games, whose extremal rays were unknown so far. Corollary 6.11 give us all the extremal rays of $\mathcal{MG}(X)$. We also study the link between $\mathcal{MG}(X)$ and $\mathcal{O}(\mathcal{P}(X)\backslash\{\emptyset, X\})$ which is $\mathcal{FM}(X)$. If we restrict the set of feasible coalitions to $\Omega \subset \mathcal{P}(X) \setminus \{\emptyset\}$ we get an order cone whose structure depends on the combinatorics of the poset associated to this order cone. In Corollary 6.16 we compute the extremal rays of $\mathcal{MG}_\Omega(X)$. Finally, we study the cone of k-symmetric measures using the results of this chapter and from Chapter 4.

Counting linear extensions become essential to understand the geometry of order polytopes. In Chapter 7.1, we develop some techniques to exploit Equation 1.5. This way we deal with the volume of $\mathcal{O}(P)$ to get recursive formulas for computing the number of linear extensions of some important families of posets. Following this line, we transform a combinatorial problem into the problem of computing some integrals defining the volume of an order polytope. In many cases, solving these integrals exactly is not easy at all. However, it is interesting to investigate the posets leading to integrable problems. As application we develop a procedure for counting the number of 2-alternating permutations. Finally, we show that the volume can be given in terms of the coefficients of some polynomials, thus obtaining a linear recurrence for the volume and the number of linear extensions. We show that this procedure can also be applied to other important families of fence-type posets to derive formulas for the number of linear extensions.

Let us review some open problems arising on each chapter. Regarding Chapter 1, we know a regular triangulation for an order polytope from Theorem 1.70. However, there is an open problem to determine all the regular triangulations of an order polytope. Conmutative algebra techniques are required for such a purpose.

Regarding Chapter 2, there are a lot of subfamilies of fuzzy measures not being order polytopes. Developing techniques to study its combinatorics structure arises as an important open problem.

In Chapter 3, there are some open problems that we aim to treat in the future. First, we have the problem of obtaining necessary and sufficient conditions on the poset to apply the procedure; following this line, we have already obtained Th. 3.21, but it could be interesting to derive other results based on the structure of P. Another interesting problem would be to compare this procedure with other procedures appearing in the literature, in the sense of their computational costs; in this sense, it should be noted that BU-algorithm is very fast once w^* is obtained; then, it seems to be an appealing solution if many random linear extensions are needed. Finally, we have seen that an equivalent system can be stated if we consider positioned antichains instead of adjacent linear extensions, leading in general to a reduction in the number of equations; on the other hand, we have already seen that the number of equations needed can be less than the number of the positioned antichains; a deeper study on how the number of equations can be reduced seems interesting. BU-algorithm is based on minimal elements. Similar procedures could be developed if attention is fixed on maximal elements or other positions in the linear extension. This could increase the range of applicability of this philosophy; for example, starting from last element would be a suitable choice for rooted trees when the root is the maximal element. Another open problem is studying extensions of Bottom-Up methods different from the Ideal-based method.

Regarding Chapter 4, next step should be extending this procedure for general k-symmetric measures. However, many technical problems arise, showing the difficulties that random generation prompts. The first one is to extend Hook theorem for three-dimensional Young diagrams; this is a very complex problem that has not been solved yet; indeed, generalizations of Hook theorem for (2-dimensional) Young diagrams where some cells inside the diagram are not considered has not been solved in general (see shifted tableaux [83]). Another interesting problem arises from Proposition 4.16 and consists in determining the minimum number $k(n)$ of 2-symmetric measures needed to write a fuzzy measure $\mu \in \mathcal{FM}(X)$ as maximum or minimum of 2-symmetric measures. In this sense, it can be shown that $k(n) = k'(n)$, $\forall n \in \mathbb{N}$ and the first three values of $k(n)$ are $k(1) = k(2) = 1$ and $k(3) = 2$. However, the problem of computing the exact value of $k(n)$ for a general natural number seems to be a deep and complex combinatorial problem. Finally, we have characterized when two vertices in $\mathcal{FM}(A_1, A_2)$ are adjacent. Thus, given a vertex $\mu \in \mathcal{FM}(A_1, A_2)$, how can we count the number of adjacent vertices to μ?

In Chapter 5, one open problem will be to translate these results for general $\mathcal{FM}^k(X)$, $k \geq 2$. For this, we have to face a different situation that leads to new problems. The most apparent one comes from the fact that for $k \geq 3$ there are vertices that are not $\{0,1\}$-valued [90]. Moreover, these vertices have not been fully described in a suitable way. Thus, this seems a complicated problem for which more research is needed. The method for random generation of 2-additive measures could be applied to the problem of identifying 2-additive measures. In a similar way to the one we saw in Chapter 4, we can use the random generation of 2-additives measures to solve the problem of identifying 2-additive measures using genetic algorithms. If the quadratic error is considered, this amounts to looking for a 2-additive fuzzy measure μ_0 minimizing

$$F(\mu) := \sum_{i=1}^{m} \left(\mathcal{C}_\mu(x_1^i, ..., x_n^i) - y_i\right)^2,$$

for some partial scores $(x_1^i, ..., x_n^i)$, and the overall scores y_i. It would also be interesting to investigate whether it is possible to identify general fuzzy measures using 2-additive measures, just as we did in Chapter 4 with 2-symmetric measures. For this, we would have to find cross-over operators in the genetic algorithms with good properties. As far as we know this problem has not been studied. Related to $\mathcal{FM}^k(X)$, another application appears if we restrict to the convex hull of $\{0,1\}$-vertices in $\mathcal{FM}^k(X)$; this leads to a subpolytope of $\mathcal{FM}^k(X)$ and it seems interesting to study if the results obtained in this chapter still hold in this case.

Refering to Chapter 6, we feel that the concept of order cone could be an interesting tool for studying several families of monotone games just focusing on the subjacent poset. Note on the other hand that the order relation is essential for order cones. This means that the definition fails if we remove monotonicity. Studying a generalization dealing with this situation seems to be a complex problem that we intend to study in the future.

For Chapter 7 we will try to generalize these integral techniques to more general families of posets. However, dealing with this problem requires more sofisticated combinatorial tools. Moreover, a deeper research about how to extend this technique to generate linear extensions randomly is still pending.

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