UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

Departamento de Matemática Aplicada

TESIS DOCTORAL

Unicidad de soluciones largas

(Uniqueness of large solutions)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Luis Maire Martín

Director

Julián López-Gómez

Madrid, 2018

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Contents

Summary

0.1 Introduction, motivation and methodology

The principal aim of this doctoral thesis is to establish uniqueness results, as much general as possible, to the following diffusive logistic elliptic system,

$$\begin{cases} -\Delta u_i = \lambda_i u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij} u_j - \mathfrak{a}_i(x) f_i(u_i) u_i & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega, \end{cases} \quad 1 \leq i \leq n, \tag{0.1}$$

where $n \in \mathbb{N}$, Ω is a bounded subdomain of $\mathbb{R}^N$, $N \in \mathbb{N}$, with regular boundary, $\lambda_i \in \mathbb{R}$, $a_{ij} > 0$ are the coupling parameters and $\mathfrak{a}_i \in \mathcal{C}^\nu(\bar{\Omega})$ for some $\nu \in (0,1)$ satisfies $\mathfrak{a}_i(x) > 0$, for all $x \in \Omega$ and $1 \leq i, j \leq n$. The singular boundary conditions should be understood in the sense that

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} u_i(x) = +\infty, \quad 1 \leq i \leq n.$$

Thus, this Thesis deals with *large*, or *explosive*, solutions. The model (0.1) is a generalization of the diffusive logistic equation studied in Chapters 6, 7 and 8 of [48], to a system with linear cooperative coupling, because $a_{ij} > 0$ for all $1 \leq i, j \leq n$, $i \neq j$. As far as concerns the single generalized logistic equation,

$$-\Delta u = \lambda u - \mathfrak{a}(x) f(u) u,$$

there is a huge amount of literature. The most pioneering results go back to L. Bieberbach [9] and H. Rademacher [69], who considered the equation $\Delta u = e^u$ in two and three dimensions, respectively, and J. B. Keller [35] and R. Osserman [66], who dealt with the equation $\Delta u = f(u)$ for some class of monotone f's. Since then, studies on solutions of single equations with boundary blow-up have followed in many different ways. For example, establishing existence and uniqueness results for more general kinds of nonlinearities, as in [8, 19, 28, 14], for models with spatial heterogeneities, [73, 7, 18, 12, 13, 41, 45], for more general elliptic operators, [73, 16], or even considering domains with irregular boundary, [58, 59]. Another usual topic in the framework of large solutions is to establish the asymptotic boundary expansion of the solution, as in [2, 14, 3]. There are also some astonishing multiplicity results in the context of superlinear indefinite problems, as the ones of [42, 60, 55, 56]. Nevertheless, the literature on systems is substantially more reduced. A pioneering work dealing with (0.1) under Dirichlet boundary conditions in case $n = 2$ is [61, 62, 63], where it is studied the existence and the uniqueness of positive solutions. One of the first papers on large solutions for systems is [32], where the authors characterized the existence of large solutions to the classical diffusive symbiotic model of Lotka-Volterra, as well as the blow-up rates of each of the components of these singular solutions. More recently, [31] showed the existence and uniqueness of large solutions for a class of *autonomous* reaction diffusion systems of cooperative type. As far as we know, the first paper that shows the existence of a solution for (0.1), in case $n = 2$, is [4]. Therefore, the problem of the uniqueness for (0.1) remained utterly open. Actually, this problem is still open, even for the single equation.

Although the uniqueness has a great interest from a mathematical point of view, one can provide an important motivation that arises in the context of Population Dynamics. Let $\mathcal{D} \subset \mathbb{R}^N$, $N \leq 3$, the inhabiting area of a species and denote by $u = u(t, x)$ the population density of the species, which varies with space, $x \in \mathcal{D}$, and time $t \geq 0$. Let us asume the next hypotheses:

- The species is divided into n groups, $u_1, \ldots, u_n$, such that each group cooperates with all others, in the sense that if u_i grows then u_j also grows, for every $1 \leq i, j \leq n$, $j \neq i$.
- Each u_i spans randomly in the inhabiting area $\mathcal{D}$, with diffusion rate measured by $d_i > 0$, $1 \leq i \leq n$.
- There exist some places in $\mathcal{D}$ with unlimited natural resources, while others have limited natural resources. This entails that the species will grow according to the Malthus law where the resources are unlimited, while it has a logistic growth in the complement. By the sake of simplicity, we suppose that $\Omega \subset \bar{\mathcal{D}}$ is the zone where the natural resources are limited.

Keeping in mind the last assumptions, a possible mathematical model for the evolution of this species might given by

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - d_i \Delta u_i = \lambda_i u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij} u_j - \mathfrak{b}_i(x) f_i(u_i) u_i & x \in \mathcal{D},\ t > 0 \\ u_i = 0 & \text{on } \partial\mathcal{D}, \\ u_i(0) = u_{0,i} > 0, & \end{cases} \tag{0.2}$$

where $1 \leq i \leq n$ and $\mathfrak{b}_i \in \mathcal{C}^{\nu}(\bar{\mathcal{D}})$ satisfies

$$\begin{cases} \mathfrak{b}_i(x) > 0 & \text{if } x \in \Omega, \\ \mathfrak{b}_i(x) = 0 & \text{if } x \neq \Omega, \end{cases} \qquad 1 \leq i \leq n.$$

Thus, we have combined in $\mathcal{D}$ both laws of population growth mentioned above. The Dirichlet homogeneous boundary conditions in (0.2) are imposed just by simplicity. They entail that every member reaching the boundary, dies. Of course, a very interesting problem would be to consider another boundary conditions.

What is the asymptotic behavior of the unique global solution of Problem (0.2)? The answer to this question is closely related to the problem of the uniqueness of (0.1), in the following way. Under some assumptions on the range of the parameters λ_i, $1 \leq i \leq n$, if we denote by $L^{\min}$ and $L^{\max}$ the *minimal* and the *maximal* solutions of (0.1), we have that

$$L^{\min} \leq \liminf_{t\to+\infty} u(t) \leq \limsup_{t\to+\infty} u(t) \leq L^{\max}, \quad x \in \Omega, \tag{0.3}$$

while $u(t) \uparrow +\infty$ in $\mathcal{D} \setminus \Omega$. Therefore, *metasolutions* do govern the dynamics of (0.2), within some open regions of the parameters λ_i. *Grosso modo*, a metasolution is a steady state of (0.2) equaling infinity somewhere. The reader is sent to [33, 4], or [48, Chapter 5], for a precise definition of metasolution. Summarizing, the uniqueness of the solution of (0.1) entails $L^{\min} = L^{\max}$ in (0.3), which characterizes the asymptotic behavior of the solutions of (0.2).

The most important technique used throughout this Thesis is the following comparison principle, derived from the theorem of characterization of the maximum principle of [54]. Let $w_1, w_2 \in \left[\mathcal{C}^{2+\nu}(\partial\Omega)\right]^n$ such that $w_{1,i} < w_{2,i}$ for every $1 \leq i \leq n$. Then, the unique positive solution of

$$\begin{cases} -\Delta u_i = \lambda_i u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij} u_j - \mathfrak{a}_i(x) f_i(u_i) u_i & \text{in } \Omega, \\ u_i = w_{k,i} > 0 & \text{on } \partial\Omega, \end{cases} \quad 1 \leq i \leq n, \tag{0.4}$$

throughout denoted by $\theta_{[\Omega,w_k]}$, $k = 1, 2$, satisfies $\theta_{[\Omega,w_1]} < \theta_{[\Omega,w_2]}$. Actually, the same principle holds if $w_2 = (+\infty, \dots, +\infty)$ on $\partial\Omega$. In order to show this, it suffices applying the previous result to

$$\Omega_\varepsilon := \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

for sufficiently small $\varepsilon > 0$ and then letting $\varepsilon \downarrow 0$. Thanks to this comparison principle, we have achieved many of the theorems of this thesis. In particular, the results of Chapters 2, 3 and 4 are deduced by applying it in a number of rather sophisticated ways.

Besides the previous comparison principle, in Chapter 5 we have used one of the most usual uniqueness techniques: Establishing explicit formulas for the *boundary blow-up rates* of the solutions of (0.1). Finally, we have adapted to our present context the localization method introduced in [41] for the single equation.

0.2 Content

The results established in this Thesis have been obtained by the author together with his superadvisor during 2015, 2016 and the first three months of 2017. Among them, those of [49, 50, 51] have already been published, while [52, 53] are in process of publication. We have ordered them not in chronological order, but according to the number of equations taken into consideration. The main contributions provided in each chapter of this Thesis are the following:

(1) The first chapter contains the results of [53], where we introduced the large solutions through the simplest model possible,

$$\begin{cases} u'' = f(u) & t \in (-T, T), \\ u(\pm T) = +\infty. \end{cases} \tag{0.5}$$

The novelty of our analysis is that we do not impose any restriction on the sign of f and f', as it is done in all previous studies, where $f \geq 0$ in $[0, \infty)$, or for sufficiently large u. This difference may change substantially the dynamics of (0.5). Indeed, the function

$$\mathcal{T}(x) := \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{du}{\sqrt{\int_x^u f}}, \qquad x := u(0),$$

provides us with the maximal existence time of any solution of $u'' = f(u)$, when it blows up. The condition $f(u) \geq 0$ for large u implies

$$\liminf_{x \to +\infty} \mathcal{T}(x) = 0,$$

which does not necessary happen without sign restrictions on f, as the counterexamples of the last section of this chapter show. We also establish a rather astonishing multiplicity result of large solutions from any given increasing positive function $f(u)$ that satisfies the Keller–Osserman condition, destroying the monotonicity of $f(u)$ on a compact set with arbitrarily small measure.

(2) Chapter 2 contains the results of [52] and it consists of three uniqueness theorems for the singular boundary value problem

$$\begin{cases} -\Delta u = \lambda u - \mathfrak{a}(x) f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{0.6}$$

where $\lambda \in \mathbb{R}$, $f \in \mathcal{C}^1[0, +\infty)$, $f(0) = 0$, $f' \geq 0$ and $\mathfrak{a}(x) > 0$ for every $x \in \Omega$. As it is usual, to get existence of solutions to (0.6), one should assume that f satisfies the following adaptation of the classical condition of Keller [35] and Osserman [66]:

(KO) For every $\alpha > 0$ there exists $u^* = u^*(\alpha) > 0$ such that

$$I(u) := \int_1^{+\infty} \frac{d\theta}{\sqrt{\int_1^\theta (\alpha \frac{f(ut)}{u} - t) dt}} < +\infty \quad \text{for all } \ u > u^*,$$

The first result of this chapter provides with some sufficient conditions for the uniqueness in star-shaped domains:

Theorem 0.1 *Suppose Ω is star-shaped, $\lambda \geq 0$ and f satisfies the next property:*

(C) *There exists $p > 1$ such that*

$$f(tu) \geq t^p f(u) \quad \textit{for all} \quad t > 1 \;\; \textit{and} \;\; u > 0. \tag{0.7}$$

Moreover, assume that there exists $\eta > 0$ such that, for every $z \in \partial\Omega$,

$$\mathfrak{a}\left(z + \tfrac{x_0 - z}{|x_0 - z|} t\right) \leq \mathfrak{a}\left(z + \tfrac{x_0 - z}{|x_0 - z|} s\right) \quad \textit{if} \quad 0 < t < s < \eta. \tag{0.8}$$

Then, (0.6) *has a unique positive solution.*

The second theorem of Chapter 2 is an adaptation of Theorem 0.1 to cover the class of domains which can be represented as an star-shaped domain with m-star-shaped holes and it can be stated as follows:

Theorem 0.2 *Suppose $\lambda = 0$ in* (0.6) *and f satisfies* (C). *Moreover, assume that there are an integer $m \geq 1$ and $m + 1$ star-shaped domains, Ω_i, $0 \leq i \leq m$, with $\partial\Omega_i$ Lipschitz continuous, such that*

$$\bar{\Omega}_i \subset \Omega_0, \quad \bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset, \quad 1 \leq i, j \leq m, \quad i \neq j,$$

and

$$\Omega = \Omega_0 \setminus \left(\bar{\Omega}_1 \cup \cdots \cup \bar{\Omega}_m\right). \tag{0.9}$$

For every $0 \leq i \leq m$, let denote by x_i the (center) point with respect to which Ω_i is star-shaped. Finally, suppose that there exists $\eta > 0$ such that, for every $1 \leq i \leq m$, $z_0 \in \partial\Omega_0$ and $z_i \in \partial\Omega_i$,

$$\left.\begin{array}{l} \mathfrak{a}\left(z_0 + \frac{x_0 - z_0}{|x_0 - z_0|} t\right) \leq \mathfrak{a}\left(z_0 + \frac{x_0 - z_0}{|x_0 - z|} s\right) \\ \mathfrak{a}\left(z_i + \frac{z_i - x_i}{|x_i - z_i|} t\right) \leq \mathfrak{a}\left(z_i + \frac{z_i - x_i}{|x_i - z_i|} s\right) \end{array}\right\} \quad \textit{if} \quad 0 < t < s < \eta, \quad 1 \leq i \leq m. \tag{0.10}$$

Then, (0.6) *has a unique positive solution.*

As far as concerns Ω, any annular region satisfies the requirements of Theorem 0.2, as well as any ball, Ω_0, perforated by finitely many closed disjoint balls, Ω_i, $1 \leq i \leq m$. The last theorem of Chapter 2 studies the uniqueness in smooth domains:

Theorem 0.3 *Let $\Omega \in \mathcal{C}^1$ such that the uniform interior sphere property is satisfied on $\partial\Omega$ and $\lambda < \sigma[-\Delta, \Omega]$. Assume that, for every $z \in \partial\Omega$, there exists $\delta > 0$ such that $|x - z| < \delta$, with $x \in \Omega$, implies*

$$\mathfrak{a}(x + \varrho n_z) \leq \mathfrak{a}(x) \quad \textit{if} \;\; x + \varrho n_z \in \Omega \;\; \textit{and} \;\; \varrho > 0, \tag{0.11}$$

where n_z stands for the outward unit normal vector to $\partial\Omega$ at z. Moreover, suppose that f is super-additive *with constant $C \geq 0$, i.e., there exists $C \geq 0$ such that*

$$f(a+b) \geq f(a) + f(b) - C \qquad \text{for all} \;\; a, b \geq 0. \tag{0.12}$$

Then,under (KO)*, the singular boundary value problem* (0.6) *possesses a unique positive solution.*

Although at first glance (0.11) might look a little bit strange, it just means that $\mathfrak{a}(x)$ is non-increasing as x approximates $\partial\Omega$ along parallel rays to the line passing through z and $z + n_z$. Naturally, it holds if either $\mathfrak{a}(x)$ is constant in a neighborhood of $\partial\Omega$, or if $\mathfrak{a} \in \mathcal{C}^1(\bar{\Omega})$ and

$$\frac{\partial \mathfrak{a}}{\partial n_z}(z) < 0 \quad \text{for all} \;\; z \in \partial\Omega.$$

The superadditivity property (0.12) goes back to Theorem 0.3 of [58], where Marcus and Véron obtain uniqueness of large solutions of

$$-\Delta u + f(u) = 0 \tag{0.13}$$

for domains whose boundary is locally the graph of a continuous function. This is an extremely weaker hypothesis on the regularity of $\partial\Omega$ than ours, although their proof needs f to be convex. Moreover, there is no linear term in (0.13), and no spatial heterogeneities can be incorporated to the model without some additional further work.
Theorem 0.3 is a new finding even in the autonomous case:

Corollary 0.4 *Suppose $\Omega \in \mathcal{C}^1$ satisfies the uniform interior sphere property on $\partial\Omega$ and $\lambda < \sigma[-\Delta, \Omega]$. Assume that* (0.12) *and* (KO) *hold. Then, if $\mathfrak{a}(x) = 1$ for every $x \in \Omega$, the singular boundary value problem* (0.6) *has a unique positive solution.*

(3) In the third chapter we introduce the system (0.1) and derive the existence of a minimal and a maximal solution of (0.1), adapting the corresponding existence theorems of [48] and [4]. Precisely, we establish the comparison principle for cooperative systems explained in the previous section, getting as a consequence that the mapping

$$m \mapsto \theta_{[\Omega, \vec{m}]}$$

is increasing, where $\vec{m} := (m, \ldots, m)$. Thus, the point-wise limit

$$\theta_{[\Omega,\infty]}(x) := \lim_{m \to +\infty} \theta_{[\Omega,\vec{m}]}(x), \qquad x \in \Omega, \tag{0.14}$$

is well defined, though it might be infinite somewhere in Ω. For this reason, it is natural to assume some Keller-Osserman condition for the system, as, for example, the existence of an increasing $F \in \mathcal{C}^1[0, +\infty)$ with $F(0) = 0$ and $f_i(u)u > F(u)$, such that F satisfies Condition (KO). Under the latest assumption, the limit given by (0.14) is finite in Ω and provides us with the minimal solution of (0.1). Moreover, the maximal solution of (0.1) is given by

$$L^{\max} = \lim_{\delta \downarrow 0} \theta_{[\Omega_\delta, \infty]},$$

where $\Omega_\delta := \{x \in \Omega \ : \ \text{dist}\ (x, \partial\Omega) > \delta\}$.

(4) Chapter 4 discusses the main result of [49], where a radially symmetric counterpart of (0.1) is studied. Its main result stands as follows.

Theorem 0.5 *Suppose that Ω is a ball or an annulus in* (0.1)*, $\lambda_i \geq 0$, and*

$$\mathfrak{a}_i(x) = \mathfrak{a}_i(\text{dist}(x, \partial\Omega)), \quad 1 \leq i \leq n,$$

are positive nonincreasing functions. Suppose the function $g(u) := f(u)u$ satisfies Condition (C) *defined in* (0.7)*. Then, Problem* (0.1) *has a unique positive solution. Moreover, it is radially symmetric.*

The proof of this theorem is based on a rather sophisticated use of the maximum principle for weakly coupled cooperative elliptic systems, without invoking to the boundary blow-up rates of the large solutions. This result is the first available uniqueness theorem in the literature for n-species cooperative systems.

(5) Lastly, in Chapter 5 we study the blow-up rates of the solutions of (0.1). The results summarized here were first established in [50] for the case $n = 2$, and later in [51] in the general case. The main result of this chapter provides us, for each $z \in \partial\Omega$, with $\alpha_i(z), A_i(z) > 0$ such that

$$\lim_{\substack{x \to z \\ x \in \Omega}} \frac{u_i(x)}{\text{dist}(x, \partial\Omega)^{-\alpha_i(z)}} = A_i(z), \quad 1 \leq i \leq n, \tag{0.15}$$

for any solution of (0.1), $u = (u_1, \ldots, u_n)$, under some special assumptions on the terms f_i and $\mathfrak{a}_i$. Concretely, we ascertain the boundary blow-up rates in case $f_i(u) = u^{p_i - 1}$ for some $p_i > 1$, assuming that $\mathfrak{a}_i(x)$ behaves like a power near $\partial\Omega$, *not necessary with fixed rate*, in the sense that there exist $\mathfrak{b}_i, \gamma_i \in \mathcal{C}(\partial\Omega)$, with $\mathfrak{b}_i(z) > 0$ for all $z \in \partial\Omega$ and $\gamma_i \geq 0$ on $\partial\Omega$, such that

$$\lim_{\substack{x \to z \\ x \in \Omega, z \in \partial\Omega}} \frac{\mathfrak{a}_i(x)}{\mathfrak{b}_i(z)[\text{dist}(x, \partial\Omega)]^{\gamma_i(z)}} = 1, \qquad 1 \leq i \leq n. \tag{0.16}$$

By [41], it is well known that, setting

$$\mu_i(z) := \frac{\gamma_i(z) + 2}{p_i - 1}, \qquad 1 \leq i \leq n, \tag{0.17}$$

for every $z \in \partial\Omega$, these $\mu_i(z)$'s provide us with the blow-up rates on $\partial\Omega$ of the positive solutions of the uncoupled singular problems

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega, \end{cases} \quad 1 \leq i \leq n.$$

Let $z \in \partial\Omega$ and suppose the n equations of (0.1) have been re-ordered so that

$$0 < \mu_n(z) \leq \mu_{n-1}(z) \leq \cdots \leq \mu_1(z). \tag{0.18}$$

Then, we have the next result, which is completely new ever in the very special case when $n = 2$.

Theorem 0.6 *Let $z \in \partial\Omega$ such that* (0.18) *is satisfied and consider the next partition of the subscripts set*

$$\begin{aligned} I_+ &:= \{i \in \{1,\dots,n\} :\ \mu_i(z) + 2 - \mu_1(z) > 0\}, \\ I_0 &:= \{i \in \{1,\dots,n\} :\ \mu_i(z) + 2 - \mu_1(z) = 0\}, \\ I_- &:= \{i \in \{1,\dots,n\} :\ \mu_i(z) + 2 - \mu_1(z) < 0\}. \end{aligned}$$

Let $k \in \{1,\dots,n\}$ be such that

$$I_M := \{i \in \{1,\dots,n\} :\ \mu_i(z) = \mu_1(z)\} = \{1,\dots,k\}.$$

Then, any positive solution of (5.1)*, $u = (u_1,\dots,u_n)$, satisfies* (0.15) *with*

$$\alpha_i(z) := \begin{cases} \mu_i(z) & \text{if} \quad i \in I_+ \cup I_0, \\ \dfrac{\mu_1(z) + \gamma_i(z)}{p_i} & \text{if} \quad i \in I_-, \end{cases}$$

and

$$A_i(z) := \begin{cases} \left(\dfrac{\mu_i(z)(\mu_i(z)+1)}{\mathfrak{b}_i(z)}\right)^{\frac{1}{p_i-1}} & \text{if} \quad i \in I_+, \\[2ex] \left[\dfrac{1}{\mathfrak{b}_i(z)}\displaystyle\sum_{j=1}^{k} a_{ij}\left(\dfrac{\mu_1(z)(\mu_1(z)+1)}{\mathfrak{b}_j(z)}\right)^{\frac{1}{p_j-1}}\right]^{\frac{1}{p_i}} & \text{if} \quad i \in I_-, \\[2ex] A_{0,i} & \text{if} \quad i \in I_0, \end{cases}$$

where $A_{0,i}$ stands for the unique positive solution of the equation

$$\mathfrak{b}_i(z)x^{p_i} - \mu_i(z)(\mu_i(z)+1)x = \sum_{j=1}^{k} a_{ij}\left(\frac{\mu_1(z)(\mu_1(z)+1)}{\mathfrak{b}_j(z)}\right)^{\frac{1}{p_j-1}}.$$

The most astonishing fact is that if the blow-up rates of the uncoupled system are sufficiently close between them, then the corresponding blow-up rates of (0.1) equal the uncoupled ones, without taking into account the size of the coupling coefficients. On the other hand, the blow-up rates of (0.1) can differ from the uncoupled blow-up rates when some of the uncoupled blow-up rate is far from the others. As our model introduces spatial heterogeneities, the uncoupled blow-up rates may vary throughout $\partial\Omega$, by (0.16). Hence, the blow-up rates of (0.1) *may behave like the uncoupled ones on some places of* $\partial\Omega$, while may be affected by the coupled coefficients on other locations. This surprising behavior was documented by the first time in [50]. No previous result on blow-up rates for n-species cooperative systems is available before [51].

0.3 Conclusions

Although problems with singular boundary conditions have an enormous difficulty, by the huge number of technicalities involved in their mathematical treatment, we have succeeded in getting a number of uniqueness theorems based on the comparison theorem. Hence, imposing $a_{ij} > 0$ for all $1 \leq i \leq n$ is imperative to carry out our analysis. Actually, if some coupling coefficient, a_{ij}, becomes negative, then the maximum principle for cooperative systems [54] fails, and therefore, this comparison technique also fails. It would be pretty interesting to consider this kind of situations in the future. A good way to approach this problem would be through the *quasi-cooperative* case, i.e. when $n = 2$ and $a_{12}a_{21} > 0$.

As a byproduct of our investigations, we are providing with some refinements and extensions of the usual uniqueness techniques as well as some new ideas. Theorem 0.5 is an adaptation of a result of [46] to cover the class of cooperative systems with radial symmetry. This adaptation is certainly not trivial in the annular case, where an auxiliary construction is required. Theorems 0.1 and 0.2 are also inspired by the main idea of [46]. It consists in a refinement that allows us to relax the hypotheses extremely. This apparently new technique should be able to apply readily to the cooperative case.

Theorem 0.3 sharpens all previous hypotheses concerning to the nonlinear term of the equation of (0.6), ever in the autonomous case, which is the result given by Corollary 0.4. The assumptions we have made on $\partial\Omega$ are rather general, but the ones made in [58, 59] are much weaker. On the other hand, we think that the technique developed in the proof of Theorem 0.3 might be adapted to cover more general cases, including non smooth domains or cooperative systems, but this is something we plan to do in another work.

The main difficulty of the proof of Theorem 0.6 is due to the presence on an arbitrary number of equations in (0.1). In our context we have ascertained the blow-up rates for the potential case, which is an important case, tough restricted. A future improvement may be reached by considering heterogeneous terms with non potential behavior on the boundary, in the spirit of [45], or even general nonlinearities. Of course, this is a truly non trivial problem! Going back to our result, the most important idea behind the proof is, probably, that we have studied the blow-up rates of (0.1) keeping in mind the uncoupled ones. We think this idea should work in larger classes of cooperative-type systems, like the n-equations counterpart of the one considered in [31].

Probably the best uniqueness theorem that one should expect is the following one for the single equation.

Conjecture. The problem

$$\begin{cases} \Delta u = \mathfrak{a}(x) f(u) & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \qquad \mathfrak{a} > 0,$$

with $\partial\Omega \in \mathcal{C}^2$, possesses a unique positive solution if and only if (0.5) has a unique positive solution for all $T > 0$

This would show that $\mathfrak{a}(x)$ does not play any important role in the uniqueness of the large solution of the equation.

Resumen

0.4 Introducción, motivación y metodología

El objetivo principal de esta tesis doctoral es establecer resultados de unicidad de solución, tan generales como sea posible, para el siguiente problema de contorno singular elíptico de tipo cooperativo,

$$\begin{cases} -\Delta u_i = \lambda_i u_i + \sum_{j=1, j\neq i}^{n} a_{ij} u_j - \mathfrak{a}_i(x) f_i(u_i) u_i & \text{en } \Omega, \\ u_i = +\infty & \text{en } \partial\Omega, \end{cases} \quad 1 \leq i \leq n, \tag{0.19}$$

donde $n \in \mathbb{N}$, Ω es un subdominio acotado de $\mathbb{R}^N$, $N \in \mathbb{N}$, cuya frontera es de clase $C^{2+\nu}$ para cierto $\nu \in (0,1)$, $\lambda_i \in \mathbb{R}$, $a_{ij} > 0$ representan los parámetros de acople y $\mathfrak{a}_i \in \mathcal{C}^\nu(\bar{\Omega})$, con $\mathfrak{a}_i(x) > 0$ para todo $x \in \Omega$ y $1 \leq i, j \leq n$. Las condiciones de frontera deben entenderse como

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} u_i(x) = +\infty, \quad 1 \leq i \leq n,$$

de ahí que sea habitual llamar a las soluciones de (0.19) *largas*, o *explosivas*. El modelo representado en (0.19) es una generalización de la ecuación logística-difusiva con término heterogéneo estudiada en los capítulos 6, 7 y 8 de [48], para cubrir el caso de un sistema con acoplamiento lineal de tipo cooperativo, e.d. con $a_{ij} > 0$ para todo $1 \leq i, j \leq n$. El caso en que (0.19) tiene una única ecuación ha sido ampliamente estudiado en la literatura. Dicho caso se remonta a los resultados pioneros de L. Bieberbach [9] y H. Rademacher [69], que consideran la ecuación $\Delta u = e^u$ en dos y tres dimensiones, respectivamente, y los de J. B. Keller [35] y R. Osserman [66], que estudian la ecuación $\Delta u = f(u)$ para cierto tipo de operadores monótonos. Desde entonces, los diferentes estudios sobre soluciones de ecuaciones con explosión en la frontera han seguido diversos caminos. Por ejemplo, estableciendo la existencia o la unicidad para términos no lineales más generales, como en [8, 19, 28, 14], para modelos con términos espaciales, [73, 7, 18, 12, 13, 41, 45], para operadores elípticos más generales, [73, 16] o incluso considerando dominios con frontera irregular, [58, 59]. Otra rama muy estudiada es la de establecer la expansión asintótica en la frontera de la solución explosiva, como en [2, 14, 3]. También hay resultados sobre multiplicidad de solución en el contexto de problemas superlineales indefinidos, [42, 60, 55, 56]. Sin embargo, la literatura existente para sistemas cooperativos con anterioridad a nuestro trabajo es realmente escasa. En efecto, un trabajo pionero en el uso de sistema cooperativos como el de (0.19) es [61, 62, 63], donde se establece la existencia y unicidad de soluciones para un sistema de dos ecuaciones con condiciones Dirichlet homogéneas en la frontera. Uno de los primeros trabajos sobre soluciones largas en sistemas es [32], donde se estudia el modelo simbiótico de Lotka-Volterra. También se puede encontrar el más reciente [31], que estudia la existencia y unicidad de un sistema de tipo cooperativo de reacción difusión, aunque en el caso *autónomo*. Hasta donde sabemos, el primer artículo que prueba la existencia de solución en (0.19), para el caso $n = 2$, es [4]. Así, cuando comenzamos a estudiar el modelo (0.19), el problema de la unicidad permanecía ampliamente abierto, incluso en el caso en que (0.19) se reduce a una única ecuación.

Desde una perspectiva matemática, el problema de estudiar la unicidad de solución en (0.19) tiene un gran interés en sí mismo. No obstante, podemos dar una motivación que parte del contexto de Dinámica de Poblaciones. Supongamos que $\mathcal{D} \subset \mathbb{R}^N$, $N \leq 3$, es un dominio acotado donde viven los individuos de una especie. Sea $u = u(t, x)$ la densidad de población de la especie en cuestión, que será una función positiva del espacio, $x \in \mathcal{D}$, y del tiempo, $t \geq 0$. Efectuemos ahora las siguiente hipótesis:

- La especie está dividida en n grupos, $u_1, \ldots, u_n$, de forma que cada grupo coopera con todos los demás, en el sentido de que el crecimiento de un grupo beneficia al crecimiento de los demás grupos.

- Cada grupo u_i se esparce aleatoriamente por el hábitat $\mathcal{D}$, con tasa de difusión d_i.

- Existen en $\mathcal{D}$ zonas con infinidad de recursos y zonas con recursos limitados. Esto es, en términos de Dinámica de Poblaciones, donde los recursos son ilimitados la especie experimenta un crecimiento malthusiano, mientras que el crecimiento es de tipo logístico si los recursos son finitos. Por simplicidad, supondremos que $\Omega \subset \bar{\mathcal{D}}$ es la zona que tiene recursos limitados.

Teniendo en cuenta todo lo anterior, un posible modelo matemático es el siguiente,

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - d_i \Delta u_i = \lambda_i u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij} u_j - \mathfrak{b}_i(x) f_i(u_i) u_i & x \in \mathcal{D},\ t > 0 \\ u_i = 0 & \text{en } \partial\mathcal{D}, \\ u(0) = u_0 > 0, \end{cases} \tag{0.20}$$

donde $1 \leq i \leq n$ y $\mathfrak{b}_i \in \mathcal{C}^\nu(\bar{\mathcal{D}})$ satisface

$$\begin{cases} \mathfrak{b}_i(x) > 0 & \text{si } x \in \Omega, \\ \mathfrak{b}_i(x) = 0 & \text{si } x \neq \Omega, \end{cases} \qquad 1 \leq i \leq n.$$

De esta manera, conseguimos combinar en $\mathcal{D}$ las leyes de crecimiento mencionadas arriba. Si hemos impuesto condiciones de frontera Dirichtlet homogéneas en (0.20) es únicamente por simplicidad; un problema muy interesante sería considerar otras condiciones de frontera.

En este punto, es natural preguntarse por el comportamiento asintótico de la única solución global de (0.20). La respuesta en muchos casos viene determinada por el estudio del problema (0.19). Suponiendo que los parámetros λ_i se encuentran dentro del rango apropiado, y llamando $L^{\min}$ y $L^{\max}$ a las soluciones *minimal* y *maximal* de (0.19), se tiene que

$$L^{\min} \leq \liminf_{t\to+\infty} u(t) \leq \limsup_{t\to+\infty} u(t) \leq L^{\max}, \quad x \in \Omega, \tag{0.21}$$

mientras que $u(t) \uparrow +\infty$ uniformemente en subconjuntos compactos de $\mathcal{D} \setminus \Omega$. Esto es, las *metasoluciones* gobiernan la dinámica de (0.20). Éstas son, *grosso modo*, soluciones estáticas de (0.20) que valen infinito en una zona de medida positiva; véase [33, 4] o [48, Chapter 5] para una definición rigurosa de metasolución. Así, si (0.19) tiene una única solución, $L^{\min} = L^{\max}$ en (0.21), por lo que podemos conocer el comportamiento de las soluciones de (0.20) para tiempos grandes.

La técnica fundamental para obtener unicidad en (0.19) es el siguiente principio de comparación, derivado de teorema de caracterización del principio del máximo, [54]. Sean $w_1, w_2 \in \left[\mathcal{C}^{2+\nu}(\partial\Omega)\right]^n$ tales que $w_{1,i} < w_{2,i}$ para cada $1 \leq i \leq n$. Entonces, si para cada $k = 1, 2$ llamamos $\theta_{[\Omega,w_k]}$ a la única solución de

$$\begin{cases} -\Delta u_i = \lambda_i u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij} u_j - \mathfrak{a}_i(x) f_i(u_i) u_i & \text{en } \Omega, \\ u_i = w_{k,i} > 0 & \text{en } \partial\Omega, \end{cases} \qquad 1 \leq i \leq n, \tag{0.22}$$

se tiene que $\theta_{[\Omega,w_1]} < \theta_{[\Omega,w_2]}$. El mismo resultado se cumple si $w_2 = (+\infty, \ldots, +\infty)$; basta con aplicar el resultado anterior al dominio

$$\Omega_\varepsilon := \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$$

para ε suficientemente pequeño y hacer $\varepsilon \downarrow 0$. Aplicando con ingenio esta técnica de comparación hemos obtenido buena parte de los resultados aquí reunidos. Concretamente, los resultados de los capítulos 2, 3 y 4 se deben enteramente a este principio, a la vez que los resultados del capítulo 5 tambíen lo utilizan.

También hemos aplicado, en el último capítulo de la tesis, la técnica más frecuente en la literatura: hallar fórmulas explícitas para las *tasas de explosión* en la frontera de las soluciones de (0.19). De hecho, este capítulo adapta la técnica de localización introducida en [41] al sistema (0.19).

0.5 Contenido

Los resultados establecidos en esta tesis doctoral han sido hallados por el autor y su director de tesis durante los años 2015, 2016 y los tres primeros meses de 2017. De estos resultados, ya han sido publicados los reunidos en [49, 50, 51], los de [52] se encuentran en proceso de publicarse y los de [53] han sido recientemente enviados a publicar. Para presentarlos en esta tesis hemos elegido un orden distinto al cronológico, clasificando los resultados en dos partes, en función de si estudian (0.19) en el caso particular $n = 1$ o en el caso general. Las aportaciones principales de cada capítulo son:

(1) El primer capítulo contiene los resultados de [53]. En él se introduce el estudio de soluciones largas a través del modelo más sencillo posible, el obtenido al reducir (0.19) a una ecuación autónoma en una dimensión,

$$\begin{cases} u'' = f(u) & t \in (-T, T), \\ u(\pm T) = +\infty. \end{cases} \tag{0.23}$$

La ventaja que tiene este modelo es que podemos estudiar sus soluciones existentes sin imponer de entrada ninguna condición sobre f, al contrario que todos los estudios disponibles, que imponen como mínimo que $f(u)$ sea *positiva para u grande*, e.d. que exista $M > 0$ tal que $f(u) > M \geq 0$ para todo $u > M$. Esta diferencia hace que podamos tener comportamientos novedosos. Por ejemplo, la función que asigna a cada solución el tiempo máximal de existencia, que cuando es finita viene representada por

$$\mathcal{T}(x) := \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{du}{\sqrt{\int_x^u f}}, \qquad x := u(0),$$

no tiene por qué cumplir que

$$\liminf_{x \to +\infty} \mathcal{T}(x) = 0,$$

al contrario que en el caso en que f es positiva para u grande. Tambíen proporcionamos un resultado de multiplicidad de soluciones largas para cualquier f positiva que satisface la condición de Keller–Osserman, rompiendo la monotonía en subconjuntos de medida arbitrariamente pequeña.

(2) El capítulo 2 contiene los resultados reunidos en [52], que consisten en tres teoremas de unicidad de soluciones largas para la ecuación logística difusiva sublineal

$$-\Delta u = \lambda u - \mathfrak{a}(x) f(u).$$

El primero de ellos establece unicidad en dominios estrellados cuando $\lambda \geq 0$ y el término no lineal, $f(u)$, cumple la siguiente condición,

$$\exists\, p > 1 \ : \ f(tu) \geq t^p f(u), \quad \text{para todo } \ t \geq 1, \ u \geq 0, \tag{0.24}$$

además de una condición de decrecimiento del término heterogéneo, $\mathfrak{a}(x)$, en un entorno de la frontera. El segundo teorema de unicidad aprovecha la misma idea de la demostración del primero para generalizar el resultado, cuando $\lambda = 0$, a una clase de dominios más general. Se trata de la clase formada por los dominios que admiten la representación

$$\Omega = \Omega_0 \setminus \bigcup_{i=1}^{k} \bar{\Omega}_i, \qquad \bar{\Omega}_i \subset \Omega_0, \;\; \bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset, \;\; 1 \leq i, j \leq k \in \mathbb{N},$$

donde cada Ω_i es un dominio estrellado. El último resultado del capítulo es de naturaleza ligeramente distinta a los anteriores. En él se prueba la unicidad en dominios cuya frontera es regular cuando $\lambda < \lambda_1[-\Delta, \Omega]$, $\mathfrak{a}(x)$ tiene un cierto decaimiento en $\partial\Omega$ y $f(u)$ es *superaditiva* con constante $C \geq 0$, e.d.

$$\exists C \geq 0 \quad : \quad f(a + b) \geq f(a) + f(b) \quad a, b \geq 0.$$

Esta propiedad de superaditividad se remonta al Teorema 0.3 de [58], donde Marcus y Véron la utilizan para obtener unicidad de soluciones largas de la ecuación

$$-\Delta u + f(u) = 0 \tag{0.25}$$

en dominios cuya frontera es localmente el grafo de una función continua. Ésta es, en efecto, una hipótesis en la regularidad de $\partial\Omega$ mucho más relajada que la nuestra, aunque, por otra parte, su prueba requiere que f sea una función convexa. De hecho, en (0.25) no hay término lineal, y no se puede añadir un término heterogéneo de manera sin hacer algún trabajo adicional.

(3) En el tercer capítulo introducimos el problema de contorno singular (0.19) y damos un esquema de la demostración de la existencia de soluciones minimal y maximal.

(4) El capitulo 4 expone el resultado principal de [49], donde se estudia el problema de unicidad para un sistema cooperativo con simetría radial, e.d. cuando

$$\mathfrak{a}_i(x) = \mathfrak{a}_i(|x|), \quad 1 \leq i \leq n,$$

y Ω es una bola o un anillo. Se prueba la unicidad cuando para cada $1 \leq i \leq n$, $\lambda_i \geq 0$, $\mathfrak{a}_i(|x|)$ es una función no creciente y $g_i(u) := f_i(u)u$ satisface la propiedad (0.24). En la demostración se utiliza únicamente el principio del máximo, sin aludir a las tasas de explosión en la frontera. Es el primer teorema que se ha publicado de unicidad de soluciones largas para un sistema de n ecuaciones.

(5) Finalmente, el capitulo 5 está dedicado al estudio de las tasas de explosión de las soluciones de (0.19). Estos resultados fueron hallados primero para el caso especial $n = 2$, [50], y después para el caso general, [51]. En concreto, hallamos las tasas de explosión cuando, para cada $1 \leq i \leq n$, $f_i(u) = u^{p_i - 1}$ para algún $p_i > 1$ y

$\mathfrak{a}_i(x)$ tiene un decaimiento tipo potencial en la frontera de Ω, *no necesariamente con tasa fija*, en el sentido de que existen $\mathfrak{b}_i, \gamma_i \in \mathcal{C}(\partial\Omega)$, tales que $\mathfrak{b}_i(z) > 0$ para cada $z \in \partial\Omega$ y $\gamma_i \geq 0$ en $\partial\Omega$, de forma que

$$\lim_{\substack{x\to z\\ x\in\Omega, z\in\partial\Omega}} \frac{\mathfrak{a}_i(x)}{\mathfrak{b}_i(z)[\operatorname{dist}(x,\partial\Omega)]^{\gamma_i(z)}} = 1, \qquad 1 \leq i \leq n. \tag{0.26}$$

El caso $n = 1$ es bien conocido desde [41]. Sin embargo, para sistemas tipo cooperativos los únicos resultados que había disponibles cuando comenzamos nuestra investigación eran los de [31]. El teorema principal del capítulo 5 establece que para cada $z \in \partial\Omega$ existen $\alpha_i(z), A_i(z) > 0$ tales que para cualquier solución de (0.19), $u = (u_1, \ldots, u_n)$, se tiene que

$$\lim_{\substack{x\to z\\ x\in\Omega}} \frac{u_i(x)}{\operatorname{dist}(x,\partial\Omega)^{-\alpha_i(z)}} = A_i(z), \quad 1 \leq i \leq n.$$

De forma más precisa, lo que dice el teorema es que las tasas de explosión de (0.19) se puede establecer en función de cómo estén de cercanas entre si las tasas de explosión del *sistema desacoplado*, que es el obtenido cuando $a_{ij} = 0$ para todo $1 \leq i, j \leq n$. Lo más sorprendente es que si todas las tasas de explosión del sistema desacoplado están suficientemente cercanas entre sí, las tasas de explosión del correspondiente sistema acoplado son las mismas, independientemente del tamaño de los términos de acople, a_{ij}. El teorema también determina cómo cambia la naturaleza de las tasas de explosión de (0.19) cuando alguna de las tasas de explosión del sistema desacoplado se aleja de las demás. Como nuestro modelo es heterogéneo, las tasas de explosión del sistema desacoplado pueden variar en función del punto $z \in \partial\Omega$, debido a (0.26). Por tanto, las tasas de explosión se pueden comportar en unas zonas de $\partial\Omega$ *como si el sistema estuviera desacoplado* y en otras zonas de $\partial\Omega$ verse afectadas por los téminos de acople a_{ij}. Este comportamiento quedó registrado por primera vez en [50]. Más aún, ningún artículo previo a [51] estudia las tasas de explosión de un sistema de n ecuaciones.

0.6 Conclusiones

Pese a que los problemas con condiciones singulares en la frontera tienen una gran dificultad, hemos podido obtener una amplia gama de teoremas de unicidad, fundamentalmente usando el teorema de comparación descrito en 0.4. Que el sistema sea de tipo cooperativo es algo crucial en el análisis realizado. En efecto, si alguno de los a_{ij} de (0.19) fuese negativo no tendríamos disponible el teorema de caracterización del principio del máximo [54], por lo que fallarían las técnicas de comparación, que son la herramienta fundamental de esta tesis. Claro que, por otra parte, sería muy interesante considerar este tipo de situaciones en el futuro. Seguramente un buen acercamiento sería a través del caso en que (0.19) es *cuasi-cooperativo*, e.d. cuando $n = 2$ y $a_{12}a_{21} > 0$, ya que en este caso también está disponible el principio del máximo (véase [48, Chapter 10]).

Como resultado de nuestra investigación, hemos refinando y extendido las técnicas de unicidad más habituales, además de aportar algunas ideas nuevas. El resultado descrito en **(4)** es una adaptación del resultado de [46] para el sistema cooperativo. Esta adaptación, más inmediata para el caso de la bola, requiere una construcción auxiliar mucho más compleja para el caso del anillo. Asimismo, los dos primeros resultados resumidos en **(2)** se deben a un refinamiento extremo de la idea fundamental de [46], que nos permite relajar enormemente las hipótesis. Esta nueva técnica se debería poder usar para extender estos resultados al caso de un sistema cooperativo sin presentar demasiadas complicaciones.

El último resultado resumido en **(2)** es de una generalidad asombrosa, pues, como se verá en el capítulo 2, la propiedad de superaditividad de f relaja todas las hipótesis que se han hecho previamente, incluso en el caso autónomo. En cuando a las hipótesis que hemos pedido a $\partial\Omega$, aunque sean bastante generales, no son tan débiles como las realizadas en [58, 59]. Por otro lado, creemos que las técnicas desarrolladas en nuestra prueba podría adaptarse para cubrir casos más generales, como dominios cuya frontera no sea necesariamente diferenciable, o sistemas cooperativos. Pero eso es algo que haremos en otro trabajo.

Respecto al último de nuestros teoremas, descrito en **(5)**, quisíeramos destacar la dificultad de determinar las tasas de explosión para una cantidad arbitraria de ecuaciones. En nuestro caso, hemos logrados dar con las fórmulas para el caso concreto en el que tanto los términos no lineales como los factores espaciales se comportan como potencias. Una posible generalización, nada obvia, sería hallar las tasas prescindiendo de alguna de estas hipótesis. Seguramente la idea más importante para obtener nuestro resultado haya sido pensar en las tasas de explosión de (0.19) en función de las tasas del sistema desacoplado. Esta forma de proceder podría intentar aplicarse a otro tipo de sistemas cooperativos, con diferente acople, como el equivalente de n ecuaciones al estudiado en [31].

Para terminar este resumen, dejamos el enunciado de lo que creemos que debería ser el teorema óptimo de unicidad para la ecuación.

Conjetura. El problema

$$\begin{cases} \Delta u = \mathfrak{a}(x) f(u) & \text{en } \Omega \\ u = +\infty & \text{en } \partial\Omega, \end{cases}$$

con $\partial\Omega \in \mathcal{C}^1$ tiene una única solución si y sólo si (0.23) admite una única solución para todo $T > 0$.

¡Quizás aún falte mucho tiempo para poder probar o refutar esta conjetura!

Part I

Large solutions for the equation

Chapter 1

Multiplicity of large solutions in one spatial dimension

The main goal of this chapter is to analyze the existence, uniqueness and multiplicity of positive solutions of the singular boundary value problem

$$\begin{cases} u'' = f(u), & t \in [0,T], \\ u'(0) = 0, \quad u(T) = +\infty, \end{cases} \tag{1.1}$$

where $T \in (0,\infty)$ and $f \in \mathcal{C}^1[0,+\infty)$ satisfies $f(0) = 0$. So, 0 is a constant solution of $u'' = f(u)$. By reflection around $t = 0$, these solutions provide us with the positive large solutions of the singular problem

$$\begin{cases} u'' = f(u), & t \in [-T,T], \\ u(-T) = u(T) = +\infty. \end{cases}$$

By a positive large solution of (1.1) it is meant any positive solution in $[0,T)$ such that

$$\lim_{t \uparrow T} u(t) = +\infty.$$

The singular problem (1.2) has been widely studied in the literature. However, almost all available results assumed $f \geq 0$ in $[0,\infty)$, or, at least, for sufficiently large u, [14, 15]. Here we are not imposing any special restriction on the sign of f.

As for arbitrary $f(u)$ the existence and multiplicity of positive solutions of (1.1) might depend on the length of the interval, $T > 0$, it is very natural to analyze the existence of positive explosive solutions of the associated Cauchy problem

$$\begin{cases} u'' = f(u), \\ u(0) = x > 0, \quad u'(0) = 0, \end{cases} \tag{1.2}$$

where $x > 0$ is regarded as parameter.

Although there is a huge interest in analyzing the existence and the uniqueness of large solutions for wide classes of singular sublinear boundary value problems, because they provide us with the limiting profiles as time grows of the solutions of large classes of diffusive logistic equations of degenerate type, where the species can growth exponentially in some protection zones of the territory, [33, 34, 18, 12, 13, 48], and the classical condition of J. B. Keller [35] and R. Osserman [66], as well as some of their variants, as those of [40] and [19], have dominated the scenario of this theory during the last two decades, except in Section 1.1 of [48], no serious effort has been made to realize the true meaning of the several Keller–Osserman conditions involved.

In most of the literature collected in our bibliography, the Keller–Osserman condition is imposed in order to guarantee the existence of large solutions of some autonomous or non-autonomous problem where the nonlinearity uses to be chosen so that the underlying semilinear elliptic equation can exhibit, at most, a unique large solution; the main aim of most of these papers being to show that any large solution must have the same blow-up rate on the edges of the domain to infer from this feature the uniqueness of the large solution by means of a rather standard comparison device. As a consequence of this severe focusing of most of experts's attention, the real meaning of the so called Keller–Osserman condition remains a true enigma!

This prompted us to focus attention in the simplest autonomous one dimensional singular problem (1.1) in order to characterize, simply, the values of T for which this singular problem admits a positive solution. Should it be the case, our second aim being either establishing uniqueness, or multiplicity results, keeping in mind, rather crucially, that, in general, (1.1) might admits positive solutions for some range of values of T but not for others! This apparently new methodology contrasts heavily with most of the available results in the literature, where the Keller–Osserman condition entails the existence of a positive solution of the singular problem (1.1) for every $T > 0$, because the function $f(u)$ is required to satisfy some additional monotonicity property to infer from it the uniqueness of the positive solution of (1.1). So, our methodology here seems completely new.

In the context of superlinear indefinite problems there are available some multiplicity results, as [29, 42, 60], but probably the most astonishing existing multiplicity results are those of [56] and [55], where it was established that if $a(x)$ changes of sign in the interval $[-T, T]$, then the problem

$$\begin{cases} -u'' = \lambda u - a(x)u^p, & \text{in } [-T, T], \\ u(-T) = u(T) = +\infty, \end{cases}$$

where $p > 1$, can admit an arbitrarily large number of positive solutions by taking $\lambda < 0$ sufficiently large. In these results the multiplicity is caused by the fact that $a(x)$ changes of sign and $\lambda < 0$ is very large, and is far from attributable to the nature of the nonlinearity, $f(u) = u^p$, with $p > 1$, for which the singular problem (1.1) has a unique positive solution for each $T > 0$. Instead, the multiplicity results of this chapter are attributable to the *oscillating* properties of the function $f(u)$ in (1.1), even when $f(u) > 0$ for all $u > 0$. Consequently, our findings here are of a great novelty and independent of all previous available multiplicity results.

Besides the introduction, this chapter consists of 3 sections. Section 1.1 studies the Cauchy problem (1.2), Section 1.2 deals with the existence, uniqueness and multiplicity of positive solutions for the singular problem (1.1), and Section 1.3 gives some interesting counterexamples to an important result of [19]. Astonishingly, our main multiplicity result in Section 1.2 shows how destroying the monotonic character of any increasing function $f(u)$ on a compact set with arbitrarily small measure can originate an arbitrarily large number of positive solutions for the singular problem (5.1).

1.1 The associated Cauchy problem

Since $f \in \mathcal{C}^1[0,+\infty)$ and $x > 0$, by the main existence theorem for $\mathcal{C}^1$ nonlinearities, it becomes apparent that, for every $x > 0$, the initial value problem (1.2) possesses a maximal positive solution, $u(t)$, $t \in [0, T_{\max}(x))$, for some $T_{\max}(x) \in (0,+\infty]$. Moreover, the following result holds.

Theorem 1.1 *The following properties are satisfied:*

(a) *If $f(x) = 0$, then x is a constant solution and hence, $T_{\max}(x) = +\infty$.*

(b) *If $f(x) > 0$ and $T_{\max}(x) = +\infty$, then, either $u(t)$ is periodic, or*

$$\lim_{t\uparrow+\infty} u(t) = \omega$$

for some $\omega > x$ such that $f(\omega) = 0$, or

$$\lim_{t\uparrow+\infty} u(t) = +\infty.$$

(c) *If $f(x) > 0$ and $T_{\max}(x) < +\infty$, then $u'(t) > 0$ for all $t \in [0, T_{\max}(x))$,*

$$\lim_{t\uparrow T_{\max}(x)} u(t) = +\infty, \tag{1.3}$$

$\int_x^\theta f > 0$ for all $\theta > x$, and

$$T_{\max}(x) = \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{d\theta}{\sqrt{\int_x^\theta f}} < +\infty. \tag{1.4}$$

(d) *If $f(x) < 0$ and $T_{\max}(x) = +\infty$, then, either $u(t)$ is periodic, or $u'(t) < 0$ for all $t > 0$ and there exists $\alpha \in [0, x)$ such that $f(\alpha) = 0$ and*

$$\lim_{t\uparrow\infty} u(t) = \alpha.$$

(e) *If* $f(x) < 0$ *and* $T_{\max}(x) < +\infty$*, then* $u'(t) < 0$ *for all* $t \in (0, T_{\max}(x)]$ *and* $u(T_{\max}(x)) = 0$.

Proof: If $f(x) = 0$, then x is a constant solution of (1.2) and hence, $T_{\max}(x) = +\infty$. In particular, $u(t)$ is periodic. This proves Part (a).

Now, suppose that $f(x) > 0$. Then,

$$u''(0) = f(u(0)) = f(x) > 0$$

and hence, since $u'(0) = 0$, there exists $\delta > 0$ such that $u'(t) > 0$ for all $t \in (0, \delta)$. Either $u'(t) > 0$ for all $t \in (0, T_{\max}(x))$, or there exists $t_0 > 0$ such that $u'(t) > 0$ for all $t \in (0, t_0)$ and $u'(t_0) = 0$. In the second case, by reflecting $u(t)$ about $t = t_0$, it becomes apparent that $T_{\max}(x) = +\infty$ and that $u(t)$ is a nontrivial periodic solution of $u'' = f(u)$. Suppose

$$u'(t) > 0 \quad \text{for all} \;\; t \in (0, T_{\max}(x)) \tag{1.5}$$

and, in addition, $T_{\max}(x) = +\infty$. Then, by (1.5),

$$\lim_{t \uparrow +\infty} u(t) = \omega \in (0, +\infty]$$

is well defined. Moreover, $f(\omega) = 0$ if $\omega < +\infty$, because

$$0 = \lim_{t \uparrow +\infty} u''(t) = \lim_{t \uparrow +\infty} f(u(t)) = f(\omega),$$

which concludes the proof of Part (b).

Suppose (1.5) and $T_{\max}(x) < +\infty$. Then,

$$\limsup_{T \uparrow T_{\max}(x)} \big(u(t) + u'(t)\big) = \infty. \tag{1.6}$$

Moreover, for each $t \in (0, T_{\max}(x))$, integrating the differential equation yields

$$u'(t) = \int_0^t f(u(s))\, ds.$$

Thus, if there is a constant C such that $u(s) \le C$ for all $s \in [0, T_{\max}(x))$, then

$$|u'(t)| \le T_{\max}(x) \max_{u \in [0,C]} |f(u)| < +\infty,$$

which contradicts (1.6). Therefore,

$$\lim_{t \uparrow T_{\max}(x)} u(t) = +\infty,$$

which provides us with (1.3). Finally, multiplying $u'' = f(u)$ by u' and integrating in $[0, t]$, $t < T_{\max}(x)$, we obtain that

$$\frac{1}{2}(u'(t))^2 = \int_x^{u(t)} f(s) ds > 0 \quad \text{for all} \;\; 0 < t < T_{\max}(x). \tag{1.7}$$

Since $u(t)$ ranges in $[x, +\infty)$ as $t \in [0, T_{\max}(x))$, (1.5) and (1.7) imply that $\int_x^\theta f > 0$ for all $\theta > x$. Moreover,

$$T_{\max}(x) = \int_0^{T_{\max}(x)} dt = \int_0^{T_{\max}(x)} \frac{u'(t)}{\sqrt{2\int_x^{u(t)} f(s)\,ds}}\,dt = \int_x^{+\infty} \frac{d\theta}{\sqrt{2\int_x^\theta f}},$$

which establishes (1.4) and ends the proof of Part (c).

Finally, suppose $f(x) < 0$. Then, since $u''(0) = f(x) < 0$, there exists $\delta > 0$ such that $u'(t) < 0$ for all $t \in (0, \delta)$. If there exists $t_0 > 0$ such that $u'(t) < 0$ for all $t \in (0, t_0)$ and $u'(t_0) = 0$, then $u(t)$ is periodic and hence $T_{\max}(x) = +\infty$. Therefore, if $u(t)$ is not periodic and $T_{\max}(x) = +\infty$, then $u'(t) < 0$ for all $t > 0$ and hence, Part (d) holds. If $T_{\max}(x) < +\infty$, necessarily $u(T_{\max}(x)) = 0$. Moreover, $u'(T_{\max}(x)) < 0$, because if it vanishes, then $u \equiv 0$, which is impossible. This ends the proof. □

By Theorem 1.1, if there exists $T > 0$ for which (1.1) possesses a positive solution, then, setting $x := u(0)$, we have that $x > 0$, $f(x) > 0$, $\int_x^\theta f > 0$ for all $\theta > x$, and

$$T = \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{d\theta}{\sqrt{\int_x^\theta f}} = T_{\max}(x).$$

Moreover, the following converse holds.

Lemma 1.2 *Let $x > 0$ be such that $f(x) > 0$ and $\int_x^\theta f > 0$ for all $\theta > x$. Then,* (1.5) *holds. If, in addition,*

$$\frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{d\theta}{\sqrt{\int_x^\theta f}} < +\infty,$$

then the unique solution of the Cauchy problem (1.2) *blows up at*

$$T_{\max}(x) = \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{d\theta}{\sqrt{\int_x^\theta f}}.$$

Therefore, the singular problem (1.1) *admits a positive solution if $T = T_{\max}(x)$.*

Proof: Since $f(x) > 0$, by continuity, there exists $\delta > 0$ such that $u''(t) = f(u(t)) > 0$ for all $t \in (0, \delta)$. Hence, u' is increasing $(0, \delta)$. So, since $u'(0) = 0$, we find that $u'(t) > 0$ for all $t \in (0, \delta)$. Consider

$$\hat{\delta} := \sup\,\{\delta > 0 \;:\; u'(t) > 0 \;\text{ for all }\; t \in (0, \delta)\}.$$

Necessary $\hat{\delta} = T_{\max}(x)$, because, otherwise, we may infer from

$$\frac{1}{2}(u'(t))^2 = \int_x^{u(t)} f(s)ds \quad \text{for every } \; t \in (0, \hat{\delta}),$$

that
$$0 = \frac{1}{2}(u'(\hat{\delta}))^2 = \int_x^{u(\hat{\delta})} f(s)ds,$$
which contradicts the assumption that $\int_x^\theta f > 0$ for all $\theta > x$, since $u(\hat{\delta}) > x$. Thus,
$$u'(t) > 0 \quad \text{for all} \quad t \in (0, T_{\max}(x)),$$
which is the first assertion of the lemma. Thanks to (1.5), for every $t \in (0, T_{\max}(x))$, we find that
$$\begin{aligned} t = \int_0^t ds &= \int_0^t \frac{u'(s)}{\sqrt{2\int_x^{u(s)} f}}\, ds \\ &= \frac{1}{\sqrt{2}} \int_x^{u(t)} \frac{d\theta}{\sqrt{\int_x^\theta f}} < \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{d\theta}{\sqrt{\int_x^\theta f}} < +\infty. \end{aligned}$$
Therefore, letting $t \uparrow T_{\max}(x)$ yields
$$T_{\max}(x) \leq \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{d\theta}{\sqrt{\int_x^\theta f}} < +\infty.$$
Consequently, by Part (c) of Theorem 1.1, the solution blows up at $T_{\max}(x)$ and (1.4) holds. This ends the proof. □

According to these results, in searching for the solutions of the singular problem (1.1), it is natural to consider the set
$$\mathcal{D} := \left\{ x > 0 \ : \ f(x) > 0 \ \text{ and } \ \int_x^\theta f > 0 \ \text{ for all } \ \theta > x \right\},$$
as well as the operator $\mathcal{T} : \mathcal{D} \to (0, +\infty]$ defined by
$$\mathcal{T}(x) := \frac{1}{\sqrt{2}} \int_x^{+\infty} \frac{d\theta}{\sqrt{\int_x^\theta f(s)ds}} \tag{1.8}$$
for all $x \in \mathcal{D}$. Indeed, in terms of $(\mathcal{D}, \mathcal{T})$, the next result holds.

Theorem 1.3 *The singular boundary value problem* (1.1) *possesses a positive solution if, and only if, there exists $x \in \mathcal{D}$ such that $T = \mathcal{T}(x)$. Moreover, in such case*
$$T = \mathcal{T}(x) = T_{\max}(x) < +\infty,$$
where $T_{\max}(x)$ stands for the blow-up time of the solution of the Cauchy problem (1.2). *Furthermore, the number of positive solutions of* (1.1), *$n(T)$, is given by*
$$n(T) = \text{cardinal of } \{x \in D \ : \ \mathcal{T}(x) = T\}.$$

Similarly, if $\mathcal{T}(x) = +\infty$ *for some* $x \in \mathcal{D}$*, then* $T_{\max}(x) = +\infty$*. In particular, the solution of the Cauchy problem* (1.2) *cannot provide us with a solution of the singular problem* (1.1) *for some* $T > 0$*.*

Proof: Given $T > 0$, suppose that $T = \mathcal{T}(x)$ for some $x \in \mathcal{D}$. Then, by Lemma 1.2, the unique solution of the Cauchy problem (1.2), $u(t)$, satisfies (1.5) and

$$\lim_{t \uparrow T} u(t) = +\infty, \qquad T = T_{\max}(x) = \mathcal{T}(x).$$

Therefore, $u(t)$ provides us with a positive solution of the singular problem (1.1).

Conversely, suppose that (1.1) admits a positive solution, $u(t)$, and set $x := u(0)$. If $x = 0$, then $u \equiv 0$, which contradicts our assumption. Thus, $x > 0$ and it follows from Theorem 1.1 that $T = T_{\max}(x) = \mathcal{T}(x)$.

The number of positive solutions of (1.1) equals $n(T)$ because one can establish a bijection between the solutions of the singular problem (1.1) and its values at $t = 0$, by the uniqueness of the solution of the initial value problem (1.2).

Finally, suppose that $\mathcal{T}(x) = +\infty$ for some $x \in \mathcal{D}$ and let $u(t)$ be the unique solution of the Cauchy problem (1.2). Since $x \in \mathcal{D}$, by the first statement of Lemma 5.26, $u'(t) > 0$ for all $t \in (0, T_{\max}(x))$. Hence,

$$t = \frac{1}{\sqrt{2}} \int_0^t \frac{u'(s)}{\sqrt{\int_x^{u(s)} f}} = \frac{1}{\sqrt{2}} \int_x^{u(t)} \frac{d\theta}{\sqrt{\int_x^{\theta} f}} \tag{1.9}$$

for all $t \in (0, T_{\max}(x))$. Suppose $T_{\max}(x) < +\infty$. Then, by (1.3),

$$\lim_{t \uparrow T_{\max}(x)} u(t) = +\infty$$

and hence, letting $t \uparrow T_{\max}(x)$ in (1.9) yields

$$T_{\max}(x) = \mathcal{T}(x) = +\infty$$

which contradicts $T_{\max}(x) < +\infty$. Therefore, $T_{\max}(x) = \mathcal{T}(x) = +\infty$, which ends the proof. □

Remark 1.4 Suppose that $f(u) \geq 0$ for all $u > 0$. Then, $\frac{d}{d\theta} \int_x^{\theta} f = f(\theta) \geq 0$ for all $\theta > 0$ and hence, $\int_x^{\theta} f > 0$ for all $\theta > x$ provided $f(x) > 0$. Thus, $\mathcal{D} = (0, +\infty) \setminus f^{-1}(0)$, though $\mathcal{T}(x)$ might be finite or infinity. Therefore, in this important case, the singular problem (1.1) admits a positive solution for some $T > 0$ if, and only if, there exists $x > 0$ such that $f(x) > 0$ and $\mathcal{T}(x) = T$. Moreover, $n(T)$ equals the number of such x's.

Naturally, we can extend the definition of $\mathcal{T}$ by setting

$$\mathcal{T}(x) = T_{\max}(x) \quad \text{for all } \ x \in (0, +\infty) \setminus \mathcal{D}.$$

According to Theorem 1.3, this implies that $\mathcal{T} \equiv T_{\max}$ in $(0, \infty)$. The following result establishes an important property of this extension.

Lemma 1.5 $T_{\max}(x) = +\infty$ *if* $x \in (0, +\infty) \setminus \mathcal{D}$ *with* $f(x) \geq 0$.

Proof: Let $x > 0$ be such that $x \notin \mathcal{D}$. If $f(x) = 0$, then x is a constant solution of $u'' = f(u)$ and hence $T_{\max}(x) = +\infty$. So, suppose $f(x) > 0$. According to Part (c) of Theorem 5.25, $T_{\max}(x)$ cannot be finite, because in that case we should have $\int_x^\theta f > 0$ for all $\theta > x$, and hence, $x \in \mathcal{D}$, which contradices our assumption. Therefore, in all possible cases, $T_{\max} = +\infty$. The proof is complete. □

By the theorem of differentiability of Peano, $T_{\max}(x)$, and hence the extended function $\mathcal{T}(x)$, is continuous with respect to $x \in (0, +\infty)$. Therefore, as soon as $\mathcal{T}(x_0) < +\infty$ for some $x_0 \in \mathcal{D}$, there exists an open subinterval $(a, b) \subset \mathcal{D}$, maximal for the inclusion, such that $x_0 \in (a, b)$, $\mathcal{T}(a) = \mathcal{T}(b) = +\infty$, and

$$\mathcal{T}(x) < +\infty \qquad \text{for all } \; x \in (a, b).$$

1.2 Existence, uniqueness and multiplicity

As the graph of the time map $\mathcal{T}$ can be as wiggle as we wish by choosing an appropriate $f(u)$, it is a challenge to analyze the general global behavior of $\mathcal{T}$, unless we impose some additional (severe) restrictions on $f(u)$, like the monotonicity of $f(u)$. The next result explains why.

Theorem 1.6 *Suppose that there exists* $x_0 \geq 0$*, with* $f(x_0) > 0$ *and* $\mathcal{T}(x_0) < +\infty$*, such that* $f(u)$ *is increasing for* $u > x_0$*. Then,* $\mathcal{T}(x)$ *is decreasing for* $x > x_0$*. In particular,* $\mathcal{T}(x) < \mathcal{T}(x_0) < +\infty$ *for all* $x > x_0$*. Moreover,*

$$\lim_{x \uparrow +\infty} \mathcal{T}(x) = 0.$$

Therefore, for every $T \in (0, \mathcal{T}(x_0))$*, the singular problem* (1.1) *possesses, at least, one positive solution.*

Proof: Since f is increasing in $[x_0, +\infty)$, we have that $f(x) > f(x_0) > 0$ for all $x > x_0$ and that $\int_x^\theta f > 0$ for all $\theta > x$. Thus, $[x_0, +\infty) \subset \mathcal{D}$ and hence, for every $x \geq x_0$, $\mathcal{T}(x)$ is given through (1.8). Consequently, performing the change of variable $\tau := \theta - x$, we find that, for every $x \geq x_0$,

$$\mathcal{T}(x) = \frac{1}{\sqrt{2}} \int_0^{+\infty} \frac{d\tau}{\sqrt{\int_x^{x+\tau} f(s)ds}} = \frac{1}{\sqrt{2}} \int_0^{+\infty} \frac{d\tau}{\sqrt{\int_0^\tau f(x+t)dt}}.$$

Suppose $x_0 \leq x < y$. Then, $f(x_0 + t) \leq f(x + t) < f(y + t)$ for all $t > 0$ and hence,

$$\sqrt{\int_0^\tau f(x_0 + t)dt} \leq \sqrt{\int_0^\tau f(x + t)dt} < \sqrt{\int_0^\tau f(y + t)dt} \qquad \text{for every } \; \tau > 0.$$

Since $\mathcal{T}(x_0) < +\infty$, the involved integrals are convergent. Therefore, we find that $\mathcal{T}(x) > \mathcal{T}(y)$. The monotonicity of $\mathcal{T}$ for $x \geq x_0$, entails the existence of the limit

$$L := \lim_{x \uparrow +\infty} \mathcal{T}(x) \geq 0.$$

The following result of S. Dumont et al. [19] guarantees that actually $L = 0$.

Lemma 1.7 *Suppose that there exists $x_0 > 0$ such that $f(u) \geq 0$ for all $u \geq x_0$, and $\mathcal{T}(x) < +\infty$ for some $x > x_0$. Then,*

$$\liminf_{x \uparrow +\infty} \int_0^{+\infty} \frac{d\tau}{\sqrt{\int_0^\tau f(x+t)dt}} = 0. \tag{1.10}$$

Finally, the last assertion of the theorem is a direct consequence from Theorem 1.3, as for every $T \in (0, T(x_0))$ there exists $x > x_0$ such that $\mathcal{T}(x) = T$. □

Naturally, the next result follows easily from Theorem 1.6.

Corollary 1.8 *Suppose $f(0) = 0$, f is increasing, and $\mathcal{T}(x_0) < +\infty$ for some $x_0 > 0$. Then, the singular problem* (1.1) *possesses a unique positive solution for each $T > 0$.*

Proof: Since $f(0) = 0$, $\mathcal{T}(0) = +\infty$. Moreover, by Theorem 1.6,

$$\lim_{x \uparrow +\infty} \mathcal{T}(x) = 0.$$

Therefore, since $\mathcal{T}$ is continuous and decreasing when it is finite, for every $T > 0$ there exists a unique $x > 0$ such that $\mathcal{T}(x) = T$. Theorem 1.3 ends the proof. □

Remark 1.9 In order to get the existence of a positive solution of the singular problem (1.1) for sufficiently small $T > 0$ one should impose

$$\liminf_{x \to +\infty} \mathcal{T}(x) = 0. \tag{1.11}$$

Nevertheless, even when $f \geq 0$ or $f(u) > 0$ for all $u > 0$, the singular problem (1.1) might admit an arbitrarily large number of positive solutions for sufficiently large $T > 0$, as established by the next result.

Theorem 1.10 *Let $x_1, \ldots, x_p \in (0, +\infty)$ be distinct and $f \in \mathcal{C}^1[0, +\infty)$ such that*

(a) *$f(0) = f(x_j) = 0$ for each $j \in \{1, ..., p\}$ and $f(u) > 0$ if $u \notin \{x_1, ..., x_p\}$.*

(b) *Setting $x_0 := 0$ and $x_{p+1} := +\infty$, for every $j \in \{0, ..., p\}$, there exists $x_0^j \in (x_j, x_{j+1})$ such that $\mathcal{T}(x_0^j) < +\infty$.*

Then, there exists $T^* > 0$ *such that* (1.1) *possesses, at least* $2p+1$ *positive solutions for every* $T > T^*$. *Moreover, there exists* $T_* > 0$ *such that* (1.1) *possesses, at least, a positive solution for each* $T < T_*$.

If, in addition, f *is increasing for sufficiently large* u, *then* $T_* > 0$ *can be shortened, if necessary, so that* (1.1) *admits a unique positive solution for every* $T < T_*$.

Proof: Since x_j, $0 \le j \le p$, are constant solutions of $u'' = f(u)$,

$$\mathcal{T}(x_j) = T_{\max}(x_j) = +\infty. \tag{1.12}$$

Moreover, since on each of the intervals (x_{j-1}, x_j), $1 \le j \le p+1$, $\mathcal{T}$ is assumed to be somewhere finite, the following values are well defined

$$\min_{x \in (x_{j-1}, x_j)} \mathcal{T}(x) < +\infty, \quad 1 \le j \le p,$$

and, thanks to Lemma 1.7,

$$\liminf_{x \uparrow +\infty} \mathcal{T}(x) = 0.$$

Therefore, combining the continuity of $\mathcal{T}$ with (1.12), it becomes apparent that for every

$$T > T^* := \max_{1 \le j \le p} \min_{x \in (x_{j-1}, x_j)} \mathcal{T}(x),$$

there exist at least $2p+1$ different points, $x \in \mathcal{D}$, such that $\mathcal{T}(x) = T$. The final assertion of the theorem is a byproduct of Theorem 1.6. This ends the proof. □

For every $x_1, \ldots, x_p \in (0, +\infty)$ with $x_i \ne x_j$ if $i \ne j$, the function

$$f(u) := u \prod_{j=1}^{p} (u - x_j)^2 \tag{1.13}$$

satisfies all the requirements of the theorem, even the monotonicity for $u > x_p$, and, since

$$\lim_{u \uparrow +\infty} \frac{f(u)}{u^{2p+1}} = 1,$$

it is easily seen that

$$\mathcal{T}(x) < +\infty \quad \text{for all } \; x \in \mathcal{D} = (0, \infty) \setminus \{x_1, ..., x_p\}.$$

Therefore, according to Theorem 1.10, there exist $0 < T_* < T^* < +\infty$ such that, for this special choice of $f(u)$, the singular problem (1.1) possesses at least $2p+1$ positive solutions if $T > T^*$ and a unique positive solution if $T < T_*$. Figure 1.1 shows the graph of the map $\mathcal{T}(x)$ associated to $f(u)$, given by (1.13), for the special choice $p = 2$, $x_1 = 4$ and $x_2 = 8$. Regarding $T > 0$ as a parameter we can easily ascertain the bifurcation diagram of the large positive solutions of the associated singular problem. As $f(u)$ is increasing for all $u > 8$, by

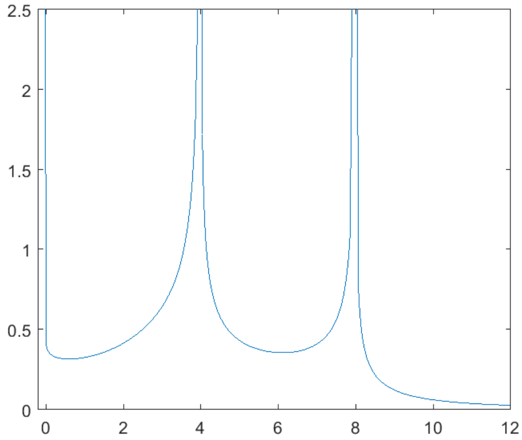


Figure 1.1: The time map $\mathcal{T}(x)$ for $f(u) = u(u-4)^2(u-8)^2$

Theorem 1.10 there exists $T_* >$ such that (1.1) admits a unique positive solution for every $T < T_*$. Moreover, from the proof of Theorem 1.6 we can infer that $u(0) = x \uparrow +\infty$ as $T \downarrow 0$. Hence, there exists a branch of large positive solutions that bifurcates from infinity. As we let T grow, two new branches of large positive solutions appear, just like shown by Figure 1.2, where we are plotting the parameter T in abscisas versus the initial data, x, in ordinates. This simple example provides us with a rather general scheme to generate as

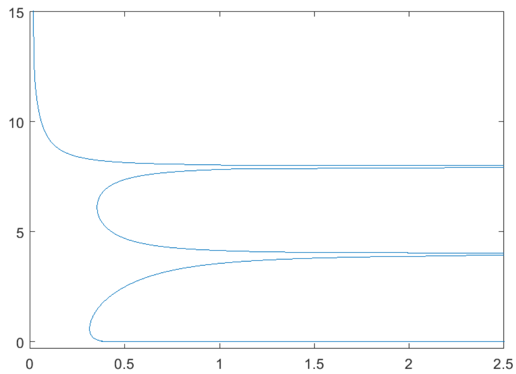


Figure 1.2: Bifurcation diagram for $f(u) = u(u-4)^2(u-8)^2$

many large positive solutions as we wish starting at an arbitrary increasing function $f(u)$. Indeed, fixed $p \geq 1$ and p distinct points, $x_1, ..., x_p > 0$, pick $\eta > 0$ such that

$$0 < x_{j-1} - \eta < x_{j-1} + \eta < x_j - \eta, \qquad 2 \leq j \leq p,$$

and then, change f inside each of the intervals

$$I_j := (x_j - \eta, x_j + \eta), \quad 1 \leq j \leq p,$$

by some smooth function g such that

$$\begin{aligned} &g^{i)}(x_j \pm \eta) = f^{i)}(x_j \pm \eta) \text{ for } i \in \{0,1\},\\ &g(u) > 0 \text{ for all } u \in I_j \setminus \{x_j\},\\ &g(x_j) = 0. \end{aligned}$$

Let f_p denote the perturbed function constructed in this way from the increasing f. As, imposing the appropriate growth to $f(u)$ at infinity, Theorem 1.10 applied to f_p establishes the existence of some interval of T's where the singular problem (1.1) with $f = f_p$ possesses at least $2p + 1$ positive solutions, and this independently on the size of η, which is arbitrarily small, it becomes apparent how breaking down the monotonic character of the original function $f(u)$ on a set with arbitrarily small measure causes an arbitrarily large number of positive solutions for the singular problem (1.1).

Naturally, for every $\varepsilon > 0$, the function

$$f_\varepsilon(u) := u \prod_{j=1}^{p} [(u - x_j)^2 + \varepsilon]$$

satisfies $f_\varepsilon(u) > 0$ for all $u > 0$ and $\mathcal{T}(x) < +\infty$ for all $x > 0$, though $\mathcal{T}(x_j)$, $1 \leq j \leq p$, might be arbitrarily large by choosing a sufficiently small $\varepsilon > 0$. Indeed, by continuous dependence with respect to ε, the associated time maps, $\mathcal{T}_\varepsilon(x)$, $\varepsilon \geq 0$, to each of the Cauchy problems

$$\begin{cases} u'' = f_\varepsilon(u),\\ u(0) = x > 0, \quad u'(0) = 0, \end{cases}$$

must be continuos also with respect to $\varepsilon \geq 0$. Thus, if we consider the singular problems

$$\begin{cases} u'' = f_\varepsilon(u), & t \in [0,T],\\ u'(0) = 0, \quad u(T) = +\infty, \end{cases} \tag{1.14}$$

where $\varepsilon \geq 0$, by continuous dependence, one can easily infer the following global properties of these problems for sufficiently small $\varepsilon > 0$:

- There exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there are $T_1^*(\varepsilon), T_2^*(\varepsilon) > 0$, with $T_1^*(\varepsilon) < T_2^*(\varepsilon)$, such that, for every $T \in (T_1^*(\varepsilon), T_2^*(\varepsilon))$, the singular problem (1.14) possesses, at least, $2p + 1$ solutions. Moreover, one can choose $T_1^*(\varepsilon)$ and $T_2^*(\varepsilon)$ in such a way that

$$\lim_{\varepsilon \downarrow 0} T_2^*(\varepsilon) = \infty, \qquad \lim_{\varepsilon \downarrow 0} T_1^*(\varepsilon) = \max_{1 \leq j \leq p} \min_{x \in (x_{j-1}, x_j)} \mathcal{T}_0(x).$$

- For every $\varepsilon > 0$, there exist $0 < T_1(\varepsilon) < T_2(\varepsilon)$ such that (1.14) has a unique positive solution if either $T > T_2(\varepsilon)$, or $T < T_1(\varepsilon)$. Moreover,

$$\lim_{\varepsilon \downarrow 0} T_2(\varepsilon) = +\infty, \qquad \lim_{\varepsilon \downarrow 0} T_1(\varepsilon) = T_1(0) > 0.$$

Figure 1.3 provides us with the profile of $\mathcal{T}$ for the particular choice

$$f_\varepsilon(u) = u[(u-4)^2 + \varepsilon][(u-8)^2 + \varepsilon],$$

with $\varepsilon = 0.01$. Similarly to the case when $\varepsilon = 0$, $f_\varepsilon(u)$ is increasing for all $\varepsilon > 0$ and

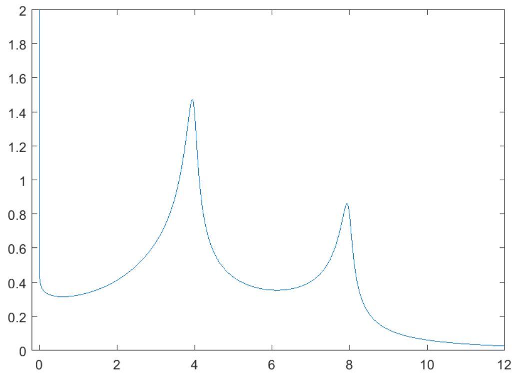


Figure 1.3: The time map $\mathcal{T}(x)$ for $f(u) = u((u-4)^2 + 0.01)((u-8)^2 + 0.01)$

$u > 8$. Therefore, by Theorem 1.6, there exists a branch of large solutions that bifurcates from infinity. By the global properties discussed above, for sufficiently small $\varepsilon > 0$ there exist $0 < T_1 < T_1^* < T_2^* < T_2$ such that (1.14) has a unique positive solution if either $0 < x < T_1$, or $x > T_2$, and five positive solutions if $x \in (T_1^*, T_2^*)$. Figure 1.4 plots the values of T, in ordinates, versus x, in abscisas, for $\varepsilon = 0.01$. These multiplicity results

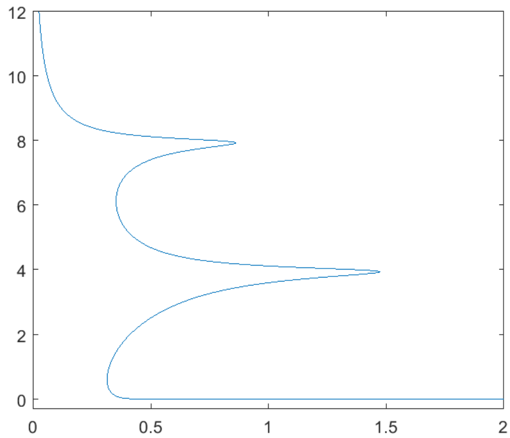


Figure 1.4: Bifurcation diagram for $f(u) = u((u-4)^2 + 0.01)((u-8)^2 + 0.01)$

for sufficiently small $\varepsilon > 0$ fail to be true for large $\varepsilon > 0$, because f_ε is increasing for sufficiently large $\varepsilon > 0$ and hence, thanks to Corollary 1.8, (1.14) possesses a unique positive large solution for every $T > 0$.

1.3 Counterexamples

In general, when $f(u)$ is not positive for large u, the assumption $\mathcal{T}(x) < +\infty$ at some $x \in \mathcal{D}$ does not necessary imply (1.11), as the following example shows. Therefore, the positivity of $f(u)$ for large u cannot be relaxed in the statement of Lemma 1.7. Consequently, the singular problem (1.1) might not admit a positive solution for sufficiently small T.

Consider the function $F : [0, +\infty) \to \mathbb{R}$ defined by

$$F(\theta) = \begin{cases} \theta^3 & \text{if } \theta \in A := [0, +\infty) \setminus \bigcup_{n \geq 2} (n - \frac{1}{n^2}, n + \frac{1}{n^2}), \\ g_n(\theta) & \text{if } \theta \in \left(n - \frac{1}{n^2}, n + \frac{1}{n^2}\right) \text{ for some } n \geq 2, \end{cases}$$

where, for every $n \geq 2$, g_n is an arbitrary smooth function such that $g_n(n) = 1$, $g_n(\theta) > 1$ if $\theta \neq n$, and $F \in \mathcal{C}^2[0, +\infty)$. Then, setting $f(u) := F'(u)$ in (1.2), we have that $\mathcal{D} = (0, 1)$. Moreover, for every $x \in \mathcal{D}$,

$$\begin{aligned} \sqrt{2}\,\mathcal{T}(x) &= \int_x^{+\infty} \frac{d\theta}{\sqrt{F(\theta) - F(x)}} \\ &= \int_{A \setminus [0,x]} \frac{d\theta}{\sqrt{\theta^3 - x^3}} + \sum_{n \geq 2} \int_{n - \frac{1}{n^2}}^{n + \frac{1}{n^2}} \frac{d\theta}{\sqrt{g_n(\theta) - x^3}} \\ &< \int_x^{+\infty} \frac{d\theta}{\sqrt{\theta^3 - x^3}} + \sum_{n \geq 2} \int_{n - \frac{1}{n^2}}^{n + \frac{1}{n^2}} \frac{d\theta}{\sqrt{1 - x^3}} \\ &= \int_x^{+\infty} \frac{d\theta}{\sqrt{\theta^3 - x^3}} + \frac{2}{\sqrt{1 - x^3}} \sum_{n \geq 2} \frac{1}{n^2} < +\infty. \end{aligned}$$

Thus, the Cauchy problem (1.2) has a large positive solution for each $x \in (0, 1)$. Furthermore, $\mathcal{T}(x) = +\infty$ if $x \geq 1$. Therefore,

$$\lim_{x \uparrow +\infty} \mathcal{T}(x) = +\infty.$$

In this example, $\mathcal{D}$ is bounded. The fact that the solutions of the Cauchy problem (1.2) are globally defined in time for all $x > 1$ prevents these solutions to blow-up, which explains why (1.1) cannot admit a solution for large x. One may wonder if, more generally, when $\mathcal{D}$ is unbounded, the condition $\mathcal{T}(x) < +\infty$ at some x should entail (1.10). Our next example shows that the answer to this question is also negative.

Consider the coefficients

$$C_n := \left(\int_{\frac{1}{n^2}}^{\frac{2}{n^2}} \frac{ds}{\sqrt{s^2 - \frac{1}{n^4}}} \right)^2, \qquad n \in \mathbb{N},$$

which are finite because

$$\int_a^b \frac{ds}{\sqrt{s^2-a^2}} < +\infty \quad \text{for all } \; 0 < a \le b < \infty.$$

Naturally, they have been defined to satisfy

$$\int_{\frac{1}{n^2}}^{\frac{2}{n^2}} \frac{ds}{\sqrt{C_n(s^2-\frac{1}{n^4})}} = 1 \quad \text{for all } n \in \mathbb{N}. \tag{1.15}$$

Now, consider the sequence of functions defined recursively by

$$\begin{aligned} &h_2(\theta) := C_2(\theta-2)^2+1, && \theta \in [\tfrac{3}{2}, \tfrac{5}{2}],\\ &h_n(\theta) := C_n(\theta-n)^2 + h_{n-1}(n-1+\tfrac{1}{(n-1)^2}), && \theta \in [n-\tfrac{2}{n^2}, n+\tfrac{2}{n^2}], \quad n \ge 3, \end{aligned}$$

as well as the function

$$F(\theta) := \begin{cases} \theta^3 & \text{if } \; \theta \in A := [0,+\infty) \setminus \bigcup_{n\ge 2} (n-\frac{3}{n^2}, n+\frac{3}{n^2}),\\ g_n(\theta) & \text{if } \; \theta \in (n-\frac{3}{n^2}, n+\frac{3}{n^2}) \; \text{ for some } \; n \ge 2, \end{cases}$$

where, for every $n \ge 2$,

$$g_n(\theta) = \begin{cases} g_{n,1}(\theta) & \text{if } \; \theta \in (n-\frac{3}{n^2}, n-\frac{2}{n^2}),\\ h_n(\theta) & \text{if } \; \theta \in [n-\frac{2}{n^2}, n+\frac{2}{n^2}],\\ g_{n,2}(\theta) & \text{if } \; \theta \in (n+\frac{2}{n^2}, n+\frac{3}{n^2}), \end{cases}$$

being $g_{n,1}$ and $g_{n,2}$ two smooth functions such that

$$g_{n,i}(\theta) > h_n(n+\tfrac{2}{n^2}) \quad \text{for } \; i = 1, 2,$$

and $F \in \mathcal{C}^2[0,+\infty)$. Set $f(u) := F'(u)$ in Problem (1.2). Then, by the construction of F,

$$\mathcal{D} = (0,1) \cup \bigcup_{n\ge 2} (n, n+\tfrac{1}{n^2}),$$

so $\mathcal{D}$ is unbounded. Moreover, for every $n \geq 2$ and $x \in (n, n+\frac{1}{n^2})$,

$$\begin{aligned}\sqrt{2}\,\mathcal{T}(x) &= \int_x^{n+\frac{2}{n^2}} \frac{d\theta}{\sqrt{F(\theta)-F(x)}} + \int_{n+\frac{2}{n^2}}^{n+\frac{3}{n^2}} \frac{d\theta}{\sqrt{F(\theta)-F(x)}} \\ &\quad + \int_{A\cap(n+\frac{3}{n^2},+\infty)} \frac{d\theta}{\sqrt{F(\theta)-F(x)}} + \int_{(n+\frac{3}{n^2},+\infty)\setminus A} \frac{d\theta}{\sqrt{F(\theta)-F(x)}} \\ &= \int_x^{n+\frac{2}{n^2}} \frac{d\theta}{\sqrt{h_n(\theta)-h_n(x)}} + \int_{n+\frac{2}{n^2}}^{n+\frac{3}{n^2}} \frac{d\theta}{\sqrt{g_{n,2}(\theta)-h_n(x)}} \\ &\quad + \int_{A\cap(n+\frac{3}{n^2},+\infty)} \frac{d\theta}{\sqrt{\theta^3-h_n(x)}} + \sum_{m>n} \int_{m-\frac{3}{m^2}}^{m+\frac{3}{m^2}} \frac{d\theta}{\sqrt{g_m(\theta)-h_n(x)}}\end{aligned}$$

and hence, there exists a constant $C>0$ such that

$$\begin{aligned}\sqrt{2}\,\mathcal{T}(x) &\leq C + \sum_{m>n} \int_{m-\frac{3}{m^2}}^{m+\frac{3}{m^2}} \frac{d\theta}{\sqrt{h_n(n+\frac{1}{n^2})-h_n(x)}} \\ &= C + \frac{6}{\sqrt{h_n(n+\frac{1}{n^2})-h_n(x)}} \sum_{m>n} \frac{1}{m^2} < +\infty.\end{aligned}$$

On the other hand, from the first previous identity, it is easily seen that, for every $n \geq 2$ and $x \in (n, n+\frac{1}{n^2})$,

$$\sqrt{2}\,\mathcal{T}(x) > \int_x^{n+\frac{2}{n^2}} \frac{d\theta}{\sqrt{h_n(\theta)-h_n(x)}} \geq \int_{n+\frac{1}{n^2}}^{n+\frac{2}{n^2}} \frac{d\theta}{\sqrt{h_n(\theta)-h_n(n+\frac{1}{n^2})}}.$$

The last inequality holds since $h_n(\theta)$ is convex, which entails the mappings

$$\alpha \mapsto \int_\alpha^{n+\frac{2}{n^2}} \frac{d\theta}{\sqrt{h_n(\theta)-h_n(\alpha)}}, \quad \alpha \in (n, n+\tfrac{2}{n^2}),\ n \geq 2,$$

to be decreasing. The proof of this feature follows the same patterns as the proof of Theorem 1.6. Hence, it follows from (1.15) that

$$\sqrt{2}\,\mathcal{T}(x) > \int_{n+\frac{1}{n^2}}^{n+\frac{2}{n^2}} \frac{d\theta}{\sqrt{C_n(\theta-n)^2 - C_n(\frac{1}{n^2})^2}} = \int_{\frac{1}{n^2}}^{\frac{2}{n^2}} \frac{ds}{\sqrt{C_n(s^2-\frac{1}{n^4})}} = 1$$

for every $n \geq 2$ and $x \in (n, n+\frac{1}{n^2})$. Therefore, (1.10) indeed fails to be true.

Chapter 2

Uniqueness of large positive solutions

In this chapter we study the uniqueness of the solution of the singular elliptic problem

$$\begin{cases} -\Delta u = \lambda u - \mathfrak{a}(x) f(u) & \text{in} \quad \Omega, \\ u = +\infty & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded subdomain of $\mathbb{R}^N$, $N \geq 1$, whose boundary, $\partial\Omega$, is a Lipschitzian $(N-1)$-surface of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$ is a parameter and $\mathfrak{a} \in \mathcal{C}(\bar{\Omega})$, $\mathfrak{a} \geq 0$ in Ω. A function u is a solution of (2.1) if it satisfies the differential equation and

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} u(x) = +\infty.$$

These solutions are called *large* or *explosive* solutions of

$$-\Delta u = \lambda u - \mathfrak{a}(x) f(u). \tag{2.2}$$

Whether (2.1) admits a positive solution has been largely studied in the specialized literature: the reader is sent to [35, 66, 7, 73, 37, 38, 18, 12, 13, 19, 10, 48] for a detailed discussion on their existence. In our context, the conditions for guaranteeing the existence of solution of (2.1) can be summarize as follows.

- (M) $f \in \mathcal{C}^1[0,\infty)$ is a nondecreasing function such that
  - $f(0) \geq 0$,
  - $f(u)/u$ is increasing if $\sigma[-\Delta,\Omega] \leq \lambda$, where $\sigma[-\Delta,\Omega]$ denotes the principal eigenvalue of $-\Delta$ in Ω under Dirichlet homogeneous boundary conditions.
- (KO) For every $\alpha > 0$ there exists $u^* = u^*(\alpha) > 0$ such that

$$I(u) := \int_1^{+\infty} \frac{d\theta}{\sqrt{\int_1^\theta (\alpha \frac{f(ut)}{u} - t)dt}} < +\infty \quad \text{for all} \;\; u > u^*,$$

which is a condition on the solution of

$$\begin{cases} -u'' = \lambda u - af(u), \\ u(0) = x, \quad u'(0) = 0, \end{cases} \qquad \lambda, a > 0,$$

in the spirit of Chapter 1.

Under the latest hypotheses it is well known that (2.1) possesses a *minimal* and a *maximal* solution, $L^{\min}$ and $L^{\max}$, in the sense that any other positive solution of (2.1), u, satisfies

$$L^{\min}(x) \leq u(x) \leq L^{\max}(x), \quad x \in \Omega.$$

Indeed, by (M) the following comparison principle for the solutions of (2.2) is available. Consider $w_1, w_2 \in \mathcal{C}^{2+\nu}(\partial\Omega)$ such that $w_2 > w_1 > 0$ on $\partial\Omega$ and let u_i, $i = 1, 2$, denote the unique solution of

$$\begin{cases} -\Delta u = \lambda u - \mathfrak{a}(x) f(u) & \text{in } \Omega, \\ u = w_i & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

If we suppose

$$\sigma[-\Delta, \Omega] > \lambda, \tag{2.4}$$

then the operator $-\Delta - \lambda$ satisfies the strong maximum principle, and hence, since f is non decreasing, we find that $u_2 \gg u_1$. When (2.4) fails but $f(u)/u$ is increasing, by [48, Th. 1.7] we also have $u_2 \gg u_1$. Throughout this work, given $u_1, u_2 \in \mathcal{C}(D) \cap \mathcal{C}^1(\partial D)$, $D \subset \mathbb{R}^N$, $\partial D \in \mathcal{C}^1$, we will set $u_2 \gg u_1$ if

$$w(x) := u_2(x) - u_1(x) > 0 \quad \text{for every } \ x \in D$$

and

$$\frac{\partial w}{\partial n_z}(z) < 0 \quad \text{for every } \ z \in w^{-1}(0) \cap \partial\Omega,$$

where n_z is the outward unit normal vector field to D at $z \in \partial D$. Similarly, for any $u_1, u_2 \in \mathcal{C}(D)$ we denote $u_2 > u_1$ if $u_2(x) \geq u_1(x)$ for every $x \in D$ but $u_1 \neq u_2$. Consequently, setting $w_i \equiv m \in \mathbb{R}^+$ in (2.3) and denoting $u_{[\Omega,m]}$ the unique solution of (2.3), we have that the mapping $m \mapsto u_{[\Omega,m]}$ is increasing. Thus,

$$u_{[\Omega,\infty]} := \lim_{m \to +\infty} u_{[\Omega,m]}$$

is well defined, and thanks to (KO) it is *finite* in Ω (see [48, Chapter 3] for the details). Moreover,

$$L^{\min} = u_{[\Omega,+\infty]}, \qquad L^{\max} = \lim_{\delta \downarrow 0} u_{[\Omega_\delta,+\infty]},$$

where

$$\Omega_\delta := \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > \delta\}. \tag{2.5}$$

Assumption (KO) goes back to the original one introduced for dealing with monotone operators by J. B. Keller [35] and R. Osserman [66]. Actually, the pioneering results of Keller and Osserman generalized some previous results of L. Bieberbach [9] and H. Rademacher [69], who considered the equation $\Delta u = e^u$ in two and three dimensions, respectively.The reader is sent to [48, Chapter 3] and [19] for a detailed discussion on Keller–Ossermann conditions, as well as to the monographs [44], [70] and [24].

As often happens in boundary blow-up problems, it is quite tricky to find uniqueness results. In fact, the problem of uniqueness in (2.1) remains largely open, even in the *autonomous* case, i.e. when $\mathfrak{a}(x)$ is assumed to be constant. In this chapter we establish three very general uniqueness results based on the global geometric properties of the underlying domain Ω and on the regularity of its boundary, $\partial\Omega$. In these results, the weight function $\mathfrak{a}(x)$ must be non-increasing as $\mathrm{dist}\,(x, \partial\Omega) \downarrow 0$, which is far from being a serious restriction if $\mathfrak{a} \equiv 0$ on $\partial\Omega$, or if $\mathfrak{a}(x)$ is constant in Ω.

Remark 2.1 In the autonomous case, (M) is not strictly necessary for the existence of solution of (2.1). In fact, setting $\mathfrak{a} \equiv 1$ in (2.1) and assuming $f \in \mathcal{C}^1[0, +\infty)$, $f > 0$, by [19] Condition (KO) is sufficient and necessary for the existence of large solutions. However, as our goal is to establish uniqueness results, (M) is far from being a serious restriction. Indeed, it suffices to have a glance at Chapter 1, which deals with the simplest possible model, to realice why the monotonicity of f is essential for the uniqueness.

The distribution of this chapter is the following. Section 2.1 shows a uniqueness result of (2.1) in star-shaped domains, which will be adapted in Section 2.2 to obtain a uniqueness result in more general domains. Finally, Section 2.3 provides us with a uniqueness result working in general smooth domains and weakening all previous available results in the autonomous case.

2.1 The star-shaped case

Our first result studies uniqueness when Ω is a star shaped domain, *i.e.*, if there exists a point $x_0 \in \Omega$ such that, for every $x \in \Omega$, the line segment from x_0 to x belongs to Ω. Actually, in such case it is said that Ω is star-shaped with respect to x_0. Balls and stars are the most paradigmatic star-shaped domains. The result can be stated as follows.

Theorem 2.2 *Suppose Ω is star-shaped, $\lambda \geq 0$ and $f \in \mathcal{C}^1[0, +\infty)$ satisfies $f(0) \geq 0$ and the next property*

(C) *There exists $p > 1$ such that*

$$f(tu) \geq t^p f(u) \quad \textit{for all} \quad t > 1 \;\; \textit{and} \;\; u > 0. \tag{2.6}$$

Moreover, assume that there exists $\eta > 0$ such that, for every $z \in \partial\Omega$,

$$\mathfrak{a}\left(z + \tfrac{x_0 - z}{|x_0 - z|} t\right) \leq \mathfrak{a}\left(z + \tfrac{x_0 - z}{|x_0 - z|} s\right) \quad \textit{if} \quad 0 < t < s < \eta. \tag{2.7}$$

Then, (2.1) *has a unique positive solution.*

Condition (C) goes back to [46, Eq. (11)] and [11, Eq. (6)]. It is related with convexity in the following way. According to Lemma 2.1 of [11], Condition (C) holds for $p = 2$ if

$$h(u) := \frac{f(u)}{u} \in \mathcal{C}[0,\infty) \cap \mathcal{C}^2(0,\infty)$$

satisfies $h(0) = 0$ and $h''(u) \geq 0$ for all $u > 0$.

Note that (C) entails (M) and (KO), so it is not necessary to include them in the hypotheses of Theorem 2.2. Indeed, by (2.6) we have that

$$\frac{f(\theta u) - f(u)}{\theta u - u} \geq \frac{\theta^p f(u) - f(u)}{\theta u - \theta u} = \frac{\theta^p - 1}{\theta - 1}\frac{f(u)}{u} > \frac{f(u)}{u}$$

for every $\theta > 1$ and $u > 0$. Then, $f'(u) \geq f(u)/u$, which entails $(f(u)/u)' \geq 0$, and hence $f(u)/u$ is increasing. Therefore (M) is satisfied. In order to show (KO), pick any $\alpha > 0$ and $u > 0$. Then,

$$\begin{aligned} I(u) &:= \int_1^{+\infty} \frac{d\theta}{\sqrt{\int_1^\theta (\alpha \frac{f(ut)}{u} - t)dt}} \leq \int_1^{+\infty} \frac{d\theta}{\sqrt{\int_1^\theta (\alpha \frac{f(u)}{u} t^p - t)dt}} \\ &= \int_1^{+\infty} \frac{d\theta}{\sqrt{\alpha \frac{f(u)}{u} \frac{\theta^{p+1}-1}{p+1} - \frac{\theta^2 - 1}{2}}} < +\infty. \end{aligned}$$

Using (2.6), $f(u) \geq u^p f(1)$ for all $u > 1$, and hence

$$\frac{f(u)}{u} \geq u^{p-1} f(1) \quad \text{for all } \ u > 1,$$

which guarantees the convergence of the integral at $\theta = 1$ and (KO).

In spite of (C) is the main hypothesis of the statement of Theorem 2.2, we will not use (C) as it is written in (2.6), but the following equivalent inequality:

$$\exists\, b > 0 \ \text{ such that } \ \varrho^{2+b} f(\varrho^{-b} v) \leq f(v) \quad \text{for all} \quad (\varrho, v) \in (1, +\infty) \times [0, +\infty). \quad (2.8)$$

Indeed, the change of variables

$$v = tu, \quad t = \varrho^b, \quad b = \frac{2}{p-1},$$

transforms (2.6) into (2.8). The condition (2.7) forces the weight function $\mathfrak{a}(x)$ to decay along all rays passing through x_0, the stelar center of Ω, as x approximates $\partial\Omega$. Even in the simplest case when Ω is a ball, Theorem 2.2 provides us with a substantial extension of [43, Th. 1.1] and [46, Th.1], because here we are far from imposing that $\mathfrak{a}(x)$ is globally decreasing along all radial directions, but only locally on a neighborhood of the boundary. Moreover, here $\mathfrak{a}(x)$ does not necessary depend on the distance to the boundary, as it occurs in the radially symmetric case. It is remarkable that the main ideas of the proof of Theorem 2.2 are strongly inspired by [43, 46].

2.1.1 Proof of Theorem 2.2

Due to the existence of a minimal and a maximal solution of (2.1), for establishing the uniqueness it suffices to show that $L^{\min} \geq L^{\max}$. By performing the change of variable $y = x - x_0$, without loss of generality, we can assume that Ω is star-shaped with respect to the origin, i.e., $x_0 = 0$. Then, for sufficiently small $\varepsilon > 0$, we consider the ε-neighborhood of $\partial\Omega$ defined by

$$\Gamma^\varepsilon := \{z + \frac{z}{|z|}\delta \;\;:\;\; z \in \partial\Omega, \;\; |\delta| < \varepsilon\}.$$

By definition,

$$\lim_{\varepsilon \downarrow 0} |\Gamma^\varepsilon| = 0,$$

where $|\Gamma^\varepsilon|$ stands for the Lebesgue measure of Γ^ε. Thus, by the Faber–Krahn inequality, [21], [36],

$$\lim_{\varepsilon \downarrow 0} \sigma[-\Delta, \Gamma^\varepsilon] = +\infty,$$

where we have denoted by $\sigma[-\Delta, \Gamma^\varepsilon]$ the first eigenvalue of $-\Delta$ in Γ^ε under homogeneous Dirichlet boundary conditions (see [39, Th. 5.1] if necessary). Thus, there exists $\hat{\varepsilon} = \hat{\varepsilon}(\lambda) > 0$ such that

$$\sigma[-\Delta, \Gamma^\varepsilon] \geq \lambda \qquad \text{for all } \; 0 < \varepsilon < \hat{\varepsilon}. \tag{2.9}$$

Subsequently, for every $\varrho > 0$, we set

$$\Omega_\varrho := \{x \in \Omega \;\;:\;\; \varrho x \in \Omega\} = \tfrac{1}{\varrho}\Omega.$$

Then, by hypothesis (2.7), there exist $0 < \varepsilon_0 < \hat{\varepsilon}$ and $\varrho_0 > 1$ such that

$$\mathfrak{a}(\varrho x) \leq \mathfrak{a}(x) \quad \text{for every } x \in \Gamma_\varrho := \Omega_\varrho \cap \Gamma^{\varepsilon_0} \quad \text{and} \;\; 1 < \varrho < \varrho_0. \tag{2.10}$$

Figure 2.1 sketches this construction. Note that the components of $\partial\Gamma_\varrho$ are

$$\partial\Gamma_\varrho = I_\varrho \cup J, \qquad \text{where} \; \begin{cases} I_\varrho = \partial\Omega_\varrho, \\ J = \partial\Gamma^{\varepsilon_0} \cap \Omega. \end{cases} \tag{2.11}$$

Let φ be a principal eigenfunction associated to $\sigma[-\Delta, \Gamma^{\hat{\varepsilon}}]$, i.e.,

$$\begin{cases} -\Delta\varphi = \sigma[-\Delta, \Gamma^{\hat{\varepsilon}}]\varphi & \text{in } \Gamma^{\hat{\varepsilon}}, \\ \varphi = 0 & \text{on } \partial\Gamma^{\hat{\varepsilon}}. \end{cases} \tag{2.12}$$

As $\varphi(x) > 0$ for all $x \in \Gamma^{\hat{\varepsilon}}$ and $\varepsilon_0 < \hat{\varepsilon}$, it is apparent that

$$\min_{\bar{\Gamma}^{\varepsilon_0}} \varphi > 0. \tag{2.13}$$

Now, for any given solution, u, of (2.1) and each $1 < \varrho < \varrho_0$, we introduce the function

$$\bar{u}_\varrho(x) := \varrho^b u(\varrho x) + \tau\varphi(\varrho x), \qquad x \in \Gamma_\varrho,$$

where $b > 0$ satisfies (2.8) and $\tau \in \mathbb{R}$ is regarded as a parameter. Then, the following result of a technical nature holds.

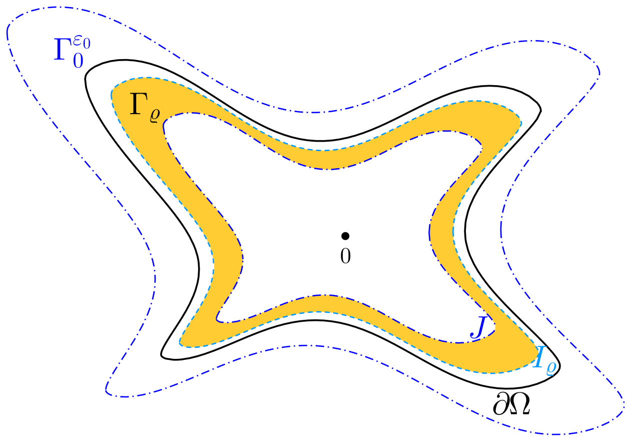


Figure 2.1: Scheme of the construction

Lemma 2.3 *There exist $\tau > 1$ such that $\bar{u}_\varrho$ is a supersolution of the singular problem*

$$\begin{cases} -\Delta w = \lambda w - \mathfrak{a}(x) f(w) & \text{in } \Gamma_\varrho, \\ w = +\infty & \text{on } I_\varrho, \\ w = L^{\max} & \text{on } J, \end{cases} \tag{2.14}$$

for every $1 < \varrho < \varrho_0$.

Proof. By (2.11), $\bar{u}_\varrho = +\infty$ on I_ϱ for all $\varrho > 1$. On the other hand, shortening $\varrho_0 > 1$ if necessary, one gets that

$$\{\varrho z \;\;:\;\; z \in J, \;\; 1 \leq \varrho \leq \varrho_0\} \subset (\Gamma^{\varepsilon_0} \cap \Omega)\,.$$

Then, thanks to (2.13), we can pick a $\tau > 1$ sufficiently large so that

$$\bar{u}_\varrho(z) = \varrho^b u(\varrho z) + \tau \varphi(\varrho z) > L^{\max}(z) \qquad \text{for all } \; z \in J, \;\; 1 < \varrho < \varrho_0,$$

because $L^{\max}$ is bounded on J. Thus, the required estimates on the boundary hold. Moreover, by definition and according to (2.9) and (2.12), we have that, for every $1 < \varrho < \varrho_0$ and $x \in \Gamma_\varrho$,

$$\begin{aligned} -\Delta \bar{u}_\varrho(x) &= -\varrho^{2+b} \Delta u(\varrho x) - \varrho^2 \tau \Delta \varphi(\varrho x) \\ &= \varrho^{2+b} \lambda u(\varrho x) - \varrho^{2+b} \mathfrak{a}(\varrho x) f(u(\varrho x)) + \varrho^2 \tau \sigma[-\Delta, \Gamma^{\hat{\varepsilon}}] \varphi(\varrho x) \\ &\geq \varrho^{2+b} \lambda u(\varrho x) + \varrho^2 \tau \lambda \varphi(\varrho x) - \varrho^{2+b} \mathfrak{a}(\varrho x) f(u(\varrho x)) \\ &\geq \varrho^b \lambda u(\varrho x) + \tau \lambda \varphi(\varrho x) - \varrho^{2+b} \mathfrak{a}(\varrho x) f(u(\varrho x)), \end{aligned}$$

because $\lambda \geq 0$ and $\varrho > 1$. Hence, invoking to (2.10) and (2.8) and taking into account that f is nondecreasing yields

$$\begin{aligned}-\Delta \bar{u}_\varrho(x) &\geq \varrho^b \lambda u(\varrho x) + \tau\lambda\varphi(\varrho x) - \varrho^{2+b}\mathfrak{a}(x) f(u(\varrho x)) \\ &= \lambda[\varrho^b u(\varrho x) + \tau\varphi(\varrho x)] - \varrho^{2+b}\mathfrak{a}(x) f(\varrho^{-b}\varrho^b u(\varrho x)) \\ &\geq \lambda[\varrho^b u(\varrho x) + \tau\varphi(\varrho x)] - \mathfrak{a}(x) f(\varrho^b u(\varrho x)) \\ &\geq \lambda[\varrho^b u(\varrho x) + \tau\varphi(\varrho x)] - \mathfrak{a}(x) f(\varrho^b u(\varrho x) + \tau\varphi(\varrho x)) \\ &= \lambda \bar{u}_\varrho(x) - \mathfrak{a}(x) f(\bar{u}_\varrho(x)),\end{aligned}$$

for every $0 < \varrho < \varrho_0$ and $x \in \Gamma_\varrho$. Therefore, $\bar{u}_\varrho$ is a supersolution of (2.14) for all $1 < \varrho < \varrho_0$. □

As, for every $x \in \bar{\Gamma}_\varrho \subset \Omega$, $L^{\max}(x) < +\infty$, according to the maximum principle, by uniqueness, we find that

$$L^{\max}(x) \leq \bar{u}_\varrho(x) = \varrho^b u(\varrho x) + \tau\varphi(\varrho x), \quad x \in \Gamma_\varrho, \quad 1 < \varrho < \varrho_0.$$

Hence, letting $\varrho \downarrow 1$ yields

$$L^{\max}(x) \leq u(x) + \tau\varphi(x), \qquad x \in \Gamma_1 := \lim_{\varrho\downarrow 1} \Gamma_\varrho = \Omega \cap \Gamma^{\varepsilon_0}.$$

In particular, making the choice $u(x) = L^{\min}(x)$ we can infer that

$$1 \leq \frac{L^{\max}(x)}{L^{\min}(x)} \leq \frac{L^{\min}(x) + \tau\varphi(x)}{L^{\min}(x)}, \qquad x \in \Gamma_1. \tag{2.15}$$

Since $\tau\varphi(x)$ is bounded in $\bar{\Gamma}_1$, we have that

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} \frac{L^{\min}(x) + \tau\varphi(x)}{L^{\min}(x)} = 1.$$

Consequently, by (2.15), the quotient function

$$q(x) := \begin{cases} \frac{L^{\max}(x)}{L^{\min}(x)} & x \in \Omega, \\ 1 & x \in \partial\Omega, \end{cases}$$

is uniformly continuous in $\bar{\Omega}$. As a byproduct, for every $\varepsilon > 0$, $\eta > 0$ exists such that

$$|q(x) - 1| = \frac{L^{\max}(x)}{L^{\min}(x)} - 1 < \varepsilon \quad \text{as soon as } \operatorname{dist}(x, \partial\Omega) < \eta.$$

So, setting

$$Q_\eta := \{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq \eta\},$$

we find that

$$L^{\max}(x) < (1+\varepsilon)L^{\min}(x), \qquad x \in Q_\eta. \tag{2.16}$$

Finally, note that $L^{\max}(x)$ is a solution of the problem

$$\begin{cases} -\Delta u = \lambda u - \mathfrak{a}(x)f(u) & \text{in} \quad \Omega \setminus Q_\eta, \\ u = L^{\max} & \text{on} \quad \partial(\Omega \setminus Q_\eta), \end{cases} \tag{2.17}$$

and that, owing to (2.16), $(1+\varepsilon)L^{\min}$ is a supersolution of (2.17). Therefore,

$$L^{\max}(x) < (1+\varepsilon)L^{\min}(x), \qquad x \in \Omega \setminus Q_\eta$$

and letting $\varepsilon \downarrow 0$ we find that

$$L^{\max} \leq L^{\min}.$$

This ends the proof. □

2.2 The generalized star-shaped case

Our second result generalizes substantially, when $\lambda = 0$, the annular uniqueness results established by [43, Th. 1.1] and [46, Th.1]. It can be stated as follows.

Theorem 2.4 *Suppose* $\lambda = 0$ *in* (2.1) *and* $f \in \mathcal{C}^1[0,+\infty)$ *satisfies* $f(0) \geq 0$ *and* (C). *Suppose there are an integer* $m \geq 1$ *and* $m+1$ *star-shaped domains,* Ω_i*,* $0 \leq i \leq m$*, with* $\partial\Omega_i$ *Lipschitz continuous, such that*

$$\bar{\Omega}_i \subset \Omega_0, \quad \bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset, \quad 1 \leq i,j \leq m, \quad i \neq j,$$

and

$$\Omega = \Omega_0 \setminus \left(\bar{\Omega}_1 \cup \cdots \cup \bar{\Omega}_m\right). \tag{2.18}$$

For every $0 \leq i \leq m$*, let denote by* x_i *the (center) point with respect to which* Ω_i *is star-shaped. Suppose, in addition, that there exists* $\eta > 0$ *such that, for every* $1 \leq i \leq m$*,* $z_0 \in \partial\Omega_0$ *and* $z_i \in \partial\Omega_i$*,*

$$\left.\begin{array}{l} \mathfrak{a}\left(z_0 + \frac{x_0-z_0}{|x_0-z_0|}t\right) \leq \mathfrak{a}\left(z_0 + \frac{x_0-z_0}{|x_0-z|}s\right) \\ \mathfrak{a}\left(z_i + \frac{z_i-x_i}{|x_i-z_i|}t\right) \leq \mathfrak{a}\left(z_i + \frac{z_i-x_i}{|x_i-z_i|}s\right) \end{array}\right\} \quad \text{if} \quad 0<t<s<\eta, \quad 1 \leq i \leq m. \tag{2.19}$$

Then, (2.1) *has a unique positive solution.*

As far as concerns Ω, any annular region satisfies the requirements of Theorem 2.4, as well as any ball, Ω_0, perforated by finitely many closed disjoint balls, Ω_i, $1 \leq i \leq m$.

Condition (2.19) entails the weight function $\mathfrak{a}(x)$ is non-increasing along the principal rays through the center x_i, $0 \le i \le m$, as x approximates $\partial\Omega_i$. Note that

$$\partial\Omega = \bigcup_{i=0}^{m} \partial\Omega_i.$$

Throughout the proof of Theorem 2.4 we will use the next inequality,

$$\varrho^{2+b} f(\varrho^{-b} v) \ge f(v) \quad \text{for all} \quad (\varrho, v) \in (0,1) \times [0,+\infty), \tag{2.20}$$

which is equivalent to (2.6) owing to the following change of variables,

$$v = u, \quad t = \varrho^{-b}, \quad b = \frac{2}{p-1}.$$

2.2.1 Proof of Theorem 2.4

Suppose $\lambda = 0$, (2.19) holds, and Ω admits the representation (2.18). For sufficiently small $\varepsilon > 0$, we will consider the following neighborhoods of the components of $\partial\Omega = \cup_{i=0}^{m} \partial\Omega_i$

$$\Gamma_i^\varepsilon := \left\{ z + \tfrac{z-x_i}{|z-x_i|}\delta \;\; : \;\; z \in \partial\Omega_i \; , \; |\delta| < \varepsilon \right\}, \qquad 0 \le i \le m.$$

Due to the local nature of the proof of Theorem 2.2, it is easily seen that

$$\lim_{\operatorname{dist}(x,\partial\Omega_0)\downarrow 0} \frac{L^{\max}(x)}{L^{\min}(x)} = 1. \tag{2.21}$$

It remains to prove that the same property holds along the remaining components of $\partial\Omega$, i.e., along $\partial\Omega_i$, $1 \le i \le m$. Fix $i \in \{1, \dots, m\}$. As in Section 2, by performing the change of variable $y = x - x_i$, without loss of generality we can assume that $x_i = 0$. Subsequently, for every $\varrho < 1$ sufficiently close to 1, we will consider the sets $\Omega_\varrho := \varrho^{-1}\Omega$ already defined in Section 2. Thanks to (2.19), $\mathfrak{a}(x)$ decays along the line through x and $x_i = 0$ as x approximates $\partial\Omega_i$. Thus, $\varepsilon_i > 0$ and $\varrho_i < 1$ exist such that

$$\mathfrak{a}(\varrho x) \le \mathfrak{a}(x) \quad \text{for every } x \in \Gamma_{i,\varrho} := \Omega_\varrho \cap \Gamma_i^{\varepsilon_i} \;\text{ and }\; \varrho_i < \varrho < 1. \tag{2.22}$$

By construction, the components of $\partial\Gamma_{i,\varrho}$ are

$$\partial\Gamma_{i,\varrho} = I_\varrho \cup J, \qquad \text{where} \left\{ \begin{array}{l} I_\varrho = \partial[(\Omega_i)_\varrho] \subset \Omega, \\ J = \partial\Gamma_i^{\varepsilon_i} \cap \Omega, \end{array} \right.$$

where

$$(\Omega_i)_\varrho := \varrho^{-1}\Omega_i \supsetneq \bar{\Omega}_i.$$

Subsequently, as in Section 2.1, for every solution of (2.1), u, and $b > 0$ satisfying (2.20), we consider a principal eigenfunction associated to $\sigma[-\Delta, \Gamma_i^{2\varepsilon_i}]$, $\varphi > 0$, as well as

$$\bar{u}_\varrho(x) := \varrho^b u(\varrho x) + \tau\varphi(\varrho x), \qquad x \in \Gamma_{i,\varrho}, \quad \varrho_i < \varrho < 1.$$

Then, there exists $\tau > 1$ such that $\bar{u}_\varrho$ is a supersolution of the singular problem

$$\begin{cases} -\Delta w = -\mathfrak{a}(x) f(w) & \text{in } \Gamma_{i,\varrho}, \\ w = +\infty & \text{on } I_\varrho, \\ w = L^{\max} & \text{on } J, \end{cases}$$

for every $\varrho_i < \varrho < 1$. Indeed, by definition $\bar{u}_\varrho = +\infty$ on I_ϱ for all $\varrho \in (\varrho_i, 1)$. On the other side, by construction, for $\varrho_i < 1$ sufficiently close to 1,

$$\{\varrho z \ : \ z \in J, \ \varrho_i \leq \varrho \leq 1\} \subset \left(\Gamma_i^{\varepsilon_i} \cap \Omega\right).$$

Thus, for sufficiently large $\tau > 1$, we also have that

$$\bar{u}_\varrho(z) = \varrho^b u(\varrho z) + \tau\varphi(\varrho z) > L^{\max}(z) \qquad \text{for all } \ z \in J, \ \ \varrho_i < \varrho < 1.$$

So, for this choice of τ, $\bar{u}_\varrho$ satisfies the required inequalities on the boundary.

Lastly, by (2.21) and (2.20), we also have that

$$\begin{aligned} -\Delta\bar{u}_\varrho(x) &= -\varrho^{2+b}\Delta u(\varrho x) - \varrho^2\tau\Delta\varphi(\varrho x) \\ &\geq -\varrho^{2+b}\mathfrak{a}(\varrho x) f(u(\varrho x)) + \varrho^2\tau\sigma[-\Delta, \Gamma_i^{2\varepsilon_i}]\varphi(\varrho x) \\ &\geq -\mathfrak{a}(\varrho x)\varrho^{2+b} f(\varrho^{-b}\varrho^b u(\varrho x)) \\ &\geq -\mathfrak{a}(x) f(\bar{u}_\varrho(x)), \end{aligned}$$

for every $\varrho_i < \varrho < 1$ and $x \in \Gamma_{i,\varrho}$, which concludes the proof of the claim above.

Since $L^{\max}$ is *finite* on I_ϱ for all $\varrho_i < \varrho < 1$, the maximum principle implies

$$L^{\max}(x) \leq \bar{u}_\varrho(x) = \varrho^b u(\varrho x) + \tau\varphi(\varrho x), \quad x \in \Gamma_{i,\varrho}, \quad \varrho_i < \varrho < 1.$$

Hence, letting $\varrho \uparrow 1$ yields

$$L^{\max}(x) \leq u(x) + \tau\varphi(x), \qquad x \in \Gamma_{i,1} := \lim_{\varrho\uparrow 1} \Gamma_{i,\varrho} = \Omega \cap \Gamma_i^{\varepsilon_i}.$$

In particular, making the choice $u(x) = L^{\min}(x)$ we can infer that

$$\lim_{\operatorname{dist}(x,\partial\Omega_i)\downarrow 0} \frac{L^{\max}(x)}{L^{\min}(x)} = 1.$$

Now, the uniqueness follows by easily adapting the final part of the proof of Theorem 2.2.

2.3 Uniqueness in smooth domains

Theorems 2.2 and 2.4 require the Condition (C), besides the Lipschitz regularity of $\partial\Omega$ and the local decaying property of $\mathfrak{a}(x)$ as x approximates $\partial\Omega$. Our third theorem establishes an extremely sharp uniqueness result, almost optimal, if $\partial\Omega$ is smooth.

Theorem 2.5 *Suppose Ω is of class $\mathcal{C}^1$ and it satisfies the uniform interior sphere property, as discussed in* [74] *and* [47, Def. 1.2]. *For every $z\in\partial\Omega$, let n_z denote the outward unit normal vector to $\partial\Omega$ at z. Suppose, in addition, that, for every $z\in\partial\Omega$, there exists $\delta>0$ such that $|x-z|<\delta$, with $x\in\Omega$, implies*

$$\mathfrak{a}(x+\varrho n_z)\le\mathfrak{a}(x)\quad\text{if }\ x+\varrho n_z\in\Omega\ \text{ and }\ \varrho>0. \tag{2.23}$$

Lastly, suppose λ satisfies (2.4) *and $f\in\mathcal{C}^1[0,+\infty)$ is nondecreasing and* super-additive *with constant $C\ge0$, i.e., there exists $C\ge0$ such that*

$$f(a+b)\ge f(a)+f(b)-C\qquad\text{for all }\ a,b\ge0. \tag{2.24}$$

Then, if in addition (KO) *holds, problem* (2.1) *possesses a unique positive solution.*

According to Definition 1.2 of [47], Ω satisfies the uniform interior sphere property if there exists $r>0$ such that, for every $z\in\partial\Omega$, there is a point $x_z\in\Omega$ for which

$$|z-x_z|=r,\qquad B_r(x_z)\subset\Omega.$$

Condition (2.23) entails $\mathfrak{a}(x)$ is non-increasing as x approximates $\partial\Omega$ along parallel rays to the line passing through z and x_z. Naturally, it holds if either $\mathfrak{a}(x)$ is constant in a neighborhood of $\partial\Omega$, or if $\mathfrak{a}\in\mathcal{C}^1(\bar\Omega)$ and

$$\frac{\partial\mathfrak{a}}{\partial n_z}(z)<0\quad\text{for all }\ z\in\partial\Omega.$$

Condition (2.24) goes back to Theorem 0.3 of [58]. Although, in [58], Marcus and Véron obtain uniqueness of large solutions of

$$-\Delta u+f(u)=0 \tag{2.25}$$

for domains whose boundary is locally the graph of a continuous function, which is an extremely weaker hypothesis on the regularity of $\partial\Omega$ than ours, their proof uses an uniqueness theorem for convex functions together with some tricky comparisons derived from the super-additivity of f and the comparison principle. Here we are not imposing convexity to f. Moreover, there is no linear term in (2.25), and no spatial heterogeneities can be incorporated to the model without some additional further work.

Before showing the proof of Theorem 2.5 let us see what it says in the autonomous case,

$$\begin{cases}-\Delta u=\lambda u-f(u) & \text{in }\ \Omega,\\ u=+\infty & \text{on }\ \partial\Omega.\end{cases} \tag{2.26}$$

Corollary 2.6 *Suppose* $\Omega \in \mathcal{C}^1$ *satisfies* (2.4) *and the uniform interior sphere property on* $\partial\Omega$*,* λ *satisfies* (2.4)*,* f *is nondecreasing and* (KO) *and* (2.24) *hold. Then, the singular problem* (2.26) *has a unique positive solution.*

Even the weakest uniqueness result provided by this corollary is substantially sharper than the strongest available results for smooth domains in the literature , where some severe additional growth conditions at infinity are needed (see [17], [28], [15] and the references there in). Indeed, Theorem 6.10 of [17], going back to [7], shows the uniqueness of (2.26) in the special case $\lambda = 0$, assuming in addition that

$$\frac{f(u)}{u} \quad \text{is increasing for large } \; u, \tag{2.27}$$

plus the requirement of regular variation at infinity for f:

$$\liminf_{u\to+\infty} \frac{\Psi(\beta u)}{\Psi(u)} > 1 \qquad \text{for all } \; \beta \in (0,1), \tag{2.28}$$

where

$$\Psi(u) := \frac{1}{\sqrt{2}} \int_u^{+\infty} \frac{dt}{\sqrt{F(t)}}, \qquad F(t) := \int_C^t f, \qquad C > 0.$$

In the same vein, also with $\lambda = 0$, Theorem 1 of [28] shows the uniqueness of a positive solution for (2.26) imposing, instead of (2.27), that there exists $p > 1$ such that

$$\frac{f(u)}{u^p} \quad \text{is increasing for large } \; u, \tag{2.29}$$

which implies (2.28) but weakens (2.27). Nevertheless, the condition (2.24) is substantially weaker that (2.27) and (2.29). Indeed, if $\frac{f(u)}{u^p}$ is increasing for some $p \geq 1$ and for sufficiently large u, there exists a constant $M \geq 0$ such that

$$f(a) + f(b) = \frac{f(a)}{a^p} a^p + \frac{f(b)}{b^p} b^p \leq \frac{f(a+b)}{(a+b)^p} a^p + \frac{f(a+b)}{(a+b)^p} b^p \leq f(a+b), \tag{2.30}$$

for every $a, b > M$. On the other side, by the monotonicity of f and (2.30), we have

$$f(a) + f(b) - f(a+b) \leq \begin{cases} f(a) & \text{if } \; a \leq M, \\ 0 & \text{if } \; a, b > M. \end{cases}$$

Then, the constant

$$C := \sup_{a,b \geq 0} f(a) + f(b) - f(a+b) \geq 0$$

is finite, and therefore

$$f(a+b) = f(a) + f(b) + f(a+b) - f(a) - f(b) \geq f(a) + f(b) - C$$

for every $a, b \geq 0$.

More recently, in [15], the uniqueness was proven assuming that $\partial\Omega$ is of class $\mathcal{C}^3$ and has nonnegative mean curvature imposing in addition that there exists $M > 0$ such that

$$\sqrt{F(u)} \quad \text{is convex for all } \ u > M. \tag{2.31}$$

Once again, (2.31) implies (2.24). Indeed, by the convexity of $\sqrt{F(u)}$, it becomes apparent that $\frac{f(u)}{\sqrt{F(u)}}$ is increasing for $u > M$. Thus, arguing as above, we find that

$$\begin{aligned} f(a) + f(b) &= \frac{f(a)}{\sqrt{F(a)}}\sqrt{F(a)} + \frac{f(b)}{\sqrt{F(b)}}\sqrt{F(b)} \\ &\leq \frac{f(a+b)}{\sqrt{F(a+b)}}(\sqrt{F(a)} + \sqrt{F(b)}), \end{aligned} \tag{2.32}$$

for all $a, b > M$. On the other hand, the extension of $\sqrt{F(u)}$ by zero,

$$G(u) := \begin{cases} 0, & 0 \leq u \leq M, \\ \sqrt{F(u)}, & u > M, \end{cases}$$

is convex in $[0, +\infty)$. Then,

$$G(\theta u) = G(\theta u + (1-\theta)0) \leq \theta G(u) + (1-\theta)G(0) = \theta G(u) \quad \text{for all } \ \theta \in (0,1),$$

and so, (2.32) yields

$$\begin{aligned} \sqrt{F(a)} + \sqrt{F(b)} &= G\left((a+b)\frac{a}{a+b}\right) + G\left((a+b)\frac{b}{a+b}\right) \\ &\leq G(a+b) = \sqrt{F(a+b)}. \end{aligned}$$

Therefore, (2.31) indeed entails (2.24). As far as concerns the regularity of Ω, the assumptions of Corollary 2.6 are obviously much weaker than those of [15]. Indeed, by Theorem 1.9 of [47], the regularity requirements of Corollary 2.6 hold if Ω is $\mathcal{C}^2$. Moreover, we are not imposing any restriction on the mean curvature.

Lastly, note that Condition (C) also implies (2.24). Making the change of variables $v = tu$ and $\theta = t^{-1}$, we deduce that (C) is equivalent to

$$f(\theta v) \leq \theta^p f(v) \quad \text{for every } \ 0 < \theta < 1 \ \text{ and } \ v > 0. \tag{2.33}$$

Then,

$$\begin{aligned} f(a) + f(b) &= f(\frac{a}{a+b}(a+b)) + f(\frac{b}{a+b}(a+b)) \\ &\leq \left[(\frac{a}{a+b})^p + (\frac{b}{a+b})^p\right] f(a+b) < f(a+b) \qquad \text{for every } \ a, b > 0. \end{aligned}$$

In the non-autonomous case our uniqueness theorems provide us with substantial improvements of most of the previous results available in the literature, where the uniqueness follows from a number of different explicit formulas providing us with the blow-up rate of all positive solutions (see. e.g., [12, 13, 41, 45, 67, 68, 10, 75, 26, 22, 76, 77, 27, 25] and the references there in). But in order to get the exact blow-up rates of the positive solutions of (2.1) one must impose some severe growth conditions on $f(u)$ at infinity, as for instance (2.27), (2.28). Essentially, those conditions require $f(u)$ to be normalized regularly varying, or growing faster than any power at infinity (see Definition 2.3 of [25]). As far as concerns the heterogeneity coefficient, $\mathfrak{a}(x)$, in most of the available references, it is imposed some additional non-oscillation property at $\partial\Omega$. Our assumption (2.23) does not require of any additional non-oscillation property, though it is slightly stronger when $\mathfrak{a}(x)$ is bounded away from zero on some piece of $\partial\Omega$. Although [3], [6] and [8] are also devoted to the existence and uniqueness of large solutions of $\Delta u = f(u)$ for some special classes of f's, the reader should be aware that dealing with the non-autonomous problem $\Delta u = \mathfrak{a}(x)f(u)$ is more sophisticated technically, since the results might depend on the behavior of $\mathfrak{a}(x)$ near $\partial\Omega$, as it is illustrated by the main findings of this paper. Crucially, the methodology adopted in this chapter differs substantially from the previous ones of [3], [6] and [8], where some of the uniqueness results are based on the curvature of $\partial\Omega$. As the uniqueness of the positive solution of $\Delta u = \mathfrak{a}(x)f(u)$ should be exclusively based on the uniqueness of the large solutions of the scalar equation $u'' = f(u)$, we conjecture that the curvature of $\partial\Omega$ should not play any role on the problem of the uniqueness. Certainly, it does not play it in Corollary 2.6, which is a new finding for the autonomous model.

2.3.1 Proof of Theorem 2.5

Throughout the proof we consider $z \in \partial\Omega$ fixed. Since Ω satisfies a uniform interior sphere property, there exists $r > 0$ such that

$$D := \bigcup_{\substack{y\in\partial\Omega \\ |y-z|<\frac{\delta}{2}}} B_r(y - rn_y) \subset B_\delta(z) \cap \Omega,$$

where $\delta > 0$ is the one arising in (2.23). Shortening $r > 0$, if necessary, there exists $\varrho_0 > 0$ such that the set

$$D_\varrho := \{x - \varrho n_z \ : \ x \in D\} = D - \varrho n_z, \quad \varrho > 0,$$

satisfies

$$D_\varrho \subset B_\delta(z) \cap \Omega \quad \text{for all } \ 0 \le \varrho < \varrho_0.$$

Subsequently, pick $\varepsilon > 0$ and consider non-negative function $\psi_\varepsilon \in \mathcal{C}^\infty(\partial D)$ such that

$$\psi_\varepsilon := \begin{cases} \frac{1}{\varepsilon} & \text{on} \quad B_{\delta/4}(z) \cap \partial\Omega, \\ 0 & \text{on} \quad \partial D \setminus \overline{(B_{\delta/2}(z) \cap \partial\Omega)}, \end{cases}$$

with

$$0<\psi_\varepsilon(x)<\tfrac{1}{\varepsilon}\quad\text{if }\ \tfrac{\delta}{4}<|x-z|<\tfrac{\delta}{2},\ \ x\in\partial\Omega,$$

and let us consider any increasing sequence $\{g_n\}_{n\geq 1}$ of positive functions defined on ∂D such that

$$\lim_{n\to\infty} g_n=\begin{cases}\frac{1}{\psi_\varepsilon} & \text{on } \ B_{\delta/2}(z)\cap\partial\Omega,\\ +\infty & \text{uniformly on } \ \partial D\setminus(B_{\delta/2}(z)\cap\partial\Omega).\end{cases}$$

Since f is increasing, from the method of sub and supersolutions it is easily seen that, for every $n\in\mathbb{N}$, the problem

$$\begin{cases}-\Delta w=\lambda w-\mathfrak{a}(x)(f(w)-C) & \text{in } \ D,\\ w=g_n & \text{on } \ \partial D,\end{cases}$$

has a unique positive solution, ℓ_{g_n}. The constant C in the equation is the one given by (2.24). Thanks to (KO), letting $n\uparrow+\infty$, ℓ_{g_n} approximates the minimal solution of the singular problem

$$\begin{cases}-\Delta w=\lambda w-\mathfrak{a}(x)(f(w)-C) & \text{in } \ D,\\ w=\frac{1}{\psi_\varepsilon} & \text{on } \ B_{\delta/2}(z)\cap\partial\Omega,\\ w=+\infty & \text{on } \ \partial D\setminus(B_{\delta/2}(z)\cap\partial\Omega).\end{cases}\tag{2.34}$$

Let us denote ℓ the minimal solution of (2.34). Let u be a positive solution of (2.1) and consider

$$\bar u_\varrho(x):=u(x+\varrho n_z)+\ell(x+\varrho n_z),\quad x\in D_\varrho,\ \ 0<\varrho<\varrho_0.$$

We claim that $\bar u_\varrho$ is a supersolution of the problem

$$\begin{cases}-\Delta w=\lambda w-\mathfrak{a}(x)f(w) & \text{in } \ D_\varrho,\\ w=+\infty & \text{on } \ \partial D_\varrho,\end{cases}\tag{2.35}$$

for all $0<\varrho<\varrho_0$. Indeed, since $\mathfrak{a}(x)$ satisfies (2.23) and f satisfies (2.24), the following inequalities hold

$$\begin{aligned}-\Delta\bar u_\varrho(x)&=-\Delta u(x+\varrho n_z)-\Delta\ell(x+\varrho n_z)\\ &=\lambda[u(x+\varrho n_z)+\ell(x+\varrho n_z)]-\mathfrak{a}(x+\varrho n_z)\left[f(u(x+\varrho n_z))+f(\ell(x+\varrho n_z))-C\right]\\ &\geq\lambda\bar u_\varrho(x)-\mathfrak{a}(x)f(u(x+\varrho n_z)+\ell(x+\varrho n_z))\\ &=\lambda\bar u_\varrho(x)-\mathfrak{a}(x)f(\bar u_\varrho(x))\end{aligned}$$

for all $x\in D_\varrho$ and $\varrho\in(0,\varrho_0)$. Moreover, by construction,

$$\bar u_\varrho=+\infty\quad\text{on }\ \partial D_\varrho$$

for all these ϱ's. Therefore, $\bar{u}_\varrho$ is a supersolution to the singular problem (2.35).

As $L^{\max}$ is finite on ∂D_ϱ for every $\varrho \in (0, \varrho_0)$, because $\bar{D}_\varrho \subset \Omega$, it follows from the maximum principle, by the monotonicity of $f(u)$, that

$$L^{\max}(x) \leq \bar{u}_\varrho(x) = u(x + \varrho n_z) + \ell(x + \varrho n_z) \quad \text{for all } \ x \in \bar{D}_\varrho.$$

Thus, making the choice $u = L^{\min}$, it is apparent that, for every $\varrho \in (0, \varrho_0)$,

$$L^{\max}(x) \leq L^{\min}(x + \varrho n_z) + \ell(x + \varrho n_z) \quad \text{for all } \ x \in \bar{D}_\varrho.$$

Consequently, letting $\varrho \downarrow 0$ yields

$$L^{\max}(x) \leq L^{\min}(x) + \ell(x) \quad \text{for all } \ x \in D.$$

Hence, thanks to (2.34),

$$\limsup_{\substack{x \to z \\ x \in \Omega}} L^{\max}(x) - L^{\min}(x) \leq \varepsilon$$

and therefore, letting $\varepsilon \downarrow 0$ we have

$$\lim_{\substack{x \to z \\ x \in \Omega}} L^{\max}(x) - L^{\min}(x) = 0.$$

Actually, as z is arbitrary, we may infer that

$$\lim_{\substack{x \to z \\ x \in \Omega, z \in \partial\Omega}} L^{\max}(x) - L^{\min}(x) = 0. \tag{2.36}$$

Lastly, denote $L := L^{\min} - L^{\max}$. Then, by the monotonicity of f and (2.36), we find

$$\begin{cases} (-\Delta - \lambda)L = f(L^{\max}) - f(L^{\min}) \geq 0 & \text{in } \ \Omega, \\ L = 0 & \text{on } \ \partial\Omega, \end{cases}$$

and applying the maximum principle, we deduce that, necessary, $L = 0$. The proof is complete. $\square$

Part II

Large solutions of cooperative systems

Chapter 3

Introduction to cooperative-logistic systems

The aim of the second part of this Thesis is to study the existence and uniqueness of the solution of the singular elliptic problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij}u_j - \mathfrak{a}_i(x)f_i(u_i)u_i & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega, \end{cases} \quad 1 \leq i \leq n, \tag{3.1}$$

where $a_{ij} > 0$, Ω is a bounded subdomain of $\mathbb{R}^N$, $N \in \mathbb{N}$, of class $C^{2+\nu}$ for some $\nu \in (0,1)$, $\lambda_i \in \mathcal{C}^{\nu}(\bar{\Omega})$, and $\mathfrak{a}_i \in \mathcal{C}^{\nu}(\bar{\Omega})$ satisfy $\mathfrak{a}_i(x) > 0$ for all $x \in \Omega$ and $1 \leq i, j \leq n$. As far as concerns the nonlinear terms of (3.1), the following conditions are imposed:

(A1) For every $1 \leq i \leq n$, $f_i \in \mathcal{C}^{1+\nu}[0,\infty)$, $f_i(0) = 0$ and $f_i'(u) > 0$ for all $u > 0$.

(A2) There exists $F \in \mathcal{C}^{1+\nu}[0,\infty)$ such that $F(0) = 0$, $F(u) > 0$, $F'(u) > 0$,

$$\min_{1 \leq i \leq n} f_i(u) \geq F(u) \quad \text{for all } u \geq 0,$$

and $G(u) := F(u)u$ satisfies Condition (KO) introduced in Chapter 2.

Problem (3.1) is the most natural way of coupling the fully uncoupled problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i - \mathfrak{a}_i(x)f_i(u_i)u_i & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega, \end{cases} \quad 1 \leq i \leq n, \tag{3.2}$$

in a cooperative way, i.e., with $a_{ij} > 0$ for all $1 \leq i, j \leq n$, $i \neq j$. Thus, the characterization of the strong maximum principle holds, which in the context of this work is Theorem 3.1 in the next section. A simpler prototype with $n = 2$ of the system in (3.1) goes back to

[62, 61, 63]. When $a_{ij} < 0$ for some $i \neq j$, the cooperative structure of (3.1) is lost and all the available comparison techniques in the context of cooperative systems might fail. Consequently, all results provided in this part of the thesis might not be true.

Since (3.2) consists of n-uncoupled singular boundary value problems of logistic type and the logistic equation is the most paradigmatic one in population dynamics and mathematical biology, [48, 64, 65], the problem of analyzing the singular problem (3.1) should deserve a significative attention in spatial ecology. Indeed, the solutions of (3.1) provide us with the asymptotic profiles of the positive solutions of wide classes of cooperative parabolic systems in the presence of spatial heterogeneities, [4, 5, 48]. A more realistic model would be to have different diffusion rates for each species, measured by $d_i > 0$,

$$-d_i \Delta u_i = \lambda_i u_i + \sum_{j=1, j\neq i}^{n} a_{ij} u_j - \mathfrak{a}_i(x) f_i(u_i) u_i \quad 1 \leq i \leq n,$$

but dividing by d_i, one is naturally driven to deal with (3.1).

Although there is a huge amount of literature devoted to the existence and uniqueness of large positive solutions for the single generalized logistic equation, as it becomes apparent by simply looking at the references given in the introduction of Chapter 2, and the rather complete list of references in [48], and even there are some fairly astonishing multiplicity results for large positive solutions, [56], the literature on systems is very short. Among the few previous results available for systems, in [32] the existence of large solutions was characterized for the classical diffusive symbiotic model of Lotka-Volterra, and the blow-up rates of each of the components of these singular solutions were ascertained. In [4] and [5] a general version of (5.1) was analyzed. Although the existence of large solutions was established there in, the problem of their uniqueness, as well as the problem of ascertaining their blow-up rates, remained fully open. Finally, [31] proves the existence and uniqueness of large solutions for a class of *autonomous* reaction diffusion systems of cooperative type. However, by the presence of the spatial heterogeneities in (3.1), the techniques developed in [31] cannot be applied here.

This part of the thesis summarizes the results already published in [49, 50, 51]. We schematize the existence of large solutions of (3.1) in the following section. The uniqueness question will be discuss in the next two chapters.

3.1 Existence of large solutions

The main goal of this section is to sketch the proof of the existence of solution for (3.1). This result was publish in [51] for a model a bit less general than (3.1), tough essentially the same. Some previous results were already found in [4] and [49]. Given a smooth subdomain $D \subset \mathbb{R}^N$, we consider the operator

$$\mathfrak{L} : [\mathcal{C}^{2+\nu}(D)]^n \longrightarrow [\mathcal{C}^{\nu}(D)]^n$$

defined by

$$(\mathfrak{L}u)_i = -\Delta u_i - \lambda_i(\cdot)u_i - \sum_{j=1, j\neq i}^{n} a_{ij}u_j, \qquad 1 \leq i \leq n.$$

By the main result of [54], there exists a unique $\sigma \in \mathbb{R}$, the *principal eigenvalue* of $\mathfrak{L}$ under Dirichlet homogeneous conditions, such that the linear eigenvalue problem

$$\begin{cases} \mathfrak{L}\varphi = \sigma\varphi & \text{in } \ D, \\ \varphi = 0 & \text{on } \ \partial D, \end{cases}$$

admits a positive eigenfunction $\varphi \in [\mathcal{C}^{2+\nu}(D)]^n$, $\varphi_i \geq 0$, $1 \leq i \leq n$, $\varphi \neq 0$. Such a value of σ will be denoted by $\sigma[\mathfrak{L}, D]$.

Throughout the rest of this chapter, for any given $u \in [\mathcal{C}(D)]^n$ it is said that $u > 0$ in D if $u_i \geq 0$ for all $1 \leq i \leq n$ but $u \neq 0$. Similarly, given $u \in [\mathcal{C}(D)]^n \cap [\mathcal{C}^1(\partial D)]^n$, it is said that u is *strongly positive* in D, $u \gg 0$, if $u_i \gg 0$ for each $1 \leq i \leq n$.

The following characterization of the strong maximum principle going back to [54] holds.

Theorem 3.1 *The following assertions are equivalent:*

(a) $\sigma\,[\mathfrak{L}, D] > 0$.

(b) *There exists* $\bar{u} \in [\mathcal{C}^2(D)]^n \cap [\mathcal{C}(\bar{D})]^n$ *such that* $\bar{u} > 0$ *in* D *and*

$$\mathfrak{L}\bar{u} \geq 0 \qquad \textit{in} \quad D,$$

and, for some $1 \leq i_0 \leq n$, *either* $\bar{u}_{i_0} > 0$ *on* ∂D, *or else*

$$(\mathfrak{L}\bar{u})_{i_0} > 0 \qquad \textit{in} \quad D.$$

Should this be the case, $\bar{u}$ *is said to be a positive strict supersolution of* $\mathfrak{L}$ *in* D.

(c) *The operator* $\mathfrak{L}$ *satisfies the strong maximum principle in* D, *in the sense that, for every* $h \in [\mathcal{C}^{\nu}(\bar{D})]^n$, $u \in [\mathcal{C}^{2+\nu}(\bar{D})]^n$ *and* $w \in [\mathcal{C}^{2+\nu}(\partial D)]^n$ *satisfying*

$$\begin{cases} \mathfrak{L}u = h \geq 0 & \textit{in } \ D, \\ u = w \geq 0 & \textit{on } \ \partial D, \end{cases}$$

with some of these inequalities $\geq$ *strict, one has that* $u \gg 0$ *in* D.

Using Theorem 3.1 one can easily show the monotonicity of the principal eigenvalue with respect to the potentials $\lambda_i(\cdot)$ and the coefficients a_{ij}, $1 \leq i, j \leq n$. Actually, this result was established by [54, Th. 3.2]. As a result, if we assume that

$$\underline{\lambda}_i(x) \leq \bar{\lambda}_i(x) \quad \text{and} \quad \underline{a}_{ij} \leq \bar{a}_{ij} \quad \text{for all } \ x \in \bar{\Omega} \quad \text{and} \quad 1 \leq i, j \leq n,$$

with some of these inequalities strict, then, setting

$$(\underline{\mathfrak{L}}u)_i := -\Delta u_i - \underline{\lambda}_i(\cdot)u_i - \sum_{j=1,j\neq i}^{n} \underline{a}_{ij}u_j, \quad (\bar{\mathfrak{L}}u)_i := -\Delta u_i - \bar{\lambda}_i(\cdot)u_i - \sum_{j=1,j\neq i}^{n} \bar{a}_{ij}u_j,$$

we find that

$$\sigma[\underline{\mathfrak{L}}, D] > \sigma[\bar{\mathfrak{L}}, D]. \tag{3.3}$$

Next, for every $w \in \left[\mathcal{C}^{2+\nu}(\partial\Omega)\right]^n$, $w > 0$, we consider the non-homogeneous Dirichlet boundary value problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i + \displaystyle\sum_{j=1,j\neq i}^{n} a_{ij}u_j - \mathfrak{a}_i(x)f_i(u_i)u_i & \text{in } \Omega, \\ u_i = w_i & \text{on } \partial\Omega, \end{cases} \quad 1 \leq i \leq n. \tag{3.4}$$

Using Theorem 3.1, the results of [61], [62] and [63] can be easily adapted to obtain the next one

Theorem 3.2 *Suppose* (3.4) *admits a subsolution* $\underline{u} \in [\mathcal{C}^{2+\nu}(\bar{\Omega})]^n$ *and a supersolution* $\bar{u} \in [\mathcal{C}^{2+\nu}(\bar{\Omega})]^n$ *satisfying* $\underline{u} \leq \bar{u}$. *Then,* (3.4) *possesses a solution* $u \in [\mathcal{C}^{2+\nu}(\bar{\Omega})]^n$ *such that* $\underline{u} \leq u \leq \bar{u}$. *Actually,* (3.4) *possesses a minimal and a maximal solution in the interval* $[\underline{u}, \bar{u}]$.

Using Theorems 3.1 and 3.2, the abstract results of [4, Section 3] can be easily adapted, almost *mutatis mutandis*, to get the next one.

Theorem 3.3 *Problem* (3.4) *has a unique positive solution,* $\theta_{[\Omega,w]}$. *Moreover, for every positive subsolution* $\underline{u}$ *(resp. supersolution* $\bar{u}$*) of* (3.4),

$$\underline{u} \leq \theta_{[\Omega,w]} \qquad (\textit{resp. } \bar{u} \geq \theta_{[\Omega,w]}). \tag{3.5}$$

Proof: By (A1), $\underline{u} := 0$ is a subsolution of (3.5). In the special case $\mathfrak{a}_i(x) > 0$ for all $x \in \bar{\Omega}$ and $1 \leq i \leq n$, one can take $\bar{u} = \vec{m} := (m, \dots, m)$ as a supersolution, for some $m > 0$ sufficiently large. In order to get a supersolution in the general case, we may proceed as follows.

Since $\partial\Omega$ is smooth, $\partial\Omega$ possesses finitely many components, Γ_k, $1 \leq k \leq m$. For each $\varepsilon > 0$ and $1 \leq k \leq m$, denote

$$\Omega_k^\varepsilon := \{x \in \mathbb{R}^N : \text{ dist}(x, \Gamma_k) < \varepsilon\}.$$

Let

$$\lambda := \max_{1\leq i\leq n} ||\lambda_i||_\infty + 1, \qquad a := \max_{1\leq i,j\leq n} a_{ij},$$

and let $\bar{\mathfrak{L}}$ be the operador

$$(\bar{\mathfrak{L}}u)_i := -\Delta u_i - \lambda u_i - \sum_{j=1,j\neq i}^{n} a u_j.$$

Thanks to (3.3),

$$\sigma[\mathfrak{L},\Omega_k^\varepsilon] > \sigma[\bar{\mathfrak{L}},\Omega_k^\varepsilon] \quad \text{for all} \quad \varepsilon>0 \quad \text{and} \quad 1\le k\le m.$$

On the other hand, by the uniqueness of the principal eigenvalue,

$$\sigma[\bar{\mathfrak{L}},\Omega_k^\varepsilon] = \sigma[-\Delta-\lambda-(n-1)a,\Omega_k^\varepsilon] = \sigma[-\Delta,\Omega_k^\varepsilon]-\lambda-(n-1)a, \qquad 1\le k\le m.$$

Thus, since

$$\lim_{\varepsilon\downarrow 0}|\Omega_k^\varepsilon| = 0, \quad 1\le k\le m,$$

where $|\Omega_k^\varepsilon|$ stands for the Lebesgue measure of Ω_k^ε, the Faber–Krahn inequality, going back to [21] and [36], yields

$$\lim_{\varepsilon\downarrow 0}\sigma[\mathfrak{L},\Omega_k^\varepsilon] = +\infty, \qquad 1\le k\le m,$$

(see Theorem 5.1 of [39]). Therefore, ε can be shortened, if necessary, so that

$$\min_{1\le k\le m}\sigma[\mathfrak{L},\Omega_k^\varepsilon] > 0.$$

Fix $\varepsilon>0$ satisfying the last inequality and, for each $1\le k\le m$, let

$$\varphi_k := (\varphi_{k,1},\dots,\varphi_{k,n})$$

be a principal eigenfunction associated to $\sigma[\mathfrak{L},\Omega_k^\varepsilon]$. As $\varphi_k\gg 0$, it is apparent that

$$\min\left\{\min_{\Gamma_k}\varphi_{k,i}:\ 1\le i\le n\right\}>0, \qquad \min\left\{\min_{\Omega\cap\partial\Omega_k^{\varepsilon/2}}\varphi_{k,i}:\ 1\le i\le n\right\}>0, \tag{3.6}$$

for all $1\le k\le m$. Subsequently, we consider the auxiliary function Φ defined through

$$\Phi := \begin{cases} \varphi_k & \text{in} \quad \bar{\Omega}_m^{\varepsilon/2}, & 1\le k\le m,\\ g & \text{in} \quad \Omega_{\text{int}} := \Omega\setminus\left(\bigcup_{k=1}^m\bar{\Omega}_k^{\varepsilon/2}\right), & \end{cases}$$

where g is any $C^{2+\nu}$-extension of the function $\varphi_1\otimes\cdots\otimes\varphi_m$ to the open set Ω_{int} with the special requirement that

$$\inf_{\Omega_{\text{int}}} g>0.$$

Such a function exists because of (3.6). Then, $\tau\Phi$ is a supersolution of (3.4) for sufficiently large $\tau>1$. Indeed, by (3.6), there exists $\tau_0\ge 1$ such that

$$\tau\Phi > w \quad \text{on} \quad \partial\Omega, \quad \text{for all} \quad \tau>\tau_0.$$

Moreover, for every $1 \leq k \leq m$ and $1 \leq i \leq n$, we find that, in $\Omega_k^{\varepsilon/2} \cap \Omega$,

$$\begin{aligned}-\Delta(\tau\Phi_i) - \lambda_i(\cdot)\tau\Phi_i - \sum_{j=1,j\neq i}^{n} a_{ij}\tau\Phi_j &= (\mathfrak{L}(\tau\Phi))_i = \tau(\mathfrak{L}(\varphi_k))_i \\ &= \tau\sigma[\mathfrak{L},\Omega_k^{\varepsilon}]\varphi_{k,i} > 0 \geq -\mathfrak{a}_i(\cdot)f_i(\tau\Phi_i)\tau\Phi_i.\end{aligned}$$

Lastly, in $\Omega_{\rm int}$, we have that $\mathfrak{a}_i$, $-\Delta g_i$ and g_i are bounded away from zero for every $1 \leq i \leq n$. Hence, thanks to (A2), taking $\tau > 1$ sufficiently large we have

$$\begin{aligned}\mathfrak{a}_i(\cdot)f_i(\tau g_i)g_i &\geq \mathfrak{a}_i(\cdot)F(\tau g_i) \\ &> -(-\Delta)g_i + \lambda_i(\cdot)g_i + \sum_{j=1,j\neq i}^{n} a_{ij}g_j, \quad 1 \leq i \leq n,\end{aligned}$$

because, as F satisfies (KO), $\lim_{u\to+\infty} F(u) = +\infty$. Therefore

$$-\Delta(\tau\Phi_i) = -\tau\Delta g_i \geq \lambda_i(\cdot)\tau g_i + \sum_{j=1,j\neq i}^{n} a_{ij}\tau g_j - \mathfrak{a}_i(\cdot)F(\tau g_i)\tau g_i$$

for sufficiently large $\tau > 1$.

By Theorem 3.2, there exists a minimal and a maximal positive solution, u_* and u^* in the interval $[0, \tau\Phi]$. Using Theorem 3.1 it is easy to see that necessary $u^* \gg 0$ and $u_* \gg 0$. Let us show the uniqueness by contradiction. Suppose $\omega := u^* - u_* > 0$. Clearly, $\omega\big|_{\partial\Omega} = 0$ and

$$-\Delta\omega_i = \lambda_i(x)\omega_i + \sum_{j=1,j\neq i}^{n} a_{ij}\omega_j - \mathfrak{a}_i(x)(f_i(u_i^*)u_i^* - f_i(u_{*i})u_{*i}), \quad 1 \leq i \leq n.$$

Setting

$$\psi_i(t) := f_i(tu_i^* + (1-t)u_{*i})(tu^* + (1-t)u_{*i}), \quad t \in [0,1], \quad 1 \leq i \leq n,$$

we deduce, for each $1 \leq i \leq n$, that

$$\begin{aligned}f_i(u_i^*)u_i^* - f_i(u_{*i})u_{*i} = \psi_i(1) - \psi_i(0) &= \int_0^1 \psi_i' \\ &= \int_0^1 f_i'(tu_i^* + (1-t)u_{*i})(tu^* + (1-t)u_{*i})dt\ \omega_i \\ &\quad + \int_0^1 f_i(tu_i^* + (1-t)u_{*i})dt\ \omega_i := V_i(x)\ \omega_i.\end{aligned}$$

Consequently ω solves

$$\begin{cases} (\mathfrak{M}\omega)_i := (\mathfrak{L}\omega)_i + \mathfrak{a}_i(x)V_i(x)\omega_i = 0 & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases} \quad 1 \leq i \leq n.$$

Necessary $\omega \gg 0$ and $\sigma[\mathfrak{M},\Omega] = 0$. But this is impossible, because we find from from (A1) that

$$V_i(x) > \int_0^1 f_i(tu_i^* + (1-t)u_{i*})dt\, \omega_i > f_i(u_{i*}),$$

and hence

$$\begin{cases} (\mathfrak{L}u_*)_i = -\mathfrak{a}_i(x)f_i(u_{*i})u_{*i} > -\mathfrak{a}(x)V_i(x)u_{*i} & \text{in } \Omega, \\ u_{*i} \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, u_* provides us with a positive strict supersolution for $\mathfrak{M}$ in Ω. The contradiction cames from the fact that, owing to Theorem (3.1), $\sigma[\mathfrak{M},\Omega] > 0$.

The latest assertion is easily deduced from the uniqueness. □

Thanks to Theorem 3.3 the mapping

$$\begin{aligned} (0,+\infty) &\longrightarrow \left[\mathcal{C}^{2+\nu}(\bar{\Omega})\right]^n \\ m &\longmapsto \theta_{[\Omega,\vec{m}]} \end{aligned}$$

is strongly increasing, in the sense that $m_1 < m_2$ yields $\theta_{[\Omega,\vec{m}_1]} \ll \theta_{[\Omega,\vec{m}_2]}$. Hence, the point-wise limit

$$\theta_{[\Omega,\infty]}(x) := \lim_{m\to+\infty} \theta_{[\Omega,\vec{m}]}(x), \qquad x \in \Omega, \tag{3.7}$$

is well defined. In fact, the next result holds.

Theorem 3.4 *There exists a minimal and a maximal positive solution of* (3.1)*, L^{min} and L^{max}, respectively, in the sense that any solution, L, of* (3.1) *satisfies*

$$L^{min}(x) \leq L(x) \leq L^{max}(x) \qquad x \in \Omega.$$

Moreover the point-wise limit (3.7) *provides us with the minimal solution*

$$L^{min} = \theta_{[\Omega,\infty]},$$

while the maximal solution is given by

$$L^{max} = \lim_{\delta\downarrow 0} \theta_{[\Omega_\delta,\infty]},$$

where Ω_δ was defined in (2.5).

Chapter 4

The radially symmetric case

In this chapter we study the uniqueness of solution of the following radially symmetric counterpart of (3.1),

$$\begin{cases} -\Delta u_i = \lambda_i u_i + \sum_{j=1, j\neq i}^{n} a_{ij} u_j - \alpha_i(d(x)) f_i(u_i) u_i & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega, \end{cases} \qquad 1 \leq i \leq n, \quad (4.1)$$

where $\lambda_i \in \mathbb{R}$, $\alpha_i \in \mathcal{C}^\nu[0,\infty)$, for some $\nu \in (0,1]$, satisfy $\alpha_i \geq 0$ in Ω for every $1 \leq i \leq n$, $\alpha := (\alpha_1, \cdots, \alpha_n) \neq 0$ and

$$d(x) := \operatorname{dist}(x, \partial\Omega), \qquad x \in \Omega,$$

is the distance to the boundary function. The kind of domains Ω considered here are the ball and the annulus. So,

$$\Omega \in \{B_R(x_0), A_{R_1,R_2}(x_0)\}, \tag{4.2}$$

where $x_0 \in \mathbb{R}^N$, $N \geq 1$, $R > 0$, $R_2 > R_1 > 0$, and

$$B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\},$$

$$A_{R_1,R_2}(x_0) := \{x \in \mathbb{R}^N : R_1 < |x - x_0| < R_2\}.$$

The main result of this chapter is the following

Theorem 4.1 *Suppose* (4.2)*,* $\lambda_i \geq 0$*,* $a_{ij} > 0$ *for all* $1 \leq i \leq j \leq n$*,* $j \neq i$ *and* α_i *are positive nondecreasing functions,*

$$0 < \alpha_i(t) \leq \alpha_i(s) \quad \textit{for all} \quad 0 < t \leq s, \quad 1 \leq i \leq n. \tag{4.3}$$

Suppose the function $g(u) := f(u)u$ *satisfies Condition* (C) *defined in* (2.6)*, i.e., in terms of* f*, there exists* $r > 0$ *such that*

$$\gamma^r f_i(u) \leq f_i(\gamma u) \quad \textit{for all} \quad \gamma > 1, \quad u > 0, \quad 1 \leq i \leq n. \tag{4.4}$$

Then, Problem (4.1) *has a unique positive solution. Moreover, it is radially symmetric.*

The condition provided by (4.4) is equivalent to the existence of $b > 0$ such that

$$\rho^2 f_i(\rho^{-b}u) \leq f_i(u) \quad \text{for all} \quad (\rho, u) \in (1,\infty)\times[0,\infty), \quad 1 \leq i \leq n, \tag{4.5}$$

because the change of variables

$$b = \frac{2}{r}, \quad \gamma = \rho^b,$$

transforms (4.4) into (4.5). According to Lemma 7.2 of [48], going back to [10], it is already known that (4.5) holds, with $r = 1$, provided $f_i \in \mathcal{C}[0,\infty) \cap \mathcal{C}^2(0,\infty)$ and

$$f_i''(u) \geq 0 \quad \text{for all} \quad u > 0 \quad \text{and} \quad 1 \leq i \leq n.$$

This result is a substantial extension of the uniqueness results of [48, Chapter 7], going back to [43], [45] and [46], to cover the class of cooperative systems dealt with in this chapter.

Our proof of Theorem 5.1 is based on a rather sophisticated use of the maximum principle for weakly coupled cooperative elliptic systems, as discussed in [54], [62], [61] and [63], and it relies on some pioneering ideas going back to [46]. The reader should appreciate how the use of the maximum principle in the proof of Theorem 5.1, specially for the case of the annulus, is based on a rather intricate choice of some auxiliary functions in order to perform some necessary comparisons through a clever use of the maximum principle for weakly coupled cooperative systems, which seems to be a rather pioneering technical device in the field.

4.1 Proof of Theorem 4.1 in case $\Omega = B_R(x_0)$

Subsequently, we set

$$L \equiv L^{\min}, \qquad \rho_\varepsilon := \frac{R}{R-\varepsilon} > 1, \quad \varepsilon \in (0,R),$$

and consider the function

$$\bar{L}_\varepsilon(x) := L(x_0 + \rho_\varepsilon(x - x_0)), \qquad 0 \leq |x - x_0| < R - \varepsilon.$$

By definition, $\bar{L}_\varepsilon = +\infty$ on $\partial B_{R-\varepsilon}(x_0)$. Moreover, for every $x \in B_{R-\varepsilon}(x_0)$ and $1 \leq i \leq n$, we have that

$$\begin{aligned}
-\Delta \bar{L}_{\varepsilon,i}(x) &= -\rho_\varepsilon^2 \Delta L_i(x_0 + \rho_\varepsilon(x-x_0)) \\
&= \rho_\varepsilon^2 \lambda_i \bar{L}_{\varepsilon,i}(x) + \rho_\varepsilon^2 \sum_{j=1, j\neq i}^n a_{ij} \bar{L}_{\varepsilon,j}(x) - \rho_\varepsilon^2 \alpha_i(R - \rho_\varepsilon|x-x_0|) f_i(\bar{L}_{\varepsilon,i}(x)) \bar{L}_{\varepsilon,i}(x) \\
&\geq \lambda_i \bar{L}_{\varepsilon,i}(x) + \sum_{j=1, j\neq i}^n a_{ij} \bar{L}_{\varepsilon,j}(x) - \rho_\varepsilon^2 \alpha_i(R - |x-x_0|) f_i(\bar{L}_{\varepsilon,i}(x)) \bar{L}_{\varepsilon,i}(x) \\
&= \lambda_i \bar{L}_{\varepsilon,i}(x) + \sum_{j=1, j\neq i}^n a_{ij} \bar{L}_{\varepsilon,j}(x) - \rho_\varepsilon^2 \alpha_i(d(x)) f_i(\bar{L}_{\varepsilon,i}(x)) \bar{L}_{\varepsilon,i}(x),
\end{aligned}$$

because $\rho_\varepsilon > 1$, $\lambda_i \geq 0$, and, thanks to (4.3),

$$\alpha_i(R - \rho_\varepsilon |x - x_0|) \leq \alpha_i(R - |x - x_0|), \qquad 1 \leq i \leq n.$$

Now, let $b > 0$ be satisfying (4.5) and consider the function

$$\hat{L}_\varepsilon(x) := \rho_\varepsilon^b \bar{L}_\varepsilon(x), \qquad x \in B_{R-\varepsilon}(x_0).$$

By definition, $\hat{L}_\varepsilon = \infty$ on $\partial B_{R-\varepsilon}(x_0)$, and, for every $x \in B_{R-\varepsilon}(x_0)$ and $1 \leq i \leq n$,

$$\begin{aligned} -\Delta \hat{L}_{\varepsilon,i}(x) &= -\rho_\varepsilon^b \Delta \bar{L}_{\varepsilon,i}(x) \\ &\geq \lambda_i \hat{L}_{\varepsilon,i}(x) + \sum_{j=1, j\neq i}^n a_{ij} \hat{L}_{\varepsilon,j}(x) - \rho_\varepsilon^2 \alpha_i(d(x)) f_i(\rho_\varepsilon^{-b} \hat{L}_{\varepsilon,i}(x)) \hat{L}_{\varepsilon,i}(x) \\ &\geq \lambda_i \hat{L}_{\varepsilon,i}(x) + \sum_{j=1, j\neq i}^n a_{ij} \hat{L}_{\varepsilon,j}(x) - \alpha_i(d(x)) f_i(\hat{L}_{\varepsilon,i}(x)) \hat{L}_{\varepsilon,i}(x), \end{aligned}$$

because of (4.5). Therefore, $\hat{L}_\varepsilon$ if a supersolution of the singular problem

$$\begin{cases} -\Delta u_i = \lambda_i u_i + \displaystyle\sum_{j=1, j\neq i}^n a_{ij} u_j - \alpha_i(d(x)) f_i(u_i) u_i & \text{in} \quad B_{R-\varepsilon}(x_0), \\ u_i = \infty & \text{on} \quad \partial B_{R-\varepsilon}(x_0), \end{cases} \qquad 1 \leq i \leq n.$$

Note that, for each $\delta < \varepsilon$, $L^{\min}_{[\lambda, B_{R-\delta}(x_0)]}$ is finite on $\partial B_{R-\varepsilon}(x_0)$. So, there exists $\varepsilon_1 > \varepsilon$ such that

$$L^{\min}_{[\lambda, B_{R-\delta}(x_0)]} < \hat{L}_\varepsilon \qquad \text{in} \ \ A_{R-\varepsilon_1, R-\varepsilon}(x_0).$$

By comparison, from the maximum principle, the last inequality must be satisfied on the entire $B_{R-\varepsilon}(x_0)$, i.e.,

$$L^{\min}_{[\lambda, B_{R-\delta}(x_0)]} < \hat{L}_\varepsilon \qquad \text{in} \quad B_{R-\varepsilon}(x_0) \quad \text{for all} \quad \delta < \varepsilon. \tag{4.6}$$

Hence, letting $\delta \downarrow 0$ in (4.6) yields

$$L^{\max}_{[\lambda, B_R(x_0)]} \leq \hat{L}_\varepsilon \quad \text{in} \quad B_{R-\varepsilon}(x_0)$$

for all $\varepsilon \in (0, R)$. Lastly, taking into account that

$$\hat{L}_\varepsilon(x) = \rho_\varepsilon^b L^{\min}_{[\lambda, B_R(x_0)]}(x_0 + \rho_\varepsilon(x - x_0))$$

and letting $\varepsilon \downarrow 0$, we find that

$$L^{\max}_{[\lambda, B_R(x_0)]} \leq L^{\min}_{[\lambda, B_R(x_0)]} \quad \text{in} \quad B_R(x_0),$$

with ends the proof of the theorem in this case.

4.2 Proof of Theorem 4.1 in case $\Omega = A_{R_1,R_2}(x_0)$

Throughout this section, we will set

$$R_m := \frac{R_1 + R_2}{2}, \qquad r := |x - x_0|.$$

Hence,

$$d(x) = \begin{cases} r - R_1 & \text{if } R_1 \le r \le R_m, \\ R_2 - r & \text{if } R_m \le r \le R_2. \end{cases}$$

Moreover, as $L^{\min}_{[\lambda,\Omega]}$ and $L^{\max}_{[\lambda,\Omega]}$ are radially symmetric,

$$L^{\min}_{[\lambda,\Omega]}(x) = \psi_{\min}(r), \quad L^{\max}_{[\lambda,\Omega]}(x) = \psi_{\max}(r), \quad x \in \Omega = A_{R_1,R_2}(x_0),$$

where $\psi_{\min}$ and $\psi_{\max}$ are the reflections about $r = R_m$ of the minimal and the maximal positive solutions, respectively, of the singular problem

$$\begin{cases} -\psi_i'' - \frac{N-1}{r}\psi_i' = \lambda_i\psi_i + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_j - \alpha_i(R_2 - r) f_i(\psi_i)\psi_i, \quad R_m < r < R_2, \\ \psi_i'(R_m) = 0, \quad \psi_i(R_2) = \infty, \end{cases} \qquad 1 \le i \le n.$$

In order to adapt the argument of the previous section, we need to show that the minimal positive solution of this problem satisfies

$$\psi'_{\min,i} \ge 0 \quad \text{in} \quad [R_m, R_2) \quad \text{for all} \quad 1 \le i \le n. \tag{4.7}$$

For this purpose, we will need the next lemma.

Lemma 4.2 *There exists a sequence, $\{r_k\}_{k\in\mathbb{N}}$, in (R_m, R_2) such that $\lim_{k\to\infty} r_k = R_2$ and, for every $k \in \mathbb{N}$,*

$$\psi_{\min,i}(r) \le \|\psi_{\min}(r_k)\|_\infty = \max_{1\le i\le n} \psi_{\min,i}(r_k) \quad \textit{for all} \quad r \in [R_m, r_k], \quad 1 \le i \le n. \tag{4.8}$$

Proof. For each $k \in \mathbb{N}$, consider the interval

$$I_k := [R_m, R_2 - (R_2 - R_m)/(k+1)].$$

Now, let $r_k \in I_k$ be such that

$$\|\psi_{\min}\|_{\mathcal{C}(I_k)} = \max_{\substack{r\in I_k \\ 1\le i\le n}} \psi_{\min,i}(r) = \psi_{i_k}(r_k)$$

for some $1 \le i_k \le n$. By construction, condition (4.8) holds with this choice of $(r_k)_{k\ge1}$. Therefore, it only remains to show that $r_k \to R_2$ as $k \to \infty$. Otherwise, there exist an

$\varepsilon > 0$ and a subsequence, $\{r_{k_m}\}_{m\in\mathbb{N}}$, with $r_{k_m} \in [R_m, R_2-\varepsilon]$ for all $m \in \mathbb{N}$. But this is impossible, because

$$\lim_{m\uparrow\infty} \psi_{\min,i_{k_m}}(r_{k_m}) = \lim_{m\uparrow\infty} \|\psi_{\min}\|_{\mathcal{C}(I_{k_m})} = \infty,$$

whereas $\|\psi_{\min}\|_{\mathcal{C}[R_m,R_2-\varepsilon]} < \infty$. $\square$

Let r_ℓ, $\ell \geq 1$, be the sequence given by Lemma 4.2 and, for each $\ell \in \mathbb{N}$, consider the associated boundary value problem

$$\begin{cases} -\psi_i'' - \frac{N-1}{r}\psi_i' = \lambda_i\psi_i - \alpha_i(R_2-r)f_i(\psi_i)\psi_i, & R_m < r < r_\ell, \\ \psi_i'(R_m) = 0, \quad \psi_i(r_\ell) = \psi_{\min,i}(r_\ell), \end{cases} \quad 1 \leq i \leq n. \tag{4.9}$$

Observe that (4.9) is uncoupled. Next, we will construct a sequence of functions according to the following scheme. Let

$$\psi_1 := (\psi_{1,1}, \cdots, \psi_{1,n}) \in \left(\mathcal{C}^{2+\nu}[R_m, r_\ell]\right)^n$$

be the unique solution of (4.9). The uniqueness follows easily from (A1) using the maximum principle. The existence can be obtained with the method of sub and supersolutions. Indeed, $\underline{\psi} := 0$ provides us with a subsolution, and $\bar{\psi} := \psi_{\min}$ is a supersolution because

$$\begin{aligned} -\psi_{\min,i}'' - \frac{N-1}{r}\psi_{\min,i}' &= \lambda_i\psi_{\min,i} + \sum_{j=1,j\neq i}^{n} a_{ij}\psi_{\min,j} - \alpha_i(R_2-r)f_i(\psi_{\min,i})\psi_{\min,i} \\ &\geq \lambda_i\psi_{\min,i} - \alpha_i(R_2-r)f_i(\psi_{\min,i})\psi_{\min,i}. \end{aligned}$$

In particular,

$$\psi_1 \leq \psi_{\min} \qquad \text{in } [R_m, r_\ell].$$

Subsequently, given some $k \geq 1$ and

$$\psi_{k-1} = (\psi_{k-1,1}, \cdots, \psi_{k-1,n}) \in \left(\mathcal{C}^{2+\nu}[R_m, r_\ell]\right)^n,$$

we will denote by

$$\psi_k = (\psi_{k,1}, \ldots, \psi_{k,n}) \in \left(\mathcal{C}^{2+\nu}[R_m, r_\ell]\right)^n$$

the unique positive solution of the boundary value problem

$$\begin{cases} -\psi_i'' - \frac{N-1}{r}\psi_i' = \lambda_i\psi_i - \alpha_i(R_2-r)f_i(\psi_i)\psi_i + \displaystyle\sum_{j=1,j\neq i}^{n} a_{ij}\psi_{k-1,j}(r), & R_m < r < r_\ell, \\ \psi_i'(R_m) = 0, \quad \psi_i(r_\ell) = \psi_{\min,i}(r_\ell), & 1 \leq i \leq n, \end{cases} \tag{4.10}$$

if it exists. The uniqueness is guaranteed by (A2). Such solutions are indeed defined for all $k \geq 1$ and they satisfy

$$\psi_1 \leq \psi_2 \leq \cdots \leq \psi_k \leq \psi_{\min}. \tag{4.11}$$

The proof proceeds by induction. For $k = 1$ we already know that ψ_1 exists and $\psi_1 \leq \psi_{\min}$. Suppose

$$\psi_1 \leq \cdots \leq \psi_{k-1} \leq \psi_{\min} \tag{4.12}$$

for some $k \geq 2$. Then, by construction and, due to (4.12),

$$\begin{aligned} -\psi''_{k-1,i} - \frac{N-1}{r}\psi'_{k-1,i} &= \lambda_i \psi_{k-1,i} - \alpha_i(R_2 - r) f_i(\psi_{k-1,i})\psi_{k-1,i} + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_{k-2,j}(r) \\ &\leq \lambda_i \psi_{k-1,i} - \alpha_i(R_2 - r) f_i(\psi_{k-1,i})\psi_{k-1,i} + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_{k-1,j}(r) \end{aligned}$$

in (R_m, r_ℓ). Moreover, by construction, ψ_{k-1} satisfies the boundary conditions. Thus, ψ_{k-1} provides us with a subsolution of (4.10). Similarly,

$$\begin{aligned} -\psi''_{\min,i} - \frac{N-1}{r}\psi'_{\min,i} &= \lambda_i \psi_{\min,i} - \alpha_i(R_2 - r) f_i(\psi_{\min,i})\psi_{\min,i} + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_{\min,j} \\ &\geq \lambda_i \psi_{\min,i} - \alpha_i(R_2 - r) f_i(\psi_{\min,i})\psi_{\min,i} + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_{k-1,j}(r) \end{aligned}$$

in (R_m, r_ℓ), because $\psi_{\min} \geq \psi_{k-1}$, by (4.12). Consequently, (4.10) indeed possesses a (unique) solution, ψ_k, such that

$$\psi_{k-1} \leq \psi_k \leq \psi_{\min},$$

which ends the proof of the claim above.

Next, we will show that

$$\psi'_{k,i}(r) \geq 0, \qquad \text{for all} \quad r \in [R_m, r_\ell), \quad 1 \leq i \leq n, \quad k \in \mathbb{N}. \tag{4.13}$$

Indeed, thanks to the proof of Theorem 7.1 in [48, Chapter 7], (4.13) holds if $k = 1$. Suppose (4.13) is true for some $k \in \mathbb{N}$. We want to show it for $k + 1$. On the contrary, suppose there exist $\tilde{r} \in (R_m, r_\ell)$ and $1 \leq i_0 \leq n$ such that $\psi'_{k+1,i_0}(\tilde{r}) < 0$. Then, by (4.11) and (4.8), we find that

$$\psi'_{k+1,i_0}(R_m) = 0, \qquad \psi_{k+1,i_0}(\tilde{r}) \leq \psi_{\min,i_0}(\tilde{r}) \leq \psi_{\min,i_0}(r_\ell).$$

So, there are $R_m \leq \varrho_0 < \tilde{r} < \varrho_1 < r_\ell$ such that

$$\begin{cases} \psi'_{k+1,i_0}(\varrho_0) = \psi'_{k+1,i_0}(\varrho_1) = 0, \quad \psi'_{k+1,i_0}(r) < 0 \ \text{ if } r \in (\varrho_0, \varrho_1), \\ \psi''_{k+1,i_0}(\varrho_0) \leq 0, \ \ \psi''_{k+1,i_0}(\varrho_1) \geq 0. \end{cases} \tag{4.14}$$

Subsequently, we consider the function H defined by

$$H(\xi) := \lambda_{i_0}\xi - \alpha_{i_0}(R_2 - \varrho_0) f_{i_0}(\xi)\xi + \sum_{j=1, j\neq i_0}^{n} a_{i_0 j}\psi_{k,j}(\varrho_0) \quad \text{for all } \xi > 0.$$

By (A1), (A2), the function $H(\xi)$ possesses a unique positive zero, ξ_0. Moreover,

$$\begin{cases} H(\xi) > 0 & \text{if } 0 < \xi < \xi_0, \\ H(\xi) < 0 & \text{if } \xi > \xi_0. \end{cases}$$

According to (4.10) and (4.14), we have that

$$0 \leq -\psi''_{k+1,i_0}(\varrho_0) = -\psi''_{k+1,i_0}(\varrho_0) - \frac{N-1}{\varrho_0}\psi'_{k+1,i_0}(\varrho_0) = H(\psi_{k+1,i_0}(\varrho_0)),$$

witch implies

$$\psi_{k+1,i_0}(\varrho_0) \leq \xi_0. \tag{4.15}$$

On the other hand, again by (4.10),

$$\begin{aligned} 0 &\geq -\psi''_{k+1,i_0}(\varrho_1) = -\psi''_{k+1,i_0}(\varrho_1) - \frac{N-1}{\varrho_0}\psi'_{k+1,i_0}(\varrho_1) \\ &= \lambda_{i_0}\psi_{k+1,i_0}(\varrho_1) + \sum_{j=1, j\neq i_0}^{n} a_{i_0 j}\psi_{k,j}(\varrho_1) - \alpha_{i_0}(R_2 - \varrho_1) f_{i_0}(\psi_{k+1,i_0}(\varrho_1))\psi_{k+1,i_0}(\varrho_1) \\ &\geq H(\psi_{k+1,i_0}(\varrho_1)), \end{aligned}$$

because $\varrho_0 < \varrho_1$ and (4.3) imply

$$-\alpha_{i_0}(R_2 - \varrho_1) \geq -\alpha_{i_0}(R_2 - \varrho_0)$$

and, thanks to the induction hypothesis,

$$\psi_{k,j}(\varrho_1) \geq \psi_{k,j}(\varrho_0) \qquad \text{for all } 1 \leq j \leq n,$$

because ψ_k is non-decreasing. Consequently,

$$H(\psi_{k+1,i_0}(\varrho_1)) \leq 0. \tag{4.16}$$

On the other hand, since $\psi'_{k+1,i_0}(r) < 0$ for all $r \in (\varrho_0, \varrho_1)$, we have $\psi_{k+1,i_0}(\varrho_1) < \psi_{k+1,i_0}(\varrho_0)$, and, thanks to (4.15),

$$H(\psi_{k+1,i_0}(\varrho_1)) > 0.$$

As this contradicts (4.16), (4.13) holds.

By (4.11), the point-wise limit

$$\psi := \lim_{k\to\infty} \psi_k \leq \psi_{\min}$$

is well defined. Actually, $\psi = \psi_{\min}$ in $[R_m, r_\ell]$. Indeed, by the Schauder estimates, the point-wise limit ψ must be a solution of

$$\begin{cases} -\psi_i'' - \frac{N-1}{r}\psi_i' = \lambda_i\psi_i - \alpha_i(R_2-r)f_i(\psi_i)\psi_i + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_j, & R_m < r < r_\ell, \\ \psi_i'(R_m) = 0, \quad \psi_i(r_\ell) = \psi_{\min,i}(r_\ell), & 1 \leq i \leq n. \end{cases} \tag{4.17}$$

As $\psi_{\min}$ is the unique solution of (4.17), $\psi_{\min} = \psi$. Lastly, owing to (4.13), we find that $\psi'_{\min,i} \geq 0$ on $[R_m, r_\ell)$ for all $1 \leq i \leq n$. Therefore, since the previous argument is independent on ℓ, (4.7) holds.

Subsequently, we set

$$\eta_\varepsilon := \frac{R_2 - R_m}{R_2 - R_m - \varepsilon} > 1, \quad \varepsilon \in (0, R_2 - R_m),$$

and consider the function $\bar\psi_\varepsilon$ defined by

$$\bar\psi_\varepsilon(r) := \psi_{\min}(\eta_\varepsilon(r - R_m) + R_m), \quad R_m \leq r < R_2 - \varepsilon.$$

By definition, we have

$$\bar\psi'_{\varepsilon,i}(R_m) = \eta_\varepsilon \psi'_{\min,i}(R_m) = 0, \qquad 1 \leq i \leq n,$$

and

$$\lim_{r\uparrow R_2-\varepsilon} \bar\psi_\varepsilon(r) = \infty.$$

Moreover, setting

$$\varrho := \eta_\varepsilon(r - R_m) + R_m, \qquad R_m \leq r \leq R_2 - \varepsilon,$$

it becomes apparent that, for every $r \in (R_m, R_2 - \varepsilon)$, $1 \leq i \leq n$,

$$-\bar\psi''_{\varepsilon,i}(r) - \frac{N-1}{r}\bar\psi'_{\varepsilon,i}(r) = -\eta_\varepsilon^2\psi''_{\min,i}(\varrho) - \frac{N-1}{\varrho}\frac{\varrho}{r}\eta_\varepsilon\psi'_{\min,i}(\varrho).$$

Thus, taking into account that

$$\lambda_i \geq 0, \quad \psi'_{\min,i} \geq 0, \quad \eta_\varepsilon > 1, \quad \frac{\varrho}{r} = \frac{R_m(1-\eta_\varepsilon) + \eta_\varepsilon r}{r} \leq \eta_\varepsilon,$$

we find that, for every $1 \leq i \leq n$,

$$\begin{aligned}
-\bar{\psi}''_{\varepsilon,i}(r) - \frac{N-1}{r}\bar{\psi}'_{\varepsilon,i}(r) &= -\eta_\varepsilon^2 \psi''_{\min,i}(\varrho) - \frac{N-1}{\varrho}\frac{\varrho}{r}\eta_\varepsilon \psi'_{\min,i}(\varrho) \\
&\geq \eta_\varepsilon^2 \big[-\psi''_{\min,i}(\varrho) - \frac{N-1}{\varrho}\psi'_{\min,i}(\varrho)\big] \\
&= \eta_\varepsilon^2 [\lambda_i \psi_{\min,i}(\varrho) + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_{\min,j}(\varrho) - \alpha_i(R_2-\varrho) f_i(\psi_{\min,i}(\varrho))\psi_{\min,i}(\varrho)] \\
&\geq \lambda_i \bar{\psi}_{\varepsilon,i}(r) + \sum_{j=1, j\neq i}^{n} a_{ij}\bar{\psi}_{\varepsilon,i}(r) - \eta_\varepsilon^2 \alpha_i(R_2-r) f_i(\bar{\psi}_{\varepsilon,i}(r))\bar{\psi}_{\varepsilon,i}(r),
\end{aligned}$$

where we have used that

$$\alpha_i(R_2-\varrho) \leq \alpha_i(R_2-r)$$

which is true because α_i is increasing and $\varrho > r$.

Let $b > 0$ be satisfying condition (4.5) and consider the function

$$\hat{\psi}_\varepsilon(r) := \eta_\varepsilon^b \bar{\psi}_\varepsilon(r), \qquad R_m \leq r < R_2 - \varepsilon.$$

Then,

$$\hat{\psi}'_{\varepsilon,i}(R_m) = 0, \qquad \lim_{r \uparrow R_2-\varepsilon} \hat{\psi}_{\varepsilon,i}(r) = \infty \quad \text{for all} \quad 1 \leq i \leq n$$

and, due to (4.5),

$$\begin{aligned}
-\hat{\psi}''_{\varepsilon,i}(r) - \frac{N-1}{r}\hat{\psi}'_{\varepsilon,i}(r) &\geq \lambda_i \hat{\psi}_{\varepsilon,i}(r) + \sum_{j=1, j\neq i}^{n} a_{ij}\hat{\psi}_{\varepsilon,i}(r) \\
&\qquad - \eta_\varepsilon^2 \alpha_i(R_2-r) f_i(\eta_\varepsilon^{-b}\hat{\psi}_{\varepsilon,i}(r))\hat{\psi}_{\varepsilon,i}(r) \\
&\geq \lambda_i \hat{\psi}_{\varepsilon,i}(r) + \sum_{j=1, j\neq i}^{n} a_{ij}\hat{\psi}_{\varepsilon,i}(r) - \alpha_i(R_2-r) f_i(\hat{\psi}_{\varepsilon,i}(r))\hat{\psi}_{\varepsilon,i}(r)
\end{aligned}$$

for every $r \in (R_m, R_2 - \varepsilon)$, $1 \leq i \leq n$. Therefore, $\hat{\psi}_\varepsilon$ is a supersolution of the singular problem

$$\begin{cases} -\psi''_i - \frac{N-1}{r}\psi'_i = \lambda_i \psi_i + \sum_{j=1, j\neq i}^{n} a_{ij}\psi_j - \alpha_i(R_2-r) f_i(\psi_i)\psi_i, & R_m < r < R_2 - \varepsilon, \\ \psi'_i(R_m) = 0, \quad \psi_i(R_2-\varepsilon) = \infty, & \end{cases} \qquad 1 \leq i \leq n.$$

Consequently, by the definition of $\psi_{\max}$, we have that

$$\eta_\varepsilon^b \psi_{\min}(\eta_\varepsilon(r - R_m) + R_m) \geq \psi_{\max}(r) \qquad R_m \leq r \leq R_2 - \varepsilon.$$

So, letting $\varepsilon \downarrow 0$ yields

$$\psi_{\min}(r) \geq \psi_{\max}(r) \qquad R_m \leq r \leq R_2.$$

This ends the proof of the theorem. $\square$

Remark 4.3 The results of this chapter can be easily generalized to cover the case when the terms a_{ij} of (4.1) are not only constants but continuous functions $a_{ij}(d(x))$ depending on the distance to the boundary, $d(x)$, such that

(i) $a_{ij} \in \mathcal{C}^{\nu}[0,\infty)$ and $a_{ij}(t) \geq 0$ for all $t \geq 0$ and $1 \leq i \leq j \leq n$, $i \neq j$.

(ii) a_{ij} are non-increasing functions, i.e.

$$a_{ij}(t) \geq a_{ij}(s) \quad \text{if} \qquad t < s.$$

Chapter 5

Boundary blow-up rates of the large solution

In this chapter we ascertain the boundary *blow-up rates* of the classical positive solution of the singular boundary value problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i + \sum_{j=1, j\neq i}^{n} a_{ij}u_j - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega, \end{cases} \quad 1 \le i \le n, \tag{5.1}$$

where $p_i > 1$ are constants and $\mathfrak{a}_i$ are assumed to behave power-like near $\mathfrak{a}_i^{-1}(0) \subset \partial\Omega$ for every $1 \le i \le n$, in the sense that there exist $\mathfrak{b}_i, \gamma_i \in \mathcal{C}(\partial\Omega)$, with $\mathfrak{b}_i(z) > 0$ for all $z \in \partial\Omega$ and $\gamma_i \ge 0$ on $\partial\Omega$, such that

$$\lim_{\substack{x \to z \\ x\in\Omega, z\in\partial\Omega}} \frac{\mathfrak{a}_i(x)}{\mathfrak{b}_i(z)[\operatorname{dist}(x,\partial\Omega)]^{\gamma_i(z)}} = 1, \qquad 1 \le i \le n. \tag{5.2}$$

Note that (5.1) is a more restricted problem than (3.1), since the functions $f_i(u) := u^{p_i - 1}$ satisfy (A1) and

$$F(u) := Cu^{p-1}, \quad p := \min\{p_i \ : \ 1 \le i \le n\},$$

satisfies (A2) for some $C > 0$ sufficiently small. In particular, the existence results of Chapter 3 hold.

Throughout this chapter, for every $z \in \partial\Omega$ we set

$$\mu_i(z) := \frac{\gamma_i(z) + 2}{p_i - 1}, \qquad 1 \le i \le n. \tag{5.3}$$

According to [18, 30, 41], $\mu_i(z)$, $1 \le i \le n$, provide us with the blow-up rates on $\partial\Omega$ of the positive solutions of the uncoupled problem,

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega, \end{cases} \quad 1 \le i \le n, \tag{5.4}$$

in the sense that for every solution of (5.4), $\ell = (\ell_1, \ldots, \ell_n)$, one has that

$$\lim_{\substack{x \to z \\ x\in\Omega, z\in\partial\Omega}} \frac{\ell_i(x)}{[\operatorname{dist}(x,\partial\Omega)]^{-\mu_i(z)}} = \left(\frac{\mu_i(z)(\mu_i(z)+1)}{\mathfrak{b}_i(z)}\right)^{\frac{1}{p_i-1}}, \qquad 1 \le i \le n.$$

The main result of this chapter provides us with the blow-up rates of all positive solutions of (5.1) in terms of the $\mu_i(z)$, $1 \le i \le n$, defined in (5.3). Precisely, for any given $z \in \partial\Omega$, suppose the n equations of (5.1) have been re-ordered so that

$$0 < \mu_n(z) \le \mu_{n-1}(z) \le \cdots \le \mu_1(z). \tag{5.5}$$

Then, the next result holds.

Theorem 5.1 *Let $z \in \partial\Omega$ such that* (5.5) *is satisfied and consider the next partition of the subscripts set*

$$\begin{aligned} I_+ &:= \{i \in \{1,\ldots,n\} : \ \mu_i(z) + 2 - \mu_1(z) > 0\},\\ I_0 &:= \{i \in \{1,\ldots,n\} : \ \mu_i(z) + 2 - \mu_1(z) = 0\},\\ I_- &:= \{i \in \{1,\ldots,n\} : \ \mu_i(z) + 2 - \mu_1(z) < 0\}. \end{aligned} \tag{5.6}$$

Let $k \in \{1,\ldots,n\}$ be such that

$$I_M := \{i \in \{1,\ldots,n\} : \ \mu_i(z) = \mu_1(z)\} = \{1,\ldots,k\}.$$

Then, any positive solution of (5.1), $u = (u_1, \ldots, u_n)$, *satisfies*

$$\lim_{\substack{x \to z \\ x\in\Omega}} \frac{u_i(x)}{[\operatorname{dist}(x,\partial\Omega)]^{-\alpha_i(z)}} = A_i(z), \qquad 1 \le i \le n, \tag{5.7}$$

where

$$\alpha_i(z) := \begin{cases} \mu_i(z) & \text{if} \quad i \in I_+ \cup I_0,\\ \dfrac{\mu_1(z)+\gamma_i(z)}{p_i} & \text{if} \quad i \in I_-, \end{cases} \tag{5.8}$$

and

$$A_i(z) := \begin{cases} \left(\dfrac{\mu_i(z)(\mu_i(z)+1)}{\mathfrak{b}_i(z)}\right)^{\frac{1}{p_i-1}} & \text{if} \quad i \in I_+,\\[2ex] \left[\dfrac{1}{\mathfrak{b}_i(z)}\displaystyle\sum_{j=1}^{k} a_{ij}\left(\dfrac{\mu_1(z)(\mu_1(z)+1)}{\mathfrak{b}_j(z)}\right)^{\frac{1}{p_j-1}}\right]^{\frac{1}{p_i}} & \text{if} \quad i \in I_-,\\[2ex] A_{0,i} & \text{if} \quad i \in I_0, \end{cases} \tag{5.9}$$

where $A_{0,i}$ stands for the unique positive solution of the equation

$$\mathfrak{b}_i(z)x^{p_i} - \mu_i(z)(\mu_i(z)+1)x = \sum_{j=1}^{k} a_{ij} \left(\frac{\mu_1(z)(\mu_1(z)+1)}{\mathfrak{b}_j(z)} \right)^{\frac{1}{p_j-1}}.$$

Essentially, Theorem 5.1 establishes that the values of the blow-up rates at z of any solution of (5.1) only depend on the precise way the blow-up rates at z of the solution of (5.4) are interrelated, rather than on the size of the coupling coefficients of the problem, a_{ij}, $i \neq j$, as one might have expected from the every beginning. Indeed, by Theorem 5.1, the coupling coefficients only alter the values of $A_i(z)$. In particular, (5.8) provides us with the components of (5.1) whose blow-up rates equal the corresponding blow-up rates of (5.4), as well as with the blow-up rates of the remaining components. Note that, although I_0 and I_- might be empty, owing to (5.5), $1 \in I_+$. Hence, by Theorem 5.1, the first component of (5.1) blows up with exactly the same rate as the first component of the uncoupled problem (5.4), i.e., $\alpha_1(z) = \mu_1(z)$. Moreover, $\mu_1(z)$ plays a significant role in the blow-up rates of $\alpha_i(z)$ for all $i \in I_-$.

To illustrate how the blow-up rates given by (5.8) depend on the ones provided by (5.3) we are going to see the special cases $n = 2$ and $n = 3$. Regarding

$$(\mu_1, \mu_2) \in [0, +\infty)^2$$

as an independent variable, we can divide the first quadrant of $\mathbb{R}^2$ into three portions, according to the following relations

$\mu_1 + 2 - \mu_2 \geq 0$	$\mu_1 + 2 - \mu_2 \geq 0$	$\mu_1 + 2 - \mu_2 < 0$
$\mu_2 + 2 - \mu_1 \geq 0$	$\mu_2 + 2 - \mu_1 < 0$	$\mu_2 + 2 - \mu_1 \geq 0$

The first case is equivalent to

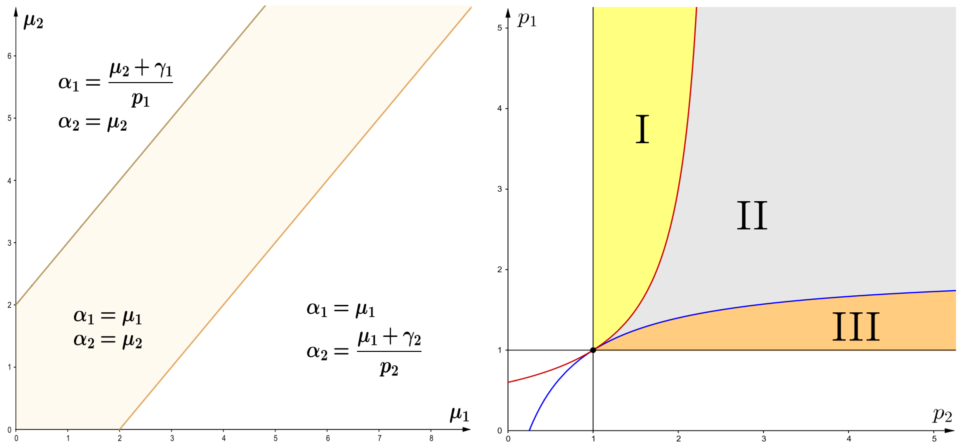
$$|\mu_1 - \mu_2| \leq 2, \tag{5.10}$$

and according to Theorem 5.1, it is the region where the blow-up rates α_1 and α_2 must equal μ_1 and μ_2. Therefore, condition (5.10) measures how close should be μ_1 and μ_2 in order to get that the coupled blow-up rates behave as the uncoupled ones. Figure 5.1 represents the region (5.10), as well as the values of the coupled blow-up rates in the complement of (5.10). Similarly, we can fix $z \in \partial\Omega$ and represent the zones

$$\begin{aligned}
&\mathrm{I} := \{(p_1,p_2) \ : \ \tfrac{\gamma_2(z)+2}{p_2-1} + 2 - \tfrac{\gamma_1(z)+2}{p_1-1} < 0\} \\
&\mathrm{II} := \Big\{(p_1,p_2); \ \ \tfrac{\gamma_1(z)+2}{p_1-1} + 2 - \tfrac{\gamma_2(z)+2}{p_2-1} \geq 0, \ \ \tfrac{\gamma_2(z)+2}{p_2-1} + 2 - \tfrac{\gamma_1(z)+2}{p_1-1} \geq 0\Big\} \\
&\mathrm{III} := \{(p_1,p_2); \ \ \tfrac{\gamma_1(z)+2}{p_1-1} + 2 - \tfrac{\gamma_2(z)+2}{p_2-1} < 0\}
\end{aligned}$$

in terms of the exponents $p_1, p_2 > 1$. According to Theorem 5.1, we have that

$$\begin{aligned}
&\alpha_1(z) = \mu_1(z) \quad \text{and} \quad \alpha_2(z) = \tfrac{\mu_1(z)+\gamma_2(z)}{p_2} \quad \text{in region I,} \\
&\alpha_1(z) = \mu_1(z) \quad \text{and} \quad \alpha_2(z) = \mu_2(z) \quad \text{in region II,} \\
&\alpha_1(z) = \tfrac{\mu_2(z)+\gamma_1(z)}{p_1} \quad \text{and} \quad \alpha_2(z) = \mu_2(z) \quad \text{in region III.}
\end{aligned}$$



(a) The zones of (μ_1, μ_2) where (α_1, α_2) changes (b) The zones I, II, III for $\gamma_1(z) = 3$, $\gamma_2(z) = 2$

Figure 5.1: The case $n = 2$

Let us see what happens if $n = 3$. By regarding $\mu := (\mu_1, \mu_2, \mu_3) \in [0, +\infty)^3$ as an independent variable, we can divide $[0, +\infty)^3$ into several portions, taking into account the relationships between the components of the variable μ. The figure provided in Table 5.1 shows a partition of the set of values of the parameters, $\mu \in [0, +\infty)^3$, into thirteen complementary zones according to the nature of the values of the blow-up rates of the solutions of (5.1),

$$\alpha_i := \alpha_i(\mu), \qquad i = 1, 2, 3,$$

depending on μ. The central portion of the figure in Table 5.1 stands for the closed hexagonal prism

$$\mu_i + 2 - \mu_j \geq 0 \quad \text{for all} \quad i, j \in \{1, 2, 3\}, \quad i \neq j,$$

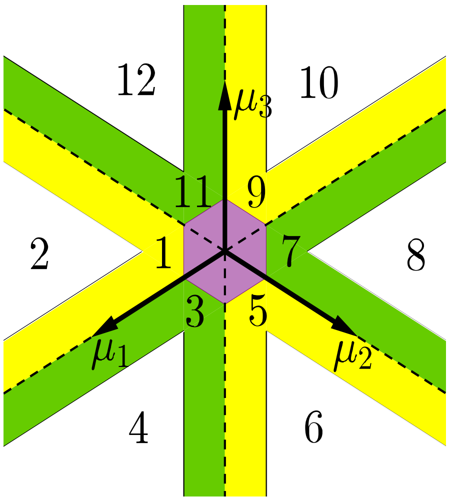
where, according to Theorem 5.1,

$$\alpha := (\alpha_1, \alpha_2, \alpha_3) = (\mu_1, \mu_2, \mu_3),$$

i.e. it consists of the set of values of μ where all the blow-up rates of the coupled cooperative problem (5.1) equal the corresponding blow-up rates of the uncoupled one (5.4). Set

$$\gamma_i := \gamma_i(\mu) = \mu_i(p_i - 1) - 2, \qquad 1 \leq i \leq n.$$

Table 5.1 collects all the values of the blow-up rates α in each of the remaining zones according to Theorem 5.1. Surrounding the central prisma and labeled by the first six odd integers, we have represented the set of values of μ for which two of the blow-up rates α_i equal the corresponding blow-up rates μ_i. Labeled by the first six even integers, we find the set of values of μ for which exactly one of the α_i equals the corresponding μ_i. As it has been already commented after the statement of Theorem 5.1, there is not any value of μ for which the three blow-up rates α_i can differ from the corresponding blow-up rates μ_i.



Order	Zone	α_1	α_2	α_3
$\mu_2 \le \mu_3 \le \mu_1$	1	μ_1	$\frac{\mu_1+\gamma_2}{p_2}$	μ_3
	2	μ_1	$\frac{\mu_1+\gamma_2}{p_2}$	$\frac{\mu_1+\gamma_3}{p_3}$
$\mu_3 \le \mu_2 \le \mu_1$	3	μ_1	μ_2	$\frac{\mu_1+\gamma_3}{p_3}$
	4	μ_1	$\frac{\mu_1+\gamma_2}{p_2}$	$\frac{\mu_1+\gamma_3}{p_3}$
$\mu_3 \le \mu_1 \le \mu_2$	5	μ_1	μ_2	$\frac{\mu_2+\gamma_3}{p_3}$
	6	$\frac{\mu_2+\gamma_1}{p_1}$	μ_2	$\frac{\mu_2+\gamma_3}{p_3}$
$\mu_1 \le \mu_3 \le \mu_2$	7	$\frac{\mu_2+\gamma_1}{p_1}$	μ_2	μ_3
	8	$\frac{\mu_2+\gamma_1}{p_1}$	μ_2	$\frac{\mu_2+\gamma_3}{p_3}$
$\mu_1 \le \mu_2 \le \mu_3$	9	$\frac{\mu_3+\gamma_1}{p_1}$	μ_2	μ_3
	10	$\frac{\mu_3+\gamma_1}{p_1}$	$\frac{\mu_3+\gamma_2}{p_2}$	μ_3
$\mu_2 \le \mu_1 \le \mu_3$	11	μ_1	$\frac{\mu_3+\gamma_2}{p_2}$	μ_3
	12	$\frac{\mu_3+\gamma_1}{p_1}$	$\frac{\mu_3+\gamma_2}{p_2}$	μ_3

Table 5.1: Blow-up rates α versus blow-up rates μ

Remark 5.2 The assignations $z \mapsto \alpha_i(z)$ are uniformly continuous for all $1 \le i \le n$.

Remark 5.3 The results of this chapter can be easily generalized to cover the case where the coefficients a_{ij} are replaced by positive Hölder continuous functions, $a_{ij} \in \mathcal{C}^\nu(\Omega)$, $1 \le i \le n$. Also in this case, the blow-up rates of the solution of (5.1) at $z \in \partial\Omega$ are given by (5.8), while the coefficients $A_i(z)$ are given by (5.9) by replacing a_{ij} by $a_{ij}(z)$ for all $1 \le i, j \le n$. Moreover, under the appropriate circumstances, the $\gamma_i(z)$ introduced in (5.2) might satisfy $\gamma_i(z) > -2$ without affecting substantially most of the results.

From Theorem 5.1 the next result follows readily

Theorem 5.4 *Problem* (5.1) *admits a unique positive solution.*

The structure of this chapter is the following. Section 5.1 illustrates how to reach heuristically the blow-up rates of (5.1) when $n = 2$. In Section 5.2 we construct a supersolution for the singular problem in any ball and a subsolution for the problem in any annulus, both with the same blow-up rates. Finally, Section 5.3 provides us with the proofs of Theorems 5.1 and 5.4, and Remark 5.2.

5.1 A natural way of finding out the blow-up rates

This section ascertains, heuristically, the blow-up rates provided by Theorem 5.1 for the simplest two-species one-dimensional prototype model

$$\begin{cases} -u_1'' = \lambda_1 u_1 + a_{12} u_2 - \beta_1(r)(R-r)^{\gamma_1} u_1^{p_1} & \\ -u_2'' = \lambda_2 u_2 + a_{21} u_1 - \beta_2(r)(R-r)^{\gamma_2} u_2^{p_2} & 0 < r < R, \\ u_1(0) = u_2(0) = 0, & \\ u_1(R) = u_2(R) = +\infty. & \end{cases} \tag{5.11}$$

where $\gamma_i \geq 0$ and $\beta_i \in \mathcal{C}^\nu[0,R]$ satisfies $\beta_i(r) > 0$ for all $r \in [0,R]$, $i \in \{1,2\}$. This argument will provide us with some keys to prove Theorem 5.1 in case $n = 2$.

A reasonable strategy is performing the change of variables

$$u_1(r) = (R-r)^{-\alpha_1}\varphi_1(r), \quad u_2(r) = (R-r)^{-\alpha_2}\varphi_2(r),$$

where $\alpha_1, \alpha_2 > 0$ have to be determined, in order to catch the blow-up rate of the solution of (5.11), (u_1, u_2), at $r = R$. Substituting in (5.11) yields

$$\begin{aligned} -\alpha_i(\alpha_i+1)(R-r)^{-\alpha_i-2}\varphi_i(r) - 2\alpha_i(R-r)^{-\alpha_i-1}\varphi_i'(r) - (R-r)^{-\alpha_i}\varphi_i''(r) \\ = \lambda_i(R-r)^{-\alpha_i}\varphi_i(r) + a_{ij}(R-r)^{-\alpha_j}\varphi_j(r) \\ - \beta_i(r)(R-r)^{\gamma_i - p_i\alpha_i}\varphi_i^{p_i}(r), \end{aligned} \tag{5.12}$$

for $i, j \in \{1,2\}$, $i \neq j$, subject to the boundary conditions

$$\varphi_1(0) = \varphi_2(0) = 0, \qquad \varphi_1(R), \varphi_2(R) \in (0, +\infty),$$

so that α_1 and α_2 provide us with the precise blow-up rates of u_1 and u_2. Multiplying by $(R-r)^{\alpha_i+2}$ in (5.12), leads to

$$\begin{aligned} -\alpha_i(\alpha_i+1)\varphi_i(r) - 2\alpha_i(R-r)\varphi_i'(r) - (R-r)^2\varphi_i''(r) \\ = \lambda_i(R-r)^2\varphi_i(r) + a_{ij}(R-r)^{\alpha_i+2-\alpha_j}\varphi_j(r) \\ - \beta_i(r)(R-r)^{\gamma_i - p_i\alpha_i + \alpha_i + 2}\varphi_i^{p_i}(r), \end{aligned} \tag{5.13}$$

for $i, j \in \{1,2\}$, $i \neq j$. Assuming that

$$\lim_{r \uparrow R}\left[(R-r)^2\varphi_i''(r)\right] = \lim_{r \uparrow R}\left[(R-r)\varphi_i'(r)\right] = 0, \qquad i \in \{1,2\},$$

and imposing

$$\gamma_i - p_i\alpha_i + \alpha_i + 2 = 0, \qquad i \in \{1,2\},$$

one is driven to

$$\alpha_i = \frac{\gamma_i + 2}{p_i - 1}, \qquad i \in \{1,2\}, \tag{5.14}$$

and, letting $r \uparrow R$ in (5.13), it becomes apparent that $\varphi_1(R)$ and $\varphi_2(R)$ must satisfy

$$-\alpha_i(\alpha_i+1)\varphi_i(R) = a_{ij}\varphi_j(R) \lim_{r\uparrow R}(R-r)^{\alpha_i+2-\alpha_j} - \beta_i(R)\varphi_i^{p_i}(R), \tag{5.15}$$

for $i,j \in \{1,2\}$, $i \neq j$, provided both limits are finite. It turns that this occurs if

$$\alpha_i + 2 - \alpha_j \geq 0, \quad i,j \in \{1,2\}, \quad i \neq j.$$

By (5.14), this happens if

$$\frac{\gamma_i+2}{p_i-1} + 2 - \frac{\gamma_j+2}{p_j-1} \geq 0, \quad i,j \in \{1,2\}, \quad i \neq j. \tag{5.16}$$

In the special case when both inequalities are strict in (5.16), the unique positive solution of (5.15) is given by

$$\varphi_i(R) = \left[\frac{\alpha_i(\alpha_i+1)}{\beta_i(R)}\right]^{\frac{1}{p_i-1}}, \quad i \in \{1,2\}. \tag{5.17}$$

On the contrary, if, for instance, $\frac{\gamma_1+2}{p_1-1} + 2 - \frac{\gamma_2+2}{p_2-1} = 0$, then $\frac{\gamma_2+2}{p_2-1} + 2 - \frac{\gamma_1+2}{p_1-1} = 2+2 > 0$ and (5.15) becomes

$$\begin{aligned} -\alpha_1(\alpha_1+1)\varphi_1(R) &= a_{12}\varphi_2(R) - \beta_1(R)\varphi_1^{p_1}(R), \\ -\alpha_2(\alpha_2+1)\varphi_2(R) &= -\beta_2(R)\varphi_2^{p_2}(R). \end{aligned} \tag{5.18}$$

From the second equation of (5.18), we find that $\varphi_2(R)$ must be given by (5.17). Consequently, $\varphi_1(R)$ is given through the unique positive zero of

$$f(x) := \beta_1(R)x^{p_1} - \alpha_1(\alpha_1+1)x - a_{12}\varphi_2(R), \qquad x \geq 0.$$

When some of the estimates in (5.16) fails, for instance

$$\frac{\gamma_1+2}{p_1-1} + 2 - \frac{\gamma_2+2}{p_2-1} < 0, \tag{5.19}$$

then

$$\frac{\gamma_2+2}{p_2-1} + 2 - \frac{\gamma_1+2}{p_1-1} > 4 > 0,$$

and, having a glance at (5.15), with the assignment (5.14) for α_1 and α_2, the first equation of (5.15) does not make sense. Consequently, we must change of strategy to capture the blow-up rates.

So, let us go back to the equations (5.12) with α_1 and α_2 to be determined again. Multiplying by $(R-r)^{p_1\alpha_1-\gamma_1}$ the first equation of (5.12) and by $(R-r)^{\alpha_2+2}$ the second one, it follows that

$$\begin{aligned}
&-\alpha_1(\alpha_1+1)(R-r)^{p_1\alpha_1-\gamma_1-\alpha_1-2}\varphi_1-2\alpha_1(R-r)^{p_1\alpha_1-\gamma_1-\alpha_1-1}\varphi_1' \\
&\qquad\qquad\qquad\qquad -(R-r)^{p_1\alpha_1-\gamma_1-\alpha_1}\varphi_1'' \\
&\qquad =\lambda_1(R-r)^{p_1\alpha_1-\gamma_1-\alpha_1}\varphi_1+a_{12}(R-r)^{p_1\alpha_1-\gamma_1-\alpha_2}\varphi_2-\beta_1(r)\varphi_1^{p_1}, \\
&-\alpha_2(\alpha_2+1)\varphi_2-2\alpha_2(R-r)\varphi_2'-(R-r)^2\varphi_2'' \\
&\qquad =\lambda_2(R-r)^2\varphi_2+a_{21}(R-r)^{\alpha_2+2-\alpha_1}\varphi_1-\beta_2(r)(R-r)^{\gamma_2-p_2\alpha_2+\alpha_2+2}\varphi_2^{p_2}.
\end{aligned}$$

If we impose

$$p_1\alpha_1-\gamma_1-\alpha_2=0 \quad \text{and} \quad \gamma_2-p_2\alpha_2+\alpha_2+2=0, \tag{5.20}$$

then we will have

$$\alpha_2=\frac{\gamma_2+2}{p_2-1} \quad \text{and} \quad \alpha_1=\frac{\alpha_2+\gamma_1}{p_1} \tag{5.21}$$

from (5.20). However, as in the previous case, the following relations should make sense

$$\begin{aligned}
-\alpha_1(\alpha_1+1)\varphi_1(R)&\lim_{r\uparrow R}(R-r)^{\alpha_1(p_1-1)-\gamma_1-2} \\
&-2\alpha_1\lim_{r\uparrow R}\left[(R-r)^{\alpha_1(p_1-1)-\gamma_1-1}\varphi_1'(r)\right] \\
&-\lim_{r\uparrow R}\left[(R-r)^{\alpha_1(p_1-1)-\gamma_1}\varphi_1''(r)\right] \\
&=\lambda_1\varphi_1(R)\lim_{r\uparrow R}(R-r)^{\alpha_1(p_1-1)-\gamma_1}+a_{12}\varphi_2(R)-\beta_1(R)\varphi_1^{p_1}(R),
\end{aligned} \tag{5.22}$$

from the first equation, and

$$-\alpha_2(\alpha_2+1)\varphi_2(R)=a_{21}\varphi_1(R)\lim_{r\uparrow R}(R-r)^{\alpha_2+2-\alpha_1}-\beta_2(R)\varphi_2^{p_2}(R). \tag{5.23}$$

from the second one. In order to ascertain all those limits, the sign of the exponents should be determined. According to (5.21), we find from (5.19) that

$$\gamma_1<(p_1-1)(\alpha_2-2)-2=p_1\alpha_2-2p_1-\alpha_2.$$

Therefore, again by (5.21),

$$\alpha_2+2-\alpha_1=\alpha_2+2-\frac{\alpha_2}{p_1}-\frac{\gamma_1}{p_1}>\alpha_2+2-\frac{\alpha_2}{p_1}-\frac{p_1\alpha_2-2p_1-\alpha_2}{p_1}=4>0,$$

and hence, (5.23) becomes

$$\alpha_2(\alpha_2+1)\varphi_2(R)=\beta_2(R)\varphi_2^{p_2}(R)$$

and consequently,

$$\varphi_2(R)=\left[\frac{\alpha_2(\alpha_2+1)}{\beta_2(R)}\right]^{\frac{1}{p_2-1}}.$$

On the other hand, according to (5.19) and the first identity of (5.21),

$$\frac{\gamma_1+2}{p_1-1}+2<\frac{\gamma_2+2}{p_2-1}=\alpha_2. \tag{5.24}$$

Thus, adding γ_1 to (5.24), taking common factor of γ_1+2, and dividing the resulting identity by p_1, yields

$$\frac{\gamma_1+2}{p_1-1}<\frac{\alpha_2+\gamma_1}{p_1}=\alpha_1,$$

and hence,

$$\alpha_1(p_1-1)-\gamma_1-2>0.$$

Therefore,

$$\alpha_1(p_1-1)-\gamma_1-1>1 \quad \text{and} \quad \alpha_1(p_1-1)-\gamma_1>2$$

and so, (5.22) becomes

$$0=a_{12}\varphi_2(R)-\beta_1(R)\varphi_1^{p_1}(R),$$

and consequently

$$\varphi_1(R)=\left[\frac{a_{12}\varphi_2(R)}{\beta_1(R)}\right]^{\frac{1}{p_1}}.$$

Analogously, if instead of (5.19),

$$\frac{\gamma_2+2}{p_2-1}+2-\frac{\gamma_1+2}{p_1-1}<0$$

holds, then

$$\alpha_1=\frac{\gamma_1+2}{p_1-1} \quad \text{and} \quad \alpha_2=\frac{\alpha_1+\gamma_2}{p_2}$$

and

$$\varphi_1(R)=\left[\frac{\alpha_1(\alpha_1+1)}{\beta_1(R)}\right]^{\frac{1}{p_1-1}}, \quad \varphi_2(R)=\left[\frac{a_{21}\varphi_1(R)}{\beta_2(R)}\right]^{\frac{1}{p_2}}.$$

Consequently, we have ascertained the values of the blow-up rates in all possible cases, in complete agreement with the result established by Theorem 5.1.

5.2 Two pivotal technical results under radial symmetry

In this section, for every $R>0$ we consider the auxiliary problem

$$\begin{cases} -\psi_i''-\frac{N-1}{r}\psi_i'=\lambda\psi_i+\displaystyle\sum_{j=1,j\neq i}^{n}a_{ij}\psi_j-b_i(R-r)^{\gamma_i}\psi_i^{p_i}, \quad 0<r<R, \\ \psi_i'(0)=0, \quad \psi_i(R)=+\infty, \end{cases} \quad 1\le i\le n \tag{5.25}$$

where $\lambda \in \mathbb{R}$, $\gamma_i \geq 0$, $p_i > 1$, $a_{ij} > 0$ and $b_i > 0$ for all $1 \leq i, j \leq n$. Without loss of generality, we can assume that the equations in (5.25) have been reordered so that

$$0 < \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1,$$

where, as in (5.3), for each $1 \leq i \leq n$, μ_i is defined by

$$\mu_i := \frac{\gamma_i + 2}{p_i - 1}.$$

As in (5.6), we consider

$$\begin{aligned} I_+ &:= \{i \in \{1, \ldots, n\} : \ \mu_i + 2 - \mu_1 > 0\}, \\ I_0 &:= \{i \in \{1, \ldots, n\} : \ \mu_i + 2 - \mu_1 = 0\}, \\ I_- &:= \{i \in \{1, \ldots, n\} : \ \mu_i + 2 - \mu_1 < 0\}. \end{aligned} \tag{5.26}$$

Let $k \geq 1$ be such that

$$I_M := \{i \in \{1, \ldots, n\} : \ \mu_i = \mu_1\} = \{1, \ldots, k\}, \tag{5.27}$$

and set

$$\alpha_i := \begin{cases} \mu_i & \text{if} \quad i \in I_+ \cup I_0, \\ \dfrac{\mu_1 + \gamma_i}{p_i} & \text{if} \quad i \in I_-, \end{cases} \tag{5.28}$$

and

$$\bar{A}_i := \begin{cases} \left(\dfrac{\mu_i(\mu_i + 1)}{b_i} \right)^{\frac{1}{p_i - 1}} & \text{if} \quad i \in I_+, \\ \left[\dfrac{1}{b_i} \displaystyle\sum_{j=1}^{k} a_{ij} \left(\dfrac{\mu_1(\mu_1 + 1)}{b_j} \right)^{\frac{1}{p_j - 1}} \right]^{\frac{1}{p_i}} & \text{if} \quad i \in I_-, \\ \bar{A}_{0,i} & \text{if} \quad i \in I_0, \end{cases} \tag{5.29}$$

where $\bar{A}_{0,i}$ stands for the unique positive solution of

$$b_i x^{p_i} - \mu_i(\mu_i + 1)x = \sum_{j=1}^{k} a_{ij} \left(\frac{\mu_1(\mu_1 + 1)}{b_j} \right)^{\frac{1}{p_j - 1}}. \tag{5.30}$$

The first result of this section reads as follows.

Lemma 5.5 *For each $\varepsilon > 0$ there exists a constant $C := C(\varepsilon, n)$ such that the function $\overline{\psi}_\varepsilon := (\overline{\psi}_{\varepsilon,1}, \cdots, \overline{\psi}_{\varepsilon,n})$ defined by*

$$\overline{\psi}_{\varepsilon,i}(r) := (1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\Big(\frac{r}{R}\Big)^2 + C, \quad 0 \le r < R, \quad 1 \le i \le n,$$

provides us with a supersolution of (5.25).

Proof: By definition, $\overline{\psi}_\varepsilon$ is smooth and satisfies the boundary conditions. Hence, $\overline{\psi}_\varepsilon$ is a supersolution of (5.25) if, and only if,

$$\begin{aligned}
&-\Big(\frac{r}{R}\Big)^2(1+\varepsilon)\bar{A}_i\alpha_i(\alpha_i+1)(R-r)^{-\alpha_i-2} \\
&-\frac{(N+3)r}{R^2}(1+\varepsilon)\bar{A}_i\alpha_i(R-r)^{-\alpha_i-1} - \frac{2N}{R^2}(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i} \\
&\quad \ge \lambda\Big[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\Big(\frac{r}{R}\Big)^2 + C\Big] \\
&\qquad + \sum_{j=1, j\neq i}^{n} a_{ij}\Big[(1+\varepsilon)\bar{A}_j(R-r)^{-\alpha_j}\Big(\frac{r}{R}\Big)^2 + C\Big] \\
&\qquad - b_i(R-r)^{\gamma_i}\Big[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\Big(\frac{r}{R}\Big)^2 + C\Big]^{p_i},
\end{aligned} \tag{5.31}$$

for every $0 < r < R$ and $0 \le i \le n$. Now, multiply (5.31) by $(R-r)^{\alpha_i+2}$ if $i \in I_+ \cup I_0$, and by $(R-r)^{-\gamma_i+\alpha_i p_i}$ if $i \in I_-$. Then, $\overline{\psi}_\varepsilon$ is a supersolution if, and only if, for every $0 < r < R$,

$$\begin{aligned}
&-\Big(\frac{r}{R}\Big)^2(1+\varepsilon)\bar{A}_i\alpha_i(\alpha_i+1) \\
&-\frac{(N+3)r}{R^2}(1+\varepsilon)\bar{A}_i\alpha_i(R-r) - \frac{2N}{R^2}(1+\varepsilon)\bar{A}_i(R-r)^2 \\
&\quad \ge \lambda\left[(1+\varepsilon)\bar{A}_i(R-r)^2\Big(\frac{r}{R}\Big)^2 + C(R-r)^{\alpha_i+2}\right] \\
&\qquad + \sum_{j=1, j\neq i}^{n} a_{ij}\left[(1+\varepsilon)\bar{A}_j(R-r)^{\alpha_i+2-\alpha_j}\Big(\frac{r}{R}\Big)^2 + C(R-r)^{\alpha_i+2}\right] \\
&\qquad - b_i(R-r)^{\alpha_i+2+\gamma_i}\left[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\Big(\frac{r}{R}\Big)^2 + C\right]^{p_i}
\end{aligned} \tag{5.32}$$

if $i \in I_+ \cup I_0$, and

$$
\begin{aligned}
&-\left(\frac{r}{R}\right)^2(1+\varepsilon)\bar{A}_i\alpha_i(\alpha_i+1)(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i-2}\\
&-\frac{(N+3)r}{R^2}(1+\varepsilon)\bar{A}_i\alpha_i(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i-1}-\frac{2N}{R^2}(1+\varepsilon)\bar{A}_i(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i}\\
&\quad\geq \lambda\Big[(1+\varepsilon)\bar{A}_i(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i}\left(\frac{r}{R}\right)^2+C(R-r)^{-\gamma_i+\alpha_i p_i}\Big]\\
&\qquad\quad+\sum_{j=1,j\neq i}^{n} a_{ij}\Big[(1+\varepsilon)\bar{A}_j(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_j}\left(\frac{r}{R}\right)^2+C(R-r)^{-\gamma_i+\alpha_i p_i}\Big]\\
&\qquad\quad-b_i(R-r)^{\alpha_i p_i}\Big[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\left(\frac{r}{R}\right)^2+C\Big]^{p_i}
\end{aligned}
\tag{5.33}
$$

if $i \in I_-$.

Let us show that

$$\alpha_i+2-\alpha_j>0 \quad \text{for all} \quad i\in I_+ \quad \text{and} \quad 1\leq j\leq n,\ \ j\neq i. \tag{5.34}$$

Indeed, if $i \in I_+$ and $j \in I_+ \cup I_0$, $j \neq i$, we have that

$$\alpha_i+2-\alpha_j=\mu_i+2-\mu_j\geq \mu_i+2-\mu_1>0,$$

by the definition of I_+ (see (5.26)). When $j \in I_-$, we may proceed as follows. According to (5.26),

$$\frac{\gamma_j+2}{p_j-1}<\mu_1-2, \qquad j\in I_-.$$

So, multiplying by p_j-1 we deduce that

$$\gamma_j<p_j\mu_1-2p_j-\mu_1, \qquad j\in I_-,$$

and dividing by p_j yields

$$\frac{\gamma_j}{p_j}+\frac{\mu_1}{p_j}<\mu_1-2, \qquad j\in I_-.$$

Equivalently,

$$\alpha_j<\mu_1-2, \qquad j\in I_-, \tag{5.35}$$

and therefore,

$$\alpha_i+2-\alpha_j=\mu_i+2-\alpha_j>\mu_i+2-\mu_1+2>2>0$$

for every $i \in I_+$ and $j \in I_-$.

Analogously, the following estimates hold

$$
\begin{aligned}
&\alpha_i+2-\alpha_j=0 \quad \text{for every} \quad i\in I_0, \quad 1\leq j\leq k,\\
&\alpha_i+2-\alpha_j>0 \quad \text{for every} \quad i\in I_0, \quad k+1\leq j\leq n, \quad j\neq i.
\end{aligned}
\tag{5.36}
$$

Indeed, by (5.28), $\alpha_i = \mu_i$ for all $i \in I_0$. Similarly, since $\{1,\dots,k\} \subset I_+$, we have $\alpha_j = \mu_j$ for all $1 \le j \le k$. Moreover, $\mu_j = \mu_1$ for all $1 \le j \le k$. Thus,

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \mu_j = \mu_i + 2 - \mu_1 = 0$$

for all $i \in I_0$ and $1 \le j \le k$, which shows the validity of the identities of (5.36). In order to check the inequalities of (5.36) we can argue as follows. Pick $i \in I_0$ and $j \in \{k+1,\dots,n\}$, $j \neq i$. Suppose $j \in I_+ \cup I_0$. Then, by (5.28) and taking into account that, by construction, $\mu_j < \mu_1$ for all $k+1 \le j \le n$, we find that

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \mu_j > \mu_i + 2 - \mu_1 = 0,$$

because $i \in I_0$. Now, suppose that $j \in I_-$. Then, thanks to (5.28) and (5.35),

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \alpha_j > \mu_i + 2 - \mu_1 + 2 = 2 > 0.$$

Therefore, (5.36) holds.

Next, we will see that

$$-\gamma_i + \alpha_i p_i - \alpha_i - 2 > 0 \quad \text{for all} \quad i \in I_-. \tag{5.37}$$

Indeed, by the definition of μ_i and since $i \in I_-$,

$$\frac{\gamma_i + 2}{p_i - 1} + 2 < \mu_1, \qquad i \in I_-.$$

Thus, adding γ_i at both sides of this inequality and taking common factor $\gamma_i + 2$ yields

$$(\gamma_i + 2)\frac{p_i}{p_i - 1} < \mu_1 + \gamma_i, \qquad i \in I_-.$$

Hence, by (5.28),

$$\frac{\gamma_i + 2}{p_i - 1} < \frac{\mu_1 + \gamma_i}{p_i} = \alpha_i, \qquad i \in I_-,$$

whence (5.37).

Lastly, we will establish that

$$\begin{aligned} &-\gamma_i + \alpha_i p_i - \alpha_j = 0 \quad \text{for all} \quad i \in I_-, \quad 1 \le j \le k, \\ &-\gamma_i + \alpha_i p_i - \alpha_j > 0 \quad \text{for all} \quad i \in I_-, \quad k+1 \le j \le n, \quad j \neq i. \end{aligned} \tag{5.38}$$

By (5.28),

$$-\gamma_i + \alpha_i p_i = \mu_1 \quad \text{for all } \ i \in I_-.$$

Thus,

$$-\gamma_i + \alpha_i p_i - \alpha_j = \mu_1 - \alpha_j = \mu_1 - \mu_j = 0$$

for all $i \in I_-$ and $1 \le j \le k$, whence the identities of (5.38). Now, pick $i \in I_-$ and $k+1 \le j \le n$, $i \ne j$. Suppose $j \in I_+ \cup I_0$. Then, due to (5.28),

$$-\gamma_i + \alpha_i p_i - \alpha_j = \mu_1 - \alpha_j = \mu_1 - \mu_j > 0$$

and hence, (5.38) holds in this case. Now, suppose $j \in I_-$. Then, by (5.35),

$$-\gamma_i + \alpha_i p_i - \alpha_j = \mu_1 - \alpha_j > 2 > 0,$$

and therefore, (5.38) is satisfied.

By (5.28) and the definition of μ_i,

$$\alpha_i + 2 + \gamma_i - \alpha_i p_i = \mu_i + 2 + \gamma_i - \mu_i p_i = 0 \quad \text{for all } i \in I_+ \cup I_0.$$

Thus,

$$\lim_{r \uparrow R} \left(b_i (R-r)^{\alpha_i + 2 + \gamma_i} \left[(1+\varepsilon) \bar{A}_i (R-r)^{-\alpha_i} \left(\frac{r}{R}\right)^2 + C \right]^{p_i} \right) = b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i} \tag{5.39}$$

for all $C \ge 0$ and $i \in I_+ \cup I_0$. Hence, thanks to (5.34), (5.36) and (5.39), we can extend (5.32) to $r = R$ by letting $r \uparrow R$. Similarly, (5.33) can be extended to $r = R$ by (5.37) and (5.38). More precisely, at $r = R$ (5.32) provides us with

$$\begin{aligned} -(1+\varepsilon)\bar{A}_i \alpha_i (\alpha_i + 1) &\ge -b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i}, && i \in I_+, \\ -(1+\varepsilon)\bar{A}_i \alpha_i (\alpha_i + 1) &\ge \sum_{j=1}^{k} a_{ij} (1+\varepsilon) \bar{A}_j - b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i}, && i \in I_0, \end{aligned} \tag{5.40}$$

while (5.33) at $r = R$ becomes

$$0 \ge \sum_{j=1}^{k} a_{ij} (1+\varepsilon) \bar{A}_j - b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i} \qquad i \in I_-. \tag{5.41}$$

Due to (5.29) and using that $(1+\varepsilon) < (1+\varepsilon)^{p_i}$ (since $p_i > 1$ for all $1 \le i \le n$) it is easily seen that (5.40) and (5.41) are true. In the derivation one should note that $\alpha_j = \mu_j = \mu_1$ for all $1 \le j \le k$, by construction. Actually, all inequalities in (5.40) and (5.41) are strict, because $(1+\varepsilon) < (1+\varepsilon)^{p_i}$. By continuity, this entails the existence of $\delta := \delta(\varepsilon, n) > 0$ such that (5.32) and (5.33) are satisfied for all $r \in [R-\delta, R)$. Therefore, choosing a sufficiently large $C > 0$, we can assume that (5.31) holds in $(0, R)$, because, for each $1 \le i \le n$, the function $b_i (R-r)^{\gamma_i}$ is positive and bounded away from zero in $[0, R-\delta]$. The proof is complete. □

The next result provides us with a universal subsolution on an annulus.

Lemma 5.6 *Let $R_2 > R_1 > 0$ and consider the problem*

$$\left\{ \begin{array}{ll} -\psi_i'' - \frac{N-1}{r}\psi_i' = \lambda\psi_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij}\psi_j - \beta_i(r)(r-R_1)^{\gamma_i}\psi_i^{p_i}, & R_1 < r < R_2, \\ \psi_i(R_1) = +\infty, \ \ \psi_i'(R_2) = 0, & \end{array} \right. \quad 1 \leq i \leq n \tag{5.42}$$

where all the coefficients satisfy the same requirements as in (5.25) *and, for every $1 \leq i \leq n$, the function $\beta_i \in \mathcal{C}^\nu[R_1, R_2]$ satisfies $\beta_i(r) > 0$ for all $r \in [R_1, R_2]$. Then, for every $\varepsilon \in (0,1)$ there exists a negative constant $D := D(\varepsilon, n) < 0$ such that the function*

$$\underline{\psi}_\varepsilon := (\underline{\psi}_{\varepsilon,1}, \cdots, \underline{\psi}_{\varepsilon,n})$$

defined by

$$\underline{\psi}_{\varepsilon,i}(r) := \max\left\{0, (1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D\right\}, \quad R_1 < r \leq R_2, \quad 1 \leq i \leq n,$$

provides us with a weak subsolution of (5.25)*, as discussed in* [1]*, where the constants α_i, $1 \leq i \leq n$, are given by* (5.28) *and*

$$\underline{A}_i := \left\{ \begin{array}{ll} \left(\dfrac{\mu_i(\mu_i+1)}{\beta_i(R_1)}\right)^{\frac{1}{p_i-1}} & \text{if} \quad i \in I_+, \\ \left[\dfrac{1}{\beta_i(R_1)}\displaystyle\sum_{j=1}^{k} a_{ij}\left(\dfrac{\mu_1(\mu_1+1)}{\beta_j(R_1)}\right)^{\frac{1}{p_j-1}}\right]^{\frac{1}{p_i}} & \text{if} \quad i \in I_-, \\ \underline{A}_{0,i} & \text{if} \quad i \in I_0, \end{array} \right. \tag{5.43}$$

where $\underline{A}_{0,i}$ stands for the unique positive solution of

$$\beta_i(R_1)x^{p_i} - \mu_i(\mu_i+1)x = \sum_{j=1}^{k} a_{ij}\left(\frac{\mu_1(\mu_1+1)}{\beta_j(R_1)}\right)^{\frac{1}{p_j-1}}.$$

Proof: As the maps $r \mapsto (1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i}$ are strictly decreasing, taking any D satisfying

$$D < -(1-\varepsilon)\underline{A}_i(R_2-R_1)^{-\alpha_i} = -\min\left\{(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} : \ R_1 < r \leq R_2\right\}$$

for all $1 \leq i \leq n$, there exist $\varrho_i(D) \in (R_1, R_2)$, $1 \leq i \leq n$, such that

$$\underline{\psi}_{\varepsilon,i} = \left\{ \begin{array}{ll} (1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D & \text{if } R_1 < r \leq \varrho_i(D), \\ 0 & \text{if } \varrho_i(D) < r \leq R_2, \end{array} \right. \quad 1 \leq i \leq n.$$

Moreover, the mappings $D \mapsto \varrho_i(D)$, $1 \leq i \leq n$, can be chosen continuous, and

$$\lim_{D\downarrow -1} \varrho_i(D) = R_1, \qquad 1 \leq i \leq n. \tag{5.44}$$

Thus, $\underline{\psi}_\varepsilon$ is a subsolution of (5.42) if, and only if, for every $1 \leq i \leq n$ and $R_1 < r \leq \varrho_i(D)$,

$$\begin{aligned}
&-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1)(r-R_1)^{-\alpha_i-2} + \tfrac{N-1}{r}(1-\varepsilon)\underline{A}_i\alpha_i(r-R_1)^{-\alpha_i-1}\\
&\quad \leq \lambda\left[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D\right] + \sum_{j=1,j\neq i}^{n} a_{ij}\left[(1-\varepsilon)\underline{A}_j(r-R_1)^{-\alpha_j} + D\right]\\
&\qquad -\beta_i(r)(r-R_1)^{\gamma_i}\left[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D\right]^{p_i}.
\end{aligned} \tag{5.45}$$

Next, we will adapt the proof of Lemma 5.5. Multiplying (5.45) by $(r-R_1)^{\alpha_i+2}$ when $i \in I_+ \cup I_0$ and by $(r-R_1)^{-\gamma_i+\alpha_i p_i}$ when $i \in I_-$, it becomes apparent that $\underline{\psi}_\varepsilon$ is a subsolution of (5.42) if, and only if,

$$\begin{aligned}
&-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1) + \frac{N-1}{r}(1-\varepsilon)\underline{A}_i\alpha_i(r-R_1)\\
&\quad \leq \lambda\left[(1-\varepsilon)\underline{A}_i(r-R_1)^2 + D(r-R_1)^{\alpha_i+2}\right]\\
&\qquad + \sum_{j=1,j\neq i}^{n} a_{ij}\left[(1-\varepsilon)\underline{A}_j(r-R_1)^{\alpha_i+2-\alpha_j} + D(r-R_1)^{\alpha_i+2}\right]\\
&\qquad - \beta_i(r)(r-R_1)^{\alpha_i+2+\gamma_i}\left[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D\right]^{p_i},
\end{aligned} \tag{5.46}$$

for all $i \in I_+ \cup I_0$ and $R_1 < r \leq \varrho_i(D)$, and

$$\begin{aligned}
&-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1)(r-R_1)^{-\alpha_i-2-\gamma_i+\alpha_i p_i}\\
&+\frac{N-1}{r}(1-\varepsilon)\underline{A}_i\alpha_i(r-R_1)^{-\alpha_i-1-\gamma_i+\alpha_i p_i}\\
&\quad \leq \lambda\left[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i-\gamma_i+\alpha_i p_i} + D(r-R_1)^{-\gamma_i+\alpha_i p_i}\right]\\
&\qquad + \sum_{j=1,j\neq i}^{n} a_{ij}\left[(1-\varepsilon)\underline{A}_j(r-R_1)^{-\gamma_i+\alpha_i p_i-\alpha_j} + D(r-R_1)^{-\gamma_i+\alpha_i p_i}\right]\\
&\qquad - \beta_i(r)(r-R_1)^{\alpha_i p_i}\left[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D\right]^{p_i},
\end{aligned} \tag{5.47}$$

for all $i \in I_-$ and $R_1 < r \leq \varrho_i(D)$. Thanks to (5.34), (5.36), (5.37) and (5.38), letting $r \downarrow R_1$ in (5.46) and (5.47) yields

$$\begin{aligned}
-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1) &\leq -\beta_i(R_1)(1-\varepsilon)^{p_i}\underline{A}_i^{p_i}, && i \in I_+,\\
-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1) &\leq \textstyle\sum_{j=1}^{k} a_{ij}(1-\varepsilon)\underline{A}_j - \beta_i(R_1)(1-\varepsilon)^{p_i}\underline{A}_i^{p_i}, && i \in I_0,\\
0 &\leq \textstyle\sum_{j=1}^{k} a_{ij}(1-\varepsilon)\underline{A}_j - \beta_i(R_1)(1-\varepsilon)^{p_i}\underline{A}_i^{p_i}, && i \in I_-.
\end{aligned} \tag{5.48}$$

As $(1-\varepsilon) > (1-\varepsilon)^{p_i}$ for all $1 \leq i \leq n$, all inequalities in (5.48) are strict. Hence, by continuity, there exists $\delta = \delta(\varepsilon, n) > 0$ such that all inequalities in (5.46) and (5.47) hold in the interval $(R_1, R_1+\delta]$. Finally, by (5.44), $D<0$ can be taken sufficiently negative so that

$$\max_{1\le i\le n} \varrho_i(D) \le R_1+\delta.$$

This ends the proof. $\square$

5.3 Proofs of the main results

As $\partial\Omega$ is smooth, the outward unit normal vector field to $\partial\Omega$ is well defined at every point of $\partial\Omega$. We will denote it by

$$\begin{aligned} n : \partial\Omega &\longrightarrow \mathbb{R}^N \\ z &\longmapsto n_z. \end{aligned}$$

Since $\partial\Omega \in \mathcal{C}^2$, Ω satisfies the *uniform interior sphere in the strong sense on* $\partial\Omega$ (see Theorem [47, Th. 1.9]). So, there exists $R_0>0$ such that for every $x\in\Omega$ with $\mathrm{dist}(x,\partial\Omega)<R_0$ there is a point $\pi(x)\in\partial\Omega$ such that

$$t(x) := \mathrm{dist}(x,\partial\Omega) = |x-\pi(x)|, \qquad B_{R_0}\left(\pi(x)+R_0\frac{x-\pi(x)}{t(x)}\right)\subset\Omega. \tag{5.49}$$

Moreover, R_0 can be shortened so that, for every $z\in\partial\Omega$,

$$\bar{B}_{R_0}(z-R_0 n_z)\cap\partial\Omega=\{z\} \quad\text{and}\quad \bar{B}_{R_0}(z+R_0 n_z)\cap\bar{\Omega}=\{z\}. \tag{5.50}$$

5.3.1 Proof of Theorem 5.1

Fix $z\in\partial\Omega$ and $0<\eta<1$. By (5.2), there exist $\delta>0$ such that, for every $1\le i\le n$,

$$(1-\eta)\mathfrak{b}_i(z)[\mathrm{dist}(x,\partial\Omega)]^{\gamma_i(z)} < \mathfrak{a}_i(x) < (1+\eta)\mathfrak{b}_i(z)[\mathrm{dist}(x,\partial\Omega)]^{\gamma_i(z)} \tag{5.51}$$

for all $x\in B_\delta(z)\cap\Omega$. Choose R_0 sufficiently small so that (5.49) and (5.50) hold, and

$$0<R_0<\delta.$$

Set

$$\Gamma := \bar{B}_{R_0/2}(z)\cap\partial\Omega.$$

It is rather clear that there exist $R>0$ and $\varrho_0>0$ such that

$$B_R(y-(R+\varrho)n_y)\subset B_\delta(z)\cap\Omega,$$

for all $y\in\Gamma$ and $0\le\varrho\le\varrho_0$. Figure 5.2 sketches this construction scheme.

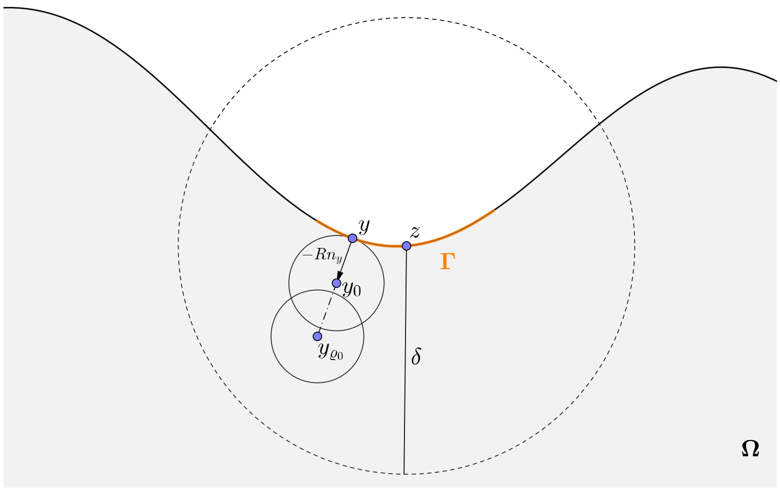


Figure 5.2: Scheme of the construction.

According to (5.51), for every $y \in \Gamma$ and $0 \leq \varrho \leq \varrho_0$,

$$\begin{aligned}\mathfrak{a}_i(x) &> (1-\eta)\mathfrak{b}_i(z)[\mathrm{dist}(x,\partial\Omega]^{\gamma_i(z)} \\ &\geq (1-\eta)\mathfrak{b}_i(z)[\mathrm{dist}(x,\partial B_R(y-(R+\varrho)n_y))]^{\gamma_i(z)}, \quad 1 \leq i \leq n,\end{aligned} \tag{5.52}$$

for all $x \in B_R(y-(R+\varrho)n_y)$. Set

$$\begin{aligned}\bar{\lambda} &:= \max\{||\lambda_i||_\infty : \ 1 \leq i \leq n\}, \\ y_\varrho &:= y-(R+\varrho)n_y, \quad y \in \Gamma, \quad \varrho \in [0,\varrho_0], \\ \bar{b}_i(z) &:= (1-\eta)\mathfrak{b}_i(z), \qquad 1 \leq i \leq n,\end{aligned}$$

and consider, for each $y \in \Gamma$ and $0 < \varrho \leq \varrho_0$, the problem

$$\begin{cases} -\Delta u_i = \bar{\lambda} u_i + \displaystyle\sum_{j=1,j\neq i}^{n} a_{ij}u_j - \bar{b}_i(z)(R-|x-y_\varrho|)^{\gamma_i(z)}u_i^{p_i} & \text{in } B_R(y_\varrho), \\ u_i = +\infty & \text{on } \partial B_R(y_\varrho), \end{cases} \quad 1 \leq i \leq n. \tag{5.53}$$

By (5.52), any positive solution of (5.1),

$$L = (L_1, \cdots, L_n) := (u_1, \cdots, u_n),$$

is a *bounded* positive subsolution of (5.53), for every $y \in \Gamma$ and $0 < \varrho \leq \varrho_0$.

Let $\varepsilon > 0$. Applying Lemma 5.5 to problem (5.25) with

$$\lambda = \bar{\lambda}, \quad \gamma_i = \gamma_i(z), \quad b_i = \bar{b}_i(z), \qquad 1 \leq i \leq n,$$

we get the functions

$$\bar{\psi}_{\varepsilon,i}(r) = (1+\varepsilon)\bar{A}_i(z)(R-r)^{-\alpha_i(z)}\left(\frac{r}{R}\right)^2 + C, \quad 0 \le r < R, \quad 1 \le i \le n,$$

where $\alpha_i(z)$, $\bar{A}_i(z)$, $1 \le i \le n$, are defined through (5.28) and (5.29). By the radial symmetry of (5.53), for every $y \in \Gamma$ and $0 < \varrho \le \varrho_0$, the functions

$$\bar{L}^{y_\varrho}_{\varepsilon,i}(x) := \bar{\psi}_{\varepsilon,i}(r), \quad x \in B_R(y_\varrho), \quad r := |x - y_\varrho|, \quad 1 \le i \le n,$$

provide us with a supersolution of (5.53). Hence, by Theorem 3.3,

$$L_i(x) \le \bar{L}^{y_\varrho}_{\varepsilon,i}(x), \quad x \in B_R(y_\varrho), \quad 1 \le i \le n,$$

for every $y \in \Gamma$ and $0 < \varrho \le \varrho_0$. Consequently, we may infer

$$L_i(x) \le (1+\varepsilon)\bar{A}_i(z)[\operatorname{dist}(x, \partial B_R(y_0))]^{-\alpha_i(z)}\left(\frac{|x-y_0|}{R}\right)^2 + C \tag{5.54}$$

for all $y \in \Gamma$, $x \in B_R(y_0)$ and $1 \le i \le n$.

On the other hand, for every x sufficiently close to Γ we have that

$$\operatorname{dist}(x, \Gamma) = \operatorname{dist}(x, \partial B_R(y_0)).$$

with $y_0 = \pi(x) - Rn_{\pi(x)}$. Thus, (5.54) implies

$$\limsup_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x,\Gamma)]^{-\alpha_i(z)}} \le (1+\varepsilon)\bar{A}_i(z), \qquad 1 \le i \le n.$$

Therefore, letting $\varepsilon \downarrow 0$ yields

$$\limsup_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x,\Gamma)]^{-\alpha_i(z)}} \le \bar{A}_i(z), \quad 1 \le i \le n. \tag{5.55}$$

Now, we will construct a subsolution of (5.1) with the appropriate growth on Γ. Due to (5.50) and taking into account that Ω is bounded, there exist $R_2 > R_1 > 0$ and $\varrho^0 > 0$ such that

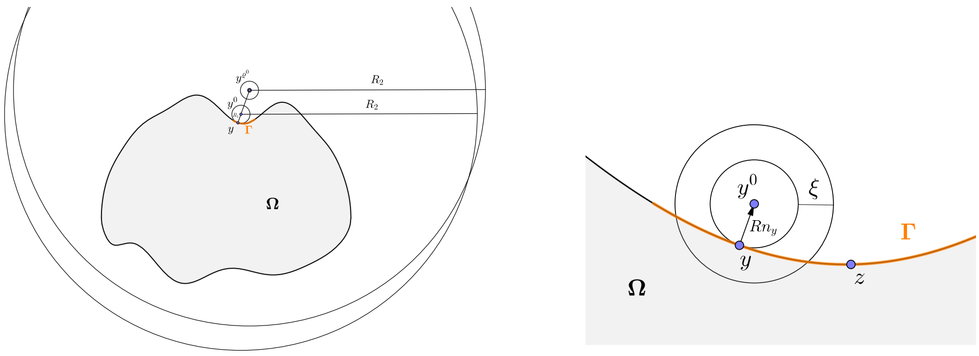
$$\Omega \subset \bigcap_{0 \le \varrho \le \varrho^0} A_{R_1,R_2}(y + (R_1+\varrho)n_y)$$

and

$$\overline{\Omega} \cap \partial A_{R_1,R_2}(y + R_1 n_y) = \{y\}$$

for all $y \in \Gamma$ and $\varrho \in [0, \varrho^0]$, where, for every $z \in \mathbb{R}^N$ and $0 < r_1 < r_2$,

$$A_{r_1,r_2}(z) := \{x \in \mathbb{R}^N : r_1 < |x - z| < r_2\}.$$



(a) Construction (b) Magnification

Figure 5.3: Scheme of the construction.

As before, we will denote

$$y^\varrho := y + (R_1 + \varrho)n_y, \quad y \in \Gamma, \quad 0 \le \varrho \le \varrho^0.$$

Figure 5.3 sketches this construction.

Shortening R_1 and ϱ^0 if necessary, it becomes apparent that there exists $\xi > 0$ such that

$$A_{R_1,R_1+\xi}(y^\varrho) \cap \Omega \subset (B_\delta(z) \cap \Omega) \quad \text{for all } y \in \Gamma \quad \text{and } \varrho \in [0, \varrho^0]. \tag{5.56}$$

Using again (5.51) and thanks to (5.56), we obtain that,

$$\begin{aligned} \mathfrak{a}_i(x) &< (1+\eta)\mathfrak{b}_i(z)[\text{dist}(x, \partial\Omega)]^{\gamma_i(z)} \\ &\le (1+\eta)\mathfrak{b}_i(z)[\text{dist}(x, \partial B_{R_1}(y^\varrho))]^{\gamma_i(z)} \end{aligned} \tag{5.57}$$

for all

$$y \in \Gamma, \quad \varrho \in [0, \varrho^0], \quad x \in A_{R_1,R_1+\xi}(y^0) \cap \Omega, \quad 1 \le i \le n.$$

For getting (5.57) in the entire Ω, we may proceed as follows. Set

$$K := \max_{1 \le i \le n} \left\{ \frac{||\mathfrak{a}_i||_\infty}{(R_1 + \xi)^{\gamma_i(z)}} + (1+\eta)\mathfrak{b}_i(z) \right\}$$

and consider, for each $1 \le i \le n$, the piecewise linear function

$$\beta_i(r) = \begin{cases} (1+\eta)\mathfrak{b}_i(z), & R_1 \le r < R_1 + \frac{\xi}{2}, \\ K + \dfrac{K - (1+\eta)\mathfrak{b}_i(z)}{\xi/2}\left(r - (R_1 + \xi)\right), & R_1 + \frac{\xi}{2} \le r < R_1 + \xi, \\ K, & R_1 + \xi \le r \le R_2. \end{cases}$$

Using the definition of β_i and (5.57), we have that

$$\mathfrak{a}_i(x) \le \beta_i(|x - y^\varrho|)[|x - y^\varrho| - R_1]^{\gamma_i(z)}, \qquad 1 \le i \le n, \tag{5.58}$$

for all $y \in \Gamma$, $0 < \varrho \leq \varrho^0$ and $x \in \Omega$.

Subsequently, we will consider the auxiliary problem

$$\begin{cases} -\Delta u_i = \underline{\lambda} u_i + \sum_{j=1, j\neq i}^{n} a_{ij} u_j - \beta_i(r)(r-R_1)^{\gamma_i(z)} u_i^{p_i} & \text{in } A_{R_1,R_2}(y_\varrho), \\ u_i = +\infty & \text{on } \partial A_{R_1,R_2}(y_\varrho), \end{cases} \quad 1 \leq i \leq n, \tag{5.59}$$

where

$$\underline{\lambda} := \min \{ -||\lambda_i||_\infty : \ 1 \leq i \leq n \}, \quad r := |x - y^\varrho|.$$

Pick $\varepsilon \in (0,1)$. Then, due to Lemma 5.6, for every $y \in \Gamma$ and $0 < \varrho \leq \varrho^0$, the function $L_\varepsilon^{y_\varrho} := (L_{\varepsilon,1}^{y_\varrho}, \dots, L_{\varepsilon,n}^{y_\varrho})$ defined, for every $1 \leq i \leq n$, by

$$L_{\varepsilon,i}^{y_\varrho}(x) := \underline{\psi}_{\varepsilon,i}(r) = \max \left\{ 0, (1-\varepsilon)\underline{A}_i(z)(r-R_1)^{-\alpha_i(z)} + D \right\}, \quad x \in A_{R_1,R_2}(y^\varrho),$$

provide us with a positive subsolution for (5.59), where $\underline{A}_i(z)$ is defined through (5.43) for every $1 \leq i \leq n$.

By (5.58), the restriction $L_\varepsilon^{y_\varrho}\big|_\Omega$ provides us with a *bounded* positive subsolution of (5.1). Hence, by Theorem 3.3,

$$L_i(x) \geq \underline{L}_{\varepsilon,i}^{y^\varrho}(x), \quad x \in \Omega, \quad 1 \leq i \leq n,$$

for all $y \in \Gamma$ and $0 < \varrho \leq \varrho^0$. Thus, we can infer that

$$L_i \geq \underline{L}_{\varepsilon,i}^{y_0}, \qquad 1 \leq i \leq n,$$

for all $y \in \Gamma$. The last inequality implies that

$$\liminf_{\mathrm{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\mathrm{dist}(x,\Gamma)]^{-\alpha_i(z)}} \geq (1-\varepsilon)\underline{A}_i(z), \qquad 1 \leq i \leq n.$$

Hence, letting $\varepsilon \downarrow 0$ yields

$$\liminf_{\mathrm{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\mathrm{dist}(x,\Gamma)]^{-\alpha_i(z)}} \geq \underline{A}_i(z), \qquad 1 \leq i \leq n. \tag{5.60}$$

Consequently, owing to (5.55) and (5.60), for each $z \in \partial\Omega$ and $0 < \eta < 1$, there exists a compact neighborhood of $z \in \partial\Omega$, Γ, such that

$$\underline{A}_i(z) \leq \liminf_{\mathrm{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\mathrm{dist}(x,\Gamma)]^{-\alpha_i(z)}} \leq \limsup_{\mathrm{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\mathrm{dist}(x,\Gamma)]^{-\alpha_i(z)}} \leq \bar{A}_i(z),$$

for every $1 \leq i \leq n$. Therefore, letting $\eta \downarrow 0$ yields

$$\lim_{\substack{x\to z \\ x\in\Omega}} \frac{L_i(x)}{[\mathrm{dist}(x,\partial\Omega)]^{-\alpha_i(z)}} = A_i(z),$$

because
$$A_i(z) = \lim_{\eta\downarrow 0} \bar{A}_i(z) = \lim_{\eta\downarrow 0} \underline{A}_i(z).$$
This ends the proof of Theorem 5.1. $\square$

5.3.2 Proof of Theorem 5.4

Let $L := (L_1, \dots, L_n)$ and $M := (M_1, \dots, M_n)$ be two positive solutions of (5.1). Using (5.7) it is easily seen that the quotients

$$q_i(x) := \begin{cases} \frac{L_i(x)}{M_i(x)} & x \in \Omega, \\ 1 & x \in \partial\Omega, \end{cases} \qquad 1 \le i \le n,$$

are uniformly continuous in $\bar{\Omega}$. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|q_i(x) - q_i(\pi(x))| = |q_i(x) - 1| < \varepsilon \quad \text{if } |x - \pi(x)| < \delta, \quad 1 \le i \le n.$$

Thus, setting
$$Q_\xi := \{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \le \xi\}, \qquad \xi > 0,$$
we find that
$$(1-\varepsilon)M_i \le L_i \le (1+\varepsilon)M_i \quad \text{in } Q_\delta, \quad 1 \le i \le n.$$
Moreover, L is a solution of the problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij}u_j - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega \setminus Q_\delta, \\ u_i = L_i & \text{on } \partial(\Omega \setminus Q_\delta), \end{cases} \qquad 1 \le i \le n, \tag{5.61}$$

$(1-\varepsilon)M$ is a subsolution of (5.61) and $(1+\varepsilon)M$ is a supersolution of (5.61). Therefore, by Theorem 3.3,

$$(1-\varepsilon)M_i \le L_i \le (1+\varepsilon)M_i \quad \text{in } \Omega \setminus Q_\delta, \quad 1 \le i \le n.$$

Letting $\varepsilon \downarrow 0$ we find that $L = M$ in Ω. This ends the proof. $\square$

5.3.3 Proof of Remark 5.2

Setting
$$\mu_{\max}(z) := \max_{1\le j\le n} \mu_j(z), \qquad z \in \partial\Omega,$$

by Theorem 5.1, for every $z \in \partial\Omega$ and $1 \le i \le n$ we have

$$\alpha_i(z) = \begin{cases} \mu_i(z) & \text{if} \quad \mu_i(z) + 2 - \mu_{\max}(z) \ge 0, \\ \dfrac{\mu_{\max}(z) + \gamma_i(z)}{p_i} & \text{if} \quad \mu_i(z) + 2 - \mu_{\max}(z) < 0. \end{cases}$$

Pick $z \in \partial\Omega$ and $1 \le i \le n$. Based on the continuity of $\mu_{\max}(z)$, it easy to check that α_i is continuous at z if

$$\mu_i(z) + 2 - \mu_{\max}(z) \neq 0.$$

Suppose

$$\mu_i(z) + 2 - \mu_{\max}(z) = 0 \tag{5.62}$$

for some $z \in \partial\Omega$ and let $(z_n)_{n\ge1} \subset \partial\Omega$ be a sequence such that $z_n \to z$ if $n \to +\infty$ and

$$\mu_i(z_n) + 2 - \mu_{\max}(z_n) < 0, \quad \text{for all} \quad n \ge 1.$$

Then, invoking (5.62) and (5.3), shows that

$$\begin{aligned} \lim_{n\to\infty} \alpha_i(z_n) = \lim_{n\to\infty} \frac{\mu_{\max}(z_n) + \gamma_i(z_n)}{p_i} &= \frac{\mu_{\max}(z) + \gamma_i(z)}{p_i} \\ = \frac{\mu_i(z) + 2 + \gamma_i(z)}{p_i} &= \frac{\frac{\gamma_i(z)+2}{p_i-1} + 2 + \gamma_i(z)}{p_i} \\ = \frac{(\gamma_i(z) + 2)\left(\frac{1}{p_i-1} + 1\right)}{p_i} &= \frac{\gamma_i(z) + 2}{p_i - 1} = \alpha_i(z). \end{aligned}$$

Therefore, α_i is also continuous at $z \in \partial\Omega$ if (5.62) holds. This ends the proof. □

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