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AGGREGATION RULES IN COMMITTEE PROCEDURES

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Abstract

Very often, decision procedures in a committee compensate potential manipulations by taking into account the ordered profile of qualifications. It is therefore rejected the standard assumption of an underlying associative binary connective allowing the evaluation of arbitrary finite sequences of items by means of a one-by-one sequential process. In this paper we develop a mathematical approach for non-associative connectives allowing a sequential definition by means of binary fuzzy connectives. It will be then stressed that a connective rule should be understood as a consistent sequence of binary connective operators. Committees should previously decide about which connective rule they will be conditioning, not just about a single operator.

Keywords: Fuzzy Connectives, Fuzzy Sets, Aggregation Operators.

1 Introduction.

Many real-life problems are solved by means of some information aggregation procedures. If we observe chunks of partial information and we have to elaborate a global opinion we have an aggregation process which can be modeled in terms of aggregation rules. In principle, the only property which is required is just the ability to transform the data set into a simpler representation. For

example, if we expect data with a fixed and known dimension n, we do not need to define how $n+1$ or $n-1$ items of information have to be aggregated into one single index. Nevertheless, in practice we rarely know in advance the dimension of the real problem we are going to be faced to. Moreover, even if we do know the dimension of the input data we have the semantic problems of using the *same* aggregation operator everytime this is the case, on inputs of different dimensions. Therefore, any aggregation rule has to be able to operate with an arbitrary dimension of the data set, even when we assume that such a data set has nice mathematical properties like homogeneity and non-redundant information.

When a committee has to decide about the qualification of candidates, the objetive is just to aggregate the opinion of each committee member into a single index. Each committee member represents a piece of information. Quite often, the number of voting members may be not fixed (e.g., unavoidable last minute absences), or the number of voting members can not be *a priori* known (this is the case when we just have a *qualitative* definition of the crisp set of voters, i.e., the list of properties giving people the right to vote). Moreover, if the number of potential voters is too big, existence of an algorithm allowing some kind of sequential reckoning will help calculus.

A standard solution to such a basic semantical and operational problem is the assumption of Associativity. Under Associativity, we can aggregate information by pieces, item by item, by applying a unique binary connective, no matter the dimension of our data set. The aggregated value can be obtained by aggregating information by means of a sequential one-by-one process. The aggregation process is fully characterized by a single binary aggregation rule, the one applied to the first couple of aggregated elements. Then the same computation is applied iteratively until the data set has been completely read. This is the case with classical fuzzy connective operators for conjunction and disjunction, t-norms and t-conorms (see, e.g., [7]).

Associativeness is a quite frequent assumption in group decision making models. Associative decision procedures are not affected by the order in which individuals express their opinions, and every aggregated opinion is considered just like another individual opinion. However, important operators are not associative.

Some decision procedures in a committee sometimes do not take into account committee highest and lowest qualifications for each candidate, and then they evaluate the mean of the remaining *middle* values. This is a particular case of *Ordered Weighted Averaging (OWA)* operators, introduced by Yager [12] in order to fill the gap between *min* (which is the maximal t-norm) and *max* (which is the minimal t-conorm). OWA operators are not associative, and their application requires that the number of items to be aggregated has been previously fixed.

Classical Mean Rule, for example, takes into account social support of each alternative, in such way that group opinions are no longer equivalent to individual opinions (see, e.g., [10]). Notice that a key OWA connective operator

like the standard mean, defined as a mapping

$$M_n : [0,1]^n \to [0,1]$$

such that

$$M_n(a_1,\ldots,a_n) = \frac{\sum_{i=1}^n a_i}{n},$$

is not associative. Each mean M_n is just the mean of n numbers (it has been defined for a fixed n). M_n is just an operator, not a *rule*. When we refer to the *Mean Rule* we refer to the rule that evaluates the above mean for every n. The *Mean Rule* is not a single mapping M_n, but the complete sequence $\{M_n\}_n$ of all those mappings.

Moreover, it should be also pointed out that an operational calculus for the *Mean Rule* would not follow the above formula, but a *left* recursive calculus

$$M_n(a_1,\ldots,a_n) = \frac{(n-1)M_{n-1}(a_1,\ldots,a_{n-1}) + a_n}{n}$$

or alternatively, a *right* recursive calculus

$$M_n(a_1,\ldots,a_n) = \frac{a_1 + (n-1)M_{n-1}(a_2,\ldots,a_n)}{n}$$

where $M_2(a,b) = (a+b)/2$ (see [5, 10] for a discussion on some ethical and computational issues).

In general, it is not so easy to talk about *OWA rules*. Although several interesting families of OWA operators have been introduced in the past, showing the great flexibility in the choice of types of OWA operators (see, e.g., [13, 14, 15]), not every family of OWA operators can be properly considered as a *rule*.

In this paper we generalize the arguments introduced by the authors in [2], where each OWA rule was represented in terms of a family of binary OWA operators (see also [3, 4, 5]). Any connective rule will be conceived as a consistent family of connectives capable of solving arbitrary dimension problems. As a consequence, it is claimed that committees should always previously fix their *coherent* connective (aggregation) rule.

2 Recursive connective rules.

A connective rule should allow an aggregated value for any possible dimension of the list of items to be aggregated. That is, a connective rule should be a sequence of connective operators

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

to be used to aggregate any finite number of items. We shall focus our attention here on those connective rules which allow the aggregation of arbitrary lists in

a recursive manner. In particular, we shall consider those families of connective operators that can be defined by means of a left or a right recursive application of binary operators, once an appropriate re-arrangement of the items to be aggregated has been previously realized. In this way, a connective rule should be understood as a family of connective operators which can be recursively evaluated.

Obviously, in order to be considered as a *rule*, some *consistency* assumption has to be imposed on the family of connectives. Not every family of connectives either defines a connective rule or can be considered to *consistent*, or allow a recursive definition.

DEFINITION 1 *An ordering rule π on a set of items A, $A \neq \emptyset$, is a family of permutations*

$$\pi_B : B \to B$$

such that for every finite sequence of items

$$B = (a_1, a_2, \ldots, a_n), \quad a_i \in A \ \forall i,$$

a new sequence is defined on B in such a way that

$$\pi_B(a) < \pi_B(b) \Rightarrow \pi_C(a) < \pi_C(b)$$

whenever $a, b \in B \cap C$.

An ordering rule tells us the exact position each new element will placed in any previously given ordered set of items.

An immediate example of ordering rule is the natural decreasing order of real numbers

$$\sigma : \Re \to \Re$$

which assigns to each list of n numbers $(a_1, \ldots, a_n)$ its sorting permutation

$$\sigma(a_1, \ldots, a_n) = (a_{[1]}, \ldots, a_{[n]})$$

such that $a_{[i]} \geq a_{[j]}$ for all $i \leq j$.

DEFINITION 2 *A left-recursive connective rule is a family of connective operators*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0,1]^2 \to [0,1]\}_{n>1}$$

verifying

$$\begin{aligned} \phi_2(a_1, a_2) &= L_2(\pi(a_1), \pi(a_2)) \text{ and} \\ \phi_n(a_1, \ldots, a_n) &= L_n(\phi_{n-1}(\pi(a_1), \ldots, \pi(a_{n-1})), \pi(a_n)) \end{aligned}$$

for some ordering rule π.

Right-recursiveness can be analogously defined.

DEFINITION 3 *A collection of connective operators*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

is said to be a right-recursive connective rule whenever

$$\phi_2(a_1, a_2) = R_2(\pi(a_1), \pi(a_2))$$

and

$$\phi_n(a_1, \ldots, a_n) = R_n(\pi(a_1), \phi_{n-1}(\pi(a_2), \ldots, \pi(a_n)))$$

hold for some family of binary operators

$$\{R_n : [0,1]^2 \to [0,1]\}_{n>1}$$

and some ordering rule π.

We immediately have that an operator $\phi_n : [0,1]^n \to [0,1]$ can be right-recursively defined if and only if it can be left-recursively defined. This can be seen as follows. Let

$$\phi_n(a_1, \ldots, a_n) = R_n(\pi(a_1), \phi_{n-1}(\pi(a_2), \ldots, \pi(a_n))).$$

Define the permutation $\hat{\pi}$ in such a way that for any π

$$\hat{\pi}(a) < \hat{\pi}(b) \Leftrightarrow \pi(a) > \pi(b).$$

Moreover, put $L_k(a, b) = R_k(b, a)$ for all k. Then we have

(a) $R_2(\pi(a_1), \pi(a_2)) = L_2(\pi(a_2), \pi(a_1))$

(b)

$$\begin{aligned} \phi_2(a_1, a_2, a_3) &= R_3(\pi(a_1), \phi_2(\pi(a_2), \pi(a_3))) \\ R_3(\pi(a_1), R_2(\pi(a_2), \pi(a_3))) &= L_3(L_2(\pi(a_3), \pi(a_2)), \pi(a_1)) = \\ & L_3(L_2(\hat{\pi}(a_1), \hat{\pi}(a_2)), \hat{\pi}(a_3)) \end{aligned}$$

(c) In general, we can see by induction that

$$\begin{aligned} & R_n(\pi(a_1), \phi_{n-1}(\pi(a_2), \ldots, \pi(a_n))) = \\ & L_n(\phi_{n-1}(\hat{\pi}(a_1), \ldots, \hat{\pi}(a_{n-1})), \hat{\pi}(a_n)) \end{aligned}$$

The ordering rule $\hat{\pi}$ is to be known as the *dual* ordering rule of π.

The existence of a right (left) recursion representation of a given operator does not imply in general the existence of an equivalent left (right) recursion representation by means of the same underlying ordering rule (see next section for an example). Of course, some rules will allow no one-side recursive definition (see also next section).

In some cases, such a recursive representation of a connective rule is fixed from the underlying ordering rule, as shown in the following result.

THEOREM 1 *Let*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

be a left-recursive connective rule with respect to the ordering rule π, such that $\phi_n(0,\ldots,0) = 0$ and $\phi_n(1,\ldots,1) = 1$, with ϕ_n continuous and strictly increasing in each coordenate, for all n. Then $\{L_n\}_{n>1}$ is unique in its range for each ordering rule π such that

$$\phi_n(a_1,\ldots,a_n) = L_n(\phi_{n-1}(\pi(a_1),\ldots,\pi(a_{n-1})),\pi(a_n)).$$

(Analogous result holds for right recursiveness).

Proof: First of all, notice that $L_n(0,0) = 0$ for all n. In fact, since $L_2(0,0) = \phi_2(0,0) = 0$, it follows that

$$0 = \phi_3(0,0,0) = L_3(L_2(0,0),0) = L_3(0,0)$$

and so on,

$$0 = \phi_n(0,\ldots,0) = L_n(L_{n-1}(\ldots L_2(0,0)\ldots,0),0) = L_n(0,0)$$

Analogously, $L_n(1,1) = 1$ for all n. Moreover, every L_n is assured to be strictly increasing too. It is direct for $n = 2$, and if we assume it is true for $n-1$, then we have

1. if $b = \pi(a_n) \in \{a_1,\ldots,a_n\}$ increases and the other values remain constant, then $\phi_n(a_1,\ldots,a_n)$ increases, and therefore $L_n(a,b)$ also increases;
2. if

   $$a = L_{n-1}(\ldots L_2(\pi(a_1),\pi(a_2))\ldots,\pi(a_{n-1})),$$

   increases, due to continuity of ϕ_{n-1} and the induction hypothesis it is assured the existence of a point $(\pi(b_1)\ldots,\pi(b_{n-1}))$ such that $\pi(b_j) \geq \pi(a_j)$ for all j, with some strict inequality, such that

   $$L_{n-1}(\ldots L_2(\pi(b_1),\pi(b_2))\ldots,\pi(b_{n-1}))$$

   takes the new value, in such a way that

   $$\phi_n(\pi(a_1),\ldots,\pi(a_n)) < \phi_n(\pi(b_1),\ldots,\pi(b_{n-1}),\pi(a_n))$$

   and therefore L_n has increased as well.

Hence, for each a, b there is at most one c such that $L_j(c,a) = b$ (it is unique whenever the pair (a,b) belongs to the range of L_j, and therefore $c \geq a$ holds). That is, from

$$\phi_n(a_1,\ldots,a_n) = L_n(c,\pi(a_n))$$

we can evaluate

$$c = L_{n-1}(\ldots L_2(\pi(a_1),\pi(a_2))\ldots,\pi(a_{n-1})).$$

Succesively, from this equation we can evaluate

$$L_{n-2}(\ldots L_2(\pi(a_1),\pi(a_2))\ldots,\pi(a_{n-2}))$$

and so on, whenever we stay in the real ranges. ■

Hence, if a recursive connective rule contains a continuous strictly increasing operator of dimension n, then consistent operators of lower dimension can be obtained according to the above result. Obviously, consistent upper dimension operators can not be freely chosen since for all n

$$\phi_n(a_1,\ldots,a_n) = L_n(\phi_{n-1}(\pi(a_1),\ldots,\pi(a_{n-1})),\pi(a_n))$$

and, analogously,

$$\phi_n(a_1,\ldots,a_n) = R_n(\pi(a_1),\phi_{n-1}(\pi(a_2),\ldots,\pi(a_n))).$$

An interesting case to analyze is the one in which left and right recursions share the same underlying ordering rule. That is, when

$$\phi_n(a_1,\ldots,a_n) = R_n(\pi(a_1),\phi_{n-1}(\pi(a_2),\ldots,\pi(a_n))) = L_n(\phi_{n-1}(\pi(a_1),\ldots,\pi(a_{n-1})),\pi(a_n))$$

holds for some ordering rule π.

DEFINITION 4 *If both left and right recursiveness hold for the same ordering rule then we have* recursive rules.

In this way, recursiveness generalizes the concept of associativity, in the sense that recursive rules are the ones that can be evaluated iteratively (both sides), after an appropriate pre-arrangement of data. This ability of being iteratively evaluated is in fact the deep reason for associativity in practice. An operational calculus algorithm usually implies an iterative reckoning. But this iterative calculus does not necessarely requieres a unique binary operator. As shown above, the *Mean Rule* allows both left and right recursive definitions, although it is not associative.

The *Mean Rule* verifies an additional property: both left and right recursive definitions do not depend on the permutation, i.e., they are the same no matter the particular sequence of permutations being chosen. Left and right recursion hold for any possible ordering rule. If such a condition holds, we can talk about *commutative* recursive rules. Commutative recursive rules will be those connective rules which do not depend on any particular ordering rule.

In some way we could say that a connective rule $\{\phi_n\}_{n>1}$ is recursive if and only if a set of general associativity equations (in the sense of Mak [9]) hold for each n, once the items have been properly ordered. In fact, recursiveness holds whenever

$$\phi_n(a_1,\ldots,a_n) = \phi_n(\pi(a_1),\ldots,\pi(a_n)) = R_n(\pi(a_1),\phi_{n-1}(\pi(a_2),\ldots,\pi(a_n))) = L_n(\phi_{n-1}(\pi(a_1),\pi(a_{n-1})),\pi(a_n))$$

for all n and some ordering rule π. If each one of these binary connecti L_n, R_n can be assumed to be defined in the cartesian product of two nontriv compact intervals on the real line, being continuous strictly increasing in ea coordenate, then it can be shown (see [8]) that they are commutative a basically additive, in such a way that

$$\phi_n(a_1,\ldots,a_n) = \psi_0^{-1}(\psi_1(a_1) + \ldots + \psi_n(a_n))$$

for some homomorphisms in the unit interval $\psi_0, \psi_1, \ldots \psi_n$. This result ca allow a particular representation of theorem 1. If we take, for example, th natural decreasing order σ as the underlying ordering rule, then each L_j defined on a simplex $a_{j-1} \geq a_j$. Assuming the above conditions in a prop extended cartesian product of two nontrivial compact intervals, plus continuit and strict continuity, would assure such an additive solution (see [11]).

Associativity appears when the ordering rule is taken as the identity (i.e., the ordering rule keeping positions as presented), and

$$L_n = L_2 = F = R_2 = R_n \quad \forall n$$

(that is, the whole recursive connective rule is characterized by a unique associative binary connective F, with no pre-arrangement of data).

Many connective rules $\{\phi_2, \ldots, \phi_n \ldots\}$ we can find in the literature are defined by means of a unique commutative and associative binary operator $\phi : [0,1]^2 \to [0,1]$ such that

$$\phi_n(a_1,\ldots,a_n) = \phi(\ldots,\phi(\phi(b_1,b_2),b_3),\ldots,b_n) = \phi(b_1,\ldots,(b_{n-3},\phi(b_{n-2},\phi(b_{n-1},b_n)))\ldots)$$

for $(b_1,\ldots,b_n)$ any permutation of $(a_1,\ldots,a_n)$. When we refer to a t-norm or a t-conorm as a *connective rule* we really mean the family of *connective operators* in such a way univocally defined (only one binary connective not depending on the ordering rule). The whole family of connective operators is fully characterized by its first connective operator of dimension 2, and no pre-arrangement of data is needed.

3 OWA recursive rules.

As pointed out above, sometimes only one underlying ordering rule is allowed by the decision maker. Perhaps there is only one natural way of ranking our data, and data reach to us previously pre-arranged according to such an ordering rule. If this is the case, the above concepts should be modified in order to meet such a restriction. Either a recursive definition is consistent with such an ordering rule, or such a recursive definition can not be applied. Either both recursive definitions make use of such an ordering rule, or it can not be applied as a recursive rule. For example, it may be the case that data are assumed to be ranked in its natural decreasing ordering, as happens with OWA rules.

Let us particularize the above ideas to the OWA case. First, we remind the reader some key concepts about OWA operators.

3.1 Basics on OWA operators.

OWA operators [12] are based upon the natural (decreasing) ordering. An OWA operator of dimension n is a connective operator

$$\phi : [0,1]^n \to [0,1]$$

such that for any list $(a_1,\ldots,a_n)$ then

$$\phi(a_1,\ldots,a_n) = \sum_{i=1}^{n} w_i a_{|i|},$$

for some associated list of weights $W = (w_1,\ldots,w_n)$ such that

1. $w_i \in [0,1]$ for all $1 \leq i \leq n$
2. $\sum_{i=1}^{n} w_i = 1$

OWA operators are therefore assuming the (decreasing) natural ordering on the real line as the underlying ordering rule σ.

OWA operators are obviously commutative, monotone and idempotent, but as pointed out above, not associative in general. In fact, a binary ($n = 2$) OWA operator is associative if and only if either it is the *min* ($w_2 = 1$) operator or the *max* ($w_1 = 1$) operator. Therefore, given an OWA operator of dimension n, it can be only applied to aggregation problems of such a dimension n. If the dimension problem is modified, such OWA operator can not be applied.

Three short comments about OWA operators before going back to our operationality problem:

- A significative measure associated with OWA operators is the *orness* which estimates how close an OWA operator is to the *max* operator. It is defined as

$$orness(\phi) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i.$$

  Dual to the measure of *orness* is the measure of *andness* defined as

$$andness(\phi) = 1 - orness(\phi),$$

  which therefore measures how close an OWA operator is to the *min* operator.

- Another important notion is *duality*. Given an OWA operator ϕ with weights $[w_1,\ldots,w_n]$, the dual $\hat{\phi}$ of ϕ is the OWA operator whose weights are $[w_n,\ldots,w_1]$. It is not difficult to see that $orness(\hat{\phi}) = andness(\phi)$.

- A particular class of OWA operator is given by the *buoyancy measures*. They are OWA operators that verify the property $w_i \geq w_j$ if $i < j$. Any buoyancy measure ϕ is such that $orness(\phi) \geq \frac{1}{2}$.

It may be the case that the existing left and right recursive definitions do not make use of the same underlying ordering rule. For example, the following two OWA operators

$$\phi_2(a_1, a_2) = \frac{1}{2}a_{|1|} + \frac{1}{2}a_{|2|}$$

and

$$\phi_3(a_1, a_2, a_3) = \frac{1}{4}a_{|1|} + \frac{1}{4}a_{|2|} + \frac{1}{2}a_{|3|}$$

allow a left recursive rule, since we can write

$$\phi_3(a_1, a_2, a_3) = \frac{1}{2}(\frac{1}{2}a_{|1|} + \frac{1}{2}a_{|2|}) + \frac{1}{2}a_{|3|}$$

But once we have chosen such a decreasing natural ordering as our ordering rule, then there is no function $h : [0, 1]^2 \to [0, 1]$ such that

$$\phi_3(a_1, a_2, a_3) = h(a_{|1|}, \frac{1}{2}a_{|2|} + \frac{1}{2}a_{|3|}).$$

Hence, they can not be together in the same right-recursive rule.

Moreover, not every family of OWA operators allows some one-side recursive definition based upon the same natural ordering rule. For example, no left or right recursive rule can be defined if we take $\phi_2' = \phi_2$ as above, but

$$\phi_3'(a_1, a_2, a_3) = \frac{1}{4}a_{|1|} + \frac{1}{2}a_{|2|} + \frac{1}{4}a_{|3|}.$$

3.2 OWA recursive rules.

Although the standard associative procedure can not be considered when dealing with OWA operators, it may be the case that a recursive analysis can be applied to the decreasing ordered list $(a_{|1|}, \ldots, a_{|n|})$. Thus, practical OWA aggregation problems where the number of values to be aggregated is not previously known, should be solved by choosing one of these consistent recursive families of OWA operators, by means only of such a natural ordering rule. Each one of these families solves every aggregation problem for any arbitrary size of the input.

DEFINITION 5 *A recursive OWA rule is a recursive connective rule of OWA operators allowing left and right recursive definitions based upon the natural decreasing ordering rule, by means of binary OWA rules.*

This recursiveness definition has the advantage that aggregation weights can be computed quickly by using a dynamic programming approach (see [1]).

Such a recursiveness should not be confused with the *ordered linkage* property, considered in [6] in order to characterize OWA operators.

Anyway, we can check that once an OWA operator of dimension n has been fixed, all OWA operators of lower dimension belonging to its right and left OWA rules are almost univocally defined. In fact, it will be shown that every OWA operator can be recursively defined, both left and right, once the values to be aggregated have been pre-arranged according to the natural order in the real line. These two recursive representations will be basically unique.

THEOREM 2 *Let us consider a fixed OWA operator ϕ of dimension n. Then there exist at least one family of $n-1$ OWA operators of dimension 2*

$$L_2, \ldots, L_n$$

and another family of $n-1$ OWA operators all of them also of dimension 2

$$R_2, \ldots, R_n$$

allowing a left recursion and a right recursion, respectively, in such a way that

$$\phi(a_1, \ldots, a_n)$$

is equivalent to

$$L_n(L_{n-1}(\ldots L_3(L_2(a_{|1|}, a_{|2|}), a_{|3|}) \ldots), a_{|n|})$$

and

$$R_n(a_{|1|}, R_{n-1}(a_{|2|}, \ldots, R_3(a_{|n-2|}, R_2(a_{|n-1|}, a_{|n|})) \ldots))$$

Moreover, each one of these binary OWA operators is either unique or it can be freely chosen.

Proof: Let us assume right recursion, for example. If

$$R_n(b_1, b_2) = (1 - f(n))b_{|1|} + f(n)b_{|2|}$$

Then,

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{|i|} = (1 - f(n))a_{|1|} + f(n)b_{|2|}$$

where

$$b_{|2|} = R_{n-1}(a_{|2|}, \ldots, R_3(a_{|n-2|}, R_2(a_{|n-1|}, a_{|n|})) \ldots))$$

in such a way that

$$f(n) = 1 - w_1.$$

Hence, if we now assume

$$R_{n-1}(b_1, b_2) = (1 - f(n-1))b_{|1|} + f(n-1)b_{|2|},$$

it must be

$$\phi(a_1,\dots,a_n) = (1-f(n))a_{|1|} + f(n)[(1-f(n-1))a_{|2|} + f(n-1)b_{|3|}]$$

where

$$b_{|3|} = R_{n-2}(a_{|3|},\dots,R_3(a_{|n-2|},R_2(a_{|n-1|},a_{|n|}))\dots))$$

in such a way that

$$f(n-1) = 1 - w_2/f(n) = 1 - w_2/(1-w_1)$$

whenever $w_1 \neq 1$. In case $w_1 = 1$, it is the *max* rule, and the remaning binary OWA operators are not relevant at all. The process continues till we reach the trivial case R_2. In particular, in each step we obtain

$$f(n-i) = 1 - \frac{w_{i+1}}{1-\sum_{j=1}^{i} w_j}$$

whenever $\sum_{j=1}^{i} w_j < 1$ (otherwise, the remanining operators will be not relevent at all). ■

Notice that right recursion is unique for the *min* rule ($w_n = 1$), and left recursion will be unique for the *max* rule ($w_1 = 1$). In case $w_i \neq 0$ for all i, our OWA operator would be strictly increasing in each coordenate and theorem 1 would apply. The above result proves that in fact every OWA operator allows both left and right recursive definitions.

OWA rules as considered in this paper will consistently allow the recursive definition of each one of its operators. In other words, our *OWA rules* will be given by a sequence of OWA operators that can be explained in terms of a sequence of binary OWA operators allowing its right or left recursive representation. It is therefore natural to characterize each *recursive OWA rule* by means of the sequence of weights associated to its right or left recursive representation (see [2]).

DEFINITION 6 *A* basis function *is any mapping f that to any integer n associates a number in the unit interval (that is, $f(n) \in [0,1]$ for all n) with $f(1) = 1$.*

Each basis function f will then allow the recursive definition of two families of OWA operators. For any $n \geq 2$, we can define L_n and R_n such that

$$L_n(b_1,b_2) = (1-f(n))b_{|1|} + f(n)b_{|2|}$$

and

$$R_n(b_1,b_2) = f(n)b_{|1|} + (1-f(n))b_{|2|}$$

Then any left recursive operation

$$L_{n+1}(L_n(\dots(L_2(a_{|1|},a_{|2|})),\dots,a_{|n|}),a_{|n+1|})$$

and any right recursive operation

$$R_{n+1}(a_{|1|},(R_n(a_{|2|},\dots,R_2(a_{|n|},a_{|n+1|}))))$$

will always lead to OWA operators, for every $n \geq 2$, as it will be shown below. Each one of these two families of OWA operators ($\Phi = \{\phi_2,\dots,\phi_n,\dots\}$ if obtained via left-recursion call and $\Phi' = \{\phi'_2,\dots,\phi'_n,\dots\}$ if obtained via right-recursion call), will be then associated to the basis function f. According to our last theorem, left recursive (LR) and right recursive (RR) families of OWA operators will be defined, in particular, as follows:

- n is the dimension of the OWA operators ϕ_n and ϕ'_n;
- the weights of ϕ_n are denoted by $w_{1,n},\dots,w_{n,n}$ and $w'_{1,n},\dots,w'_{n,n}$ will denote the weights of ϕ'_n;
- for every $n \geq 2$ and every $i = 1,2,\dots,n$ we define

$$w_{i,n} = \begin{cases} f(n) & \text{if } i = n \\ (1-f(n))w_{i,n-1} & \text{if } i < n \end{cases}$$

- for every $n \geq 2$ and every $i = 1,2,\dots,n$ we define

$$w'_{i,n} = \begin{cases} f(n) & \text{if } i = 1 \\ (1-f(n))w'_{i-1,n-1} & \text{if } i > 1 \end{cases}$$

Therefore,

- $w_{i,n} = f(i)\prod_{j=i+1}^{n}(1-f(j))$ for every $n \geq 2$ and every $i = 1,2,\dots,n$.
- $w'_{i,n} = f(n-i+1)\prod_{j=n-i+2}^{n}(1-f(j))$ for every $n \geq 2$ and every $i = 1,2,\dots,n$.

In view of the above equations it is immediate to check that ϕ_n and ϕ'_n are in fact OWA operators, since for any $n \geq 2$ we have

$$\sum_{i=1}^{n} w_{n,i} = \sum_{i=1}^{n} w'_{n,i} = 1$$

It is also easy to check now that not every family $\{\phi_2,\dots,\phi_n,\dots\}$ of OWA operators can be recursively defined by means of binary OWA operators on the basis of the decreasing natural ordering. Recursive consistency can be easily characterized by means of the weights of the OWA operators. For example, if $w_{i,k} = 0$, in order to be able to provide a left recursive characterization it must also be $w_{i,k+1} = 0$. Analogously, $w_{i+1,k+1} = 0$ must hold for a right recursive definition, whenever $w_{i,k} = 0$.

¿From theorem 2 it is implied that ϕ_n of dimension n being fixed, then all left-recursive and right-recursive *consistent* OWA operators with lower dimension $\phi_2,\dots,\phi_{n-1}$ are univocally defined. More in general, we have the following result, which also gives a formal characterization of recursive consistency for OWA rules.

THEOREM 3 *Let us consider a family of OWA operators $\{\phi_2, \ldots, \phi_n, \ldots\}$. Then it can be defined by LR (i.e., it is* LR consistent*) if and only if $w_{i,k}w_{j,k+1} = w_{j,k}w_{i,k+1}$ for all $i, j = 1, 2, \ldots, k$ and every k. Analogously, such a family of OWA operators can be defined by RR (i.e., it is* RR consistent*) if and only if $w_{i,k}w_{j+1,k+1} = w_{j,k}w_{i-1,k+1}$ for all $i, j = 1, 2, \ldots, k$ and every k.*

Proof: Direct since every weight of the OWA operator of dimension k is multiplied by the same weight of the next binary OWA operators in order to allow the OWA operator of dimension $k+1$. In case of right recursion, for example,

$$\sum_{i=1}^{k+1} w_{i,k+1}a_{|i|} = f(k+1)a_{|1|} + (1 - f(k+1)) \sum_{i=1}^{k} w_{i,k}a_{|i+1|}$$

Hence,

$$\frac{w_{i+1,k+1}}{w_{i,k}} = 1 - f(k+1)$$

for all $i = 1, 2, \ldots, k$. ■

Therefore, once any ϕ_n has been chosen, clear restrictions are implied in order to obtain families of OWA operators which are consistent with ϕ_n, both with respect to left recursion and right recursion. But if a left (right) recursion exists, the associated LR (RR) basis function is basically unique. Thus, each basis function is characterizing a LR (RR) consistent family of OWA operators. Moreover, it has been already pointed out above that right (left) recursive consistency for a given family $\Phi = \{\phi_2, \ldots, \phi_n, \ldots\}$ of OWA operators does not imply left (right) recursive consistency.

Being the underlying ordering rule fixed as the decreasing natural order, it may be the case that both left and right recursions exist for a given family of OWA operators $\{\phi_2, \ldots, \phi_n, \ldots\}$. The associated left and right basis functions f and g will in this case define the same family of OWA operators $\{\phi_2, \ldots, \phi_n, \ldots\}$. Many standard families of OWA operators do belong to this class, as shown in the next section.

The following theorem refers to duality, and the next one can be quite useful in estimating the orness of recursive OWA operators.

THEOREM 4 *Let f be a basis function with associated LR family of OWA operators Φ, and Φ' its associated RR family. Then $\phi'_n \equiv \hat{\phi}_n$ for all n, i.e. ϕ'_n is the dual of ϕ_n for all n.*

Proof: Immediate. ■

THEOREM 5 *Let f be basis function such that $(1 - f(n))f(n-1) \geq f(n)$ for all n, and let us denote by Φ and Φ' the corresponding LR and RR families of OWA operators, respectively. Then every ϕ_n is a buoyancy measure. Therefore, $orness(\phi_n) \geq \frac{1}{2}$ and $andness(\phi'_n) \geq \frac{1}{2}$.*

Proof: Also immediate from the expression of weights in terms of the basis function. ■

4 Some examples.

We will now provide some interesting examples of recursive families of OWA operators.

4.1 Mean Rule.

The following result characterizes the Mean Rule as a commutative OWA rule.

THEOREM 6 *Let f be a basis function with associated LR and RR families of OWA operators being identical ($\Phi \equiv \Phi'$, that is, $\phi_n \equiv \hat{\phi}_n$ for all n). Then it must be $f(n) = 1/n$ for all n, and in turn the weights of each ϕ_n are $w_{i,n} = 1/n$ for all $i = 1, 2, \ldots, n$.*

Proof: We shall prove it by induction, just for the RR case. The result is obvious for $n = 2$. Let us assume $w_{i,n} = 1/n$ for all $i = 1, 2, \ldots, n$. Then it must be

$$w_{1,n+1} = f(n+1) = (1 - f(n+1))w_{n,n} = (1 - f(n+1))/n$$

in such a way that $f(n+1) = 1/(n+1)$. Therefore, since

$$f(n+1) = 1 - \frac{w_{i+1,n+1}}{w_{i,n}}$$

we obtain

$$w_{i,n+1} = (1 - f(n+1))(w_{i,n}) = \frac{1}{n+1}$$

■

4.2 Constant basis function.

Together with the *Mean Rule*, another case *a priori* deserving our attention are those rules characterized by a constant basis function (i.e., when there exists a value $a \in [0, 1]$ such that $f(n) = a$ for all $n \geq 2$).

In case $f(n) = a$ for all $n \geq 2$, each LR OWA operator ϕ_n will have weights

$$w_{1,n} = (1 - a)^{n-1},$$

and

$$w_{i,n} = (1 - a)^{n-i}a$$

for all $i = 2, \ldots, n$. Analogously, weights for each RR OWA operator ϕ'_n will be

$$w_{i,n} = (1 - a)^{i-1}a$$

for all $i = 1, 2, \ldots, n-1$ and

$$w_{n,n} = (1 - a)^{n-1}.$$

When $a = 1$ ($a = 0$) in left (right) recursion we obtain the *min* rule, and in right (left) recursion we obtain the *max* rule.

4.3 Harmonic OWA operators.

We recall that the n-th harmonic number is

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

Harmonic OWA operators are obtained by taking

$$f(n) = \frac{1/n}{H_n}.$$

Thus,

$$1 - f(n) = \frac{H_{n-1}}{H_n}.$$

By using theorem 5 it is immediate to see that the family of $LR-Harmonic$ OWA operators is a class of buoyancy measures. For example, its first OWA operators will have the following weights:

$$w_{1,2} = \frac{2}{3}\ w_{2,2} = \frac{1}{3}$$

$$w_{1,3} = \frac{6}{11},\ w_{2,3} = \frac{3}{11},\ w_{3,3} = \frac{2}{11}$$

$$w_{1,4} = \frac{12}{25},\ w_{2,4} = \frac{6}{25},\ w_{3,4} = \frac{4}{25},\ w_{4,4} = \frac{3}{25}$$

4.4 A monotone fuzzy quantifier.

In [12, 13] it is shown how to obtain the evaluation of monotone fuzzy quantifiers by means of OWA operators. In particular, given a monotone non decreasing fuzzy quantifier Q such that $Q(0) = 0$ and $Q(1) = 1$, the weights $w_{i,n}$ for $i = 1, 2, \ldots, n$ of an OWA operator of dimension n to evaluate Q are defined as

$$w_{i,n} = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)$$

In case $Q(a) = a^r$ for some $r > 0$ we obtain that taking

$$f(n) = 1 - Q\left(\frac{n-1}{n}\right)$$

for all n, the associated left-recursive family of OWA operators does verify such a property. Hence, such a monotone fuzzy quantifier allows a left recursive definition. But it can not be right-recursively defined.

5 Final Comments.

OWA rules do play a main role in group decision making, since many aggregation procedures in practice are just particular cases. This paper generalizes previous results obtained just for OWA operators in [2]. A general approach to non-associative connective rules allowing an *operational* definition has been proposed. By *operational* we understand the ability of a recursive one-by-one evaluation, on the basis of a previous re-arrangement of the data set.

As a consequence, it has been stressed the fact that a connective rule, in order to be considered a rule, should be able to deal with any arbitrary number of items. An OWA operator is just an operator as the mean of n numbers is. None of them are *connective rules*, but single connectives. Considerably many real life decision processes require at different times to aggregate (possibly very large) lists of inputs of different dimensions. Connective rules have to be defined before knowing such a list. A connective rule is in general a rule allowing aggregation of any list, no matter its dimension.

Connective rules have been conceived here as consistent families of connective operators, allowing a representation in terms of right or left recursion of binary connective operators. Associativity is just an easy way of assuring such an operational representation.

Obviously, there are families of OWA operators that represent rules in the sense that they allow the evaluation of any arbitrary number of items, not allowing the recursive approach as developed in this paper, but being *consistent* in some other alternative sense. This is the case, for example, of the *Binomial* OWA rule $\{\phi_n(a_1, \ldots, a_n)\}_{n>1}$ where each ϕ_n is an OWA operator of dimension n with weights

$$w_{i,n} = \binom{n-1}{i-1} a^{i-1}(1-a)^{n-i} \quad \forall i = 1, \ldots, n$$

for some fixed $a \in (0, 1)$. Each one of these operators can be recursively defined, but the family itself does not verify the recursive OWA rule condition given in definition 5, neither the more general recursiveness definition 4. An operative description of this family of OWA operators, still by means of a sequence of binary OWA operators and the natural decreasing ordering, can be based upon the ordered linkage property of OWA operators (see [6]).

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