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Smoothable locally non Cohen–Macaulay multiple structures on curves

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Abstract In this article we show that a wide range of multiple structures on curves arise whenever a family of embeddings degenerates to a morphism φ of degree n. One could expect to see, when an embedding degenerates to such a morphism, the appearance of a locally Cohen–Macaulay multiple structure of certain kind (a so-called *rope* of multiplicity n). We show that this expectation is naive and that locally non Cohen–Macaulay multiple structures also occur in this situation. In seeing this we find out that many multiple structures can be *smoothed*. When we specialize to the case of double structures we are able to say much more. In particular, we find numerical conditions, in terms of the degree and the arithmetic genus, for the existence of many locally Cohen–Macaulay and non Cohen–Macaulay smoothable double structures. Also, we show that the existence of these double structures is determined, although not uniquely, by the elements of certain space of vector bundle homomorphisms, which are related to the first order infinitesimal deformations of φ. In many instances, we show that, in order to determine a double structure uniquely, looking merely at a first order deformation of φ is not enough; one needs to choose also a formal deformation.

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Introduction

In this article we study projective *multiple structures* embedded in projective space, not necessarily locally Cohen–Macaulay, on curves. We then specialize our study to the case of *double structures* to obtain very concrete results. We do this following two motivations. Our first motivation is to give evidence that multiplicity n structures of a certain kind arise naturally when a family of embeddings degenerates to a morphism of degree n or, more precisely, as the flat limit of the images of the embeddings of such families. Multiple structures are everywhere non reduced schemes. Still, some of them, the so-called *ropes* are relatively nice (a rope on a smooth, irreducible projective curve Y of $\mathbf{P}^r$, is locally Cohen–Macaulay and contained in the first infinitesimal neighborhood of Y). When the multiple structure is a rope, and more so, when, in addition, its multiplicity is 2 (we call it in this case a *ribbon*) this phenomenon has been studied in several situations (see e.g. [2], [3] and [4]) –in contrast, locally non Cohen–Macaulay schemes have rarely been studied in the literature as they are often considered "wild" structures–. A family of embeddings degenerating to a finite morphism is a natural occurrence in algebraic geometry, and considering the past evidence, it is reasonable to expect the natural limit of such a degeneration to be a rope. Our results show that this expectation is naive and far from true and this is one of the surprising novelties of this article: to show (see Theorem 1.5 for multiplicity n structures and Corollary 2.1, Proposition 2.4 and Theorems 3.1, 4.1 and 4.2 for double structures) that a wide range of locally non Cohen–Macaulay multiple structures of multiplicity n that are *generically ropes* do appear naturally as limits of images of embeddings that degenerate to an n-to-one morphism. In particular, we prove that all those multiple structures are *smoothable* (i.e., can be deformed to smooth, irreducible curves). Observe that in the case $n = 2$, all multiple structures, i.e., all double structures, are generically ribbons. Thus when we focus on multiplicity 2 in Sections 2 to 5 we are looking at double structures in all generality. Why is that sometimes the limit of images of a family of embeddings is a locally non Cohen–Macaulay multiple structure, instead of a rope? Being locally non Cohen–Macaulay is, obviously, a local question and indeed, the appearance of locally non Cohen–Macaulay multiple structures as limits should be detected by local computations. However, our results show that it is the global geometry that dictates the local algebra and that the appearance of the locally non Cohen–Macaulay structures is the direct result of such a dictate. More precisely, this is the global reason for the appearance of locally non Cohen–Macaulay multiple structures: the genus of the smooth, embedded curves do not always match the arithmetic genus of the expected rope. If such is the case, the flat limit has to contain necessarily not only this rope but also some embedded points so that the genera can match. Theorems 1.5, 3.1, 4.1 and 4.2, Corollary 2.1 and Propositions 2.4 and 2.5 show that this situation happens very often.

Our second motivation is to give a geometric interpretation of the non surjective homomorphisms belonging to $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$. Before going on, let us say what $\mathscr{I}$ and $\mathscr{E}$ are. Let X and Y be smooth, irreducible projective curves, let i be an embedding of Y in $\mathbf{P}^r$ and let $\mathscr{I}$ be the ideal sheaf of $i(Y)$ in $\mathbf{P}^r$. Assume there exists a morphism φ from X to $\mathbf{P}^r$ and a morphism π of degree n from X to Y such that φ factors as $i \circ \pi$ and let $\mathscr{E}$ be the trace zero module of π (which is a vector bundle of rank $n-1$). It is well known (see for instance [8] for $n = 2$; the arguments for arbitrary n are essentially the same) that a surjective element of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ corresponds to a rope of multiplicity n in $\mathbf{P}^r$ whose reduced structure is $i(Y)$. In this direction, the second author generalized this fact in [6, Proposition 2.1] and showed that an element of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ corresponds to a pair $(\widehat{Y}, \hat{i})$, where $\widehat{Y}$ is a rope on Y and $\hat{i}$ is a morphism from $\widehat{Y}$ to $\mathbf{P}^r$ extending i. There is another geometric interpretation of the surjective elements μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}, \mathscr{E})$. In [4, Theorems 1.5, 2.2] we showed that, in a very general setting and under weak natural conditions (namely: essentiality assuming φ is unobstructed) such surjective elements mean not only the existence of the embedded ropes mentioned before, but also the fact that these ropes can be smoothed. Thus, a natural question to ask is whether a similar picture exists when μ is not surjective. In this article we show (see Theorem 1.5) that, under quite general and natural conditions on $\mathscr{E}$ and $i(Y)$ (although these conditions seem stronger than the unobstructedness condition on [4, Theorems 1.5, 2.2], the assumption on μ is weaker) the nonzero non surjective homomorphisms $\mu \in \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ with image of rank $n-1$ bear witness to the existence of locally non Cohen–Macaulay, rope-like multiple structures on Y which can be smoothed. Moreover, if $n = 2$, Corollary 2.1 shows that the nonzero non surjective homomorphisms $\mu \in \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ bear witness to the existence of locally non Cohen–Macaulay double structures on Y which can be smoothed.

When we specialize to multiplicity 2 structures, our geometric study of the elements of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ goes further and reveals that the assignment to μ of a smoothable double structure is quite subtle. We suggested in the previous paragraph that, according to Corollary 2.1, μ "produces" a smoothable double structure. However, as Proposition 5.1 shows, μ can be related to many different double structures. To understand the process we mention first that the connection of an algebraic object such as μ (a vector bundle homomorphism) with geometric objects such as deformations of morphisms and double structures is made through the study of the first order infinitesimal deformations of φ (for details on this connection see [6, §3]). Indeed, to each μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ we can associate (not in a unique way) a first infinitesimal deformation of φ. Proposition 5.1 tells that neither the information encoded by μ nor even the first order infinitesimal deformation of φ chosen are enough to determine a unique double structure. Instead, in order to determine a unique double structure we need to look not only at a first order infinitesimal deformation of φ but also at how this first order infinitesimal deformation extends to higher orders. More geometrically put, in order to determine a double structure starting from a given tangent vector v to the base $\mathscr{V}$ of the semiuniversal deformation space of φ we need to choose a way to integrate v along a path of $\mathscr{V}$ (for the definition of algebraic formally semiuniversal deformation, see [10, 2.5.7]).

1 Smoothable rope-like multiple structures on curves

The purpose of this section is to show that under quite general conditions there exist smoothable multiple structures (which are generically ropes but not necessarily locally Cohen–Macaulay) on curves. We do so in Theorem 1.5, where we see how these multiple structures appear naturally when we make degenerate a family of embeddings to an n-to-one morphism. First we set up the notation to be used and recall the definitions of the objects and concepts we will study in this section.

1.1 Notation and set-up

Throughout the article, unless otherwise explicitly stated, we will use the following notation and set-up:

(1) We work over an algebraically closed field of characteristic 0.
(2) Y is a smooth, irreducible projective curve embedded in $\mathbf{P}^r$, $r \geq 3$, of genus g and degree d.
(3) $\mathscr{I}$ is the ideal sheaf of Y in $\mathbf{P}^r$.
(4) $\mathscr{E}$ is a vector bundle on Y of rank $n-1$ and degree $-e$, and $\tilde{g} = g - \chi(\mathscr{E})$.
(5) μ is a homomorphism of $\operatorname{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$, $\mathscr{E}'$ is the image of μ and $\hat{g} = g - \chi(\mathscr{E}')$.

Definition 1.1 A subscheme $\widetilde{Y}$ of $\mathbf{P}^r$ is a *multiple structure* on Y of multiplicity n if the reduced structure of $\widetilde{Y}$ is Y and the degree of $\widetilde{Y}$ is n times the degree of Y. If $n = 2$ we say that $\widetilde{Y}$ is a *double structure* on Y.

Note that a multiple structure $\widetilde{Y}$ need not be locally Cohen–Macaulay. Among locally Cohen–Macaulay multiple structures we are interested in those ones called *ropes* and among locally non Cohen–Macaulay multiple structures we are interested in those who are generically ropes. We give now the precise definitions (the definition of rope we give here is [1, Definition 1]; for another equivalent definition, see [9]):

Definition 1.2 Let $\widetilde{Y}$ be a scheme.

(1) We say that $\widetilde{Y}$ is a *rope* of multiplicity n on Y with conormal bundle $\mathscr{E}$ (of rank $n-1$) if the reduced structure of $\widetilde{Y}$ is Y and
   (a) $\mathscr{I}^2 = 0$ and
   (b) $\mathscr{I}$ and $\mathscr{E}$ are isomorphic as $\mathscr{O}_Y$–modules.

   Note that a rope of multiplicity n is a multiple structure of multiplicity n. Note also that we can give the same definition of rope if Y is a smooth curve non necessarily complete.
(2) A rope of multiplicity 2 on Y is called a *ribbon* on Y.
(3) We say that $\widetilde{Y}$ is a *rope-like* multiple structure of multiplicity n on Y if $\widetilde{Y}$ is generically a rope of multiplicity n on Y, i.e., if there exists a non-empty open set $\widetilde{U}$ of $\widetilde{Y}$ such that $\widetilde{U}$ is a rope of multiplicity n on $\widetilde{U} \cap Y$.

Definition 1.3 Let $\widetilde{Y}$ be a subscheme of $\mathbf{P}^r$. By a *smoothing* of $\widetilde{Y}$ in $\mathbf{P}^r$ we mean a flat, integral family $\mathscr{Y}$ of subschemes of $\mathbf{P}^r$ over a smooth, irreducible algebraic curve T, such that over a closed point $0 \in T$, $\mathscr{Y}_0 = \widetilde{Y}$ and over the remaining closed points t of T, $\mathscr{Y}_t$ is a smooth, irreducible projective curve.

We introduce a homomorphism defined in [6, Proposition 3.7]:

Proposition 1.4 *Let C be a smooth irreducible projective curve, let $\pi : C \longrightarrow Y$ be a morphism of degree n, let $\mathscr{E}$ be the trace zero module of π and let $\varphi : C \longrightarrow \mathbf{P}^r$ be the composition of π followed by the inclusion of Y in $\mathbf{P}^r$. There exists a homomorphism*

$$H^0(\mathscr{N}_\varphi) \xrightarrow{\Psi} \operatorname{Hom}(\pi^*(\mathscr{I}/\mathscr{I}^2), \mathscr{O}_X),$$

that appears when taking cohomology on the commutative diagram [6, (3.3.2)]. Since

$$\operatorname{Hom}(\pi^*(\mathscr{I}/\mathscr{I}^2), \mathscr{O}_X) = \operatorname{Hom}(\mathscr{I}/\mathscr{I}^2, \pi_*\mathscr{O}_X) = \operatorname{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y) \oplus \operatorname{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E}),$$

the homomorphism Ψ has two components,

$$H^0(\mathscr{N}_\varphi) \xrightarrow{\Psi_1} \operatorname{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y) \text{ and}$$
$$H^0(\mathscr{N}_\varphi) \xrightarrow{\Psi_2} \operatorname{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E}).$$

We also recall this sequence of the commutative diagram [6, (3.3.2)]):

$$0 \longrightarrow \mathcal{N}_\pi \longrightarrow \mathcal{N}_\varphi \longrightarrow \pi^*\mathcal{N}_{Y,\mathbf{P}^N} \longrightarrow 0. \tag{1}$$

Theorem 1.5 *Let $Y, r, \mathcal{E}, \tilde{g}, \mu, \mathcal{E}'$ and $\hat{g}$ be as in 1.1 and assume that the rank of the image $\mathcal{E}'$ of μ is $n-1$. If*

(1) There exist a smooth irreducible projective curve C and a morphism $\pi : C \longrightarrow Y$ of degree n whose trace zero module is $\mathcal{E}$;
(2) $h^0(\mathcal{O}_Y(1)) + h^0(\mathcal{E} \otimes \mathcal{O}_Y(1)) \geq r+1$; and
(3) $h^1(\mathcal{O}_Y(1)) = h^1(\mathcal{E} \otimes \mathcal{O}_Y(1)) = 0$,

then there exist

(i) A rope-like multiple structure $\widetilde{Y}$ supported on Y, of multiplicity n and arithmetic genus $\tilde{g}$, embedded in $\mathbf{P}^r$;
(ii) A rope $\widehat{Y}$ supported on Y, of multiplicity n and arithmetic genus $\hat{g}$ and with conormal bundle $\mathcal{E}'$, contained in $\widetilde{Y}$;
(iii) A smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathcal{C}$ over T and a T-morphism $\Phi : \mathcal{C} \longrightarrow \mathbf{P}^{\tilde{g}}_T$ such that

(a) $\mathcal{C}_t$ is a smooth irreducible curve of genus $\tilde{g}$,
(b) Φ_t is an embedding for all $t \neq 0$, and
(c) Φ_0 is of degree n onto Y and $[\Phi(\mathcal{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ is smoothable in $\mathbf{P}^r$).

Proof Let φ be the composition of the inclusion of Y in $\mathbf{P}^r$ followed by π. Thus C is a smooth curve of genus $\tilde{g}$. Since C is a curve, $H^1(\mathcal{N}_\pi) = 0$, so the map Ψ_2 introduced in Proposition 1.4 surjects. Let $\nu \in H^0(\mathcal{N}_\varphi)$ be such that $\Psi_2(\nu) = \mu$. Such ν corresponds to an infinitesimal deformation $\widetilde{\varphi}$ of φ. Now we want to apply [3, Theorem 1.1]. By (3), $h^1(\mathcal{O}_Y(1)) = h^1(\mathcal{E} \otimes \mathcal{O}_Y(1)) = 0$ and by (2), $h^0(\varphi^*\mathcal{O}_{\mathbf{P}^r}(1)) \geq r+1$. Then, according to [3, Theorem 1.1] there exist a smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathcal{C}$ over T and a T-morphism $\Phi : \mathcal{C} \longrightarrow \mathbf{P}^r_T$ such that

(a) $\mathcal{C}_t$ is a smooth, irreducible curve;
(b) The restriction of Φ to the first infinitesimal neighborhood of 0 is $\tilde{\varphi}$ (and hence $\Phi_0 = \varphi$); and
(c) For any $t \in T$, $t \neq 0$, Φ_t is an embedding into $\mathbf{P}^r$.

In particular, $\mathcal{C}_0 = C$. Then, for all $t \in T$, $\mathcal{C}_t$ is a smooth, irreducible curve of genus $\tilde{g}$, so $\mathcal{Y} = \Phi(\mathcal{C})$ is a flat family over T of 1-dimensional subschemes of $\mathbf{P}^r$ of arithmetic genus $\tilde{g}$. Moreover, the reduced part of $\mathcal{Y}_0$ is Y and, because of (b) and (c) above, the degree of $\mathcal{Y}_0$ is n times the degree of Y. Thus $\mathcal{Y}_0$ is a multiple structure of multiplicity n and arithmetic genus $\tilde{g}$, supported on Y. On the other hand, because of (b) $\mathcal{Y}_0$ contains $(\text{im}\tilde{\varphi})_0$. By [6, Proposition 2.1], μ corresponds to a pair $(\overline{Y}, \overline{i})$, where $\overline{Y}$ is a rope of multiplicity n on Y and $\overline{i}$ a morphism (though not necessarily a closed embedding), from $\overline{Y}$ to $\mathbf{P}^r$, that extends the inclusion of Y in $\mathbf{P}^r$. In addition, $\overline{i}(\overline{Y})$ is a rope of multiplicity n whose conormal bundle is $\mathcal{E}'$ and we set $\widehat{Y} = \overline{i}(\overline{Y})$. Now, by [6, Theorem 3.8 (1)], $(\text{im}\widetilde{\varphi})_0$ equals $\overline{i}(\overline{Y})$. In particular, $\mathcal{Y}_0$ is a rope-like multiple structure of multiplicity n and arithmetic genus $\tilde{g}$. Thus we may set $\widetilde{Y} = \mathcal{Y}_0$. □

Observation 1.6 *The following are equivalent:*

(1) $\mathcal{E} = \mathcal{E}'$ (i.e, μ is surjective),
(2) $\tilde{g} = \hat{g}$,

(3) $\widetilde{Y} = \widehat{Y}$,
(4) $\widetilde{Y}$ *is a rope of multiplicity* n.

If $\widetilde{Y}$ *is not a rope, then* $\widetilde{Y}$ *contains* $\widehat{Y}$ *and some embedded points.*

Proof (1) and (2) are equivalent by the definition of $\tilde{g}$ and $\hat{g}$. Since $\widehat{Y}$ is contained in $\widetilde{Y}$ and both are multiple structures on Y with the same multiplicity and, respectively, of arithmetic genus $\hat{g}$ and $\tilde{g}$, (2) implies (3). On the other hand, (3) obviously implies (2). Since $\widehat{Y}$ is a rope of multiplicity n, (3) implies (4). Finally, if $\widetilde{Y}$ is a rope of multiplicity n, since $\widehat{Y}$ is also a rope of multiplicity n and is contained in $\widetilde{Y}$, then $\widetilde{Y} = \widehat{Y}$, so (4) implies (3). □

2 Conditions for the existence of smoothable double structures on curves

In the remaining of the article we focus on double structures in general. It is well-known that a double structure on Y is always rope-like (or better yet, ribbon-like), since a locally Cohen–Macaulay double structure on Y is necessarily a ribbon, so we will be able to use Theorem 1.5. Thus in this section we apply Theorem 1.5 to give sufficient numerical conditions to guarantee the existence of smoothable double structures, not necessarily locally Cohen–Macaulay, on curves. We summarize these numerical conditions in Proposition 2.5. We keep using 1.1, but we give an extra piece of convention to be used in this section and in the remaining of the article.

2.1 Notation and set-up

From now on, $\mathscr{E}$ is a line bundle on Y.

First of all, we give this corollary of Theorem 1.5 for double structures in general, in which the assumption on the rank imposed on μ is superfluous:

Corollary 2.1 *Let* $Y, r, \mathscr{E}, \tilde{g}, \mu, \mathscr{E}'$ *and* $\hat{g}$ *be as in 1.1 and 2.1 and assume* $\mu \neq 0$. *If*

(1) $|\mathscr{E}^{-2}|$ *possesses a smooth, effective, non empty divisor;*
(2) $h^0(\mathscr{O}_Y(1)) + h^0(\mathscr{E} \otimes \mathscr{O}_Y(1)) \geq r + 1$*; and*
(3) $h^1(\mathscr{O}_Y(1)) = h^1(\mathscr{E} \otimes \mathscr{O}_Y(1)) = 0$,

then there exist

(i) A double structure $\widetilde{Y}$ *supported on* Y *of arithmetic genus* $\tilde{g}$*, embedded in* $\mathbf{P}^r$*;*
(ii) A ribbon $\widehat{Y}$*, supported on* Y *of arithmetic genus* $\hat{g}$ *and with conormal bundle* $\mathscr{E}'$*, contained in* $\widetilde{Y}$*;*
(iii) A smooth irreducible algebraic curve T *with a distinguished closed point* 0*, a flat family* $\mathscr{C}$ *over* T *and a* T*-morphism* $\Phi : \mathscr{C} \longrightarrow \mathbf{P}^{\tilde{g}}_T$ *such that*

(a) $\mathscr{C}_t$ *is a smooth irreducible curve of genus* $\tilde{g}$*,*
(b) Φ_t *is an embedding for all* $t \neq 0$*, and*
(c) Φ_0 *is of degree* 2 *onto* Y *and* $[\Phi(\mathscr{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ *is smoothable in* $\mathbf{P}^r$*).*

Proof We just need to apply Theorem 1.5 for $n = 2$. First note that in this case $\mathscr{E}$ is a line bundle, so if μ is nonzero, the rank of the image of μ is 1. Conditions (2) and (3) of Theorem 1.5 and Corollary 2.1 are stated exactly in the same way. Condition (1) of Corollary 2.1 implies

Condition (1) Theorem 1.5, since Condition (1) of Corollary 2.1 implies the existence of a double cover $\pi : C \longrightarrow Y$ branched along a smooth, effective, non empty divisor B in $|\mathscr{E}^{-2}|$. Then C is smooth and irreducible and $\mathscr{E}$ is the trace zero module of π. Finally, as already pointed out, locally Cohen–Macaulay double structures are ribbons, so double structures are always ribbon-like and, in (i) of Theorem 1.5 we can substitute "rope-like multiple structure" for "ribbon" and, obviously, in (ii) and (iii) of Theorem 1.5, "rope" for "ribbon" and "degree n" for "degree 2". □

In the next two remarks we give simple numerical conditions which guarantee the existence of nonzero surjective and non surjective homomorphisms μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$.

Remark 2.2 Assume that (1) and (3) of Corollary 2.1 hold. In addition,

(1) if $g = 0$, let $3 \leq e \leq d - 1$; and
(2) if $g \geq 1$, let $d - e \geq g$,

then there exists a nonzero, non surjective homomorphism μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$.

Proof The existence of a nonzero, non surjective homomorphism in $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ is equivalent to the existence of a non zero global section in $H^0(\mathscr{N}_{Y,\mathbf{P}^r} \otimes \mathscr{E})$ with non empty vanishing locus. We consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{E} & = & \mathscr{E} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{F} \otimes \mathscr{E} & \longrightarrow & (\mathscr{E} \otimes \mathscr{O}_Y(1))^{\oplus r+1} & \longrightarrow & \mathscr{N}_{Y,\mathbf{P}^r} \otimes \mathscr{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{T}_Y \otimes \mathscr{E} & \longrightarrow & \mathscr{T}_{\mathbf{P}^r|Y} \otimes \mathscr{E} & \longrightarrow & \mathscr{N}_{Y,\mathbf{P}^r} \otimes \mathscr{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array} \tag{2}$$

where $\mathscr{F}$ is the kernel of the composite surjective map

$$H^0(\mathscr{O}_Y(1))^\vee \otimes \mathscr{O}_Y(1) \twoheadrightarrow \mathscr{T}_{\mathbf{P}^r|Y} \twoheadrightarrow \mathscr{N}_{Y,\mathbf{P}^r}.$$

Recall that $\mathscr{E}$ has no global sections. We see also that $H^0(\mathscr{T}_Y \otimes \mathscr{E}) = 0$. Corollary 2.1, (1) implies $e \geq 0$, so, by (1) the degree of $\mathscr{T}_Y \otimes \mathscr{E}$ is negative except if $g = 1$ and $e = 0$. Since $\mathscr{E}$ is not trivial, in any case $H^0(\mathscr{T}_Y \otimes \mathscr{E}) = 0$. On the other hand, Corollary 1.5, (3) implies that $h^0(\mathscr{E} \otimes \mathscr{O}_Y(1)) = d - e - g + 1$, which is a positive number by (1) and (2). Moreover, (1) and (2) also imply that the degree of $\mathscr{E} \otimes \mathscr{O}_Y(1)$ is positive, so $\mathscr{E} \otimes \mathscr{O}_Y(1)$ has non zero global sections with non empty vanishing locus. Then taking global sections in the middle exact sequence of (2) we obtain the desired non zero global sections of $H^0(\mathscr{N}_{Y,\mathbf{P}^r} \otimes \mathscr{E})$ with non empty vanishing locus. □

Remark 2.3 Assume that (1) and (3) of Corollary 2.1 hold. If $\mathscr{E} \otimes \mathscr{O}_Y(1)$ is globally generated (hence $1 \leq e \leq d$ when $g = 0$ and $d - e \geq g + 1$ otherwise), then $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ possesses a surjective homomorphism.

Proof Since $\mathscr{E} \otimes \mathscr{O}_Y(1)$ is globally generated, the result follows from diagram (2). □

Corollary 2.1 and Remarks 2.2 and 2.3 yield the results that follow. The proof of Proposition 2.4 is straightforward. Proposition 2.5 summarizes sufficient conditions on the genus and degree of a linearly normal curve Y of $\mathbf{P}^r$ to guarantee the existence of smoothable double structures.

Proposition 2.4 *Assume that (1), (2) and (3) of Corollary 2.1 hold.*

(1) If (1) or (2) of Remark 2.2 holds, then there exist a double structure $\widetilde{Y}$ and a ribbon $\widehat{Y}$ satisfying (i), (ii) and (iii) of Corollary 2.1.
(2) If $\mathscr{E} \otimes \mathscr{O}_Y(1)$ is globally generated, then there exist ribbons $\widetilde{Y}$ on Y with conormal bundle $\mathscr{E}$ (hence, of arithmetic genus $e + 2g - 1$) satisfying (i) and (iii) of Corollary 2.1.

Proposition 2.5 *Let $h^1(\mathscr{O}_Y(1)) = 0$ and let $\tilde{\gamma}$ be an integer.*

(1) If

(1.1) $2 \le \tilde{\gamma} \le min(d-2, 2d-r)$ when $g = 0$; or
(1.2) $\frac{5g-1}{2} \le \tilde{\gamma} \le min(d, 2d-r)$ when $g \ge 1$,

then there exist

(i) locally non Cohen–Macaulay double structures $\widetilde{Y}$ supported on Y of arithmetic genus $\tilde{\gamma}$, embedded in $\mathbf{P}^r$;
(ii) for some $\hat{\gamma} > \tilde{\gamma}$, ribbons $\widehat{Y}$ of arithmetic genus $\hat{\gamma}$ supported on Y and contained in $\widetilde{Y}$.

(2) If

(2.1) $2 \le \tilde{\gamma} \le min(d-1, 2d-r)$ when $0 \le g \le 1$; or
(2.2) $\frac{5g-1}{2} \le \tilde{\gamma} \le min(d, 2d-r)$ when $g \ge 2$,

then there exist ribbons $\widetilde{Y}$ on Y of arithmetic genus $\tilde{\gamma}$, embedded in $\mathbf{P}^r$.

Moreover, under hypothesis (1) or (2), there exist a smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathscr{C}$ over T and a T-morphism $\Phi : \mathscr{C} \longrightarrow \mathbf{P}^{\tilde{g}}_T$ such that

(a) $\mathscr{C}_t$ is a smooth irreducible curve of genus $\tilde{\gamma}$,
(b) Φ_t is an embedding for all $t \neq 0$, and
(c) Φ_0 is of degree 2 onto Y and $[\Phi(\mathscr{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ is smoothable in $\mathbf{P}^r$).

Proof We prove (1) first. Let $\epsilon = \tilde{\gamma} - 2g + 1$. The lower bounds on $\tilde{\gamma}$ assumed in (1.1) and (1.2) imply that $2\epsilon \ge g + 1$. Thus we may choose a non trivial line bundle $\mathscr{E}$ on Y of degree $-\epsilon$, for which $\epsilon = e$ and $\tilde{\gamma} = \tilde{g}$, and such that $|\mathscr{E}^{-2}|$ possesses a smooth, effective, non empty divisor, so (1) of Corollary 2.1 is satisfied. By hypothesis, $h^1(\mathscr{O}_Y(1)) = 0$. In addition, the upper bounds in (1.1) and (1.2) imply $d \ge \tilde{g}$; this is equivalent to $d - e \ge 2g - 1$, so $h^1(\mathscr{E} \otimes \mathscr{O}_Y(1)) = 0$ and $\mathscr{E}$ satisfies (3) of Corollary 2.1. On the other hand, (1.1) and (1.2) imply $\tilde{g} \le 2d - r$, which is the same as $\mathscr{E}$ satisfying hypothesis (2) of Corollary 2.1. Finally (1.1) and (1.2) also imply (1) and (2) of Remark 2.2, so the thesis of (1) follows from Proposition 2.4, (1).

Now we prove (2). Let again $\epsilon = \tilde{\gamma} - 2g + 1$. The lower bounds assumed for $\tilde{\gamma}$ on (2.1) and (2.2) allow us (as (1.1) and (1.2) did before) to choose a non trivial line bundle $\mathscr{E}$ on Y of degree $-\epsilon$, for which $\epsilon = e$ and $\tilde{\gamma} = \tilde{g}$, and such that $|\mathscr{E}^{-2}|$ possesses a smooth, effective, non empty divisor B. Thus (1) of Corollary 2.1 is satisfied. Moreover, (2.1) and (2.2) imply that $d-e \geq 0$ if $g=0$ and $d-e \geq g+1$ otherwise, so we may further assume $\mathscr{E} \otimes \mathscr{O}_Y(1)$ to be globally generated, thus satisfying the hypothesis of Remark 2.3. Finally, (2.1) and (2.2) also imply hypotheses (2) and (3) of Corollary 2.1, so the thesis of (2) follows from Proposition 2.4, (2). □

3 Double structures on rational normal curves

In this section we study smoothable double structures supported on rational normal curves. For this purpose we apply Theorem 1.5 (or, rather, Corollary 2.1) to obtain Theorem 3.1. The most well known example of smoothable double structures on rational normal curves are canonical ribbons. Among other things, Theorem 3.1 says that, in addition to canonical ribbons, in $\mathbf{P}^3$ or in projective spaces of higher dimension there exist smoothable double structures, both locally Cohen–Macaulay and non Cohen–Macaulay, supported on rational normal curves, of any arithmetic genus and of any degree in the non special range. In this particular case of Y being a rational normal curve, Theorem 3.1 is much stronger than Proposition 2.5.

Theorem 3.1 *Let $\tilde{\gamma}$ be a non negative integer and let Y be a smooth rational normal curve of degree d in $\mathbf{P}^d$ (i.e, $d=r$ in this case), where either $d=\tilde{\gamma}-1$ and $\tilde{\gamma} \geq 3$ or $d \geq max(\tilde{\gamma}, 3)$. For any integer $\hat{\gamma}$ such that $\tilde{\gamma} \leq \hat{\gamma} \leq d+1$ there exist*

(i) Double structures $\widetilde{Y}$ supported on Y of arithmetic genus $\tilde{\gamma}$, embedded in $\mathbf{P}^d$;
(ii) Ribbons $\widehat{Y}$, supported on Y of arithmetic genus $\hat{\gamma}$, contained in $\widetilde{Y}$;
(iii) A smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathscr{C}$ over T and a T-morphism $\Phi : \mathscr{C} \longrightarrow \mathbf{P}^{\tilde{g}}_T$ such that

(a) $\mathscr{C}_t$ is a smooth irreducible curve of genus $\tilde{\gamma}$,
(b) Φ_t is an embedding for all $t \neq 0$, and
(c) Φ_0 is of degree 2 onto Y and $[\Phi(\mathscr{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ is smoothable in $\mathbf{P}^r$). These double structures $\widetilde{Y}$ are locally Cohen–Macaulay if and only if $\hat{\gamma}=\tilde{\gamma}$. If $\hat{\gamma} > \tilde{\gamma}$, then $\widetilde{Y}$ contains $\widehat{Y}$ and some embedded points.

Proof It is well known that the conormal bundle of Y inside $\mathbf{P}^d$ is

$$\mathscr{I}/\mathscr{I}^2 \simeq \mathscr{O}_{\mathbf{P}^1}(-d-2)^{\oplus d-1}. \tag{3}$$

If $d=\tilde{\gamma}-1$, then $\hat{\gamma}=\tilde{\gamma} \geq 3$ and Y has degree $\tilde{\gamma}-1$ in $\mathbf{P}^{\tilde{\gamma}-1}$. Then

$$\mathscr{I}/\mathscr{I}^2 \simeq \mathscr{O}_{\mathbf{P}^1}(-\tilde{\gamma}-1)^{\oplus \tilde{\gamma}-2}.$$

Thus there are double structures $\widetilde{Y}$ on Y with arithmetic genus $\tilde{\gamma}$ and these are necessarily canonical ribbons. It is well known (see e.g. [2]) that a canonical ribbon $\widetilde{Y}$ is the flat limit of smooth canonical curves that approach the hyperelliptic locus and that, if we set the family of morphisms Φ in (iii) to be the relative canonical morphism of such family of curves, then $[\Phi(\mathscr{X})]_0 = \widetilde{Y}$. This proves the result when $d=\tilde{\gamma}-1$.

Now assume $d \geq \tilde{\gamma}$, $d \geq 3$ and fix an integer $\hat{\gamma}$ such that $\tilde{\gamma} \leq \hat{\gamma} \leq d+1$. Set $\mathscr{E} \simeq \mathscr{O}_{\mathbf{P}^1}(-\tilde{\gamma}-1)$ (thus $e=\tilde{\gamma}+1$ and $\tilde{g}=\tilde{\gamma}$). We claim that $\mathscr{E}$ satisfies (1), (2) and (3) of Corollary 2.1. Indeed, $\mathscr{E}^{-2}$ has positive degree, so $|\mathscr{E}^{-2}|$ possesses a smooth, effective, non empty divisor, so (1) of Corollary 1.5 is satisfied. On the other hand, $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{P}^1}(d)$ and $\mathcal{E} \otimes \mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{P}^1}(d - \tilde{g} - 1)$, so (3) of Corollary 2.1 holds and

$$h^0(\mathcal{O}_Y(1)) + h^0(\mathcal{E} \otimes \mathcal{O}_Y(1)) = 2d - \tilde{g} + 1 \geq d + 1 = r + 1,$$

so (2) of Corollary 2.1 also holds. Finally, since $\hat{\gamma} \leq d + 1$ and $d \geq 3$, there exist $d - 1$ global sections of $\mathcal{O}_{\mathbf{P}^1}(d - \hat{\gamma} + 1)$ which do not vanish simultaneously at any given point of $\mathbf{P}^1$. Then (3) yields the existence of a nonzero homomorphism $\mu \in \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$ whose image is the line subbundle $\mathcal{O}_{\mathbf{P}^1}(-\hat{\gamma} - 1)$, which we will call $\mathcal{E}'$ (thus $\hat{g} = \hat{\gamma}$). Then the result follows from Corollary 2.1 and Observation 1.6. □

4 Double structures on elliptic normal curves

In this section we apply Theorem 1.5 (or, rather, Corollary 2.1) to study smoothable double structures on elliptic normal curves. Precisely, in Theorems 4.1 and 4.2 we show the existence of smoothable double structures, both locally Cohen–Macaulay and non Cohen–Macaulay, on elliptic normal curves in $\mathbf{P}^3$ or $\mathbf{P}^4$. Theorems 4.1 and 4.2 do not follow from Proposition 2.5.

Theorem 4.1 *Let Y be a smooth elliptic normal curve of degree 4 in $\mathbf{P}^3$.*

(1) If $\mathcal{E}$ is a line bundle of degree -4 on Y such that $\mathcal{E}^{-1} \neq \mathcal{O}_Y(1)$, then there are nonzero elements μ of Hom$(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$. For each one of these μ, there exist

(i) A double structure $\widetilde{Y}$ supported on Y of arithmetic genus 5, embedded in $\mathbf{P}^3$;
(ii) A ribbon $\widehat{Y}$, supported on Y of arithmetic genus $\hat{g}$ and with conormal bundle $\mathcal{E}'$ ($\mathcal{E}' = \mathrm{im}\,\mu$), contained in $\widetilde{Y}$; and
(iii) A smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathcal{C}$ over T and a T-morphism $\Phi : \mathcal{C} \longrightarrow \mathbf{P}^3_T$ such that

(a) $\mathcal{C}_t$ is a smooth irreducible curve of genus 5,
(b) Φ_t is an embedding for all $t \neq 0$, and
(c) Φ_0 is of degree 2 onto Y and $[\Phi(\mathcal{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ is smoothable in $\mathbf{P}^3$).

Moreover, there are ribbons $\widehat{Y}$ as in (i), (ii), (iii), of arithmetic genus 5, 6, 7 and 9 and with conormal bundle $\mathcal{E}''$, for any line bundle $\mathcal{E}''$ of degree -5 or -6 or satisfying $\mathcal{E}'' = \mathcal{E}$ or satisfying $\mathcal{E}'' = \mathcal{O}_Y(-2)$. Conversely, if $\widetilde{Y}$ and $\widehat{Y}$ are as in (i), (ii), then $\hat{g} = 5, 6, 7$ or 9.

The double structures $\widetilde{Y}$ above are ribbons if and only its arithmetic genus is 5. In contrast, if $\hat{g} = 6, 7$ or 9, then $\widetilde{Y}$ has embedded points.

(2) If $\mathcal{E}$ is a line bundle on Y such that $\mathcal{E}^{-1} = \mathcal{O}_Y(1)$, then there are surjective elements μ of Hom$(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$. For each of these μ there exist

(iv) A ribbon $\widetilde{Y}$, supported on Y of arithmetic genus 5 and with conormal bundle $\mathcal{E}$; and
(v) A smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathcal{C}$ over T and a T-morphism $\Phi : \mathcal{C} \longrightarrow \mathbf{P}^3_T$ such that

(a) $\mathcal{C}_t$ is a smooth irreducible curve of genus 5,
(b) Φ_t is an embedding for all $t \neq 0$, and
(c) Φ_0 is of degree 2 onto Y and $[\Phi(\mathcal{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ is smoothable in $\mathbf{P}^3$).

Proof First we see that there are ribbons $\widehat{Y}$ as in (i), (ii), of arithmetic genus 5, 6, 7 and 9 and with conormal bundle $\mathscr{E}''$, for any line bundle $\mathscr{E}''$ of degree -5 or -6 or satisfying $\mathscr{E}'' = \mathscr{E}$ or satisfying $\mathscr{E}'' = \mathscr{O}_Y(-2)$. Note first that such a line bundle $\mathscr{E}''$ is a subbundle of $\mathscr{E}$. Then for our purpose it suffices to prove the existence of an element μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ with $\mathrm{im}\,\mu = \mathscr{E}''$. Any smooth elliptic normal curve of degree 4 in $\mathbf{P}^3$ is the complete intersection of two quadrics, so $\mathscr{N}_{Y,\mathbf{P}^3} = \mathscr{O}_Y(2) \oplus \mathscr{O}_Y(2)$. The line bundle $\mathscr{O}_Y(2) \otimes \mathscr{E}$ is of degree 4 on Y so, for any subbundle $\mathscr{O}_Y(2) \otimes \mathscr{E}''$ of $\mathscr{O}_Y(2) \otimes \mathscr{E}$ of degree 2, 3 or 4, there are nonzero global sections σ_1 and σ_2 of $\mathscr{O}_Y(2) \otimes \mathscr{E}''$ with disjoint zero loci. Then, for any line bundle $\mathscr{E}''$ of degree -5 or -6 or satisfying $\mathscr{E}'' = \mathscr{E}$ or $\mathscr{E}'' = \mathscr{O}_Y(-2)$ there exists an element $\mu \in \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ whose image is $\mathscr{E}''$.

Now we see that if $\widetilde{Y}$ and $\widehat{Y}$ are as in (i), (ii), then $\hat{g} = 5, 6, 7$ or 9. If $\mathscr{E}''$ is a subbundle of $\mathscr{E}$, then $\deg\mathscr{E}'' \leq -4$ (and if $\deg\mathscr{E}'' = -4$, then $\mathscr{E} = \mathscr{E}''$). If $\deg\mathscr{E}'' \leq -9$, then $\mathscr{N}_{Y,\mathbf{P}^3} \otimes \mathscr{E}''$ does not have nonzero global sections, so if μ is a nonzero element of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ its image could not be such $\mathscr{E}''$. On the other hand, if $\deg\mathscr{E}'' = -7$, any two global sections of $\mathscr{O}_Y(2) \otimes \mathscr{E}''$ have a common zero, so there are no $\mu \in \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ whose image is $\mathscr{E}''$.

Now assume $\mathscr{E}^{-1} \neq \mathscr{O}_Y(1)$. Obviously there are nonzero elements μ in $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ and, since (1), (2) an (3) of Corollary 2.1 are satisfied, then (i), (ii) and (iii) follow from Corollary 2.1. Moreover, the double structures $\widetilde{Y}$ and the ribbons $\widehat{Y}$ whose existence we showed in the first paragraph of this proof also satisfy (iii). Finally, last claim of (1) follows from Observation 1.6.

Now assume $\mathscr{E}^{-1} = \mathscr{O}_Y(1)$. Since one can choose two sections of $\mathscr{O}_Y(1)$ having no common zeros, there exist surjective elements in $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$. Let μ be one of them. Then μ corresponds to a ribbon on Y of arithmetic genus 5, embedded in $\mathbf{P}^3$. Since in this case $h^1(\mathscr{E} \otimes \mathscr{O}_Y(1)) = h^1(\mathscr{O}_Y) \neq 0$, we cannot use Corollary 2.1. Instead we will apply [4, Theorem 1.5]. To do so, let B be a smooth divisor in $|\mathscr{E}^{-2}|$, let $\pi : C \longrightarrow Y$ be the double cover of Y branched along B and with trace zero module $\mathscr{E}$ and let φ be the composition of π followed by the inclusion of Y in $\mathbf{P}^3$. Now consider the homomorphism Ψ_2 associated to φ defined in Proposition 1.4. By (1) and the fact that $H^1(\mathscr{N}_\pi) = 0$ (because Y is a curve), Ψ_2 is surjective. Let ν be a counterimage of μ and let $\tilde{\varphi}$ be the first order infinitesimal deformation. To apply [4, Theorem 1.5] we observe that φ has an algebraic formally semiuniversal deformation because Y is a curve. Finally we need to check that φ is unobstructed. This holds if we show that $H^1(\mathscr{N}_\varphi) = 0$. Using the sequence (1), $H^1(\mathscr{N}_\varphi) = 0$ follows from $H^1(\mathscr{N}_{Y,\mathbf{P}^3}) = 0$, $H^1(\mathscr{N}_{Y,\mathbf{P}^3} \otimes \mathscr{E}) = 0$ and $H^1(\mathscr{N}_\pi) = 0$. This proves (2). □

Theorem 4.2 *Let Y be a smooth elliptic normal curve of degree 5 in $\mathbf{P}^4$.*

(1) If $\mathscr{E}$ is a line bundle of degree -5 on Y such that $\mathscr{E}^{-1} \neq \mathscr{O}_Y(1)$, then there are nonzero elements μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$. For each of these μ there exist

(i) A double structure $\widetilde{Y}$ supported on Y of arithmetic genus 6, embedded in $\mathbf{P}^4$;
(ii) A ribbon $\widehat{Y}$, supported on Y of arithmetic genus $\hat{g}$ and with conormal bundle $\mathscr{E}'$, ($\mathscr{E}' = \mathrm{im}\,\mu$) contained in $\widetilde{Y}$; and
(iii) A smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathscr{C}$ over T and a T-morphism $\Phi : \mathscr{C} \longrightarrow \mathbf{P}^4_T$ such that

(a) $\mathscr{C}_t$ is a smooth irreducible curve of genus 6,
(b) Φ_t is an embedding for all $t \neq 0$, and
(c) Φ_0 is of degree 2 onto Y and $[\Phi(\mathscr{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ is smoothable in $\mathbf{P}^4$).

Moreover, there are ribbons $\widehat{Y}$ *as above of arithmetic genus* $\hat{g} = 6, 7, 8$ *and* 9 *and with conormal bundle* $\mathscr{E}''$*, for any line bundle* $\mathscr{E}''$ *of degree* $-6, -7$ *or* -8 *or such that* $\mathscr{E}'' = \mathscr{E}$*. Conversely, if* $\widetilde{Y}$ *and* $\widehat{Y}$ *are as in (i), (ii), then* $\hat{g} = 6, 7, 8$ *or* 9*.*

The double structures $\widetilde{Y}$ *above are ribbons if and only its arithmetic genus is* 6*. In contrast, if* $\hat{g} = 7, 8$ *or* 9*, then* $\widetilde{Y}$ *has embedded points.*

(2) If $\mathscr{E}$ *is a line bundle on* Y *such that* $\mathscr{E}^{-1} = \mathscr{O}_Y(1)$*, then there are surjective elements* μ *of* $Hom(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$*. For each one of these* μ*, there exist*

(iv) A ribbon $\widetilde{Y}$*, supported on* Y *of arithmetic genus* 6 *and with conormal bundle* $\mathscr{E}$*; and*
(v) A smooth irreducible algebraic curve T *with a distinguished closed point* 0*, a flat family* $\mathscr{C}$ *over* T *and a* T*-morphism* $\Phi : \mathscr{C} \longrightarrow \mathbf{P}^4_T$ *such that*

(a) $\mathscr{C}_t$ *is a smooth irreducible curve of genus* 6*,*
(b) Φ_t *is an embedding for all* $t \neq 0$*, and*
(c) Φ_0 *is of degree* 2 *onto* Y *and* $[\Phi(\mathscr{X})]_0 = \widetilde{Y}$

(in particular, $\widetilde{Y}$ *is smoothable in* $\mathbf{P}^4$*).*

Proof To prove both (1) and (2), first we need to study $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ for any line bundle $\mathscr{E}$ on Y of degree -5. Let O be a point of Y such that $\mathscr{O}_Y(1) = \mathscr{O}_Y(5O)$. Let E'_Y be the unique indecomposable rank 2 vector bundle on Y whose determinant is $\mathscr{O}_Y(O)$ and let E_Y be the rank 3 vector bundle given by the unique non split extension

$$0 \longrightarrow \mathscr{O}_Y \longrightarrow E_Y \longrightarrow E'_Y \longrightarrow 0.$$

It follows from [7, Corollary V.1.4] that $\mathscr{N}_{Y,\mathbb{P}^4} = E_Y(8O)$. Thus we have the exact sequences

$$\begin{aligned} &0 \longrightarrow \mathscr{O}_Y(8O) \otimes \mathscr{E} \longrightarrow E'_Y(8O) \otimes \mathscr{E} \longrightarrow \mathscr{O}_Y(9O) \otimes \mathscr{E} \longrightarrow 0, \\ &0 \longrightarrow \mathscr{O}_Y(8O) \otimes \mathscr{E} \longrightarrow \mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E} \longrightarrow E'_Y(8O) \otimes \mathscr{E} \longrightarrow 0. \end{aligned} \tag{4}$$

Since $\mathscr{E}$ has degree -5, $\mathscr{O}_Y(8O) \otimes \mathscr{E}$ and $\mathscr{O}_Y(9O) \otimes \mathscr{E}$ are globally generated and $H^1(\mathscr{O}_Y(8O) \otimes \mathscr{E}) = 0$. Then (4) implies that $\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}$ is globally generated. Therefore $\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}$ has nowhere vanishing global sections, so there are surjective elements μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$.

Now we are going to prove (1) so we assume $\mathscr{E}^{-1} \neq \mathscr{O}_Y(1)$. We have just seen that there are μ which are surjective, so in particular, there are homomorphisms μ in $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ which are nonzero. Now we see that if $\widetilde{Y}$ and $\widehat{Y}$ are as in (i), (ii), then $\hat{g} = 5, 6, 7$ or 9. For this we study the possible images of a nonzero element μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$. Recall that in this case the image of μ is $\mathscr{E}'$, which is a line subbundle of $\mathscr{E}$. Since the degree of $\mathscr{E}$ is -5, then the $\deg \mathscr{E}' \leq -5$. On the other hand, (4) and the fact that E'_Y does not split implies that if $\deg \mathscr{E}' \leq -9$, then $\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}'$ does not have nonzero global sections, so no such $\mathscr{E}'$ can be the image of μ.

Now we see that there are ribbons $\widehat{Y}$ as in (i), (ii), of arithmetic genus 6, 7, 8 and 9 and with conormal bundle $\mathscr{E}''$, for any line bundle $\mathscr{E}''$ of degree $-6, -7$ or -8 or satisfying $\mathscr{E}'' = \mathscr{E}$. For this it suffices to show the existence of homomorphisms μ whose image is such a line bundle $\mathscr{E}''$. If $\mathscr{E}'' = \mathscr{E}$, we did already see this when we proved the existence of surjective elements in $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$. If $\mathscr{E}''$ is of degree $-6, -7$ or -8, then $\mathscr{E}''$ is a subbundle of $\mathscr{E}$ so for each line bundle $\mathscr{E}''$ of degree $-6, -7$ or -8 there is an effective divisor D on Y, respectively of degree 1, 2 or 3, such that $\mathscr{E}'' = \mathscr{E} \otimes \mathscr{O}_Y(-D)$. Then, such $\mathscr{E}''$ being the image of some μ is equivalent to the existence of a nowhere vanishing global section of $\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}''$. For the latter, it suffices to show the existence of a global section of $\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}$ vanishing exactly along D. To prove the existence of this global section we start arguing for $\mathscr{E}'' = \mathscr{O}_Y(-8O)$. In this case $\mathscr{O}_Y(8O) \otimes \mathscr{E} = \mathscr{O}_Y(D)$, so there is global section of $\mathscr{O}_Y(8O) \otimes \mathscr{E}$ that maps to a nonzero global section of $\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}$ vanishing exactly along the divisor D.

Now suppose $\mathscr{E}'' \neq \mathscr{O}_Y(-8O)$ and is of degree -6, -7 or -8. Then it suffices to prove that

(*) there is a nowhere vanishing section $t \in H^0(E'_Y(8O) \otimes \mathscr{E}'')$.

Indeed, we consider the exact sequences

$$\begin{aligned} 0 \longrightarrow \mathscr{O}_Y(8O) \otimes \mathscr{E}'' \longrightarrow E'_Y(8O) \otimes \mathscr{E}'' \longrightarrow \mathscr{O}_Y(9O) \otimes \mathscr{E}'' \longrightarrow 0, \\ 0 \longrightarrow \mathscr{O}_Y(8O) \otimes \mathscr{E}'' \longrightarrow \mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}'' \longrightarrow E'_Y(8O) \otimes \mathscr{E}'' \longrightarrow 0. \end{aligned} \tag{5}$$

Thus, if (*) holds, since $h^1(\mathscr{O}_Y(8O) \otimes \mathscr{E}'') = 0$, then the section t can be lifted to a section $s \in H^0(\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}'')$. Thus s is a nowhere vanishing global section of $\mathscr{N}_{Y,\mathbb{P}^4} \otimes \mathscr{E}''$ as wished.

Now let us prove (*). If $\mathscr{E}''$ is a subbundle of $\mathscr{E}$ of degree -6, then the first sequence of (5) shows that $E'_Y(8O) \otimes \mathscr{E}''$ is globally generated. This implies (*) in this case. If $\mathscr{E}''$ is a subbundle of $\mathscr{E}$ of degree -7, then $\mathscr{O}_Y(8O) \otimes \mathscr{E}'' = \mathscr{O}_Y(P)$ for some $P \in Y$. Let $t' \in H^0(E'_Y(8O) \otimes \mathscr{E}'')$ be a counterimage of a section $t'' \in H^0(\mathscr{O}_Y(9O) \otimes \mathscr{E}'')$ not vanishing at P. Then, for a suitable $r \in H^0(\mathscr{O}_Y(8O) \otimes \mathscr{E}'')$ the zero locus of $t = t' + r \in H^0(E'_Y(8O) \otimes \mathscr{E}'')$ is empty. Finally consider a subbundle $\mathscr{E}''$ of $\mathscr{E}$ of degree -8 such that $\mathscr{E}'' \neq \mathscr{O}(-8O)$. In this case, $h^0(E'_Y(8O) \otimes \mathscr{E}'') = h^0(\mathscr{O}_Y(9O) \otimes \mathscr{E}'') = 1$, so $E'_Y(8O) \otimes \mathscr{E}''$ has nonzero global sections which are either nowhere vanishing or vanish exactly along one point of Y. We claim that only the former happens. Suppose the contrary, i.e, suppose there exists a section $t \in H^0(E'_Y(8O) \otimes \mathscr{E}'')$ whose zero locus is exactly one point of Y. Then, since $\mathscr{E}''$ is a degree -8 subbundle of $\mathscr{E}$, which has degree -5, t gives rise to a global section m of $E'_Y(8O) \otimes \mathscr{E}$ that vanishes on a subscheme of Y of length exactly 4. We see now that this is impossible. Indeed, consider the first sequence of (4). If m maps to $0 \in H^0(\mathscr{O}_Y(9O) \otimes \mathscr{E})$, then m comes from a nonzero global section of $\mathscr{O}_Y(8O) \otimes \mathscr{E}$, which is a line bundle of degree 3, so m would vanish at a subscheme of Y of length exactly 3, not 4. Thus m should map to a nonzero global section m' of $\mathscr{O}_Y(9O) \otimes \mathscr{E}$. Since $\mathscr{O}_Y(9O) \otimes \mathscr{E}$ is a line bundle of degree 4, m vanishes along a subscheme of Y of length 4 if and only if the zero loci of m and m' are the same. In that case the exact sequence (4) will split, so E'_Y would be decomposable and this is a contradiction. Thus all the nonzero global sections of $E'_Y(8O) \otimes \mathscr{E}'$ are nowhere vanishing.

Now we finish the proof of (1). Since (1), (2) and (3) of Corollary 2.1 are satisfied, (i), (ii) and (iii) follow from Corollary 2.1 and the last claim of (1) follows from Observation 1.6.

Now we are going to prove (2), so assume $\mathscr{E}^{-1} = \mathscr{O}_Y(1)$. As seen before, there are surjective homomorphisms in $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$. In this case we cannot apply Corollary 2.1 because $\mathscr{E} \otimes O_Y(1)$ equals $\mathscr{O}_Y$, which is special. Then (2) follows from [4, Theorem 1.5] arguing as we did in the proof of Theorem 4.1, (2). □

5 Hom $(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ and the semiuniversal deformation of φ

In Sect. 1 we gave a geometric interpretation of a given nonzero element μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$: μ "produces" a multiple structure $\widetilde{Y}$ on Y in $\mathbf{P}^r$ that appears when a family of embeddings (more generally, a family of morphisms of degree 1) degenerates to a double cover of Y (in fact, in Sect. 1 we gave this interpretation for homomorphisms μ whose image has rank n and families of embeddings degenerating to an n-to-one morphism, but recall that from Sect. 2 onwards we are assuming $\mathscr{E}$ to be a line bundle). In this section we explore further how this process takes place. There are specific conditions under which the double structure "produced" by μ is determined uniquely. E.g., this happens if μ is surjective (see Observation 1.6). However, in general, the assignment of a double structure to an element μ of $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ is far from unique. Indeed, the next Proposition 5.1 shows that $\widetilde{Y}$ is not uniquely determined by μ, not even by the counterimage $\nu \in H^0(\mathscr{N}_\varphi)$ chosen in the construction made in the proof of Theorem 1.5. Instead, μ only determines the minimal primary component of the ideal sheaf of $\widetilde{Y}$. To determine $\widetilde{Y}$ completely we need to specify not only a tangent vector v to $\mathscr{V}$ but also the way in which we extend v to an algebraic or, at least, formal curve T, tangent to v, in the process to produce a family of morphisms deforming φ ($\mathscr{V}$ is the base of an algebraic formally semiuniversal deformation of φ; see [10, 2.5.7]). I.e, in order to determine $\widetilde{Y}$ we need not only to look at a first order infinitesimal deformation of φ but also possibly at higher order infinitesimal deformations of φ.

Proposition 5.1 *Let C be a smooth, irreducible hyperelliptic curve of genus $\tilde{g}$, let $\pi : C \longrightarrow \mathbf{P}^1$ be its associated hyperelliptic double cover and let $\varphi : C \longrightarrow \mathbf{P}^{\tilde{g}}$ be the morphism induced by $H^0(\pi^*\mathscr{O}_{\mathbf{P}^1}(\tilde{g}))$. Let $\mathscr{V}$ be the base of an algebraic formally semiuniversal deformation of φ and let 0 denote the point of $\mathscr{V}$ corresponding to φ. Let $\mathscr{U}$ be the locus of $\mathscr{V}$ parameterizing embeddings from smooth irreducible curves to $\mathbf{P}^{\tilde{g}}$, let $\mathscr{Z}$ be the complement of $\mathscr{U}$ in $\mathscr{V}$ and let $\mathscr{H}$ be the locus of $\mathscr{V}$ parameterizing morphisms of degree 2 from smooth irreducible curves onto their images in $\mathbf{P}^{\tilde{g}}$. There exists*

(1) A nonzero, non surjective element ν of $H^0(\mathscr{N}_\varphi)$;
(2) Two smooth irreducible algebraic curves T_1 and T_2, passing through 0 and tangent to v (where v is the vector of the tangent space to $\mathscr{V}$ at 0 corresponding to ν), T_1 contained in $\mathscr{Z} \setminus \mathscr{H}$ and T_2 contained in $\mathscr{U}$ except for 0; and
(3) Two flat families $\mathscr{C}_1$ and $\mathscr{C}_2$ of smooth irreducible curves of genus $\tilde{g}$ such that the T_1-morphism $\Phi_1 : \mathscr{C}_1 \longrightarrow \mathbf{P}^{\tilde{g}}_{T_1}$ and the T_2-morphism $\Phi_2 : \mathscr{C}_2 \longrightarrow \mathbf{P}^{\tilde{g}}_{T_2}$ obtained as pullbacks from the semiuniversal deformation of φ satisfy that
   (a) $[\Phi_1(\mathscr{C}_1)]_0$ is a ribbon of genus $\tilde{g}+1$ and, for all $t \in T_1, t \neq 0$, $(\Phi_1)_t$ is of degree 1 and $(\Phi_1(\mathscr{C}_1))_t$ is a reduced and irreducible curve of arithmetic genus $\tilde{g}+1$ with one singular point; and
   (b) $[\Phi_2(\mathscr{C}_2)]_0$ is a locally non Cohen–Macaulay double structure of arithmetic genus $\tilde{g}$ containing a ribbon of arithmetic genus $\tilde{g}+1$ and an embedded point and, for all $t \in T_2, t \neq 0$, $(\Phi_2)_t$ is an embedding.

Proof Let T_1 be a smooth, irreducible, algebraic curve with a distinguished closed point 0 and let $\mathscr{C}_1$ be a flat family over T_1 of smooth irreducible curves such that $(\mathscr{C}_1)_0 = C$ and $(\mathscr{C}_1)_t$ is non hyperelliptic of genus $\tilde{g}$ for all $t \in T_1, t \neq 0$. Shrinking T_1 if necessary, we may consider a relative line bundle $\mathscr{L}$ on $\mathscr{C}_1$ such that $\mathscr{L}_0 = \varphi^*\mathscr{O}_{\mathbf{P}^{\tilde{g}}}(1)$ and, for all $t \in T_1$, $\mathscr{L}_t$ is $\omega_{(\mathscr{C}_1)_t}$ twisted by an effective line bundle of degree 2. The relative global sections of $\mathscr{L}$ induce a T_1-morphism $\Phi_1 : \mathscr{C}_1 \longrightarrow \mathbf{P}^{\tilde{g}}_{T_1}$. Let Δ_1 be the first infinitesimal neighborhood of 0 in T_1, let $\tilde{\varphi} = (\Phi_1)_{\Delta_1}$, let ν be the element of $H^0(\mathscr{N}_\varphi)$ that corresponds to $\tilde{\varphi}$ and let $\mu = \Psi_2(\nu)$. In addition, $\mathscr{C}_1$ can be taken so that ν, and therefore μ, is nonzero. Let $Y = \varphi(C)$, which is a smooth rational normal curve of degree $\tilde{g}$. The image $\mathscr{Y}_1$ of Φ_1 is a flat family of 1-dimensional subschemes of $\mathbf{P}^{\tilde{g}}$ such that, if $t \neq 0$, $(\mathscr{Y}_1)_t$ is an irreducible and reduced singular curve of degree $2\tilde{g}$ and arithmetic genus $\tilde{g}+1$, having exactly one singular point. Thus $(\mathcal{Y}_1)_0$ is a double structure on Y of arithmetic genus $\tilde{g}+1$ that contains $\mathrm{im}\tilde{\varphi}$. Let $\mathcal{E}$ be the trace zero module of π. Then $\mathcal{E}=\mathcal{O}_{\mathbf{P}^1}(-\tilde{g}-1)$ and the conormal bundle of Y in $\mathbf{P}^{\tilde{g}}$ is $\mathcal{I}/\mathcal{I}^2=\mathcal{O}_{\mathbf{P}^1}(-\tilde{g}-2)^{\oplus \tilde{g}-1}$. Then the image of the homomorphism μ of $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2,\mathcal{E})$ is either $\mathcal{O}_{\mathbf{P}^1}(-\tilde{g}-1)$ or $\mathcal{O}_{\mathbf{P}^1}(-\tilde{g}-2)$. By [6, Theorem 3.8 (1)], in the first case $\mathrm{im}\tilde{\varphi}$ is a ribbon on Y of arithmetic genus $\tilde{g}$ and this would contradict the fact that $(\mathcal{Y}_1)_0$ is a double structure on Y of arithmetic genus $\tilde{g}+1$. Thus necessarily the image of μ is $\mathcal{O}_{\mathbf{P}^1}(-\tilde{g}-2)$. Then $\mathrm{im}\tilde{\varphi}$ is a ribbon on Y of arithmetic genus $\tilde{g}+1$ and $\mathrm{im}\tilde{\varphi}=(\mathcal{Y}_1)_0$. This proves the existence of the desired family Φ_1 satisfying (a). Moreover, by formal semiuniversality, since ν is nonzero, after possibly shrinking T_1 again and taking a suitable étale cover, we may assume that T_1 admits an embedding to $\mathcal{V}$ that maps 0 to $[\varphi]$ and the tangent vector to T_1 at 0 to v in such a way that the pullback of the semiuniversal deformation of φ to T_1 is Φ_1. Then the image of $T_1 \smallsetminus \{0\}$ in $\mathcal{V}$ is obviously contained in $\mathcal{Z} \smallsetminus \mathcal{H}$ by the construction of Φ_1 so, by an abuse of notation, we may identify T_1 with its image in $\mathcal{V}$.

Now, to construct Φ_2 recall that, since $\mathcal{O}_Y(1)$ and $\mathcal{E}\otimes\mathcal{O}_Y(1)$ are non special, by (1) and (2), φ is unobstructed. Then it is possible to find a smooth, irreducible algebraic curve T_2 in $\mathcal{V}$, passing through 0, tangent to the vector v corresponding to ν and contained in the locus $\mathcal{U}$ except for 0. Pulling back the semiuniversal deformation of φ to T_2 we get a flat family $\mathcal{C}_2$ of smooth irreducible curves of genus $\tilde{g}$ and a T_2-morphism $\Phi_2:\mathcal{C}_2\longrightarrow \mathbf{P}^{\tilde{g}}_{T_2}$ such that $(\Phi_2)_t$ is an embedding for all $t\in T_2, t\neq 0$ and $(\Phi_2)_{\Delta_2}=\tilde{\varphi}$. Let $\mathcal{Y}_2=\Phi_2(\mathcal{C}_2)$. Then $(\mathcal{Y}_2)_t$ is a smooth irreducible curve in $\mathbf{P}^{\tilde{g}}$ of degree $2\tilde{g}$ and genus $\tilde{g}$ and $(\mathcal{Y}_2)_0$ is a double structure on Y also of arithmetic genus $\tilde{g}$. As before, $\mathrm{im}\tilde{\varphi}$, which is a ribbon of arithmetic genus $\tilde{g}+1$, is contained in $(\mathcal{Y}_2)_0$, so $(\mathcal{Y}_2)_0$ is the union of $\mathrm{im}\tilde{\varphi}$ and a double point supported on an embedded point of $(\mathcal{Y}_2)_0$. This proves the existence of a family Φ_2 satisfying (b). □

Proposition 5.1 provides an example that shows that, in general, μ does not uniquely determine a double structure. However, we saw (cf. Observation 1.6) that, if μ is surjective, the double structure associated to μ is uniquely determined by μ (it is a ribbon with conormal bundle $\mathcal{E}$). The next proposition tells the geometric reason behind this: if μ is surjective, any ν in $H^0(\mathcal{N}_\varphi)$ lying over μ corresponds to a tangent vector to $\mathcal{V}$ that only extends to paths contained in $\mathcal{U}$.

Proposition 5.2 *Let μ be a homomorphism of $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2,\mathcal{E})$. Let C be a smooth irreducible curve of genus $\tilde{g}$ and let $\varphi: C\longrightarrow \mathbf{P}^r$ be a morphism from C to $\mathbf{P}^r$ which factors through a double cover $\pi: C\longrightarrow Y$ with trace zero module $\mathcal{E}$. Let $\mathcal{V}$ be the base of an algebraic formally semiuniversal deformation of φ and let $[\varphi]\in\mathcal{V}$ be the point of $\mathcal{V}$ corresponding to φ. Let ν be a counterimage of μ in $H^0(\mathcal{N}_\varphi)$, let $\tilde{\varphi}$ be the first order infinitesimal deformation corresponding to ν and let v the tangent vector of $\mathcal{V}$ corresponding to $\tilde{\varphi}$. Let $\mathcal{U}$ be the locus of $\mathcal{V}$ parameterizing embeddings from smooth irreducible curves to $\mathbf{P}^r$ and let $\mathcal{Z}$ be the complement of $\mathcal{U}$ in $\mathcal{V}$. Suppose that φ is unobstructed, then*

(1) v is tangent to an algebraic, smooth, irreducible curve passing through $[\varphi]$ and contained in $\mathcal{U}$ except for $[\varphi]$; and

(2) v cannot be tangent to any algebraic, smooth, irreducible curve passing through $[\varphi]$ and contained in $\mathcal{Z}$.

Proof If φ is unobstructed, then the base $\mathcal{V}$ of an algebraic formally semiuniversal deformation of φ is smooth at the point $[\varphi]$ corresponding to φ. By semiuniversality, there exist a smooth algebraic irreducible curve T with a distinguished point 0, a flat family $\mathcal{C}$ over T and a T-morphism $\Phi:\mathcal{C}\longrightarrow \mathbf{P}^{\tilde{g}}_T$ such that $\mathcal{C}_t$ is a smooth irreducible curve, $\Phi_0=\varphi$

and $\Phi_{\Delta} = \tilde{\varphi}$. [4, Proposition 1.4] implies that, after shrinking T if necessary, Φ_t is an embedding for all $t \in T, t \neq 0$ (since we are working with 1-dimensional schemes, there is an alternate argument for this, based on studying what possible arithmetic genera the flat limit of $\Phi_t(\mathscr{C}_t)$ might have). This proves (1). Now suppose there exists an algebraic smooth irreducible curve T passing through $[\varphi]$ and contained in $\mathscr{Z}$ whose tangent vector at $[\varphi]$ is v. Then pulling back the semiuniversal deformation of φ to T we would obtain a family of morphisms contradicting [4, Proposition 1.4]. □

Remark 5.3 The unobstructedness of φ required in Proposition 5.2 is not a very strong condition. For instance, φ is unobstructed if both $\mathscr{O}_Y(1)$ and $\mathscr{E} \otimes \mathscr{O}_Y(1)$ are nonspecial (this is condition (3) of Theorem 1.5 and Corollary 2.1). On the other hand, Proposition 5.2 remains true if, instead of assuming φ to be unobstructed, we assume the existence of a smooth irreducible algebraic curve T with a distinguished closed point 0, a flat family $\mathscr{C}$ over T and a T-morphism $\Phi : \mathscr{C} \longrightarrow \mathbf{P}^{\tilde{g}}_T$ such that $\mathscr{C}_t$ is a smooth irreducible curve, $\Phi_0 = \varphi$ and $\Phi_{\Delta} = \tilde{\varphi}$, where Δ is the first infinitesimal neighborhood of 0 in T.

Appendix

We take advantage of this opportunity to fix a gap in the article [5], written by the first and third author. The gap concerns the arguments used there to prove [5, Theorem 3.5]. [5, Theorem 3.5] says that $K3$ carpets supported on rational normal scrolls can be smoothed. [5, Theorem 3.5] is nevertheless true and a different, independent proof of it was given in [4, Corollary 2.9].

We explain first what the problem is with the argument in [5] and then we outline the way to fix it. Precisely, the problem lies in [5, Lemma 3.2], which is false as stated. The main thesis of [5, Lemma 3.2] is this:

Let $\mathscr{X}$ be a flat family of irreducible varieties over a smooth irreducible algebraic curve T which is mapped to relative projective space by a morphism Φ induced by a relatively complete linear series. Assume that Φ_t is an embedding for all $t \neq 0$ $(0 \in T)$ and Φ_0 is a finite morphism of degree 2. Let H be a hyperplane in projective space. Then $\Phi(\mathscr{C}) \cap (H \times T)$ is flat.

This claim is false in general. Indeed, if it were true, using [5, Theorem 2.1] and arguing like in [5, Corollary 3.3] we would show that the double structure $\widetilde{Y}$ that appears in Corollary 2.1 is always a ribbon and this is false as remarked in Observation 1.6. The mistake in the proof of [5, Lemma 3.2] is that, when we tensor the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathscr{Y}} \xrightarrow{\alpha} \Phi_* \mathscr{O}_{\mathscr{C}} \longrightarrow \mathscr{F} \longrightarrow 0$$

with $\mathscr{O}_{\mathbf{P}^n_T}/\mathscr{I}(H \times T)$, the resulting sequence does not necessarily remain exact on the left.

We point out now how to avoid using [5, Lemma 3.2] when proving [5, Theorem 3.5]. The proof of [5, Theorem 3.5] is based on [5, Proposition 3.4], which says that the flat limit at $t = 0$ of a family of $K3$ surfaces $\mathscr{Y}_t$ embedded by a very ample polarization ζ_t is a $K3$ carpet provided that $(\mathscr{Y}_0, \zeta_0)$ is a hyperelliptic polarized $K3$ surface. [5, Proposition 3.4] was proved using [5, Theorem 2.1] and [5, Theorem 2.1] essentially says the following:

A double structure D of dimension m, supported on D_{red} is locally Cohen–Macaulay if and only if through every closed point of D there exists locally a Cartier divisor h that cuts out on D a locally Cohen–Macaulay double structure of dimension $m - 1$, supported on the restriction of h to D_{red}.

It was in the process of applying [5, Theorem 2.1] to prove [5, Proposition 3.4] that we used [5, Lemma 3.2]. Thus we outline now a different argument avoiding the use of [5, Lemma 3.2]. If the flat family $(\mathscr{C}, \zeta)$ over T and the relative hyperplane section $H \times T$ of [5, Proposition 3.4] are suitably chosen, then [5, Theorem 2.1] can be applied to $\mathscr{Y}_0$ in the same way as in the proof of [5, Proposition 3.4]. Precisely, let $\tilde{\varphi}$ be the restriction of Φ_ζ to $H \times \Delta$. If one is able to choose $(\mathscr{C}, \zeta)$ and the relative hyperplane section $H \times T$ so that $\tilde{\varphi}$ corresponds to a surjective homomorphism in $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$, then $\mathscr{Y}_0 \cap H$ will be a canonical ribbon. Then [5, Proposition 3.4] would follow from [5, Theorem 2.1].

References

1. Chandler, K.A.: Geometry of dots and ropes. Trans. Amer. Math. Soc. **347**, 767–784 (1995)
2. Fong, L.Y.: Rational ribbons and deformation of hyperelliptic curves. J. Algebraic Geom. **2**, 295–307 (1993)
3. Gallego, F.J., González, M., Purnaprajna, B.P.: Deformation of finite morphisms and smoothing of ropes. Compos. Math. **144**, 673–688 (2008)
4. Gallego, F.J., González, M., Purnaprajna, B.P.: An infinitesimal criterion to assure that a finite morphism can be deformed to an embedding. Rev. Mat. Complut. **26**, 253–269 (2013)
5. Gallego, F.J., Purnaprajna, B.P.: Degenerations of K3 surfaces in projective space. Trans. Amer. Math. Soc. **349**, 2477–2492 (1997)
6. González, M.: Smoothing of ribbons over curves. J. Reine Angew. Math. **591**, 201–235 (2006)
7. Hulek, K.: Projective geometry of elliptic curves, Astérisque vol. **137**, Société Mathématique de France (1986)
8. Hulek, K., Van de Ven, A.: The Horrocks–Mumford bundle and the Ferrand construction. Manuscripta Math. **50**, 313–335 (1985)
9. Migliore, J.C., Peterson, C., Pitteloud, Y.: Ropes in projective space. J. Math. Kyoto Univ. **36**, 251–278 (1996)
10. Sernesi, E.: Deformations of Algebraic Schemes, vol. **334**, Springer, Grundlehren der mathematischen Wissenschaften (2006)