Generalized grating imaging using an extended monochromatic light source

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It is a well-known fact that one grating can act as an imaging element for another grating when the first is illuminated with an extended monochromatic light source. The conditions for image formation in such a system are studied when the finite size and position of the broad light source are considered. From the presented analysis, expressions for the location and the depth of focus of such images can be derived. [S0740-3232(00)00907-8]

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1. INTRODUCTION

There are several processes for creating images of a grating by using only gratings as imaging elements. The best known of these processes is the Talbot effect[1,2]; self-images of a grating appear at certain distances from it when the grating is illuminated with collimated monochromatic light. In this case the wavelike nature of light, jointly with the periodic structure of the grating, produces its self-images.

There are other cases in which one grating acts as an imaging element for another grating. The Lau effect[3] is the best-known example of such a process. In this case two gratings of the same pitch placed parallel to each other and separated by a certain distance are illuminated by using an extended monochromatic source, and a pseudoimage of the first grating is formed at infinity. The term pseudoimage is used because the intensity distribution on the observation plane (in this case located at infinity) is a set of well-modulated fringes but with a fringe profile that could be different from that of the first grating. For example, if the first and second gratings are Ronchi gratings, a set of triangular fringes appears at infinity. The formation of pseudoimages at infinity with two gratings of the same pitch has also been studied for partially coherent and coherent illumination.[4] Also, a study of the Lau effect within the theory of partial coherence has been presented by Gori.[5]

In the case in which the second grating has a slightly different pitch from that of the first, pseudoimages of the first grating may appear at a certain number of discrete distances from the second grating.[6,7] The formation of these pseudoimages is well understood when the source is considered of infinite extent and placed at infinity or at the focal plane of a lens. In this case the light incident on the system is treated as an incoherent sum of planar wave fronts with different orientations; this approach substantially simplifies the following analysis. But when this formalism is used, the depth of focus of the images formed is exactly zero, which means that on an observation plane infinitely close to the theoretical plane of image formation we would get no fringes at all. This instability of the pseudoimages is a consequence of considering the light source to be of infinite extent and placed at infinity, as will be shown in this work.

We have used a different approach, which allows us to estimate the depth of focus of the pseudoimages. We have obtained expressions for the total intensity distribution on the observation plane that take into account the limited size of the source, equivalent to those derived by Olszak and Wronkowski[8] with a different method. We show how it is possible to obtain, from these expressions, information about the conditions for the formation of pseudoimages and their focal depth. We have done this work motivated by the importance that the depth of focus has for any practical application that tries to take advantage of this grating imaging phenomenon.

In this work an analysis is presented of the process of pseudoimage formation considering the use of a finite light source placed at a finite distance from the gratings. The light incident on the system is treated as the incoherent sum of the spherical waves emerging from each point on the emitting surface. With this approach it is possible to derive expressions for the depth of focus of the pseudoimages that take into account all the relevant parameters of the optical system.

In Section 2 we derive the general expressions for the intensity distribution on the observation plane when the limited size of the light source is considered. In Section 3 we determine the conditions for the existence of pseudoimages and derive expressions for their focal depth and stability. In Section 4 we present a complete study of the particular case when both of the gratings in our system are Ronchi gratings. This example is useful for better understanding the conditions for the formation of the pseudoimages and their properties. We have included in Appendix A some calculations that are necessary to obtain the results presented in Section 3.

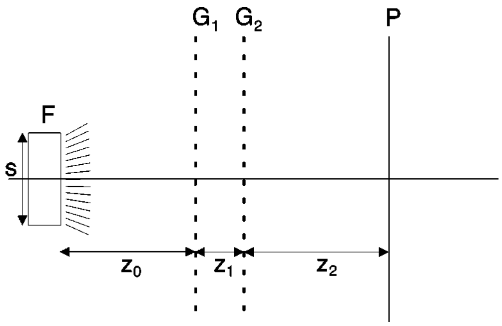

Fig. 1. Optical system used to obtain pseudoimages of the first grating G_1 on the observation plane P.

2. GENERAL ANALYSIS

The optical system for pseudoimage formation is the one depicted in Fig. 1. It consists of an extended monochromatic light source F and two periodic gratings G_1 and G_2, each of which can have a different pitch. At first, no assumption is made on the amplitude or phase profile of these gratings, besides the fact that they are periodic. The distance between the light source and the first grating is z_0, and the distance between both gratings is z_1. The size of the light source along the axis perpendicular to the pitch of the gratings is S, the source size along the gratings' lines not being relevant to our discussion.

The goal of this analysis is to obtain the light distribution on a plane P located at a distance z_2 from the second grating. The method employed in the analysis will be a series of propagations using the Fresnel approximation from one element of the system to the next, the effect of each grating on the incident wave front being introduced through its complex transmittance.

Because of the symmetry of the arrangement that we are going to analyze, the extended light source can be treated as one dimensional. Each point from the source emits a wave front that generates a certain field distribution on the plane where the first grating is located. It will be considered that all the points from the source are emitting light incoherently from each other. It will also be considered that they are all emitting light with identical angular distributions, so the field distribution on the plane of the first grating corresponding to the light emitted from a single point can be written as

$$\Psi(x, x_i) = \phi(x - x_i), \tag{1}$$

where x_i is the spatial coordinate of a point on the source plane and x is the spatial coordinate of a point on the plane where the first grating is located. Here $\phi(x)$ would be the field originating from a point located on the origin of the source coordinate system and propagated to the plane of the first grating.

The first grating is characterized by its complex transmittance. Since the grating is periodic with period p_1, its transmittance can be expressed by its Fourier expansion as

$$t_1(x) = \sum_n a_n \exp(inq_1x), \tag{2}$$

where $q_1 = 2\pi/p_1$ is the spatial frequency of the first grating.

The field distribution right behind the first grating that is due to the light emitted from a single point on the source plane will be given by

$$\begin{aligned} f_1(x, x_i) &= t_1(x)\phi(x - x_i) \\ &= \sum_n a_n \exp(inq_1x)\phi(x - x_i). \end{aligned} \tag{3}$$

The Fourier transform of this field distribution will be necessary for the following calculations. It will be useful to write this Fourier transform in the following way:

$$\begin{aligned} F_1(k_x, x_i) &= \mathrm{FT}[f_1(x, x_i)] \\ &= \sum_n a_n \exp[-i(k_x - nq_1)x_i] \\ &\quad \times \Phi(k_x - nq_1), \end{aligned} \tag{4}$$

where $\Phi(k_x) = \mathrm{FT}[\phi(x)]$.

If we use expression (4) and the Fresnel approximation to study the propagation of the light field along the z axis, the field distribution right before the second grating will be given by

$$\begin{aligned} f_2(x, x_i) = \sum_n a_n \int_{-\infty}^{\infty} &\exp\left[i\left(k_x x - k_x^2\frac{z_1}{2k}\right)\right] \\ &\times \exp[-i(k_x - nq_1)x_i]\Phi(k_x - nq_1)\mathrm{d}k_x. \end{aligned} \tag{5}$$

The second grating, G_2, has a period p_2 and a spatial frequency $q_2 = 2\pi/p_2$. The field transmittance of this grating can also be written as a Fourier series:

$$t_2(x) = \sum_m b_m \exp(imq_2x). \tag{6}$$

When use is made of the previous two expressions, the field distribution right after the second grating will be

$$\begin{aligned} f_3(x, x_i) &= f_2(x, x_i)t_2(x) \\ &= \sum_n \sum_m a_n b_m \exp(imq_2x) \\ &\quad \times \int_{-\infty}^{\infty} \exp\left[i\left(k_x x - k_x^2\frac{z_1}{2k}\right)\right] \\ &\quad \times \exp[-i(k_x - nq_1)x_i]\Phi(k_x - nq_1)\mathrm{d}k_x. \end{aligned} \tag{7}$$

The Fourier transform of this expression can be written after some manipulations as

$$\begin{aligned} F_3(k_x, x_i) &= \mathrm{FT}[f_3(x, x_i)] \\ &= \sum_n \sum_m a_n b_m \exp[-i(k_x - nq_1 - mq_2)x_i] \\ &\quad \times \Phi(k_x - nq_1 - mq_2) \\ &\quad \times \exp\left[-i(k_x - mq_2)^2\frac{z_1}{2k}\right]. \end{aligned} \tag{8}$$

The Fresnel approximation is used again to obtain the light field propagated to the plane of observation P that is located at a distance z_2 from the second grating. This new field distribution will be given by the expression

$$f_4(x, x_i) = \sum_n \sum_m a_n b_m \exp[i(nq_1 + mq_2)x] \times \exp\left[-i(nq_1 + mq_2)^2 \frac{z_2}{2k}\right] \exp\left(-in^2 {q_1}^2 \frac{z_1}{2k}\right) \times \int_{-\infty}^{\infty} \exp\left\{ik_x\left[x - (nq_1 + mq_2) \times \frac{z_2}{k} - nq_1 \frac{z_1}{k}\right]\right\} \exp\left(-i{k_x}^2 \frac{z_1 + z_2}{2k}\right) \times \exp(-ik_x x_i)\Phi(k_x)\mathrm{d}k_x . \tag{9}$$

Finally, the intensity distribution on the observation plane P that is due to the light emitted from a single point on the source will be

$$I_i(x, x_i) = |f_4(x, x_i)|^2 = \sum_n \sum_{n'} \sum_m \sum_{m'} a_n a^*_{n'}, b_m b^*_{m'} \exp\{i[q_1(n - n') + q_2(m - m')]x\} \exp\left[-i(q_1 n + q_2 m)^2 \frac{z_2}{2k}\right] \times \exp\left(-i{q_1}^2 n^2 \frac{z_1}{2k}\right) \exp\left[i(q_1 n' + q_2 m')^2 \times \frac{z_2}{2k}\right] \exp\left(i{q_1}^2 n'^2 \frac{z_1}{2k}\right) \times \int_{-\infty}^{\infty} \exp\left\{ik_x\left[x - (nq_1 + mq_2) \times \frac{z_2}{k} - nq_1 \frac{z_1}{k}\right]\right\} \exp\left(-i{k_x}^2 \frac{z_1 + z_2}{2k}\right) \times \exp(-ik_x x_i)\Phi(k_x)\mathrm{d}k_x \times \int_{-\infty}^{\infty} \exp\left\{-ik'_x\left[x - (n'q_1 + m'q_2) \times \frac{z_2}{k} - n'q_1 \frac{z_1}{k}\right]\right\} \exp\left(ik'^2_x \frac{z_1 + z_2}{2k}\right) \times \exp(ik'_x x_i)\Phi^*(k'_x)\mathrm{d}k'_x . \tag{10}$$

At this point we cannot go further without any assumption for the function $\phi(x)$ and its Fourier transform. The paraxial (Fresnel) approximation for a spherical wave is

$$\phi(x) = \exp\left(ix^2 \frac{k}{2z_0}\right), \tag{11}$$

and its Fourier transform (ignoring multiplicative constants) is

$$\Phi(k_x) = \exp\left(-i{k_x}^2 \frac{z_0}{2k}\right). \tag{12}$$

The substitution of Eq. (12) into Eq. (10) allows us to solve the integrals, so we finally get

$$I_i(x, x_i) = \sum_n \sum_{n'} \sum_m \sum_{m'} a_n a^*_{n'} b_m b^*_{m'} \times \exp\{i[q_1(n - n') + q_2(m - m')]x\} \times \exp\left[i \frac{z_0}{z_T}(n^2 - n'^2){q_1}^2 \frac{z_1 + z_2}{2k}\right] \times \exp\left[i \frac{z_0 + z_1}{z_T}(m^2 - m'^2){q_2}^2 \frac{z_2}{2k}\right] \times \exp\left[i \frac{z_0}{z_T}(nm - n'm')q_1 q_2 \frac{z_2}{2k}\right] \times \exp\left\{-i \frac{x - x_i}{z_T}[(n - n')q_1(z_1 + z_2) + (m - m')q_2 z_2]\right\}, \tag{13}$$

where $z_T = z_0 + z_1 + z_2$ is the total distance from the source to the observation plane.

The incoherent light source has a linear size S, and it is placed at a distance z_0 from the first grating. We will also consider that it is centered on the origin of coordinates. Accordingly, the total intensity distribution on the observation plane P will be

$$I_S(x) = \int_{-S/2}^{S/2} I_i(x, x_i)\mathrm{d}x_i . \tag{14}$$

Substituting Eq. (13) into Eq. (14), we obtain the following expression for the total intensity on the observation plane:

$$I_S(x) = \sum_n \sum_{n'} \sum_m \sum_{m'} a_n a^*_{n'} b_m b^*_{m'} \times \exp\left\{ix\left[q_1 \frac{z_0}{z_T}(n - n') + q_2 \frac{z_0 + z_1}{z_T}(m - m')\right]\right\} \times \exp\left[i \frac{z_0}{z_T}(n^2 - n'^2){q_1}^2 \frac{z_1 + z_2}{2k}\right] \times \exp\left[i \frac{z_0 + z_1}{z_T}(m^2 - m'^2){q_2}^2 \frac{z_2}{2k}\right] \times \exp\left[i2 \frac{z_0}{z_T}(nm - n'm')q_1 q_2 \frac{z_2}{2k}\right] \times \operatorname{sinc}\left\{\frac{S}{z_T}[(n - n')q_1(z_1 + z_2) + (m - m')q_2 z_2]\right\}, \tag{15}$$

where the notation $\operatorname{sinc}(x) = \sin(x)/x$ is used.

It is convenient now to introduce the following dimensionless quantities:

$$R = \frac{q_2}{q_1}, \qquad Z_0 = \frac{{q_1}^2 z_0}{\pi k}, \qquad Z_1 = \frac{{q_1}^2 z_1}{\pi k},$$

$$Z_2 = \frac{{q_1}^2 z_2}{\pi k}, \qquad Z_T = \frac{{q_1}^2 z_T}{\pi k}, \tag{16}$$

where R is the ratio between the spatial frequencies of the first and the second grid and Z_0, Z_1, Z_2, and Z_T are, respectively, the distances z_0, z_1, z_2, and z_T normalized to half the Talbot distance[9] of the first grating.

Using these dimensionless quantities, we can write the total intensity distribution on the observation plane as

$$\begin{aligned} I_S(x) = {} & \sum_n \sum_{n'} \sum_m \sum_{m'} a_n a_{n'}^* b_m b_{m'}^* \\ & \times \exp\left\{ ix \frac{q_1}{Z_t} [Z_0(n - n') + R(Z_0 + Z_1) \right. \\ & \left. \times (m - m')] \right\} \exp\left[i \frac{\pi}{2} \frac{Z_0}{Z_T} (n^2 - n'^2)(Z_1 + Z_2) \right] \\ & \times \exp\left[i \frac{\pi}{2} \frac{Z_0 + Z_1}{Z_T} (m^2 - m'^2) R^2 Z_2 \right] \\ & \times \exp\left[i\pi \frac{Z_0}{Z_T} (nm - n'm') R Z_2 \right] \\ & \times \operatorname{sinc}\left\{ \frac{S}{Z_T} q_1 [(n - n')(Z_1 + Z_2) \right. \\ & \left. + (m - m') R Z_2] \right\}. \end{aligned} \tag{17}$$

This expression is a general result that provides an accurate value for $I_S(x)$. This is the same result as the one obtained with a different approach by Olszak and Wronkowski.[8] We decided to use a different method for deriving this last expression because it provides intermediate expressions that can be useful for studying other cases of practical interest. In particular, one could consider the case when the light source is placed in the focal plane of a collimating lens by substituting expression (11) with a planar wave front.

We recall that the only assumptions made up to now to derive expression (17) are the infinite extensions of the gratings along the x and y directions, the translation symmetry along the y axis, and the Fresnel approximation applied to propagation calculations. In the remainder of this paper, we study this expression under particular situations that allow us to simplify it and to extract useful information from it. The first effort in this direction is the modification of previous results obtained by Swanson and Leith.[10] Let us make S/z_0 and z_0 take very large values in expression (17). The total intensity distribution can then be approximated by the following expression:

$$\begin{aligned} I(x) = {} & \sum_n \sum_{n'} \sum_m \sum_{m'} a_n a_{n'}^* b_m b_{m'}^* \exp\{ i q_1 x [(n - n') \\ & + R(m - m')]\} \exp\left[i \frac{\pi}{2} (Z_1 + Z_2)(n^2 - n'^2) \right] \\ & \times \exp\left[i \frac{\pi}{2} Z_2 R^2 (m^2 - m'^2) \right] \\ & \times \exp[i\pi R Z_2 (nm - n'm')] \\ & \times \delta([(Z_1 + Z_2)(n - n') + R Z_2 (m - m')]). \end{aligned} \tag{18}$$

This is the same expression that is obtained by Swanson and Leith with a different procedure when the light source is considered of infinite extent and placed at an infinite distance from the first grating.[10] In that case the light incident on the system is treated as an incoherent superposition of planar wave fronts. It is also shown in Ref. 6 that, under these assumptions (infinite source, infinitely far), from expression (18) it can be derived that pseudoimages of the gratings are formed when some conditions on R, Z_1, and Z_2 are met.

3. FORMATION OF PSEUDOIMAGES

In the case of an infinite source located infinitely far from the first grating, one of the conditions that must be satisfied for the existence of a pseudoimage is

$$\frac{Z_1 + Z_2}{R Z_2} = Q, \tag{19}$$

where $Q = n/m$ must be a rational number.[6] The fulfillment of this equation is a necessary condition to obtain fringes in the observation plane. In effect, if Eq. (19) is not satisfied, the delta functions in Eq. (18) are all zero except when $n - n' = 0$ and $m - m' = 0$, in which case all the arguments of the oscillatory functions vanish and the intensity on the observation plane becomes constant.

Let us consider now a certain set of values of R, Z_1, and Z_2 that satisfy Eq. (19). Considering that we can find irrational numbers $\Re$ arbitrarily close to any rational number, there will exist values Z_2', infinitely close to Z_2, such that the quotient $(Z_1 + Z_2')/RZ_2'$ is irrational. There will also exist values Z_1' arbitrarily close to Z_1 such that $(Z_1' + Z_2)/RZ_2$ is irrational too. This means that given a certain set of values of R, Z_1, and Z_2 for which there exists a pseudoimage on the observation plane, if an infinitesimally small change is given to the value of either Z_1 or Z_2, there will be no pseudoimage at all. Equations (18) and (19) then imply that the focal depth of the pseudoimages is exactly zero. Obviously, this cannot be true; it is a consequence of using the approximation that the source has infinite extent and that it is placed infinitely far from the gratings.

The rigorous analysis for pseudoimage formation based on Eq. (17) is rather tedious, so to lighten the present discussion, we postpone it to Appendix A. From the results derived in Appendix A, it can be stated that, when the light source is not considered infinite, the condition for the existence of a pseudoimage is

$$(Z_1 + Z_2)n_I + RZ_2 m_I \simeq 0, \tag{20}$$

where n_I and m_I are small integers such that

$$|n_I| \ll \frac{Sq_1}{\pi Z_T} RZ_2, \qquad |m_I| \ll \frac{Sq_1}{\pi Z_T}(Z_1 + Z_2). \tag{21}$$

There is an additional condition that limits the range of values of Z_1 and Z_2 for which relation (20) will be true and can be expressed as

$$\frac{|n_I|}{|m_I|}, \frac{|m_I|}{|n_I|} \ll \left[\frac{Z_T}{Sq_1}\left|\frac{1}{(Z_1 + Z_2)n_I + RZ_2 m_I}\right|\right]^2. \tag{22}$$

Under these conditions a pseudoimage will be formed, which will be called a pseudoimage of order (n_I, m_I). The location of the pseudoimages is the same as that predicted in the case in which an infinite source is considered. An important difference is that, now, not all of the pseudoimages that were predicted by Eq. (19) will exist; the number of images that will exist for given values of S, q_1, and R will be limited by relations (21). The range of values of Z_1 and Z_2 for which the image of order (n_I, m_I) will exist will be limited by relations (22).

In the model presented by Swanson and Leith, there are an infinite number of images infinitely close to one another. In our model, there is a finite number of images, each with a certain focal depth. The pseudoimages that, according to Swanson and Leith, should appear for large values of $|n_I|$ and $|m_I|$ will not be observed because they will be formed too close to pseudoimages corresponding to lower values of $|n_I|$ and $|m_I|$ with a larger modulation and with a large focal depth.

When the conditions given by relations (20)–(22) are satisfied, the intensity distribution on the observation plane is

$$I_S \simeq I_0 + \sum_{j\neq 0} d_j \exp\left(ix\frac{Z_1}{Z_2}q_1, jn_I\right) \times \operatorname{sinc}\left\{j\frac{Sq_1}{Z_T}[(Z_1 + Z_2)n_I + RZ_2 m_I]\right\}. \tag{23}$$

This equation describes a set of fringes of period $p_I = p_1 Z_2/(Z_1 n_I)$, which will have a modulation given mainly by the first harmonics in its Fourier series; this modulation will be

$$M \simeq \left|\operatorname{Re}(d_1)\operatorname{sinc}\left\{\frac{Sq_1}{Z_T}[(Z_1 + Z_2)n_I + RZ_2 m_I]\right\}\right|. \tag{24}$$

The half-width at half-maximum of the sinc( ) term appearing in relation (24) will be given by

$$\frac{Sq_1}{Z_T}[(Z_1 + Z_2)n_I + RZ_2 m_I] = \frac{\pi}{2}. \tag{25}$$

It follows from this last equation and from the previous discussion that there will be an interval of values of Z_1 given by $[Z_{10} - \Delta Z_1, Z_{10} + \Delta Z_1]$ and an interval of values of Z_2 given by $[Z_{20} - \Delta Z_2, Z_{20} + \Delta Z_2]$ for which a pseudoimage of order (n_I, m_I) will have a good modulation. The values of Z_{10}, Z_{20}, ΔZ_1, and ΔZ_2 can be derived from the following expressions:

$$(Z_{10} + Z_{20})n_I + RZ_{20}m_I = 0, \tag{26}$$

$$\Delta Z_1 = \frac{\pi}{2}\frac{Z_T}{Sq_1}\frac{1}{n_I}, \tag{27}$$

$$\Delta Z_2 = \frac{\pi}{2}\frac{Z_T}{Sq_1}\frac{1}{n_I + Rm_I}, \tag{28}$$

where we have supposed that Z_T remains constant for small changes of Z_1 and Z_2. The depth of focus of the pseudoimages will be given by ΔZ_2. The stability of the pseudoimages with respect to changes in Z_1 will be related to ΔZ_1. Both quantities will be larger for a smaller size of the source, S, and for a larger size of the optical system, Z_T. Both quantities will also be larger in a pseudoimage corresponding to smaller values of n_I and m_I.

Aside from this, expression (24) will also be affected by the behavior of d_1 as a function of Z_1. From Appendix A we know that

$$d_1 = \exp\left(i\frac{\pi}{2}RZ_1 n_I m_I\right) \times \sum_{n'} a_{n'+n_I}a^*_{n'}\sum_{m'} b_{m'+m_I}b^*_{m'} \times \exp(-i\pi RZ_1 n_I m'); \tag{29}$$

this is a periodic function of Z_1 with a period given by

$$p_d = \frac{2}{Rn_I}. \tag{30}$$

Summarizing all these conclusions, the modulation of the fringe distribution will be, as a function of Z_2, a sinc( ) function centered on Z_{20} with a half-width of the central maximum given by ΔZ_2. And as a function of Z_1, it will be a sinc( ) function, centered on Z_{10} with a half-width of ΔZ_1, modulated by a periodic function of period p_d. To obtain a clearly visible pseudoimage, it will be necessary that Z_{10} be close to a maximum of $|\operatorname{Re}(d_1)|$. The tolerance of the pseudoimage to changes in Z_2 (the focal depth) will be given by ΔZ_2. The tolerance of the pseudoimage to changes in Z_1 will be given by ΔZ_1 or $|p_d|/4$, whichever is smaller.

For a better understanding of the properties of the pseudoimages, the particular case when the gratings used in the system are both Ronchi gratings will be studied in Section 4.

4. PARTICULAR CASE: TWO RONCHI GRATINGS

We will now consider that the first and second gratings are both Ronchi gratings; this means that the coefficients of the Fourier series describing the first and second gratings, a_n and b_n, are

$$a_0 = b_0 = \frac{1}{2}, \qquad a_{2n+1} = b_{2n+1} = \frac{1}{\pi}\frac{(-1)^n}{2n+1},$$

$$a_{2n} = b_{2n} = 0 \qquad (n \neq 0). \tag{31}$$

With those particular values of the Fourier expansion coefficients for both gratings, we will have to distinguish three different cases depending on the order of each pseudoimage, given by n_I and m_I:

- If n_I is an even integer, then $d_j = 0$ for any value of j. There will not be pseudoimages of order (n_I, m_I), where n_I is an even integer.
- If n_I and m_I are both odd integers, then the modulation of the fringes, according to relation (24), will be given by

$$M \simeq \left| \frac{1}{\pi^2 n_I m_I} \cos\left(\frac{\pi}{2} R Z_1 n_I m_I\right) \times \operatorname{sinc}\left\{ \frac{S q_1}{Z_T} [(Z_1 + Z_2) n_I + R Z_2 m_I] \right\} \right|. \quad (32)$$

- If n_I is an odd integer and m_I is even, then the modulation of the fringes will be given by

$$M \simeq \left| \frac{1}{\pi n_I} \operatorname{sinc}\left\{ \frac{S q_1}{Z_T} [(Z_1 + Z_2) n_I + R Z_2 m_I] \right\} \times \sum_{m'} \frac{1}{\pi^2 (2m + 1' + m_I)(2m' + 1)} \times \cos[\pi R Z_1 n_I (2m' + 1 - m_I/2)] \right|. \quad (33)$$

In any of these cases, the maximum value of the modulation is inversely proportional to $|n_I m_I|$, so the best pseudoimages will be obtained for small values of n_I and m_I.

When m_I is an odd integer, one of the conditions to get maximum modulation is

$$Z_1 = \frac{2l}{R n_I m_I} \qquad (l \in N). \quad (34)$$

That is, the maximum modulation is obtained when the distance between the first and the second grating is an even multiple of the Talbot distance, defined as ${p_1}^2/(2\lambda)$, but affected by a factor $1/(R n_I m_I)$. This condition and the one stated in Eq. (26) completely define the location of the pseudoimages for odd values of n_I and m_I.

When m_i is an even integer, the periodic dependency of the modulation with Z_1 is given by the rather complicated expression

$$\left| \frac{1}{\pi n_I} \sum_{m'} \frac{1}{\pi^2 (2m + 1' + m_I)(2m' + 1)} \times \cos[\pi R Z_1 n_I (2m' + 1 - m_I/2)] \right|. \quad (35)$$

It is difficult to determine from this expression that value of Z_1 for which the modulation reaches its highest value.

In Fig. 2 we show three different maps of the modulation of the pseudoimages as a function of Z_1 (horizontal axis) and Z_2 (vertical axis). Figure 2(a) shows the modulation of images of order (1, −1), Fig. 2(b) shows the modulation of images of order (1, −2), and Fig. 2(c) shows the modulation of images of order (1, −3). In the three

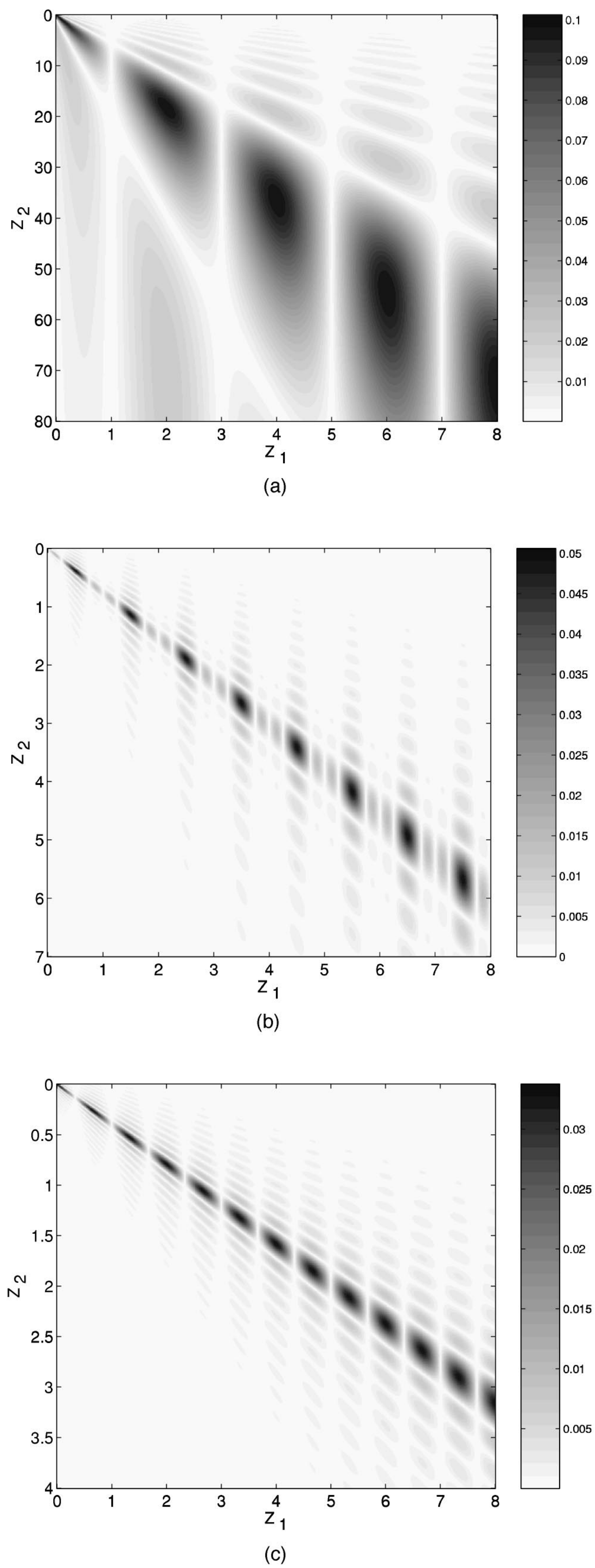

Fig. 2. Maps of the modulation of the pseudoimages as a function of Z_1 (horizontal axis) and Z_2 (vertical axis): (a) images of order (1, −1), (b) images of order (1, −2), (c) images of order (1, −3).

cases, we have used the values $R = 1.1$, $Sq_1 = 20\pi$, and $Z_0 = 1$. These maps were calculated by using expressions (32) and (33).

The maxima for the modulation are located in each case along the lines defined by $(Z_1 + Z_2)n_I + RZ_2 m_I = 0$ (it is important to note the different scales on the vertical axes for each figure), and the values of this maxima are proportional in each case to $1/|n_I m_I|$. The periodic dependency of the modulation on Z_1 can be clearly seen along the horizontal direction. It is clear as well that the areas of good modulation get wider for higher values of $Z_1 + Z_2$ (that is, higher values of Z_T) and for smaller values of $1/|n_I m_I|$.

5. CONCLUSIONS

In this work we have studied the process of pseudoimage formation when two gratings of different pitch are illuminated by using an extended monochromatic light source. The aim of this paper is to show the effect of the limited size of the light source on the existence and the properties of such pseudoimages.

First, we have derived general expressions for the intensity distribution that would be obtained on an observation plane located at an arbitrary distance from the second grating. Those expressions take into account the limited size of the light source.

From the expressions previously obtained, we have derived the conditions on the relevant parameters of the system (size of the light source, distances between the different elements in the system, and relationship between periods of the gratings) that must be satisfied to get well-modulated pseudoimages. We have proved that the number of images that it is possible to obtain depends on the size of the light source and the total length of the optical system. We have also proved that the depth of focus of the existing pseudoimages is inversely proportional to the size of the source and directly proportional to the total length of the optical system. We have shown as well that there is a periodic dependency of the modulation of the pseudoimages on the distance between gratings, just as there is in a standard moiré setting that uses collimated light. As an example, we have studied in more detail a particular case for two Ronchi gratings.

APPENDIX A

To study the conditions for the formation of pseudoimages when we are considering the limited size of the source, it is convenient to write expression (17) in a simplified manner as

$$I_S(x) = \sum_{n,m} \exp[iq_1(\alpha n + \beta m)x]\operatorname{sinc}(An + Bm)c_{nm}, \tag{A1}$$

where

$$\alpha = Z_0/Z_T, \tag{A2}$$

$$\beta = R(Z_0 + Z_1)/Z_T, \tag{A3}$$

$$A = Sq_1(Z_1 + Z_2)/Z_T, \tag{A4}$$

$$B = Sq_1 RZ_2/Z_T, \tag{A5}$$

$$\begin{aligned} c_{nm} = {} & \exp\left[i\frac{\pi}{2}\frac{Z_0}{Z_T}(Z_1 + Z_2)n^2\right] \\ & \times \exp\left(i\frac{\pi}{2}\frac{Z_0 + Z_1}{Z_T}R^2 Z_2 m^2\right)\exp\left(i\pi\frac{Z_0}{Z_T}RZ_2 nm\right) \\ & \times \sum_{n',m'} a_{n'+n}a_{n'}^* b_{m'+m}b_{m'}^* \\ & \times \exp\left[-i\pi\frac{Z_0}{Z_T}(Z_1 + Z_2)nn'\right] \\ & \times \exp\left(-i\pi\frac{Z_0 + Z_1}{Z_T}R^2 Z_2 mm'\right) \\ & \times \exp\left[-i\pi\frac{Z_0}{Z_T}RZ_2(nm' - n'm)\right], \end{aligned} \tag{A6}$$

which, in terms of the parameters defined above, can be written as

$$\begin{aligned} c_{nm} = {} & \exp\left(i\frac{\pi}{2}\frac{AZ_0}{Sq_1}n^2\right)\exp\left[i\frac{\pi}{2}RZ_2(\beta m^2 + 2\alpha nm)\right] \\ & \times \sum_{n',m'} a_{n'+n}a_{n'}^*, b_{m'+m}b_{m'}^* \\ & \times \exp\left(-i\pi\frac{AZ_0}{Sq_1}nn'\right) \\ & \times \exp\{-i\pi RZ_2[\beta mm' + \alpha(nm' - n'm)].\} \end{aligned} \tag{A7}$$

We will suppose that the system is arranged in such a way that the conditions

$$\frac{S}{Z_T} \gg \frac{p_1}{Z_1 + Z_2}, \quad \frac{S}{Z_T} \gg \frac{p_2}{Z_2} \tag{A8}$$

are satisfied. As shown in Fig. 3, there is a geometrical meaning to these inequalities. On every point of the observation plane, there should be light arriving from a large number of periods of the first grating and from a possibly different large number of periods of the second grating. This would mean that the light distribution on the observation plane depends on the periodic structure of each grating and not only on the structure of each single period of the gratings. If we apply the conditions stated in relations (A8) to the definitions of the parameters A and B, we deduce that the absolute value of both A and B should be much greater than π. This can be stated as follows:

$$|A| > \frac{\pi}{\epsilon_1}, \quad |B| > \frac{\pi}{\epsilon_2}, \quad \epsilon_1, \epsilon_2 \ll 1; \tag{A9}$$

that is, for any integers n, $m \neq 0$,

$$|\operatorname{sinc}(nA)| < \frac{\epsilon_1}{n\pi} \approx 0, \qquad |\operatorname{sinc}(mB)| < \frac{\epsilon_2}{m\pi} \approx 0, \tag{A10}$$

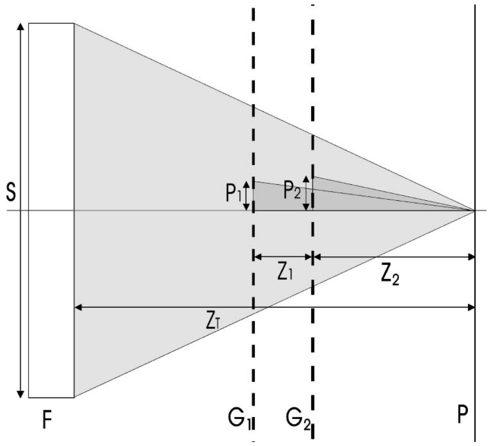

Fig. 3. Angle subtended on any point of the observation plane by the light source, a single period of the first grating, and a single period of the second grating. The first of these angles must be much greater than the other two in order to obtain clearly visible self-images.

and henceforth the terms with $n = 0$ or $m = 0$ in the double summation (A1) are negligible except for that with $n = m = 0$. This fact allows us to write expression (A1) as

$$I_S(x) \simeq I_0 + \sum_{n,m \neq 0} \exp(i\alpha xn) \times \exp(i\beta xm)\operatorname{sinc}(An + Bm)c_{nm}, \quad \text{(A11)}$$

where $I_0 = c_{00}$ is a constant-intensity background and the double summation is extended only to those values of n and m different from zero.

We will also suppose that the absolute value of the coefficients c_{nm} decreases at least as $1/nm$ when n and m increase. That is to say,

$$|c_{nm}| \leqslant K/|nm|, \quad \text{(A12)}$$

where K is a constant and n and m are different from zero. It is easy to show that when the coefficients a_n and b_m have alternating sign and their absolute values decrease at least as $1/n$ and $1/m$, respectively, then condition (A12) is satisfied. These last conditions on a_n and b_m are fulfilled in the case of Ronchi gratings and in most grating profiles with practical interest.

Let us suppose that there exists a set of values for the parameters A, B, α, and β that allows for the formation of a pseudoimage, in which case there will be a periodic distribution of irradiance on the observation plane. We will call q_I the fundamental frequency of this pseudoimage. Because of the functional form of expression (A1), there must exist two unique numbers n_I and m_I such that the fundamental frequency can be expressed as

$$q_I = q_1(\alpha n_I + \beta m_I). \quad \text{(A13)}$$

According to this, the irradiance distribution on the observation plane can be written as

$$I(x) = I_0 + \sum_j c_{jn_i, jm_I} \operatorname{sinc}[j(An_I + Bm_I)] \times \exp(ijq_I x) + I_2(x), \quad \text{(A14)}$$

where the term $I_2(x)$ comprises all the terms for which $(n, m) \neq j(n_I, m_I)$. We will demonstrate now that $I_2(x)$ is negligible when the parameters A, B, α and β take values that allow for the formation of a pseudoimage, and we will also find what these values and their tolerances are.

First, to have $I_2(x)$ negligible with respect to the Fourier series, at least the first terms of this series should have significant values. This implies that

$$\operatorname{sinc}[j(An_I + Bm_I)] \simeq 1, \quad \text{(A15)}$$

where j is any integer smaller than a certain n_L, such that $n_L \gg 1$. We could also write this last equation as the condition

$$|An_I + Bm_I| < \frac{\pi}{2n_L}, \qquad n_L \gg 1; \quad \text{(A16)}$$

this will be the first of the conditions for the existence of a pseudoimage.

Now we must study the terms of $I_2(x)$ and see the conditions under which they are negligible. There are different cases that we will treat separately:

1. $|n| \leqslant |m|$

We can write $n = r_1 n_I$ and $m = r_1 m_I + r_2$, where the numbers r_1 and r_2 are rational and r_2, which is given by

$$r_2 = \frac{mn_I - nm_I}{n_I}, \quad \text{(A17)}$$

will always be different from zero for any term that belongs to I_2.

a. $r_1 < n_L$

In this case

$$|An + Bm| = |r_1(An_I + Bm_I) + r_2 B| > \|r_2 B| - |r_1(An_I + Bm_I)\|. \quad \text{(A18)}$$

From Eq. (A17) we derive

$$|r_2 B| > \left|\frac{B}{n_I}\right| \quad \text{(A19)}$$

If we suppose that $|B/n_I| > |r_1(An_I + Bm_I)|$, we now obtain

$$|An + Bm| > \left|\frac{B}{n_I}\right| - |r_1(An_I + Bm_I)|. \quad \text{(A20)}$$

Also, from expression (A16), and taking into account that $r_1 < n_L$, we know that

$$|r_1(An_I + Bm_I)| < \frac{\pi}{2}, \quad \text{(A21)}$$

so

$$|An + Bm| > \left|\frac{B}{n_I}\right| - \frac{\pi}{2}. \quad \text{(A22)}$$

With the use of expression (A9), this last equation is transformed into

$$|An + Bm| > \left|\frac{\pi}{\epsilon_2 n_I}\right| - \frac{\pi}{2}. \tag{A23}$$

From this equation we can obtain the condition under which the considered terms will be negligible. In effect, if

$$|n_I| \ll \frac{1}{\epsilon_2}, \tag{A24}$$

then $|An + Bm| \gg \pi$ and $\mathrm{sinc}(An + Bm) \simeq 0$. Also when relation (A24) is true, it is also true that $|B/n_I| > |r_1(An_I + Bm_I)|$. We have proved that the condition given by relation (A24) is enough to make negligible any of the terms belonging to I_2 with $|n| \leq |m|$ and $r_1 < n_L$.

b. $r_1 \geq n_L$

In this case, and since $|n| \leq |m|$, we can consider that the following expression will be true:

$$|nm| \geq n^2 = r_1{}^2 n_I{}^2 \geq n_L{}^2 n_I{}^2. \tag{A25}$$

If the condition

$$n_L{}^2 n_I{}^2 \gg |n_I m_I|, \tag{A26}$$

is met, then from expression (A12) it follows that $|c_{nm}| \ll |c_{n_I m_I}|$ and therefore we can consider the term negligible. We have proved that the condition given by

$$\frac{|m_I|}{|n_I|} \ll n_L{}^2 \tag{A27}$$

is enough to make negligible any of the terms belonging to I_2 with $|n| \leq |m|$ and $r_1 \geq n_L$.

2. $|n| \geq |m|$

This is the same case as the one just studied if we switch the roles of n and m and those of A and B. So, following the same procedure as that above, we obtain the following conditions:

$$|m_I| \ll 1/\epsilon_1, \tag{A28}$$

$$|n_I|/|m_I| \ll n_L{}^2. \tag{A29}$$

These two conditions are enough to ensure that the terms belonging to I_2 such that $|n| \geq |m|$ will be negligible.

Summarizing these results, we have proved that, in a general case, the intensity distribution on the observation plane is close to a periodic distribution with a spatial frequency $q_I = \alpha n_I + \beta m_I$ if the following conditions are met:

$$|An_I + Bm_I| < \frac{\pi}{2n_L}, \qquad n_L \gg 1, \tag{A30}$$

$$|n_I| \ll 1/\epsilon_2, \qquad |m_I| \ll 1/\epsilon_1, \tag{A31}$$

$$|m_I|/|n_I|, |n_I|/|m_I| \ll n_L{}^2. \tag{A32}$$

In the work presented by Swanson and Leith,[10] the equivalent condition derived for the formation of the pseudoimages was

$$|An_I + Bm_I| = 0. \tag{A33}$$

This is equivalent in our scheme to the case when $n_L \to \infty$; the only conditions for the existence of a pseudoimage are relations (A31). These expressions determine the total number of images that will be observed in a system with a given source size and a given total length. The number of images is not infinite when we consider the limited size of the source.

Now let us suppose that $A = A_0$ and $B = B_0$ satisfy Eq. (A33) and that n_I and m_I satisfy relations (A31). The values of A and B close to A_0 and B_0 can be expressed as

$$A = A_0 + \Delta A, \qquad B = B_0 + \Delta B. \tag{A34}$$

In this case we can write

$$|An_I + Bm_I| = |\Delta A n_I + \Delta B m_I| = \frac{\pi}{2n_L}. \tag{A35}$$

The conditions to keep having a pseudoimage will be given by relations (A30) and (A32) and will be

$$|\Delta A n_I + \Delta B m_I| \ll \pi/2, \tag{A36}$$

$$|\Delta A n_I + \Delta B m_I|^2 \ll \frac{\pi^2 |n_I|}{|m_I|}, \tag{A37}$$

respectively. These last two relations indicate the intervals ΔA and ΔB around A_0 and B_0 for which the pseudoimage will exist. When the finite size of the source is considered, these intervals are different from zero and they depend on n_I and m_I.

Substituting the values of A and B, we could express the conditions for the existence of a pseudoimage in a general case as

$$(Z_1 + Z_2)n_I + RZ_2 m_I \simeq 0, \tag{A38}$$

$$|n_I| \ll \frac{Sq_1}{\pi Z_T} RZ_2, \qquad |m_I| \ll \frac{Sq_1}{\pi Z_T}(Z_1 + Z_2), \tag{A39}$$

$$\frac{|n_I|}{|m_I|}, \frac{|m_I|}{|n_I|} \ll \left[\frac{Z_T}{Sq_1}\left|\frac{1}{(Z_1 + Z_2)n_I + RZ_2 m_I}\right|\right]^2. \tag{A40}$$

We also obtain for the intensity distribution of the pseudoimages the expression

$$I_S \simeq I_0 + \sum_{j \neq 0} d_j \exp\left(ix \frac{Z_1}{Z_2} q_1 j n_I\right) \times \mathrm{sinc}\left\{j \frac{Sq_1}{Z_T}[(Z_1 + Z_2)n_I + RZ_2 m_I]\right\}, \tag{A41}$$

where

$$d_j = \exp\left(i \frac{\pi}{2} RZ_1 n_I m_I j^2\right) \sum_{n'} a_{n'+jn_I} a^*_{n'} \times \sum_{m'} b_{m'+jm_I} b^*_{m'} \exp(-i\pi RZ_1 n_I m' j). \tag{A42}$$

Expression (A41) represents a set of fringes of period $p_I = p_1 Z_2 / (Z_1 n_I)$.

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