ON THE RELEVANCE OF OWA RULES

J. Montero, A. del Amo
Faculty of Mathematics
Complutense University
Madrid, Spain

E. Molina
Faculty of Applied Sciences
Miguel Hernández University
Elche, Spain

V. Cutello
Department of Mathematics
University of Catania
Catania, Italy

Abstract

In this paper we discuss the relevance of some Ordered Weighted Average (OWA) operators, as far as they define a recursive rule. In particular, it is shown how important are these OWA rules under certain assumptions leading to an additive representation of a recursive aggregation procedure.

Key words: Fuzzy Connective Rules, Recursiveness

1 Introduction.

OWA operators where introduced by Yager (see, e.g., [10]), in order to capture key aggregation operators in between the *min* and the *max* rules. Among these rules we find, for example, the regular mean. But such a mean is not a binary operator, like those two classical operators *min* and the *max*, or more in general, most classical connectives considered in fuzzy set theory (t-norms and t-conorms).

Taking into account the key role that associativity plays in order to assure a consistent aggregation by means of a unique binary operators, the concept of *recursiveness* has been introduced [2, 3, 4, 5], generalizing associativity and allowing a consistent definition of the concept of *rule*. Indeed, classical approaches assume the existence of a unique commutative binary operator (see, e.g., [6, 7, 9, 11]).

Recursiveness shows that *consistency* can be also assured allowing the binary operators evolve. This is the case, for example, when dealing with the mean rule: any mean can be evaluated by means of a sequence of binary connectives, obviously depending on the number of items already aggregated in each step.

A *recursive rule* is a *consistent* family of connectives

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

being each operator ϕ_n the one to be applied when the number of items is n (all through out this paper, all those connectives will be assumed continuous and non-decreasing in each coordinate).

2 Re-arrangement of data.

Many aggregation procedures require a previous re-arrangement of data, by means of an *ordering rule*. It may because there is a *natural* order in the evaluation space (this is the case in OWA operators). But data can also be ordered according to an *external* order (real time, for example).

Definition 2.1. Let us denote

$$\pi_n(a_1, a_2, \ldots, a_n) = (a_{\pi_n(1)}, a_{\pi_n(2)}, \ldots, a_{\pi_n(n)})$$

An *ordering rule* π is a *consistent* family of permutations $\{\pi_n\}_{n>1}$ such that for any possible finite collections of numbers, each extra item a_{n+1} is allocated keeping relative positions of items, i.e.,

$$\pi_{n+1}(a_1, a_2, \ldots, a_n, a_{n+1}) = (a_{\pi_n(1)}, \ldots, a_{\pi_n(j-1)}, a_{\pi_{n+1}(j)}, a_{\pi_n(j+1)}, \ldots)$$

for some $j \in \{1, \ldots, n+1\}$.

3 Recursiveness.

The following definition was then proposed in [5].

Definition 3.1. A left-recursive connective rule is a family of connective operators

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0,1]^2 \to [0,1]\}_{n>1}$$

verifying

$$\phi_2(a_1, a_2) = L_2(a_{\pi(1)}, a_{\pi(2)})$$

and for all $n > 2$

$$\phi_n(a_1, \ldots, a_n) = L_n(\phi_{n-1}(a_{\pi(1)}, \ldots, a_{\pi(n-1)}), a_{\pi(n)})$$

for some ordering rule π.

Right recursiveness can be analogously defined, and a *recursive rule* is a left-and-right recursive rule (both being based on the same ordering rule).

Of course, ordering may not by relevant at all for some particular aggregation procedures. Those are *commutative* recursive rules.

By means of simplicity, main result of this paper is restricted to recursive rules based upon the identity ordering rule (*standard recursive rules*).

4 Additive recursive rules.

Standard additive recursive rules are defined by means of weighted means. Lets check which particular weighted means. We just introduce a small restriction in order to get the main result of this paper.

Definition 4.1. A *regular recursive connective rule* is a family of connective operators

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0,1]^2 \to [0,1]\}_{n>1}$$

and

$$\{R_n : [0,1]^2 \to [0,1]\}_{n>1}$$

which are surjective continuous mappings verifying the following conditions:

1. If $x' \leq x''$ and $y' \leq y''$, then $L_n(x', y') \leq L_n(x'', y'')$ and $R_n(x', y') \leq R_n(x'', y'')$

   If $x' < x''$ and $y' < y''$, then $L_n(x', y') < L_n(x'', y'')$ and $R_n(x', y') < R_n(x'', y'')$

2. If $x' < x''$, then $L_n(x', y) < L_n(x'', y)$ for some y and $R_n(x', y) < R_n(x'', y)$ for some y

   If $y' < y''$, then $L_n(x, y') < L_n(x, y'')$ for some x and $R_n(x, y') < R_n(x, y'')$ for some x

3. $\nexists x_i, x'_i \in (0,1)$ such that $L_n(x_i, \bar{x}) = R_n(x'_i, \bar{x}) = 0$ $\forall \bar{x} \in (0,1)$

   $\nexists x_i, x'_i \in (0,1)$ such that $L_n(\bar{x}, x_i) = R_n(\bar{x}, x'_i) = 0$ $\forall \bar{x} \in (0,1)$

4. $L_n(0, y') = 0, \quad \forall y' \in [0,1] \iff L_n(y', 0) = 0 \quad \forall y' \in [0,1]$

   $R_n(0, y'') = 0 \quad \forall y'' \in [0,1] \iff R_n(y'', 0) = 0 \quad \forall y'' \in [0,1]$

$$L_n(1,y') = 1 \quad \forall y' \in [0,1] \iff$$
$$L_n(y',1) = 1 \quad \forall y' \in [0,1]$$
$$R_n(1,y'') = 1 \quad \forall y'' \in [0,1] \iff$$
$$R_n(y'',1) = 1 \quad \forall y'' \in [0,1]$$

Theorem 4.1. *Let*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

be a regular standard recursive rule. If ϕ_n is strictly increasing in each coordinate for all $n > 1$, then there exists:

(a) $p : [0,1] \to \mathbb{R}_+$*, continuous and strictly increasing function*

(b) $\{\delta_n : [0,1] \to \mathbb{R}_+\}_{n>1}$*, family of continuous and strictly increasing functions, and*

(c) $\{c_n\}_{n\geq 1}$*, sequence of positive real numbers*

in such a way that

$$\phi_n(a_1,\ldots,a_n) = \delta_n^{-1}\left(\prod_{j=2}^{n-2} c_j \sum_{k=1}^{n} c_1^{k-1} p(a_k)\right)$$

for all $(a_1,\ldots,a_n) \in [0,1]^n$ *and for all* $n \geq 2$*, where* $\prod_{j=2}^{\ell} c_j$ *is taken as 1 whenever* $\ell \leq 2$*.*

Moreover, given $\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$ a recursive connective rule in the above theorem conditions, then $\{L_n\}_{n>1}$ and $\{R_n\}_{n>1}$, are unique in their range (see [1]).

5 Idempotent standard additive recursive rules.

We can then expect that additional conditions assumed to our standard additive recursive rule may evantually fix functions $\{\delta_n\}_{n>1}$ and p, and parameters $\{c_n\}_{n\geq 1}$.

An usual assumption is indeed *idempotency* (i.e., in case aggregated values are all the same, then the aggregated value is exactly such a value).

Theorem 5.1. *Let*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

be a regular standard recursive rule such that ϕ_n is strictly increasing and idempotent, for all $n > 1$. Then there exists $p : [0,1] \to \mathbb{R}_+$, continuous and strictly increasing function, and a real number $c > 0$, such that for all $(a_1,\ldots,a_n) \in [0,1]^n$ and all $n \geq 2$,

$$\phi_n(a_1,\ldots,a_n) = p^{-1}\left(\frac{1-c}{1-c^n}\sum_{k=1}^{n} c^{k-1} p(a_k)\right)$$

if $c \neq 1$, and

$$\phi_n(a_1,\ldots,a_n) = p^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} p(a_k)\right)$$

if $c = 1$

Hence, being $\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$ a rule verifying the above theorem conditions, for each $n > 2$, there exists a weights system $w_{i,n}$, $i = 1,\ldots,n$, and a function p, such that:

$$\phi_n(a_1,\ldots,a_n) = p^{-1}\left(w_{1,n}p(a_1) + \cdots + w_{n,n}p(a_n)\right)$$

for all $(a_1,\ldots,a_n) \in [0,1]^n$, where

(i) For all $i = 1,\ldots,n$, and for all $n \geq 2$

$$w_{i,n} = \begin{cases} \frac{1-c}{1-c^n}c^{i-1} & \text{if } c \neq 1 \\ \frac{1}{n} & \text{if } c = 1 \end{cases}$$

(ii) $\displaystyle\sum_{i=1}^{n} w_{i,n} = 1$.

Furthermore, $\forall n > 1$, there exist:

$$\ell_n = \begin{cases} \frac{1-c^{n-1}}{1-c^n} & \text{if } c \neq 1 \\ 1-\frac{1}{n} & \text{if } c = 1 \end{cases}$$

and

$$r_n = \begin{cases} \frac{1-c}{1-c^n} & \text{if } c \neq 1 \\ \frac{1}{n} & \text{if } c = 1 \end{cases}$$

such that

$$L_n(a,b) = p^{-1}\big(\ell_n p(a) + (1-\ell_n)p(b)\big)$$
$$R_n(a,b) = p^{-1}\big(r_n p(a) + (1-r_n)p(b)\big)$$

Therefore, every regular standard recursive rule that is strictly increasing and idempotent, is made up of aggregation operators that are *quasilinear means* with *generator function* p and weights ℓ_n, $1-\ell_n$ and r_n, $1-r_n$ for L_n and R_n, respectively.

Being π the natural decreasing order, a recursive rule family of special interest shows up. Concretely, being $\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$ the regular recursive rule and π the underlying natural ordering rule in the unit interval, all the connective operators are *quasi–OWA* operators (see Fodor-Marichal-Roubens [8]).

Moreover, it must be pointed out that once the first aggregation operator ϕ_2 is a known mean, then the whole rule is fully settled (there is only one family of weights being *recursively consistent*).

6 Final comments

As shown above, a rule is actually quite determined from the first choices. Each time we choose a particular operator (being consistent to previous choices), less consistent options are left. It may be the even case that the whole rule is fixed once the first aggregation has been defined. This result is coherent with a true intuitive fact: if the first two items are aggregated according to a regular mean, we do not expect that the next aggregation is made according to a maximum or any other aggregation different than a weighted mean (and indeed not an arbitrary one).

Under the above conditions, decision maker will be defining a recursive rule based upon weighted means *once data have been properly ordered*. In this sense, we find out that such a recursive rule is an *OWA-like* recursive rule (an OWA rule in case the ordering rule is the natural order in the real unit interval).

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