Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2022) 116:113
https://doi.org/10.1007/s13398-022-01256-y

ORIGINAL PAPER

Lineability, spaceability, and latticeability of subsets of $C([0, 1])$ and Sobolev spaces

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Received: 6 July 2021 / Accepted: 28 April 2022

Abstract
This work is a contribution to the ongoing search for algebraic structures within a nonlinear setting. Here, we shall focus on the study of lineability of subsets of continuous functions on the one hand and within the setting of Sobolev spaces on the other (which represents a novelty in the area of research).

Keywords Lineability · Algebrability · Continuous function · Sobolev space · Banach lattice

Mathematics Subject Classification 15A03 · 46B87 · 46E10 · 46E99

1 Introduction and preliminaries

Since its appearance in 2005, the terminology lineability and spaceability has attracted the attention of many researchers and, just recently, the American Mathematical Society introduced this terminology in its 2020 Mathematical Subject Classification under the references

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Published online: 21 May 2022

15A03 and 46B87. In a nutshell, this notion consists on finding (when possible) large algebraic structures within non-linear subsets of a topological vector space.

Some early examples of results within this theory are due to V. I. Gurariy (1935–2005), who proved that the set of continuous nowhere differentiable functions on $\mathbb{R}$ contains (except for $\{0\}$) infinite dimensional linear spaces. Furthermore, in 2005 [1], he proved that the set of differentiable nowhere monotone functions also contains (except for $\{0\}$) infinite dimensional linear spaces. After these seminal works, a lot has been done linking many areas of mathematics, such as Set Theory [8, 9, 11], Real and Complex Analysis [10, 18], Linear and Multilinear Algebra [5], Linear Dynamics [15], or Statistics [12]. Let us recall some terminology we shall need throughout this work (which can be found in [3, 4, 22, 24]). Assume that X is a vector space and α is a cardinal number. Then a subset $A \subset X$ is said to be:

- *lineable* if there is an infinite dimensional vector space M such that $M\setminus\{0\} \subset A$.
- α-*lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M\setminus\{0\} \subset A$. In the case $\dim(M) = \dim(X)$, the set A is called maximal lineable.

  If, in addition, X is a topological vector space, then the subset A is said to be:
- *spaceable* in X whenever there is a closed infinite-dimensional vector subspace M of X such that $M\setminus\{0\} \subset A$. In the case $\dim(M) = \dim(X)$, the set A is called maximal spaceable.
- α-*latticeable* if there exists a Riesz space M such that $M\setminus\{0\} \subset A$ and M is an α-dimensional vector space. If, in addition, M is closed then A is said to be α-spaceable latticeable. Even more, it is said to be maximal whenever $\dim(M) = \dim(X)$.
- M is said to be μ-*dense-lineable* if $M \cup \{0\}$ contains a dense vector space of dimension μ.

This paper is arranged as follows. Section 2 focuses on studying the lineability properties of certain subsets of continuous functions on bounded intervals. Within this class we consider, among several others: (i.) $C^\infty(]0, 1])$, i.e., the class of all elements of $C([0, 1])$ which are infinitely differentiable on $]0, 1]$, (ii.) $\mathcal{D}_0$, i.e., the class of all functions $h \in C([0, 1])$ which are differentiable λ-almost everywhere with derivative 0 but are not Lipschitz continuous, or (iii.) $\mathcal{H}$, that is, the family of all functions $h \in C([0, 1])$ which are α-Hölder continuous for every $\alpha \in]0, 1[$ but not Lipschitz continuous.

Next, if we fix an integer $m \geq 0$, a real number $1 \leq p < \infty$ and a vector space X, Sect. 3 studies the problem of finding lineable/latticeable subsets of the Sobolev Space $W^{m,p}$ $(]0, 1[)$. We shall provide a brief background on theory of Sobolev Spaces over one dimensional bounded intervals in order to have this section self-contained. Section 4 considers the unbounded counterpart of Sect. 3 for Sobolev Spaces. The notation used throughout the paper shall be rather usual.

2 Lineability and spaceability in $C([0, 1])$

We consider the Banach space $(l^\infty, \|\cdot\|_\infty)$ of all bounded sequences in $\mathbb{R}$ as well as the Banach space $(\mathrm{Bd}([0, 1]), d_\infty)$ of all bounded functions, and the Banach space $(C([0, 1], d_\infty)$ of all real-valued continuous functions on $[0, 1]$, endowed with the uniform distance d_∞, respectively (we write d_∞ instead of $\|\cdot\|_\infty$ to avoid the usage of one symbol for two different objects). In the sequel we will also view each of these Banach spaces as Banach algebras and lattices (with the usual coordinate-wise/pointwise operations). The same holds for the closed subspace c_0 of l^∞ containing all sequences in l^∞ converging to 0.

Throughout this section $f : [0, 1] \to [0, 1]$ will denote a function for which there exists $x_0 \in]0, 1[$ with $f(x_0) = 1$. The only exception is the proof of Theorem 2.10 in which f is $[-1, 1]$-valued. Writing $\mathbf{s} = (s_1, s_2, \ldots) \in l^\infty$ and $I_n =]\frac{1}{2^n}, \frac{1}{2^{n-1}}[$ for every $n \in \mathbb{N}$), define the operator $\Phi_f : l^\infty \to \mathrm{Bd}([0, 1])$ by

$$\Phi_f(\mathbf{s})(x) = \begin{cases} s_n f(2^n x - 1) \text{if } x \in I_n \text{ for some } n \in \mathbb{N}, \\ 0 \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{2.1}$$

The subsequent lemma gathers the most important properties of Φ_f and is straightforward to prove:

Lemma 2.1 *Φ_f is a linear isometric embedding of $(l^\infty, \|\cdot\|_\infty)$ in $(Bd([0, 1]), d_\infty)$ and at the same time an algebra- and lattice isomorphism. Moreover, $\Phi_f(l^\infty)$ is a closed subspace (algebra, lattice) of $(Bd([0, 1]), d_\infty)$. The same holds for Φ_f if $(l^\infty, \|\cdot\|_\infty)$ is replaced by $(c_0, \|\cdot\|_\infty)$. Additionally, for $\mathbf{s} \in c_0$ the function $\Phi_f(\mathbf{s})$ is continuous at 0.*

Proof The fact that Φ_f is a structure-preserving isometry on $(l^\infty, \|\cdot\|_\infty)$ (and hence also on $(c_0, \|\cdot\|_\infty)$) is straightforward to verify (the fact that $f(x_0) = 1$ is crucial to guarantee this since, otherwise, we would not have an isometry, but a contraction). Considering that $(l^\infty, \|\cdot\|_\infty)$ is complete and that Φ_f is an isometry it follows that $\Phi_f(l^\infty)$ is complete and consequently closed in $(\mathrm{Bd}([0, 1]), d_\infty)$. The same reasoning applies to $(c_0, \|\cdot\|_\infty)$.

Finally, continuity of $\Phi_f(\mathbf{s})$ at 0 for $\mathbf{s} \in c_0$ is a straightforward consequence of the fact that $|\Phi_f(\mathbf{s})(x)\mathbf{1}_{I_n}(x)| \leq |s_n|$ holds for every $n \in \mathbb{N}$ and for every $x \in [0, 1]$. □

Selecting f adequately yields various results on spaceability/lineability/latticeability of seemingly 'small' subfamilies of $(\mathrm{Bd}([0, 1]), d_\infty)$ and $(C([0, 1]), d_\infty)$. In order to simplify notation we will let $\mathcal{N}$ denote the class of non-measurable, bounded functions on $[0, 1]$. Recall that the notation $C^\infty(]0, 1])$, $\mathcal{D}_0$, and $\mathcal{H}$ was already presented in the previous section.

Theorem 2.2 *The set $C^\infty(]0, 1])$ is maximal spaceable and latticeable in $(C([0, 1]), d_\infty)$.*

Proof Let $g : \mathbb{R} \to [0, 1]$ be defined by $g(t) = e^{-\frac{1}{(t(1-t))^2}}$ and set $f = \frac{1}{g(\frac{1}{2})} g$. Then f maps $[0, 1]$ into $[0, 1]$ and fulfills $f(0) = f(1) = 0$ as well as $f^{(n)}(0) = f^{(n)}(1) = 0$ for every $n \in \mathbb{N}$ where $f^{(n)}$ denotes the derivative of order $n \in \mathbb{N}$. For every $\mathbf{s} \in l^\infty$ it therefore follows that $\Phi_f(\mathbf{s})$ is infinitely differentiable on $(0, 1]$. According to Lemma 2.1 $\Phi_f(\mathbf{s})$ is continuous at 0 for every $\mathbf{s} \in c_0$, which, again using Lemma 2.1, altogether yields that $\Phi_f(c_0)$ is a closed subspace and sublattice of $(C([0, 1]), d_\infty)$. This completes the proof. □

The next theorem has already been established and proved in [20]—our approach using Φ_f, however, allows for an alternative very short and simple proof.

Theorem 2.3 *The set $\mathcal{D}_0$ is maximal spaceable and latticeable in $(C([0, 1]), d_\infty)$.*

Proof Let $F : [0, 1] \to [0, 1]$ denote a non-decreasing continuous function with $F(0) = 0$, $F(1) = 1$ fulfilling $F'(x) = 0$ for λ-almost every $x \in [0, 1]$. We could, for instance, choose F as the famous Cantor function (a.k.a. evil's staircase, e.g., [14]) or Minkowski's questions mark function (see, e.g., [17, 20] and the references therein) or work with fractal interpolation (see [26]). Obviously each such F corresponds to a probability measure which is singular w.r.t. λ, and is not Lipschitz continuous. Choose such a function F and define the function $f : [0, 1] \to [0, 1]$ by

$$f(x) = F(2x)\mathbf{1}_{[0,\frac{1}{2}]}(x) + F(2 - 2x)\mathbf{1}_{(\frac{1}{2},1]}(x). \tag{2.2}$$

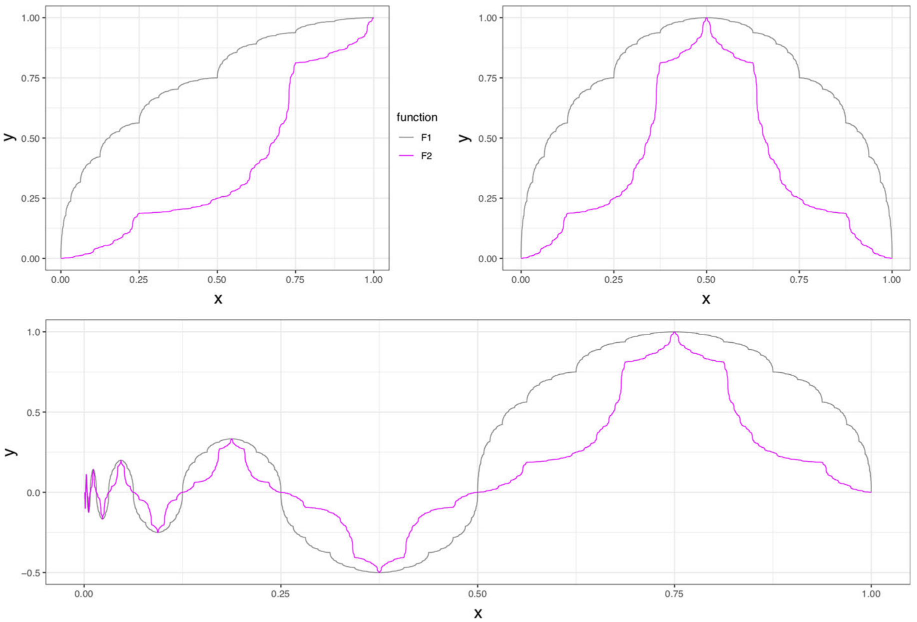

Fig. 1 Two choices for the singular function F (upper left panel), the functions f according to Eq. 2.2 (upper right panel), and the resulting functions $\Phi_f(\mathbf{s})$ for $\mathbf{s} = (s_1, s_2, s_3, \ldots)$ with $s_i = \frac{1}{i}(-1)^{i+1}$ (lower panel). Both functions F were constructed via fractal interpolation as discussed in [26] (see, also, [13])

Then $f \in C([0,1])$, f is differentiable λ-everywhere with derivative 0, and f is not Lipschitz continuous. As direct consequence of the construction of Φ_f the same is true for $\Phi_f(\mathbf{s})$ if we consider $\mathbf{s} \in c_0$. Figure 1 depicts two examples of singular functions F, as well as the corresponding functions f and $\Phi_f(\mathbf{s})$ for some $\mathbf{s} \in c_0$. According to Lemma 2.1 $\Phi_f(c_0)$ is a closed subspace and sublattice of $(C([0,1]), d_\infty)$, so the proof is complete. $\square$

Theorem 2.4 *The set $\mathcal{H}$ is maximal spaceable and latticeable in* $(C([0,1]), d_\infty)$.

Proof Letting $T : [0,1] \to [0,1]$ denote Takagi function (see [25]), defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} d(2^n x, \mathbb{Z}),$$

whereby $d(y, \mathbb{Z}) := \min\{d(y,z) : z \in \mathbb{Z}\}$ and setting $f := \frac{3}{2} T$ yields a function $f : [0,1] \to [0,1]$ with $\max_{x \in [0,1]} f(x) = 1$, which is α-Hölder continuous for every $\alpha \in (0,1)$ but not Lipschitz continuous (again see [25]). Considering Φ_f and proceeding as in the last two proofs yields the desired result. $\square$

Theorem 2.5 *The set $\mathcal{N}$ is maximal spaceable and latticeable in* $(Bd([0,1]), d_\infty)$.

Proof Letting $N \subseteq]0,1[$ denote a non-measurable set, setting $f = \mathbf{1}_N$ as well as Φ_f and proceeding as before directly yields the desired result. $\square$

Considering the following slight modification of Φ_f allows for an alternative simple proof of the fact that the set of all functions $f \in C([0,1])$ whose graph has Hausdorff- and Box-Counting dimension equal to some fixed $s \in]1,2]$ is $\mathfrak{c}$-lineable and latticeable in $C([0,1])$ (see [6]).

Using the same notation as for Φ_f define the operator $\Psi_f : l^\infty \to \mathrm{Bd}([0,1])$ by

$$\Psi_f(\mathbf{s})(x) = \begin{cases} \frac{s_n}{2^n} f(2^n x - 1) \text{if } x \in I_n \text{ for some } n \in \mathbb{N} \\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise} . \end{cases} \tag{2.3}$$

Then Ψ_f is well-defined, obviously linear, injective, and Lipschitz continuous with Lipschitz constant $L = 1$ but no isometry. Before focusing on the aforementioned result we prove the following simple lemma which will be used afterwards.

Lemma 2.6 *Suppose that* $f : [0,1] \to [0,1]$ *fulfills that its graph* $\Gamma(f)$ *has Hausdorff and Box–Counting dimension equal to* $\alpha \in (1,2]$ *and let* Ψ_f *be defined according to* Eq. (2.3). *Then for every* $\mathbf{s} \in l^\infty$ *the same is true for* $\Psi_f(\mathbf{s})$, *i.e.,*

$$dim_H(\Gamma(\Psi_f(\mathbf{s}))) = dim_B(\Gamma(\Psi_f(\mathbf{s}))) = \alpha$$

holds.

Proof Fix $\mathbf{s} \neq \mathbf{0}$ and set $M := \max\{\|\mathbf{s}\|_\infty, 1\}$. Some bi-Lipschitz argument in combination with countable stability of the Hausdorff dimension implies $dim_H(\Gamma(\Psi_f(\mathbf{s}))) = \alpha$. It therefore suffices to show that the upper Box-Counting dimension[1] (see [16]) fulfills $\overline{dim_B}(\Gamma(\Psi_f(\mathbf{s}))) \leq \alpha$ which can be done as follows: According to [16] in the calculation of the box-counting dimension it suffices to work with $\delta_k = \frac{M}{2^k}$ meshes and $k \in \mathbb{N}$. Since the set

$$\Gamma(\Psi_f(\mathbf{s})) \cap \left[0, \frac{1}{2^{k+1}}\right] \times \mathbb{R}$$

can be covered by one square of side length δ_k, the minimum number $N_{\delta_k}\left(\Gamma(\Psi_f(\mathbf{s}))\right)$ of squares of side length δ_k needed to cover $\Gamma(\Psi_f(\mathbf{s}))$ fulfills

$$\begin{aligned} N_{\delta_k}\left(\Gamma(\Psi_f(\mathbf{s}))\right) &\leq 1 + \sum_{i=1}^{k+1} N_{\delta_k}(\Gamma(\Psi_f(\mathbf{s})) \cap I_i \times \mathbb{R}) \\ &\leq 1 + (k+1) N_{\delta_k}\left(\Gamma(f \cdot M)\right), \end{aligned}$$

which yields

$$\begin{aligned} \limsup_{k\to\infty} \frac{\ln\left(N_{\delta_k}\left(\Gamma(\Psi_f(\mathbf{s}))\right)\right)}{-\ln(\delta_k)} &\leq \limsup_{k\to\infty} \frac{\ln\left((k+2) N_{\delta_k}\left(\Gamma(f \cdot M)\right)\right)}{-\ln(\delta_k)} \\ &= \limsup_{k\to\infty} \frac{\ln(k+1)}{-\ln(\delta_k)} + \limsup_{k\to\infty} \frac{\ln\left(N_{\delta_k}\left(\Gamma(f \cdot M)\right)\right)}{-\ln(\delta_k)} \\ &= 0 + \overline{dim_B}(\Gamma(f \cdot M)) = \overline{dim_B}(\Gamma(f)) = \dim_B(\Gamma(f)) = \alpha \end{aligned}$$

□

Lemma 2.6 directly yields the following result already proved in a different manner in [6]. Notice that (contrary to the results on the previous pages) we do not get spaceability since Ψ_f is not an isometry and we can not simply conclude that the subspace $\Psi_f(l^\infty)$ is closed.

[1] Given any non-empty subset M of $\mathbb{R}^n$, we let $N_\delta(M)$ denote the smallest number of sets, of diameter at most δ, needed to cover M. Then

$$\overline{dim_B}(M) := \limsup_{\delta\to 0} \frac{\log N_\delta(M)}{-\log \delta}.$$

Theorem 2.7 *The family of all functions* $f \in C([0,1])$ *whose graph has Hausdorff- and Box-Counting dimension equal to some fixed* $s \in]1,2]$ *is* $\mathfrak{c}$ *lineable and latticeable in* $(C([0,1]), d_\infty)$.

Focusing exclusively on the Hausdorff dimension, working with $\Phi_f : c_0 \to \mathrm{Bd}([0,1])$ for some continuous $f : [0,1] \to [0,1]$ whose graph $\Gamma(f)$ fulfills $dim_H(\Gamma(f)) = s \in [1,2]$, and again using some bi-Lipschitz argument together with countable stability of the Hausdorff dimension even yields spaceability:

Theorem 2.8 *The family of all functions* $f \in C([0,1])$ *whose graph has Hausdorff dimension equal to some fixed* $s \in]1,2]$ *is spaceable and latticeable in* $(C([0,1]), d_\infty)$.

Considering yet another small modification of Φ_f allows for quick alternative proofs for some of the results going back to [18, 19]. In fact, setting

$$\hat{\Psi}_f(\mathbf{s})(x) = \begin{cases} \frac{s_n}{4^n} f(2^n x - 1) & \text{if } x \in I_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} , \end{cases} \tag{2.4}$$

analogously to Ψ_f the new operator $\hat{\Psi}_f$ is well-defined, linear, injective, and Lipschitz continuous with Lipschitz constant $L = 1$ (but not an isometry). Using $\hat{\Psi}_f$ we can show the subsequent result (Theorem 2.3 combined with Corollary 2.1 in [18])—thereby $\mathcal{D}_{dis}$ denotes the set of all functions $h \in C([0,1])$ which are differentiable on $[0,1]$ (at 0 and 1 we consider the one-sided derivates) with a derivative that is discontinuous at every point of a set with positive λ-measure, and $\mathcal{D}_{\neg R}$ the family of all functions $h \in C([0,1])$ which are differentiable on $[0,1]$ with a derivative that is bounded but not Riemann integrable.

Theorem 2.9 $\mathcal{D}_{dis}$ *and* $\mathcal{D}_{\neg R}$ *are* $\mathfrak{c}$*-lineable and latticeable in* $(C([0,1]), d_\infty)$.

Proof (i) The assertion concerning $\mathcal{D}_{dis}$ can be proved as follows: Let f_h denote one of the functions constructed in the proof of Theorem 2.3. in [18] and proceed as follows. Defining $g : [0,1] \to [0,\infty)$ by

$$g(x) = x^2(1-x)^2\,(f_h(x) - \min\{f_h(z) : z \in [0,1]\})$$

and setting $f(x) := \frac{g(x)}{\max\{g(z): z\in[0,1]\}}$ yields a non-negative, continuous function $f : [0,1] \to [0,1]$ with $f(0) = f(1) = 0$, which attains its maximum 1 in $(0,1)$, which is differentiable on $[0,1]$ and fulfills that its derivative f' is discontinuous on a fat Cantor set C. As a direct result, the function $\hat{\Psi}_f(\mathbf{s})$ is obviously differentiable on $(0,1]$. Considering that the very definition of $\hat{\Psi}_f$ implies that $\hat{\Psi}_f(\mathbf{s})$ is also differentiable with (right-hand) derivative 0 at 0 it follows that $\hat{\Psi}_f(\mathbf{s})$ is differentiable on $[0,1]$. Moreover, if $s_n \neq 0$ for some $n \in \mathbb{N}$, then $(\hat{\Psi}_f(\mathbf{s}))'$ is discontinuous on some fat Cantor set contained in I_n, implying $\hat{\Psi}_f(\mathbf{s}) \in \mathcal{D}_{dis}$.

(ii) The assertion concerning $\mathcal{D}_{\neg R}$ follows in the same fashion. □

We now turn to the family $\mathcal{M}_s \subseteq C([0,1])$ of all functions f fulfilling that the sets $\underline{U}_h, \overline{U}_h$, defined by

$$\underline{U}_h = \{x \in [0,1] : h(x) = \min_{z\in[0,1]} h(z)\}, \quad \overline{U}_h = \{x \in [0,1] : h(x) = \max_{z\in[0,1]} h(z)\}$$

both have Hausdorff dimension $s \in]0,1[$ and, again using Φ_f for some appropriately chosen f, show that $\mathcal{M}_s$ is spaceable in $C([0,1])$. Notice that in this context we do not obtain latticeability since the range of the constructed function f is $[0,1]$.

Theorem 2.10 $\mathcal{M}_s$ *is spaceable in* $(C([0,1]), d_\infty)$ *for every* $s \in [0,1]$.

Proof Since the assertion is trivial for $s \in \{0, 1\}$ it suffices to consider $s \in (0, 1)$. Fix $\beta \in]0, \frac{1}{2}[$ and define the contractions $w_1, w_2 : [0, 1] \to [0, 1]$ by $w_1(x) = \beta x$ and $w_2(x) = \beta x + 1 - \beta$. Considering the Iterated Function System (IFS, for short) $\{w_1, w_2\}$ and using the standard properties of IFSs (see [16] and [21]) it follows that there exists a unique non-empty compact subset C^*_β of $[0, 1]$ fulfilling

$$C^*_\beta = w_1(C^*_\beta) \cup w_2(C^*_\beta).$$

To simplify notation we will write $\mathcal{W}(K) = w_1(K) \cup w_2(K)$ for every non-empty compact subset K of $[0, 1]$ and refer to $\mathcal{W}$ as Hutchinson operator induced by the IFS. Again following [16] and [21] and considering that w_1, w_2 are similarities the set C^*_β is self-similar and its Hausdorff dimension $\dim_H(C^*_\beta)$ is the unique solution s of the equation $2\beta^s = 1$, i.e., $\dim_H(C^*_\beta) = \frac{-\log(2)}{-\log(\beta)} \in (0, 1)$.

As next step we construct a continuous function $g : [0, 1] \to [0, 1]$ fulfilling $\overline{U}_g = C^*_\beta$ in several steps. Set $g_1(x) = 1$ for every $x \in [0, 1]$ and define the function $g_2 : [0, 1] \to [0, 1]$ by

$$g_2(x) = \begin{cases} g_1(x) & \text{if } x \in \mathcal{W}([0, 1]), \\ \frac{1}{\frac{1}{2}-\beta}\,(x - \frac{1}{2}) & \text{if } x \in [\beta, \frac{1}{2}], \text{ and} \\ \frac{1}{\frac{1}{2}-\beta}\,(\frac{1}{2} - x) & \text{if } x \in [\frac{1}{2}, 1 - \beta]. \end{cases} \tag{2.5}$$

The basic idea for the construction of g_2 is to replace g_1 on the (open) interval $[0, 1]\backslash\mathcal{W}([0, 1])$ by a reflected tent map. Modifying g_2 on each of the intervals constituting $[0, 1]\backslash\mathcal{W}^2([0, 1])$ in the same manner yields the function g_3. Proceeding analogously yields a sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions that converges uniformly to a continuous function g, which obviously fulfills $\overline{U}_g = C^*_\beta$. The first and the second panel in the first row of Fig. 2 depict g_2 and g_7. Shrinking $[0, 1]$ to $[\beta, 1 - \beta]$ and extending linearly on both sides to 0 yields the continuous function g^* fulfilling $g^*(0) = 0 = g^*(1)$ (third upper panel in Fig. 2). Finally defining $f : [0, 1] \to [-1, 1]$ by (see fourth panel in the first row of Fig. 2)

$$f(x) = \begin{cases} g^*(x) & \text{if } x \in [0, \frac{1}{2}], \\ -g^*(x) & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{2.6}$$

it follows that f is continuous, fulfills $f(0) = 0 = f(1)$, and attains its maximum 1 and its minimum -1 both in sets of Hausdorff dimension $\frac{-\log(2)}{-\log(\beta)}$. Considering the induced operator $\Phi_f : c_0 \to \mathrm{Bd}([0, 1])$ it follows that $\Phi_f(\mathbf{s}) \in C([0, 1])$ for every $\mathbf{s} \in c_0$ and the assertion of the theorem follows for $s = \frac{-\log(2)}{-\log(\beta)}$. Since $\{\frac{-\log(2)}{-\log(\beta)} : \beta \in]0, \frac{1}{2}[\} =]0, 1[$ this completes the proof. The lower panel in Fig. 2 depicts $\Phi_f(\mathbf{s})$ for $\beta = \frac{3}{8}$ and $\mathbf{s} = (s_1, s_2, s_3, \ldots)$ with $s_i = \frac{1}{i}(-1)^{i+1}$ □

Remark 2.11 An alternative simple way for constructing the (Lipschitz) continuous function g^* used in the proof of Theorem 2.10 would be as follows: Letting $\mathcal{F}$ denote the family of functions in $C([0, 1])$ fulfilling $f(0) = f(1) = 1$ it follows that $\mathcal{F}$ is closed in $(C([0, 1]), d_\infty)$. Defining the operator $T_\beta : \mathcal{F} \to \mathcal{F}$ by

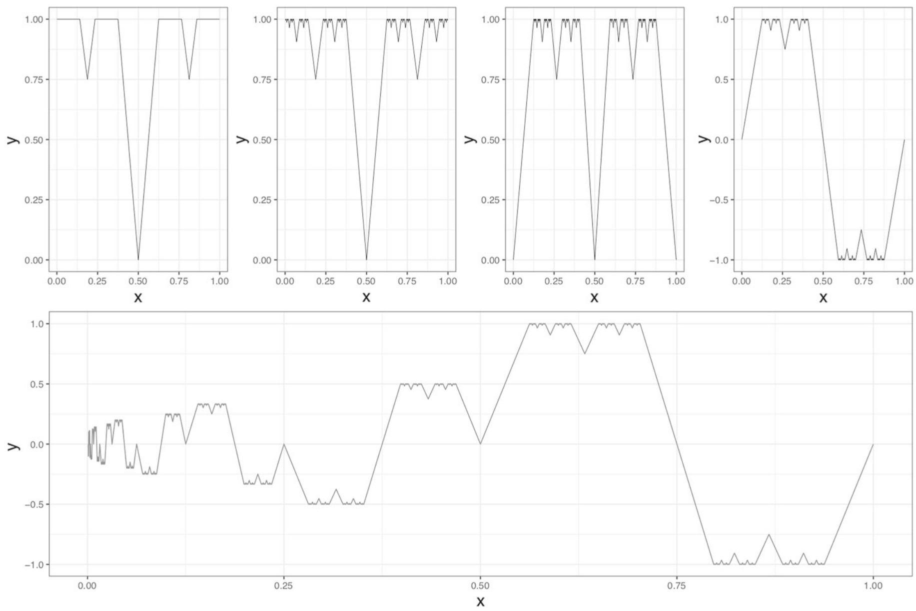

Fig. 2 The construction used in the proof of Theorem 2.10 for the case $\beta = \frac{3}{8}$. The first row depicts the construction of the function f, the lower panel the function $\Phi_f(\mathbf{s})$ for $\mathbf{s} = (s_1, s_2, s_3, \ldots)$ with $s_i = \frac{1}{i}(-1)^{i+1}$

$$T_\beta(h)(x) = \begin{cases} \beta h(\dfrac{x}{\beta}) & \text{if } x \in [0, \beta], \\ \dfrac{1}{\frac{1}{2} - \beta}(x - \dfrac{1}{2}) & \text{if } x \in [\beta, \frac{1}{2}], \\ \dfrac{1}{\frac{1}{2} - \beta}(\dfrac{1}{2} - x) & \text{if } x \in [\frac{1}{2}, 1 - \beta], \text{ and} \\ \beta\left(\dfrac{x - (1 - \beta)}{\beta}\right) + 1 - \beta & \text{if } x \in [1 - \beta, 1]. \end{cases} \tag{2.7}$$

it is straightforward to verify that T_β is well-defined and a contraction on $(\mathcal{F}, d_\infty)$, so Banach's Fixed Point Theorem implies the existence of a unique, globally attractive fixed point, which is easily verified to coincide with g^*.

We conclude this section by carrying over the main results from $C([0, 1])$ established so far to

$$L_p([0, 1], \mathcal{B}([0, 1]), \lambda) =: L^p([0, 1])$$

and start with the L^p-version Φ_f^p of the operator Φ_f. For $p \in [1, \infty)$, $f \in L^p([0, 1])$ and $\mathbf{s} \in l^p$ define $\Phi_f^p : l^p \to L^p([0, 1])$ by

$$\Phi_f^p(\mathbf{s})(x) = \begin{cases} s_n 2^{\frac{n}{p}} f(2^n x - 1) \text{ if } x \in I_n \text{ for some } n \in \mathbb{N}, \\ \qquad 0 \text{ otherwise}, \end{cases} \tag{2.8}$$

where the right-hand side is to be interpreted as equivalence class in $L^p([0, 1])$. In the sequel we will write $\|\cdot\|_p$ both for the norm on l^p and the norm on $L^p([0, 1])$ since no confusion will arise. Considering that for every $f \in L^p([0,1])$ with $\|f\|_p = 1$ we have

$$\|\Phi_f^p(\mathbf{s})\|_p^p = \int_{[0,1]} |\Phi_f^p(\mathbf{s})(x)|^p \, d\lambda(x) = \sum_{n=1}^{\infty} |s_n|^p 2^n \int_{I_n} |f(2^n x - 1)|^p \, d\lambda(x)$$
$$= \sum_{n=1}^{\infty} |s_n|^p = \|\mathbf{s}\|_p^p$$

it follows that for each such f the operator $\Phi_f^p : l^p \to L^p([0,1])$ is a linear isometry and a lattice isomorphism. As a direct consequence, $\Phi_f^p(l^p)$ is a closed subspace of $L^p([0,1])$. Notice, however, that although obviously $l^p \subseteq c_0$ holds, in general $\Phi_f^p(\mathbf{s})$ is not necessarily continuous at 0 (and the same holds true for each representative of $\Phi_f^p(\mathbf{s})$). Working with the operator Φ_f^p and preceding as before yields the following complementing results on lineability/spaceability of subsets of $L^p([0,1])$:

Theorem 2.12 *The following assertions hold for every fixed* $p \in [1, \infty)$*:*

1. *The set of equivalence classes in* $L^p([0,1])$ *which contain some representative contained in* $C^\infty(]0,1])$ *is spaceable and latticeable in* $L^p([0,1])$.
2. *The set of equivalence classes in* $L^p([0,1])$ *which contain some representative that is differentiable* λ*-almost everywhere with derivative* 0 *but not Lipschitz continuous is spaceable and latticeable in* $L^p([0,1])$.
3. *The set of equivalence classes in* $L^p([0,1])$ *which contain some representative that is* α*-Hölder continuous for every* $\alpha \in]0,1[$ *but not Lipschitz continuous is spaceable and latticeable in* $L^p([0,1])$.
4. *The set of equivalence classes in* $L^p([0,1])$ *which contain some representative whose graph has Hausdorff- and Box–Counting dimension equal to some* $s \in]1,2]$ *is* $\mathfrak{c}$*-lineable and latticeable in* $L^p([0,1])$.
5. *The set of equivalence classes in* $L^p([0,1])$ *which contain some representative whose graph has Hausdorff dimension equal to some* $s \in]1,2]$ *is spaceable and latticeable in* $L^p([0,1])$.

3 Lineability, spaceability and latticeability in Sobolev spaces over bounded intervals

In this section we consider the problem of finding lineable/latticeable subsets of Sobolev Spaces over one-dimensional bounded intervals. Thus, fixed an integer $m \geq 0$ and a real number $1 \leq p < \infty$ we consider, as vector space X, the Sobolev Space $W^{m,p}(]0,1[)$. Before recalling basic facts about Sobolev spaces over intervals we start with some preparations which will be used subsequently.

Let $(t_n)_{n\in\mathbb{N}}$ denote the Thue–Morse sequence (also known as Prouhet–Thue–Morse sequence) defined to be zero if the sum of the digits in the binary expansion of n is even and $t_n = 1$ otherwise. Let us recall that t_n is a binary digits sequence for which the sequence $d_n = t_0 t_1 \cdots t_{2^n-1}$ satisfies that d_{n+1} is the concatenation of d_n and is Boolean complementary, i.e. $d_0 = 0, d_1 = 01, d_2 = 0110, d_3 = 01101001, d_4 = 0110100110010110, \ldots$

Since binary sequences are identified with $\mathbb{Z}_2$ numbers, the number $t = \sum_{n=0}^{\infty} t_n 2^n$ associated to the sequence t_n is the unique fixed point of the contraction

$$x = \sum_{n=0}^{\infty} x_n 2^n \to \sum_{n=0}^{\infty} (x_n + (1 - x_n)2)\, 2^{2n}.$$

For each fixed integer $m \geq 1$, we will use the following functions in $\mathbb{R}^{[0,1]}$:

$$s_{m,0}(x) := \begin{cases} 1 - 2t_n, & \text{if } x \in]\frac{n}{2^m}, \frac{n+1}{2^m}] \text{ with } n \in \{1, \dots, 2^m - 1\}, \\ 1, & \text{if } x \in \left[0, \frac{1}{2^m}\right]. \end{cases}$$

In addition, for each $j \in \{1, \dots, m\}$, we recursively define the function $s_{m,j} \in \mathbb{R}^{[0,1]}$ by

$$s_{m,j}(x) := \int_{[0,x]} s_{m,j-1}(t)dt.$$

In the following lemma we collect some elementary properties of the functions $s_{m,j}$.

Lemma 3.1 *For every integer $m \geq 1$ the function $s_{m,m}$ is non-negative and*

(a) $s_{m,j}$ *is bounded for every* $j \in \{0, 1, \dots, m\}$,
(b) $s_{m,j}(0) = s_{m,j}(1) = 0$ *for every* $j \in \{1, \dots, m\}$.

We now recall some basic definitions and main properties of Sobolev spaces in one dimension, see [7] for an extended study using weak derivatives (for a distributional point of view see also [23]). We recall that, given $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ the Sobolev space $W^{m,p}\,(]a, b[)$ can be defined as follows

$$W^{m,p}\,(]a, b[) = \left\{ \begin{array}{l} u \in L^p(]a, b[) : \text{for each } j \in \{1, \dots, m\} \text{ there exists } g_j \in L^p(]a, b[) \text{ with} \\ \displaystyle\int_{]a,b[} u(x)\varphi^j(x)dx = (-1)^j \int_{]a,b[} g_j(x)\varphi(x)dx,\ \forall \varphi \in C_c^\infty(]a, b[) \end{array} \right\},$$

where $C_c^\infty(]a, b[)$ denotes the space of compactly supported, infinitely differentiable functions on $]a, b[$.

Also, the function g_j involved in the previous definition is the well known weak derivative of j-order of the function u and is denoted as usual by $g_j \equiv u^{j)}$.

The standard norm in the Sobolev space $W^{m,p}\,(]a, b[)$ is given by

$$\|u\|_{m,p} = \|u\|_p + \sum_{j=1}^{m} \|u^{j)}\|_p,$$

where $\|\cdot\|_p$ denotes the usual L^p-norm. The space $W^{m,p}\,(]a, b[)$ endowed with the norm $\|\cdot\|_{m,p}$ satisfies the following basic properties:

Lemma 3.2 *Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$ and $]a, b[\subset \mathbb{R}$. Then:*

1. $W^{m,p}\,(]a, b[)$ *is a Banach space.*
2. $W^{m,p}\,(]a, b[)$ *is reflexive for* $1 < p < \infty$.
3. $W^{m,p}\,(]a, b[)$ *is separable for* $1 \leq p < \infty$.
4. *For every* $1 \leq p < \infty$, $C_c^\infty(\mathbb{R})$ *is dense* $W^{m,p}\,(]a, b[)$ *with respect to the norm* $\|\cdot\|_{m,p}$.

In the particular case of $]a, b[$ being bounded we have, in addition, that:

5. *For every* $1 \leq p < \infty$*, the polynomials are dense in* $W^{m,p}\,(]a,b[)$.[2]
6. $W^{m,p}\,(]a,b[)$ *is continuously embedded in* $C^{m-1}([a,b])$*. Moreover the embedding is compact for* $1 < p \leq \infty$.

The clousure of $C_c^\infty(]a,b[)$ in the space $W^{m,p}\,(]a,b[)$, for $1 \leq p < \infty$ is denoted by $W_0^{m,p}\,(]a,b[)$ and satisfies the following properties:

Lemma 3.3 *Let* $m \in \mathbb{N}$, $1 \leq p < \infty$ *and* $]a,b[\subset \mathbb{R}$. *Then:*

1. $W_0^{m,p}\,(]a,b[)$ *is a separable Banach space.*
2. $W_0^{m,p}\,(]a,b[)$ *is reflexive for* $1 < p$.
3. $u \in W^{m,p}\,(]a,b[) \cap C^{m-1}\left(\overline{]a,b[}\right)$ *belongs to* $W_0^{m,p}\,(]a,b[)$ *if and only if* $u = u' = \cdots = u^{m-1)} = 0$ *on the boundary of* $]a,b[$.
4. *When* $]a,b[$ *is bounded,* $\|u^{m)}\|_p$ *for* $u \in W_0^{m,p}\,(]a,b[)$ *is a norm equivalent to* $\|u\|_{m,p}$.

The main result in this section is the following.

Theorem 3.4 *For every* $1 \leq p < +\infty$ *the set* $W^{m,p}\,(]0,1[) \setminus \bigcup_{q>p} W^{m,q}\,(]0,1[)$ *is spaceable latticeable.*

Proof Fixed m and p, we consider the function $f_n \in \mathbb{R}^{[0,1]}$ defined by

$$f_n(x) := \begin{cases} s_{m,m}\,(2^n x - 1) \cdot 2^{-\left(m-\frac{1}{p}\right)n}, & \text{if } x \in]\frac{1}{2^n}, \frac{1}{2^{n-1}}[, \\ 0, & \text{otherwise.} \end{cases}$$

We observe that f_n has derivatives up to order m and $f_n^{j)}$ is continuous for every $j \in \{0,1,\ldots,m-1\}$ and $f_n^{j)}\left(\frac{1}{2^n}\right) = f_n^{j)}\left(\frac{1}{2^{n-1}}\right) = 0$. Therefore, for every $j = 1,2,\ldots,m$

$$\int_{\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]} f_n^{j)}(x)\varphi(x)dx = (-1)^j \int_{\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]} f_n(x)\varphi^{j)}(x)dx,$$

holds for all sufficiently regular φ, implying

$$\int_{]0,1[} f_n^{j)}(x)\varphi(x)dx = (-1)^j \int_{]0,1[} f_n(x)\varphi^{j)}(x)dx, \quad \text{for every } \varphi \in C_c^\infty(]0,1[).$$

Now we consider, for every $d \in \mathbb{N}$, an infinite subset $A_d \subset \mathbb{N}$ such that $\mathbb{N} = \bigcup_{d\in\mathbb{N}} A_d$ and any two of these subsets are disjoint. Let us denote by $d(n)$ the element in the position n of A_d with the usual order. Moreover, we define the function $\gamma_d \in \mathbb{R}^{[0,1]}$ by

$$\gamma_d(x) := \begin{cases} f_{d(n)}(x)a_n, & \text{if } x \in]\frac{1}{2^{d(n)}}, \frac{1}{2^{d(n)-1}}[, \\ 0, & \text{otherwise,} \end{cases}$$

where $a_n := \left(n \ln^2(n+1)\right)^{-1/p}$.

We claim that $\gamma_d \in W^{m,p}\,(]0,1[) \setminus \bigcup_{q>p} W^{m,q}\,(]0,1[)$. In order to prove the claim we first show that γ_d admits weak derivatives up to order m. Indeed, given a function φ regular enough with $\operatorname{supp}\varphi \cap]\frac{1}{2^{d(n)}}, \frac{1}{2^{d(n)-1}}[\neq \emptyset$, we have, using the properties of $f_{d(n)}$, that

$$\int_{\left[\frac{1}{2^{d(n)}}, \frac{1}{2^{d(n)-1}}\right]} \gamma_d^{j)}(x)\varphi(x)dx = (-1)^j \int_{\left[\frac{1}{2^{d(n)}}, \frac{1}{2^{d(n)-1}}\right]} \gamma_d(x)\varphi^{j)}(x)dx.$$

[2] Item (5) in Lemma 3.2 follows straightforwardly by using item (4) and Bernstein's proof of the Weierstrass theorem, where Bernstein's polynomials approximate $C^k([a,b])$ functions in $C^k([a,b])$.

In particular, given $\varphi \in C_c^\infty(]0,1[)$ we deduce that

$$\int_{]0,1[} \gamma_d^{j)}(x)\varphi(x)dx = (-1)^j \int_{]0,1[} \gamma_d(x)\varphi^{j)}(x)dx.$$

Since γ_d and its derivatives up to order $m-1$ are bounded functions we have that $\gamma_d^{j)} \in L^p(]0,1[)$ for every $j \in \{0,1,\ldots,m-1\}$. With respect to $\gamma_d^{m)}$ we know that

$$\gamma_d^{m)}(x) = s_{m,0}(2^{d(n)}x-1)2^{d(n)/p}a_n, \quad \text{for every } x \in]\frac{1}{2^{d(n)}}, \frac{1}{2^{d(n)-1}}[.$$

In particular $\gamma_d \in W^{m,p}(]0,1[)$ since

$$\int_{]0,1[} \left|\gamma_d^{m)}(x)\right|^p = \sum_{n=1}^{\infty} 2^{d(n)} a_n^p \frac{1}{2^{d(n)}} = \sum_{n=1}^{\infty} \left(n\ln^2(n+1)\right)^{-1} < +\infty.$$

However, $\gamma_d^{m)}$ does not belong to $L^q(]0,1[)$ when $q > p$ since

$$\int_{]0,1[} \left|\gamma_d^{m)}(x)\right|^q = \sum_{n=1} \frac{1}{2^{d(n)}} \left(\frac{2^{d(n)}}{n\ln^2(n+1)}\right)^{q/p} = +\infty.$$

This proves the claim, i.e. $\gamma_d \in W^{m,p}(]0,1[) \setminus \bigcup_{q>p} W^{m,q}(]0,1[)$.

Let us write $h_d := \gamma_d / \|\gamma_d\|_{m,p}$ and consider the set $H \subset W^{m,p}(]0,1[)$ given by

$$H = \left\{h \in W^{m,p}(]0,1[) : h = \sum_{d=1}^{\infty} c_d h_d\right\}.$$

By construction, the coefficients c_d associated to any function $h \in H$ are uniquely determined and H is a nontrivial vector space. We prove now that $H\setminus\{0\} \subset W^{m,p}(]0,1[) \setminus \bigcup_{q>p} W^{m,q}(]0,1[)$, H is closed in $W^{m,p}(]0,1[)$ and that it is a lattice.

In order to prove that H is closed we assume that $g_r \in H$ and that $\lim_{r\to\infty} \|g_r - g\|_{m,p} = 0$ for some $g \in W^{m,p}(]0,1[)$. We may assume, if necessary, that g_r and g are continuous according to the Sobolev embedding (see (6), Lemma 3.2). In particular, since $\|g_r - g\|_p \leq \|g_r - g\|_{m,p}$, we have that

$$\lim_{r\to\infty} \int_{]2^{-d(n)},2^{-d(n)+1}[} |g_r(x) - g(x)|^p\, dx = 0, \quad \text{for every } d, n \in \mathbb{N}.$$

For each $d \in \mathbb{N}$ there exists a unique coefficient $c_{d_r} \in \mathbb{R}$ such that, given $n \in \mathbb{N}$ we have that

$$g_r(x) = c_{d_r} \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n, \quad \text{for every } x \in]2^{-d(n)}, 2^{-d(n)+1}[.$$

Therefore it follows that

$$\lim_{r\to\infty} \int_{]2^{-d(n)},2^{-d(n)+1}[} \left| c_{d_r} \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n - g(x)\right|^p dx = 0.$$

Now we claim that, fixed d, the sequence $\{c_{d_r}\}$ is bounded. Otherwise we may assume - up to a subsequence - that c_{d_r} is increasing and unbounded (observe that we may replace g by $-g$). Using the definition of $f_{d(n)}$ we may choose $\omega > 0$ such that $S_\omega = f_{d(n)}^{-1}([\omega, +\infty[)$ is a subset of $]2^{-d(n)}, 2^{-d(n)+1}[$ with positive measure. Moreover, g is bounded and $f_{d(n)}(x)a_n \geq 0$ in the interval $]2^{-d(n)}, 2^{-d(n)+1}[$ and, as a particular case, in S_ω. Thus, given $M > 0$ there exists $r_M > 0$ such that $\left| c_{d_r} \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n - g(x) \right|^p > M$ for every $x \in S_\omega$ and every $r > r_M$. This implies, using that the measure of S_ω is positive, that

$$\begin{aligned} 0 &= \lim_{r\to\infty} \int_{]2^{-d(n)},2^{-d(n)+1}[} \left| c_{d_r} \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n - g(x) \right|^p dx \\ &\geq \lim_{r\to\infty} \int_{S_\omega} \left| c_{d_r} \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n - g(x) \right|^p dx = +\infty. \end{aligned}$$

This is a contradiction and this completes the proof that $\{c_{d_r}\}$ is bounded. In addition, we may assume that - up to a subsequence - $\{c_{d_r}\}$ converges to some $c_d \in \mathbb{R}$. In particular

$$\lim_{r\to\infty} \int_{]2^{-d(n)},2^{-d(n)+1}[} \left| c_{d_r} \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n - c_d \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n \right|^p dx = 0.$$

This implies that the restriction of the function g to the interval $]2^{-d(n)}, 2^{-d(n)+1}[$ is equal to $c_d \frac{f_{d(n)}(x)}{\|\gamma_d\|_{m,p}} a_n$ for every $n \in \mathbb{N}$ (observe that c_{d_r} and c_d do not depend on n). Therefore g restricted to the set

$$\bigcup_{n\in\mathbb{N}}]2^{-d(n)}, 2^{-d(n)+1}[$$

is equal to $c_d h_d$ or equivalently $g = \sum_{d=1}^{\infty} c_d h_d$ and we have proved that H is closed.

Observe also that, by construction, given $h \in H \backslash \{0\}$ then $h = \sum_{d=1}^{\infty} c_d h_d$ with some $c_d \neq 0$, thus, if $h \in W^{m,q}(]0,1[)$ for some $q > p$ then the corresponding function h_d belongs to $W^{m,q}(]0,1[)$ which is a contradiction, i.e. $H \backslash \{0\} \subset W^{m,p}(]0,1[) \setminus \bigcup_{q>p} W^{m,q}(]0,1[)$.

Moreover H is infinite dimensional, actually we may define the injective map $T : l_\infty \to H$ given by $T(c) = \sum_{d=1}^{\infty} c_d h_d$ for every $c = \{c_d\} \in l^\infty$. We observe that

$$\begin{aligned} \|T(c)\|_{m,p} &= \sum_{j=0}^{m} \left(\int_{]0,1[} \left| \sum_{r=1}^{\infty} c_r h_r^{j)}(x) \right|^p \right)^{1/p} dx = \sum_{j=0}^{m} \left(\int_{]0,1[} \sum_{r=1}^{\infty} \left| c_r h_r^{j)}(x) \right|^p dx \right)^{1/p} \\ &\leq \|c\|_\infty \sum_{j=0}^{m} \left(\int_{]0,1[} \sum_{r=1}^{\infty} \left| h_r^{j)}(x) \right|^p dx \right)^{1/p} = \|c\|_\infty, \end{aligned}$$

which implies that T is well defined and even more, that l^∞ is continuously embedded in H.

In particular, H contain $\mathfrak{c}$ independent vectors.

We finally show that H is a lattice. Indeed, given $z, g \in H$ with $z(x) = \sum_{d=1}^{\infty} c_d h_d(x)$ and $g(x) = \sum_{d=1}^{\infty} g_d h_d(x)$ then $z \vee g$ is given by $(z \vee g)(x) = \sum_{d=1}^{\infty} (c_d \vee g_d) h_d(x)$. Observe that, for φ regular enough we have

$$\int_{\left[\frac{1}{2^{d(n)}}, \frac{1}{2^{d(n)-1}}\right]} (c_d \vee g_d) h_d^{j)}(x)\varphi(x)dx = (-1)^j \int_{\left[\frac{1}{2^{d(n)}}, \frac{1}{2^{d(n)-1}}\right]} (c_d \vee g_d) h_d^{j)}(x)\varphi^{j)}(x)dx.$$

In particular,

$$\int_{]0,1[} (z \vee g)^{j)}(x)\varphi(x)dx = (-1)^j \int_{]0,1[} (z \vee g)(x)\varphi^{j)}(x)dx, \quad \text{for every } \varphi \in C_c^\infty(]0,1[).$$

Moreover, since

$$\left|(z \vee g)^{j)}\right|^p (x) = \sum_{d=1}^{\infty} |c_d \vee g_d|^p \left|h_d^{j)}(x)\right|^p \leq \sum_{d=1}^{\infty} \left(|c_d|^p + |g_d|^p\right) \left|h_d^{j)}(x)\right|^p,$$

we have that $z \vee g \in W^{m,p}(]0,1[)$ and $\|z \vee g\|_{m,p}^p \leq \|z\|_{m,p}^p + \|g\|_{m,p}^p$. Therefore $z \vee g \in H$.

Finally the maximality is deduced from the fact that $W^{m,p}(]0,1[)$ is continuously embedded in $C(]0,1[)$. Thus the space $W^{m,p}(]0,1[)$ has cardinality $\mathfrak{c}$. □

Remark 3.5 Observe that, since the functions h_d in the proof above vanish at the boundary of $]0,1[$, from Lemma 3.3 it is deduced that in fact we have proved that, for every $1 \leq p < +\infty$ the set

$$W_0^{m,p}(]0,1[) \setminus \bigcup_{q>p} W_0^{m,q}(]0,1[)$$

is maximal spaceable latticeable.

We conclude this section proving maximal dense-lineability by using the sufficient condition in [2] (see Theorem 3.7 bellow). Let us also recall the following definition from that paper.

Definition 3.6 Let A, B be subsets of a vector space X. We say that A is stronger than B if $A + B \subseteq A$.

Theorem 3.7 [2] *Let X be a separable Banach space, and consider two subsets A, B of X such that A is lineable and B dense-lineable. If A is stronger than B, then A is dense-lineable.*

Now we can prove the maximal dense-lineability of $W^{m,p}(]0,1[) \setminus \bigcup_{q>p} W^{m,q}(]0,1[)$.

Corollary 3.8 *For every $1 \leq p < +\infty$ the set $W^{m,p}(]0,1[) \setminus \bigcup_{q>p} W^{m,q}(]0,1[)$ is maximal dense-lineable.*

Proof The proof follows immediately from Theorem 3.7 with $X = W^{m,p}(]0,1[)$, the set A is given by $W^{m,p}(]0,1[) \setminus \bigcup_{q>p} W^{m,q}(]0,1[)$ which, according to Theorem 3.4, is in particular lineable, and finally the set B is the vector space of polynomials in $]0,1[$ which, by Lemma 3.2, is dense in $W^{m,p}(]0,1[)$. Thus we only have to prove that A is stronger than B. i.e., $A + B \subseteq A$. In order to prove that we observe that $A + B \subseteq W^{m,p}(]0,1[)$ and given $f \in A$ and $p_0 \in B$ we have that $(f + p_0) \notin \bigcup_{q>p} W^{m,q}(]0,1[)$. Otherwise, for some $q_0 > p$, $(f + p_0) \in W^{m,q_0}(]0,1[)$ and, since $p_0 \in W^{m,q_0}(]0,1[)$, we have that $f = (f + p_0) - p_0 \in W^{m,q_0}(]0,1[)$ which contradicts that $f \in A$. □

4 Lineability, spaceability and latticeability in Sobolev spaces over unbounded intervals

In this section we focus on the problem of finding latticeable subsets of Sobolev Spaces over one dimensional unbounded intervals, namely, the Sobolev Space $W^{m,p}(]1,+\infty[)$.

The analogues to the functions $s_{m,j}$ are given by $k_{m,j} \in \mathbb{R}^{[0,2^m]}$ with

$$k_{m,0}(x) := \begin{cases} 1 - 2t_n, & \text{if } x \in]n, n+1] \text{ with } n \in \{1, \ldots, 2^m - 1\}, \\ 1, & \text{if } x \in [0,1], \end{cases}$$

and, for $j \in \{1, \dots, m\}$,

$$k_{m,j}(x) := \int_{[0,x]} k_{m,j-1}(t)dt.$$

The properties of functions $k_{m,j}$ are analogous to those of $s_{m,j}$ and are collected in the following lemma.

Lemma 4.1 *For every integer $m \geq 1$ the function $k_{m,m}$ is non-negative and*

(a) *$k_{m,j}$ is bounded for every $j \in \{0, 1, \dots, m\}$,*
(b) *$k_{m,j}(0) = k_{m,j}(2^m) = 0$ for every $j \in \{1, \dots, m\}$.*

The main result about latticeability on Sobolev Spaces over unbounded intervals is the following.

Theorem 4.2 *For every $1 < p < +\infty$ the set $W^{m,p}\,(]1, +\infty[) \setminus \bigcup_{q<p} W^{m,q}\,(]1, +\infty[)$ is spaceable latticeable.*

Proof Let us consider, for every $e \in \mathbb{N}$, an infinite subset $B_e \subset \mathbb{N}$ such that $\mathbb{N} = \bigcup_{e\in\mathbb{N}} B_e$ and any two of these subsets are disjoint. Let us denote by $e(n)$ the element in the position n of B_e with the usual order. The proof follows as in Theorem 3.4 replacing functions γ_d by

$$\rho_e(x) := \begin{cases} k_{m,m}(2^{m+1-e(n)}x - 2^m)2^{-\frac{e(n)}{p}}a_n, & \text{if } x \in]2^{e(n)-1}, 2^{e(n)}[, \\ 0, & \text{otherwise,} \end{cases}$$

where, as before, $a_n := \left(n \ln^2(n+1)\right)^{-1/p}$.
Write $\eta_e := \rho_e / \|\rho_e\|_{m,p}$ and consider now the set $B \subset W^{m,p}\,(]1, +\infty[)$ given by

$$B = \left\{ \rho \in W^{m,p}\,(]1, +\infty[) : \rho = \sum_{e=1}^{\infty} c_e \eta_e \right\}.$$

Arguing as in the previous section the coefficients c_e associated to any function $\rho \in B$ are uniquely determined and B is a vector space with $B\setminus\{0\} \subset W^{m,p}\,(]1, +\infty[) \setminus \bigcup_{q<p} W^{m,q}\,(]1, +\infty[)$. Let us show that B is closed in $W^{m,p}\,(]1, +\infty[)$. Indeed, let us assume that $h_r \in B$ and that $\lim_{r\to\infty} \|h_r - h\|_{m,p} = 0$ for some $h \in W^{m,p}\,(]1, +\infty[)$.

We may assume, if necessary, that h_r and h are continuous according to the Sobolev embedding. In particular, since $\|h_r - h\|_p \leq \|h_r - h\|_{m,p}$, we have that

$$\lim_{r\to\infty} \int_{]2^{e(n)-1}, 2^{e(n)}[} |h_r(x) - h(x)|^p \, dx = 0, \quad \text{for every } e, n \in \mathbb{N}.$$

For each $e \in \mathbb{N}$ there exists a unique coefficient $c_{e_r} \in \mathbb{R}$ such that, given $n \in \mathbb{N}$ we have that

$$h_r(x) = c_{e_r} \frac{k_{m,m}(2^{m+1-e(n)}x - 2^m)2^{-\frac{e(n)}{p}}}{\|\rho_e\|_{m,p}} a_n, \quad \text{for every } x \in]2^{e(n)-1}, 2^{e(n)}[.$$

Therefore we have that, for every $e, n \in \mathbb{N}$,

$$0 = \lim_{r\to\infty} \int_{]2^{e(n)-1}, 2^{e(n)}[} \left| c_{e_r} \frac{k_{m,m}(2^{m+1-e(n)}x - 2^m)2^{-\frac{e(n)}{p}}}{\|\rho_e\|_{m,p}} a_n - h(x) \right|^p dx$$

$$= \lim_{r\to\infty}\int_{]0,2^m[}\left|c_{e_r}\frac{k_{m,m}(t)2^{-\frac{e(n)}{p}}}{\|\rho_e\|_{m,p}}a_n - h((2^m+t)2^{e(n)-m-1})\right|^p 2^{e(n)-m-1}dt$$

$$= \frac{1}{2^{m+1}}\lim_{r\to\infty}\int_{]0,2^m[}\left|c_{e_r}\frac{k_{m,m}(t)}{\|\rho_e\|_{m,p}}a_n - h(2^{e(n)-1}+t2^{e(n)-m-1})2^{\frac{e(n)}{p}}\right|^p dt.$$

Now we claim that, fixed $e \in \mathbb{N}$, the sequence $\{c_{e_r}\}$ is bounded. We will use that for every fixed $n \in \mathbb{N}$ we have

$$\lim_{r\to\infty}\int_{]0,2^m[}\left|c_{e_r}\frac{k_{m,m}(t)}{\|\rho_e\|_{m,p}}a_n - h(2^{e(n)-1}+t2^{e(n)-m-1})2^{\frac{e(n)}{p}}\right|^p dt = 0.$$

Assume on the contrary that - up to a subsequence - c_{e_r} is increasing and unbounded (observe that we may replace h by $-h$). Since $k_{m,m}$ is non-negative and non-trivial we may choose $\omega > 0$ such that $S_\omega = k_{m,m}^{-1}([\omega, +\infty[)$ is a positive measure subset of $]0, 2^m[$. Moreover, since h is bounded on compact sets,

$$|h(2^{e(n)-1}+t2^{e(n)-m-1})2^{\frac{e(n)}{p}}| \le C_{n,e}$$

for some positive constant $C_{n,e} \in \mathbb{R}$ not depending on r. Thus, given $M > 0$ there exists $r_0 \equiv r_0(M, e, n) > 0$ such that, for every $t \in S_\omega$,

$$\left|c_{e_r}\frac{k_{m,m}(t)}{\|\rho_e\|_{m,p}}a_n - h(2^{e(n)-1}+t2^{e(n)-m-1})2^{\frac{e(n)}{p}}\right|^p > \left|c_{e_r}\frac{\omega}{\|\rho_e\|_{m,p}}a_n - C_{n,e}\right|^p > M,\ \forall r \ge r_0.$$

This implies, considering that the measure of S_ω is positive, that

$$0 = \lim_{r\to\infty}\int_{]0,2^m[}\left|c_{e_r}\frac{k_{m,m}(t)}{\|\rho_e\|_{m,p}}a_n - h(2^{e(n)-1}+t2^{e(n)-m-1})2^{\frac{e(n)}{p}}\right|^p dt$$

$$\ge \lim_{r\to\infty}\int_{S_\omega}\left|c_{e_r}\frac{\omega}{\|\rho_e\|_{m,p}}a_n - C_{n,e}\right|^p dx = +\infty.$$

This is a contradiction and thus we have the claim proved, i.e. $\{c_{e_r}\}$ is bounded. In addition, we may assume that - up to a subsequence - $\{c_{e_r}\}$ converges to some $c_e \in \mathbb{R}$. In particular

$$\lim_{r\to\infty}\int_{]0,2^m[}\left|c_{e_r}\frac{k_{m,m}(t)}{\|\rho_e\|_{m,p}}a_n - c_e\frac{k_{m,m}(t)}{\|\rho_e\|_{m,p}}a_n\right|^p dt = 0.$$

This implies that the restriction of the function h to the interval $]2^{e(n)-1}, 2^{e(n)}[$ is equal to

$$c_e\frac{k_{m,m}(2^{m+1-e(n)}x - 2^m)2^{-\frac{e(n)}{p}}}{\|\rho_e\|_{m,p}}a_n$$

for every $n \in \mathbb{N}$ (observe that c_{e_r} and c_e do not depend on n). Therefore h restricted to the set

$$\bigcup_{n\in\mathbb{N}}]2^{e(n)-1}, 2^{e(n)}[$$

is equal to $c_e\eta_e$ or equivalently $h = \sum_{e=1}^{\infty} c_e\eta_e$ and we have proved that B is closed. □

The next result is obtained by combining the results of Theorems 3.4 and 4.2.

Theorem 4.3 *For every* $1 < p < +\infty$ *the set* $W^{m,p}\left(]0,+\infty[\right) \setminus \bigcup_{q\neq p} W^{m,q}\left(]0,+\infty[\right)$ *is latticeable.*

Remark 4.4 Arguing as was pointed out in Remark 3.5 the same result is also true for the set

$$W_0^{m,p}\left(]0,+\infty[\right) \setminus \bigcup_{q\neq p} W_0^{m,q}\left(]0,+\infty[\right).$$

Proof Let us denote by λ_s with $s \in [0,1]$ to a family of $\mathfrak{c}$ linearly independent functions in the set $W^{m,p}\left(]0,1[\right) \setminus \bigcup_{q>p} W^{m,q}\left(]0,1[\right)$ whose existence is guaranteed by Theorem 3.4. Moreover, we may assume that $\lambda_s \in C([0,1])$ and $\lambda_s(0) = \lambda_s(1) = 0$. Similarly we denote by υ_s with $s \in [0,1]$ to a family of $\mathfrak{c}$ linearly independent functions in the set $W^{m,p}\left(]1,+\infty[\right) \setminus \bigcup_{q<p} W^{m,q}\left(]1,+\infty[\right)$ whose existence ensure Theorem 4.2. In this case we may assume that $\upsilon_s \in C([1,+\infty[)$ and $\upsilon_s(1) = 0$.
Next we define the functions

$$\Gamma_s(x) := \begin{cases} \lambda_s(x), & \text{if } x \in]0,1[, \\ \upsilon_s(x), & \text{if } x \in]1,+\infty[, \\ 0, & \text{if } x = 1, \end{cases}$$

which generate a vector space contained in $W^{m,p}\left(]0,+\infty[\right) \setminus \bigcup_{q\neq p} W^{m,q}\left(]0,+\infty[\right)$. In particular, this set is maximal latticeable. □

As a corollary we deduce the same result for Sobolev Spaces in the real line.

Corollary 4.5 *For every* $1 < p < +\infty$ *the set* $W^{m,p}(\mathbb{R}) \setminus \bigcup_{q\neq p} W^{m,q}(\mathbb{R})$ *is latticeable.*

Corollary 4.6 *Assume that I is an unbounded open interval and* $1 \leq p < +\infty$. *Then, the set*

$$W^{m,p}(I) \setminus \bigcup_{q\neq p} W^{m,q}(I)$$

is maximal dense-lineable.

Proof Assume without loss of generality that $I = \mathbb{R}$. We use again Theorem 3.7 with $X = W^{m,p}(\mathbb{R})$, $A = W^{m,p}(\mathbb{R}) \setminus \bigcup_{q\neq p} W^{m,q}(\mathbb{R})$ which, according to Corollary 4.5, is in particular lineable, and $B = C_c^\infty(\mathbb{R})$ which, by Lemma 3.2, is dense in $W^{m,p}(\mathbb{R})$. We now show that A is stronger than B. i.e., $A+B \subseteq A$. First we observe that $A+B \subseteq W^{m,p}(\mathbb{R})$ and, given $f \in A$ and $g \in B$ we have that $(f+g) \notin \bigcup_{q\neq p} W^{m,q}(\mathbb{R})$. Otherwise, for some $q_0 \neq p$, $(f+g) \in W^{m,q_0}(\mathbb{R})$ and, since $g \in W^{m,q_0}(\mathbb{R})$, it follows that $f = (f+g) - g \in W^{m,q_0}(\mathbb{R})$ which contradicts that $f \in A$. □

We conclude this section by extending the obtained results to $\mathbb{R}^N$. In fact, the results of the previous section can be easily extended to the case of Sobolev spaces $W^{m,p}(I)$ where $I = I_1 \times \cdots \times I_N$ is a N-dimensional bounded cube of $\mathbb{R}^N$. Indeed, since I is bounded we have $W^{m,p}(I_i) \subset W^{m,p}(I)$ by considering extensions $\tilde{w}(x_1,\ldots,x_N) = w(x_i)$ for each $w \in W^{m,p}(I_i)$. In addition $\|\tilde{w}\|_{m,p} \leq C\|w\|_{m,p}$. Thus, since $W^{m,p}(I_i) \setminus \bigcup_{q>p} W^{m,q}(I_i)$ is spaceable latticeable and

$$W^{m,p}(I_i) \setminus \bigcup_{q>p} W^{m,q}(I_i) \subset W^{m,p}(I) \setminus \bigcup_{q>p} W^{m,q}(I),$$

we have that $W^{m,p}(I)\setminus\bigcup_{q>p}W^{m,q}(I)$ is spaceable latticeable.

Moreover, for any N-dimensional cube I, non necessarily bounded, we have that $W_0^{m,p}(I_1)\times\cdots\times W_0^{m,p}(I_N)\subset W_0^{m,p}(I)$ by considering $\tilde{w}(x_1,\ldots,x_N)=\prod_{i=1}^N w_i(x_i)$ for every $(w_1,\ldots,w_N)\in W_0^{m,p}(I_1)\times\cdots\times W_0^{m,p}(I_N)$. In addition, $\|\tilde{w}\|_{m,p}\leq C\prod_{i=1}^N\|w_i\|_{m,p}$. Thus, if I_1 is bounded, using that

$$\left(W_0^{m,p}(I_1)\setminus\bigcup_{q>p}W_0^{m,q}(I_1)\right)\times W_0^{m,p}(I_2)\times\ldots\times W_0^{m,p}(I_N)\subset W_0^{m,p}(I)\setminus\bigcup_{q>p}W_0^{m,q}(I),$$

it follows that $W_0^{m,p}(I)\setminus\bigcup_{q>p}W_0^{m,q}(I)$ is spaceable latticeable.

Analogously, if I_1 is unbounded we have that $W_0^{m,p}(I)\setminus\bigcup_{q\neq p}W_0^{m,q}(I)$ is spaceable latticeable for every $1<p<\infty$. These results are also true for a general open subset $\Omega\subset\mathbb{R}^N$ since for every N-dimensional cube $I\subset\Omega$, $W_0^{m,p}(I)\subset W_0^{m,p}(\Omega)$ by means of extending functions by zero.

Collecting everything we have proved the following result.

Theorem 4.7 *Assume $I=I_1\times\cdots\times I_N$ that for some real intervals, $I_1,\ldots,I_N$. Then*

1. *If I is bounded, $W^{m,p}(I)\setminus\bigcup_{q>p}W^{m,q}(I)$ is spaceable latticeable.*
2. *If I is bounded in one direction, $W_0^{m,p}(I)\setminus\bigcup_{q>p}W_0^{m,q}(I)$ is spaceable latticeable.*
3. *If I is unbounded in one direction, $W_0^{m,p}(I)\setminus\bigcup_{q\neq p}W_0^{m,q}(I)$ is spaceable latticeable.*
4. *If $\Omega\subset\mathbb{R}^N$ is open then $W_0^{m,p}(\Omega)\setminus\bigcup_{q>p}W_0^{m,q}(\Omega)$ is spaceable latticeable.*
5. *If $\Omega\subset\mathbb{R}^N$ is open and it contains an unbounded cube I then*

$$W_0^{m,p}(\Omega)\setminus\bigcup_{q\neq p}W_0^{m,q}(\Omega)$$

   is spaceable latticeable.

Acknowledgements José Carmona Tapia is partially supported by Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and (FEDER) Fondo Europeo de Desarrollo Regional under Research Project PGC2018-096422-B-I00, Junta de Andalucía, Consejería de Transformación Económica, Industria, Conocimiento y Universidades-Unión Europea Grant P18-FR-667, FQM194 and CDTIME. Juan B. Seoane-Sepúlveda gratefully acknowledges the support of Grant PGC2018-097286-B-I00. Wolfgang Trutschnig gratefully acknowledges the support of the WISS 2025 project 'IDA-lab Salzburg' (20204-WISS/225/197-2019 and 20102-F1901166-KZP).

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

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