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ABSTRACT

Using the contraction of the centrally extended Schrödinger algebra $\widehat{\mathcal{S}}(N)$ onto the Lie algebra $\mathcal{S}(N)\oplus\mathbb{R}$ in combination with the Newton identities associated with the characteristic polynomial of a matrix, we derive explicit expressions for the Casimir operators of the unextended Schrödinger algebra $\mathcal{S}(N)$ in terms of trace operators. It is shown that these operators can be defined independently of the contraction from which a direct method for the computation of the $\mathcal{S}(N)$-invariants is deduced.

I. INTRODUCTION

The maximal kinematical invariance algebra of the free Schrödinger equation $\left(\frac{\hbar}{2m}\Delta + i\partial_t\right)\psi(t,\mathbf{x}) = 0$, commonly known as the Schrödinger algebra $\mathcal{S}(N)$, and its central extension $\widehat{\mathcal{S}}(N)$ arising naturally in the context of projective representations of $\mathcal{S}(N)$ are certainly among the most studied physically relevant algebras that contain the Galilean algebra.[1–4] Although $\widehat{\mathcal{S}}(N)$ and $\mathcal{S}(N)$ are not the result of a contraction of the conformal Lie algebra $\mathfrak{so}(2,4)$, a significant connection between these algebras has been established, relating the notions of free relativistic particles in spaces of constant curvature to nonrelativistic particles in external fields.[5–9] In a more ample context, the extended Schrödinger algebra appears for the value $\ell=\frac{1}{2}$ as a particular case within the hierarchy of the so-called generalized conformal Galilean algebras $\mathfrak{Gal}_\ell(N)$.[10–15] Considered from a structural point of view, the extended Schrödinger algebra $\widehat{\mathcal{S}}(N)$ is the semidirect product of the semisimple Lie algebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(N)$ and the Heisenberg algebra $\mathfrak{h}_N$, thus belonging to a large class of Lie algebras whose fundamental properties have been analyzed in some detail.[16] In particular, the Casimir operators of such Lie algebras can be determined explicitly using either analytical or purely algebraic procedures (see, e.g., Refs. 17–19 and references therein). As the contraction of $\widehat{\mathcal{S}}(N)$ onto the decomposable Lie algebra $\mathcal{S}(N)\oplus\mathbb{R}$ preserves the number of Casimir operators, it is reasonable to derive the invariants of the unextended Schrödinger algebra $\mathcal{S}(N)$ as a limit of the invariants of $\widehat{\mathcal{S}}(N)$. This procedure, albeit valid, is not entirely satisfactory as it requires the explicit consideration of the Lie algebra contraction and a suitable choice of invariants in $\widehat{\mathcal{S}}(N)$ in order to ensure that the corresponding limits provide independent invariants of the contraction.

The objective of this work is to propose a constructive method for the invariants of the Lie algebra $\mathcal{S}(N)$ based uniquely on the traces of suitably chosen matrices, so that a complete set of invariants can be computed directly, without using first the central extension $\widehat{\mathcal{S}}(N)$. In order to derive such a method, we first reformulate the well-known construction of the Casimir operators of $\widehat{\mathcal{S}}(N)$ in terms of the so-called Newton identities.[20] Combining appropriately the resulting expressions with the Inönü–Wigner contraction of $\widehat{\mathcal{S}}(N)$ onto $\mathcal{S}(N)\oplus\mathbb{R}$, we can write the invariants of $\widehat{\mathcal{S}}(N)$ as trace polynomials dependent on the contraction parameter. Using the contraction of invariants,[21] we obtain the Casimir operators of the unextended Schrödinger algebra $\mathcal{S}(N)$ as the sum of trace operators. As these traces are obtained from matrices that solely depend upon the commutators of $\mathcal{S}(N)$, it follows that the invariants can be constructed directly and independently of the contraction.

Unless otherwise stated, any Lie algebra considered in this work is finite-dimensional and defined over the field of real numbers.

Given a basis $\{X_1, \ldots, X_n\}$ and the structure tensor $\{C_{ij}^k\}$ of the Lie algebra $\mathfrak{g}$, its elements can be realized as differential operators in the space $C^\infty(\mathfrak{g}^*)$,

$$\widehat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{1}$$

where $[X_i, X_j] = C_{ij}^k X_k$ $(1 \le i < j \le n)$ and $\{x_1, \ldots, x_n\}$ are the coordinates in a dual basis of $\{X_1, \ldots, X_n\}$. A smooth function $F \in C^\infty(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system of partial differential equations (PDEs),

$$\widehat{X}_i F = 0, \; 1 \le i \le n. \tag{2}$$

The number $\mathcal{N}(\mathfrak{g})$ of functionally independent solutions of such a system is an invariant of the algebra and is well-known to be given by

$$\mathcal{N}(\mathfrak{g}) := \dim\, \mathfrak{g} - \mathrm{rank}\Big(C_{ij}^k x_k\Big)_{1 \le i < j \le \dim \mathfrak{g}}, \tag{3}$$

where $A(\mathfrak{g}) := \big(C_{ij}^k x_k\big)$ is the matrix that represents the commutator table of $\mathfrak{g}$ over the given basis. For polynomials F in the commuting variables $\{x_1, \ldots, x_n\}$, the operator $\widehat{X}_i F$ represents the action of the generator X_i of $\mathfrak{g}$ on F. If the polynomial is invariant by this action, i.e., satisfies $\hat{X}_i F = 0$, then it corresponds to an element in the center $Z(\mathfrak{U}(\mathfrak{g}))$ of the enveloping algebra of $\mathfrak{g}$.[22] The explicit correspondence between the set of polynomial solutions of (2) and $Z(\mathfrak{U}(\mathfrak{g}))$ is established using the following symmetrization map, defined on monomials $x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_p}$ as

$$\mathrm{Sym}_p(x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p}) := \frac{1}{p!} \sum_{\sigma \in \Sigma_p} X_{\alpha_{\sigma(1)}} X_{\alpha_{\sigma(2)}} \cdots X_{\alpha_{\sigma(p)}}, \tag{4}$$

where Σ_p is the symmetric group in p letters. The image of a polynomial $F(x_1, \ldots, x_n)$ is easily obtained by linear extension of (4).

We briefly recall the elementary notions about contractions of Lie algebras that will be used in the following.[23] Consider a Lie algebra $\mathfrak{g}$ and a family of automorphisms Ψ_ε with $\varepsilon \in (0, 1]$ such that $\Psi_1 = \mathrm{Id}_{\mathfrak{g}}$. For arbitrary elements $X, Y \in \mathfrak{g}$, we define

$$[X, Y]_\varepsilon := \Phi_\varepsilon^{-1}[\Phi_\varepsilon(X), \Phi_\varepsilon(Y)] \tag{5}$$

that obviously corresponds to the Lie bracket over the transformed basis. If the limit

$$[X, Y]_0 := \lim_{\varepsilon \to 0} \Phi_\varepsilon^{-1}[\Phi_\varepsilon(X), \Phi_\varepsilon(Y)] \tag{6}$$

exists for all $X, Y \in \mathfrak{g}$, Eq. (6) defines a Lie algebra $\mathfrak{g}'$ that we call the contraction of $\mathfrak{g}$ (by Φ_ε). If $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}'$, then the contraction is said to be non-trivial. For physical applications, the most relevant type of contraction is given by the generalized Inönü–Wigner contractions,[24,25] corresponding to the case where the contraction matrix A_Φ is diagonal on some basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$. In this situation, the matrix elements and the automorphisms are given, respectively, by

$$(A_\Phi)_{ij} = \delta_i^j \varepsilon^{n_j}; \quad \Phi_\varepsilon(X_j) = \varepsilon^{n_j} X_j, \quad n_j \in \mathbb{Z},\; 1 \le j \le n. \tag{7}$$

The limiting process can also be extended to invariant functions of Lie algebras.[21] Given a non-trivial contraction $\mathfrak{g} \rightsquigarrow \mathfrak{g}'$, the inequality $\mathcal{N}(\mathfrak{g}) \le \mathcal{N}(\mathfrak{g}')$ is always satisfied. In particular, for a given contraction of the type (7) and a Casimir operator $F(X_1, \ldots, X_n) = \alpha^{i_1 \cdots i_p} X_{i_1} \ldots X_{i_p}$ of $\mathfrak{g}$ having degree p, it is straightforward to verify that the transformed invariant is given by

$$F(\Phi_\varepsilon^{-1}(X_1), \ldots, \Phi_\varepsilon^{-1}(X_n)) = \varepsilon^{-n_{i_1} - \cdots - n_{i_p}} \alpha^{i_1 \ldots i_p} X_{i_1} \ldots X_{i_p}. \tag{8}$$

Defining

$$\Omega = \max\{n_{i_1} + \cdots + n_{i_p} \quad | \quad \alpha^{i_1 \ldots i_p} \ne 0\}, \tag{9}$$

it follows at once that the limit

$$F'(X_1, \ldots, X_n) = \lim_{\varepsilon \to 0} \varepsilon^\Omega F(\Phi_\varepsilon^{-1}(X_1), \ldots, \Phi_\varepsilon^{-1}(X_n)) = \sum_{n_{i_1} + \cdots + n_{i_p} = \Omega} \alpha^{i_1 \ldots i_p} X_{i_1} \ldots X_{i_p} \tag{10}$$

is a Casimir operator of the contraction $\mathfrak{g}'$. Starting from a suitable fundamental system of invariants $\{C_1, \ldots, C_r\}$ of $\mathfrak{g}$, we can always obtain r independent invariants of $\mathfrak{g}'$, although the primitiveness of the contracted operators cannot be ensured (see, e.g., Refs. 21 and 26).

II. THE CONTRACTION $\hat{\mathcal{S}}(N) \rightsquigarrow \mathcal{S}(N) \oplus \langle M \rangle$

The centrally extended Schrödinger algebra $\widehat{\mathcal{S}}(N)$ structurally corresponds to the semidirect product of the $(2N+1)$-dimensional Heisenberg algebra with the semisimple Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(N)$, where the action of the latter on the former algebra is given by the representation $(D_{\frac{1}{2}} \otimes \Lambda) \oplus D_0$, with $D_{\frac{1}{2}} \otimes \Lambda$ being the tensor product of the fundamental representations $D_{\frac{1}{2}}$ of $\mathfrak{sl}(2,\mathbb{R})$ and Λ of $\mathfrak{so}(N)$ and D_0 being the trivial multiplet.[2,9,12,27] It is convenient to consider the ordered basis $\{D, K, P_t, J_{ij}, P_k, G_k, M\}$, where D is the generator of scale transformations, K is the generator of conformal Galilean transformations, and P_t is the time translation, while P_μ and G_μ are the infinitesimal generators of the spatial translation and special Galilei transformations, respectively. The skew-symmetric operators $J_{\mu\nu} + J_{\nu\mu} = 0$ are rotations, while M is a central mass generator.[2,5] Over this basis, the non-trivial commutators are

$$\begin{array}{ll}
[J_{\mu\nu}, J_{\lambda\sigma}] = \delta_{\mu\lambda}J_{\nu\sigma} + \delta_{\nu\sigma}J_{\mu\lambda} - \delta_{\mu\sigma}J_{\nu\lambda} - \delta_{\nu\lambda}J_{\mu\sigma}, & \\
[J_{\mu\nu}, P_\lambda] = \delta_{\mu\lambda}P_\nu - \delta_{\nu\lambda}P_\mu, & [J_{\mu\nu}, G_\lambda] = \delta_{\mu\lambda}G_\nu - \delta_{\nu\lambda}G_\mu, \\
[P_t, G_\mu] = P_\mu, & [K, P_\mu] = -G_\mu, \\
[D, G_\mu] = G_\mu, & [D, P_\mu] = -P_\mu, \\
[D, K] = 2K, & [D, P_t] = -2P_t, \\
[K, P_t] = -D, & [P_\mu, G_\nu] = \delta_{\mu\nu}M.
\end{array} \tag{11}$$

We observe that the unextended Schrödinger algebra $\mathcal{S}(N)$ corresponds to the factor of $\widehat{\mathcal{S}}(N)$ by its center, generated by M, although it is more convenient to describe $\mathcal{S}(N)$ by means of a contraction procedure.[21] For $\varepsilon \in (0,1]$, define the automorphism Ψ_ε of $\widehat{\mathcal{S}}(N)$ given by

$$\begin{aligned} \Psi_\varepsilon(D) = D,\ \Psi_\varepsilon(K) = K,\ \Psi_\varepsilon(P_t) = P_t,\ \Psi_\varepsilon(J_{\mu\nu}) = J_{\mu\nu}, \\ \Psi_\varepsilon(G_\mu) = \varepsilon G_\mu,\ \Psi_\varepsilon(P_\mu) = \varepsilon P_\mu,\ \Psi_\varepsilon(M) = M. \end{aligned} \tag{12}$$

Over the transformed basis, it can be easily seen that the brackets are given by

$$\begin{array}{lll}
[J_{\mu\nu}, J_{\lambda\sigma}]_\varepsilon = [J_{\mu\nu}, J_{\lambda\sigma}], & [J_{\mu\nu}, P_\lambda]_\varepsilon = [J_{\mu\nu}, P_\lambda], & [J_{\mu\nu}, G_\lambda]_\varepsilon = [J_{\mu\nu}, G_\lambda], \\
[P_t, G_\mu]_\varepsilon = [P_t, G_\mu], & [D, G_\mu]_\varepsilon = [D, G_\mu], & [D, P_\mu]_\varepsilon = [D, P_\mu], \\
[D, K]_\varepsilon = [D, K], & [D, P_t]_\varepsilon = [D, P_t], & [K, P_t]_\varepsilon = [K, P_t], \\
[P_\mu, G_\nu]_\varepsilon = \varepsilon^2[P_\mu, G_\nu]. & &
\end{array} \tag{13}$$

For any $\varepsilon \neq 0$, these brackets are clearly those of $\widehat{\mathcal{S}}(N)$, while in the limit $\varepsilon \to 0$, we obtain the commutators of the decomposable Lie algebra $\mathcal{S}(N) \oplus \langle M \rangle$. The Levi decomposition of the latter is given by

$$\mathcal{S}(N) \simeq (\mathfrak{so}(N) \oplus \mathfrak{sl}(2,\mathbb{R})) \vec{\oplus}_{D_{\frac{1}{2}} \otimes \Lambda} \mathbb{R}^{2N}, \tag{14}$$

and hence, $\dim \mathcal{S}(N) = \frac{1}{2}\left(N^2 + 3N + 6\right)$. The contraction (12) preserves the number of invariants so that the identity

$$\mathcal{N}(\mathcal{S}(N)) = \mathcal{N}\left(\widehat{\mathcal{S}}(N)\right) - 1 \tag{15}$$

holds, suggesting the construction of the Casimir operators of $\mathcal{S}(N)$ using the limit formula (10).

III. CASIMIR OPERATORS OF $\hat{\mathcal{S}}(N)$ AND THE NEWTON IDENTITIES

As any semidirect product $\mathfrak{g} \vec{\oplus}_R \mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ and the Heisenberg algebra $\mathfrak{h}_N$ satisfies the identity $\mathcal{N}(\mathfrak{g} \vec{\oplus}_R \mathfrak{h}) = \operatorname{rank}(\mathfrak{g}) + 1$, it follows, in particular, that $\mathcal{N}\left(\widehat{\mathcal{S}}(N)\right) = 2 + \left[\frac{N}{2}\right]$. The invariants of $\widehat{\mathcal{S}}(N)$ can further be constructed explicitly using the following prescription (see, e.g., Refs. 26 and 27):

1. Define

$$\widetilde{j}_{\mu\nu} = m j_{\mu\nu} + g_\mu p_\nu - g_\nu p_\mu,\ 1 \le \mu < \nu \le N, \tag{16}$$

and let $\hat{\mathbf{A}}$ be the skew-symmetric polynomial matrix with entries $\hat{\mathbf{A}}_{\mu\nu} = \widetilde{j}_{\mu\nu}$ for $\mu < \nu$. Compute the characteristic polynomial of $\hat{\mathbf{A}}$,

$$\chi(T) = \left|\hat{\mathbf{A}} - T\mathrm{Id}_N\right| = T^{\frac{1-(-1)^N}{2}}\left(T^{2\left[\frac{N}{2}\right]} + \sum_{s=1}^{\left[\frac{N}{2}\right]} \hat{C}_{2s} T^{2\left(\left[\frac{N}{2}\right]-s\right)}\right), \tag{17}$$

and determine the $\left[\frac{N}{2}\right]$ coefficient functions $\hat{C}_{2s}$.

2. Define $C'_4 = \widetilde{d}^2 - 4\widetilde{k}\widetilde{p_t}$, where

$$\widetilde{d} = md - \sum_{\mu=1}^{N} g_\mu p_\mu, \quad \widetilde{k} = mk - \frac{1}{2}\sum_{\mu=1}^{N} g_\mu^2, \quad \widetilde{p_t} = mp_t - \frac{1}{2}\sum_{\mu=1}^{N} p_\mu^2. \tag{18}$$

Through the symmetrization map (4), the Casimir operators of the Lie algebra $\widehat{\mathcal{S}}(N)$ are obtained as the symmetric representatives $\mathrm{Sym}(\widetilde{C}'_4)$ and $\mathrm{Sym}(\widetilde{C}_{2s})$ for $1 \le s \le \left[\frac{N}{2}\right]$ to which the central charge M must be added.

A. Invariants of $\hat{\mathcal{S}}(N)$ in terms of traces

The functions $\hat{C}_{2s}$ in (17) can alternatively be described in terms of the traces $\mathrm{Tr}(\hat{\mathbf{A}}^k)$ using the so-called Newton identities,[28] a procedure that will enable us to give also a description of the invariants of the unextended Schrödinger algebra in terms of traces. The characteristic polynomial of $\hat{\mathbf{A}}$ can formally be written as

$$\chi(T) = (-1)^N\left[T^N + \lambda_1 T^{N-1} + \lambda_2 T^{N-2} + \cdots + \lambda_{N-1}T + \lambda_N\right], \tag{19}$$

where $\lambda_1 = -\mathrm{Tr}(\hat{\mathbf{A}})$ and $\lambda_N = (-1)^N \det(\hat{\mathbf{A}})$. For any $k \ge 1$, we define the trace of the kth-power of $\hat{\mathbf{A}}$ as $\sigma_k = \mathrm{Tr}(\hat{\mathbf{A}}^k)$. Then, the following relations, known as the Newton identities, are satisfied:[28]

$$\lambda_1 + \sigma_1 = 0, \quad \sigma_k + \sum_{\ell=1}^{k-1}\lambda_\ell\sigma_{k-\ell} + k\lambda_k = 0,\ k = 2,\ldots,N. \tag{20}$$

The explicit expressions for the coefficients λ_k are obtained solving recursively the preceding system, leading to trace polynomials of the type (see, e.g., Refs. 20 and 28)

$$\lambda_k = -\frac{1}{k}\sigma_k + \frac{1}{2!}\sum_{i_1+i_2=k}\frac{\sigma_{i_1}\sigma_{i_2}}{i_1 i_2} - \frac{1}{3!}\sum_{i_1+i_2+i_3=k}\frac{\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}}{i_1 i_2 i_3} + \cdots + \frac{(-1)^k}{k!}\sigma_1^k. \tag{21}$$

IV. CASIMIR OPERATORS OF $\hat{\mathcal{S}}(N)$ AND $\mathcal{S}(N)$ AS TRACE OPERATORS

In this section, we reformulate the invariants of the extended Schrödinger algebra using the Newton identities (20), a procedure that, in particular, enables us to obtain an explicit construction of the invariants of $\mathcal{S}(N)$ that turns out to be independent of the contraction.

We first consider the case of invariants of $\widehat{\mathcal{S}}(N)$ that do not depend on the variables of the $\mathfrak{sl}(2,\mathbb{R})$-generators. If we evaluate Eq. (17) over the basis (12) transformed by the automorphism Ψ_ε, the coefficient functions $\hat{C}_{2s}$ can be rewritten as a function with respect to the contraction parameter ε. Expanding the characteristic polynomial $\chi(T)$ in terms of the variables (16), it is straightforward to verify that for any $s \ge 1$, the Casimir invariant $\hat{C}_{2s}$ is given by the following sum (see, e.g., Refs. 15 and 18):

$$\hat{C}_{2s} = m^{2s}\Phi_{2s,1} + 2\varepsilon^{-2}m^{2s-1}\Phi_{2s+1,2} + \varepsilon^{-4}m^{2s-2}\Phi_{2s+2,3}, \tag{22}$$

where the coefficient functions $\Phi_{2s,1}$, $\Phi_{2s+1,2}$, and $\Phi_{2s+2,3}$ can be expressed as a sum of terms of the generic form

$$\prod_{\alpha=1}^{2s} j_{\mu_\alpha\nu_\alpha},\ \prod_{\beta=1}^{2s-1} j_{\mu_\beta\nu_\beta}p_{k_1}g_{k_2},\ \prod_{\gamma=1}^{2s-2} j_{\mu_\gamma\nu_\gamma}p_{k_1}p_{k_2}g_{k_3}g_{k_4}, \tag{23}$$

respectively. The first subindex in Φ describes the degree of the polynomial in the variables distinct from the central charge m. From the decomposition (22) and formula (10), we immediately obtain that the functions $C_{2s} = \lim_{\varepsilon\to 0}\varepsilon^4\,\hat{C}_{2s} = m^{2s-2}\Phi_{2s+2,3}$ are invariants of the contraction $S(N) \oplus \langle M\rangle$. Without loss of generality we can set $m = 1$, as the generator M is central, so that we conclude that the functions $\Phi_{2s+2,3}$ are invariants of the unextended Schrödinger algebra $\mathcal{S}(N)$. This approach is however inconvenient to some extent, as it first requires to compute the Casimir operators of $\widehat{\mathcal{S}}(N)$ and then to contract them using (10). Our purpose is thus to describe each of the functions $\Phi_{2s,1}$, $\Phi_{2s+1,2}$, $\Phi_{2s+2,3}$ in terms of matrix traces in analogy to the formulas in (21), but using a sum of matrices instead of the matrix $\hat{\mathbf{A}}$.

First of all, the structure of the products in (23) suggests us to separate the matrix $\hat{\mathbf{A}}$ into two submatrices, one corresponding to the generators of the orthogonal subalgebra $\mathfrak{so}(N)$ and the other comprising the generators in the representation space. Specifically, let $\hat{\mathbf{A}} = m\mathbf{A} + \mathbf{B}$, with $\mathbf{A}$ and $\mathbf{B}$ being the skew-symmetric matrices having the following entries:

$$(\mathbf{A})_{\mu\nu} = j_{\mu\nu}, \quad (\mathbf{B})_{\mu\nu} = g_\mu p_\nu - p_\mu g_\nu, \quad 1 \le \mu < \nu \le N. \tag{24}$$

As both $\mathbf{A}$ and $\mathbf{B}$ are skew-symmetric, the trace of their odd powers is zero, implying that

$$\mathrm{Tr}(\mathbf{A}^{2p})\,\mathrm{Tr}(\mathbf{B}^{2r+1}) = \mathrm{Tr}(\mathbf{A}^{2r+1})\,\mathrm{Tr}(\mathbf{B}^{2p}) = 0 \tag{25}$$

for any positive integers $p, r \geq 1$. Using the similarity properties of matrices (see, e.g., Ref. 29), it can be routinely verified that for arbitrary indices $p, q \geq 0$, the following relations hold:

$$\begin{aligned} \mathrm{Tr}\big(\mathbf{A}^p\mathbf{B}\big) &= \mathrm{Tr}\big(\mathbf{A}^{p-1}\mathbf{B}\mathbf{A}\big) = \cdots = \mathrm{Tr}\big(\mathbf{A}^{p-q}\mathbf{B}\mathbf{A}^q\big) = \cdots = \mathrm{Tr}\big(\mathbf{B}\mathbf{A}^p\big), \\ \mathrm{Tr}\big(\mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}\big) &= \mathrm{Tr}\big(\mathbf{B}\mathbf{A}^p\mathbf{B}\mathbf{A}^q\big) = \cdots = \mathrm{Tr}\big(\mathbf{A}^q\mathbf{B}\mathbf{A}^p\mathbf{B}\big) = \cdots = \mathrm{Tr}\big(\mathbf{B}\mathbf{A}^q\mathbf{B}\mathbf{A}^p\big). \end{aligned} \tag{26}$$

Now, as a consequence of Eq. (23), we have that for any indices $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2$, the identities

$$\frac{\partial^5 \widehat{C}_{2s}}{\partial p_{\mu_1}\partial p_{\mu_2}\partial p_{\mu_3}\partial g_{\nu_1}\partial g_{\nu_2}} = \frac{\partial^5 \widehat{C}_{2s}}{\partial g_{\mu_1}\partial g_{\mu_2}\partial g_{\mu_3}\partial p_{\nu_1}\partial p_{\nu_2}} = 0 \tag{27}$$

are satisfied. This means that for any positive integers $p, r \geq 1$,

$$\mathrm{Tr}(\mathbf{A}^{2p}\mathbf{B}^{2r+1}) = \mathrm{Tr}(\mathbf{A}^{2r+1}\mathbf{B}^{2p}) = 0, \tag{28}$$

and hence, we do not need to consider the powers $\mathbf{B}^k$ for $k \geq 3$. We further observe that the matrix elements $(\mathbf{B})_{\mu\nu} = g_\mu p_\nu - g_\nu p_\mu$, besides the obvious skew-symmetry, possess an additional property that will be useful in the following. Specifically, for any indices $1 \leq \alpha < \mu \leq N, 1 \leq \gamma < \beta \leq N$, the identity

$$(\mathbf{B})_{\alpha\mu}(\mathbf{B})_{\gamma\beta} + (\mathbf{B})_{\alpha\beta}(\mathbf{B})_{\mu\gamma} + (\mathbf{B})_{\alpha\gamma}(\mathbf{B})_{\beta\mu} = 0 \tag{29}$$

is satisfied, as can be immediately verified. This actually shows that the matrix elements of $\mathbf{B}$ can be seen as the components of a bivector.

Lemma 1. For any $N > 1$, the matrices $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy the relation*

$$\mathrm{Tr}(\mathbf{ABAB}) - \frac{1}{2}(\mathrm{Tr}(\mathbf{AB}))^2 = 0. \tag{30}$$

We prove the assertion by direct verification. Denote $(\mathbf{B})_{\mu\nu}$ by $R_{\mu\nu}$, and consider the matrix product $\mathbf{M} = \mathbf{AB}$. The matrix elements of $\mathbf{M}$ are given by $\mathbf{M}_{\mu\nu} = j_{\mu\alpha}R_{\alpha\nu}$, while those of $\mathbf{M}^2$ have the form

$$\big(\mathbf{M}^2\big)_{\mu\nu} = j_{\mu\alpha}R_{\alpha\beta}j_{\beta\gamma}R_{\gamma\nu}. \tag{31}$$

Therefore, the trace is given by $\mathrm{Tr}\big(\mathbf{M}^2\big) = j_{\mu\alpha}R_{\alpha\beta}j_{\beta\gamma}R_{\gamma\mu}$. If we now take the square of the trace $\mathrm{Tr}(\mathbf{M}) = j_{\mu\alpha}R_{\alpha\mu}$ and use the relation (29), we obtain

$$\begin{aligned} (\mathrm{Tr}(\mathbf{M}))^2 &= \big(j_{\mu\alpha}R_{\alpha\mu}\big)\big(j_{\beta\gamma}R_{\gamma\beta}\big) = j_{\mu\alpha}j_{\beta\gamma}R_{\alpha\mu}R_{\gamma\beta} = -j_{\mu\alpha}j_{\beta\gamma}\big(R_{\alpha\beta}R_{\mu\gamma} + R_{\alpha\gamma}R_{\beta\mu}\big) \\ &= j_{\mu\alpha}j_{\beta\gamma}R_{\alpha\beta}R_{\gamma\mu} + j_{\mu\alpha}j_{\gamma\beta}R_{\alpha\gamma}R_{\beta\mu} = 2\,\mathrm{Tr}\big(\mathbf{M}^2\big). \end{aligned} \tag{32}$$

Let us now inspect the degrees of the functions $\Phi_{2s,1}$, $\Phi_{2s+1,2}$, and $\Phi_{2s+2,3}$ in each of the variables $j_{\mu\nu}$, p_ρ, and g_σ. In view of the previous relations, it is not difficult to see that in terms of traces, these functions can be described as follows:

(i) As $\Phi_{2s,1}$ is homogeneous of degree $2s$ in the variables $j_{\mu\nu}$, it is obtained from the trace $\mathrm{Tr}(\mathbf{A}^{2s})$. It corresponds naturally to an invariant of the orthogonal subalgebra $\mathfrak{so}(N)$.

(ii) The function $\Phi_{2s+1,2}$ is homogeneous of degree $2s-1$ in the variables $j_{\mu\nu}$ and linear in both the variables p_ρ and g_σ. Such terms can be described as sums of traces of the type $\mathrm{Tr}\big(\mathbf{A}^{2p-1}\mathbf{B}\big)\prod_{\ell=1}^r \mathrm{Tr}\big(\mathbf{A}^{2q_\ell}\big)$ with $p + q_1 + \cdots + q_r = s$.

(iii) Finally, the function $\Phi_{2s+2,3}$ is homogeneous of degree $2s-2$ in the variables $j_{\mu\nu}$ and of degree two in the variables p_ρ and g_σ, respectively. Taking into account the identity (30), we see that the terms of $\Phi_{2s+2,3}$ can be obtained combining the following three different types of products:

  (a) Traces of the type $\mathrm{Tr}\big(\mathbf{A}^p\mathbf{B}\mathbf{A}^q\mathbf{B}\big)$ with $p + q = 2s - 2$.
  (b) Products of the type $\mathrm{Tr}\big(\mathbf{A}^{2p-r}\mathbf{B}\mathbf{A}^r\mathbf{B}\big)\prod_{\ell=1}^r \mathrm{Tr}\big(\mathbf{A}^{2q_\ell}\big)$ with $p > 0$, $r \geq 0$, $q_\ell \geq 0$ and $p + q_1 + \cdots + q_r = s - 1$.
  (c) Products of the form $\mathrm{Tr}\big(\mathbf{B}^2\big)\prod_{\ell=1}^r \mathrm{Tr}\big(\mathbf{A}^{2q_\ell}\big)$ with $q_\ell \geq 0$ and $q_1 + \cdots + q_r = s - 1$.

In particular, if $s = 1$, the function $\Phi_{2s+2,3}$ reduces to a multiple of $\mathrm{Tr}\big(\mathbf{B}^2\big)$.

With the preceding types of traces in $\mathbf{A}$ and $\mathbf{B}$, we can completely decompose the characteristic polynomial $\chi(T)$ of (17). We compare with Eq. (21), where the invariants $\hat{C}_{2s}$ are given in terms of the traces σ_k. As for each index k, the function σ_k can be decomposed in terms of $\Phi_{2s,1}$, $\Phi_{2s+1,2}$, and $\Phi_{2s+2,3}$, it merely remains to determine the expressions of these functions as a linear combination of the different trace types enumerated above. In this context, it is worth observing that, as the powers of the matrix $\mathbf{B}$ are either one or two, the coefficients of such linear combinations will depend on the indices $p, q_1, \ldots, q_r$ of powers in $\mathbf{A}$ appearing in the different trace types in (ii) and (iii) and the constraints to which they are subjected. This naturally suggests to consider partitions of integers.[30]

Let r be a positive integer and let $[\lambda] = [\lambda_1, \ldots, \lambda_r]$ be a partition of r. Then, the conditions $\lambda_1 + \cdots + \lambda_r = r$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ are satisfied. In order to take into account, the multiplicities of integers within a partition, let $\sigma([\lambda])$ denote the number of different integers λ_k intervening in the partition and let $\nu_k \geq 1$ be its multiplicity. Reordering the indices if necessary, we can write

$$r = \nu_1\lambda_1 + \cdots + \nu_{\sigma([\lambda])}\lambda_{\sigma([\lambda])}; \;\; \lambda_1 > \lambda_2 > \ldots > \lambda_{\sigma([\lambda])} > 0 \tag{33}$$

so that the partition can be described as $[\lambda] = \left[(\lambda_1)^{\nu_1}, \ldots, \left(\lambda_{\sigma([\lambda])}\right)^{\nu_{\sigma([\lambda])}}\right]$. For each entry in such a partition, we define the scalar function

$$\varphi(\lambda_j, \nu_j) = \frac{(-1)^{\nu_j}}{(2\lambda_j)^{\nu_j}\Gamma(\nu_j + 1)}. \tag{34}$$

To the partition $[\lambda]$, we associate a trace function as follows:

$$[\lambda] = \left[(\lambda_1)^{\nu_1}, \ldots, \left(\lambda_{\sigma([\lambda])}\right)^{\nu_{\sigma([\lambda])}}\right] \rightarrow \Psi([\lambda]) = \prod_{j=1}^{\sigma([\lambda])} \varphi(\lambda_j, \nu_j)\left(\mathrm{Tr}\left(\mathbf{A}^{2\lambda_j}\right)\right)^{\nu_j}. \tag{35}$$

It is clear that $\Psi([\lambda])$ is a homogeneous polynomial of degree $2r$ in the variables $j_{\mu\nu}$ of $\mathbf{A}$. If we now consider the set $\mathfrak{M}_r$ of all partitions of the integer r, we define the function

$$Q_{2r} = \sum_{[\lambda]\in\mathfrak{M}_r} \Psi([\lambda]), \quad r \geq 1, \tag{36}$$

where for notational convenience, we also set $Q_0 = 1$. The number of terms of the homogeneous polynomial Q_{2r} (i.e., the cardinal of the set $\mathfrak{M}_r$) is given by the total number $\varrho(r)$ of partitions of r that is determined by the generating function,

$$\sum_{m=0}^{\infty} \varrho(m) z^n = \prod_{m=1}^{\infty} \frac{1}{(1 - z^m)}, \quad |z| < 1. \tag{37}$$

To describe efficiently the traces of mixed products of $\mathbf{A}$ and $\mathbf{B}$, we further define the trace function

$$\begin{aligned} R_0 &= \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{B}^2\right), \\ R_{2q} &= \sum_{\ell=0}^{q-1} (-1)^{\ell}\,\mathrm{Tr}\left(\mathbf{A}^{2q-\ell}\mathbf{B}\mathbf{A}^{\ell}\mathbf{B}\right) + \frac{(-1)^q}{2}\,\mathrm{Tr}\left(\mathbf{A}^q\mathbf{B}\mathbf{A}^q\mathbf{B}\right), \; q \geq 1. \end{aligned} \tag{38}$$

We observe that, as a function of the variables $j_{\mu\nu}, p_\rho, g_\sigma$, the operator R_{2q} has order $2q + 4$ for any $q \geq 0$.

From a routine but long and cumbersome comparison of the traces σ_k in (21) with Eq. (22), we conclude that the suitable combinations of traces providing $\Phi_{2s,1}$, $\Phi_{2s+1,2}$, and $\Phi_{2s+2,3}$ are given by

$$\begin{aligned} \Phi_{2s,1} &= \mathrm{Tr}\left(\mathbf{A}^{2s}\right), \\ \Phi_{2s+1,2} &= \sum_{\ell=0}^{s-1} \mathrm{Tr}\left(\mathbf{A}^{2s-1-2\ell}\mathbf{B}\right) Q_{2\ell}, \\ \Phi_{2s+2,3} &= \sum_{\ell=0}^{s-1} R_{2s-2\ell-2} Q_{2\ell}. \end{aligned} \tag{39}$$

In summary, the Casimir invariants of the Lie algebra $\widehat{\mathcal{S}}(N)$ can be written in terms of traces as

$$\hat{C}_{2s} = \mathrm{Tr}\left(\mathbf{A}^{2s}\right) m^{2s} + \varepsilon^{-2} \sum_{\ell=0}^{s-1} \mathrm{Tr}\left(\mathbf{A}^{2s-1-2\ell}\mathbf{B}\right) Q_{2\ell} m^{2s-1} + \varepsilon^{-4} \sum_{\ell=0}^{s-1} R_{2s-2\ell-2} Q_{2\ell} m^{2s-2}. \tag{40}$$

Comparing with the expansion (22), we see that the last term in (40), setting $m = 1$, corresponds to a $\mathcal{S}(N)$-Casimir invariant of degree $2s + 2$. The key observation is that both the matrices $\mathbf{A}$ and $\mathbf{B}$ in (24) are defined in terms of the generators of $\mathcal{S}(N)$ only so that the operators

$$C_{2s} = \Phi_{2s+2,3} = \sum_{\ell=0}^{s-1} R_{2s-2\ell-2} Q_{2\ell}, \quad 1 \leq s \leq \left[\frac{N}{2}\right] \tag{41}$$

can always be determined using Eqs. (36) and (38), independently of the central extension $\widehat{\mathcal{S}}(N)$. This provides a direct construction method of the Casimir operators of $\mathcal{S}(N)$ that does no more rely on the contraction (12). We must however warn that the trace operators (41) are not themselves obtainable as the coefficients of a characteristic polynomial, as follows at once from the decomposition (22).

In order to illustrate the specific form of the trace operators (41), we enumerate C_{2s} for the first values $1 \leq s \leq 4$,

$$\begin{aligned}
C_2 &= \frac{1}{2}\operatorname{Tr}\left(\mathbf{B}^2\right),\\
C_4 &= \operatorname{Tr}\left(\mathbf{A}^2\mathbf{B}^2\right) - \frac{1}{2}\operatorname{Tr}(\mathbf{ABAB}) - \frac{1}{4}\operatorname{Tr}\left(\mathbf{A}^2\right)\operatorname{Tr}\left(\mathbf{B}^2\right),\\
C_6 &= \operatorname{Tr}\left(\mathbf{A}^4\mathbf{B}^2\right) - \operatorname{Tr}\left(\mathbf{A}^3\mathbf{BAB}\right) + \frac{1}{2}\operatorname{Tr}\left(\mathbf{A}^2\mathbf{BA}^2\mathbf{B}\right) - \frac{1}{2}\operatorname{Tr}\left(\mathbf{A}^2\mathbf{B}^2\right)\operatorname{Tr}\left(\mathbf{A}^2\right)\\
&\quad + \frac{1}{4}\operatorname{Tr}(\mathbf{ABAB})\operatorname{Tr}\left(\mathbf{A}^2\right) - \frac{1}{8}\operatorname{Tr}\left(\mathbf{A}^4\right)\operatorname{Tr}\left(\mathbf{B}^2\right) + \frac{1}{16}\operatorname{Tr}\left(\mathbf{A}^2\right)^2\operatorname{Tr}\left(\mathbf{B}^2\right),\\
C_8 &= \operatorname{Tr}\left(\mathbf{A}^6\mathbf{B}^2\right) - \operatorname{Tr}\left(\mathbf{A}^5\mathbf{BAB}\right) + \operatorname{Tr}\left(\mathbf{A}^4\mathbf{BA}^2\mathbf{B}\right) - \frac{1}{2}\operatorname{Tr}\left(\mathbf{A}^3\mathbf{BA}^3\mathbf{B}\right) - \frac{1}{2}\operatorname{Tr}\left(\mathbf{A}^2\right)\operatorname{Tr}\left(\mathbf{A}^4\mathbf{B}^2\right)\\
&\quad + \frac{1}{2}\operatorname{Tr}\left(\mathbf{A}^2\right)\left(\operatorname{Tr}\left(\mathbf{A}^3\mathbf{BAB}\right) - \frac{1}{2}\operatorname{Tr}\left(\mathbf{A}^2\mathbf{BA}^2\mathbf{B}\right)\right) + \frac{1}{16}\left(\operatorname{Tr}\left(\mathbf{A}^2\right)^2 - 2\operatorname{Tr}\left(\mathbf{A}^4\right)\right)\times\\
&\quad \left(2\operatorname{Tr}\left(\mathbf{A}^2\mathbf{B}^2\right) - \operatorname{Tr}(\mathbf{ABAB})\right) - \operatorname{Tr}\left(\mathbf{B}^2\right)\left(\frac{1}{12}\operatorname{Tr}\left(\mathbf{A}^6\right) - \frac{1}{16}\operatorname{Tr}\left(\mathbf{A}^4\right)\operatorname{Tr}\left(\mathbf{A}^2\right) + \frac{1}{96}\operatorname{Tr}\left(\mathbf{A}^2\right)^3\right).
\end{aligned}$$

In general, for each fixed value of s, the length of C_{2s}, i.e., the number of terms as a sum of traces, is given by the combinatorial formula

$$\mathfrak{l}(C_{2s}) = \sum_{\ell=0}^{s-1}(\ell+1)\varrho(s-\ell). \tag{42}$$

In particular, for even values of N, the invariant $\hat{C}_N$ is a perfect square[31] so that there is an invariant $C'_{\frac{N}{2}}$ of degree $\frac{N+2}{2}$ such that $C_N = \left(C'_{\frac{N}{2}}\right)^2$. This function can be described adequately using those permutations of the symmetric group Σ_N that are products of $\frac{N}{2}$ disjoint transpositions, i.e., permutations associated with the partition $\left[\left(\frac{N}{2}\right)^2\right]$. If $\mathcal{T}_N$ denotes the set of such permutations, it can be verified easily that the following relation holds:

$$C'_{\frac{N}{2}} = \sum_{\sigma\in\mathcal{T}_N} j_{\sigma(1)\sigma(2)}\dots j_{\sigma(N-3)\sigma(N-2)}R_{\sigma(N-1)\sigma(N)}. \tag{43}$$

From this expression, we further see that the degree of $C'_{\frac{N}{2}}$ in the $\mathfrak{so}(N)$-variables is given by $\frac{N-2}{2}$, while the terms $R_{\mu\nu} = g_\mu p_\nu - g_\nu p_\mu$ appear only linearly. As a consequence of the identities (26), the function $C'_{\frac{N}{2}}$ itself cannot be written in terms of traces of products of the matrices **A** and **B**. This however has no effect on the general description, as the invariant is generally obtained factorizing the trace operator C_N. To complete the description of the $\mathcal{S}(N)$-invariants that do not depend on the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra, we still have to prove that the Casimir invariants determined by (41) are independent.

Proposition 1. For any $N > 1$ and $s = 1,\dots,\left[\frac{N}{2}\right]$, the trace operators

$$C_{2s} = \sum_{\ell=0}^{s-1} R_{2s-2\ell-2}Q_{2\ell} \tag{44}$$

are functionally independent Casimir invariants of the unextended Schrödinger algebra $\mathcal{S}(N)$.

Without loss of generality, for even N, we can consider the square C_N instead of $C'_{\frac{N}{2}}$ and proceed directly computing an appropriate Jacobian. Let $q = \left[\frac{N}{2}\right]$ and consider the set of variables $\mathcal{A} = \{j_{34},\dots,j_{2q-1,2q},g_1\}$. Computing the Jacobian with respect to these variables and isolating the terms that depend solely on these variables and p_2, we can write the Jacobian as

$$\begin{aligned}
&\frac{\partial\{C_2,C_4,\dots,C_{2q}\}}{\partial\{j_{34},\dots,j_{2q-1,2q},g_1\}} = (-1)^N 2^q g_1^{\frac{2N-3+(-1)^N}{2}} p_2^{\frac{2N+1+(-1)^N}{2}}\times\\
&\prod_{a=1}^{\left[\frac{N-2}{2}\right]} j_{2a+1,2a+2}\prod_{1\leq a<b\leq q-1}\left(j_{2a+1,2a+2}^2 - j_{2b+1,2b+2}^2\right) + \mathcal{V}(j_{\mu\nu},R_{\sigma\tau}),
\end{aligned} \tag{45}$$

where the terms in $\mathcal{V}(j_{\mu\nu},R_{\sigma\tau})$ depend on at least one variable that is not contained in $\mathcal{A}\cup\{p_2\}$. If we now evaluate the Jacobian on the slice defined by the conditions

$$j_{12} = 0,\ g_\mu = 0,\ p_\nu = 0,\quad 2\leq\mu\leq N,\ 1\leq\nu\neq 2\leq N, \tag{46}$$

then only the leading term survives, showing that the Jacobian does not vanish.

A. The cubic invariant of $\mathcal{S}(N)$

Besides the invariants obtained from the contraction of the functions $\hat{C}_{2s}$, we have an additional invariant of $\mathcal{S}(N)$ that arises from the contraction of the function C_4' associated with $\mathfrak{sl}(2,\mathbb{R})$-variables. Using (18), on the transformed basis, this invariant adopts the form

$$C_4' = \left(md - \sum_{\mu=1}^{N} \varepsilon^{-2} g_\mu p_\mu\right)^2 - 4\left(mk - \frac{1}{2}\sum_{\mu=1}^{N} \varepsilon^{-2} g_\mu^2\right)\left(mp_t - \frac{1}{2}\sum_{\mu=1}^{N} \varepsilon^{-2} p_\mu^2\right). \tag{47}$$

According to (10), the term giving rise to an invariant of $\mathcal{S}(N)$ is

$$\left(\sum_{\mu=1}^{N} g_\mu p_\mu\right)^2 - \left(\sum_{\mu=1}^{N} g_\mu^2\right)\left(\sum_{\mu=1}^{N} p_\mu^2\right) = -\sum_{\mu<\nu} (g_\mu p_\nu - g_\nu p_\mu)^2 = \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{B}^2\right). \tag{48}$$

However, from Eq. (22), we have that

$$\hat{C}_2 = -m^2\sum_{\mu<\nu} j_{\mu\nu}^2 + 2\varepsilon^{-2} m \sum_{\mu<\nu} j_{\mu\nu}(g_\mu p_\nu - g_\nu p_\mu) - \varepsilon^{-4}\sum_{\mu<\nu} (g_\mu p_\nu - g_\nu p_\mu)^2, \tag{49}$$

and, hence,

$$\lim_{\varepsilon\to 0} \varepsilon^4 C_4' = \lim_{\varepsilon\to 0} \varepsilon^4 C_4 = \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{B}^2\right), \tag{50}$$

showing that the contracted quartic invariants are functionally dependent. To obtain a new Casimir operator of $\mathcal{S}(N)$ that also depends on the variables of the simple subalgebra $\mathfrak{sl}(2,\mathbb{R})$, we consider the linear combination $C_4' - C_4$ so that the transformed invariant by the automorphism Ψ_ε equals

$$C_4' - C_4 = m^2\left(d^2 - 4kp_t + \sum_{\mu<\nu} j_{\mu\nu}^2\right) + 2m\varepsilon^{-2}\sum_{1=\mu<\nu}^{N} \left(R_{\mu\nu} j_{\mu\nu} + g_\mu^2 p_t + p_\mu^2 k - d g_\mu p_\mu\right). \tag{51}$$

Applying (10) to this expression and setting $m = 1$, we get the following cubic polynomial:

$$C_3 = \sum_{\mu=1}^{N} \left(g_\mu^2 p_t + p_\mu^2 k - d g_\mu p_\mu\right) + \sum_{\mu<\nu} j_{\mu\nu}(g_\mu p_\nu - g_\nu p_\mu). \tag{52}$$

As $\frac{\partial C_3}{\partial z} \neq 0$ for any $z \in \mathfrak{sl}(2,\mathbb{R})$, it follows at once that C_3 is functionally independent of the invariants C_{2s} $(1 \le s \le \left[\frac{N}{2}\right])$ computed previously from which we deduce that $\mathcal{F} = \{C_3\} \cup \left\{C_2, C_4, \ldots, C_{2\left[\frac{N}{2}\right]}\right\}$ is a complete set of invariants for the unextended Schrödinger algebra $\mathcal{S}(N)$.

We finally remark that, albeit the function C_3 cannot be expressed as a trace function of the matrices $\mathbf{A}$ and $\mathbf{B}$, we can also write it as the trace of an appropriate matrix. If we define the matrix $\mathbf{F}$ of order $(N+3)$ having the entries

$$\begin{gathered} \mathbf{F}_{\mu\nu} = j_{\mu\nu},\ \mathbf{F}_{\nu\mu} = R_{\mu\nu},\ 1 \le \mu < \nu \le N, \\ \mathbf{F}_{N+1,N+2} = p_t,\ \mathbf{F}_{N+1,N+3} = k,\ \mathbf{F}_{N+2,N+3} = -d, \\ \mathbf{F}_{N+2,N+1} = \sum_{\mu=1}^{N} g_\mu^2, \mathbf{F}_{N+3,N+1} = \sum_{\mu=1}^{N} p_\mu^2, \mathbf{F}_{N+3,N+2} = \sum_{\mu<\nu}^{N} R_{\mu\nu}, \end{gathered} \tag{53}$$

a quick computation shows that $C_3 = \mathrm{Tr}\left(\mathbf{F}^2\right)$. It should however taken into account that the traces of higher powers of $\mathbf{F}$ are no more invariants of the Lie algebra $\mathcal{S}(N)$.

V. CONCLUSIONS

Decomposing the polynomial matrix $\hat{\mathbf{A}}$ whose characteristic polynomial provides the Casimir operators of the centrally extended Schrödinger algebra $\widehat{\mathcal{S}}(N)$ as the sum of two matrices $\mathbf{A}$, $\mathbf{B}$ and applying the Newton identities in combination with the natural contraction $\widehat{\mathcal{S}}(N) \rightsquigarrow \mathcal{S}(N) \oplus \langle M\rangle$, the Casimir invariants of the unextended Schrödinger algebra $\mathcal{S}(N)$ have been obtained as trace operators. As both the matrices $\mathbf{A}$ and $\mathbf{B}$ depend only on the commutators of the latter Lie algebra, the procedure can be seen as a direct method for the computation of the $\mathcal{S}(N)$-invariants that does no longer requires the explicit use of the contraction.

It should be observed that the result can be easily extended to the semidirect products $\mathcal{S}(N-p,p) = (\mathfrak{so}(N-p,p) \oplus \mathfrak{sl}(2,\mathbb{R}))\overrightarrow{\oplus}_{D_{\frac{1}{2}}\otimes\Lambda}\mathbb{R}^{2N}$ by merely replacing the matrix elements $j_{\mu\nu}$ of $\mathbf{A}$ in Eq. (24) by $\gamma_{\nu\nu} j_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the diagonal metric tensor whose eigenvalues consist of +1 with multiplicity $N-p$ and -1 with multiplicity p. Indeed, if we extend the base field from the real to the complex numbers, then formula (44) provides the Casimir invariants of the complex Lie algebra $\mathcal{S}(N)\otimes_{\mathbb{R}}\mathbb{C}$ on a new basis where the $\mathfrak{so}(N,\mathbb{C})$-generators satisfy the commutators

$$[J_{\mu\nu}, J_{\lambda\sigma}] = \gamma_{\mu\lambda}J_{\nu\sigma} + \gamma_{\nu\sigma}J_{\mu\lambda} - \gamma_{\mu\sigma}J_{\nu\lambda} - \gamma_{\nu\lambda}J_{\mu\sigma},$$
$$[J_{\mu\nu}, P_\lambda] = \gamma_{\mu\lambda}P_\nu - \gamma_{\nu\lambda}P_\mu,\ [J_{\mu\nu}, G_\lambda] = \gamma_{\mu\lambda}G_\nu - \gamma_{\nu\lambda}G_\mu.$$

The validity of formula (44) follows from the observation that for any $p \geq 0$, the Lie algebras $\mathfrak{so}(N-p,p)\otimes_{\mathbb{R}}\mathbb{C}$ are isomorphic. As the coefficients of the trace operators are rationals, the restriction of scalars shows that the trace operators correspond to the Casimir operators of $\mathcal{S}(N-p,p)$ as a real Lie algebra (see also Refs. 15 and 26).

The ansatz can even be extrapolated to other types of semidirect sums of semisimple algebras $\mathfrak{s}$ and $\mathfrak{h}_N$. As a matter of fact, whenever we have a semidirect product $\mathfrak{g} = \mathfrak{s}\vec{\oplus}_{\Lambda\oplus\Gamma_0}\mathfrak{h}_N$ such that the contraction $\mathfrak{g} \rightsquigarrow \mathfrak{g}' \oplus Z(\mathfrak{g})$ with $\mathfrak{g}' = \mathfrak{s}\vec{\oplus}_{\Lambda}\mathbb{R}^{2N}$ preserves the number of invariants, that is, satisfying the equality $\mathcal{N}(\mathfrak{g}') = \text{rank}(\mathfrak{s})$, a complete set of Casimir invariants for the contraction can be obtained using formula (10). As the invariants of Lie algebras of type $\mathfrak{g}$ can always be determined by means of the characteristic polynomial of a functional matrix $\mathbf{A}$ (see, e.g., Ref. 16), the trace method presented in this work is potentially adaptable to such cases, decomposing adequately $\mathbf{A}$ into a sum of matrices whose entries depend solely on generators of $\mathfrak{g}'$. For the case of contractions such that the inequality $\mathcal{N}(\mathfrak{g}') > \text{rank}(\mathfrak{s})$ holds, the procedure is, in general, of very limited or no practical use, as it happens, for example, for the class of generalized conformal Galilean algebras $\mathfrak{Gal}_\ell(N)$ for any value $\ell > \frac{1}{2}$. Whether for such types of Lie algebras an alternative trace method for the computation of the Casimir operators exists is currently an open question deeply related to the representation theory of such algebras that deserves further analysis.

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[31] This is a direct consequence of the skew-symmetry of the matrix $\hat{A}$ defined in (16).