

# SURJECTION AND INVERSION FOR LOCALLY LIPSCHITZ MAPS BETWEEN BANACH SPACES

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ABSTRACT. We study the global invertibility of non-smooth, locally Lipschitz maps between infinite-dimensional Banach spaces, using a kind of Palais-Smale condition. To this end, we consider the Chang version of the weighted Palais-Smale condition for locally Lipschitz functionals in terms of the Clarke subdifferential, as well as the notion of pseudo-Jacobians in the infinite-dimensional setting, which are the analog of the pseudo-Jacobian matrices defined by Jeyakumar and Luc. Using these notions, we derive our results about existence and uniqueness of solution for nonlinear equations. In particular, we give a version of the classical Hadamard integral condition for global invertibility in this context.

## 1. INTRODUCTION

Surjectivity and invertibility of maps is an important issue in nonlinear analysis. In a smooth setting, if  $f : X \rightarrow Y$  is a  $C^1$  map between Banach spaces, such that its derivative  $f'(x)$  is an isomorphism for every  $x \in X$ , from the classical Inverse Function Theorem we have that  $f$  is locally invertible around each point. If, in addition,  $f$  satisfies the so-called *Hadamard integral condition*:

$$\int_0^\infty \inf_{\|x\| \leq t} \|f'(x)^{-1}\|^{-1} dt = \infty,$$

then  $f : X \rightarrow Y$  is globally invertible, and thus a global diffeomorphism from  $X$  onto  $Y$  (see e.g. the paper by Plastock [31] for a proof of this result). This sufficient condition for global invertibility was first considered by Hadamard [15] for maps between finite-dimensional spaces, and has been widely used since then. We refer to the recent survey paper [12] for an extensive information about this and other conditions for global invertibility of smooth maps between Banach spaces and, more generally, between Finsler manifolds.

In a nonsmooth setting, if  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a locally Lipschitz map, Pourciau obtained in [33] and [34] suitable versions of the Hadamard integral condition, using the *Clarke generalized Jacobian*. These results have been extended to the setting of finite-dimensional Finsler manifolds in [22]. For continuous maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which are not assumed to be locally Lipschitz, Jeyakumar and Luc introduced in [24] the concept of *approximate Jacobian matrix*, which was later called *pseudo-Jacobian matrix* (see [25]). A global inversion theorem, with a version of the Hadamard integral condition in this context, is given in [23].

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If  $f : X \rightarrow Y$  is a nonsmooth map between infinite-dimensional Banach spaces, the problem of local invertibility of  $f$  is more delicate. Assuming that  $f$  is a local homeomorphism, F. John obtained in [26] a global inversion theorem with a suitable version of the Hadamard integral condition in terms of the *lower scalar Dini derivative* of  $f$ . Later on, Ioffe obtained in [17] a global inversion result for a continuous map  $f$  which is locally one-to-one, using an analog of Hadamard integral condition, defined in terms of the so-called *constant surjection* of  $f$  at every point. Further results along this line have been obtained in [14] and [10] in the more general setting of maps between metric spaces. More recently, in [21], the authors consider the notion of *pseudo-Jacobian* for a continuous map between Banach spaces, which is an extension of pseudo-Jacobian matrices of Jeyakumar and Luc to this setting, and obtain different global inversion results in this context.

From the purely topological point of view, the Banach-Mazur Theorem states that a local homeomorphism  $f$  between Banach spaces is a global one if and only if it is a *proper* map, that is, the preimage  $f^{-1}$  sends compact sets to compact sets [3]. The proof of this result and the classical global inversion theorems cited above are generally addressed through the use of the path-lifting property or some other similar approach which includes a monodromy argument. Actually, Rabier makes a masterful use of the path-lifting arguments in the smooth setting and get the complete geometric picture leading to a generalization of the classical Ehresmann theorem of differential geometry; see Theorem 4.1 in [35]. In particular, the Banach-Mazur Theorem and the Hadamard integral condition were non-trivially related by Rabier via a sort of “uniform-strong” Palais-Smale condition so called *strong submersion* with uniformly splits kernels.

Some conditions intimately related to strong submersions seem to escape from the monodromy technique. Nevertheless, for functions between metric spaces, Katriel [27] proposes a completely different approach based on an abstract mountain-pass theorem and the Ekeland variational principle, in order to obtain global inversion theorems in a non-smooth setting, via the study of the critical points of the functional  $x \mapsto d(y, f(x))$  for all  $y$  in  $Y$ . Idczak *et al.* [16] set down the Katriel approach in the Banach space setting proving that a local diffeomorphism with a Hilbert target space is a global diffeomorphism if the functional  $F_y(x) = \frac{1}{2}|f(x) - y|^2$  satisfies the Palais-Smale condition for all  $y \in Y$ . We refer to [7] and [8] for further developments in this direction. See also [11], an earlier reference in this line in the finite-dimensional setting.

The generalization of Idczak *et al.* result for functions between two real Banach spaces and the relationship of this condition with the Hadamard integral condition and the strong submersions of Rabier were established in [13] for  $C^1$  maps. However the connection of the Palais-Smale condition for  $F_y$  with the nonsmooth global inversion results cited above for a locally Lipschitz map  $f$  are the pending issues just addressed in this article.

## 2. CALCULUS WITH PSEUDO-JACOBIANS

Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be real Banach spaces and  $U$  be a nonempty open subset of  $X$ . As usual,  $L(X, Y)$  and  $X^*$  will denote the space of bounded linear operators from  $X$  into  $Y$  and the topological dual of  $X$ , respectively.

**Pseudo-Jacobians.** Let  $f : U \rightarrow Y$  be a continuous map. Some important examples of a derivative-like objects for continuous maps can be included in a general frame so called pseudo-Jacobians (see [21]). The definition of a pseudo-Jacobian of  $f$  at a point  $x$  involves an approximation of the “scalarized” functions  $y^* \circ f$ , through all directions  $y^* \in Y$  by means of upper Dini directional derivatives and a sublinearization of the approximations by a set of operators. Recall, if  $\phi : U \rightarrow \mathbb{R}$  is a real-valued function and  $x$  is a point in  $U$ , the *upper right-hand Dini derivative* of  $\phi$  at  $x$  with respect to a vector  $v \in X$  is defined as:

$$\phi'_+(x; v) = \limsup_{t \rightarrow 0^+} \frac{\phi(x + tv) - \phi(x)}{t}.$$

A nonempty subset  $Jf(x) \subset L(X, Y)$  is said to be a *pseudo-Jacobian* of  $f$  at  $x \in U$  if, for every  $y \in Y^*$  and  $v \in X$ :

$$(y^* \circ f)'_+(x; v) \leq \sup\{ \langle y^*, T(v) \rangle : T \in Jf(x) \} \quad (1)$$

A set-valued mapping  $Jf : U \rightarrow 2^{L(X, Y)}$  is called a *pseudo-Jacobian mapping* for  $f$  on  $U$  if for every  $x \in U$  the set  $Jf(x)$  is a pseudo-Jacobian of  $f$  at  $x$ .

The following examples of pseudo-Jacobians for a continuous function  $f : U \rightarrow Y$  between Banach spaces are explained with detail in [21]; see also [24] for the finite-dimensional theory of pseudo-Jacobians.

**Example 1.** If  $f$  is *Gâteaux differentiable* at  $x$ , then the singleton  $Jf(x) := \{df(x)\}$  is a pseudo-Jacobian of  $f$  at  $x$ . In particular, this holds if  $f$  is *Fréchet differentiable* or *strictly differentiable*. Recall, a function  $f$  is strictly differentiable at  $x$  if there is a continuous linear map, denoted by  $df(x)$ , such that for every  $\epsilon > 0$  there is  $\rho > 0$  such that if  $u, w \in B(x; \rho)$  then:

$$\|f(u) - f(w) - df(x)(u - w)\| \leq \epsilon|u - w|.$$

Now suppose that  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^m$ . If  $f$  admits a singleton pseudo-Jacobian at  $x$  then  $f$  is *Gâteaux differentiable* at  $x$  and its derivative coincides with the pseudo-Jacobian matrix. In the infinite-dimensional setting, a continuous map between Banach spaces admits a singleton pseudo-Jacobian at a point  $x$  if, and only if, it is *weakly Gâteaux differentiable* at  $x$ , see p. 23 in [1].

**Example 2.** Suppose  $f$  is a *locally Lipschitz map*, namely, for every  $x \in U$  there exist  $L, r > 0$  such that, whenever  $u, w \in B(x; r) \subset U$ :

$$\|f(u) - f(w)\| \leq L|u - w|$$

Consider the *local Lipschitz constant* of  $f$  at  $x \in U$ , given by:

$$\text{Lip } f(x) = \inf_{r>0} \sup \left\{ \frac{\|f(u) - f(w)\|}{|u - w|} : u, w \in B(x, r) \text{ and } u \neq w \right\}.$$

Then the set  $Jf(x) := \text{Lip } f(x) \cdot \overline{B}_{L(X, Y)}$ , defined as the unit ball centered at zero of radius  $\text{Lip } f(x)$  in the space  $L(X, Y)$ , is a pseudo-Jacobian of  $f$  at  $x$ .

**Example 3.** Let  $f : U \subset X \rightarrow Y$  be a locally Lipschitz map. Suppose that  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^m$ . The *Clarke generalized Jacobian* of  $f$  at  $x$  is a pseudo-Jacobian of  $f$  at  $x$ . Recall that the Clarke generalized Jacobian is equivalent to the generalized Jacobian proposed by Pourciau in [32]. An extension of Clarke generalized Jacobian, enjoying all the fundamental properties desired from a derivative set, was proposed

by Páles and Zeidan in [30] to the case when  $X$  and  $Y$  are infinite-dimensional Banach spaces, and  $Y$  is a dual space satisfying the Radon-Nikodym property, in particular when  $Y$  is reflexive. Let us recall the definition in this case. Given a finite-dimensional linear subspace  $L \subset X$ , we say that  $f$  is *L-Gâteaux-differentiable* at a point  $z \in U$  if there exists a continuous linear map  $D_L f(z) : L \rightarrow Y$  such that

$$\lim_{t \rightarrow 0} \frac{f(z + tv) - f(z)}{t} = D_L f(z)(v), \quad \text{for every } v \in L.$$

Denote by  $\Omega_L(f)$  the set made up of all points  $z \in U$  such that  $f$  is *L-Gâteaux-differentiable* at  $z$ , and let  $\partial_L f(x)$  be the subset of  $\mathcal{L}(L, Y)$  given by the formula

$$\partial_L f(x) := \bigcap_{\delta > 0} \overline{\text{co}}^{WOT} \{D_L f(z) : z \in B(x, \delta) \cap \Omega_L(f)\},$$

where  $\overline{\text{co}}^{WOT}$  denotes the closed convex hull for the weak operator topology on  $\mathcal{L}(X, Y)$ . The *Páles-Zeidan generalized Jacobian of  $f$  at the point  $x$*  is then defined as the set:

$$\partial f(x) = \{T \in \mathcal{L}(X, Y) : T|_L \in \partial_L f(x), \text{ for each finite dimensional subspace } L \subset X\}.$$

In particular, if  $Y$  is reflexive, the Páles-Zeidan generalized Jacobian is indeed a pseudo-Jacobian.

**Example 4.** If  $\phi : U \subset X \rightarrow \mathbb{R}$  is a locally Lipschitz map the *Clarke generalized directional derivative of  $\phi$  at  $x$  with direction  $v$*  is defined by:

$$\phi^\circ(x, v) := \limsup_{z \rightarrow x, t \rightarrow 0^+} \frac{\phi(z + tv) - \phi(z)}{t}$$

It is well known that the map  $v \mapsto \phi^\circ(x; v)$  is convex and continuous. The *Clarke subdifferential of  $\phi$  at  $x$*  is the non-empty  $w^*$ -compact convex subset of  $X^*$  defined as:

$$\partial\phi(x) = \{x^* \in X^* : x^*(v) \leq \phi^\circ(x; v), \quad \text{for all } v \in X\}$$

Suppose that  $Y = \mathbb{R}$ . Then the Clarke subdifferential of  $f$  at  $x$  is a pseudo-Jacobian of  $f$  at  $x$ .

The theory of pseudo-Jacobians includes a sort of mean value theorem, an optimality condition for real-valued functions and some partial results concerning the chain rule (see [21]). In order to get desirable local and global surjection and inversion theorems it is necessary the validity of the chain rule for the composition with distance functions.

**Chain Rule and Strong Chain Rule conditions.** Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be real Banach spaces,  $U$  be an open subset of  $X$  and  $f : U \rightarrow Y$  be a continuous map with a pseudo-Jacobian map  $Jf$ . For every  $y \in Y$ , consider the functional:

$$F_y(x) := |f(x) - y| \tag{2}$$

For every  $x \in U$  and  $y \neq f(x)$ , we shall define the subset of  $X^*$ :

$$\Delta F_y(x) := \partial|\cdot|(f(x) - y) \circ \overline{\text{co}}(Jf(x)).$$

According to [21], we say that that  $Jf$  satisfies the *chain rule condition* on  $U$  if, for each  $x \in U$  and  $y \neq f(x)$ , the set  $\Delta F_y(x)$  is a  $w^*$ -closed and convex subset of  $X^*$  and is a pseudo-Jacobian of the functional  $F_y$ . We shall say that  $Jf$  satisfies the *strong chain rule condition* if, in addition to the above requirements, we also have

that  $f$  is locally Lipschitz and  $\Delta F_y(x)$  contains the Clarke subdifferential of  $F_y$  at  $x$ .

**Example 5.** If  $f : U \subset X \rightarrow Y$  is continuous and Gâteaux differentiable on all of  $U$  then, for every  $x \in U$ , the pseudo-Jacobian map  $Jf(x) = \{df(x)\}$  satisfies the chain rule condition; see Proposition 2.17 of [21]. Furthermore, by Theorem 2.3.10 (Chain Rule II) of [5], we have that if  $f$  strictly differentiable, in particular  $C^1$ , then  $Jf$  satisfies the strong chain rule condition.

**Example 6.** Let  $f : U \subset X \rightarrow Y$  be a locally Lipschitz map, where  $X$  and  $Y$  are reflexive Banach spaces and  $Y$  is endowed with a  $C^1$ -smooth norm. Consider  $Jf(x) := \partial f(x)$  the Páles-Zeidan generalized Jacobian of  $f$  at  $x$ . From Corollary 2.18 of [21] we have that  $Jf$  satisfies the chain rule condition. Furthermore, taking into account that  $\partial f(x)$  is a closed convex subset of  $L(X, Y)$  and using Theorem 5.2 in [30] we deduce that  $Jf$  satisfies in fact the strong chain rule condition. Indeed, given  $y \in Y$  we denote  $g(z) := |z - y|$ . Since  $g$  is  $C^1$  on the set  $\{z \in Y : z \neq y\}$  we have that, for  $f(x) \neq y$ :

$$\partial F_y(x) = \partial(g \circ f)(x) = dg(f(x)) \circ \partial f(x) = \partial|\cdot|(f(x) - y) \circ \overline{\text{co}}(\partial f(x)) = \Delta F_y(x).$$

**Example 7.** Let  $f : U \subset X \rightarrow Y$  be a locally Lipschitz map, where  $X$  and  $Y$  are reflexive Banach spaces and  $Y$  is endowed with a  $C^1$ -smooth norm, and consider the pseudo-Jacobian  $Jf(x) := \text{Lip } f(x) \cdot \overline{B}_{L(X, Y)}$  considered in Example 2. Again from Corollary 2.18 of [21] we have that  $Jf$  satisfies the chain rule condition. Furthermore,  $Jf$  also satisfies the strong chain rule condition. Indeed, given  $y \in Y$ , if we denote  $g(z) := |z - y|$  as before, taking into account that  $\partial f(x) \subset \text{Lip } f(x) \cdot \overline{B}_{L(X, Y)}$  (see Theorem 3.8 in [30]) we have for  $f(x) \neq y$ :

$$\partial F_y(x) = \partial(g \circ f)(x) = dg(f(x)) \circ \partial f(x) \subset \partial|\cdot|(f(x) - y) \circ Jf(x) = \Delta F_y(x).$$

### 3. PSEUDO-JACOBIANS AND LOCAL INVERSE THEOREMS

Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be real Banach spaces,  $U$  be an open subset of  $X$ , and let  $f : U \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  satisfying the chain rule condition. If  $T : X \rightarrow Y$  is a bounded linear operator, we consider its *Banach constant*, see p. 4 in [20]:

$$C(T) = \inf_{|v^*|_{Y^*} = 1} |T^* v^*|_{X^*}.$$

The Banach constant coincides with the quantity  $\sigma(T)$  in [29] and also with the number  $\tau_T$  in [2], see also [35]. Recall, the Banach constant  $C(T)$  is positive if and only if  $T$  is onto; in such case by the Open Mapping Theorem,  $T$  is an open map. There are a large number of nonlinear versions of this openness criterion e.g. for a strictly differentiable map  $f$  (see [6]): if  $C(df(x_0)) > 0$  then  $f$  is open with linear rate around  $x_0$ , namely, there exist a neighborhood  $V$  of  $x_0$  and a constant  $\alpha > 0$  such that for every  $x \in V$  and  $r > 0$  with  $B(x; r) \subset V$ :

$$B(f(x); \alpha r) \subset f(B(x; r)). \quad (3)$$

A natural quantity to consider in the pseudo-Jacobian frame is the following:

$$\text{Sur } Jf(x) = \sup_{r > 0} \inf \{C(T) : T \in \text{co } Jf(B(x; r))\}.$$

Of course, if  $f$  is strictly differentiable and  $Jf(x) := \{df(x)\}$  then we have that  $\text{Sur } Jf(x) = C(df(x))$ .

From the very definition, it is clear that the functional  $\text{Sur } Jf : U \rightarrow [0, \infty)$  is lower semicontinuous. On the other hand, if the set valued map  $Jf : U \rightarrow 2^{L(X,Y)}$  is upper semicontinuous at a point  $x$ , from Proposition 3.4 in [21] we have that

$$\text{Sur } Jf(x) = \inf\{C(T) : T \in \text{co}Jf(x)\}.$$

**Example 8.** Let  $X$  and  $Y$  be reflexive Banach spaces, where  $Y$  is endowed with a  $C^1$ -smooth norm. Consider a map  $f : U \subset X \rightarrow Y$  of the form  $f = f_1 + f_2$ , where  $f_1 : U \rightarrow Y$  is  $C^1$ -smooth and  $f_2 : U \rightarrow Y$  is locally Lipschitz. From Example 5 and Example 7, we see that  $Jf(x) := df_1(x) + \text{Lip } f_2(x) \cdot \overline{B}_{L(X,Y)}$  is a pseudo-Jacobian of  $f$  on  $U$ , satisfying the chain rule condition. Furthermore, it can be checked as before that in fact  $Jf$  satisfies the strong chain condition. Indeed, if we denote  $\eta(z) := |z|$ , using Theorem 5.2, Corollary 5.4 and Theorem 3.8 in [30]) we have that, for  $f(x) \neq y$ :

$$\begin{aligned} \partial F_y(x) &= d\eta(f(x) - y) \circ \partial f(x) = d\eta(f(x) - y) \circ (\partial f_1(x) + \partial f_2(x)) \\ &= d\eta(f(x) - y) \circ (df_1(x) + \partial f_2(x)) \subset \partial|\cdot|(f(x) - y) \circ Jf(x) = \Delta F_y(x). \end{aligned}$$

On the other hand, it is not difficult to check (see Example 2.6 in [21]) that the set-valued map  $Jf : U \rightarrow 2^{L(X,Y)}$  is upper semicontinuous on  $U$ , so from Proposition 3.4 in [21] we obtain that for each  $x$  in  $U$ :

$$\text{Sur } Jf(x) = \inf\{C(T) : T \in Jf(x)\} = \inf\{C(df_1(x) + R) : \|R\| \leq \text{Lip } f_2(x)\}.$$

Now, from Theorem 3.1 of [21] we have:

**Theorem 9 (local openness).** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces,  $U$  be an open subset of  $X$  and let  $f : U \rightarrow Y$  be a continuous map with pseudo-Jacobian  $Jf$  satisfying the chain rule condition on  $U$ . If  $\text{Sur } Jf(x_0) > 0$  then  $f$  is open with linear rate around  $x_0$ . More precisely, for each  $0 < \alpha < \text{Sur } Jf(x_0)$  there is a neighborhood  $V$  of  $x_0$  such that for every  $x \in V$  and  $r > 0$  with  $B(x; r) \subset V$  the inclusion (3) holds.*

As a counterpart to the Banach constant, we consider the *dual Banach constant* of a bounded linear operator  $T : X \rightarrow Y$  (see [20], p. 5) defined by:

$$C^*(T) = \inf_{|u|_X=1} |Tu|_Y.$$

Note that  $C^*(T)$  coincides with  $\|T\|$ , the co-norm of  $T$  considered by Pourciau [33], [34] in a finite-dimensional setting, and also in [21]. If  $C^*(T) > 0$  then  $T$  is one-to-one. Furthermore, if  $T$  is an isomorphism it is not difficult to check that:

$$C(T) = C^*(T) = \|T^{-1}\|^{-1}.$$

The natural quantity to consider in the pseudo-Jacobian frame is the following:

$$\text{Inj } Jf(x) = \sup_{r>0} \inf\{C^*(T) : T \in \text{co}Jf(B(x; r))\}.$$

Using the Mean Value Property given in Theorem 2.7 of [21] and proceeding as in the proof of Lemma 3.8 of [21], we have the following:

**Theorem 10 (local injectivity).** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces,  $U$  be an open subset of  $X$ , and let  $f : U \rightarrow Y$  be a continuous map with pseudo-Jacobian  $Jf$  satisfying the chain rule condition on  $U$ . If  $\text{Inj } Jf(x_0) > 0$  then  $f$  is locally one-to-one at  $x_0$ . More precisely, for each  $0 < \alpha < \text{Inj } Jf(x_0)$  there exists a neighborhood  $V$  of  $x_0$  such that for every  $u, w \in V$  we have:*

$$|f(u) - f(w)| \geq \alpha|u - w|.$$

Combining these previous results we obtain the local inversion result below (see Theorem 3.5 of [21]). It is useful to introduce first the notion of *regularity*.

**Regular pseudo-Jacobian and regularity index.** Let  $f : U \subset X \rightarrow Y$  be a continuous map between Banach spaces. We shall say that the pseudo-Jacobian  $Jf$  is *regular* at a point  $x_0 \in U$  if, for some  $r > 0$ , every operator  $T \in \text{co}Jf(B(x_0; r))$  is an isomorphism and  $\text{Reg} Jf(x_0) > 0$ , where  $\text{Reg} Jf(x_0)$  is the *regularity index* of  $f$  at  $x_0$  defined as:

$$\text{Reg} Jf(x_0) := \text{Sur} Jf(x_0) = \text{Inj} Jf(x_0).$$

**Theorem 11 (Inverse Mapping Theorem).** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces,  $U$  be an open subset of  $X$ ,  $x_0 \in U$ , and let  $f : U \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  satisfying the chain rule condition on  $U$ . Suppose  $Jf$  is regular at  $x_0$ . Then  $f$  is a bi-Lipschitz homeomorphism around  $x_0$ . More precisely, for each  $0 < \alpha < \text{Reg} Jf(x_0)$  there is an neighborhood  $V$  of  $x_0$  contained in  $U$  with the following properties:*

- *The set  $W := f(V)$  is open in  $Y$ .*
- *The map  $f|_V : V \rightarrow W$  is a bi-Lipschitz homeomorphism.*
- *The map  $f|_V^{-1}$  is  $\alpha^{-1}$ -Lipschitz on  $V$ .*

#### 4. METRIC REGULARITY, LIPSCHITZ RATE OF $f^{-1}$ , IOFFE CONSTANT OF SURJECTION AND SOME GLOBAL INVERSION CONDITIONS

Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be real Banach spaces,  $U$  be an open subset of  $X$ ,  $x_0 \in U$ , and let  $f : U \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  satisfying the chain rule condition on  $U$ . Suppose  $Jf$  is regular at  $x_0$ . By Inverse Mapping Theorem above,  $f$  is a bi-Lipschitz homeomorphism around  $x_0$  and by Theorem 9 (local openness)  $f$  is open with linear rate at  $x_0$ . The supremum of the nonnegative real numbers  $\alpha$  such that for some neighborhood  $V_{x_0}$ ,  $B(f(x); \alpha r) \subset f(B(x; r))$ , for all  $x \in V_{x_0}$  and all  $r > 0$  with  $B(x; r) \subset V_{x_0}$  is called the *rate of surjection of  $f$  near  $x_0$*  [19] or *exact covering bound of  $f$  around  $x_0$*  [28]—denoted by  $\text{cov}f(x_0)$ . So, in this context,  $\text{cov}f(x_0) \geq \text{Reg} Jf(x_0)$ . On the other hand, in [18] Ioffe introduced the modulus of surjection of  $f$  at  $x_0$ , defined for every  $r > 0$  by

$$S(f, x_0)(r) = \sup\{\rho \geq 0 : B(f(x_0); \rho) \subset f(B(x_0; r))\}.$$

Ioffe considers also the quantity now called *Ioffe constant of surjection of  $f$  at  $x_0$* :

$$\text{sur}(f, x_0) = \sup_{\epsilon > 0} \left( \inf \left\{ \frac{S(f, x_0)(r)}{r} : 0 < r < \epsilon \right\} \right).$$

Let  $V_{x_0}$  be a neighborhood of  $x_0$  and a constant  $\alpha > 0$  such that for every  $x \in V_{x_0}$  and  $r > 0$  with  $B(x; r) \subset V_{x_0}$  we have that  $B(f(x); \alpha r) \subset f(B(x; r))$ . In particular, there is an  $\epsilon > 0$ , depending on  $\alpha$ , such that for every  $0 < r < \epsilon$ :

$$B(f(x_0); \alpha r) \subset f(B(x_0; r)).$$

Therefore, for all  $0 < r < \epsilon$  we have that  $\alpha r \leq S(f, x_0)(r)$ . So,

$$\alpha \leq \inf \left\{ \frac{S(f, x_0)(r)}{r} : 0 < r < \epsilon \right\} \leq \text{sur}(f, x_0).$$

Finally we have that if  $\text{Reg } Jf(x_0) > 0$  then:

$$\text{sur}(f, x_0) \geq \text{cov}f(x_0) \geq \text{Reg } Jf(x_0). \quad (4)$$

If  $f$  is  $C^1$  and  $df(x_0)$  is a linear isomorphism, actually we have:

$$\text{sur}(f, x_0) = \text{cov}f(x_0) = \text{Reg } Jf(x_0) = C(df(x_0)) = \|df(x_0)^{-1}\|^{-1}.$$

There exists a close relationship between the rate of surjection of  $f$  and the so called *Lipschitz rate* of the multivalued function  $f^{-1} : Y \rightarrow X$  given by

$$f^{-1}(y) = \{x \in X : y = f(x)\}.$$

Recall,  $f^{-1}$  has the *Aubin property* (or *pseudo-Lipschitz property*, or *Lipschitz-like property*) around  $(y_0, x_0)$  if there exists neighborhoods  $W_{y_0}$  and  $V_{x_0}$  and a number  $\mu > 0$  such that  $\text{dist}(x, f^{-1}(y)) \leq \mu|y - z|$  provided  $y, z \in W_{y_0}, x \in f^{-1}(z) \cap V_{x_0}$ . The infimum of such  $\mu$  is the *Lipschitz rate of  $f^{-1}$  near  $(y_0, x_0)$*  —denoted by  $\text{lip}f^{-1}(y_0|x_0)$ . In general we have that  $f$  is open with linear rate around  $x_0$  if and only if  $f^{-1}$  has the Aubin property near  $(f(x_0), x_0)$  if and only if  $\text{cov}f(x_0) > 0$ . In this case, see Proposition 2.2 in [19]:

$$\text{lip}f^{-1}(f(x_0)|x_0) = \text{cov}f(x_0)^{-1} \quad (5)$$

Now, let  $m$  be a positive lower semicontinuous function on  $[0, \infty)$  such that

$$(A) \text{ For all } x \in X, \text{Sur } Jf(x) \geq m(|x|).$$

The essential example that comes from Hadamard's original global inversion theorem of 1906 [15] is associated to the nonincreasing function  $\mu$  on  $[0, \infty)$  given by

$$\mu(\rho) = \inf_{|x| \leq \rho} \text{Sur } Jf(x). \quad (6)$$

Indeed,  $\mu$  has countably many (jump) discontinuities; we then set  $m(\rho) := \mu(\rho)$  if  $\mu$  is continuous at  $\rho$  and set  $m(\rho) := \lim_{t \rightarrow \rho^+} \mu(t)$  if  $\mu$  has a jump discontinuity at  $\rho$ . So, the mapping  $m$  is lower semicontinuous on  $[0, \infty)$  and satisfies (A) since  $\mu(\rho) \geq m(\rho)$  for all  $\rho > 0$ . If in addition,  $\mu(\rho) > 0$  for all  $\rho > 0$  then  $m$  is positive on  $[0, \infty)$ .

This argument shows that, if  $\mu$  is a positive nonincreasing function such that, for all  $x \in X$ ,  $\text{Sur } Jf(x) \geq \mu(|x|)$ , then there exists an associated positive lower semicontinuous function  $m$ , constructed as above, such that (A) holds.

In general, if  $m$  is a positive lower semicontinuous function satisfying condition (A), then for all  $r > 0$  we have  $\alpha_r := \inf\{m(\rho) : 0 \leq \rho \leq r\} > 0$ . Therefore for all  $x \in B(0; r)$ ,  $\text{Sur } Jf(x) \geq \alpha$ . Thus  $f$  is open with linear rate around every  $x \in B(0; r)$  with uniform lower bound of the rate of surjection of  $f$  on  $B(0; r)$ . Since  $r$  is arbitrary, condition (A) is actually a global condition.

Furthermore, by (4) and Theorem 1 of [17] we have the following:

**Theorem 12 (Global Surjection Theorem).** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces and let  $f : X \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  satisfying the chain rule condition on  $X$ . Suppose that condition (A) holds for some positive lower semicontinuous function  $m$  on  $[0, \infty)$ . Then  $f$  is open with linear rate at every  $x \in X$ , and for each  $r > 0$  we have:*

$$B(f(0); \varrho(r)) \subset f(B(0; r)), \quad (7)$$

where

$$\varrho(r) = \int_0^r m(\rho) d\rho.$$

Furthermore,  $f : X \rightarrow Y$  is surjective provided that, in addition,

$$\int_0^\infty m(\rho) d\rho = \infty.$$

Note that if  $\mu$  is the nonincreasing function given by (6), and  $m$  is the associated lower semicontinuous function defined as above, then  $\varrho(r) = \int_0^r m(\rho) d\rho = \int_0^r \mu(\rho) d\rho$ . Besides,  $\mu(\rho) > 0$  for all  $\rho > 0$  if

$$(B) \int_0^\infty \inf_{|x| \leq \rho} \text{Sur } Jf(x) d\rho = \infty.$$

Therefore we conclude:

**Corollary 13.** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces and let  $f : X \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  satisfying the chain rule condition on  $X$ . Suppose that condition (B) is satisfied. Then  $f$  is a surjective map, open with linear rate at every  $x \in X$ , such that for every  $r > 0$  inclusion (7) holds with  $\varrho(r) = \int_0^r \inf_{|x| \leq \rho} \text{Sur } Jf(x) d\rho$ .*

Now, suppose that  $f$  is also locally one-to-one at every  $x \in X$ , e.g. under hypothesis of Inverse Mapping Theorem above, then by Theorem 2 of [17]  $f$  is actually a global homeomorphism onto  $Y$ . Note that if  $f$  is a global homeomorphism, then for all  $y \in Y$ :

$$\text{lip } f^{-1}(y) |f^{-1}(y)| \geq \text{Lip } f^{-1}(y). \quad (8)$$

So we get the following extension of Theorem 3.9 of [21]. Note that if  $f$  has a global inverse, property by (7) implies that  $f$  is a *norm-coercive map*, namely  $\lim_{|x| \rightarrow \infty} |f(x)| = \infty$ .

**Theorem 14 (Global Inverse Theorem I).** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces and let  $f : X \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  satisfying the chain rule condition on  $X$ . Suppose that  $Jf$  is regular at every  $x \in X$  and:*

$$(A') \text{ For all } x \in X, \text{Reg } Jf(x) \geq m(|x|).$$

*for some positive lower semicontinuous function  $m$  such that  $\int_0^\infty m(\rho) d\rho = \infty$ . Then  $f$  is a norm-coercive global homeomorphism onto  $Y$  and the inverse  $f^{-1}$  is Lipschitz on bounded subsets of  $Y$ , and such that for every  $y = f(x) \in Y$ :*

$$\text{Lip } f^{-1}(y) \leq (m(|f^{-1}(y)|))^{-1}.$$

*Proof.* It only remains to show that the global inverse map  $f^{-1}$  is Lipschitz on bounded subsets of  $Y$ . Let  $R > 0$  be given, and consider  $r > 0$  such that

$$\int_0^r m(\rho) d\rho > R.$$

From 7, we have that  $B(f(0); R) \subset f(B(0; r))$ . As we have remarked before,  $\alpha_r := \inf\{m(\rho) : 0 \leq \rho \leq r\} > 0$ , and thus  $\text{Reg } Jf(x) \geq m(|x|) \geq \alpha_r > 0$  whenever  $|x| \leq r$ . Therefore, if we fix  $0 < \alpha < \alpha_r$ , we obtain from Theorem 11 that  $f^{-1}$  is locally  $\alpha^{-1}$ -Lipschitz on the open ball  $B(f(0); R)$ . Using the convexity of the ball, a standard argument gives that  $f^{-1}$  is in fact  $\alpha^{-1}$ -Lipschitz on the ball  $B(f(0); R)$ , and this concludes the proof.  $\square$

If  $f : X \rightarrow Y$  is a locally Lipschitz map between reflexive Banach spaces such that the Páles-Zeidan generalized Jacobian  $\partial f$  is upper semicontinuous, then the hypotheses of Global Inverse Theorem I are satisfied if for some positive lower semicontinuous function  $m$  and each  $x \in X$ , every  $T \in \partial f(x)$  is an isomorphism and satisfies  $C^*(T) \geq m(|x|)$ . In particular Corollary 3.10 of [21] can be deduced from above result. For a  $C^1$  map  $f : X \rightarrow Y$ , the hypotheses of Global Inverse Theorem I are satisfied if for some positive lower semicontinuous function  $m$  and for each  $x \in X$ , we have that  $df(x)$  is a linear isomorphism and  $\text{cov}f(x) = C^*(df(x)) \geq m(|x|)$ .

##### 5. PALAIS-SMALE CONDITION AND LOCALLY BI-LIPSCHITZ HOMEOMORPHISMS

Let  $(X, |\cdot|)$  be a real Banach space and let  $F : X \rightarrow \mathbb{R}$  be a locally Lipschitz functional. We define the lower semicontinuous function:

$$\lambda_F(x) = \min_{w^* \in \partial F(x)} |w^*|_{X^*}$$

By a *weight* we mean a continuous nondecreasing function  $h : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$\int_0^\infty \frac{1}{1+h(\rho)} d\rho = +\infty.$$

**Weighted Chang-Palais-Smale condition.** Following Chang [4], we say that the functional  $F : X \rightarrow \mathbb{R}$  satisfies the weighted Chang-Palais-Smale condition with respect to a weight  $h$  if any sequence  $\{x_n\}$  in  $X$  such that  $\{F(x_n)\}$  is bounded and

$$\lim_{n \rightarrow \infty} \lambda_F(x_n)(1+h(|x_n|)) = 0 \tag{9}$$

contains a (strongly) convergent subsequence.

Naturally, the limit of a converging weighted Chang-Palais-Smale sequence must be a critical point, in the sense that  $\lambda_F(x) = 0$ . Furthermore, if  $F$  is bounded from below then, by the Ekeland Variational Principle there exists always a minimizing weighted Chang-Palais-Smale sequence [13]. In other words, for any weight  $h$ :

- If  $\{x_n\} \subset X$  is a sequence such that  $\lim_{n \rightarrow \infty} x_n = \hat{x}$  and satisfying (9) for  $h$ , we have that  $\lambda_F(\hat{x}) = 0$ .
- If  $F$  is bounded from below then for  $h$  then there is a sequence  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} F(x_n) = \inf_X F$  and satisfying (9).

Let  $f : X \rightarrow Y$  be a locally Lipschitz function and  $y \in Y$  be fixed. Consider the functional  $F_y(x) := |f(x) - y|$  defined in (2). Suppose that:

- (C) The locally Lipschitz functional  $F_y$  satisfies the weighted Chang-Palais-Smale condition for some weight  $h$ .

Then the first property above gives us the existence of a minimizing sequence converging to a critical point of  $F_y$ , so there exists a solution of the non-linear equation  $f(x) = y$ . The global injectivity comes from the second property above and a mountain-pass theorem if  $f$  has appropriate local properties e.g. under hypothesis of Inverse Mapping Theorem above. So we have the following extension of Theorem 1 of [13] given for  $C^1$  mappings, in turn, a generalization of Theorem 3.1 of [16] for  $Y$  Hilbert space and  $h = 0$ ; see Remark 17 below.

**Theorem 15.** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces and let  $f : X \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  regular at every  $x \in X$  and satisfying the strong chain rule condition on  $X$ . Suppose that for some  $y \in Y$ , the functional  $F_y(x) = |f(x) - y|$  satisfies (C). Then there exists a unique solution of the nonlinear equation  $f(x) = y$ .*

*Proof. Injectivity:* Let  $y \in Y$  be fixed. Suppose that there are two different points  $u$  and  $e$  in  $X$  such that  $f(u) = f(e) = y$ . Since  $f$  is open with linear rate around  $u$ , there exist  $\alpha > 0$  and  $\epsilon > 0$  such that:

$$B(y; \alpha r) \subset f(B(u; r)), \text{ for all } 0 < r < \epsilon. \quad (10)$$

Let  $r \in (0, \epsilon)$  be small enough such that  $f|_{B_r(u)} : B_r(u) \rightarrow f(B_r(u))$  is a homeomorphism, and set  $\rho = \alpha r > 0$ . Suppose first that  $u = 0$ . We have that:

- $F_y(0) = 0 \leq \rho$  and  $F_y(e) = 0 \leq \rho$ .
- $|e| \geq r$ , since  $f|_{B_r(0)}$  is injective.
- $F_y(x) \geq \rho$  for  $|x| = r$ , in view of (10).

By Schechter-Katriel Mountain-Pass Theorem (see Theorem 7.2 in [27]), there is a sequence  $\{x_n\} \subset X$  such that  $\lim_{n \rightarrow \infty} F_y(x_n) = c$  for some  $c \geq \rho$  and satisfying (9). Since  $F_y$  satisfies the weighted Chang-Palais-Smale-condition, the sequence  $\{x_n\}$  has a convergent subsequence  $\{x_{n_k}\}$  with limit  $\hat{x}$ . Therefore  $\lambda_{F_y}(\hat{x}) = 0$ , and  $f(\hat{x}) \neq y$  since  $\lim_{k \rightarrow \infty} F_y(x_{n_k}) = F_y(\hat{x}) = c \geq \rho > 0$ . Therefore, we get a contradiction since:

**Claim 16.** For every  $x \in X$ ,  $\lambda_{F_y}(x) = 0$  implies  $f(x) = y$ .

In the case that  $u \neq 0$ , we can consider  $G_y(x) = F_y(u - x)$  instead of  $F_y(x)$  and carry on an analogous reasoning.

*Surjectivity:* Let  $y \in Y$  be fixed. As we pointed out before, there is a minimizing sequence  $\{x_n\} \subset X$  such that  $\lim_{n \rightarrow \infty} F_y(x_n) = \inf_X F_y$  and satisfying (9). Since  $F_y$  satisfies weighted Chang-Palais-Smale-condition, the sequence  $\{x_n\}$  has a convergent subsequence  $\{x_{n_k}\}$  with limit  $\hat{x}$ . As before, we have that  $\hat{x}$  is a critical point of  $F_y$ . By Claim 16 we have that  $f(\hat{x}) = y$ .  $\square$

*Proof of Claim.* Suppose that  $\lambda_{F_y}(x) = 0$  and  $f(x) \neq y$ . Let  $w^* \in \partial F_y(x)$ . Since  $Jf$  satisfies the strong chain rule condition  $\partial F_y(x) \subset \Delta F_y(x)$ . Then there is  $y^* \in \partial |\cdot|(f(x) - y)$  and  $T \in \overline{\text{co}} Jf(x)$  such that  $w^* = y^* \circ T$ . Since  $f(x) - y \neq 0$  we have that  $|y^*|_{Y^*} = 1$ . We have that

$$|w^*|_{X^*} = |T^* y^*|_{X^*} \geq \inf_{|v^*|_{Y^*} = 1} |T^* v^*|_{X^*} = C(T).$$

Now, for every  $\epsilon > 0$  there is  $T_\epsilon \in \text{co} Jf(x)$  such that  $\|T - T_\epsilon\| < \epsilon$ . Therefore,  $C(T) = C^*(T) \geq C^*(T_\epsilon) - \epsilon \geq \text{Reg} Jf(x) - \epsilon$ . So, we have that  $|w^*|_{X^*} \geq \text{Reg} Jf(x)$ . Taking the minimum over  $\partial F_y(x)$  we obtain that

$$\lambda_{F_y}(x) \geq \text{Reg} Jf(x). \quad (11)$$

Therefore  $\text{Reg} Jf(x) = 0$  and we get contradiction.  $\square$

**Remark 17.** In [13] and [16] the functional  $G_y(x) = \frac{1}{2} F_y(x)^2$  is considered instead of  $F_y$ . Suppose that  $G_y$  satisfies the weighted Chang-Palais-Smale condition for some weight  $h$ . Let  $\{x_n\}$  any sequence in  $X$  such that  $\{F_y(x_n)\}$  is bounded and  $\lambda_{F_y}(x_n)(1 + h(|x_n|)) = 0$ . Since  $\lambda_{G_y} = F_y(x) \cdot \lambda_{F_y}(x)$  for all  $x \in X$  and  $y \in Y$ , then  $G_y(x_n)$  is bounded and  $\lambda_{G_y}(x_n)(1 + h(|x_n|)) = 0$ . Therefore  $\{x_n\}$  contains a

(strongly) convergent subsequence. So, if  $G_y$  satisfies the weighted Chang-Palais-Smale condition for some weight  $h$  then  $F_y$  satisfies the Chang-Palais-Smale condition with the same weight  $h$ .

By Theorem 15, equations (8) and (4) we have:

**Theorem 18 (Global Inverse Theorem II).** *Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be Banach spaces and let  $f : X \rightarrow Y$  be a locally Lipschitz map with a pseudo-Jacobian  $Jf$  regular at every  $x \in X$  and satisfying the strong chain rule condition on  $X$ . Suppose that for every  $y \in Y$  the locally Lipschitz functional  $F_y(x) = |f(x) - y|$  satisfies (C). Then  $f$  is a norm-coercive homeomorphism locally bi-Lipschitz onto  $Y$  with:*

$$\text{Lip } f^{-1}(y) \leq (\text{Reg } Jf(f^{-1}(y)))^{-1}.$$

**Remark 19.** Let  $f : X \rightarrow Y$  be a locally Lipschitz map between Banach spaces with a pseudo-Jacobian  $Jf$  regular at every  $x \in X$  and satisfying the strong chain rule condition on  $X$ . If  $f$  satisfies condition (A') for a positive nonincreasing and continuous function  $m$  such that  $\int_0^\infty m(\rho) d\rho = \infty$  then, for every  $y \in Y$ , the functional  $F_y$  satisfies (C) for the weight

$$h(\rho) := \frac{m(0)}{m(\rho)} - 1.$$

Indeed, it is easy to verify that  $h$  is actually a weight, namely, it is positive, nondecreasing, continuous map such that  $\int_0^\infty \frac{1}{1+h(\rho)} d\rho = \infty$ . Furthermore, for all  $x \in X$ :

$$0 < m(0) < \text{Reg } Jf(x)(1 + h(|x|)).$$

By the proof of (11) in the Claim above we have that, if  $y \in Y$  and  $f(x) \neq y$ , then  $\lambda_{F_y}(x) \geq \text{Reg } Jf(x)$ . Therefore, if  $f(x) \neq y$  then:

$$\lambda_{F_y}(x)(1 + h(|x|)) \geq m(0) > 0. \quad (12)$$

Suppose that there is a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} F_y(x_n) = c > 0$  for some  $y \in Y$ . Then, without loss of generality, we can assume that  $f(x_n) \neq y$  for all natural  $n$ . Therefore, by (12)  $\lim_{n \rightarrow \infty} \lambda_{F_y}(x_n)(1 + h(x_n))$  can't be zero. In other words, for each  $y \in Y$ , there is no sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} F_y(x_n) = c > 0 \text{ and } \lim_{n \rightarrow \infty} \lambda_{F_y}(x_n)(1 + h(x_n)) = 0.$$

Note that, as we conclude in the first part of the proof of Theorem 15, this implies that  $f$  is injective. Now, let  $\{x_n\} \subset X$  be such that  $\lim_{n \rightarrow \infty} F_y(x_n) = 0$  and  $\lim_{n \rightarrow \infty} \lambda_{F_y}(x_n)(1 + h(|x_n|)) = 0$ . Then, by (12) there exists  $m > 0$  such that  $f(x_n) = y$  for all  $n \geq m$ . Since  $f$  is injective, this means that  $x_n = f^{-1}f(x_n) = f^{-1}(y)$  for all  $n \geq m$ , so  $\{x_n\}$  converges to  $f^{-1}(y)$ . Therefore (C) is fulfilled.

## 6. EXAMPLE

In this section we give an example where the conditions for global invertibility introduced in the previous sections can be easily checked. We will be concerned with the following integro-differential equation, which has been considered, with several variants, in [16], [8] and [9]:

$$x'(t) + \int_0^t \Phi(t, \tau, x(\tau)) d\tau = y(t), \quad \text{for a. e. } t \in [0, 1] \quad (13)$$

with initial condition:

$$x(0) = 0. \quad (14)$$

Here  $y$  is a given function in the space  $L^p[0,1]$ , where  $1 < p < \infty$  is fixed. It is natural to consider in this setting the space  $W_0^{1,p}[0,1]$  of all absolutely continuous functions  $x : [0,1] \rightarrow \mathbb{R}$  with  $x(0) = 0$  and such that  $x' \in L^p[0,1]$ . The space  $W_0^{1,p}[0,1]$  is complete for the norm:

$$\|x\|_{W_0^{1,p}} := \left( \int_0^1 |x'(\tau)|^p d\tau \right)^{1/p}$$

Then by a *solution* of the equation (13) with initial condition (14) we mean a function  $x \in W_0^{1,p}[0,1]$  satisfying (13) almost everywhere in  $[0,1]$ .

We denote  $\Delta := \{(t, \tau) \in [0,1] \times [0,1] : \tau \leq t\}$ , and we will assume that the function  $\Phi : \Delta \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following conditions:

- (i)  $\Phi(\cdot, \cdot, u)$  is measurable in  $\Delta$  for all  $u \in \mathbb{R}$ .
- (ii) There exist functions  $a, b \in L^p(\Delta)$  such that

$$|\Phi(t, \tau, u)| \leq a(t, \tau) \cdot |u| + b(t, \tau) \quad \text{for a. e. } (t, \tau) \in \Delta \text{ and all } u \in \mathbb{R}.$$

- (iii) There exists a continuous function  $\theta : [0, \infty) \rightarrow (0, 1)$  with the property that  $\int_0^\infty (1 - \theta(r)) dr = \infty$ , and such that

$$|\Phi(t, \tau, u) - \Phi(t, \tau, v)| \leq \theta(r) |u - v| \quad \text{for a. e. } (t, \tau) \in \Delta \text{ and all } |u|, |v| \leq r.$$

Note that Condition (iii) is fulfilled, in particular, if there exists a constant  $0 < \theta < 1$  such that  $\Phi$  is globally  $\theta$ -Lipschitz in the third variable.

**Example 20.** Let  $1 < p < \infty$ , and suppose that the function  $\Phi$  satisfies conditions (i), (ii) and (iii) above. Then for each  $y \in L^p[0,1]$  there exists a unique solution of equation (13) with initial condition (14) in the space  $W_0^{1,p}[0,1]$ .

*Proof.* Consider the Banach spaces  $X = W_0^{1,p}[0,1]$  and  $Y = L^p[0,1]$ , and the map

$$f : W_0^{1,p}[0,1] \rightarrow L^p[0,1]$$

defined as  $f = T + g$ , where  $T, g : W_0^{1,p}[0,1] \rightarrow L^p[0,1]$  are given respectively by

$$T(x) = x'$$

and

$$g(x)(t) = \int_0^t \Phi(t, \tau, x(\tau)) d\tau.$$

It is clear that  $T$  is a linear isomorphism which is, in fact, an isometry; that is,

$$\|T(x)\|_{L^p} = \|x'\|_{L^p} = \left( \int_0^1 |x'(\tau)|^p d\tau \right)^{1/p} = \|x\|_{W_0^{1,p}}.$$

Thus we have that  $C(T) = \|T^{-1}\|^{-1} = 1$ .

On the other hand, we are next going to check that  $g$  is Lipschitz on bounded subsets of  $W_0^{1,p}[0,1]$ . First note that, given  $x \in W_0^{1,p}[0,1]$ , for every  $t \in [0,1]$  we have:

$$|x(t)| = \left| \int_0^t x'(\tau) d\tau \right| \leq \int_0^1 |x'(\tau)| d\tau = \|x'\|_{L^1} \leq \|x'\|_{L^p} = \|x\|_{W_0^{1,p}},$$

and therefore  $\|x\|_\infty \leq \|x\|_{W_0^{1,p}}$ .

Now let  $r \geq 0$  and consider  $u, v \in W_0^{1,p}[0, 1]$  with  $\|u\|_{W_0^{1,p}} \leq r$  and  $\|v\|_{W_0^{1,p}} \leq r$ . For each  $t \in [0, 1]$ :

$$\begin{aligned} |g(u)(t) - g(v)(t)| &\leq \int_0^t |\Phi(t, \tau, u(\tau)) - \Phi(t, \tau, v(\tau))| d\tau \leq \int_0^1 \theta(r) \cdot |u(\tau) - v(\tau)| d\tau \\ &\leq \theta(r) \cdot \|u - v\|_\infty \leq \theta(r) \cdot \|u - v\|_{W_0^{1,p}}. \end{aligned}$$

Then

$$\|g(u) - g(v)\|_{L^p} = \left( \int_0^1 |g(u)(t) - g(v)(t)|^p dt \right)^{1/p} \leq \theta(r) \cdot \|u - v\|_{W_0^{1,p}}.$$

This implies in particular that, for every  $x \in W_0^{1,p}[0, 1]$  with  $\|x\|_{W_0^{1,p}} \leq r$  we have that  $\text{Lip } g(x) \leq \theta(r + \epsilon)$  for every  $\epsilon > 0$ . By the continuity of  $\theta$  we deduce that  $\text{Lip } g(x) \leq \theta(r)$  whenever  $\|x\|_{W_0^{1,p}} \leq r$ .

From Example 8 we have that  $Jf(x) := T + \text{Lip } g(x) \cdot \overline{B}_{L(X,Y)}$  is a pseudo-Jacobian of  $f$ , satisfying the strong chain rule condition. Let us see that  $Jf$  is also regular at every  $x \in W_0^{1,p}[0, 1]$ . Indeed, suppose that  $\|x\|_{W_0^{1,p}} \leq r$ . For each  $R \in \mathcal{L}(X, Y)$  with  $\|R\| \leq \text{Lip } g(x)$  we have  $\|R \circ T^{-1}\| \leq \text{Lip } g(x) \leq \theta(r) < 1$ . In particular, the operator  $\text{Id}_Y + R \circ T^{-1}$  is an isomorphism on  $Y$ . In this way we obtain that  $T + R$  is an isomorphism. On the other hand, also using Example 8, we have

$$\begin{aligned} \text{Reg } Jf(x) = \text{Sur } Jf(x) &= \inf \{C(T + R) : \|R\| \leq \text{Lip } g(x)\} \\ &\geq \inf \{C(T + R) : \|R\| \leq \theta(r)\} \geq 1 - \theta(r) = m(r) > 0, \end{aligned}$$

where the continuous function  $m(r) := 1 - \theta(r)$  satisfies that  $\int_0^\infty m(r) dr = \infty$ .

Therefore, condition (A') and all the requirements of Theorem 14 are satisfied, and the desired conclusion follows. Also, from Remark 19 we see that, for each  $y \in L^p[0, 1]$ , the functional  $F_y$  defined in (2) satisfies the weighted Chang-Palais-Smale condition (C) for the weight

$$h(r) = \frac{\theta(r) - \theta(0)}{1 - \theta(r)}.$$

Thus Theorem 18 also applies in this case.  $\square$

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