Head-Media Interaction in Magnetic Recording

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Abstract

The tape-head interaction in magnetic recording is modelled by a coupled system of a second order differential equation for the pressure and a fourth-order differential equation for the tape deflection. There is also the constraint that the spacing between the head and tape remains positive. In this paper, we study the stationary one-dimensional case and establish the existence of a smooth solution.

1 The Model

Figure 1 shows the magnetic tape given by $\hat{y} = \hat{u}(\hat{x})$, $0 < \hat{x} < \hat{L}$, and the magnetic head profile $\hat{y} = \hat{\delta}(\hat{x})$, $\hat{L}_1 \leq \hat{x} \leq \hat{L}_2$. The spacing between the head and the tape is denoted by $\hat{h}(\hat{x})$, i.e.,

$$\hat{h}(\hat{x}) = \hat{u}(\hat{x}) - \hat{\delta}(\hat{x}),\ \hat{h}(\hat{x}) > 0,\ \hat{L}_1 \leq \hat{x} \leq \hat{L}_2.$$

The tape is driven with velocity V, and its motion entrains air in the space between the head and the tape, with pressure $\hat{p}(\hat{x})$, $\hat{L}_1 \leq \hat{x} \leq \hat{L}_2$. At the endpoints $\hat{x} = \hat{L}_1$, $\hat{x} = \hat{L}_2$ the pressure is equal to the atmospheric pressure.

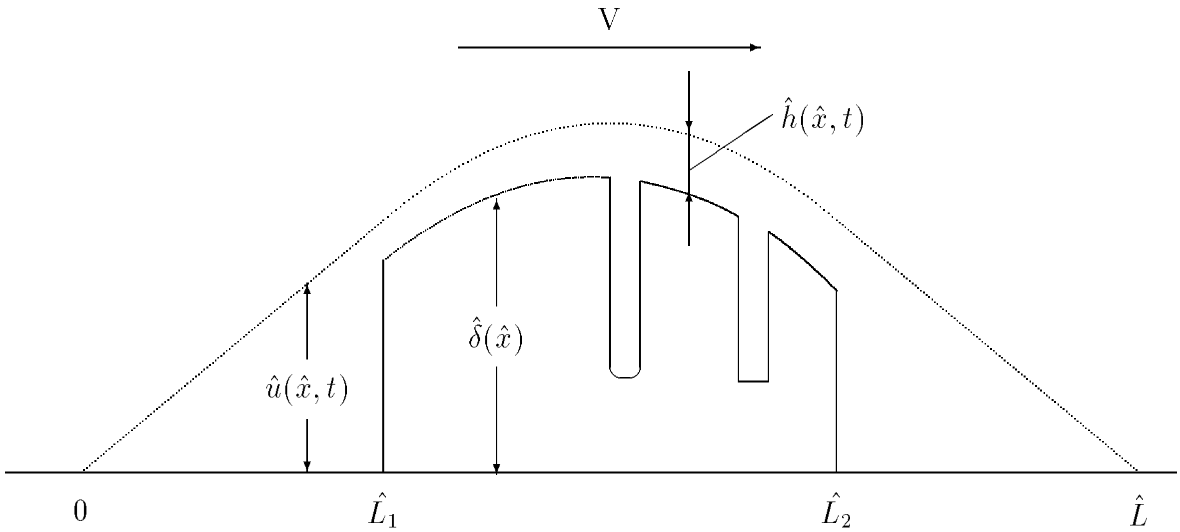

Figure 1

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After nondimensionalization one obtains the following system for the tape $y = u(x)$ and the pressure $p(x)$ [1][3;Chap.6]:

$$\frac{\partial(ph)}{\partial x} - \epsilon\frac{\partial}{\partial x}(\alpha h^2\frac{\partial p}{\partial x} + \beta h^3 p\frac{\partial p}{\partial x}) = 0,\ L_1 < x < L_2, \tag{1.1}$$

$$-\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^4 u}{\partial x^4} = K(p-1)\chi_{[L_1,L_2]},\ 0 < x < L, \tag{1.2}$$

$$u(x) = h(x) + \delta(x),\ h(x) > 0 \ \text{ if } \ L_1 \le x \le L_2 \tag{1.3}$$

where $0 < L_1 < L_2 < L$, $\chi_{[L_1,L_2]}$ is the characteristic function of the interval $[L_1, L_2]$, and, typically,

$$\alpha \sim \frac{1}{10},\ \beta \sim 1,\ L_2 - L_1 \sim 1,\ L \sim 10,$$
$$K \sim 10^4,\ \epsilon \sim 10^{-2},\ \mu \sim 10^{-3};$$

L_1 and L_2 lie near of the middle of the interval $(0, L)$.

The boundary conditions are

$$p(L_1) = p(L_2) = 1, \tag{1.4}$$

$$u = \frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L. \tag{1.5}$$

We assume throughout the paper that

(1.6) δ is piecewise continuous with jump discontinuous at $\xi_1, \xi_2, \ldots, \xi_s$ where $\xi_0 = L_1 < \xi_1 < \ldots < \xi_s < L_2 = \xi_{s+1}$, and $\delta \in C^1[\xi_i, \xi_{i+1}]$ for $0 \le i \le s$,

and

$$\delta(L_1) < \delta'(L_1)L_1,\ \ \delta(L_2) < (L_2 - L)\delta'(L_2); \tag{1.7}$$

the case where $\delta(x)$ has no discontinuities may be considered as a special case of (1.6) (with s=0).

Note that the inequality $\delta(L_1) < \delta'(L_1)L_1$ means that the tangent to the head at $x = L_1$ intersects the $x-$axis in the interval $(0, L_1)$. Similarly, the second inequality in (1.7) means that the tangent to the head at $x = L_2$ intersects the $x-$axis in the interval (L_2, L).

Note also that ϵ and μ are small numbers. In this paper we prove:

Theorem 1.1 *Assume that (1.6) and (1.7) are satisfied. Then there exist positive constants ϵ_*, μ_* such that if $0 < \epsilon < \epsilon_*$, $0 < \mu < \mu_*$, then the system (1.1)-(1.5) has a solution with p in $W^{1,\infty}[L_1, L_2]$, $u \in W^{4,\infty}[0, L]$ and $p > 0$, $h > 0$ in $[L_1, L_2]$.*

Theorem 1.1 was proved in [4] under the assumption that

$$\delta \in C^2 \text{ and } \delta''(x) < 0,\ L_1 \le x \le L_2. \tag{1.8}$$

This assumption is very restrictive, not only mathematically, but also physically: Magnetic heads do not generally satisfy the concavity condition (1.8). Indeed, in order to reduce the effect of air entrainment (which causes boundary layer for the pressure p near $x = L_2$), trenches are dug into the head (see [2]) and, of course , $\delta(x)$ is discontinuous at the edges of the trench. But even if a trench is smoothened near the edges so that δ is a smooth function in a neighbourhood of trench, $\delta''(x)$ will change sign across the trench.

For clarity we shall first prove Theorem 1.1 replacing (1.6) by the stronger assumption:

$$\delta \in C^2[L_1, L_2]; \tag{1.9}$$

the proof for this special case is given in Sections 2-4. In Section 2 we establish the existence of a solution in the case $\epsilon = \mu = 0$ and in Section 3 we prove that the problem for $\epsilon = \mu = 0$ can be written as a variational inequality. The approach we use to establish these results is entirely different from the approach in [4] ; instead of the shooting method used in [4] we use here a method based in sub- and super-solutions. In Section 4 we prove Theorem 1.1 (under the stronger assumption (1.9)) by combining the method used in [4] with the results of Section 2 and 3. The proof of Theorem 1.1 in the general case is given in Section 5.

2 The case $\epsilon = 0$, $\mu = 0$

In the special case $\epsilon = \mu = 0$, the system (1.1)-(1.3) reduces to

$$\frac{\partial (ph)}{\partial x} = 0, \ L_1 < x < L_2, \tag{2.1}$$

$$-\frac{\partial^2 u}{\partial x^2} = K(p-1)\chi_{[L_1,L_2]}, \ 0 < x < L. \tag{2.2}$$

Some of the boundary conditions in (1.4), (1.5) need to the dropped, and we take

$$p(L_1) = 1, \tag{2.3}$$

$$u(0) = u(L) = 0. \tag{2.4}$$

From (2.1) we see that $ph = constant = C$ and, since $p(L_1) = 1$, $C = h(L_1) = u(L_1) - \delta(L_1)$, so that

$$p(x) = \frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)}.$$

Hence (2.2) becomes

$$-\frac{\partial^2 u}{\partial x^2} = K(\frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]}, \ 0 < x < L. \tag{2.5}$$

Theorem 2.1 *There exists a solution $u(x)$ of (2.5),(2.4).*

The proof requires several Lemmas. Let

$$A = \|\delta'\|_{L^\infty(L_1,L_2)} \ (A > 0), \ \ B = \sup_{L_1 \le x \le L_2} [-\delta''(x)] \ (B > 0).$$

For any

$$\lambda \in I \equiv (\delta(L_1), AL_1],$$

consider the problem

$$(2.6)\qquad \begin{aligned} &-\frac{\partial^2 u}{\partial x^2} = K(\frac{\lambda - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]}, \quad 0 < x < L, \\ &u(0) = u(L) = 0, \\ &u(x) - \delta(x) > 0 \qquad \text{if } L_1 \le x \le L_2. \end{aligned}$$

Lemma 2.1 *For any $\lambda \in I$ there exists a unique solution $u(x,\lambda)$ of (2.6).*

<u>Proof</u>. Set

$$F(u,x) = K(\frac{\lambda - \delta(L_1)}{u - \delta(x)} - 1)\chi_{[L_1,L_2]}, \ u > \delta(x)$$

and note that

$$(2.7)\qquad F_u < 0.$$

Consider the function

$$\underline{u}(x) = \bar{\delta}(x) + \frac{\lambda - \delta(L_1)}{1 + c(\lambda)}$$

where

$$\bar{\delta}(x) = \begin{cases} \delta(L_1) + \delta'(L_1)(x - L_1), & 0 \le x < L_1 \\ \delta(x), & L_1 \le x \le L_2 \\ \delta(L_2) + \delta'(L_2)(x - L_2), & L_2 < x \le L, \end{cases}$$

$$c(\lambda) = max\{\frac{B}{K}, \frac{\lambda - \delta(L_1)}{-\bar{\delta}(0)}, \frac{\lambda - \delta(L_1)}{-\bar{\delta}(L)}\}.$$

Then

$$-\underline{u}_{xx} - F(\underline{u}(x), x) \le \{B - K(\frac{\lambda - \delta(L_1)}{(\delta(x) + \frac{\lambda - \delta(L_1)}{1 + c(\lambda)}) - \delta(x)} - 1)\}\chi_{[L_1,L_2]} =$$

$$= [B - Kc(\lambda)]\chi_{[L_1,L_2]}$$

so that, by the choice of $c(\lambda)$,

$$-\underline{u}_{xx}(x) \le F(\underline{u}(x), x).$$

Furthermore,

$$\underline{u}(0) = \delta(L_1) - \delta'(L_1)L_1 + \frac{\lambda - \delta(L_1)}{1 + c(\lambda)} < 0,$$

and, similarly, $\underline{u}(L) < 0$. Thus $\underline{u}$ is a subsolution for the problem (2,6).

The function $\bar{u}(x) = Ax$ is a supersolution. Indeed, since $(\bar{u} - \delta)' = A - \delta' \ge 0$,

$$\bar{u}(x) - \delta(x) \ge \bar{u}(L_1) - \delta(L_1) = AL_1 - \delta(L_1) \ge \lambda - \delta(L_1)$$

if $L_1 \le x \le L_2$, so that

$$-\bar{u}_{xx}(x) - K(\frac{\lambda - \delta(L_1)}{\bar{u}(x) - \delta(x)} - 1)\chi_{[L_1,L_2]} \ge -\bar{u}_{xx}(x) = 0.$$

Furthermore, $\bar{u}(0) = 0$ and $\bar{u}(L) \ge 0$.

Introduce the convex set of functions

$$G = \{\tilde{u} \in C^0[0, L], \underline{u}(x) \le \tilde{u} \le \bar{u}\}.$$

For any $\tilde{u} \in G$ we consider the problem

$$-u_{xx} = F(\tilde{u}, x),\ 0 < x < L,$$

$$u(0) = u(L) = 0.$$

Using (2.7) and the maximum principle (or comparison) we deduce that the solution u satisfies

$$\underline{u} \le u \le \bar{u},\ 0 \le x \le L.$$

If we define a mapping T by $T(\tilde{u}) = u$, then T maps G into itself. It is easily seen that T is continuous and $T(G)$ lies in a compact subset of G. Appealing to the Schauder fixed point theorem we conclude that T has a fixed point , which is clearly the solution to (2.6). Finally, if u_1 is another solution, then the function $w = u - u_1$ satisfies

$$w_{xx} + F_u w = 0$$

where F_u is evaluated at some intermediate point. Since $F_u < 0$, $w \equiv 0$ and thus $u_1 = u$. □

We denote the solution of (2.6) by $u(x, \lambda)$ and introduce the function

$$f(\lambda) = u(L_1, \lambda),\ \lambda \in I. \tag{2.8}$$

Lemma 2.2 *The function f is continuous.*

<u>Proof</u>. If $\lambda_n \longrightarrow \lambda_0 \in I$ then any subsequence of λ_n has a sub-subsequence $\lambda_{n'}$ for which $u(x, \lambda_{n'})$ is uniformly convergent to a function $u_0(x)$, and $u_0(x)$ is the solution of (2.6) for $\lambda = \lambda_0$ (by uniqueness). It follows that

$$f(\lambda_n) = u(L_1, \lambda_n) \longrightarrow u(L_1, \lambda_0) = f(\lambda_0).$$ □

<u>Proof of Theorem 2.1.</u> We need to show that the mapping $\lambda \longrightarrow f(\lambda)$ has a fixed point. If $f(\lambda_1) \ge \lambda_1$ for some $\lambda_1 \in I$, then, since $f(AL_1) \le AL_1$ and $f(\lambda)$ is continuous, f will have a fixed point in the interval $[\lambda_1, AL_1]$. Thus it remains to prove that the inequality

$$f(\lambda) < \lambda \text{ for all } \lambda \in I \tag{2.9}$$

cannot hold. We shall assume that (2.9) holds and proceed to derive the contradiction. To do that we define

$$\lambda_{n+1} = f(\lambda_n),\ n > 1$$

for some $\lambda_0 \in I$. Then, in view of (2.9), $\lambda_n \downarrow \lambda_*$, and $\lambda_* = \delta(L_1)$ since, otherwise, $f(\lambda_*) = \lambda_*$ by Lemma 2.2. Thus

$$\lambda_n \downarrow \delta(L_1) \text{ as } n \longrightarrow \infty. \tag{2.10}$$

Recall that $u_{xx}(x, \lambda_n) = 0$ if $0 < x < L_1$, so that

$$u_x(L_1, \lambda_n) = \frac{u(L_1, \lambda_n)}{L_1} = \frac{\lambda_{n+1}}{L_1} \longrightarrow \frac{\delta(L_1)}{L_1}.$$

Since, by (1.7),

$$\delta'(L_1) > \frac{\delta(L_1)}{L_1} + 3\epsilon_0$$

for some $\epsilon_0 > 0$, it follows that

$$u_x(L_1, \lambda_n) < \delta'(L_1) - 2\epsilon_0 \tag{2.11}$$

if n is sufficiently large. We also have

$$u_{xx}(L_1 + 0, \lambda_n) = K(1 - \frac{\lambda_n - \delta(L_1)}{\lambda_{n+1} - \delta(L_1)}) < 0 \tag{2.12}$$

since $\lambda_{n+1} = f(\lambda_n) < \lambda_n$.

Let

$$\bar{x} = max\{x \in (L_1, L_2]; \delta'(L_1) - \delta'(x') \leq \epsilon_0, \text{ for all } L_1 \leq x' \leq x\}. \tag{2.13}$$

We claim that

$$u_{xx}(x, \lambda_n) < 0 \text{ if } L_1 \leq x < \bar{x}. \tag{2.14}$$

Indeed, if this is not true then, setting for simplicity $u(x) = u(x, \lambda_n)$, we have, by (2.12),

$$\begin{aligned} &u_{xx}(x) < 0 \text{ if } L_1 \leq x < x^*, \\ &u_{xx}(x^*) = 0 \end{aligned} \tag{2.15}$$

for some $x^* = x_n^* \in (L_1, \bar{x})$, and then also

$$u_{xxx}(x^*) \geq 0.$$

From the differential equation in (2.6) we get

$$u_{xxx}(x) = K\frac{\lambda - \delta(L_1)}{(u(x) - \delta(x))^2}(u_x(x) - \delta'(x)) \text{ in } (L_1, L_2), \tag{2.16}$$

so that, by (2.11),

$$u_{xxx}(L_1 + 0) < 0.$$

If follows that exists a point $x^{**} = x_n^{**}$ in $(L_1, x^*]$ such that

$$u_{xxx}(x) < 0 \text{ if } L_1 < x < x^{**},$$

$$u_{xxx}(x^{**}) = 0.$$

Appealing again to (2.16) we deduce that

$$u_x(x^{**}) - \delta'(x^{**}) = 0$$

and then by (2.15) and (2.11)

$$\delta'(x^{**}) = u_x(x^{**}) < u_x(L_1) < \delta'(L_1) - 2\epsilon_0$$

which is a contradiction to (2.13) since $x^{**} < \bar{x}$.

From (2.14) and (2.11),(2.13) we deduce that

$$u_x(x, \lambda_n) < u_x(L_1, \lambda_n) < \delta'(L_1) - 2\epsilon_0 \le \delta'(x) - \epsilon_0 \text{ if } L_1 < x \le \bar{x}.$$

Hence

$$u(\bar{x}, \lambda_n) - \delta(\bar{x}) < \lambda_{n+1} - \delta(L_1) - \epsilon_0(\bar{x} - L_1).$$

Recalling (2.10) and the fact that $\bar{x}$ is independent of n, we get

$$u(\bar{x}, \lambda_n) - \delta(\bar{x}) < 0$$

if n is sufficiently large, which contradicts the inequality $u(x, \lambda_n) > \delta(x)$ in $[L_1, L_2]$. □

3 Variational formulation for (2.5), (2.4)

Theorem 2.1 can be extended to the system

(3.1)
$$\begin{aligned} &-\frac{\partial^2 g}{\partial x^2} = K\Big(\frac{g(L_1) - \delta(L_1)}{g(x) - \delta(x)} - 1\Big)\chi_{[L_1,L_2]}, \quad 0 < x < L, \\ &g(0) = g(L) = -\gamma, \\ &g(x) - \delta(x) > 0 \qquad\qquad \text{if } L_1 \le x \le L_2, \end{aligned}$$

provided

(3.2)
$$max\{\delta(L_1) - \delta'(L_1)L_1, \delta(L_2) - \delta'(L_2)(L_2 - L)\} < -\gamma < 0.$$

Note that the construction of the solution to

(3.3)
$$\begin{aligned} &-\frac{\partial^2 g}{\partial x^2} = K\Big(\frac{\lambda - \delta(L_1)}{g(x) - \delta(x)} - 1\Big)\chi_{[L_1,L_2]}, \quad 0 < x < L, \\ &g(0) = g(L) = -\gamma, \\ &g(x) - \delta(x) > 0 \qquad\qquad \text{if } L_1 \le x \le L_2, \end{aligned}$$

we use the same sub- and super-solutions $\underline{u}(x)$ and $\bar{u}(x)$, respectively, as before. We shall denote the solution of (3.3) by $g(x, \lambda, \gamma)$.

Set

$$F(u,x,\lambda) = K(\frac{\lambda - \delta(L_1)}{u - \delta(x)} - 1)\chi_{[L_1,L_2]},\ u > \delta(x).$$

Clearly

$$F_u < 0,\ F_\lambda > 0.$$

By the strong maximum principle we then get

$$g(x,\lambda_1,\gamma_1) > g(x,\lambda_2,\gamma_2) \text{ in } (0,L) \text{ if } \lambda_1 \geq \lambda_2,\ 0 \leq \gamma_1 < \gamma_2. \tag{3.4}$$

Lemma 3.1 *Let $g(x)$ the any solution of (3.1). Then there exists a solution $u(x)$ of (2.5), (2.4) such that*

$$u(x) > g(x) \text{ if } 0 < x < L \tag{3.5}$$

and, consequently,

$$u(x) \geq g(x) + \epsilon_0, \quad \text{if } L_1 \leq x \leq L_2. \tag{3.6}$$

for some $\epsilon_0 > 0$.

<u>Proof</u>. By (3.4) and the maximum principle,

$$u(x,\lambda) > g(x,\lambda,\gamma) \text{ if } 0 < x < L. \tag{3.7}$$

Take $\lambda = \lambda_0$ such that $g(x,\lambda_0,\gamma)$ is the solution $g(x)$. Then

$$u(L_1,\lambda_0) > g(L_1,\lambda_0,\gamma) = \lambda_0,$$

so that $f(\lambda_0) > \lambda_0$. It follows that there is a fixed point λ_* of the mapping $\lambda \longrightarrow f(\lambda)$ in the interval $(\lambda_0, AL_1]$. The function $u(x) = u(x,\lambda_*)$ is the solution of (2.5), (2.4) and by (3.4), (3.7)

$$u(x) = u(x,\lambda_*) \geq u(x,\lambda_0) > g(x) \text{ in } (0,L). \qquad \square$$

Introduce the function

$$\psi(s) = \begin{cases} s & s < 1 + AL_1 \\ 1 + AL_1 & s \geq 1 + AL_1. \end{cases}$$

Take any solution $g(x)$ of (3.1) and consider the variational inequality

$$\begin{aligned}
&-\frac{\partial^2 u}{\partial x^2} \geq K(\frac{\psi(u(L_1)) - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]}, \\
&u(x) \geq g(x), \\
&(u-g)[\frac{\partial^2 u}{\partial x^2} + K(\frac{\psi(u(L_1)) - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]}] = 0 \quad \text{in } [0,L], \\
&u(0) = u(L) = 0.
\end{aligned} \tag{3.8}$$

The truncation ψ is introduced for a technical reason, so that we can carry out the proof of Lemma 4.1 in Section 4; see also [4].

Note that the solution u of (2.5), (2.4) established in Lemma 3.1 satisfies the variational inequality (3.8) since $\psi(u(L_1)) = u(L_1)$ (as $u(L_1) \le AL_1$). We now prove the converse:

Theorem 3.1 *Any solutions $u(x)$ of the variational inequality (3.8) is a solution of (2.5),(2.4), and it satisfies the inequalities (3.5),(3.6).*

<u>Proof</u>. Since $u(L_1) \ge g(L_1)$, $\psi(s)$ is monotone increasing in s, and $g(L_1) \le AL_1$,

$$\psi(u(L_1)) - \delta(L_1) \ge \psi(g(L_1)) - \delta(L_1) = g(L_1) - \delta(L_1).$$

It follows that

$$-\frac{\partial^2 u}{\partial x^2} \ge K(\frac{g(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]}.$$

Since, furthermore,

$$u(0) = u(L) > -\gamma = g(0) = g(L),$$

the strong maximum principle yields the inequalities (3.5), (3.6). It follows that

$$-\frac{\partial^2 u}{\partial x^2} = K(\frac{\psi(u(L_1)) - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]}. \tag{3.9}$$

From the inequality $\psi(s) \le s$ we then have that

$$-\frac{\partial^2 u}{\partial x^2} \le K(\frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]},$$

and, since $\bar{u}(x) = Ax$ is a supersolution, we get, by comparison, $u(x) \le Ax$. This implies that $\psi(u(L_1)) = u(L_1)$, so that (3.9) reduces to (2.5). Thus $u(x)$ is a solution of (2.5), (2.4) satisfying (3.5), (3.6). □

4 Proof of Theorem 1.1 (special case)

Consider the system

$$\frac{\partial(ph)}{\partial x} - \epsilon\frac{\partial}{\partial x}(\alpha h^2\frac{\partial p}{\partial x} + \beta h^3 p\frac{\partial p}{\partial x}) = 0, \ L_1 < x < L_2, \tag{4.1}$$

$$\begin{aligned} &-\tfrac{\partial^2 u}{\partial x^2} + \mu\tfrac{\partial^4 u}{\partial x^4} \ge K(p-1)\chi_{[L_1,L_2]}, \quad u(x) \ge g(x), \\ &(u-g)[-\tfrac{\partial^2 u}{\partial x^2} + \mu\tfrac{\partial^4 u}{\partial x^4} - K(p-1)\chi_{[L_1,L_2]}] = 0 \quad \text{in } [0,L], \end{aligned} \tag{4.2}$$

$$h(x) = u(x) - \delta(x), \ L_1 \le x \le L_2 \tag{4.3}$$

with the boundary conditions

$$p(L_1) = p(L_2) = \frac{\psi(h(L_1))}{h(L_1)}, \tag{4.4}$$

$$(4.5) \qquad u = u_x = 0 \text{ at } x = 0 \text{ and } x = L.$$

Let

$$(4.6) \qquad G = \{p \in C^0[L_1, L_2], 0 \leq p \leq 1 + \frac{1 + AL_1}{l}\}$$

where

$$l = \inf_{L_1 \leq x \leq L_2} [g(x) - \delta(x)] > 0.$$

As in [4] one shows that for any $p \in G$ there exists a unique solution u of (4.2),(4.5) and

$$|u_x| \leq C$$

where C is a constant independent of p

Set $h = u - \delta$ and denote by $\hat{p}(x)$ the solution of (4.1),(4.4). It is obtained by solving the initial value problem

$$(4.7) \qquad \begin{aligned} &\hat{p}h - \epsilon(\alpha h^2 \tfrac{\partial \hat{p}}{\partial x} + \beta h^3 \hat{p} \tfrac{\partial \hat{p}}{\partial x}) = \eta, \quad L_1 < x < L_2, \\ &\hat{p}(L_1) = \tfrac{\psi(h(L_1))}{h(L_1)} \end{aligned}$$

for any constant $\eta \geq 0$, and then choosing η in the unique way such that

$$(4.8) \qquad \hat{p}(L_2) = \frac{\psi(h(L_1))}{h(L_1)};$$

for details see [4].

We define the mapping S by

$$S(p) = \hat{p}$$

and wish to show that S has a fixed point in G.

Lemma 4.1 *If ϵ is small enough then S maps G into itself and the parameter η determined by (4.7),(4.8) satisfies:*

$$(4.9) \qquad |\eta - \psi(h(L_1))| \leq C\epsilon$$

where C is a constant independent of ϵ.

Proof. We introduce the operator

$$\mathcal{L}(w) \equiv \frac{\partial(wh)}{\partial x} - \epsilon \frac{\partial}{\partial x}(\alpha h^2 \frac{\partial w}{\partial x} + \beta h^3 w \frac{\partial w}{\partial x}),\ L_1 < x < L_2$$

and consider the function

$$v(x) = \frac{\psi(h(L_1))}{h(L_1)} - \frac{C}{l}(x - L_1) - \nu,\ L_1 \leq x \leq L_2$$

for C positive and large and ν positive and arbitrarily small. Denote by $\hat{x}$ the point where $v(\hat{x}) = 0$. Then, as in [4],

$$\mathcal{L}(v) < 0 \text{ in } [L_1, \hat{x}]$$

if ϵ is small enough. A comparison argument used in [4] then shows that

$$\hat{p}(x) \geq v(x) \text{ in } [L_1, \hat{x}].$$

Taking $\nu \downarrow 0$ we deduce that

$$p_x(L_1) > -\frac{C}{l}$$

and, then, by (4.7),

$$\eta - \psi(h(L_1)) \leq C\epsilon.$$

The complementary inequality

$$\eta - \psi(h(L_1)) \geq -C\epsilon.$$

is derived in the same way by working with the function

$$\frac{\psi(h(L_1))}{h(L_1)} + \frac{C}{l}(x - L_1) + \nu. \qquad \square$$

Since

$$\psi(h(L_1)) \leq 1 + AL_1,$$

it follows from (4.9) that

$$\eta \leq 1 + AL_1 + C\epsilon.$$

(This is where we need the truncation ψ, since we do not have a uniform bound on $h(L_1)$.) But, then, from (4.7) we conclude that

$$\hat{p} \leq 1 + \frac{1 + AL_1}{l},$$

if ϵ is sufficient small.

From (4.7) we also deduce that $\hat{p}$ cannot take negative minimum, and thus $\hat{p} \in G$ and S maps G into itself. $\square$

PROOF OF THEOREM 1.1. By Lemma 4.1, S maps G into itself. It is also easy to show that S is continuous and $S(G)$ is contained in a compact subset of G. Hence, by Schauder's fixed point theorem, S has a fixed point in G, which is solution (u, p) of the system (4.1)-(4.5). If we can prove that

$$u(x) > g(x) \ \text{ in } \ (0, L), \tag{4.10}$$

$$u(L_1) < 1 + AL_1 \tag{4.11}$$

for $0 < \epsilon < \epsilon_*$ and $0 < \mu < \mu_*$, then (u, p) and h form a solution of (1.1)-(1.5) as asserted in Theorem 1.1.

But if either (4.10) or (4.11) is not satisfied, then, by taking sequences $\epsilon_j \downarrow 0, \mu_j \downarrow 0$ as in [4], we obtain limit functions (u_0, p_0, h_0) with

$$p_0(x)h_0 = \frac{\psi(h_0(L_1))}{h_0(L_1)} \ \text{ (using (4.9))}$$

and $u_0(x)$ satisfying the variational inequality (3.8). By Theorem 3.1,

$$u_0(x) \geq g(x) + \epsilon_0 \text{ for } L_1 \leq x \leq L_2$$

and clearly also

$$u_0(x) \leq AL_1.$$

But then, for the solution of (4.1)-(4.5), with $\epsilon = \epsilon_j$, $\mu = \mu_j$ small,

$$u(x) \geq g(x) + \frac{1}{2}\epsilon_0 \;\text{ for }\; L_1 \leq x \leq L_2, \tag{4.12}$$

and $u(L_1) < 1 + AL_1$. Since (4.12) implies (4.10), we get a contradiction to both inequalities (4.10) and (4.11). □

5 Proof of Theorem 1.1

We introduce the function

$$\hat{\delta}(x) = \delta(L_1) + ((\xi_1 - L_1)\|\delta'\|_{L^\infty[L_1,\xi_1]} + \|\delta\|_{L^\infty[\xi_1,L_2]})\frac{x - L_1}{\xi_1 - L_1} \tag{5.1}$$

for $0 \leq x \leq L$ and note that

$$\delta(x) \leq \hat{\delta}(x) \;\text{ if }\; L_1 \leq x \leq L_2. \tag{5.2}$$

We first prove Theorem 2.1 (under the assumption (1.6)). Since $-\delta''$ is not bounded from above (if $s \geq 1$ in (1.6)), we cannot construct a subsolution as before. We therefore first approximate $\delta(x)$ by functions δ_n in $C^2[L_1, L_2]$ such that

$$\int_{L_1}^{L_2} |\delta_n(x) - \delta(x)|^p dx \longrightarrow 0 \;\text{ for any }\; 1 < p < \infty, \tag{5.3}$$

$$\delta_n(L_1) = \delta(L_1), \tag{5.4}$$

$$\sup_{\substack{L_1 \leq x \leq L_1 + \eta_0 \\ L_2 - \eta_0 \leq x \leq L_2}} |\delta_n'(x) - \delta'(x)| \longrightarrow 0 \tag{5.5}$$

for some small $\eta_0 > 0$,

$$\delta_n'(x) \longrightarrow \delta'(x) \;\text{ for all }\; x \neq \xi_i \tag{5.6}$$

as $n \longrightarrow \infty$, and

$$\delta_n(x) \leq \hat{\delta}(x). \tag{5.7}$$

Consider the problem

$$\begin{aligned} -v'' &= K(\frac{\lambda - \delta(L_1)}{v(x) - \delta_n(x)} - 1)\chi_{[L_1,L_2]}, \quad 0 < x < L, \\ v(0) &= v(L) = 0. \end{aligned} \tag{5.8}$$

Lemma 5.1 *The function $\bar{u}(x) = \hat{\delta}(x) - \hat{\delta}(0)$ is a supersolution of (5.8) provided $\lambda \leq \bar{u}(L_1)$.*

<u>PROOF</u>. Clearly

$$\bar{u}(x) - \delta_n(x) = \hat{\delta}(x) - \delta_n(x) - \hat{\delta}(0) \geq -\hat{\delta}(0), \text{ by (5.7).}$$

Also

$$\hat{\delta}(L_1) - \delta(0) = \bar{u}(L_1) \geq \lambda$$

so that

$$-\hat{\delta}(0) \geq \lambda - \hat{\delta}(L_1) = \lambda - \delta(L_1).$$

Hence

$$\bar{u}(x) - \delta_n(x) \geq \lambda - \delta(L_1)$$

and, consequently,

$$-\bar{u}_{xx}(x) - K(\frac{\lambda - \delta(L_1)}{\bar{u}(x) - \delta_n(x)} - 1)\chi_{[L_1,L_2]} \geq -\bar{u}_{xx}(x) = 0.$$

Since also $\bar{u}(0) = 0$ and $\bar{u}(L) > 0$, the lemma follows. □

It seems difficult to construct a subsolution to the u_n which is independent of n. But we can nevertheless apply Theorem 2.1 (for the case where (1.9) is satisfied) to deduce that there exists a solution $u_n(x)$ of

$$(5.9) \qquad \begin{aligned} &-u_n''(x) = K(\frac{u_n(L_1) - \delta(L_1)}{u_n(x) - \delta_n(x)} - 1)\chi_{[L_1,L_2]}, \quad 0 < x < L, \\ &u_n(0) = u_n(L) = 0, \end{aligned}$$

and

$$(5.10) \qquad \delta_n(x) < u_n(x) \leq \bar{u}(x), \ 0 < x < L.$$

We may assume that

$$(5.11) \qquad u_n(L_1) \longrightarrow \lambda_*$$

for some $\delta(L_1) \leq \lambda_* \leq \bar{u}(L_1)$.

Lemma 5.2 *There holds:*

$$(5.12) \qquad \lambda_* > \delta(L_1).$$

<u>PROOF</u>. Suppose $\lambda_* = \delta(L_1)$. Then, for any small $\epsilon_0 > 0$,

$$u_n(L_1) < \delta(L_1) + L_1\epsilon_0 \ \text{ if } \ n \geq n_0(\epsilon_0),$$

and

$$(5.13) \qquad u_n'(L_1) = \frac{u_n(L_1)}{L_1} < \frac{\delta(L_1)}{L_1} + \epsilon_0 < \delta'(L_1) - 2\epsilon_0 < \delta_n'(L_1) - \epsilon_0$$

if ϵ_0 is small enough, by (1.7), (5.5).

We can now argue as in the proof of (2.14) (with $u = u_n$, $\delta = \delta_n$ and $\lambda_n = u_n(L_1)$) to deduce that

$$u_n''(x) < 0 \ \text{ if } \ L_1 \le x \le \bar{x}$$

where $\bar{x}$ is such that $\bar{x} \le \eta_0$ and $\delta_n'(L_1) - \delta_n'(x) < \frac{1}{2}\epsilon_0$ for all $L_1 \le x \le \bar{x}$, $n \ge n_0(\epsilon_0)$. Using also (5.13), we deduce that

$$u_n'(x) < \delta_n'(x) - \frac{\epsilon_0}{2} \ \text{ if } \ L_1 \le x \le \bar{x}.$$

It follows that

$$u_n(\bar{x}) - \delta_n(\bar{x}) < (u_n(L_1) - \delta_n(L_1)) - \frac{\epsilon_0}{2}(\bar{x} - L_1).$$

Hence $u_n(\bar{x}) - \delta_n(\bar{x}) < 0$ if n is sufficiently large, a contradiction. □

Since $u_n''(x) = 0$ and $0 < u_n(x) \le \bar{u}(x)$ for $0 < x < L_1$ and $L_2 < x < L$, we have

(5.14) $$0 \le u_n'(0) \le \bar{u}'(0), \quad \text{and } 0 > u_n'(L) = \frac{u_n(L_2)}{L_2 - L} \ge \frac{\bar{u}(L_2)}{L_2 - L}.$$

Integrating the differential equation in (5.9) and using (5.14) we obtain the inequality

$$\int_{L_1}^{L_2} \frac{u_n(L_1) - \delta(L_1)}{u_n(x) - \delta_n(x)} dx \le C.$$

In view of (5.10) and (5.11), (5.12), the last inequality implies that

(5.15) $$\int_{L_1}^{L_2} \frac{dx}{u_n(x) - \delta_n(x)} \le C.$$

Integrating (5.9) and using (5.15), (5.14) we see that

$$|u_n'(x)| \le C, \ 0 < x < L.$$

We may then assume that

(5.16) $$u_n(x) \longrightarrow u(x) \text{ uniformly in } \ x \in [0, L],$$

(5.17) $$|u'(x)| \le C \ \text{ a.e.}$$

We next deduce, by Fatou's Lemma and (5.15), that

(5.18) $$\int_{L_1}^{L_2} \frac{dx}{u(x) - \delta(x)} \le C.$$

Lemma 5.3 *There holds:*

(5.19) $$u(x) - \delta(x) > 0 \ \text{ if } \ L_1 \le x \le L_2.$$

PROOF. Suppose $u(x_0) - \delta(x_0) = 0$ for some $x_0 \in [L_1, L_2]$. Then, by the Mean Value Theorem,

$$|u(x) - \delta(x)| \leq C|x - x_0|$$

in an interval with endpoint at x_0, and this contradicts the estimate (5.18). □

Having proved Lemmas 5.2, 5.3, we can now pass to the limit in (5.9) and conclude that

$$-u'' = K(\frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1)\chi_{[L_1,L_2]}$$

for $0 < x < L$, $x \neq \xi_i$, and this completes the proof of Theorem 2.1. Note that $u(x)$ is continuously differentiable in $0 \leq x \leq L$ and $u''(x)$ is piecewise continuous with jump discontinuities at $\xi_1, \xi_2, \ldots, \xi_s, \xi_{s+1}$.

In the same way we can proceed to construct a solution g to (3.1).

We next claim that Lemma 3.1 extends to the present case. Indeed, since both $u(x)$ and $g(x)$ are larger than $\delta(x)$ and $u''(x)$ and $g''(x)$ are piecewise continuous, the maximum principle can be applied to deduce (3.7).

The proofs of Theorem 3.1 and of Theorem 1.1 can now proceed exactly as before.

REMARK. As in [4], the solutions asserted in Theorem 1.1 exhibit a boundary layer behavior at $x = L_2$.

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