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APPROXIMATIONS FOR MULTIVARIATE CHARACTERISTICS OF CLASSICAL RISK RUIN PROCESSES

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ABSTRACT Multivariate characteristic of risk processes are of high interest to academic actuaries. In such models the probability of ruin is obtained not only considering initial reserves u but the severity of ruin y and the surplus before ruin x.
This ruin probability can be expressed using an integral equation that can be efficiently solved using Gaver-Stehfest method of invertig Laplace transforms.

1. INTRODUCTION

Defining a classical risk process in continuous time $\{Z_t\}_{t\geq 0}$ with U_k claim sizes and premium c per time unit,

$$Z_t = u + ct - \sum_{k=1}^{N_t} U_k$$

where u are the initial reserves and N_t the total number of claims up to time t following a homogeneous Poisson process of parameter $\lambda>0$. Let B denote the distribution function of claim sizes U_k with mean $\mu^{-1}$and $c = \lambda$ $\mu^{-1}(1+\theta)$, where θ is the premium loading factor.

Let us now define $\tau = \inf\{t > 0 : Z_t < 0\}$ as the ruin time and $Y = -Z_\tau$ as the deficit at ruin time or severity of ruin and $X = Z_{\tau^-}$ as the surplus just before the ruin(X>u, initial reserves).

The probability of ultimate ruin with initial reserves u, parameter λ and severity of ruin less than y and surplus less than x is defined,

$$P\{\tau < \infty, X \leq x, Y \leq y\} = \Psi_{x,y}(u)$$

using a similar renewal argument as Gerber et al.(1987) or Frey and Schmidt(1996) the former probability can be expressed as the following defective renewal equation,

$$\Psi_{x,y}(u) = \frac{\lambda}{c} F_{x,y}(u) + \frac{\lambda}{c}\int_0^u \Psi_{x,y}(u-w)dG(w) \qquad (1.1)$$

where $G(w) = \int_0^w (1 - B(z))dz$ and

$$\begin{aligned} F_{x,y}(w) &= \int_w^{\max\{x,w\}} (B(z+y) - B(z))\, dz \qquad w \in [0, x) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad w \geq x \qquad (1.2) \end{aligned}$$

for $x \in (u, \infty) \quad y < \infty$

$$F_{\infty,y}(w) = \int_w^{w+y} (1 - B(z))\, dz \quad y < \infty \qquad (1.3)$$

$$F_{\infty,\infty}(w) = \int_w^{\infty} (1 - B(z))\, dz \qquad (1.4)$$

The result used by Frey and Schmidt(1996), expression 10, is just a particular case when the premium loading factor was defined,

$$\theta = \frac{1 - \lambda\mu^{-1}}{\lambda\mu^{-1}}, \qquad \theta > 0 \qquad (1.5)$$

2. Laplace transforms approach

We will introduce now the Laplace transform operator of a general function ξ(w) with support in the non-negative real axis,

$$L(s,\xi(w)) = \int_0^{\infty} e^{-sw}\xi(w)dw \quad \mathrm{Re}(s) > 0$$

applying it to equation 1.1 and using the properties of Laplace transform we can get,

$$L(s, \Psi_{x,y}(u)) = \frac{\lambda}{c}L(s,F_{x,y}(u)) + \frac{\lambda}{c}L(s, \Psi_{x,y}(u))L(s, g(x))$$

and finally,

$$L(s, \Psi_{x,y}(u)) = \frac{\frac{\lambda}{c}L(s,F_{x,y}(u))}{1 - \frac{\lambda}{c}L(s, g(x))} \quad \mathrm{Re}(s) > 0 \qquad (2.1)$$

where,

$$\begin{aligned} L(s, g(x)) &= \int_0^{\infty} e^{-sw}(1 - B(w))dw \\ &= \frac{1}{s} - \frac{L(s, b(x))}{s} \\ &= \frac{1 - L(s, b(x))}{s} \end{aligned}$$

bearing in mind that,

$$g(w) = \frac{dG(w)}{dw} = 1 - B(w)$$

and,

$$\begin{aligned} L(s,F_{x,y}(u)) &= \int_0^x e^{-sw} \int_w^{\max\{x,w\}} (B(z+y) - B(z))\, dz\, dw \\ &= \left(\frac{1-e^{-sx}}{s}\right) \digamma(x+y) - \left(\frac{1-e^{-sx}}{s}\right) \digamma(x) \\ &\quad - \int_0^x e^{-sw} \digamma(w+y) dw + \int_0^x e^{-sw} \digamma(w) dw \end{aligned}$$

for $x \in (u, \infty) \quad y < \infty$
where

$$\digamma(w) = \int_0^w B(z) dz \qquad w \geq 0$$

and after some operations we get,

$$\begin{aligned} L(s,F_{\infty,y}(u)) &= \int_0^\infty e^{-sw} \int_w^{w+y} (1 - B(z))\, dz\, dw \\ &= \frac{y}{s} - \int_0^\infty e^{-sw} \digamma(w+y) dw + L(s,\digamma(w)) \\ &= \frac{y}{s} - L(s,\digamma(w+y)) + \frac{L(s, b(w))}{s^2} \\ &= \frac{sy - s^2\, L(s,\digamma(w+y)) + L(s, b(w))}{s^2} \end{aligned}$$

for $y < \infty$
and also, shifting the integration limits,

$$\begin{aligned} L(s,F_{\infty,\infty}(u)) &= \int_0^\infty e^{-sw} \int_w^\infty (1 - B(z))\, dz\, dw \\ &= \frac{s\mu^{-1} - 1 + L(s, b(w))}{s^2} \end{aligned}$$

getting finally,

$$L(s, \Psi_{x,y}(u)) = \frac{\frac{\lambda}{c}\, s\, L(s,F_{x,y}(u))}{s - \frac{\lambda}{c}(1 - L(s, b(x)))} \qquad \text{Re}(s) > 0 \tag{2.2}$$

for $x \in (u, \infty) \quad y < \infty$.

$$L(s, \Psi_{\infty,y}(u)) = \frac{\frac{\lambda}{c}\ \left(sy - s^2\ L(s,F(w+y)) + L(s, b(w))\right)}{s\left(s - \frac{\lambda}{c}\left(1 - L(s, b(w))\right)\right)}$$
$$\mathrm{Re}(s) > 0 \tag{2.3}$$

for $y < \infty$.

$$L(s, \Psi_{\infty,\infty}(u)) = \frac{\frac{\lambda}{c}\ \left(s\mu^{-1} - 1 + L(s, b(w))\right)}{s\left(s - \frac{\lambda}{c}\left(1 - L(s, b(w))\right)\right)} \quad \mathrm{Re}(s) > 0 \tag{2.4}$$

3. Approximations of the ruin probability

Laplace transforms of functions $\Psi_{x,y}(u)$, $\Psi_{\infty,y}(u)$ and $\Psi_{\infty,\infty}(u)$ are obtained by 2.2, 2.3 and 2.4 respectively, we can then approximate their values using numerical approximations of Laplace transforms.

In a former "Documento de Trabajo" entitled: *"Applications of Gaver-Stehfest method to Risk Theory"* we showed that the so-called Gaver-Stehfest method of inverting Laplace transforms was a very useful tool when approximating ruin probabilities. It is clear that the same numerical technique could be applied to the multivariate case.

This approach was tested for combination of exponential and Pareto distributions for the claim size and compared with the results obtained by Gerber et al.(1987) and Dickson and Waters(1992) obtaining a high degree of accurancy. This last fact is not surprising bearing in mind that Gaver-Stehfest method converges very quickly to the exact value for increasing values of parameter N.

It is also clear that the point of view stated in the "Documento de trabajo" : *Calculating ultimate non-ruin probabilities when claim sizes follow a generalized* $\Gamma-$*convolution distribution function"* can be considered as a suitable methodology for future research in the multivariate case.

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