Local existence of classical solutions for the Einstein–Euler system using weighted Sobolev spaces of fractional order

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Resumen

The existence of a class]of local in time solution of the Einstein–Euler system is proven, which include static solutions. This result is the relativistic counterpart of a similar result for the Euler–Poisson system obtained by Gamblin [6]. As in his case the initial data of the density do not have compact support but fall off at infinity in an appropriate manner. An essential tool of the proof is the construction and use of weighted Sobolevspace of fractional order. Moreover this tools allow to improve the regularity conditions for the solutions of the constraint and evolution equation.

1. Introduction

Some progress has been made in the mathematical theory of selfgravitation perfect fluids describing compact bodies, such as stars. We will briefly resume the situation: For the Euler-Poisson system Makino [10] proved a local in time existence theorem for solutions with compact support of the density and for which the density at the boundary vanishes. Since the Euler equation are singular for $\rho = 0$ Makino had to regularize the system by introducing a new matter variable ($w = M(\rho)$). His solution however, suffered some inconveniences such as they do not contain static solutions and moreover the connection between the physical density and the new matter density remained obscure.

Makino's work has been generalised by Rendall [13] to the relativistic case of the Einstein–Euler equations. His result however suffers from the same inconveniences as Makino's result. Moreover it is restricted to initial data with moment of time symmetry and which are C^∞ regular. This regularity condition implies a severe restriction on the equation of state $p = K\epsilon^\gamma$, namely $\gamma \in \mathbb{N}$.

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On the other hand Gamblin [6] proved a local in time existence theorem in which the initial density does not have compact support but falls of at infinity in an appropriate way. Even though a Makino variable had to be used but his solution includes a one parameter class of spherically symmetric static solutions with $\gamma = \frac{6}{5}$.

Our aim is to generalise Gamblin's result to the relativistic case, and furthermore we want to get rid of Rendall's restriction of the moment of time symmetry and the C^∞ regularity.

Moreover our approach is motivated by the following observation. As it turns out, the system of evolution equation have the following form.

$$A^0\partial_t U + A^k\partial_k U = Q(\epsilon, ..) \qquad U = (w, \dots) \tag{1}$$

Where the unknown U consists besides the gravitational field and the velocity of the fluid of the Makino variable w, while the lower order term Q consists of the energy density ϵ. So we have to estimate, in the corresponding norm of our function spaces, ϵ by w, which is a algebraic function of ϵ of the form $w = K\epsilon^{\frac{\gamma-1}{2}}$ and this results in an algebraic relation between the order of the functional space k and the coefficient γ of the equation of state

$$1 < \gamma \leq \frac{2+k}{k} \tag{2}$$

This relation can be easily derived by considering $\|D^\alpha w\|_{L_2}$, $|\alpha| \leq k$. Moreover it can be interpreted either as a restriction on γ or on k. Contrary to earlier works we want to interpret it as an restriction on k and even further allow the differentiability of fractional order.

So in our theorems considering the evolution equations we will be faced with differentiability conditions of the sort $\frac{5}{3} < s < \frac{2}{\gamma-1}$, or in other words the differentiability is bounded from above and below.

Moreover numerically investigations performed by Uggla and Nilson [11] suggest that static spherically solutions of the Einstein–Euler system correspond to values of γ between $1{,}2 < \gamma \leq 1{,}29949$. Recently these results have been confirmed analytically by Uggla and Heinzle [17] who also derived the fall of conditions of the density for $r \to \infty$. We come back to their results in section 7.

Now in order to sharpen existing regularity conditions for solutions and not to impose conditions on the equation of state we are lead to the conclusion of considering function spaces of fractional order. One the other hand the Einstein equations consist of quasi linear hyperbolic and elliptic equations. So far the only function spaces which are known to be useful for existence theorem in the asymptotically flat case, are weighted Sobolevspaces $H_{k,\delta}$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, introduced by Cantor [2] and others. So we are forced to consider new function spaces $H_{s,\delta}$ $s \in \mathbb{R}$ which generalise $H_{k,\delta}$.

We proceed as follows. In the next section we discuss the introduction of these spaces and their main properties, such as Banach algebra, isomorphism for elliptic operators, estimates for non-linear function which confirm the intuitive argument presented above, and moreover results concerning quasi linear symmetric hyperbolic systems. We then give the initial value formulation of the Einstein–Euler system, where the treatment of the Euler equation is a different from the one of Rendall. This is followed by our main theorem the local existence theorem together with an outline of the proof. In the last section we discuss the fact that our class of solutions contains static solution.

Finally in the appendix we present details of the proof of the main theorem together with some properties of our new function spaces.

2. Preliminaries

2.1. Function spaces

We just remind the definition of weighted Soblevspaces of integer order:

Definition 1 ***(Weighted Sobolevspaces)*** *For $-\infty < \delta < \infty$ and k a nonnegative integer, the Sobolev weighted space $H^2_{k,\delta}$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm*

$$\|u\|_{k,\delta} = \left(\sum_{|\alpha|\leq k} \int |\langle x\rangle^{\delta+|\alpha|} \partial^\alpha u|^2 dx \right)^{\frac{1}{2}}, \tag{3}$$

where $\langle x\rangle = 1 + |x|$.

Remark 1 *In the definition above the weight varies which each derivative. This property is used in order to show isomorphism of certain elliptic operators. We emphasis this, since in the literature, the term weighted Sobolevspaces usually is used for spaces in which every derivative obtains the same weight.*

While it is straightforward to define Sobolev spaces of fractional order without weights by using the Fourier transformation, the case of weights complicates the situation. However the following definition which goes back to Triebel [16] is a starting point:

Definition 2 ***(Weighted fractional Sobolevspace: double integral)*** *For $-\infty < \delta < \infty$ and k a nonnegative integer, $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, $s = k + \lambda$ the Sobolev weighted space $H_{s,\delta}$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm*

$$\begin{aligned} \|u\|^{*\,2}_{k+\lambda,\delta} &= \sum_{|\alpha|\leq k} \int |\langle x\rangle^{\delta+|\alpha|} \partial^\alpha u|^2 dx \\ &+ \sum_{|\alpha|=k} \int\int \frac{|\langle x\rangle^{k+\lambda+\delta+} \partial^\alpha u(x) - \langle y\rangle^{k+\lambda+\delta} \partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}} dx dy\,. \end{aligned} \tag{4}$$

Remark 2

1. *These spaces have been introduced by Triebel [16], although not much properties have been specified or proven.*

2. *If $\lambda = 0$, then $s = k \in \mathbb{N}$ and this definition concides with the definition of weighted Sobolev spaces $H_{k,\delta}$ which have been introduced by Cantor [2] and others.*

3. *It turns out that the definition above 2 is not useful for proving the tools we need such as the Banach algebra property, the Moser inequalities, the Energy estimates etc. We found however a more suitable norm which is equivalent to 2 and which we present in the following.*

First we introduce the following notations:

Definition 3 $K_j = \{x : 2^{j-3} \leq |x| \leq 2^{j+2}\}$, $(j = 1, 2, ...)$ *and* $K_0 = \{x : |x| \leq 4\}$. *Let* $\{\psi_j\}_{j=0}^{\infty}$ *be a sequence of* $C_0^\infty(\mathbb{R}^n)$ *such that* $\psi_j(x) = 1$ *on* K_j, $\mathrm{supp}(\psi_j) \subset \cup_{l=j-4}^{j+3} K_l$, *for* $j \geq 1$, $\mathrm{supp}(\psi_0) \subset K_0 \cup K_1$ *and*

$$|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha| j}. \tag{5}$$

Remark 3 ***(on Definition 3)*** *The above mollifiers have the following properties: for any* $x \in \mathbb{R}^n$ *belongs at most to five* K_j *and there are at most seven* j *such that* $\psi_j(x) \neq 0$. *In addition, there are two constants, independent of* j, *such that*

$$c_0 2^j \leq \langle x \rangle \leq c_1 2^j, \quad \textit{for} \qquad x \in \mathrm{supp}(\psi_j). \tag{6}$$

The second norm is then defined in the following way.

Definition 4 ***(Weighted fractional Sobolevspaces (using infinite norms))*** *For* $-\infty < \delta < \infty$ *and* $s \in \mathbb{R}$, *the Sobolev weighted space* $H_{s,\delta}$ *is defined as the completion of* $C_0^\infty(\mathbb{R}^n)$ *under the norm*

$$(\|u\|_{s,\delta})^2 = \sum_{j=0}^{\infty} 2^{\delta 2j} \|\psi_j u\|_2^2 + \sum_{j=0}^{\infty} 2^{(\delta+m)2j} \sum_{|\alpha|=m} \|\partial^\alpha(\psi_j u)\|_2^2. \tag{7}$$

The desired equivalence we state in the following proposition, whose proof we have placed in the appendix.

Proposition 1 ***(Equivalence of the norms 2 and 4)*** *For* $u \in C_0^\infty(\mathbb{R}^n)$ *there holds*

$$C_1 \|u\|_{s,\delta} \leq \|u\|_{s,\delta}^* \leq C_2 \|u\|_{s,\delta} \tag{8}$$

where $C_1 = C_2 = C(s, \delta)$.

The proof of this proposition as well as certain properties of these functions spaces will be presented in a forthcoming articles by the authors [1]. This article will include also the proof of existence theorem for the elliptical constraints.

Definition 5 *The space $H_{s,\delta}$ as defined above has the following equivalent norm*

$$\|u\|^2_{H_{s,\delta}} \sim \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\Psi_j u)_{(2^j)}\|_{H^s} \tag{9}$$

We have used the notation H^s to denote the Bessel potential space which is defined whose norm is defined via the Fourier transformation

Definition 6 ***(Bessel potential spaces)*** *The norm of the the Bessel potential space is given as*

$$\|u\|^2_{H^s} \;=\; \int \widehat{u}^2 \left(1+\xi^2\right)^s d\xi \tag{10}$$

2.2. Properties of the function spaces

Theorem 1 ***(Banach Algebra of the fractional spaces)*** *Let $u \in H_{s_1,\delta_1}$, $v \in H_{s_2,\delta_2}$ then*

$$\|uv\|_{H_{s,\delta}} \leq C\|u\|_{H_{s_1,\delta_1}}\|v\|_{H_{s_2,\delta_2}} \tag{11}$$

Provided that $s+\frac{3}{2} < s_1+s_2$, $s \leq s_i$ and $\delta \leq \delta_1+\delta_2$

Of central importance is the following estimates for powers of function in the spaces $H_{s,\delta}(\mathbb{R}^3)$

Theorem 2 ***(Nonlinear estimate for power of functions)*** *Let $u \in H_{s,\delta} \cap L^\infty$, $F : \mathbb{R}^l \to \mathbb{R}$ be a C^n map, with $F(0) = 0$ of the form $F(u) = u^\mu$ ($1 \leq \mu$), then there exists a constant C such that*

$$\|u^\mu\|_{H_{s,\delta}} \leq C\left(|u|_{L^\infty}\right)\|u\|_{H_{s,\delta}} \tag{12}$$

In order to prove the lemma above, we will use the following estimate in the case of non weighted Bessel potential spaces, which was proven by Runst and Sickel (prop 1, p 363 ff)[14]

2.3. Symmetric hyperbolic systems and local existence theorems

We start with a general definition of symmetric hyperbolic systems.

Definition 7 ***(Uniform symmetric–hyperbolic systems)*** *A quasilinear, symmetric–hyperbolic system is a system of differential equations of the form*

$$L[U] \sum_{\alpha=0}^{n} A^\alpha(U)\partial_\alpha U + B(U;x,t) = 0 \tag{13}$$

where the matrices A^α are symmetric and for every arbitrary $U \in G$ there exists a covector ξ different from zero

$$A^\alpha(U)\xi_\alpha \tag{14}$$

is positive definite. The covectors ξ_α are spacelike with respect to the equation (13). Both matrices A^α, B satisfy certain regularity conditions, which are going to be formulated later.

Usually ξ is chosen to be the vector $(1,0,0,0)$ which implies that, via the condition (14), the matrix A^0 has to be is positive definite.

A principal tool is the following existence theorem.

Theorem 3 ***(Local existence for quasilinear symmetric–hyperbolic systems)*** *Let $A^0, A^k \in C^N(V \times G; \mathcal{M}_{l\times l})$, $B \in C^N(V\times G;\mathbb{R}^l)$ be coefficients which define the quasilinear symmetric–hyperbolic system*

$$A^0_{\alpha\beta}(U,x,t)\frac{\partial U^\beta}{\partial t} + \sum_{k=1}^{n} A^k_{\alpha\beta}(U,x,t)\frac{\partial U^\beta}{\partial x^k} + B_{\alpha\beta}(U;x,t) = 0 \tag{15}$$

Let $U(x,0)\in H_{s,\delta}(\mathbb{R})$ ($\frac{5}{2}\leq s$) and let the initial conditions be chosen such that the condition

$$C\delta_{\alpha\beta}U^\alpha U^\beta \leq A^0_{\alpha\beta}U^\alpha U^\beta \leq C^{-1}\delta_{\alpha\beta}U^\alpha U^\beta \qquad C\in\mathbb{R}^+ \tag{16}$$

is satisfied. Then there exists a $T>0$ which depends on the $H_{s,\delta}$ norm of the initial data and there exists a unique solution $U(x,t)\in C^{0,s}\left([0,T),H_{s,\delta}\right)\cap C^{1,s-1}\left([0,T),H_{s-1,\delta+1}\right)$ of equation (15).

3. The initial value problem for the Euler–Einstein system

We consider the Einstein-Euler system describing a relativistic self-gravitating perfect fluid. The unknowns in the equations are functions of t and x^a where x^a $(a=1,2,3)$ are Cartesian coordinates on $\mathbb{R}^3$. The alternative notation $x^0=t$ will also be used and Greek indices will take the values $0,1,2,3$ in the following. The evolution of the gravitational field is described by the Einstein equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{17}$$

where $G_{\alpha\beta}$ is the Einstein tensor of the spacetime metric $g_{\alpha\beta}$ and $T_{\alpha\beta}$ is the energy-momentum tensor of the matter. In the case of a perfect fluid the latter takes the form

$$T^{\alpha\beta} = (\epsilon+p)u^\alpha u^\beta + pg^{\alpha\beta} \tag{18}$$

where ϵ is the energy density, p is the pressure and u^α is the four-velocity. The quantity u^α is required to satisfy the normalisation condition

$$g_{\alpha\beta}u^\alpha u^\beta = -1. \tag{19}$$

The Euler equations describing the evolution of the fluid take the form

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{20}$$

where ∇ denotes the covariant derivative associated to the metric $g_{\alpha\beta}$. To get a determined system of equations it is necessary to specify a relation between ϵ and p (equation of state). The choice we make here is one which has been used for astrophysical problems. It is an analogue of the well known polytropic equation of state of the non-relativistic theory given by:

$$p = f(\epsilon) = K\epsilon^{\gamma} \qquad K, \gamma \in \mathbb{R}^{+} \qquad 1 < \gamma \tag{21}$$

The sound velocity is denoted by

$$\sigma^2 = \frac{\partial p}{\partial \epsilon} \tag{22}$$

The new matter variable $w = M(\epsilon)$ which regularise the Euler equations even for $\epsilon = 0$ is given by the expression.

$$w = M(\epsilon) = \epsilon^{\frac{2}{\gamma - 1}} \tag{23}$$

For this setting Rendall [12] proved a local in time existence theorem for initial data with compact support for the density generalising a result obtained by Makino [10] for the non relativistic Euler Poisson system.

3.1. The Euler equations written as a symmetric hyperbolic system

It is not obvious that the Euler equations written in the conservative form $\nabla_\alpha T^{\alpha\beta} = 0$ are symmetric hyperbolic. In fact these equations have to be transformed in order to be expressed in a symmetric hyperbolic form. Rendall presented such a transformation of the equations, however it's geometrical meaning is not entirely clear and it might be difficult to generalise is to the non time symmetric case. Hence we will present a different decomposition of the Euler equations in discuss it in some details, for we have not seen in anywhere in the literature. The basic idea is to perform the standard *fluid decomposition*, and then modify the equation by adding, in an appropriate manner, the normalisation condition (19), which will be considered as a constraint equation.

1. the equation $\nabla_\alpha T^{\alpha\beta} = 0$ will once be projected orthogonal to u^α which leads to

$$u_\beta \nabla_\alpha T^{\alpha\beta} = 0. \tag{24}$$

2. Furthermore the equation $\nabla_\alpha T^{\alpha\beta} = 0$ will be projected into the rest pace $\mathcal{O}$ orthogonal to u^α of a fluid particle gives us.

$$P_{\alpha\beta} \nabla_\nu T^{\nu\beta} = 0 \qquad \text{with} \quad P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \quad P_{\alpha\beta} u^\beta = 0. \tag{25}$$

The resulting system is of the form:

$$u^\alpha \nabla_\alpha \epsilon + \mu \nabla_\alpha u^\alpha \;=\; 0 \tag{26a}$$

$$. \, (\epsilon + p)\, u^\alpha \nabla_\alpha u_\alpha + P^{\beta}{}_{\alpha} \nabla_\beta p \;=\; 0 \tag{26b}$$

Note that we have besides the evolution equations (26a) and (26b) the following constraint equation: $g_{\alpha\beta}u^{\alpha}u^{\beta} = -1$. We will show later, in subsection 3.1.1 that this constraint equation is conserved by the evolution equation, that is if it holds initially at $t = t_0$ it will hold for $t > t_0$.

However in order to obtain a symmetric hyperbolic system it turns out that we have to modify the system in the following way. We add $u_{\alpha}\nabla_{\beta}u^{\alpha} = 0$ to equation (26a) and $2u_{\beta}u_{\alpha}\nabla_{\beta}u^{\alpha} = 0$ to (26b) and, which results in.

$$u^{\nu}\nabla_{\nu}\epsilon + (\epsilon + p)\, P^{\nu}{}_{\alpha}\nabla_{\nu}u^{\alpha} = 0 \tag{27a}$$

$$\Gamma_{\alpha\beta}u^{\nu}\nabla_{\nu}u^{\beta} + \frac{\sigma^2}{\epsilon + p}P^{\beta}{}_{\alpha}\nabla_{\beta}\epsilon = 0 \tag{27b}$$

With $\Gamma_{\alpha\beta} = g_{\alpha\beta} + 2u_{\alpha}u_{\beta}$, note that $\Gamma_{\alpha\beta}$ is positive definite. As mentioned above we will introduce a new nonlinear matter variable which is given by (23). The idea which is behind this is the following: system (27a) and (27b) is almost of symmetric hyperbolic form, we just have to multiply the equations by appropriate factor of $\epsilon\frac{\partial f}{\partial \epsilon}$, however when doing so we will be faced which a system in which the coefficients in the term A^0 will either tend to zero or to infinity, as $\epsilon \rightarrow 0$. The central point is now to introduce a new variable $w = M(\epsilon)$ by multiplying equation (27a) with $\frac{\partial w}{\partial \epsilon}$ resulting in the system

$$u^{\nu}\nabla_{\nu}w + \frac{\partial w}{\partial \epsilon}(\epsilon + p)\, P^{\nu}{}_{\alpha}\nabla_{\nu}u^{\alpha} = 0 \tag{28a}$$

$$\Gamma_{\alpha\beta}u^{\nu}\nabla_{\nu}u^{\beta} + \frac{1}{\epsilon + p}\frac{\partial f}{\partial w}P^{\beta}{}_{\alpha}\nabla_{\beta}\epsilon = 0 \tag{28b}$$

Where we have used the fact $\sigma^2 = \frac{\partial f}{\partial \epsilon}$ and we have considered now f as a function of w instead of ϵ. The system above is symetric if we demand that

$$\frac{\partial w}{\partial \epsilon} = \frac{1}{\epsilon + p}\frac{\partial f}{\partial w}. \tag{29}$$

Moreover we can in equation (29) solve for w, obtaining

$$w = \int_0^{\epsilon'} \frac{1}{\epsilon\epsilon + p}\left(\frac{\partial f}{\partial \epsilon}\right) \tag{30}$$

This integral however symmetrisises system (28a)–(28b) and gives us a functional equation for the variable w. However it turns out that it is more convenient (see [13]) to multiply equation (27a) with $g^2\frac{\partial w}{\partial \epsilon}$, where g^2 is a appropriately chosen function. So we obtain instead of (30)

$$w = \int_0^{\epsilon'} \frac{1}{g}\frac{1}{(\epsilon + p)}\left(\frac{\partial f}{\partial \epsilon}\right) \tag{31}$$

Choosing $g = \frac{(\epsilon+p)}{\epsilon}$ leads us to the expression (21).

The Euler equation in the new variable take the final form:

$$u^\alpha \nabla_\alpha w + \sigma P^\nu_\alpha \nabla_\nu u^\alpha \quad = \quad 0 \tag{32a}$$

$$\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \sigma P^\beta{}_\alpha \nabla_\beta w \quad = \quad 0. \tag{32b}$$

Here we still have used the symbol ∇_β which denotes the covariant derivate. This derivative takes in local coordinates the form $\nabla_\beta = \partial_\beta + G(g^{\mu\nu}, \partial g_{\kappa\gamma})$ which expresses the fact that the fluid is coupled to the gravitational field $g_{\alpha\beta}$.

Now we want to show that our system (26a) is indeed symmetric hyperbolic. We do this in several steps. Recall the definition of a symmetric hyperbolic system:

We now proceed in the following way:

1. We first show that u_α is a space like covector with respect to the equations.
2. Then we show that even the covector $\xi_\alpha = (1, 0, 0, 0)$ ie. t_α is a space like covector with respect to the equation, which according to the above definition implies that A^0 is positive definite. For this last step we have to know the characteristics explicitly.

So let us start with:

$$\underbrace{\left(\begin{array}{c|c} u^\nu & (\epsilon + p)\, P^\nu{}_\beta \\ \hline \frac{\sigma^2}{(\epsilon+p)} P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array}\right)}_{A^\nu} \nabla_\nu \begin{pmatrix} \epsilon \\ u^\beta \end{pmatrix} = 0, \tag{33}$$

The principal part becomes:

$$A^\nu \xi_\nu = \left(\begin{array}{c|c} u^\nu \xi_\nu & (\epsilon + p)\, P^\nu{}_\beta \xi_\nu \\ \hline \frac{\sigma^2}{(\epsilon+p)} P^\nu{}_\alpha \xi_\nu & \Gamma_{\alpha\beta} u^\nu \xi_\nu \end{array}\right). \tag{34}$$

It is a significant simplification to introduce a three dimensional basis such that $g_{ab} = \delta_{ab}$ and $u^a \delta^a{}_1$ holds. The then characteristic polynom is given by

$$Q(\xi) := \det(A^\nu \xi_\nu) = (\xi_\nu u^\nu)^3 \left\{ (\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha{}_\beta \xi_\alpha \xi^\beta \right\}. \tag{35}$$

The characteristic covectors are on the one hand given by equation

$$\xi_\nu u^\nu = 0 \qquad \text{and on the other hand by equation} \tag{36}$$

$$(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha{}_\beta \xi_\alpha \xi^\beta = 0 \tag{37}$$

Remark 4 ***(The structure of the conormal cone)*** *The conormal cone is therefore a union of two hypersurfaces in $T^*_x V$. One of these hypersurfaces is given by the condition (36). It is a three dimensional hyperplane S with the normal u^α. The other hypersurface is given by the conditions (37) and forms a three dimensional cone the so called sound cone.*

Remark 5 *Equation (37) plays an essential role in determining whether the equations form a symmetric hyperbolic system.*

Let us now consider timelike vector u_ν the linear combination $-u_\nu A^\nu$, with A^ν from equation (34) we then get.

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma_{\alpha\beta} \end{array}\right), \tag{38}$$

is positive definite. Hence $-u_\nu$ is for the hydrodynamical equations a spacelike covector in the sense of partial differential equations. Herewith one has show relatively elegant and elementary that the relativistic hydrodynamical equations are symmetric–hyperbolic.

Now we want however to show that system (32) is also symmetric–hyperbolic with respect to the vector t_α. The equation for the sound cone (37) states that as long as the condition $\sigma < c$ is satisfied the sound cone is inside the light cone. This is implies that the expression $A^\nu \chi_\nu$ is positive for all vectors which are timelike with respect to the metric. Since the timelike vector t_α can be obtained by a continues deformation of the vector u_α $A^\nu t_\nu$ is also positive definite.

3.1.1. Conservation of the constraint equation $g_{\alpha\beta}u^\alpha u^\beta = -1$

Now it will be shown that the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$, which acts as a constraint equation for the evolution equation, is conserved along stream lines u^α. Because, if for $t = t_0$ the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds and if it is conserved a long stream lines then $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds also for $t > t_0$. In order to see that one considers

$$u^\alpha \nabla_\alpha (u_\beta u^\beta) = 0, \tag{39}$$

which leads to

$$u_\beta u^\nu \nabla_\nu u^\beta = 0. \tag{40}$$

Now the modified Euler equation (32b) will be multiplied with u^α where one gets:

$$u^\alpha (g_{\alpha\beta} + 2u_\alpha u_\beta) u^\nu \nabla_\nu u^\beta + \sigma u^\alpha P^\beta{}_\alpha \nabla_\beta w = -2u_\alpha u^\nu \nabla_\nu u^\alpha + \sigma u^\alpha P^\beta{}_\alpha \nabla_\beta w = 0. \tag{41}$$

Therefore it holds:

$$2u_\alpha u^\nu \nabla_\nu u^\alpha = g^2 \sigma u^\alpha P^\beta{}_\alpha \nabla_\beta w. \tag{42}$$

Because of equation (40) and because $u^\alpha P^\beta{}_\alpha = 0$ holds, the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$ is conserved along the stream lines. ■

3.1.2. The Einstein evolution equations

The initial value problem for the Einstein-Euler system will be treated by writing the equations as a symmetric hyperbolic system in harmonic coordinates. The harmonic condition is that

$$g^{\alpha\beta}g^{\gamma\delta}(\partial_\gamma g_{\beta\delta} - \frac{1}{2}\partial_\delta g_{\beta\gamma}) = 0 \tag{43}$$

When this condition is imposed the Einstein equations imply a system of quasilinear wave equations. To get a symmetric hyperbolic system these are reduced to first order by introducing auxiliary variables

$$h_{\alpha\beta\gamma} = \partial_\gamma g_{\alpha\beta} \tag{44}$$

They can then be written in the following form

$$\begin{array}{rcl} \partial_t g_{\alpha\beta} & = & h_{\alpha\beta 0} \\ g^{ab}\partial_t h_{\gamma\delta a} & = & g^{ab}\partial_a h_{\gamma\delta 0} \\ -g^{00}\partial_t h_{\gamma\delta 0} & = & 2g^{0a}\partial_a h_{\gamma\delta 0} + g^{ab}\partial_a h_{\gamma\delta b} \\ & & +C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\gamma\gamma\gamma\gamma\gamma\delta\alpha\beta\rho\sigma} h_{\epsilon\zeta\eta} h_{\kappa\lambda\mu} g^{\alpha\beta} g^{\rho\sigma} - 16\pi T_{\gamma\delta} + 8\pi g^{\rho\sigma} T_{\rho\sigma} g_{\gamma\delta} \end{array} \tag{45}$$

The object C is a combination of Kronecker deltas with integer coefficients. The quantities w and u^α will be used to describe the fluid. This results in the following first order system:

$$A^0\partial_t V + A^k\partial_k V = F(V, \rho, p) \tag{46}$$

Where $V = (g_{\alpha\beta}, h_{\gamma\delta a}, h_{\gamma\delta 0}, w, u^\alpha)$, parts of the equation are given by (45), parts by (32) and (32b). We see here the problem described in the introduction, the matter variable in the principal part of the equations is the new variable w, but the lower order term contains $\rho = w^{\frac{2}{\gamma-1}}$ and $p = w^{\frac{2\gamma}{\gamma-1}}$ which are algebraic functions of the variable w. For the existence theorem, we need to estimate these terms by w. Theorem 2 takes care of this and gives relation between the differentiability s and the power $\alpha = \frac{2\gamma}{\gamma-1}$, similar to the intuitive argument presented in the introduction.

4. The elliptic constraints

The solution of the Einstein equations coupled to matter fields is usually done in two steps. Initial data for the Einstein equations cannot be given freely; there are constraint equations intrinsic to the initial hypersurface which must be satisfied. So the first step is to construct solutions of these constraints. The second step is then to solve the evolution equations (in the present case the symmetric hyperbolic system just described) with these initial data. To define the harmonic coordinates uniquely it is necessary to supplement the condition (43) with some conditions on the initial hypersurface defined by the equation $t = 0$. The standard choice is that on the initial hypersurface $g_{00} = -1$ and $g_{0a} = 0$. To write down the constraint equations it is convenient to introduce the second fundamental form of the initial surface. When the conditions just introduced hold this object is given by

$$K_{ab} = -\frac{1}{2}\partial_t g_{ab} \tag{47}$$

Let n^α denote the unit normal to the hypersurface and define

$$\begin{aligned} z &= T_{\alpha\beta}n^\alpha n^\beta && (48)\\ j^\alpha &= T_{\beta\gamma}(g^{\alpha\beta}+n^\alpha n^\beta)n^\gamma && (49)\end{aligned}$$

The vector j^α is tangent to the initial surface and so can be identified with a vector j^a intrinsic to this surface. More explicit expressions for z and j^a can be given using the projection $(\delta^\alpha_\beta - n^\alpha n_\beta)u^\alpha$ of the velocity onto this surface. It can be identified with a vector $\bar{u}^a$ intrinsic to the surface. Then

$$\begin{aligned} z &= \rho(1+g_{ab}\bar{u}^a\bar{u}^b)+pg_{ab}\bar{u}^a\bar{u}^b && (50)\\ j^a &= (\rho+p)\bar{u}^a(1+g_{bc}\bar{u}^b\bar{u}^c)^{1/2} && (51)\end{aligned}$$

If R_{ab} denotes the Ricci tensor of the induced metric on the initial hypersurface, $R = g^{ab}R_{ab}$ is its scalar curvature and ${}^{(3)}\nabla$ its associated covariant derivative then the constraints are

$$\begin{aligned} R - K_{ab}K^{ab} + (g^{ab}K_{ab})^2 &= 16\pi z && (52)\\ {}^{(3)}\nabla_b K^{ab} - \nabla^b(g^{bc}K_{bc}) &= -8\pi j^a && (53)\end{aligned}$$

Now the solution of the constraints by the conformal method can be discussed. This method has been discussed in detail in the literature see for example Cantor [3] and Christodoulou [4] and reference therein. So we will just briefly outline the procedure.

The initial data are split into the free initial data and those which are to be determined by the transformed constraints. The free initial data are:

$$\left(\bar{h}_{ab}, \bar{A}^{ab}_*, \bar{z}, \bar{j}^b\right). \tag{54}$$

The transformed constraints are to be solved for the scalar function ϕ and the vector field W^b

$$\begin{aligned} \bar{\Delta}\phi - \frac{1}{8}\bar{R}\phi + \frac{1}{8}\left(\bar{A}_{ab}\bar{A}^{ab}\right)\phi^{-7} &= -2\phi^{-3}\bar{z} && (55)\\ (\Delta_L W)^b &= \bar{j}^b && (56)\end{aligned}$$

Where

$$(\Delta_L W)^b = (\Delta W)^b + \frac{1}{3}\bar{D}^b\left(\bar{D}_a W^a\right) + \bar{R}^b_a W^a \tag{57}$$

Once the solution $\left(\phi, W^b\right)$ of is found the rest of the initial data are constructed as follows:

$$\begin{aligned} h_{ab} &= \phi^4\bar{h}_{ab} && (58)\\ K_{ab} &= \phi^{-2}\left(\bar{A}_{*ab} + \bar{D}^a W^b + \bar{D}^b W^a - \frac{2}{3}\bar{h}^{ab}\bar{D}_k W^k\right) && (59)\\ z &= \phi^{-8}\bar{z} && (60)\\ j^b &= \phi^{-10}\bar{j}^b && (61)\end{aligned}$$

Where we have performed a conformal transformation of the metric:

$$h_{ab} = \phi^4 \bar{h}_{ab}, \tag{62}$$

As for K_{ab} a decomposition into a trace free and into a trace part

$$A^{ab} = K^{ab} - \frac{1}{3} h^{ab} (\mathrm{K}) \tag{63}$$

And

$$A^{ab} = \phi^{-10} \bar{A}^{ab} \tag{64}$$

moreover

$$\bar{A}^{ab} = \bar{A}^{ab}_{*} + \bar{D}^a W^b + \bar{D}^b W^a - \frac{2}{3} \bar{h}^{ab} \bar{D}_k W^k \tag{65}$$

Furthermore

$$j^b = \phi^{-10} \bar{j}^b \qquad z = \phi^{-8} \bar{z}. \tag{66}$$

5. The principal result

Theorem 4 ***(Main result)***

1. ***Solution of the constrains (30) and (31):*** *Let* $\delta, s \in \mathbb{R}$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ *and* $\frac{7}{2} < s$. *Given free initial data* $\bar{h}_{ab} - \delta_{ab} \in H_{s,\delta}(\mathbb{R}^3)$, $\bar{A}^{ab}_{*} \in H_{s-1,\delta+1}(\mathbb{R}^3)$, $\bar{z}(\rho) \in H_{s-1,\delta+2}(\mathbb{R}^3)$, $j^b(u^\alpha) \in H_{s-1,\delta}(\mathbb{R}^3)$, *there exists an unique solution* ϕ, K_{ab} *of the constraint such that* $\phi - 1 \in H_{s,\delta}(\mathbb{R}^3)$, $K_{ab} \in H_{s-1,\delta+1}(\mathbb{R}^3)$.

2. ***Solution of the evolution equations (46):*** *Given the solutions of the constraints (30) and (31), let* $\frac{5}{2} < s < \frac{2}{\gamma-1}$, $-\frac{3}{2} < \delta < -\frac{1}{2}$, *then there exists a* $T > 0$ *and a unique solution* U^ν *of the Einstein–Euler system, with*

$$\begin{aligned} &g_{\alpha\beta} - \eta_{\alpha\beta} \in C^0([0,T), H_{s+1,\delta}(\mathbb{R}^3)) \cap C^1([0,T), H_{s,\delta}(\mathbb{R}^3)) \\ &w, u^\alpha \in C^0([0,T), H_{s,\delta}(\mathbb{R}^3)) \cap C^1([0,T), H_{s-1,\delta}(\mathbb{R}^3)) \end{aligned}$$

6. Proof sketch

The proof consists of three parts:

1. First we solve the elliptic constraints using the established methods introduced by [3], and Christodoulou and O'Murchadha [4] in our spaces. That will be done in a forthcoming paper, we present the result here in form of theorem 5 and 6.

2. Construction of the initial data for the fluid equations, starting with the initial data for the constrains (theorem 2).

3. Local existence of the evolution equation given by theorem 4.

6.1. Solution of the elliptic constraints

The well known strategy for the constrains goes as follows

1. First the vectorial constrains is solved (31) for W^b with given j^b. See theorem 5.
2. Secondly using W^b and and the free initial data A^{ab}_*, A^{ab} is constructed using (65).
3. Finally with z and A^{ab} given the Lichnerowicz equation is solved fro ϕ, see theorem. 6.

The results we present in the following are straitforward generalisations of results obtained by Christodoulou and O'Murchadha [4]

Theorem 5 ***(Existence and uniquness of solutions for the York equation)*** *Let $\frac{7}{2} < s$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ and $\bar{j}^b \in H_{s-1,\delta+2}$ then there exists a unique solution $W^b \in H_{s+1,\delta}$ of equation (56) with $(\Delta_L w)^b := (\Delta W)^b + \frac{1}{3}\bar{D}^b\left(\bar{D}_a W^a\right) + \bar{R}^b_a W^a$*

Theorem 6 ***(Existence and uniqueness for the solutions of the Lichnerowicz equation)*** *Let $\frac{7}{2} < s$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ and g a Riemann metric, and K a tensorfield in $\mathbb{R}^3$ such that $h_{ab} - \delta_{ab} \in H_{s,\delta}(\mathbb{R}^3)$. Let $\bar{z} \in H_{s-1,\delta+2}$ then there exists a unique solution of the equation (30).*

6.2. The compatibility problem of the initial data for the fluid and the gravitational field

In the case of a fluid there is a problem with this because the quantities which can be rescaled in a way which is consistent with the general scheme are z and j^a and not the quantities w and $\bar{u}^a$ which occur in the initial data for the evolution equations.

The matter initial data for the Einstein–Euler system are on the one hand w and u^α for the Euler equations (28a) and (28b). One the other hand $z = F(w, u^\alpha)$ and $j^a = H(w, u^\alpha)$ appear as sources in the constaint equations, namely $R - K_{ab}K^{ab} + (g^{ab}K_{ab})^2 = 16\pi z$ And ${}^{(3)}\nabla_b K^{ab} - \nabla^b(g^{bc}K_{bc}) = -8\pi j^a$

We have the possibility of either

1. Consider w and u^α as the fundamental quantites and construct then z and j^a.
2. Vice versa: consider z and j^a as the fundamental quantites and construct then w and u^α.

The first possibility does not work, because the geometric quantities which occur on the left hand side of the constraint equations are supposed to scale with some power of ψ So z and j^a, which are the sources in the constraint equations, must also scale with some definite power of ψ. If ϵ scaled with some power of ψ then so would r. But in the expression for z a sum of different powers of r occurs. Thus the power with which r scaled would have to be zero and ϵ would be left unchanged by the rescaling. Similarly it can be seen that $\bar{u}^a$ would remain unchanged. So in fact z would be unchanged and this is inconsistent with the scalings used in the conformal method.

Because of this problem we proceed as follows:

1. First geometrical and matter quantities are constructed which satisfy the constraints, whereby the matter quantities are $y = z^{\frac{\gamma-1}{2}}$ (the power $\frac{\gamma-1}{2}$ comes from the use of the Makino variable) and j^a.

2. Then w and $\bar{u}^a$ are reconstructed from these. The reconstruction is based on the following lemmas:

Lemma 1 ***(Hadamard's Lemma)*** *Let X and Y be manifolds with X connected and Y simply connected. Let $f : X \to Y$ be a mapping which is continuous, proper and open. Then f is a bijection.*

Lemma 2 ***(Reconstruction lemma for the initial data)*** *Let Φ be the mapping from $\mathbb{R}^4$ to $\mathbb{R}^4$ defined by*

$$\Phi(w, \bar{u}^a) = \Big(w[(1 + Knw^2)(1 + g_{ab}\bar{u}^a\bar{u}^b) + Kg_{ab}u^a u^b]^{1/2n}, \tag{67}$$

$$\frac{(1 + K(n+1)w^2)(1 + g_{bc}\bar{u}^b\bar{u}^c)\bar{u}^a}{(1 + Knw^2)(1 + g_{bc}\bar{u}^b\bar{u}^c) + Kw^2 g_{bc}\bar{u}^a\bar{u}^b}\Big) \tag{68}$$

Then Φ is an analytic diffeomorphism from $\mathbb{R}^4$ onto the region $G = \{(y, x^a) : \delta_{ab}x^a x^b < 1\}$.

Proof (of Lemma 2) It is obvious that the mapping Φ is analytic. It is also a straightforward calculation to show that the derivative $D\Phi$ is invertible when $w = 0$. The invertibility of $D\Phi$ away from $w = 0$ is equivalent to that of the derivative of the mapping Φ' where

$$\Phi'(\epsilon, \bar{u}^a) = (\epsilon(1 + g_{ab}\bar{u}^a\bar{u}^b) + f(\epsilon)g_{ab}\bar{u}^a\bar{u}^b, (\epsilon + f(\epsilon))\bar{u}^a(1 + g_{bc}\bar{u}^b\bar{u}^c)^{1/2}) \tag{69}$$

and f is the function introduced in section 3 in connection with the equation of state. To calculate the determinant of $D\Phi'$ it is convenient to note that the mapping Φ' is equivariant with respect to the action of the orthogonal group of g_{ab} and so when doing the calculation it can be assumed without loss of generality that the only non-vanishing component of $\bar{u}^a$ is $\bar{u}^1$. When this simplification has been made it is easy to see that this determinant is always positive. It can now be concluded from the inverse function theorem that Φ is an analytic local diffeomorphism. In particular it is continuous and open. Define $|x| = \sqrt{g_{ab}x^a x^b}$. Using the elementary inequality

$$|x|^2 - |x|(1 + |x|^2)^{1/2} \geq -\frac{1}{2} \tag{70}$$

it is possible to derive the estimate

$$|\bar{U}|^2 \leq \frac{1}{2}(1 - |j/z|)^{-1}. \tag{71}$$

Let H be a compact subset of G. Then y and $(1 - |x/y|)^{-1}$ are bounded on H. It follows from the second of these facts that $|U|$ is bounded on $\Phi^{-1}(H)$. On the other hand $w \leq y$ and so w is bounded on $\Phi^{-1}(H)$. It follows that $\Phi^{-1}(H)$ is compact and hence that Φ is a proper mapping. It can now be concluded from 1 that Φ is a bijection and this completes the proof. ∎

7. Static solutions

As mentioned in the introduction one of the motivation is to have static solutions included in the class of solutions provided by theorem 4. The fall off conditions of these solutions is given by [7]

$$\rho = \text{const}\, r^{-\frac{1}{(\gamma-1)}}(1+O(r^{-\epsilon})) \quad \text{for} \quad r \to \infty \tag{72}$$

$$\epsilon = \frac{(4-3\gamma)}{(\gamma-1)} \tag{73}$$

Which in the static case results in the following relations

$$\rho(r) \sim r^{-5}\left(1+r^{-2}\right) \qquad \gamma = 1{,}2 \tag{74}$$

$$\rho(r) \sim r^{-3{,}339}\left(1+r^{-0{,}339}\right) \qquad \gamma = 1{,}293003 \tag{75}$$

According to the existence theorem 4 $w \in H_{s,\delta}$ ($-\frac{3}{2} < \delta < -\frac{1}{2}$. As we have already discussed in section 3.1.2, we can estimate ρ by w in $H_{s,\delta}$ using theorem 2. This implies that also $\rho \in H_{s,\delta}$. Now a spherical symmetric function $\rho(r) \sim r^p$ which satisfies these regularity conditions has fall off condition of the form

$$p < -2{,}25 \qquad -\frac{3}{2} < \delta \tag{76}$$

$$p < -1{,}75 \qquad \delta < -\frac{1}{2} \tag{77}$$

And therefore fulfils the requirement of (74) and (75)

Acknowledgment: We would like to thank Alan Rendall for discussion the reconstruction problem of the initial data.

A. Nonlinear estimates

Theorem 7 ***(Nonlinear estimate for power of functions)*** *Let $u \in H_{s,\delta} \cap L^\infty$, $F : \mathbb{R}^l \to \mathbb{R}$ be a C^n map, with $F(0) = 0$ of the form $F(u) = u^\mu$ ($1 \leq \mu$), then there exists a constant C such that*

$$\|u^\mu\|_{H_{s,\delta}} \leq C\left(|u|_{L^\infty}\right)\|u\|_{H_{s,\delta}} \tag{78}$$

In order to prove the lemma above, we will use the following estimate in the case of non weighted Bessel potential spaces, which was proven by Runst and Sickel (prop 1, p 363 ff)[14]

Lemma 3 ***(Non linear estimate in Bessel potential space)***

Let $u \in H_s \cap L^\infty$, $F : \mathbb{R}^l \to \mathbb{R}$ be a C^n map, with $F(0) = 0$ of the form $F(u) = u^\mu$ ($1 \leq \mu$), then there exists a constant C such that

$$\|u^\mu\|_{H^s} \leq C\left(|u|_{L^\infty}\right)\|u\|_{H^s} \tag{79}$$

Proof (of theorem 7) PROOF SKETCH: WE FOLLOW OUR ESSENTIAL IDEA OF USING THE NORM BASED ON THE INFINITE SUM. So we start with.

$$\tilde{\Psi}_k(x) = \frac{1}{\sum \Psi_j(x)}\Psi_k(x) \tag{80}$$

Then the (s, δ) norm reads

$$\|F\|^2_{H_{s,\delta}} = \sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\|\left(\Psi_j^2 F(u)\right)_{2j} F\|^2_{H^s} \tag{81}$$

Now fix j such that

$$\Psi_j^2 F(u) = \Psi_j^2\left(F\left(\sum \tilde{\Psi}_k\right)\right) \tag{82}$$

Since $\mathrm{supp}\Psi_j \cap \mathrm{supp}\Psi_k \neq \emptyset$. Only for $k = j-3, \ldots, j+4$ we have

$$\Psi_j^2\left(F\left(\sum \tilde{\Psi}_k\right)\right) = \Psi_j^2\left(F\left(\sum_{k=j-3}^{j+4}\tilde{\Psi}_k\right)\right) \tag{83}$$

Using now estimate (79) from lemma 3 we obtain:

$$\begin{aligned}
\left\|\left(\Psi_j^2 F(u)\right)_{2^j}\right\|_{H^s} &\leq C\left\|\sum_{k=j-3}^{j+4}\left(\tilde{\Psi}_k u\right)_{2^j}\right\|_{L^\infty}\left\|\sum_{k=j-3}^{j+4}\left(\tilde{\Psi}_k u\right)_{2^j}\right\|_{H^s} && (84)\\
&\leq \left(\|u\|_{L^\infty}\right)\sum_{m=-3}^{4}\left\|\left(\tilde{\Psi}_k u\right)_{2^j}\right\|_{H^s} && (85)
\end{aligned}$$

The last term will be estimated as follows:

$$\begin{aligned}
\left\|\left(\tilde{\Psi}_j u\right))_{2^j}\right\|_{H^s} &= \left\|(f(x)\Psi_{j+m}u)_{2^j}\right\|_{H^s} && (86)\\
&\leq C\left\|(\Psi_{j+m}u)_{2^j}\right\|_{H^s} = C\left\|\left((\Psi_{j+m}u)_{2^{-m}}\right)2^{j+m}\right\|_{H^s} && (87)\\
&\leq C\frac{C\left(2^{-m}\right)}{\left(2^{-m}\right)^{\frac{3}{2}}}\left\|\left((\Psi_{j+m}u)_{2^{-m}}\right)2^{j+m}\right\|_{H^s} && (88)
\end{aligned}$$

Fixing j gives us now

$$\left\|\left(\Psi_j^2 F(u)\right)_{2^j}\right\|_{H^s}^2 \leq C\left(C\left(\|u\|_{L^\infty}\right)\sum_{m=-3}^{4}\left\|\left(\tilde{\Psi}_{j+k}u\right)_{2^j}\right\|_{H^s}^2\right) \tag{89}$$

$$\leq 2C\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\|\left(\Psi_j^2 F(u)\right)_{2j}\|_{H^s}^2 \tag{90}$$

$$\leq 2C\left(\|u\|_{L^\infty}\right)\sum_{m=-3}^{4}\sum_j 2^{\left(\frac{3}{2}+\delta\right)(j+m)}\left\|\left(\tilde{\Psi}_{m+j}u\right)_{2^{j+m}}\right\|_{H^s} \tag{91}$$

$$\leq \left(\|u\|_{L^\infty}\right)\|u\|_{H_{s,\delta}} \tag{92}$$

■

The space we use is characterised by a norm, which makes uses of a infinite sum and has the following form (we assume here $p=2$)

$$\|u\|_{H_{s,\delta}}^2 \sim \sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\|(\psi_j u)\circ\tau_{2^j}\|_{H_s^2} \tag{93}$$

Furthermore we use

Lemma 4 ***(An improved estimate of Gagliardo Nierenberg type for the weighted spaces)***

$$\|u\|_{H_{s',\delta}} \leq \|u\|_{H_{s,\delta}}^{\frac{s'}{s}}\|u\|_{L_{2,\delta}}^{1-\frac{s'}{s}} \tag{94}$$

Proof (of lemma 4) Starting with the definition given by (93) we obtain:

$$\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\|(\psi_j u)\circ\tau_{2^j}\|_{H_s^2} \tag{95}$$

$$\leq \sum_j 2^{\left(\frac{3}{2}+\delta\right)2j\left(\frac{s'}{s}\right)}\|(\psi_j u)\circ\tau_{2^j}\|_{H_s}^{2\frac{s'}{s}} \quad 2^{\left(\frac{3}{2}+\delta\right)2j\left(\frac{s-s'}{s}\right)}\|(\psi_j u)\circ\tau_{2^j}\|_{L_2}^{2\frac{s-s'}{s}} \tag{96}$$

$$\leq \left(\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\|(\psi_j u)\circ\tau_{2^j}\|_{H_s}^2\right)^{\frac{s'}{s}} \quad \left(\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\|(\psi_j u)\circ\tau_{2^j}\|_{L_2}^2\right)^{\frac{s-s'}{s}} \tag{97}$$

$$= \left(\|u\|_{H_{s,\delta}}\right)^{\frac{2s'}{s}} \quad \left(\|u\|_{L_{2,\delta}}\right)^{\frac{2(s'-1)}{s}} \tag{98}$$

B. The density of the C_0^∞ functions

For the local existence theorem we will need that C_0^∞ functions are dense in the $H_{s,\delta}$ spaces.

Theorem 8 ***(Density of C_0^∞ functions in $H_{s,\delta}$)*** *The set of C_0^∞ functions is dense in $H_{s,\delta}$ and furthermore given $s < s'$ and ϵ_0 there is a constant $C(s', \epsilon)$ such that*

$$\|J_\epsilon * u_N\|_{H_{s',\delta}} \leq C(s', \epsilon)\|u\|_{H_{s,\delta}} \tag{99}$$

Where $u_N = \sum_{k=0}^{N} \tilde{\psi}_k u$ and $J_\epsilon \in C_0^\infty$, $0 \leq J(\epsilon) \leq 1$ and $\int J(x)dx = 1$

The proof of this theorem is not complicated however for the sake of brevity we decided to include it in a forthcoming article [1].

C. The Energy estimates

The aim of this section is to derive the essential energy estimates for quasi linear symmetric hyperbolic system in our spaces.

D. Energy estimates in the fractional weighted spaces

We start with some preliminaries:

$$\widehat{(\Lambda^s f)} = \left(1 + |\xi|^2\right)^{\frac{s}{2}} \widehat{f} \tag{100}$$

$$H^s := \{\|\Lambda^s u\|_{L^2}\} < \infty \qquad u \in L^\infty \tag{101}$$

$$\langle u, v\rangle_s = (\Lambda^s u, \Lambda^s v)_{L^2} \tag{102}$$

Where $(u, v)_{L^2} = \int fgdx$

$$\partial_i \Lambda^s u, = \Lambda^s \partial_i u, \qquad u \in H^{s+1} \tag{103}$$

Which follows by taking the Fourier transform let us just mention

Proposition 2 ***(Interpolation property of H^s)*** *Let $s_\theta = \theta s_0 + (1-\theta)s_1$, $0 \leq \theta \leq 1$ then we have*

$$H^{s_{s\theta}} = [H^{s_0}, H^{s_1}] \tag{104}$$

This is a well known property for a proof see for example [15]

Proposition 3 *let $u \in H^s$, $\psi \in C^\infty(\mathbb{R}^n)$, then*

$$\|\psi u\|_{H^s} \leq \|u\|_{H^s} \tag{105}$$

Where C depends on ψ and s.

For $s = k$, $k \in \mathbb{N}$, then H^k is the standard Sobolev space and the estimate (105) can be easily seen, if $s \in \mathbb{R}$ then interpolation has to be used.

Let $\{\psi_j\}$ be a sequence in the definition of the norm $H_{s,\delta}$. That is $\psi_j \in C_0^\infty(\mathbb{R}^n)$,

Proposition 4 ***(A useful estimate)*** *We have*

$$|\partial_i \psi_j(x)| \leq C 2^{-K} \tag{106}$$

Useful is the following weighted space

Definition 8 ***(Weighted L^∞ space)***

$$\|u\|_{L^\infty_\delta} = \sup_{\mathbb{R}^n} \Big((1+|x|)^\delta \, |u(x)| \Big) = \| \, (1+|x|)^\delta \, |u(x)| \|_{L^\infty} \tag{107}$$

Proposition 5 *For $c_0 2^j \leq |x| \leq c_{12^j}$*

$$\sum_j j 2^{\delta j} \sup |u(x)| \approx \|u\|_{L^\infty_\delta} \tag{108}$$

The following theorem will be used

Theorem 9 ***(Kato and Ponce)*** *For $s > 0$, $f \in H^s \cap C^1$, $g \in H^{s-1} \cap L^\infty$ we have*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \leq C \left[\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|f\|_{H^s} \|g\|_{L^\infty} \right] \tag{109}$$

Let us just recall the $H_{s,\delta}$ spaces which cary the following norm and scalar product:

$$\|u\|_{H_{s,\delta}} = \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j^2 u\right)_{2j} \|^2_{H^s} \tag{110}$$

$$\langle u, v \rangle_{H_{s,\delta}} = \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\langle \left(\psi_j^2 u\right)_{2^j}, \left(\psi_j^2 v\right)_{2^j} \right\rangle_s \tag{111}$$

$$= \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\langle \Lambda^s \left(\psi_j^2 u\right)_{2^j}, \Lambda^s \left(\psi_j^2 v\right)_{2^j} \right\rangle^2_L \tag{112}$$

Here $u_\epsilon(x) = u(\epsilon x)$

Remark 6 ***(About the $H_{s,\delta}$ norm and scalar product)*** *$\langle \cdot, \cdot \rangle$ is the inner product over the complex. Hence $\langle v, u \rangle_s = \overline{\langle v, u \rangle_s}$ $\langle v, u \rangle_{s,\delta} = \overline{\langle v, u \rangle_{s,\delta}}$*

Proposition 6 *For any positive γ we have*

$$\|u\|^2_{H_{s,\delta}} \sim \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j^\gamma u\right)_{2j} \|^2_{H^s} \tag{113}$$

Now we come to a central lemma

Lemma 5 ***(Useful energy estimate for the linear symmetric hyperbolic system)*** *Let $v \in C_0^\infty$ and u be a C_0^∞ solution of*

$$u_t = A_0^{-1}(v) \sum A_j(v) \partial_j u + A_0^{-1}(v) B(v) u \tag{114}$$

Then the following estimates holds

$$\frac{d}{dt} \|u\|^2_{H_{s,\delta}} \leq C \left(\|u\|^2_{H_{s,\delta}} + 1 \right) \tag{115}$$

Where C depends on the C^n, $n \in \mathbb{N}$ norm of A_0^{-1}, A_k and B, $\|v\|_{H_{s,\delta}}$, $\|v\|_{L^\infty}$, $\|u\|_{L^\infty}$ and $n > s$.

Proof (of lemma 5) In order to simply the proof we do not treat the terms involving B, since the estimation is very similar to the other terms, moreover we set $A^0 = Id$. The case $A^0 \neq Id$ requires the use of the uniform estimate $C\delta_{\alpha\beta} U^\alpha U^\beta \leq A^0_{\alpha\beta} U^\alpha U^\beta \leq C^{-1} \delta_{\alpha\beta} U^\alpha U^\beta \qquad C \in \mathbb{R}^+$ and we skip the details. We start in traditional way, deriving the norm and insert the

$$\begin{aligned} \frac{d}{dt} \langle u, u \rangle_{s,\delta} &= 2 \langle u_t, u \rangle_{s,\delta} && (116) \\ &= \sum_{i=1}^m s \left\langle A^j \partial_j u, u \right\rangle_{s,\delta} && (117) \\ &= 2 \sum_{i=1}^m \sum_{j=0}^\infty 2^{(\frac{3}{2}+\delta)2j} \left(\Lambda^s (\psi_j^2 A^j \partial_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} \end{aligned}$$

In the following we will just estimate the first term on equation (118) since the other term can be dealt in the same manner.

Now fix j and i we obtain:

$$\begin{aligned} & \left(\Lambda^s (\psi_j^2 A^j \partial_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} && (118) \\ &= \left(\Lambda^s (A^j \psi_j^2 \partial_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} && (119) \\ &= \left(\Lambda^s (\sum_0^\infty (\tilde{\Psi}_k(x))_{2^j} A^j \psi_j^2 \partial_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} && (120) \\ &= \sum_{k=j-3}^{j+4} \left(\Lambda^s ((\tilde{\Psi}_k(x))_{2^j} A^j \psi_j^2 \partial_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} && (121) \\ &= \sum_{k=j-3}^{j+4} \left(\Lambda^s ((\tilde{\Psi}_k(x))_{2^j} A^j)_{2^j} (\psi_j^2 \partial_j u) 2^j - ((\tilde{\Psi}_k(x))_{2^j} \Lambda^s A^j)_{2^j} (\psi_j^2 \partial_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} \\ &+ \sum_{k=j-3}^{j+4} \left(((\tilde{\Psi}_k(x))_{2^j} \Lambda^s A^j)_{2^j} (\psi_j^2 \partial_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} && (122) \\ &= I + II && (123) \end{aligned}$$

Note that in the step from equation (120) to (121) we have used the fact that $\sum\limits_0^\infty \tilde{\psi}_k = 1$.

So we will estimate the terms separately and start with the first one. We fix the index k and use the Cauchy Schwarz inequality and the theorem of Kato and Ponce.

$$
\begin{aligned}
& \left(\Lambda^s((\tilde{\Psi}_k(x))_{2^j}A^j)_{2^j}(\psi_j^2\partial_j u)2^j - ((\tilde{\Psi}_k(x))_{2^j}\Lambda^s A^j)_{2^j}(\psi_j^2\partial_j u)2^j, \Lambda^s(\psi_j^2 u)_{2^j}\right)_{L^2} \\
\leq \ & \left\|\Lambda^s((\tilde{\Psi}_k(x))_{2^j}A^j)_{2^j}(\psi_j^2\partial_j u)2^j - ((\tilde{\Psi}_k(x))_{2^j}\Lambda^s A^j)_{2^j}(\psi_j^2\partial_j u)2^j\right\|_{L^2} \left\|\Lambda^s(\psi_j^2 u)_{2^j}\right\|_{L^2} \\
\leq \ & \left[\|\nabla(\tilde{\Psi}_k(x))_{2^j}A^j)\|_{L^\infty}\|\tilde{\Psi}_k(x))_{2^j}\|_{L^\infty}\|(\tilde{\Psi}_k(x)\partial_j)_{2^j}\|_{H^{s-1}} + \|(\tilde{\Psi}_k(x)A^j)_{2^j}\|_{H^s}\|(\tilde{\Psi}_j^2(x)\partial_j u)_{2^j}\|_{L^\infty}\right]
\end{aligned}
$$

The estimation of the term $\nabla(\tilde{\Psi}_k(x)A^i)_{2^j}$ we start with

$$
\partial_l\left(\tilde{\Psi}_k(x)A^i\right)_{2^j} = 2^j + \left(\partial_l\tilde{\Psi}_k(x)\right)(2^j x)A_i(2^j x) + 2^j\left(\tilde{\Psi}_k(x)(2^j x)\right)(\partial_l A_i)(2^j x) \tag{124}
$$

So we have using 4

$$
\left|2^j\left(\partial_l\tilde{\Psi}_k(x)\right)(2^j x)\right| \leq 2^j 2^{-k} \tag{125}
$$

Hence $\left|2^j\left(\partial_l\tilde{\Psi}_k(x)\right)(2^j x)\right| \leq C$ for $k = j-3, j-2, \ldots, j+4$.

Also

$$
\text{supp}\left\{(\tilde{\psi}_{j-3}), \tilde{\psi}_{j-2}, \ldots \tilde{\psi}_{j+4})\right\} \tag{126}
$$

Hence we have

$$
\left|\partial_l(\tilde{\Psi}_k(x)A^i)\right| \leq C2^j\frac{1}{2}\sup_{d_0 2^j\leq|x|\leq d_1 2^j}|A_j(x)| + \sup_{d_0 2^j\leq|x|\leq d_1 2^j}\left|\partial_l A^i\right| \tag{127}
$$

For the next estimation we need the following fact about the H^s norm

Proposition 7 *Let $f_\epsilon(x) = f(\epsilon x)$, $\epsilon > 0$ then the following estimate holds:*

$$
\|f(\epsilon x)\|_{H^s} \leq \frac{C(\epsilon)}{\epsilon^{\frac{n}{2}}}\|f\|_{H^s} \tag{128}
$$

Where

$$
C(\epsilon) = \begin{cases} 1 & \epsilon \leq 1 \\ \epsilon^s & 1 \leq \epsilon \end{cases}
$$

Estimation of $\|\left(\tilde{\psi}_k A_i\right)\|_{H^s}$

We start with

$$
\tilde{\psi}_k(x) = \psi_k f(x) \qquad \text{where} \quad f(x) = \frac{1}{\sum\psi_j} \tag{129}
$$

It is quite straightforward to see $|\partial^\alpha f| \leq C_\alpha$ for any ϵ. Therefore by proposition 3 we obtain

$$\|(f\psi_k A_i)_{2j}\|_{H^s} \leq C\|(\psi_k A_i)_{2^j}\|_{H^s} \tag{130}$$

Where the constant C depends on f and s.

Now chose $k = j + m$, $m = -3, -2, \ldots, 4$, then

$$(\psi_{j+m} A_i)_{2^j} = \left((\psi_{j+m} A_i)_{2^{-m}}\right)_{2^{j+m}} \tag{131}$$

Now by proposition 7 we obtain

$$\| \left(\psi_{j+m} A_i\right)_{2^j} \|_{H^s} \| \left((\psi_{j+m} A_i)_{2^{-m}}\right)_{2^{j+m}} \|_{H^s} \tag{132}$$

$$\leq \frac{C(2^{-m})}{(2^{-m})^{\frac{3}{2}}} \| \left(\psi_{j+m} A_i\right)_{2^{j+m}} \|_{2^{j+m}} \tag{133}$$

And

$$\sum_{k=j-3}^{j+3} \| \left(\psi_k A_i\right)_{2^j} \|_{H^s} \leq C \sum_{m=-3}^{4} \| \left(\psi_{j+m} A_i\right)_{2^{j+m}} \|_{H^s} \frac{C(2^{-m})}{(2^{-m})^{\frac{3}{2}}} \tag{134}$$

Now we will sum over all the j's using the equation (134) and (127), which we will treat separately:

$$\sum 2^{\left(\frac{3}{2}+\delta\right)2j} \left(\frac{1}{2^j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} |A_j(x)| \right) 2^j \| \left(\psi_j \partial_i u\right)_{2^j} \|_{H^s} \| \left(\psi_j u\right)_{2^j} \|_{H^s} \tag{135}$$

$$= \sum \left(\frac{1}{2^j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} |A_j(x)| \right) \left(2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j \partial_i u\right)_{2^j} \|_{H^{s-1}} \right) \left(2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j u\right)_{2^j} \|_{H^s} \right)$$

We use now the Hölder inequality in order to obtain

$$\leq \left(\sum \left(\frac{1}{2^j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} |A_j(x)| \right)^2 \right)^{\frac{1}{2}} \tag{136}$$

$$\left(\sum \left(2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j \partial_i u\right)_{2^j} \|_{H^{s-1}} \right)^2 \right)^{\frac{1}{4}} \left(\sum \left(2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j u\right)_{2^j} \|_{H^s} \right)^2 \right)^{\frac{1}{4}} \tag{137}$$

$$\leq \left(\sum \left(\frac{1}{2^j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} |A_j(x)| \right)^2 \right) \tag{138}$$

$$\left(\sum \left(2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j \partial_i u\right)_{2^j} \|_{H^{s-1}} \right)^2 \right)^{\frac{1}{2}} \left(\sum \left(2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j u\right)_{2^j} \|_{H^s} \right)^2 \right)^{\frac{1}{2}} \tag{139}$$

Using proposition 3 the last inequality leads to

$$\begin{aligned} &\leq C\|A_i\|_{L^\infty,-1}\|\partial_i u\|_{H_{s-1,\delta+1}}\|u\|_{H_{s,\delta}} &(140)\\ &\leq C\|A_i\|_{L^\infty,-1}\|u\|^2_{H_{s,\delta}} &(141)\end{aligned}$$

The other term goes as follows

$$\sum 2^{\left(\frac{3}{2}+\delta\right)2j}\sup_{d_0 2^j\leq|x|\leq d_1 2^j}|\nabla A_j(x)|2^j\|\left(\psi_j\partial_i u\right)_{2^j}\|_{H^{s-1}}\|\left(\psi_j^2 u\right)_{2^j}\|_{H^s} \tag{142}$$

$$\leq \|\nabla A\|_{L^\infty}\sum 2^{\left(\frac{3}{2}+\delta+1\right)2j}\|\left(\psi_j\partial_i u\right)_{2^j}\|_{H^{s-1}}2^{\left(\frac{3}{2}+\delta\right)2j}\|\left(\psi_j^2 u\right)_{2^j}\|_{H^s} \tag{143}$$

$$\leq \|\nabla A\|_{L^\infty}\left(\sum 2^{\left(\frac{3}{2}+\delta+1\right)2j}\|\left(\psi_j\partial_i u\right)_{2^j}\|^2_{H^{s-1}}\right)^{\frac{1}{2}}\left(\sum 2^{\left(\frac{3}{2}+\delta+1\right)2j}2^{\left(\frac{3}{2}+\delta\right)2j}\|\left(\psi_j^2 u\right)_{2^j}\|^2_{H^s}\right)^{\frac{1}{2}}$$

$$= C\|\nabla A_i\|_{L^\infty,-1}\|\partial_i u\|_{H_{s-1,\delta+1}}\|u\|_{H_{s,\delta}} \tag{144}$$

$$\leq C\|\nabla A_i\|_{L^\infty}\|u\|^2_{H_{s,\delta}} \tag{145}$$

Now fix $m\in\{-3,-2,\ldots,4\}$

$$\sum 2^{\left(\frac{3}{2}+\delta+1\right)2j}\|\left(\tilde{\psi}_{j+m}A_i\right)_{2^j}\|_{H^s}\|\left(\left(\psi_j\partial_j u\right)_{2^j}\right)_{2^{j+m}}\|_{L^\infty}\|\left(\psi_j^2 u\right)_{2^j}\|_{H^s} \tag{146}$$

$$\leq C\sum\frac{C(2^{-m})}{(2^{-m})^{\frac{3}{2}}}2^{\left(\frac{3}{2}+\delta-1\right)2j}\|\left(\tilde{\psi}_{j+m}A_i\right)_{2^j}\|_{H^s}\|\left(\left(\psi_j\partial_j u\right)_{2^j}\right)_{2^{j+m}}\|_{L^\infty}\|\left(\psi_j^2 u\right)_{2^j}\|_{H^s} \tag{147}$$

$$\leq C\frac{C(2^{-m})}{(2^{-m})^{\frac{3}{2}}}\sum 2^{\left(\frac{3}{2}+\delta-1\right)2j}\left\|\left(\tilde{\psi}_{j+m}A_i\right)_{2^j}\right\|_{H^s}2^j\left\|\left(\left(\psi_j\partial_j u\right)_{2^j}\right)_{2^{j+m}}\right\|_{L^\infty}2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}$$

Applying Hölder we obtain

$$\leq C\frac{C(2^{-m})}{(2^{-m})^{\frac{3}{2}}}\left(\sum\left(2^{\left(\frac{3}{2}+\delta-1\right)2j}\left\|\left(\tilde{\psi}_{j+m}A_i\right)_{2^j}\right\|^2_{H^s}\right)^2\right)^{\frac{1}{4}} \tag{148}$$

$$\left(\sum\left(2^j\left\|\left(\left(\psi_j\partial_j u\right)_{2^j}\right)_{2^{j+m}}\right\|_{L^\infty}\right)^2\right)^{\frac{1}{2}}\left(\sum\left(2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^2 u\right)_{2^j}\right\|^2_{H^s}\right)^2\right)^{\frac{1}{4}} \tag{149}$$

$$\leq C\frac{C(2^{-m})}{(2^{-m})^{\frac{3}{2}}}\left(2^{\frac{3}{2}+\delta-1}\right)^{-m}\left(\sum\left(2^{\left(\frac{3}{2}+\delta-1\right)j+m2j}\left\|\left(\tilde{\psi}_{j+m}A_i\right)_{2^j}\right\|^2_{H^s}\right)^2\right)^{\frac{1}{4}} \tag{150}$$

$$\left(\sum\left(2^j\left\|\left(\left(\psi_j\partial_j u\right)_{2^j}\right)_{2^j}\right\|_{L^\infty}\right)^2\right)\|u\|_{H_{s,\delta}} \tag{151}$$

$$\leq C\frac{C(2^{-m})}{(2^{-m})^{\frac{3}{2}}}\|A_i\|_{H_{s,\delta-1}}\|\partial_i u\|_{L^\infty,-11}\|u\|_{H_{s,\delta}} \tag{152}$$

$$\leq C\left(\|A_i\|^2_{H_{s,\delta-1}}\|\partial_i u\|^2_{L^\infty,-1}+\|u\|^2_{H_{s,\delta}}\right) \tag{153}$$

From which we conclude that

$$|I| \leq C\left(\|\nabla A_i\|_{L^\infty} + \|A_i\|_{L^\infty,-1}\right)\|u\|^2_{H_{s,\delta}} + C\left(\|A_i\|^2_{H_{s,\delta-1}}\|\partial_i u\|^2_{L^\infty,-1} + \|u\|^2_{H_{s,\delta}}\right) \quad (154)$$

Estimation of the second term II:

Fix, i, j and k

$$\begin{aligned} 0 &= \int \partial_i\left(\left(\tilde{\psi}_k A_i\right)_{2^j}\Lambda^s\left(\psi_j^2 u\right)_{2^j}, \Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)dx && (155)\\ &= 2^j\left(\partial_i\left(\tilde{\psi}_k A_i\right)_{2^j}\Lambda^s\left(\psi_j^2 u\right)_{2^j}, \Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)_{L^2} && (156)\\ &= \left(\left(\tilde{\psi}_k A_i\right)_{2^j}\partial_i\Lambda^s\left(\psi_j^2 u\right)_{2^j}, \Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)_{L^2} + \left(\left(\tilde{\psi}_k A_i\right)_{2^j}\Lambda^s\left(\psi_j^2 u\right)_{2^j}, \partial_i\Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)_{L^2} \end{aligned}$$

By symmetry of A_i the last term is equal to

$$\left(\Lambda^s\left(\tilde{\psi}_k A_i\right)_{2^j}\left(\psi_j^2 u\right)_{2^j}, \partial_i\Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)_{L^2} \quad (157)$$

In addition we have

$$\begin{aligned} \partial_i\Lambda^s\left(\psi_j^2 u\right)_{2^j} &= \Lambda^s\left(\partial_i\psi_j^2 u\right)_{2^j} && (158)\\ &= 2^j\Lambda^s\left(2\left(\partial_i\psi_j\right)_{2^j}\left(\psi_j u\right)_{2^j}\right)2^j\Lambda^s\left(\left(\psi_j^2\right)_{2^j}\left(\partial_i u\right)_{2^j}\right) && (159) \end{aligned}$$

Summing those terms we have

$$\begin{aligned} &\left(\left(\tilde{\psi}_k A_i\right)_{2^j}\left(\Lambda^s\psi_j^2\partial_i u\right)_{2^j}, \Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)_{L^2} && (160)\\ =\ & -\left(\partial_i\left(\tilde{\psi}_k A_i\right)_{2^j}\left(\Lambda^s\psi_j^2 u\right)_{2^j}, \Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)_{L^2} && (161)\\ & -4\left(\left(\tilde{\psi}_k A_i\right)_{2^j}\left(\Lambda^s\partial_i\psi_j\right)_{2^j}\left(\psi_j u\right)_{2^j}, \Lambda^s\left(\psi_j^2 u\right)_{2^j}\right)_{L^2} && (162)\\ =\ & II_1 + II_2 && (163) \end{aligned}$$

Now

$$\partial_i\left(\tilde{\psi}_k A_i\right)_{2^j} = \partial_i\tilde{\psi}_k A_i + \tilde{\psi}_k\partial_i A_i \quad (164)$$

By proposition 4 we have then

$$\begin{aligned} \left|\partial_i\tilde{\psi}_k A_i\right| &\leq C2^{-k}\sup\left\{|A_i|, x \in \text{supp}\tilde{\psi}_k\right\} && (165)\\ &\leq C2^{-j}\sup_{d_0 2^j \leq |x| \leq d_1 2^j}\left|A^i\right| && (166) \end{aligned}$$

So

$$|II_1| \leq \left(C2^{-j}\sup_{d_0 2^j \leq |x| \leq d_1 2^j}\left|A^i\right| + \sup_{d_0 2^j \leq |x| \leq d_1 2^j}\left|\partial_i A^i\right|\right)\|\left(\psi_j^2 u\right)_{2^j}\|_{H^s} \quad (167)$$

$$|II_2| \leq \sup_{d_0 2^j \leq |x| \leq d_1 2^j}\left|A^i\right|\left\|\left(\left(\partial_j\psi_j\right)_{2^j}\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \quad (168)$$

By algebra we obtain

$$\left\| ((\partial_j \psi_j)_{2^j} \left(\psi_j^2 u\right)_{2^j} \right\|_{H^s} \leq \left\| ((\partial_j \psi_j)_{2^j} \right\|_{H^s} \left\| \left(\psi_j^2 u\right)_{2^j} \right\|_{H^s} \tag{169}$$

Using interpolation one obtains

$$\left\| ((\partial_j \psi_j)_{2^j} \right\|_{H^s} \leq C 2^j \tag{170}$$

Now we have just to pick up our intermediate results and sum the j's.

$$\sum 2^{\left(\frac{3}{2}+\delta\right)2j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} \left|A^i\right| \| \left(\psi_j^2 u\right)_{2^j} \|_{H^s} \tag{171}$$

$$\leq \left(\sum 2^{\left(\frac{3}{2}+\delta\right)2j} \left(\sup_{d_0 2^j \leq |x| \leq d_1 2^j} \left|A^i\right| \right)^2 \right)^{\frac{1}{2}} \left(\sum \left(2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j^2 u\right)_{2^j} \|_{H^s} \right)^2 \right)^{\frac{1}{2}} \tag{172}$$

$$\leq \left(\sum 2^{\left(\frac{3}{2}+\delta\right)2j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} \left|A^i\right| \right) \left(\sum 2^{\left(\frac{3}{2}+\delta\right)2j} \| \left(\psi_j^2 u\right)_{2^j} \|_{H^s} \right) \tag{173}$$

$$= \|A\|_{L^\infty,-1} + \|u\|_{H_{s,\delta}} \tag{174}$$

Furthermore

$$\left(\sum 2^{\left(\frac{3}{2}+\delta\right)2j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} \left|\partial_i A^i\right| \right) \| \left(\psi_j^2 u\right)_{2^j} \|_{H^s} \tag{175}$$

$$\leq \|\partial_i A\|_{L^\infty} \|u\|_{H_{s,\delta}}^2 \tag{176}$$

Moreover

$$\left| \sum 2^{\left(\frac{3}{2}+\delta\right)2j} \left((\tilde{\Psi}_k(x))_{2^j} \Lambda^s (\partial_i \psi)_{2^j} (\psi_j u) 2^j, \Lambda^s (\psi_j^2 u)_{2^j} \right)_{L^2} \right| \tag{177}$$

$$\leq \sum 2^{\left(\frac{3}{2}+\delta\right)2j} \sup_{d_0 2^j \leq |x| \leq d_1 2^j} |A_i| \| (\partial_i \psi)_{2^j} (\psi_j u) 2^j, (\psi_j^2 u)_{2^j} \|_{H^s} \tag{178}$$

$$\leq \sum \frac{1}{2} j \sup_{d_0 2^j \leq |x| \leq d_1 2^j} |A_i| 2^{\left(\frac{3}{2}+\delta\right)j} \|(\psi_j u)_{2^j}\|_{H^s} \quad 2^{\left(\frac{3}{2}+\delta\right)j} \|(\psi_j^2 u)_{2^j}\|_{H^s} \tag{179}$$

Again using Hölder we obtain

$$\leq \left(\sum \left(\frac{1}{2} j \sup_{d_0 2^j \leq |x| \leq d_1 2^j} |A_i| \right)^2 \right)^{\frac{1}{2}} \tag{180}$$

$$\left(\sum \left(2^{\left(\frac{3}{2}+\delta\right)j} \|(\psi_j u)_{2^j}\|_{H^s}^2 \right)^2 \right)^{\frac{1}{4}} \quad \left(\sum \left(2^{\left(\frac{3}{2}+\delta\right)j} \|(\psi_j^2 u)_{2^j}\|_{H^s}^2 \right)^2 \right)^{\frac{1}{4}} \tag{181}$$

$$\leq \|A\|_{L^\infty,-1} \|u\|_{H_{s,\delta}}^2 \tag{182}$$

Thus

$$|II| \leq C \left(\|A\|_{L^\infty,-1} + \|\nabla A\|_{L^\infty} \right) \|u\|_{H_{s,\delta}}^2 \tag{183}$$

E. Local existence for hyperbolic equations

This section is devoted to present a local in time existence for quasi linear symmetric hyperbolic system in the $H_{s,\delta}$ spaces. Our approach is very close to the classical one as can be found in Majda [9], for a slightly different approach see for example, Taylor [15].

We consider the quasi linear (uniform) symmetric hyperbolic system of the form

$$A^0(t,x,u)\partial_t u + \sum_{i=1}^{n} A^i(t,x,u)\partial_i u + B(t,x,u) = 0 \tag{184}$$

where A^0 satisfies the condition

$$C\delta_{\alpha\beta}U^\alpha U^\beta \leq A^0_{\alpha\beta}U^\alpha U^\beta \leq C^{-1}\delta_{\alpha\beta}U^\alpha U^\beta \qquad C \in \mathbb{R}^+ \tag{185}$$

For which we want to prove a local in time existence theorem.

E.1. Strategy

We will proceed with the following strategy.

1. A proof of existence and uniqueness of the corresponding linear system. The linear system is achieved by *freezing* the coefficients, see below.
2. *Cutoff:* At first place the data and the coefficient are cutoff in the sense that they will be approximated by C_0^∞ functions, which are dense in the function spaces $H_{s,\delta}$, see theorem 8.
3. Starting from the solution of the linear equation, a suitable iteration is constructed. This iteration has the well known particularity that the boundness of the iteration is shown in the norm of order s say, while the convergence of the iteration is shown in the *weaker* $s-1$ norm.

E.2. Construction of the iteration

The initial data u_0 will be approximated by a sequence $\{u_0^{k+1}\}$ of smooth functions with compact support, which converge in $H_{s,\delta}(\mathbb{R}^n)$ to u_0. The iteration is then defined as follows: if u^k is given then u^{k+1} is solution of the equation

$$A^0(t,x,u^k)\partial_t u^{k+1} + \sum_{i=1}^{n} A^i(t,x,u^k)\partial_i u^{k+1} + B(t,x,u^k)u^{k+1} = 0 \tag{186}$$

Now to the formal details of the iteration. Let u_0 be a function with values in $\mathbb{R}^k$, belonging to the Sobolev space $H_{k,\delta}(\mathbb{R}^n)$. Therefore there exists a sequence u_0^{k+1} in $C_0^\infty(\mathbb{R}^n)$ with $\|u_0^{k+1} - u_0\|_{H_{k,\delta}} \to 0$ for $k \to \infty$. Let u^0 be a function on $\mathbb{R} \times \mathbb{R}^n$ defined by $u^0(t,x) = u_0^0(x)$. Define recursively a sequence u^{k+1}: the range of definition of u^{k+1} is $[0, T_k)$, where

$$T_k = \sup\{0 < t \leq T_k : u^k([0,t) \times \mathbb{R}^n) \subset G\} \tag{187}$$

The function u^{k+1} is the unique solution of (186) with initial data u_0^{k+1}, which exist according to theorem 10. Every function u^{k+1} is smooth and has support on every closed subinterval of $[0, T_k)$ which is included in a region of the form $|x| < C$. This allows us to consider integration by parts and the interchange of integration and differentiation.

Essential is therefore the following existence theorem for linear systems:

Theorem 10 ***(Existence of classical solutions of a linear symmetric–hyperbolic system)*** *Let u_0 be a smooth initial datum with compact support for the linear symmetric–hyperbolic system of the form*

$$A^0(t,x,v)\partial_t u + \sum_{i=1}^n A^i(t,x,v)\partial_i u + B(t,x,v)u = 0 \tag{188}$$

Let $A^0 - Id$, A^i, B_1 and B_2 smooth with compact support on every finite time interval, then there exists a unique smooth solution with given initial data on the interval $(-\infty, \infty)$.

For the proof we refer to John [8] and Evans [5].

An important tool is the energy estimate as given by lemma 5

For given $u_o \in H_{s,\delta}$, we take $u_0^0 \in C_0^\infty(\mathbb{R}^3)$ to obtain

$$\|u_0^0\|_{H_{s+1,\delta}} \leq C(1 + \|u_o\|_{H_{s,\delta}}) \tag{189}$$
$$\|u_0^0 - u_0\|_{H_{s+1,\delta}} \leq 1 \tag{190}$$

And by (189) $\{u_0^k\} \subset C_0^\infty(\mathbb{R}^3)$ furthermore

$$\|u_0^k - u_0\|_{H_{s+1,\delta}} \leq 2^{-k} \tag{191}$$

Where we set $u^0 = u_0^0$ and as stated u^k is a solution of (186). Furthermore we frequently write $R = \|u_0\|_{H_{s,\delta}}$.

E.3. Boundness in the $H_{s,\delta}$ norm

The main result of this subsection is

Lemma 6 ***(Boundness in the $H_{s,\delta}$ norm)*** *There exists a T^* such that*

$$\|u_0^k - u_0\|_{H_{s+1,\delta}} \leq 4\|u_o\|_{H_{s,\delta}} \qquad 0 \leq t \leq T^* \tag{192}$$

Proof (of lemma 6) The proof is based on finite induction. We denote $V^{k+1} = u^{k+1} - u_0^0$. Inserting it in the equation (186) we obtain.

$$A^0(t,x,u^k)\partial_t u^{k+1} + \sum_{i=1}^{n} A^i(t,x,u^k)\partial_i u^{k+1} + B(t,x,u^k) = \quad (193)$$

$$A^0(t,x,u^k)\partial_t V^{k+1} + \sum_{i=1}^{n} A^i(t,x,u^k)\partial_i u^{k+1} + B(t,x,u^k) = \quad (194)$$

$$A^0(t,x,u^k)\partial_t V^{k+1} + \sum_{i=1}^{n} A^i(t,x,u^k)\partial_i V^{k+1} + B(t,x,u^k) + \quad (195)$$

$$\sum_{i=1}^{n} A^i(t,x,u^k)\partial_i u_0^0 + B(t,x,u^k)u_0^0 \quad (196)$$

By assumption we have

$$\|u^k\|_{H_{s,\delta}} \leq \|u^k - u_0^0\|_{H_{s,\delta}} + \|u_0^0\|_{H_{s,\delta}} \leq 4\|u_0\|_{H_{s,\delta}} + \|u_0\|_{H_{s,\delta}} + 1 \quad (197)$$

Using the embedding property for $\frac{3}{2} < s-1$ and $-\frac{3}{2} \leq \delta$, namely

$$\|\nabla u^k\|_{L^\infty} \leq C\|\nabla u^k\|_{H_{s-1,\delta+1}} \leq C\|u^k\|_{H_{s,\delta}} \quad (198)$$

In order to obtain the energy estimates of equation (193), we have besides using lemma 5 is to multiply equation 193 by A_0^{-1} and estimates the following terms

$$\| \left(A^0(u^k)\right)^{-1} A^i(u^k)\partial_i u_0^0\|_{H_{s,\delta}} \quad (199)$$

$$\leq \| \left(A^0(u^k)\right)^{-1} \|_{H_{s,\delta}} \|A^i(u^k)\|_{H_{s,\delta}} \|\partial_i u_0^0\|_{H_{s,\delta}} \quad (200)$$

$$\leq C\left(\|u^k\|_{L^\infty}\right)\|u^k\|_{H_{s,\delta}} C\left(\|u^k\|_{L^\infty}\right)\|u^k\|_{H_{s,\delta-1}} \|u_0^0\|_{H_{s+1,\delta}} \quad (201)$$

$$\leq C\left(\|u^k\|_{L^\infty}\right)\|u^k\|^2_{H_{s,\delta}} \|u_0^0\|_{H_{s+1,\delta}} \quad (202)$$

$$\leq C(\|u_0\|_{H_{s,\delta}}) \quad (203)$$

Furthermore we have to consider

$$\| \left(A^0(u^k)\right)^{-1} B(u^k)u_0^0\|_{H_{s,\delta}} \quad (204)$$

$$\leq \| \left(A^0(u^k)\right)^{-1} \|_{H_{s,\delta}} \|B(u^k)\|_{L^\infty} \|u_0^0\|_{H_{s,\delta}} \quad (205)$$

$$\leq C(\|u_0\|_{H_{s,\delta}}) \quad (206)$$

Applying lemma 5 we obtain

$$\frac{d}{dt}\|V^{k+1}\|^2_{H_{s,\delta}} \leq C(\|u_0\|_{H_{s,\delta}})\left(\|V^{k+1}\|^2_{H_{s,\delta}}\right) \quad (207)$$

Applying Gronwall's inequality results in

$$\begin{aligned}\|V^{k+1}\|^2_{H_{s,\delta}} &\leq e^{C(\|u_0\|_{H_{s,\delta}})t}\left(\|V^{k+1}(0)\|^2_{H_{s,\delta}}+1\right) &(208)\\ &\leq e^{C(\|u_0\|_{H_{s,\delta}})t}\left(\left(2^{-k}\right)^2+1\right)\leq\left(4\|u_0\|_{H_{s,\delta}}\right)^2 &(209)\\ & &(210)\end{aligned}$$

The last inequality holds if,

$$t\leq\frac{\log(4\|u_0\|_{H_{s,\delta}})^2}{C(\|u_0\|_{H_{s,\delta}})}=:T^* \tag{211}$$

■

Lemma 7 ***(A bound for*** $\|u_t^k\|_{s-1,\delta}$ ***)*** *There exists a constant* $L(\|u_0\|_{H_{s,\delta}})$*, depending on the initial data such that*

$$\|u_t^k\|^2_{H_{s-1,\delta}}\leq L(\|u_0\|_{H_{s,\delta}})\qquad 0\leq t\leq T^* \tag{212}$$

For all k*.*

Proof (of lemma 7) We start with the equation

$$\partial_t u^{k+1}+\left(A^0(u^k)\right)^{-1}\sum_{i=1}^{n}A^i(u^k)\partial_i u^{k+1}+\left(A^0(u^k)\right)^{-1}B(u^k)=0 \tag{213}$$

By algebra and estimate (197) we obtain

$$\begin{aligned}\|\left(A^0(u^k)\right)^{-1}A^i(u^k)\partial_i u^{k+1}\|^2_{H_{s-1,\delta}} &\leq \|\left(A^0(u^k)\right)^{-1}A^i(u^k)\|^2_{H_{s-1,\delta-1}}\|\partial_i u^{k+1}\|^2_{H_{s-1,\delta+1}} &(214)\\ &\leq \|\left(A^0(u^k)\right)^{-1}A^i(u^k)\|^2_{H_{s-1,\delta}}\|u^{k+1}\|^2_{H_{s,\delta}} &(215)\\ &\leq C\left(\|u^k\|_{L^\infty}\right)\|u^k\|^2_{H_{s,\delta}}\|u^{k+1}\|^2_{H_{s,\delta}} &(216)\\ &\leq L(R) &(217)\end{aligned}$$

In a similar way we estimate the other terms

In order to proceed we define

$$|||u|||_{s,\delta,T}=\sup_{0\leq t\leq T}\|u\|_{H_{s,\delta}} \tag{218}$$

There might be missing something:

We have shown that

$$\left\{u^k\right\}\subset L^\infty\left([0,T^*;H_{s,\delta}]\right)\qquad\text{where}\quad \frac{3}{2}+1<s\quad -\frac{3}{2}\leq\delta \tag{219}$$

E.4. Contraction in the lower norm

We show here that $\{u^k\}$ has a contraction property in $\|\cdot\|_{s,\delta,T^{**}}$ for a positive T^{**}. Since

$$\|u\|_{H_{0,\delta}} \sim \|u\|_{L^2_\delta} = \left(\int \sigma^{2\delta}|u|^2 dx\right)^2 \tag{220}$$

It will be convenient to work in $\|u\|_{L^2_\delta}$ and in the following we introduce

$$\|u\|_{\delta,T} = \sup_{0\le t\le T} \|u\|_{L^2_\delta} \tag{221}$$

The following lemma establish the existence of T^{**}

Lemma 8 ***(Existence of a contraction)*** *There is a positive T^{**}, $0<\Lambda<1$ and a positive sequence $\{\beta_k\}$ with $\sum\beta_k<\infty$ such that*

$$\|u^{k+1}-u^k\|_{\delta,T^{**}} \le \Lambda\|u^k-u^{k-1}\|_{\delta,T^{**}} + \beta_k \tag{222}$$

Proof (of lemma 8) Let $0\le t$, using the definition of $L_{2,\delta}$then we consider

$$\frac{d}{dt}\|u(t)\|^2_{L_{2,\delta}} = \frac{d}{dt}\int \sigma^{2\delta}\left(u(t)\right)^2 dx \tag{223}$$

$$= 2\int \sigma^{2\delta}\left(u_t(t)u(t)\right) dx \tag{224}$$

$$\frac{d}{dt}\|u^{k+1}(t)-u^k(t)\|^2_{L_{2,\delta}} = \frac{d}{dt}\int \sigma^{2\delta}\left(u^{k+1}(t)-u^k(t)\right)^2 dx \tag{225}$$

Since $\{u^k\}$ is a solution of (213) we have

$$\begin{aligned} A^0(u^k)\left(u^{k+1}(t)-u^k(t)\right)_t &= A^0(u^k)\left(u^{k+1}(t)\right)_t - A^0(u^{k-1})\left(u^k(t)\right)_t \qquad (226)\\ &\quad -\left(A^0(u^k)-A^0(u^{k-1})\right)u^k_t \\ &= \sum_{i-1}^n A^i(u^k)\partial_i u^{k+1} - A^i(u^{k+1})\partial_i u^k \qquad (227)\\ &\quad +B(u^k)u^{k+1} - B(u^{k+1})u^k - \left(A^0(u^k)-A^0(u^{k-1})\right)u^k_t \\ &= \sum_{i-1}^n A^i(u^k)\left(\partial_i u^{k+1}-\partial_i u^k\right) + B(u^k)\left(u^{k+1}-u^k\right) \qquad (228)\\ &\quad \sum_{i-1}^n \left(A^i(u^k)-A^i(u^{k-1})\right)\partial_i u^k + \left(B(u^k)-B(u^{k-1})\right)u^k \\ &\quad -\left(A^0(u^k)-A^0(u^{k-1})\right)u^k_t \end{aligned}$$

Multiplying now by $\left(A^0\right)^{-1}$ gives

$$\begin{aligned}\left(u^{k+1}(t)-u^k(t)\right)_t &= \left(A^0\right)^{-1}\sum_{i-1}^{n} A^i(u^k)\left(\partial_i u^{k+1}-\partial_i u^k\right)+\left(A^0\right)^{-1}B(u^k)\left(u^{k+1}-u^k\right) \quad (229)\\ &\left(A^0\right)^{-1}\sum_{i-1}^{n}\left(A^i(u^k)-A^i(u^{k-1})\right)\partial_i u^k+\left(A^0\right)^{-1}\left(B(u^k)-B(u^{k-1})\right)u^k\\ &-\left(A^0\right)^{-1}\left(A^0(u^k)-A^0(u^{k-1})\right)u_t^k\\ &= E_1+\ldots+E_5 \quad (230)\end{aligned}$$

Which we will estimate by each term separately:

Let us start with E_1**:** Using again the definition of $L_{2,\delta}$ we consider:

$$\int\left(\sigma^{2\delta}\left(\widetilde{A}(v)\partial_i u\right)u\right)dx \quad (231)$$

for the symmetric matrix $\widetilde{A}(v)$, integration by parts gives:

$$\begin{aligned}2\int\sigma^{2\delta}\left(\left(\widetilde{A}(v)\partial_i u\right)u\right)dx &= \int\partial_i\left(\sigma^{2\delta}\left(\widetilde{A}(v)\partial_i u\right)u\right)dx \quad (232)\\ &-\int\partial_i\left(\sigma^{2\delta}\right)\left(\widetilde{A}(v)\partial_i u\right)udx\\ &-\int\sigma^{2\delta}\partial_i\left(\left(\widetilde{A}(v)u\right)u\right)dx\end{aligned}$$

We use the fact that

$$\|\partial_i\sigma^{2\delta}| \leq \sigma^{2\delta-1} \quad (233)$$

Proceeding

$$\begin{aligned}\left|\int\partial_i\left(\sigma^{2\delta}\right)\left(\widetilde{A}(v)\partial_i u\right)udx\right| &\leq \int\sigma^{2\delta-1}\left(\widetilde{A}(v)\partial_i u\right)udx \quad (234)\\ \leq&\ \left(\int\sigma^{2\delta-1}\left(\widetilde{A}(v)\partial_i u\right)udx\right)^{\frac{1}{2}}\left(\int\sigma^{2\delta}|u|^2\right)^{\frac{1}{2}}\\ \leq&\ \left\|\frac{\widetilde{A}}{\sigma}\right\|_{L^\infty}\|u\|^2_{L_{2,\delta}} \quad (235)\end{aligned}$$

Now

$$\begin{aligned}&\left|\int\sigma^{2\delta}\partial_i\left(\left(\widetilde{A}(v)u\right)\right)\right| \quad (236)\\ \leq&\ \int\sigma^{2\delta}\left|\left(\left(\partial_i\widetilde{A}(v)u\right)\right|^2\right)^{\frac{1}{2}}\left(\int\sigma^{2\delta}|u|^2dx\right)^{\frac{1}{2}} \quad (237)\\ \leq&\ \|\partial_i\widetilde{A}(v)\|_{L^\infty}\|u\|^2_{L_{2,\delta}} \quad (238)\end{aligned}$$

Now by Lemma 6 we have

$$\|u^k\|_{H_{2,\delta}} \le \|u^k - u_0^0\|_{H_{2,\delta}} - \|u_0^0\|_{H_{2,\delta}} \le 4R + R + 1 \tag{239}$$

And by the embedding we obtain

$$\|u\|_\infty \le C\|u\|_{H_{s-1,\delta}} \qquad \frac{3}{2} < s \qquad -\frac{3}{2} \le \delta \tag{240}$$

And so

$$\begin{aligned} \|\partial_i u\|_\infty &\le \|\sigma \partial_i u\|_\infty \le C\|\partial_i u\|_{H_{s-1,\delta+1}} & (241)\\ &\le Cu\|_{H_s,\delta} \qquad \frac{3}{2} < s-1 \qquad -\frac{3}{2} \le \delta + 1 & (242) \end{aligned}$$

By applying the above to the terms $u^{k+1} - u^k$ we obtain

$$\begin{aligned} &\left|2\int \sigma^{2\delta}\left(A^0(u^k)\right)^{-1}\left(A^i(u^k)\right)\left(\partial_i u^{k+1} - \partial_i u^k\right)\left(u^{k+1} - u^k\right)\right| & (243)\\ \le &\left(\left\|\frac{\left(A^0(u^k)\right)^{-1}\left(A^i(u^k)-\right)}{\sigma}\right\|_{L^\infty} + \left\|\partial_i\left(A^0(u^k)\right)^{-1}\left(A^i(u^k)-\right)\right\|_{L^\infty}\right) + \left\|\left(u^{k+1}-u^k\right)\right\|_{L_{2,\delta}}^2\\ \le &\; C_1(R)\left\|\left(u^{k+1}-u^k\right)\right\|_{L_{2,\delta}}^2 \end{aligned}$$

Where $C_1(R)$ depends also on $\sup|A_i(v)|$, $\sup\left|(A^0)^{-1}\right|$. Now to the next term E_2 we have

$$\begin{aligned} &\left|\int \sigma^{2\delta}\left(A^0(u^k)\right)^{-1}\left(B(u^k)\right)\left(u^{k+1}-u^k\right)\left(u^{k+1}-u^k\right)\right| & (244)\\ \le &\left(\int \sigma^{2\delta}\left|\left(A^0(u^k)\right)^{-1}\left(B(u^k)\left(u^{k+1}-u^k\right)\right)\right|^2\right)^{\frac{1}{2}}\left(\int \sigma^{2\delta}\left|\left(u^{k+1}-u^k\right)\right|^2\right)^{\frac{1}{2}} & (245)\\ \le &\left\|\left(A^0(u^k)\right)^{-1}B\left((u^k)\right)\right\|_{L^\infty}\left\|\left(u^{k+1}-u^k\right)\right\|_{L_{2,\delta}}^2 & (246)\\ \le &\; C_2(R)\left\|\left(u^{k+1}-u^k\right)\right\|_{L_{2,\delta}}^2 & (247) \end{aligned}$$

Where $C_2(R)$ depends on the sup norm of $\left(A^0(u^k)\right)^{-1}$ and B.

The term E_3: Let $F \in C^\infty$, then

$$\begin{aligned} \|F(u)-F(v)\|_{L_{2,\delta}} &= \left(\sigma^{2\delta}\left(\int_0^1 \nabla F\left(su+(1-s)v\right)(u-v)ds\right)^2 dx\right)^{\frac{1}{2}} & (248)\\ &\le \|\nabla F\|_{L^\infty}\|u-v\|_{L_{2,\delta}} & (249) \end{aligned}$$

So

$$\left| \int \sigma^{2\delta} \left(A^0(u^k) \right)^{-1} \left(A_i(u^k) - A_i(u^{k-1}) \right) \left(\partial_i u^k \right) \left(u^{k+1} - u^k \right) dx \right| \tag{250}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \left(A_i(u^k) - A_i(u^{k-1}) \right) \left(\partial_i u^k \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{251}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \right\|_{L^\infty} \left\| \partial_i u^k \right\|_{L^\infty} \left\| \left(A_i(u^k) - A_i(u^{k-1}) \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{252}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \right\|_{L^\infty} \left\| \partial_i u^k \right\|_{L^\infty} \left\| (\nabla A_i) \right\|_{L^\infty} \left\| \left(u^k - u^{k-1} \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{253}$$

$$\leq C_3(R) \left(\left\| \left(u^k - u^{k-1} \right) \right\|_{L_{2,\delta}}^2 \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}}^2 \right) \tag{254}$$

Note that we need in this step $\frac{3}{2} < s - 1$

The term E_4

$$\left| \int \sigma^{2\delta} \left(A^0(u^k) \right)^{-1} \left(B(u^k) - B(u^{k-1}) \right) \left(u^k \right) \left(u^{k+1} - u^k \right) dx \right| \tag{255}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \left(B(u^k) - B(u^{k-1}) \right) \left(u^k \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{256}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \right\|_{L^\infty} \left\| u^k \right\|_{L^\infty} \left\| \left(B(u^k) - B(u^{k-1}) \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{257}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \right\|_{L^\infty} \left\| u^k \right\|_{L^\infty} \left\| (\nabla B) \right\|_{L^\infty} \left\| \left(u^k - u^{k-1} \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{258}$$

$$\leq C_4(R) \left(\left\| \left(u^k - u^{k-1} \right) \right\|_{L_{2,\delta}}^2 \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}}^2 \right) \tag{259}$$

And now to the final term E_5. By Lemma 7 we have

$$\left\| \partial_t u^k \right\|_{L^\infty} \leq \left\| \partial_t u^k \right\|_{H_{s-1,\delta}} \leq L(R) \tag{260}$$

Therefore

$$\left| \int \sigma^{2\delta} \left(A^0(u^k) \right)^{-1} \left(A_0(u^k) - A_0(u^{k-1}) \right) \left(\partial_t u^k \right) \left(u^{k+1} - u^k \right) dx \right| \tag{261}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \left(A_0(u^k) - A_0(u^{k-1}) \right) \left(\partial_t u^k \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{262}$$

$$\leq \left\| \left(A^0(u^k) \right)^{-1} \right\|_{L^\infty} \left\| \partial_t u^k \right\|_{L^\infty} \left\| \left(A_0(u^k) - A_0(u^{k-1}) \right) \right\|_{L_{2,\delta}} \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}} \tag{263}$$

$$\leq C_5(R) \left(\left\| \left(u^k - u^{k-1} \right) \right\|_{L_{2,\delta}}^2 \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}}^2 \right) \tag{264}$$

Put $f_k = \left\| \left(u^k - u^{k-1} \right) \right\|_{L_{2,\delta}}^2$ and $g_k = \left\| \left(u^{k+1} - u^k \right) \right\|_{L_{2,\delta}}^2$ we showed that

$$f_k' \leq K_1(R) f_k + K_2(R) g_k \tag{265}$$

Where the constants K_1 and K_2 contains sums of the maximus of the relevant constants. Multiply the last equation by e^{-K_1Rt} we obtain

$$\left(e^{-K_1Rt}f_k'\right)' \leq e^{-K_1Rt}K_2(R)g_k \tag{266}$$

Integration leads to

$$e^{-K_1Rt}f_k - f_k(0) \leq \int_0^t e^{-K_1Rs}K_2(R)g_k ds \tag{267}$$

$$\leq tK_2(R) \sup_{o\leq s\leq t} g(s) \tag{268}$$

Or

$$f_k \leq e^{K_1(R)t}f_k(0) + e^{K_1(R)t}tK_2(R) \sup_{o\leq s\leq t} g(s) \tag{269}$$

Take T^{**} such that $\alpha^2 = T^{**}K_2(R)e^{-K_1(R)T^{**}} < 1$, then we have

$$\begin{aligned} e^{K_1RT^{**}}f_k(0) = e^{-K_1RT^{**}}\|u_0^{k+1} - u_0^k\|_{L_{2,\delta}} &\leq 2e^{K_1RT^{**}}\left(\|u_0^{k+1} - u_0\|_{H_{s,\delta}} + \|u_0^k - u_0\|_{H_{s,\delta}}\right) \\ &\leq 2e^{K_1RT^{**}}\left(2^{-2(k+1)} + 2^{-2k}\right) \\ &= 2e^{K_1RT^{**}}\frac{5}{4}2^{-2k} = \beta_k^2 \end{aligned} \tag{270}$$

Which completes the proof of this lemma ■

Lemma 9 ***(Convergence)*** *We have in addition*

$$\sum |||u^{k+1} - u^k|||_{0,\delta,T^{**}} < \infty \tag{271}$$

Proof (of lemma 9) We start with

$$|||u^{k+1} - u^k||| \leq \Lambda|||u^k - u^{k-1}||| + \beta_k \tag{272}$$

$$\leq \Lambda^2|||u^k - u^{k-1}||| + \Lambda\beta_{k-1} + \beta_k \tag{273}$$

$$\ldots \tag{274}$$

$$\leq \Lambda^k|||u^1 - u^0||| + \Lambda^{k-1}\beta_2 + ... + \beta_{k-1} + \beta_k \tag{275}$$

Now

$$\Lambda^{k-1}\beta_1 + \Lambda^{k-2}\beta_2 + ... + \Lambda\beta_{k-1} + \beta_k = a_k$$

Is a therm which appear in the multiplication of tow terms, namely $\sum \Lambda^k$ and $\sum \beta_k$. Since both series converges so $\sum c_k$ converges too. So indeed this gives

$$\sum \|u^{k+1} - u^k\|_{0,\delta,T^{**}} \leq \sum \Lambda^k\|u^1 - u^0\|_{0,\delta,T^{**}} + c_k < \infty$$

■

This lemma implies that $u_k \to u \in L^\infty\left([0,T^*;H_{0,\delta}]\right)$ and u is unique.

By the inequality

$$\|u\|_{H_{s',\delta}} \leq \|u\|_{H_{s,\delta}}^{\frac{s'}{s}} \|u\|_{L_{2,\delta}}^{1-\frac{s'}{s}} \tag{276}$$

for $0 < s' < s$ (See ?? for a proof) and lemma 6 we conclude that $u_k \to u \in H_{s',\delta}$. take $1+\frac{3}{2} < s' <$, then $u_k \to u \in C^1$. It remains to be shown that $u \in L^\infty\left([0,T^*;H_{s,\delta}]\right)$

Remark 7 ***(of lemma 9)*** *We might obtain*

$$u \in L^\infty\left([0,T^*;H_{s,\delta}]\right) \tag{277}$$

By using the Banach Alaglou Theorem, and weak * convergence. However this is not effective for the continuity in the $H_{s,\delta}$ norm, since the spaces used in this argument are no Hilbert spaces

Remark 8 ***(on the remark)*** *It is not stated that it is not possible to use Banach Alaglou in order to obtain equation (277).?*

Proposition 8 ***(some estimates for the contiuity of the norm)*** *Let $s < \frac{s_1+s_2}{2}$, $u \in H^{s_1}$, $v \in H^{s_2}$. Then we have*

$$|\left\langle u,v\right\rangle_s \leq \|u\|_{H^{s_1}}\|u\|_{H^{s_2}} \tag{278}$$

Proof (of proposition 8) The basic idea is to use the Fourier integral

$$\begin{aligned}
|\left\langle u,v\right\rangle_s &= \int \left(1+|\xi|^2\right)^s \widehat{u}(\xi)\widehat{v}(\xi)d\xi && (279)\\
&\leq \int \left(1+|\xi|^2\right)^{\frac{s_1+s_2}{2}} \widehat{u}(\xi)\widehat{v}(\xi)d\xi && (280)\\
&= \int \left(1+|\xi|^2\right)^{\frac{s_1}{2}} \widehat{u}(\xi)\left(1+|\xi|^2\right)^{\frac{s_2}{2}} \widehat{v}(\xi)d\xi && (281)\\
&\leq C\left(\int \left(1+|\xi|^2\right)^{s_1} |\widehat{u}(\xi)|^{\frac{1}{2}}\right) C\left(\int \left(1+|\xi|^2\right)^{s_2} |\widehat{v}(\xi)|^{\frac{1}{2}}\right) && (282)\\
&= \|u\|_{H^{s_1}}\|u\|_{H^{s_2}} && (283)
\end{aligned}$$

■

As a straightforward generalization we present

Proposition 9 ***(some estimates for the contiuity of the norm)*** *Let $s < \frac{s_1+s_2}{2}$, $u \in H_{s_1,\delta}$, $v \in H_{s_2,\delta}$. Then we have*

$$\left|\left\langle u,v\right\rangle_{s,\delta}\right| \leq \|u\|_{H_{s_1,\delta}}\|u\|_{H_{s_2,\delta}} \tag{284}$$

Proof (of proposition 9) We now use

$$\left|\langle u,v\rangle_{s,\delta}\right| \leq \;=\; \sum 2^{\left(\frac{3}{2}+\delta\right)2j}\left|\left\langle (\Psi_j u)_{2^j}\,,(\Psi_j v)_{2^j}\,,\right\rangle\right| \tag{285}$$

$$\leq \sum 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|(\Psi_j u)_{2^j}\right|_{H^{s_1}}\left\|(\Psi_j v)_{2^j}\right|_{H^{s_2}} \tag{286}$$

$$\leq \left(\sum 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|(\Psi_j u)_{2^j}\right|^2_{H^{s_1}}\right)^{\frac{1}{2}}\left(\sum 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|(\Psi_j v)_{2^j}\right|^2_{H^{s_2}}\right)^{\frac{1}{2}} \tag{287}$$

$$= \|u\|_{H_{s_1,\delta}}\|u\|_{H_{s_2,\delta}} \tag{288}$$

■

Lemma 10 ***(Limes for a test function)*** *For any $\phi \in H_{s,\delta}$ we have*

$$\lim_k \left\langle u^k,\phi\right\rangle_{s,\delta} = \left\langle u^k,\phi\right\rangle_{s,\delta} \tag{289}$$

Uniformly for $0 \leq t \leq T^{**}$ and consequently

$$\|u(t)\|_{H_{s,\delta}} \leq \lim_k \inf \|u^k(t)\|_{H_{s,\delta}} \tag{290}$$

Proof (of lemma 10) Let $s' < s < s''$ $s < \frac{s'+s''}{2}$. Let $\phi \in H_{s,\delta}$. For a given $\epsilon > 0$ there is a $\tilde{\phi} \in H_{s'',\delta}$ such that

$$\|\tilde{\phi}\|_{H_{s'',\delta}} \leq C(s'',\epsilon)\left(\epsilon + \|\phi\|_{H_{s,\delta}}\right) \tag{291}$$

$$\text{and} \quad \|\tilde{\phi}-\phi\|_{H_{s,\delta}} \leq \frac{\epsilon}{2\cdot 8R} \tag{292}$$

Now we consider

$$\left\langle u^k-u,\phi\right\rangle_{S,\delta} = \left\langle u^k-u,\tilde{\phi}\right\rangle_{S,\delta} + \left\langle u^k-u,\left(\phi-\tilde{\phi}\right)\right\rangle_{S,\delta} \tag{293}$$

$$= I + II \tag{294}$$

The first term will be estimated like

$$|I| \leq \|u^k-u\|_{H_{s',\delta}}\|\tilde{\phi}\|_{H_{s'',\delta}} \tag{295}$$

$$\leq \|u^k-u\|_{H_{s',\delta}}C(s'',\epsilon)\left(\epsilon+\|\phi\|_{H_{s,\delta}}\right) \tag{296}$$

While the second one is

$$|II| \leq \|u^k-u\|_{H_{s,\delta}}\|\phi-\tilde{\phi}\|_{H_{s,\delta}} \tag{297}$$

$$\leq \left(\|u^k-u^0_0\|_{H_{s',\delta}}-\|u-u^0_0\|_{H_{s',\delta}}\right)\|\phi-\tilde{\phi}\|_{H_{s,\delta}} \leq \frac{8R\epsilon}{16R} = \frac{\epsilon}{2} \tag{298}$$

Since $s' < s$ both estimates for I and II are uniformly in $t \in [0,T^{**}]$. On the other hand since $II < \frac{\epsilon}{2}$ and $\tilde{\phi}$ does not depend on k, there is a k_1 such that

$$\|u^k-u\|_{H_{s',\delta}} \leq C(s'',\epsilon)\left(\epsilon+\|\phi\|_{H_{s,\delta}}\right)^{-1}\frac{\epsilon}{2} \tag{299}$$

■

With this last lemma we can present

Theorem 11 ***(Solution in*** L^∞***)*** *Let* A^0, A^k, B *be the coefficients which define the quasilinear symmetric–hyperbolic system (184) . Let* $U(x,0) \in H_{s,\delta}(\mathbb{R}^3)$ *(*$\frac{5}{2} < s$*) and let the initial conditions be chosen such that the* A^0 *condition (185) is satisfied. Then there exists a* $T > 0$ *which depends on the* $H_{s,\delta}$ *norm of the initial data and there exists a unique solution* $u \in L^\infty\left([0,T^*; H_{s,\delta}]\right) \cap u \in L_{ip}\left([0,T^*; H_{s,\delta}]\right)$ *which is a classical solution of equation (184)*

Finally the regularity conditions of theorem 11 can be improved, resulting in.

Theorem 12 ***(Classical*** C^0 ***solutions of system (184))*** *Let* A^0, A^k, B *be the coefficients which define the quasilinear symmetric–hyperbolic system (184) . Let* $U(x,0) \in H_{s,\delta}(\mathbb{R}^3)$ *(*$\frac{5}{2} < s$*) and let the initial conditions be chosen such that the* A^0 *condition (185) is satisfied. Then there exists a* $T > 0$ *which depends on the* $H_{s,\delta}$ *norm of the initial data and there exists a unique solution* $U(x,t) \in C^0\left([0,T^*; H_{s,\delta}]\right) \cap C^1\left([0,T^*; H_{s-1,\delta+1}]\right)$ *of equation (184).*

Proof (of theorem 12) It is sufficient to show that

$$\lim_{t\to 0^+} \|u(t)\|_{H_{s,\delta}} = \|u(0)\|_{H_{s,\delta}} = \|u_0\|_{H_{s,\delta}} \tag{300}$$

Put $f_k(t) = \|u^k(t)\|_{H_{s,\delta}}$. Now u^k satisfies

$$u_t^{k+1} = \left(A^0\right)^{-1}\left(u^k \sum A_i(u^k)\partial_i u^{k+1} + B(u^k)\right) \tag{301}$$

So by the Energy estimates we have

$$f_k'(t) \le C(R)\left(f_k(t) + 1\right) \tag{302}$$

And thus

$$f_k(t) \le C(R)e^{C(R)t}\left(f_k(0) + C(R)t\right) \tag{303}$$

Given $\epsilon > 0$, there is a k_2 such that

$$\|u_k(t)\|_{H_{s,\delta}} \le e^{\frac{C(R)t}{2}}\left(\left(\|u_0\| + \epsilon\right)^2 + C_R t\right)^{\frac{1}{2}} \tag{304}$$

Or

$$\|u_k(t)\|_{H_{s,\delta}} \le \|u_0\| + \epsilon r(t) \tag{305}$$

For $k \ge k_2$ and where $r(t)$ does not depend on k and $\lim_{t\to 0} r(t) = 0$. By weak convergence for a fixed $t \in [0, T^{**}]$ we consider

$$\|u(t)\|_{H_{s,\delta}} \le \liminf \|u^k(t)\|_{H_{s,\delta}} \le \|u_0\| + \epsilon + r(t) \tag{306}$$

Take $\{t_m\}$ such that $t_m \ge 0$, then (306) implies

$$\limsup\left(|u(t_m)|_{H_{s,\delta}}\right) \le \|u_0\| + \epsilon + \lim r(t_m) \le \|u_0\| + \epsilon \tag{307}$$

Since ϵ is arbitrary we obtain

$$\limsup \|u(t_m)\|_{H_{s,\delta}} \le \|u_0\| \tag{308}$$

Which implies (300). ∎

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