BRAUER GROUP OF MODULI SPACES OF PAIRS

INDRANIL BISWAS, MARINA LOGARES, AND VICENTE MUÑOZ

Abstract. We show that the Brauer group of the moduli space of stable pairs with fixed determinant over a curve is zero.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ over the complex numbers. A *holomorphic pair* (also called a *Bradlow pair*) is an object of the form (E, ϕ), where E is a holomorphic vector bundle over X, and ϕ is a nonzero holomorphic section of E. The concept of stability for pairs depends on a parameter $\tau \in \mathbb{R}$. Moduli spaces of τ-stable pairs of fixed rank and degree were first constructed using gauge theoretic methods in [4], and subsequently using Geometric Invariant Theory in [3]. Since then these moduli spaces have been extensively studied.

Fix an integer $r \geq 2$ and a holomorphic line bundle Λ over X. Let $d = \deg(\Lambda)$. Let $\mathfrak{M}_\tau(r, \Lambda)$ be the moduli space of stable pairs (E, ϕ) such that $\mathrm{rk}\,(E) = r$ and $\det(E) = \bigwedge^r E = \Lambda$. This is a smooth quasi-projective variety; it is empty if $d \leq 0$. Therefore, $H^2_{\acute{e}t}(\mathfrak{M}_\tau(r, \Lambda), \mathbb{G}_m)$ is torsion, and it coincides with the Brauer group of $\mathfrak{M}_\tau(r, \Lambda)$, defined by the equivalence classes of Azumaya algebras over $\mathfrak{M}_\tau(r, \Lambda)$. Let $\mathrm{Br}(\mathfrak{M}_\tau(r, \Lambda))$ denote the Brauer group of $\mathfrak{M}_\tau(r, \Lambda)$.

We prove the following (see Theorem 3.3 and Corollary 3.5):

Theorem 1.1. *Assume that* $(r, g, d) \neq (3, 2, 2)$. *Then* $\mathrm{Br}(\mathfrak{M}_\tau(r, \Lambda)) = 0$.

Let $M(r, \Lambda)$ be the moduli space of stable vector bundles over X of rank r and determinant Λ. There is a unique universal projective bundle over $X \times M(r, \Lambda)$. Restricting this projective bundle to $\{x\} \times M(r, \Lambda)$, where x is a fixed point of X, we get a projective bundle $\mathbb{P}_x$ over $M(r, \Lambda)$. We give a new proof of the following known result (see Corollary 3.4).

Corollary 1.2. *Assume* $(r, g, d) \neq (2, 2, even)$. *The Brauer group of* $M(r, \Lambda)$ *is generated by the Brauer class of* $\mathbb{P}_x$.

This was first proved in [2]. We show that it follows as an application of Theorem 1.1.

For convenience, we work over the complex numbers. However, the results are still valid for any algebraically closed field of characteristic zero.

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2. Moduli spaces of pairs

We collect here some known results about the moduli spaces of pairs, taken mainly from [4], [5], [9], [11] and [14].

Let X be a smooth projective curve defined over the field of complex numbers, of genus $g \geq 2$. A *holomorphic pair* (E, ϕ) over X consists of a holomorphic bundle on X and a nonzero holomorphic section $\phi \in H^0(E)$. Let $\mu(E) := \deg(E)/\operatorname{rk}(E)$ be the slope of E. There is a stability concept for a pair depending on a parameter $\tau \in \mathbb{R}$. A holomorphic pair (E, ϕ) is *τ-stable* whenever the following conditions are satisfied:

- for any subbundle $E' \subset E$, we have $\mu(E') < \tau$,
- for any subbundle $E' \subset E$ such that $\phi \in H^0(E')$, we have $\mu(E/E') > \tau$.

The concept of *τ-semistability* is defined by replacing the above strict inequalities by the weaker inequalities "$\leq$" and "$\geq$". A *critical value* of the parameter $\tau = \tau_c$ is one for which there are strictly τ-semistable pairs. There are only finitely many critical values.

Fix an integer $r \geq 2$ and a holomorphic line bundle Λ over X. Let d be the degree of Λ. We denote by $\mathfrak{M}_\tau(r, \Lambda)$ (respectively, $\overline{\mathfrak{M}}_\tau(r, \Lambda)$) the moduli space of τ-stable (respectively, τ-semistable) pairs (E, ϕ) of rank $\operatorname{rk}(E) = r$ and determinant $\det(E) = \Lambda$. The moduli space $\overline{\mathfrak{M}}_\tau(r, \Lambda)$ is a normal projective variety, and $\mathfrak{M}_\tau(r, \Lambda)$ is a smooth quasi-projective variety contained in the smooth locus of $\overline{\mathfrak{M}}_\tau(r, \Lambda)$ (cf. [11, Theorem 3.2]). Moreover, $\dim \mathfrak{M}_\tau(r, \Lambda) = d + (r^2 - r - 1)(g - 1) - 1$.

For non-critical values of the parameter, there are no strictly τ-semistable pairs, so $\mathfrak{M}_\tau(r, \Lambda) = \overline{\mathfrak{M}}_\tau(r, \Lambda)$ and it is a smooth projective variety. For a critical value τ_c, the variety $\overline{\mathfrak{M}}_{\tau_c}(r, \Lambda)$ is in general singular.

Denote $\tau_m := \frac{d}{r}$ and $\tau_M := \frac{d}{r-1}$. The moduli space $\mathfrak{M}_\tau(r, \Lambda)$ is empty for $\tau \notin (\tau_m, \tau_M)$. In particular, this forces $d > 0$ for τ-stable pairs. Denote by $\tau_1 < \tau_2 < \ldots < \tau_L$ the collection of all critical values in (τ_m, τ_M). Then the moduli spaces $\mathfrak{M}_\tau(r, \Lambda)$ are isomorphic for all values τ in any interval (τ_i, τ_{i+1}), $i = 0, \ldots, L$; here $\tau_0 = \tau_m$ and $\tau_{L+1} = \tau_M$.

However, the moduli space changes when we cross a critical value. Let τ_c be a critical value (note that for us, a critical value $\tau_c \neq \tau_m, \tau_M$). Denote $\tau_c^+ := \tau_c + \epsilon$ and $\tau_c^- := \tau_c - \epsilon$ for $\epsilon > 0$ small enough such that (τ_c^-, τ_c^+) does not contain any critical value other than τ_c. We define the *flip loci* $\mathcal{S}_{\tau_c^\pm}$ as the subschemes:

- $\mathcal{S}_{\tau_c^+} = \{(E, \phi) \in \mathfrak{M}_{\tau_c^+}(r, \Lambda) \mid (E, \phi) \text{ is } \tau_c^-\text{-unstable}\}$,
- $\mathcal{S}_{\tau_c^-} = \{(E, \phi) \in \mathfrak{M}_{\tau_c^-}(r, \Lambda) \mid (E, \phi) \text{ is } \tau_c^+\text{-unstable}\}$.

When crossing τ_c, the variety $\mathfrak{M}_\tau(r,\Lambda)$ undergoes a birational transformation:

$$\mathfrak{M}_{\tau_c^-}(r,\Lambda) \setminus \mathcal{S}_{\tau_c^-} = \mathfrak{M}_{\tau_c}(r,\Lambda) = \mathfrak{M}_{\tau_c^+}(r,\Lambda) \setminus \mathcal{S}_{\tau_c^+} .$$

Proposition 2.1 ([10, Proposition 5.1]). *Suppose $r \geq 2$, and let τ_c be a critical value with $\tau_m < \tau_c < \tau_M$. Then*

- $\operatorname{codim} \mathcal{S}_{\tau_c^+} \geq 3$ *except in the case $r = 2$, $g = 2$, d odd and $\tau_c = \tau_m + \frac{1}{2}$ (in which case* $\operatorname{codim} \mathcal{S}_{\tau_c^+} = 2$*),*
- $\operatorname{codim} \mathcal{S}_{\tau_c^-} \geq 2$ *except in the case $r = 2$ and $\tau_c = \tau_M - 1$ (in which case* $\operatorname{codim} \mathcal{S}_{\tau_c^-} = 1$*). Moreover we have that* $\operatorname{codim} \mathcal{S}_{\tau_c^-} = 2$ *only for $\tau_c = \tau_M - 2$.*

The codimension of the flip loci is then always positive, hence we have the following corollary:

Corollary 2.2. *The moduli spaces $\mathfrak{M}_\tau(r,\Lambda)$, $\tau \in (\tau_m, \tau_M)$, are birational.*

The moduli spaces for the extreme values of the parameter τ_m^+ and τ_M^- are known explicitly. Let $M(r,\Lambda)$ be the moduli space of *stable* vector bundles or rank r and fixed determinant Λ. Define

$$\mathcal{U}_m(r,\Lambda) = \{(E,\phi) \in \mathfrak{M}_{\tau_m^+}(r,\Lambda) \mid E \text{ is a stable vector bundle}\} , \tag{2.1}$$

and denote

$$\mathcal{S}_{\tau_m^+} := \mathfrak{M}_{\tau_m^+}(r,\Lambda) \setminus \mathcal{U}_m(r,\Lambda)$$

(not to be confused with the definition above for $\mathcal{S}_{\tau_c^\pm}$, which refers only to critical values $\tau_c \neq \tau_m, \tau_M$). Then there is a map

$$\pi_1 : \mathcal{U}_m(r,\Lambda) \longrightarrow M(r,\Lambda), \qquad (E,\phi) \mapsto E , \tag{2.2}$$

whose fiber over E is the projective space $\mathbb{P}(H^0(E))$. When $d \geq r(2g-2)$, we have that $H^1(E) = 0$ for any stable bundle, and hence (2.2) is a projective bundle (cf. [9, Proposition 4.10]).

Regarding the rightmost moduli space $\mathfrak{M}_{\tau_M^-}(r,\Lambda)$, we have that any τ_M^--stable pair (E,ϕ) sits in an exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\phi} E \longrightarrow F \longrightarrow 0 ,$$

where F is a semistable bundle of rank $r-1$ and $\det(F) = \Lambda$. Let

$$\mathcal{U}_M(r,\Lambda) = \{(E,\phi) \in \mathfrak{M}_{\tau_M^-}(r,\Lambda) \mid F \text{ is a stable vector bundle}\} ,$$

and denote

$$\mathcal{S}_{\tau_M^-} := \mathfrak{M}_{\tau_M^-}(r,\Lambda) \setminus \mathcal{U}_M(r,\Lambda) .$$

Then there is a map

$$\pi_2 : \mathcal{U}_M(r,\Lambda) \longrightarrow M(r-1,\Lambda), \qquad (E,\phi) \mapsto E/\phi(\mathcal{O}) , \tag{2.3}$$

whose fiber over $F \in M(r-1,\Lambda)$ is the projective spaces $\mathbb{P}(H^1(F^*))$ (cf. [6]). Note that $H^0(F^*) = 0$ since $d > 0$. So (2.3) is always a projective bundle.

In the particular case of rank $r = 2$, the rightmost moduli space is

$$\mathfrak{M}_{\tau_M^-}(2,\Lambda) = \mathbb{P}(H^1(\Lambda^{-1})) , \tag{2.4}$$

since $M(1,\Lambda)=\{\Lambda\}$. In particular, Corollary 2.2 shows that all $\mathfrak{M}_\tau(2,\Lambda)$ are rational quasi-projective varieties.

We have the following:

Lemma 2.3 ([11, Lemma 5.3])**.** *Let S be a bounded family of isomorphism classes of strictly semistable bundles of rank r and determinant Λ. Then $\dim M(r,\Lambda)-\dim S\geq (r-1)(g-1)$.*

Proposition 2.4. *The following two statements hold:*

- *Suppose $d>r(2g-2)$. Then $\operatorname{codim}\mathcal{S}_{\tau_m^+}\geq 2$ except in the case $r=2$, $g=2$, d even (in which case $\operatorname{codim}\mathcal{S}_{\tau_m^+}=1$).*
- *Suppose $r\geq 3$. Then $\operatorname{codim}\mathcal{S}_{\tau_M^-}\geq 2$ except in the case $r=3$, $g=2$, d even (in which case the $\operatorname{codim}\mathcal{S}_{\tau_M^-}=1$).*

Proof. Let $(E,\phi)\in\mathfrak{M}_{\tau_m^+}(r,\Lambda)$, then E is a semistable bundle. As $d>r(2g-2)$, $H^1(E)=H^0(E^*\otimes K_X)^*=0$, since $E^*\otimes K_X$ is semistable and has negative degree. Therefore, $\dim H^0(E)$ is constant. Let $\mathcal{F}$ be the family of strictly semistable bundles E such that there exists some ϕ with $(E,\phi)\in\mathcal{S}_{\tau_m^+}$. Then $\operatorname{codim}\mathcal{S}_{\tau_m^+}=\dim\mathfrak{M}_{\tau_m^+}(r,\Lambda)-\dim\mathcal{S}_{\tau_m^+}\geq\dim M(r,\Lambda)-\dim\mathcal{F}\geq(r-1)(g-1)$ (by Lemma 2.3). The first statement follows.

As the dimension $\dim H^1(F^*)$ is constant, the codimension of $\mathcal{S}_{\tau_M^-}$ in $\mathfrak{M}_{\tau_M^-}(r,\Lambda)$ is at least the codimension of a locus of semistable bundles. Applying Lemma 2.3 to $M(r-1,\Lambda)$ we have $\operatorname{codim}\mathcal{S}_{\tau_M^-}\geq(r-2)(g-1)$. The second item follows. □

3. Brauer group

The Brauer group of a scheme Z is defined as the equivalence classes of Azumaya algebras on Z, that is, coherent locally free sheaves with algebra structure such that, locally on the étale topology of Z, are isomorphic to a matrix algebra $\mathrm{Mat}(\mathcal{O}_Z)$. If Z is a smooth quasi-projective variety, then the Brauer group $\mathrm{Br}(Z)$ coincides with $H^2_{\acute{e}t}(Z)$, and $H^2_{\acute{e}t}(Z)$ is a torsion group.

Theorem 3.1. [8, VI.5 (Purity)] *Let Z be a smooth complex variety and $U\subset Z$ be a Zariski open subset whose complement has codimension at least 2. Then $\mathrm{Br}(Z)=\mathrm{Br}(U)$.*

On the moduli space of stable vector bundles $M(r,\Lambda)$, there are three natural projective bundles. We will describe them.

We first note that there is a unique universal projective bundle over $X\times M(r,\Lambda)$. Fix a point $x\in X$. Restricting the universal projective bundle to $\{x\}\times M(r,\Lambda)$ we get a projective bundle

$$\mathbb{P}_x\longrightarrow M(r,\Lambda)\,. \tag{3.1}$$

Secondly, if $d\geq r(2g-2)$, then we have the projective bundle

$$\mathcal{P}_0\longrightarrow M(r,\Lambda)\,, \tag{3.2}$$

whose fiber over any $E \in M(r, \Lambda)$ is the projective space $\mathbb{P}(H^0(E))$; note that we have $H^1(E) = 0$ because $d \geq r(2g-2)$.

Finally, assuming $d > 0$, let

$$\mathcal{P}_1 \longrightarrow M(r, \Lambda) \tag{3.3}$$

be the projective bundle whose fiber over any $E \in M(r, \Lambda)$ is the projective space $\mathbb{P}(H^1(E^*))$.

Proposition 3.2. *The Brauer class* $\mathrm{cl}(\mathbb{P}_x) \in \mathrm{Br}(M(r, \Lambda))$ *is independent of* $x \in X$. *Moreover,*

$$\mathrm{cl}(\mathbb{P}_x) = \mathrm{cl}(\mathcal{P}_0) = -\mathrm{cl}(\mathcal{P}_1)\,,$$

when they are defined.

Proof. The moduli space $M(r, \Lambda)$ is constructed as a Geometric Invariant Theoretic quotient of a Quot scheme $\mathcal{Q}$ by the action of a linear group $\mathrm{GL}_N(\mathbb{C})$ (see [13]). The isotropy subgroup for a stable point of $\mathcal{Q}$ is the center $\mathbb{C}^* \subset \mathrm{GL}_N(\mathbb{C})$. There is a universal vector bundle

$$\mathcal{E} \longrightarrow X \times \mathcal{Q}\,.$$

Let $Z(\mathrm{GL}_N(\mathbb{C}))$ be the center of $\mathrm{GL}_N(\mathbb{C})$. The action of the subgroup $Z(\mathrm{GL}_N(\mathbb{C}))$ on $\mathcal{Q}$ is trivial. Therefore, $Z(\mathrm{GL}_N(\mathbb{C}))$ acts on each fiber of $\mathcal{E}$. Identify $Z(\mathrm{GL}_N(\mathbb{C}))$ with $\mathbb{C}^*$ by sending any $\lambda \in \mathbb{C}^*$ to $\lambda \cdot \mathrm{Id}$. We note that $\lambda \in Z(\mathrm{GL}_N(\mathbb{C}))$ acts on $\mathcal{E}$ as multiplication by λ.

Let $\mathcal{Q}^s \subset \mathcal{Q}$ be the stable locus. The restriction of $\mathcal{E}$ to $X \times \mathcal{Q}^s$ will be denoted by $\mathcal{E}^s$. Let

$$\mathcal{E}_x := \mathcal{E}^s|_{\{x\} \times \mathcal{Q}^s} \longrightarrow \mathcal{Q}^s$$

be the restriction. Let $p_2 : X \times \mathcal{Q}^s \longrightarrow \mathcal{Q}^s$ be the natural projection. Define the vector bundles

$$\mathcal{E}_0 := p_{2*}\,\mathcal{E}^s \quad \text{and} \quad \mathcal{E}_1 := R^1 p_{2*}((\mathcal{E}^s)^*)\,.$$

We noted that any $\lambda \in \mathbb{C}^* = Z(\mathrm{GL}_N(\mathbb{C}))$ acts on $\mathcal{E}_x$ as multiplication by λ. Therefore, λ acts on $(\mathcal{E}^s)^*$ as multiplication by $1/\lambda$. Hence λ acts on $\mathcal{E}_1$ as multiplication by $1/\lambda$. Consequently, the action of $Z(\mathrm{GL}_N(\mathbb{C}))$ on $\mathcal{E}_x \otimes \mathcal{E}_1$ is trivial. Hence $\mathcal{E}_x \otimes \mathcal{E}_1$ descends to a vector bundle over the quotient $M(r, \Lambda)$ of $\mathcal{Q}^s$. Therefore,

$$\mathrm{cl}(\mathbb{P}_x) = -\mathrm{cl}(\mathcal{P}_1)\,.$$

Any $\lambda \in \mathbb{C}^* = Z(\mathrm{GL}_N(\mathbb{C}))$ acts on $\mathcal{E}_0$ as multiplication by λ. Indeed, this follows immediately from the fact that λ acts as multiplication by λ on $\mathcal{E}^s$. As noted earlier, λ acts on $\mathcal{E}_1$ as multiplication by $1/\lambda$. Hence the action of $Z(\mathrm{GL}_N(\mathbb{C}))$ on $\mathcal{E}_0 \otimes \mathcal{E}_1$ is trivial. Thus $\mathcal{E}_0 \otimes \mathcal{E}_1$ descends to $M(r, \Lambda)$, implying

$$\mathrm{cl}(\mathcal{P}_0) = -\mathrm{cl}(\mathcal{P}_1)\,.$$

Finally, note that it follows that $\mathrm{cl}(\mathbb{P}_x)$ is independent of $x \in X$ for $d > 0$. For $d \leq 0$, $\mathcal{P}_0$ and $\mathcal{P}_1$ are not defined. In this case, we take a line bundle μ or large degree, and use the isomorphism $M(r, \Lambda \otimes \mu^r) \cong M(r, \Lambda)$. For any pair $x, x' \in X$, since $\mathrm{cl}(\mathbb{P}_x) = \mathrm{cl}(\mathbb{P}_{x'})$ in $\mathrm{Br}(M(r, \Lambda \otimes \mu^r))$, the same holds for $\mathrm{Br}(M(r, \Lambda))$. $\square$

Theorem 3.3. *Assume that $d > r(2g-2)$. Then for the moduli space $\mathfrak{M}_\tau(r,\Lambda)$ of stable pairs, we have that*

$$\mathrm{Br}(\mathfrak{M}_\tau(r,\Lambda)) \,=\, 0\,.$$

Proof. We will first prove it for $r = 2$. Recall from (2.4) that $\mathfrak{M}_{\tau_M^-}(2,\Lambda)$ is a projective space, hence

$$\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(2,\Lambda)) = 0\,.$$

Moreover, all $\mathfrak{M}_\tau(2,\Lambda)$ are rational varieties. Thus

$$\mathrm{Br}(\mathfrak{M}_\tau(2,\Lambda)) = 0$$

for non-critical values $\tau \in (\tau_m, \tau_M)$, since the Brauer group of a smooth rational projective variety is zero [1, p. 77, Proposition 1]. For a critical value τ_c, we have

$$\mathfrak{M}_{\tau_c}(2,\Lambda) = \mathfrak{M}_{\tau_c^+}(2,\Lambda) \setminus \mathcal{S}_{\tau_c^+}\,.$$

By Proposition 2.1, $\operatorname{codim} \mathcal{S}_{\tau_c^+} \geq 2$, so the Purity Theorem implies that

$$\mathrm{Br}(\mathfrak{M}_{\tau_c}(2,\Lambda)) = 0\,.$$

Now we assume that $r \geq 3$. From Proposition 2.1 and Theorem 3.1 it follows that the Brauer group $\mathrm{Br}(\mathfrak{M}_\tau(r,\Lambda))$ does not depend on the value of the parameter τ (for fixed r and Λ).

As $d \geq r(2g-2)$, we have a projective bundle

$$\pi_1 : \mathcal{U}_m(r,\Lambda) \longrightarrow M(r,\Lambda)$$

(see (2.2)). Note that this projective bundle coincides with the projective bundle $\mathcal{P}_0$ in (3.2). The projective bundle π_1 gives an exact sequence

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) \longrightarrow \mathrm{Br}(M(r,\Lambda)) \longrightarrow \mathrm{Br}(\mathcal{U}_m(r,\Lambda)) \longrightarrow 0 \tag{3.4}$$

(see [7, p. 193]). As $d > r(2g-2)$, Proposition 2.4 and the Purity Theorem give

$$\mathrm{Br}(\mathcal{U}_m(r,\Lambda)) \,=\, \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda))\,,$$

so we have

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) \longrightarrow \mathrm{Br}(M(r,\Lambda)) \longrightarrow \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda)) \longrightarrow 0\,. \tag{3.5}$$

We will show that the theorem follows from (3.5) if we use [2]. From Proposition 3.2 we know that $\mathrm{cl}(\mathcal{P}_0) = \mathrm{cl}(\mathbb{P}_x)$, and from [2, Proposition 1.2(a)] we know that $\mathrm{cl}(\mathbb{P}_x)$ generates $\mathrm{Br}(M(r,\Lambda))$. Therefore, from (3.5) it follows that

$$\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda)) \,=\, 0\,.$$

Since $\mathrm{Br}(\mathfrak{M}_\tau(r,\Lambda))$ is independent of τ, this completes the proof using [2]. But we shall give a different proof without using [2], because we want to show that the above mentioned result of [2] can be deduced from our Theorem 3.3 (see Corollary 3.4).

Consider the projective bundle $\pi_2 : \mathcal{U}_M(r-1,\Lambda) \longrightarrow M(r-1,\Lambda)$ from (2.3). Note that this projective bundle coincides with the projective bundle $\mathcal{P}_1$ in (3.3) for rank $r-1$. The projective bundle π_2 gives an exact sequence

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_1) \longrightarrow \mathrm{Br}(M(r-1,\Lambda)) \longrightarrow \mathrm{Br}(\mathcal{U}_M(r,\Lambda)) = \mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda)) \longrightarrow 0\,, \tag{3.6}$$

using Proposition 2.4, with the exception of the case $(r,g,d)=(3,2,\mathrm{even})$. Let us leave this "bad" case aside for the moment.

Let

$$\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) \longrightarrow \mathrm{Br}(M(r-1,\Lambda)) \longrightarrow \mathrm{Br}(\mathcal{U}_m(r-1,\Lambda)) = \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r-1,\Lambda)) \longrightarrow 0 \tag{3.7}$$

be the exact sequence obtained by replacing r with $r-1$ in (3.5); the last equality holds as $(r-1,g,d)\neq(2,2,\mathrm{even})$, by Proposition 2.4.

Since $\mathrm{cl}(\mathcal{P}_1) = -\mathrm{cl}(\mathcal{P}_0)$ (see Proposition 3.2), comparing (3.6) and (3.7) we conclude that the two quotients of $\mathrm{Br}(M(r-1,\Lambda))$, namely

$$\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda)) \quad \text{and} \quad \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r-1,\Lambda))\,,$$

coincide. In particular, $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda))$ is isomorphic to $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r-1,\Lambda))$. Therefore, using induction, the group $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda))$ is isomorphic to $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(2,\Lambda))$. We have already shown that $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(2,\Lambda)) = 0$. Hence the proof of the theorem is complete for $d>r(2g-2)$ and $(r,g,d)\neq(3,2,\mathrm{even})$.

Let us now investigate the missing case of $(r,g,d)=(3,2,2k)$. Take a line bundle ν of degree 1. Using (3.4) twice, we have

$$\begin{array}{ccccccc}
\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) & \longrightarrow & \mathrm{Br}(M(3,\Lambda)) & \longrightarrow & \mathrm{Br}(\mathcal{U}_m(3,\Lambda)) & \longrightarrow & 0\\
\downarrow\cong & & || & & & & \\
\mathbb{Z}\cdot \mathrm{cl}(\mathcal{P}_0) & \longrightarrow & \mathrm{Br}(M(3,\Lambda\otimes\nu^3)) & \longrightarrow & \mathrm{Br}(\mathcal{U}_m(3,\Lambda\otimes\nu^3)) & \longrightarrow & 0
\end{array}$$

The second vertical map is induced by the isomorphism $M(3,\Lambda)\longrightarrow M(3,\Lambda\otimes\nu^3)$ defined by $E\mapsto E\otimes\nu$, hence it is an isomorphism. This isomorphism preserves the class $\mathrm{cl}(\mathbb{P}_x)$, and hence the class $\mathrm{cl}(\mathcal{P}_0)$, by Proposition 3.2. Therefore, $\mathrm{Br}(\mathcal{U}_m(3,\Lambda)) = \mathrm{Br}(\mathcal{U}_m(3,\Lambda\otimes\nu^3))$. But $\deg(\Lambda\otimes\nu^3)$ is odd, hence

$$\mathrm{Br}(\mathcal{U}_m(3,\Lambda)) = \mathrm{Br}(\mathcal{U}_m(\Lambda\otimes\nu^3)) = 0\,.$$

By the Purity Theorem, $\mathrm{Br}(\mathfrak{M}_\tau(3,\Lambda))=0$ for any τ. □

Note that the proof of Theorem 3.3 works in the following cases:

- $r=2$, any d ;
- $(r,g,d)\neq(3,2,\mathrm{even})$, $d>(r-1)(2g-2)$; and
- $r=3$, $g=2$, $d>6$.

Before proceeding to remove the assumption $d>r(2g-2)$ in Theorem 3.3, we want to show that Theorem 3.3 implies Proposition 1.2(a) of [2].

Corollary 3.4. *Suppose that $(r,g,d)\neq(2,2,\mathit{even})$. The Brauer group $\mathrm{Br}(M(r,\Lambda))$ is generated by the Brauer class $\mathrm{cl}(\mathbb{P}_x)\in\mathrm{Br}(M(r,\Lambda))$ in (3.1).*

Proof. Without loss of generality we can assume that d is large (since we have an isomorphism $M(r,\Lambda)\xrightarrow{\sim} M(r,\Lambda\otimes\mu^r)$, $E\mapsto E\otimes\mu$, where μ is a line bundle.

First, we have $\mathrm{Br}(\mathcal{U}_m(r,\Lambda)) = \mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda))$ by the Purity Theorem and Proposition 2.4. Second, $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(r,\Lambda))=0$ by Theorem 3.3, so $\mathrm{Br}(\mathcal{U}_m(r,\Lambda))=0$. Finally, we use the exact sequence in (3.4) to see that $\mathrm{cl}(\mathcal{P}_0)$ generates $\mathrm{Br}(M(r,\Lambda))$. Now from Proposition 3.2 it follows that $\mathrm{cl}(\mathbb{P}_x)$ generates $\mathrm{Br}(M(r,\Lambda))$. □

Corollary 3.5. *Suppose* $(r,g,d)\neq(3,2,2)$. *Then we have that* $\mathrm{Br}(\mathfrak{M}_\tau(r,\Lambda))=0$.

Proof. For $r=2$, this result is proved as in Theorem 3.3. As we know it for $d>r(2g-2)$, we assume that $d\leq r(2g-2)$.

Let $r\geq 3$. Suppose first that $(r,g,d)\neq(3,2,\text{even})$, that is, $(r,g,d)\neq(3,2,2)$, $(3,2,4)$, $(3,2,6)$. As $d>0$, we still have a projective bundle $\pi_2:\mathcal{U}_M(r,\Lambda)\longrightarrow M(r-1,\Lambda)$. Therefore there is an exact sequence as in (3.6). Note that Proposition 2.4 and the Purity Theorem imply that $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda))=\mathrm{Br}(\mathcal{U}_M(r,\Lambda))$. Now using Proposition 3.2 and Corollary 3.4 and (3.6) it follows that $\mathrm{Br}(\mathfrak{M}_{\tau_M^-}(r,\Lambda))=0$. The result follows.

Let us deal with the missing cases $(r,g,d)=(3,2,4)$, $(3,2,6)$. We start with the case $(r,g,d)=(3,2,4)$. Let

$$Z=\{E\in M(3,\Lambda)\,|\,H^1(E)\neq 0\}.$$

For $E\in M(3,\Lambda)\setminus Z$, we have that $\dim H^0(E)=4+3(1-g)=1$. So the projective bundle

$$\pi_1:\mathcal{U}_m(3,\Lambda)\setminus\pi_1^{-1}(Z)\longrightarrow M(3,\Lambda)\setminus Z$$

is actually an isomorphism. In this situation, the exact sequence

$$\mathbb{Z}\cdot\mathrm{cl}(\mathcal{P}_0)\longrightarrow\mathrm{Br}(M(3,\Lambda)\setminus Z)\longrightarrow\mathrm{Br}(\mathcal{U}_m(3,\Lambda)\setminus\pi_1^{-1}(Z))\longrightarrow 0 \tag{3.8}$$

satisfies that $\mathrm{cl}(\mathcal{P}_0)=0$. The proof of Proposition 3.2 works also for $M(3,\Lambda)\setminus Z$, so $\mathrm{cl}(\mathbb{P}_x)=0$. We shall see below that

$$\operatorname{codim} Z\geq 2\qquad\text{and}\qquad\operatorname{codim}\pi_1^{-1}(Z)\geq 2\,. \tag{3.9}$$

From this, $\mathrm{Br}(M(3,\Lambda)\setminus Z)=\mathrm{Br}(M(3,\Lambda))$ and $\mathrm{Br}(\mathcal{U}_m(3,\Lambda)\setminus\pi_1^{-1}(Z))=\mathrm{Br}(\mathcal{U}_m(3,\Lambda))=\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(3,\Lambda))$. By Corollary 3.4, $\mathrm{cl}(\mathbb{P}_x)=0$ generates $\mathrm{Br}(M(3,\Lambda))$, so $\mathrm{Br}(M(3,\Lambda))=0$ and $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(3,\Lambda))=0$, as required.

To see the codimension estimates (3.9), we work as follows. Let $E\in Z\subset M(3,\Lambda)$. So $H^1(E)\neq 0$, i.e. $H^0(E^*\otimes K_X)\neq 0$. Thus there is an exact sequence

$$0\longrightarrow\mathcal{O}\longrightarrow E'=E^*\otimes K_X\longrightarrow F\longrightarrow 0\,, \tag{3.10}$$

for some sheaf F. Note that $\deg(F)=\deg(E')=2$, and E' is stable (since E stable $\Longrightarrow$ E^* stable $\Longrightarrow$ $E'=E^*\otimes K_X$ stable). Here F must be a rank 2 semistable sheaf, since any quotient $F\to Q$ with $\mu(Q)<\mu(F)=1$, would satisfy that $\mu(Q)<\mu(E')=\frac{2}{3}$, violating the stability of E'. In particular, F is a (semistable) bundle, and it is parametrized by an irreducible variety of dimension $\dim M(2,\Lambda)=3(g-1)=3$ (recall that $\dim M(r,\Lambda)=(r^2-1)(g-1)$). Now the bundle E' in (3.10) is given by an extension in $\mathbb{P}(H^1(F^*))$. As $H^0(F^*)=0$ (by semistability), we have that $\dim\mathbb{P}(H^1(F^*))=-(-2+2(1-g))-1=3$. So the bundles E' are parametrized by a 6-dimensional variety, and therefore $\dim Z=6$ and $\operatorname{codim} Z=3$.

Now let us see that $\dim\pi_1^{-1}(Z)\leq 7$. Let $E\in Z$ and F as in (3.10), and note that the determinant of F is fixed. Recalling that $\dim H^1(F^*)=4$, we see that we have to check that

$$\dim\mathcal{F}+3+\dim H^0(E)-1\leq 7\,,$$

where $\mathcal{F}$ is the family of the bundles F. Now $\dim H^0(E) = \dim H^1(E)+1 = \dim H^0(E')+1 \leq \dim H^0(F) + 2$. Hence we only need to show that

$$\dim \mathcal{F}_i + \dim H^0(F) \leq 3\,, \tag{3.11}$$

for $F \in \mathcal{F}_i$, where $\mathcal{F} = \bigsqcup \mathcal{F}_i$ is the family (suitably stratified) of the possible bundles F.

We have the following possibilities:

(1) $F = L_1 \oplus L_2$, where L_1, L_2 are line bundles of degree one, $L_2 = \det(F) \otimes L_1^{-1}$. The generic such F moves in a 2-dimensional family, and $H^0(F) = 0$. If $\dim H^0(F) \neq 0$, then it should be either $L_1 = \mathcal{O}(p)$ or $L_2 = \mathcal{O}(q)$, $p, q \in X$. In this case F moves in a 1-dimensional family, and $\dim H^0(F) \leq 2$, so (3.11) holds.

(2) F is a non-trivial extension $L \to F \to L$, where L is a line bundle of degree one. As $\det(F) = L^2$ is fixed, then there are finitely many possible L. Now $\dim \mathrm{Ext}^1(L, L) = 2$, so the bundles F move in a 1-dimensional family. Again $\dim H^0(F) \leq 2$, so (3.11) is satisfied.

(3) F is a non-trivial extension $L_1 \to F \to L_2$, where L_1, L_2 are non-isomorphic line bundles of degree one. As $\dim \mathrm{Ext}^1(L_2, L_1) = 1$, we have that F moves in 2-dimensional family. If $\dim H^0(F) = 1$ then (3.11) holds. Otherwise, it must be $L_1 = \mathcal{O}(p)$ and $L_2 = \mathcal{O}(q)$, hence F moves in a 1-dimensional family and $\dim H^0(F) \leq 2$. So (3.11) holds again.

(4) F a rank 2 stable bundle and $H^0(F) = 0$. This is clear, since $\dim M(2, \Lambda) = 3$.

(5) F a rank 2 stable bundle and $H^0(F) = 1$. Then we have an exact sequence $\mathcal{O} \to F \to L$, where L is a (fixed) line bundle of degree two. As $\dim H^1(L^*) = 3$, we have that F moves in a 2-dimensional family and (3.11) holds.

(6) F a rank 2 stable bundle, $\mathcal{O} \to F \to L$, $\dim H^0(L) = 1$ and $\dim H^0(F) = 2$. The connecting map $H^0(L) = \mathbb{C} \to H^1(\mathcal{O})$ is given by multiplication by the extension class in $H^1(L^*)$ defining F. To have $\dim H^0(F) = 2$, this connecting map must be zero, hence the extension class is in $\ker(H^1(L^*) \to H^1(\mathcal{O}))$. This kernel is one-dimensional (since the map is surjective). So the family of such F is zero-dimensional, and (3.11) is satisfied.

(7) F a rank 2 stable bundle, $\mathcal{O} \to F \to L$, $\dim H^0(L) = 2$ and $\dim H^0(F) \geq 2$. Now it must be $L = K_X$. The connecting map
$$c_\xi : H^0(K_X) \to H^1(\mathcal{O}) = H^0(K_X)^*$$
is given by multiplication with the extension class ξ in $H^1(L^*) = H^0(K_X^2)^*$ defining F. So $c_\xi \in \mathrm{Hom}(H^0(K_X), H^0(K_X)^*) = H^0(K_X)^* \otimes H^0(K_X)^*$ is the image of ξ under $H^0(K_X^2)^* \to H^0(K_X)^* \otimes H^0(K_X)^*$. But this map is the inclusion $H^0(K_X^2)^* = \mathrm{Sym}^2 H^0(K_X)^* \subset \bigotimes^2 H^0(K_X)^*$. This means that $c_\xi \in \mathrm{Sym}^2 H^0(K_X)^*$.

If $\dim H^0(F) = 2$, then c_ξ is not an isomorphism. The condition $\det(c_\xi) = 0$ gives a 2-dimensional family of $\xi \in H^1(L^*)$. So the family of such bundles F is one-dimensional and (3.11) is satisfied. If $\dim H^0(F) = 3$, then $c_\xi = 0$, and so $\xi = 0$, which is not possible (since F does not split).

Finally, we tackle the case $(r, g, d) = (3, 2, 6)$. Now $\mathcal{U}_m(3, \Lambda) \to M(3, \Lambda)$ is a projective fibration (with fibers $\mathbb{P}^2$), so Corollary 3.4 and the exact sequence (3.4) imply that

$\mathrm{Br}(\mathcal{U}_m(3,\Lambda)) = 0$. To complete the proof that $\mathrm{Br}(\mathfrak{M}_{\tau_m^+}(3,\Lambda)) = 0$, it only remains to show that $\operatorname{codim} \mathcal{S}_{\tau_m^+} \geq 2$.

Consider the family $\mathcal{F}$ of strictly semistable bundles E with $(E,\phi) \in \mathcal{S}_{\tau_m^+}$. We stratify $\mathcal{F} = \bigsqcup \mathcal{F}_j$, such that $\dim H^0(E)$ is constant on each $\mathcal{F}_j$. We have to prove that

$$\dim \mathcal{F}_j + \dim H^0(E) - 1 \leq \dim \mathfrak{M}_{\tau_m^+}(3,\Lambda) - 2 = 10 - 2 = 8.$$

For E strictly semistable, we have either an exact sequence $L \to E \to F$ or $F \to E \to L$, where $L \in \operatorname{Jac}^2 X$, and F is a semistable bundle of rank 2 and determinant $\Lambda' = \Lambda \otimes L^{-1}$ (which is of degree 4). Both cases are similar, so we assume the first one. There are three possibilities:

(1) Suppose that $\dim \operatorname{Hom}(F,L) = 0$. Then $\dim \operatorname{Ext}^1(F,L) = 2$. We stratify $\operatorname{Jac}^2 X$ depending on $\dim H^0(L)$. For $L \neq K_X$, $\dim H^0(L) = 1$; for $L = K_X$, $\dim H^0(L) = 2$. So for each stratum $\mathcal{F}' \subset \operatorname{Jac}^2 X$, we have that $\dim \mathcal{F}' + \dim H^0(L) \leq 3$. We also stratify the family of rank 2 semistable bundles F, according to $\dim H^0(F)$. For any such stratum $\mathcal{F}''$, we have that

$$\dim \mathcal{F}'' + \dim H^0(F) - 1 \leq 4\,. \tag{3.12}$$

Assuming (3.12), and noting that $\dim H^0(E) \leq \dim H^0(L) + \dim H^0(F)$, we have that, for the corresponding stratum $\mathcal{F}_0$, $\dim \mathcal{F}_0 + \dim H^0(E) - 1 \leq 3+4+2-1 = 8$.

Let us prove (3.12). For F stable, we have that $\dim \mathcal{F}'' + \dim H^0(F) - 1 \leq \dim \mathfrak{M}_{\tau_m^+}(2,\Lambda') = 4$. For F strictly semistable, there is an exact sequence $L' \to F \to \Lambda' \otimes L'^{-1}$, with $L' \in \operatorname{Jac}^2 X$. If L' is generic, then $\dim H^0(L') = \dim H^0(\Lambda' \otimes L'^{-1}) = 1$ and $\dim \operatorname{Ext}^1(\Lambda' \otimes L'^{-1}, L') = 1$. So $\dim \mathcal{F}'' + \dim H^0(F) - 1 \leq 2+2-1 = 3$. For $L' = K_X$, $\Lambda' \otimes L'^{-1} = K_X$ or $L'^2 = \Lambda'$, we have the bounds $\dim H^0(L') \leq 2$, $\dim H^0(\Lambda' \otimes L'^{-1}) \leq 2$ and $\dim \operatorname{Ext}^1(\Lambda' \otimes L'^{-1}, L') \leq 2$, giving that $\dim \mathcal{F}'' + \dim H^0(F) - 1 \leq 1 + 4 - 1 = 4$.

(2) Suppose that $\dim \operatorname{Hom}(F,L) = 1$. Then $\dim \operatorname{Ext}^1(F,L) = 3$. There is an exact sequence $\Lambda \otimes L^{-2} \to F \to L$. If $\dim H^0(L) = 1$ and $\dim H^0(\Lambda \otimes L^{-2}) = 1$ then $\dim H^0(E) = 3$. This case is covered by Lemma 2.3. Otherwise $L = K_X$ or $\Lambda \otimes L^{-2} = K_X$, so there are finitely many choices for L. Using that $\dim H^0(E) \leq 6$ and $\dim \operatorname{Ext}^1(L, \Lambda \otimes L^{-2}) \leq 2$, we get that, for the corresponding stratum $\mathcal{F}_1$, it is $\dim \mathcal{F}_1 + \dim H^0(E) - 1 \leq 1 + 2 + 6 - 1 = 8$.

(3) Suppose that $\dim \operatorname{Hom}(F,L) = 2$. Then $F = L \oplus L$ and $\dim \operatorname{Ext}^1(F,L) = 4$. The extension is unique, because the group of automorphisms of such F is of dimension 4. Note also that there are finitely many choices for L. So for the corresponding family $\mathcal{F}_2$, we have $\dim \mathcal{F}_2 + \dim H^0(E) - 1 \leq 6 - 1 = 5$.

This completes the proof of the corollary.

□

Remark 3.6. Note that $\mathrm{Br}(\mathcal{U}_M(r,\Lambda)) = 0$ for $(r,g,d) \neq (3,2,\text{even})$ (use Corollary 3.5 and Proposition 2.4).

Also, if $d > r(2g-2)$, then $\mathrm{Br}(\mathcal{U}_m(r,\Lambda)) = 0$ for $(r,g,d) \neq (2,2,\text{even})$ (use Corollary 3.5 and Proposition 2.4). Actually, in the range $d > r(2g-2)$, the proof of Theorem 3.3 shows that $\mathrm{Br}(\mathcal{U}_m(r,\Lambda)) = \mathrm{Br}(\mathcal{U}_M(r+1,\Lambda))$, for any (r,g,d).

Remark 3.7. Our techniques for proving Theorem 1.1 do not cover the case $(r, g, d) = (3, 2, 2)$. So this case remains open at the moment.

This is due to the following. Working as in the proof of Corollary 3.5, in the case $(r, g, d) = (3, 2, 2)$, we could try two approaches. First, we could look at the map $\pi_1 : \mathcal{U}_m(3, \Lambda) \to M(3, \Lambda)$. We see that whereas $\dim \mathcal{U}_m(3, \Lambda) = d + (r^2 - r - 1)(g - 1) - 1 = 6$, it is $\dim M(3, \Lambda) = (r^2 - 1)(g - 1) = 8$. Therefore π_1 is generically an immersion, and there is not much hope to recover the Brauer group of $\mathcal{U}_m(3, \Lambda)$ out of that of $M(3, \Lambda)$.

Second, we could look at the map $\pi_2 : \mathcal{U}_M(3, \Lambda) \to M(2, \Lambda)$, which is a projective fibration with fiber $\mathbb{P}^3$. The moduli space of S-equivalence classes of semistable bundles $\overline{M}(2, \Lambda)$ is isomorphic (for $g = 2$, $d \equiv 0 \pmod 2$) to $\mathbb{P}^3$ (see [12]). The locus of properly semistable bundles $Z \subset \mathbb{P}^3$ is a Kummer variety: $Z = \mathrm{Jac}^1 X/\mathbb{Z}_2$, whose elements are of the form $E = L \oplus (L^{-1} \otimes \Lambda)$, $L \in \mathrm{Jac}^1 X$. Then $\operatorname{codim} Z = 1$, so we do not get the vanishing of the Brauer group of $M(2, \Lambda) = \overline{M}(2, \Lambda) \setminus Z$.

We can still try to study the map π_2 over a larger open subset of $\overline{M}(2, \Lambda)$, recalling that π_2 extends to a map $\pi_2 : \mathfrak{M}_{\tau_M^-}(3, \Lambda) \to \overline{M}(2, \Lambda)$. We denote $\tilde{Z} = \pi_2^{-1}(Z)$. Consider a pair $(E, \phi) \in \tilde{Z}$. Then $\mathcal{O} \to E \to F$, where F is a semistable rank 2 bundle. So F sits in an exact sequence $L \to F \to L^{-1} \otimes \Lambda$, where $L \in \mathrm{Jac}^1 X$. Then $\dim \mathrm{Ext}^1(L^{-1} \otimes \Lambda, L) = 1$ if $L^2 \not\cong \Lambda$, and $\dim \mathrm{Ext}^1(L^{-1} \otimes \Lambda, L) = 2$ if $L^2 \cong \Lambda$. The family of *non-split* properly semistable bundles is then parametrized by $P_1 = Bl_{\mathrm{Fix}\,\tau}(\mathrm{Jac}^1 X)$, the blow-up of $\mathrm{Jac}^1 X$ at the fixed points of the involution $\tau : L \mapsto L^{-1} \otimes \Lambda$. There is an obvious map $q : P_1 \to Z$. The family of *split* semistable bundles is parametrized by $P_2 \cong Z$. Now consider the embedding $\imath : X \hookrightarrow \mathrm{Jac}^1 X$, given as $p \mapsto \mathcal{O}(p)$. This produces maps $\imath_1 : X \hookrightarrow P_1$ and $\imath_2 : X \to P_2$. Then for any $L \in \big(P_1 \setminus \imath_1(X) \cup \tau(\imath_1(X))\big) \sqcup \big(P_2 \setminus \imath_2(X)\big)$, we have that $H^0(F) = 0$ and $\dim H^1(F) = 4$. As a conclusion, if $F_0 = L \oplus (L^{-1} \otimes \Lambda) \in Z \setminus q \circ \imath_1(X) \subset \overline{M}(2, \Lambda)$, then the fiber

$$\pi_2^{-1}(F_0) \cong \mathbb{P}^3 \sqcup \mathbb{P}^3 \sqcup (\mathbb{P}^1 \times \mathbb{P}^1),$$

where the first $\mathbb{P}^3$ corresponds to the space of sections of F for the non-trivial extension $L \to F \to L^{-1} \otimes \Lambda$, the second $\mathbb{P}^3$ corresponds to the space of sections of F for the non-trivial extension $L^{-1} \otimes \Lambda \to F \to L$, and the $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to the sections of $F_0 = L \oplus (L^{-1} \otimes \Lambda)$ (taking the quotient by the automorphisms of the bundle).

This means that the map π_2 is not a projective fibration over any open subset larger than $M(2, \Lambda) \subset \overline{M}(2, \Lambda)$, ruling out any hope of determining the Brauer group of $\mathfrak{M}_{\tau_M^-}(3, \Lambda)$ without determining it for $M(2, \Lambda)$ first.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Serrano 113bis, 28006 Madrid, Spain

E-mail address: marina.logares@icmat.es

Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza Ciencias 3, 28040 Madrid Spain

E-mail address: vicente.munoz@mat.ucm.es