A survey on strong reflexivity of abelian topological groups

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Abstract

An Abelian topological group is called strongly reflexive if every closed subgroup and every Hausdorff quotient of the group and of its dual group are reflexive. In the class of locally compact Abelian groups (LCA) there is no need to define "strong reflexivity": it does not add anything new to reflexivity, which by the Pontryagin - van Kampen Theorem is known to hold for every member of the class. In this survey we collect how much of "reflexivity" holds for different classes of groups, with especial emphasis in the classes of pseudocompact groups, ω-groups and P-groups, in which some reflexive groups have been recently detected. In section 3.5 we complete the duality relationship between the classes of P-groups and ω-bounded groups, already outlined in [26].

By no means we can claim completeness of the survey: just an ordered view of the topic, with some small new results indicated in the text.

Introduction

Locally Compact Abelian groups (LCA groups) constitute the corner stone for several mathematical theories, mainly for Abstract Harmonic Analysis. They were initially studied by Pontryagin as the natural class of groups embracing Lie groups. In his remarkable book "Topological groups" he already touches the main topics nowadays called "duality theory for abelian groups".

The central tool for duality theory is the theorem of Pontryagin and van Kampen which establishes that the natural evaluation mapping from an LCA-group into its bidual is a topological isomorphism (See Theorem 32 of [37]). The contribution of van Kampen was to withdraw the "separability", a constrain in Pontryagin's claim.

A topological abelian group G is called reflexive if it satisfies Pontryagin - van Kampen theorem, namely if the canonical mapping α_G from G into its bidual $G^{\wedge\wedge}$ is a topological isomorphism. Obviously when we deal with reflexive groups we are within the class of abelian

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Hausdorff groups. The possibility of finding classes of reflexive groups beyond that of LCA groups was already considered in the 40's. For example,

- Kaplan proved that arbitrary products of LCA groups are reflexive ([32], 1948).
- Smith proved that any reflexive (in the usual sense of Functional Analysis) locally convex space considered as a topological group is a reflexive group. On the other hand, every Banach space is a reflexive group ([40], 1952).
- Banaszczyk proved that any metrizable complete locally convex space is a reflexive group, and observed that there exist noncomplete reflexive groups ([8, (15.2)], 1991).
- More recently Nickolas, Pestov, Galindo and Hernández, Aussenhofer have dealt with reflexivity in the class of free abelian topological groups $A(X)$, on a Tychonoff space X. They give conditions on X so that the corresponding group $A(X)$ is reflexive. ([35] 1977; [38] 1995; [24] 1999; [3] 1999).
- Aussenhofer proved that the group $C(X, \mathbb{T})$ of continuous functions on a hemicompact k-space X into the circle group $\mathbb{T}$ endowed with the compact-open topology is a reflexive group ([3], 1999)
- Hernández and Uspenkij found reflexive groups of the form $C_p(X)$, the additive group of continuous real functions on some spaces X, with the pointwise topology ([29], 2000).
- In the last two years many classes of reflexive precompact groups have been found by Ardanza-Trevijano, Bruguera, Chasco, Domínguez, Galindo, Macario and Tkachenko, [1], [15], [26], [25].

The groups belonging to the classes described above are locally quasi-convex groups. The latter were defined by Vilenkin in [42] in 1951, and popularized by Banaszczyk in his monograph [8]. The simple observation that closed subgroups and Hausdorff quotients of LCA groups are again LCA, and therefore reflexive, leads to another point of view for extending the duality properties of LCA groups: just to consider classes of reflexive groups in which the closed subgroups and the Hausdorff quotients are again reflexive. Varopoulos in [41] already studies the duality properties of subgroups and quotients of a class of reflexive non locally compact groups. In this line, an enforcement of the concept of reflexivity for groups was done for the first time in a remarkable paper by Brown, Higgins, Morris (1975) [12], where they introduce the so called "strong duality". A simplified definition of their notion is given in [8], which states the following: A topological abelian group G is *strongly reflexive* if the closed subgroups and the Hausdorff quotients of G and of its dual group $G^\wedge$ are reflexive. Hofmann and Morris in [31] consider another strengthening of the concept of reflexivity, namely: a reflexive topological group G has *sufficient duality* if for every closed subgroup H of G the quotient G/H has sufficiently many continuous characters and the quotient $G^\wedge/H^\triangleright$ is reflexive. Clearly, a strongly reflexive group has sufficient duality, and in the converse direction it can be proved that for a reflexive group G with sufficient duality, all its closed subgroups are dually closed, dually embedded and have reflexive dual.

Along this survey we will mainly consider the known positive results on strongly reflexive groups, as well as the obstructions found in several classes of reflexive groups to be strongly reflexive. We will only give proofs when we consider them simpler than the original ones, or when they have been isolated from a more general context.

1 Preliminaries

All groups considered are Abelian, therefore we omit this word in the sequel. For a topological group G, the symbol $G^\wedge$ denotes the group of all continuous characters that is, continuous homomorphisms from G into $\mathbb{T}$, the multiplicative group of complex numbers with modulus 1. The zero neighborhood $\mathbb{T}_+ := \{x \in \mathbb{T} : \mathrm{Re} \geq 0\}$ is a pivot set for duality. The group $G^\wedge$ endowed with the compact-open topology is a Hausdorff topological Abelian group called the *dual group* of G. The bidual group $G^{\wedge\wedge}$ is defined as $(G^\wedge)^\wedge$ and $\alpha_G : G \to G^{\wedge\wedge}$ stands for the canonical evaluation mapping.

Theorem 1.1 *(***Pontryagin, van-Kampen** *1935). If G denotes a locally compact Abelian group, the canonical mapping $\alpha_G : G \to G^{\wedge\wedge}$ defined by $\alpha_G(g)(\kappa) := \kappa(g)$, for all $g \in G$ and $\kappa \in G^\wedge$, is a topological isomorphism.*

A subgroup H of a topological group G is said to be:

- *dually closed* if, for every element x of $G \setminus H$, there is a continuous character φ in $G^\wedge$ such that $\varphi(H) = 1$ and $\varphi(x) \neq 1$.
- *dually embedded* if every continuous character defined on H can be extended to a continuous character on G.
- *h-embedded* if every character defined on H can be extended to a continuous character on G.

The *annihilator of* a subgroup $H \subset G$ is defined as the subgroup $H^\triangleright := \{\varphi \in G^\wedge : \varphi(H) = \{1\}\}$. If L is a subgroup of $G^\wedge$ the *inverse annihilator* is defined by $L^\triangleleft := \{g \in G : \varphi(g) = 1, \forall \varphi \in L\}$.

Annhilators are the particularizations for subgroups of the more general notion of polars of subsets. Namely, for $A \subset G$ and $B \subset G^\wedge$, the polar of A is $A^\triangleright := \{\varphi \in G^\wedge : \varphi(A) \subset \mathbb{T}_+\}$ and the inverse polar of B is $B^{\triangleleft} := \{g \in G : \varphi(g) \in \mathbb{T}_+, \forall \varphi \in B\}$. For a topological Abelian group G, it is not difficult to prove that a set $E \subset G^\wedge$ is *equicontinuous* if there exists a neighborhood U of the neutral element in G such that $E \subset U^{\triangleright}$.

Let $f : G \to E$ be a continuous homomorphism of topological groups. The *dual mapping* $f^\wedge : E^\wedge \to G^\wedge$ defined by $(f^\wedge(\chi))(g) := (\chi \circ f)(g)$ is a continuous homomorphism. If f is onto, then $f^\wedge$ is injective. For a closed subgroup H of a topological group G, denote by $p : G \to G/H$ the canonical projection and by $i : H \to G$ the inclusion. The dual mappings $p^\wedge$ and $i^\wedge$ give rise to the natural continuous homomorphisms $\varphi : (G/H)^\wedge \to H^{\triangleright}$ and $\psi : G^\wedge/H^{\triangleright} \to H^\wedge$. Observe that if H is dually embedded, ψ is onto: in general it is not a topological isomorphism, due to the fact that $G^\wedge/H^{\triangleright}$ may not be locally quasi-convex (defined below). It is easy to prove that a closed subgroup H of a topological group G is dually closed if and only if the quotient group G/H has sufficiently many continuous characters to separate points.

If all closed subgroups and Hausdorff quotients of a topological group G and of its dual group $G^\wedge$ are reflexive - for instance, if G is LCA- we obtain four topological isomorphisms derived from the natural inclusions $i : H \hookrightarrow G$ and $i' : L \hookrightarrow G^\wedge$, and the projections $p : G \to G/H$, $p' : G^\wedge \to G^\wedge/L$:

$$\varphi : (G/H)^\wedge \to H^{\triangleright}, \quad \psi : G^\wedge/H^{\triangleright} \to H^\wedge; \quad \varphi' : (G^\wedge/L)^\wedge \to L^{\triangleleft}, \quad \psi' : G/L^{\triangleleft} \to L^\wedge.$$

Reflexive groups are locally quasi-convex, a property introduced by Vilenkin in [42], which we now explain.

- A subset A of a topological group G is called *quasi-convex* if for every $g \in G \setminus A$, there is some $\chi \in A^{\triangleright}$ such that $Re\chi(g) < 0$; equivalently, if $A^{\triangleright\triangleleft} = A$. It is easy to prove that $\alpha_G(A^{\triangleright\triangleleft}) = A^{\triangleright\triangleright} \cap \alpha_G(G)$. In particular, if A is a subgroup, A is quasi-convex if and only if A is dually closed.

- The group G is *locally quasi-convex* if it has a basis of zero neighborhoods whose elements are quasi-convex subsets.

A Hausdorff topological vector space E is locally convex if and only if E considered as an additive group is locally quasi-convex [8, 2.4]. Thus, local quasi-convexity is an extension for groups of the notion of local convexity in vector spaces. Observe that although quasi-convexity can be considered as the counterpart of convexity, it is a property linked to the topology.

Other examples of locally quasi-convex groups are provided by any dual group, say $G^\wedge$. It can be easily proved that the sets $K^{\triangleright}$ where $K \subset G$ is compact, constitute a zero-neighborhood basis for the compact-open topology in $G^\wedge$. On the other hand, the sets of the form $K^{\triangleright}$ are quasi-convex in $G^\wedge$. Thus, any reflexive group is locally quasi-convex, since it is the dual of its character group.

It is straightforward to prove that any subgroup of a locally quasi-convex group is quasi-convex; a Hausdorff quotient of a locally quasi-convex group may not be locally quasi-convex [3, 12.8]. However, in the class of locally compact groups, and in the more general class of nuclear groups, every Hausdorff quotient is locally quasi-convex, see [8, 7.5]. Also quotients of locally quasi-convex groups by compact subgroups are locally quasi-convex [6].

For an abelian Hausdorff topological group G, the Bohr topology $\omega(G, G^\wedge)$ is the topology on G induced by the elements of $G^\wedge$. Comfort and Ross showed in their paper [19] that the topology of a precompact Abelian group G always coincides with $\omega(G, G^\wedge)$. In particular $\omega(G, Hom(G, \mathbb{T}))$ is the maximal precompact group topology on a given abstract Abelian group G; it is the Bohr topology for G endowed with the discrete topology. The following are well-known results which we label for future reference.

(A) For any topological group G, the mapping α_G is k-continuous. In other words, the restriction of α_G to any compact subset $K \subset G$ is continuous. Thus, if the group G is a k-space, then α_G is continuous.

In general, the evaluation mapping α_G is continuous if and only if the compact subsets of $G^\wedge$ are equicontinuous, [3, Proposition 5.10].

(B) If G is locally quasi-convex, α_G is relatively open. (Just observe that for a quasi-convex zero-neighborhood V, $\alpha_G(V) = V^{\triangleright\triangleright} \cap \alpha_G(G)$, where $V^{\triangleright}$ is compact in the compact-open topology of $G^\wedge$).

(C) If G is locally quasi-convex and Hausdorff, then $G^\wedge$ separates the points of G, thus α_G is a one to one homomorphism. Topological groups with sufficiently many continuous characters to separate points are frequently called MAP (maximally almost periodic) groups.

(D) If the only compact subsets of G are the finite ones, then α_G is onto. This derives from the fact that the compact-open topology on $G^\wedge$ coincides with the pointwise convergence topology $\omega(G^\wedge, G)$.

2 Strongly reflexive groups

The best suited groups concerning reflexivity are those groups G such that their closed subgroups and their Hausdorff quotients are reflexive. Formally let us define:

- An abelian topological group G is called *strongly reflexive (s.r.)* if every closed subgroup and every Hausdorff quotient of G and of $G^\wedge$ are reflexive.

The first examples of non locally compact strongly reflexive groups were given in [12], as countable products and sums of real lines and circles. In a natural way Banaszczyk extended this result proving that all countable products and sums of LCA groups are strongly reflexive [9]. He observed that all these examples were included in the larger class of nuclear groups, defined by him and thoroughly studied in [8]. Finally, Aussenhofer proved that all Čech-complete nuclear groups are strongly reflexive [3, 20.40], thus extending the same property obtained in [8] for complete metrizable nuclear groups. As a matter of fact, we do not know any example of strongly reflexive group outside the class formed by Čech-complete nuclear groups and their duals. Außenhofer constructed in 2007 [4] a non reflexive quotient of an uncountable product of integers. With this result she answered a question possed by Banaszczyk in 1990.

Next we study general properties of closed subgroups and Hausdorff quotients and what is lacking in order that a reflexive group be strongly reflexive.

2.1 Duality properties for subgroups of reflexive groups

Let G be a reflexive group, H a closed subgroup of G and L a closed subgroup of $G^\wedge$, then the following facts hold:

(1) The evaluation mapping α_H is relatively open and injective.

(2) The evaluation mapping $\alpha_{G/H}$ is continuous.

(3) If H is dually closed, $\alpha_G(H) = H^{\triangleright\triangleright}$.

(4) If H is dually closed and dually embedded, α_H is open and bijective.

(5) If H is dually closed, $\alpha_{G/H}$ is injective .

(6) $\alpha_{G/H}$ surjective implies $H^\triangleright$ is dually embedded ([17, 1.4]).

(7) If H is dually closed and $\alpha_{G^\wedge/H^\triangleright}$ surjective, then H is dually embedded ([17, 1.4]).

(8) If L is dually closed, there exists a closed subgroup H of G such that $H^\triangleright = L$ ([8, 14.2]) .

The open subgroups and the compact subgroups of a topological group G characterize the reflexivity (or the strong reflexivity) of the original group in the following way:

(9) Let $H \subset G$ be an open subgroup. Then, G is reflexive (s.r.) iff H is reflexive (s.r.).

(10) Assume G has sufficiently many continuous characters and let $K \subset G$ be a compact subgroup. Then, G is reflexive (s.r.) iff G/K is reflexive (s.r.).

Item (1) follows from the fact that subgroups of locally quasi-convex groups are locally quasi-convex and (B) and (C). The proof of (2) is straightforward. (3) holds because α_G is surjective. In order to prove (4), consider the following commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\iota} & G \\ \downarrow{\scriptstyle \alpha_H} & & \downarrow{\scriptstyle \alpha_G} \\ H^{\wedge\wedge} & \xrightarrow{\iota^{\wedge\wedge}} & G^{\wedge\wedge} \end{array}$$

Let $\gamma \in H^{\wedge\wedge}$. Since G is reflexive, $\gamma \circ \iota^{\wedge} = \alpha_G(x)$, for some $x \in G$. Let us see that $x \in H$. Otherwise, there would exist $\chi \in G^{\wedge}$ such that $\chi(H) = \{1\}$ and $\chi(x) \neq 1$. Hence $\iota^{\wedge}(\chi) = 1$ but $\gamma \circ \iota^{\wedge}(\chi) = \alpha_G(x)(\chi) = \chi(x) = 1$, which is a contradiction. (Item (4) is also proved in [36].)

Item (9) is [11, 2.3 and 3.3] and the proof of (10) follows from [11, 2.6, 3.3, 3.4] and [14, 1.4].

3 Reflexivity for subgroups and quotients in some classes of topological groups

3.1 The class of metrizable groups

The Pontryagin duality theory for metrizable groups has some special features that we develop next.

Let G be a metrizable topological group. Then:

(11) The canonical homomorphisms α_G and $\alpha_{G^{\wedge}}$ are continuous.

(12) If G is reflexive, every dually closed and dually embedded subgroup of G is reflexive.

(13) The mappings $\varphi : (G/H)^{\wedge} \to H^{\triangleright}$ and $\psi' : (G^{\wedge}/H^{\triangleright})^{\wedge} \to H^{\triangleright\triangleright}$ are topological isomorphisms, [17, 1.2].

(14) If all the Hausdorff quotients of G and of $G^{\wedge}$ are reflexive, the group G is is strongly reflexive, [17, 2.2].

Item (11) follows from the fact that the dual group $G^{\wedge}$ of a metrizable group G is a k-space ([3] and [16]) and from the simple observation that any metrizable space is a k-space. Now (A) implies that α_G and $\alpha_{G^{\wedge}}$ are continuous.

Item (12) is a corollary of (11) and (4).

3.2 The class of nuclear groups

The class of nuclear groups was formally introduced by Banaszczyk in [8]. A source for inspiration was his previous work [10], where he studied the behaviour of closed subgroups and quotients by closed subgroups of nuclear vector spaces. Earlier he had studied similar questions for Banach spaces, and he was aware that, from some point of view, nuclear spaces -rather than Banach spaces- are natural generalization of finite dimensional vector spaces. So he set out to find a class of topological groups embracing nuclear spaces and locally compact abelian groups (as natural generalizations of finite-dimensional vector spaces). This was the origin of the class of nuclear groups: the definition of the latter in [8] is very technical, as could be expected from its virtue of joining together objects of such different classes. A nice survey on nuclear groups is also provided by L. Aussenhofer in [5]. The following are outstanding facts from the class of nuclear groups:

(E) Nuclear groups are locally quasi-convex, [8, 8.5].

(F) Products, subgroups or quotients of nuclear groups are again nuclear, [8, 7.5].

(G) Every locally compact abelian group is nuclear, [8, 7.10].

(H) A nuclear locally convex space is a nuclear group [8, 7.4]. Furthermore, if a topological vector space E is a nuclear group, then it is a locally convex nuclear space, [8, 8.9].

Let G be a nuclear group and $H \subset G$ a closed subgroup. The following duality results hold:

(15) The canonical homomorphisms α_G and α_H, and $\alpha_{G/H}$ are injective and relatively open.

(16) Closed subgroups of G are dually closed and dually embedded, [8, 8.3] and [8, 8.6].

(17) If the group G is complete, α_G is an open isomorphism, [3, 21.5].

(18) If the group G is a complete k-space, it is reflexive and its closed subgroups are also reflexive.

(19) If the group G is Čech-complete, its dual group $G^\wedge$ is also nuclear [3, 20.31].

(20) If the group G is Čech-complete, it is strongly reflexive, [3, 20.35].

Item (15) derives from (B), (C), (E) and (F). Item (18) can be obtained from (A) and (17).

3.3 The class of precompact groups

A well-known theorem of A. Weil [43] states that precompact groups are subgroups of compact groups. The converse is obvious. Hence, by (F) and (G) precompact groups are nuclear, and thereof the statements of the previous section hold for them. As said above, the topology of a precompact Abelian group G always coincides with $\omega(G, G^\wedge)$, the weak topology corresponding to its character group. In other words, a precompact Abelian group carries the Bohr topology.

In [39] a sort of "reflexivity" is obtained within the class of precompact abelian groups, just considering always the pointwise convergence topology as the natural topology for the character groups. We turn now to the "standard" reflexivity.

Let G be a precompact group:

(21) The equicontinuous subsets in $G^\wedge$ are only the finite subsets.

(22) The evaluation mapping α_G is continuous if and only all the compact subsets of $G^\wedge$ are finite.

(23) If the only compact sets of G and of $G^\wedge$ are finite, the group G is reflexive.

(24) **Question** Are there strongly reflexive precompact noncompact groups?

In order to prove (21), consider an equicontinuous subset A in $G^\wedge$. Then $A^\triangleleft$ is a neighborhood of zero in G and its closure in the completion $\tilde{G}$ of G is also is a neighborhood of zero in $\tilde{G}$, which we call $\overline{A^\triangleleft}$. The set $(\overline{A^\triangleleft})^\triangleright = (A^\triangleleft)^\triangleright$ is compact in $(\tilde{G})^\wedge$ and thus finite. Item (22) is a corollary of (21) and (A). Item (23) is derived from (15), (21) and (D). It can also be derived from the comments opening this subsection.

3.4 The class of pseudocompact groups

A Hausdorff topological group G is said to be pseudocompact if it is pseudocompact as a topological space. Namely, if every continuous real function defined on G is bounded. This property matched with the algebraic structure of the supporting set produces the highly interesting class of pseudocompact groups. It has been intensively studied by Comfort, Dikranjan, Galindo, van Mill, Shakmatov, and some others. The first relevant properties of this class of groups are the following:

- Every pseudocompact group is precompact, [20, 1.1].
- A Hausdorff topological group G is pseudocompact iff it is G_δ-dense in its Stone-Čech compactification βG, [30, Theorem 28].

- The Bohr compactification of a pseudocompact group G coincides with its Stone-Čech compactification βG. Thus, the group operation of G can be continuously extended to βG, [20, Theorem 4.1].

Examples of pseudocompact groups are the Σ-products of compact groups. It can be easily proved that a locally finite family of open sets in a pseudocompact group must be finite. Thus a paracompact pseudocompact group must be already compact. In particular, a metrizable pseudocompact group must be compact, and a pseudocompact group is uncountable.

There is a subclass of pseudocompact groups which deserves special attention, and will be treated in the next section, namely the class of ω-bounded groups. We enumerate now some properties of pseudocompact groups related to reflexivity:

(25) A precompact topological group G is pseudocompact if and only if the countable subgroups of $G^\wedge$ are h-embedded, $[1, 2.3]$ and [27, 3.4].

(26) If G is a pseudocompact group then α_G is continuous, injective and relatively open. Thus, G is reflexive if and only if α_G is surjective.

(27) If G is pseudocompact, then G is a dual group. In fact, G is topologically isomorphic to $(G^\wedge, w(G^\wedge, G))^\wedge$.

(28) If G is pseudocompact and the only compact sets of G are finite, the group G is reflexive, ($[1, 2.5]$ and $[25, 6.1]$).

(29) If G is pseudocompact and its countable subgroups are h-embedded, $G^\wedge$ is also pseudocompact with countable subgroups h-embedded, $[1, 1.2]$.

(30) There exist pseudocompact reflexive non strongly reflexive groups.

(31) **Question** Are there strongly reflexive pseudocompact non-compact groups?

The following facts are very useful in this context:

(I) If a topological group has the property that its countable subgroups are h-embedded, then its compact subsets must be finite $[1, 2.1]$.

(J) If G is a pseudocompact group, then every $w(G^\wedge, G)$-compact subset of $G^\wedge$ is finite [27, 4.4]. Consequently, every compact subset of $G^\wedge$ is finite.

In order to prove (26) take a pseudocompact group G. By (J) (or by (25) + (I)) it follows that the compact subsets of $G^\wedge$ are finite, and therefore α_G is continuous. Since G is locally quasi-convex, α_G is injective and relatively open (see (B) and (C)); thus, the reflexivity of G is reduced to confirm that α_G is surjective. For (27) consider first that $(G^\wedge, w(G^\wedge, G))^\wedge$ may be algebraically identified with G for any topological group G. If G is moreover pseudocompact, it follows from (J) that the compact-open topology in $(G^\wedge, w(G^\wedge, G))^\wedge$ coincides with $w(G, G^\wedge)$.

For the proof of (30) observe that any pseudocompact group G, can be identified with a quotient H/L, where H is a pseudocompact group such that all its countable subgroups are h-embedded and L is a closed pseudocompact subgroup of H [23, 5.5]. By (28) H is reflexive, and the quotient H/L may be non-reflexive. This already proves (30), but it can be seen further that the closed subgroups of reflexive pseudocompact groups neither inherit reflexivity. In fact, take a non reflexive pseudocompact group G and let H and L be as above. Observe that $L^\triangleright$ is a closed subgroup of $H^\wedge$ and $(L^\triangleright)^\wedge \cong \frac{H^{\wedge\wedge}}{L^{\triangleright\triangleright}}$. Since L is dually closed and H is reflexive, $\frac{H^{\wedge\wedge}}{L^{\triangleright\triangleright}} \cong H/L \cong G$ and consequently $H^\wedge$ is a pseudocompact reflexive group with a non reflexive closed subgroup $L^\triangleright$.

Observe that if (31) had a positive answer, witnessed by a group G in the situation of item (29), all countable subgroups of G would be discrete. [Countable subgroups are h-embedded and are endowed with the maximal precompact topology, hence their dual groups are compact. On the other hand if the countable subgroups are h-embedded, they must be closed [1, 2.1], and the assumption of strong reflexivity of G implies that they are reflexive, thus discrete]. Moreover G must be sequentially complete, [1, 2.1].

3.5 ω-bounded groups and P-groups

We study together the two classes mentioned in the title because they are closely interrelated by duality, in the following sense: the dual of a P-group is ω-bounded and any ω-bounded group is the dual of some P-group (see (32) and (42)).

A topological group G is said to be ω-bounded if every countable subset $M \subset G$ is contained in a compact subset of G. Clearly, "countable subset" may be replaced by "countable subgroup" in the definition of ω-bounded group.

The following facts about ω-bounded groups have straightforward proofs:

- Every ω-bounded group is pseudocompact, and hence precompact.
- If G is a separable ω-bounded group, then G is compact.

Every ω-bounded group G carries the topology $w(G, G^\wedge)$ and, as stated in (27), G can be identified with the dual of $(G^\wedge, w(G^\wedge, G))$.

We recall that a topological space X is a P-space if all of its G_δ-sets are open. An Abelian topological group which is a P-space is called a P-group. For general properties on P-spaces and P-groups the reader can consult [2], where a whole section is devoted to them. We only mention here what is needed for our aims.

A P-group has a basis of neighborhoods of the neutral element consisting of open subgroups and hence, it can be embedded in a product of discrete groups. By (F) and (G) it can be asserted that the class of P-groups is included in that of nuclear groups. Thus, for a P-group G, the corresponding mapping α_G is injective and relatively open (by (15)). It is easy to check that the compact subsets of a P-group G are finite, and hence the compact-open topology in $G^\wedge$ coincides with the pointwise convergence topology $w(G^\wedge, G)$.

Any topological group (G, τ) gives rise to a P-group. In fact, let $P\tau$ denote the topology generated by the G_δ subsets of τ. The pair $(G, P\tau)$ will be called the *P-modification* of (G, τ), or simply the P-modification of G if there is no confusion on which topology we are dealing with. We now state the facts about reflexivity known to hold for P-groups.

Let G be a P-group.

(32) The closure of every countable subset of $G^\wedge$ is equicontinuous and therefore compact in the compact-open topology. Consequently, the dual group $G^\wedge$ is ω-bounded.

(33) The evaluation mapping $\alpha_{G^\wedge}$ is continuous.

(34) If G is reflexive, then the countable subgroups of G are h-embedded.

(35) There are nondiscrete reflexive P-groups. The P-modification of a product of c-many discrete groups is an example, [26].

(36) Hausdorff quotients of reflexive P-groups are reflexive, [26].

(37) A reflexive nondiscrete P-group may not be strongly reflexive.

(38) **Question** Are there nondiscrete strongly reflexive P-groups?

In order to prove (32) take a countable subset of $G^\wedge$, say $S := \{\psi_n, n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, $\psi_n^{-1}(\mathbb{T}_+)$ is a neighborhood of zero in G, and $V := \bigcap_{n\in\mathbb{N}} \psi_n^{-1}(\mathbb{T}_+)$ is also a neighborhood of zero in the P-group G. Hence $V^\triangleright$ is an equicontinuous subset of $G^\wedge$ which contains S. By Ascoli theorem $V^\triangleright$ is compact in the compact-open topology, which as mentioned above coincides with $w(G^\wedge, G)$. Thus $\overline{S}$ is equicontinuous and compact, and the last assertion of (32) is also proved. Item (33) derives from the fact that $G^\wedge$ is ω-bounded, in particular pseudocompact and (26) applies. For (34) just observe that $G^\wedge$ pseudocompact implies that the countable subgroups of $G^{\wedge\wedge}$ are h-embedded, and G is topologically isomorphic to $G^{\wedge\wedge}$.

Item (37) can be derived from a classical example of Leptin in [34], recalled in [26]. It provides a non reflexive group which is a closed subgroup of the P-modification of the product of c copies of the binary group $\{0,1\}$. Hence, the P-modification itself is an example of reflexive non strongly reflexive P-group.

3.5.1 The duality between ω-bounded groups and P-groups

Some aspects of this duality have been treated in [26]. We give in (41) a sufficient condition for the reflexivity of ω-bounded groups, and we prove in (42) that a topological group is ω-bounded if and only if it is the dual of a P-group. Apparently these results were not known. The following assertion will be a tool for our proofs:

Proposition 3.1 *For an ω-bounded group G denote by $G^{\wedge}_w := (G^{\wedge}, w(G^{\wedge}, G))$ and by $G^{\wedge}_{Pw} := (G^{\wedge}, Pw(G^{\wedge}, G))$. Then $G^{\wedge}_w$ and $G^{\wedge}_{Pw}$ have the same dual group, namely $(G, w(G, G^{\wedge}))$. Further $Pw \leq \tau_{co}$, where the latter is the compact-open topology in $G^{\wedge}$.*

Proof.

The precompactness of $G^{\wedge}_w$ gives the identification between G and $(G^{\wedge}_w)^{\wedge}$. In [21] it is proved that any character $\psi : G^{\wedge} \to \mathbb{T}$ which is $Pw(G^{\wedge}, G)$-continuous is also $w(G^{\wedge}, G)$-continuous. This and the inequality $w(G^{\wedge}, G) \leq Pw(G^{\wedge}, G)$ prove that $G^{\wedge}_w$ and $G^{\wedge}_{Pw}$ have the same character group, G. Further, the corresponding compact-open topologies coincide: in fact, the only $w(G^{\wedge}, G)$-compact subsets of $G^{\wedge}$ are the finite ones by (J) and the only $Pw(G^{\wedge}, G)$-compact subsets are also finite because $G^{\wedge}_{Pw}$ is a P-group. So, in both cases the compact-open topology coincides with the pointwise convergence topology $w(G, G^{\wedge})$.

Let us prove now that any G_δ subset of $G^{\wedge}_w$ is open in τ_{co}. Take $V := \bigcap_{n\in\mathbb{N}} F_n^{\triangleright}$, with $F_n \subset G$ finite. Then $V = (\bigcup F_n)^{\triangleright}$. Since $\bigcup F_n$ is contained in a compact subset of G, we get that V is a zero neighborhood in $(G^{\wedge}, \tau_{co})$.

Clearly if the compact subsets of G_w are separable, then $Pw = \tau_{co}$.

□

Remark. 1) Let G be an ω-group. Observe that $w(G^{\wedge}, G)$ and $Pw(G^{\wedge}, G)$ are in general different topologies in $G^{\wedge}$. However, $(G^{\wedge}, w(G^{\wedge}, G))^{\wedge} = (G^{\wedge}, Pw(G^{\wedge}, G))^{\wedge}$, as proved in Proposition 3.1. For the original group G the situation is totally different: $w(G, G^{\wedge})$ is the only locally quasi-convex topology on G for which the character group is $G^{\wedge}$ ([33], Theorems 8.44 and 8.50). This holds in a more general context, as we state in the next Proposition.

Proposition 3.2 *Let G be a precompact group such that all compact subsets of $(G^\wedge, w(G^\wedge, G))$ are finite. Then $w(G, G^\wedge)$ is the only locally quasi-convex topology on G with character group $G^\wedge$.*

Let G be an ω-bounded group. Then,

(39) The evaluation mapping α_G is continuous open and injective. Therefore G is reflexive if and only if α_G is surjective.

(40) If every compact subset of G is contained in the closure of a countable set, then $Pw(G^\wedge, G) = \tau_{co}$, where τ_{co} denotes the compact-open topology in $G^\wedge$.

(41) G is reflexive if every compact subset of G is contained in the closure of a countable set.

(42) G is the dual of a P-group. On the other hand, the dual group of a P- group is ω- bounded.

(43) If G is reflexive, the closed subgroups of G are reflexive.

(44) If every compact subset of G is contained in the closure of a countable set then G has sufficient duality.

(45) A reflexive noncompact ω-bounded group may not be strongly reflexive.

(46) **Question** Are there noncompact strongly reflexive ω-bounded groups?

The proof of (39) is covered by (26) since the ω-bounded group G is in particular pseudocompact. The proof of (40) is is included in that of (41).

In order to prove item (41), assume that G is an ω-bounded group for which every compact subset is contained in the closure of a countable set. We must check that α_G is surjective. To that end, take a continuous character $\varphi : G^\wedge \to \mathbb{T}$. There exists then a compact subset K in G such that $\varphi(K^\triangleright) \subset \mathbb{T}_+$. By the assumption, we can find a countable set $L = \{x_n, n \in \mathbb{N}\} \subset G$ such that $K \subset \overline{L}$. Now

$$K^\triangleright \supset \overline{\{x_n, n \in \mathbb{N}\}}^\triangleright = \{x_n, n \in \mathbb{N}\}^\triangleright = \bigcap_{n \in \mathbb{N}} \{x_n\}^\triangleright,$$

which is a zero neighborhood in $Pw(G^\wedge, G)$. Since $\varphi(\bigcap_{n\in\mathbb{N}} \{x_n\}^\triangleright) \subset \mathbb{T}+$, we obtain that φ is a continuous character on $(G^\wedge, Pw(G^\wedge, G))$. By Proposition 3.1, $\varphi \in (G^\wedge, w(G^\wedge, G))^\wedge = G$ and thus $\varphi = \alpha_G(g)$ for some $g \in G$.

Item (42): If G is ω- bounded, then $G = (G^\wedge, Pw(G^\wedge, G))^\wedge$ by Proposition 3.1. The last assertion is (32) Item(43): Closed subgroups H are dually closed and dually embedded by (16), and α_H is open and bijective by (4). Moreover, since H is ω-bounded, α_H is continuous (39). Therefore H is reflexive.

Item (44): Since ω-bounded groups are nuclear, every closed subgroups H of G is dually closed by (16) or equivalently the continuous characters of G/H separate points. On the other hand, G is reflexive by (41), and this implies that $G^\wedge$ is also reflexive. The condition that every compact subset of G is contained in the closure of a countable set implies by (40) that $G^\wedge$ is a P-group, and now from (42) we obtain that all the Hausdorff quotients of $G^\wedge$ are reflexive. Thus, G has sufficient duality.

Item (45): Take the reflexive P-group G of item (37). Then $G^\wedge$ is reflexive but not strongly reflexive, for otherwise its dual group G would be also strongly reflexive.

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