UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

Departamento de Álgebra, Geometría y Topología

TESIS DOCTORAL

Orbifolds and geometric structures

Orbifolds y estructuras geométricas

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Director

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Madrid
Ed. electrónica 2019

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PhD Thesis

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Memoria presentada para optar al grado de Doctor en Investigación Matemática.

Abstract.

In this thesis we study geometric structures on orbifolds. Our main interest lies in the relationship between such structures in orbifolds and corresponding geometric structures of associated manifolds.

One instance of this is the symplectic resolution of orbifold singularities in which we associate a symplectic manifold (the resolution) to a symplectic orbifold. Resolution of symplectic orbifolds is a natural extension to the symplectic category of the classical problem of resolution of singularities in algebraic geometry. Apart from the intrinsic interest of the problem of resolution of singularities in symplectic geometry, resolution of symplectic orbifolds also gives a powerful method to construct symplectic manifolds starting from symplectic orbifolds. With this idea in mind, we develop a method to resolve a certain type of symplectic orbifolds, which we call homegenous isotropy orbifolds. These do not cover orbifolds in full generality, but they suffice to construct interesting manifolds.

Another instance of the interplay between geometric structures of orbifolds and manifolds comes from Sasakian and K-contact geometry. There is a strong relationship between K-contact (Sasakian) structures on $(2n+1)$-manifolds, and almost-Kahler (Kahler) structures on $2n$-orbifolds. The former are basically Seifert circle bundles over the latter. In this way, the problem of finding K-contact non-Sasakian manifolds can be translated to a corresponding problem of finding symplectic non-Kahler orbifolds satisfying some specific properties. We exploit this fact in our construction of a K-contact manifold with first homology trivial and with no semi-regular Sasakian structure.

Keywords: **Orbifold, Symplectic Geometry, Sasakian Geometry.**

Resumen.

En esta tesis estudiamos estructuras geométricas en orbifolds, y su relación con estructuras geométricas en variedades. Concretamente, estamos interesados en la relación entre dichas estructuras en orbifolds y otras estructuras geométricas en variedades asociadas a éstos.

Un ejemplo de esta relación es la resolución simpléctica de singularidades de orbifolds, en la cual se asocia una variedad simplética (la resolución) a un orbifold simpléctico. El problema de encontrar una resolución simpléctica de un orbifold es una extensión natural a la categoría simpléctica del problema clásico de resolución de singularidades en geometría algebraica. Aparte del interés intrínseco de dicho problema, la resolución simpléctica de orbifolds también aporta un método muy útil para construir variedades simplécticas a partir de orbifolds simplécticos. Con esta última idea en mente, desarrollamos un método para resolver cierto tipo de orbifolds simplécticos, que llamamos orbifolds con isotropía homogénea. Este tipo de orbifolds no abarca a los orbifolds en toda su generalidad, pero sí sirven para construir variedades con interesantes propiedades.

Otro ejemplo de la relación de estructuras geométricas en variedades y en orbifolds viene de las geometrías Sasakiana y de K-contacto. Hay, en efecto, una fuerte relación entre estructuras de K-contacto (Sasakianas) en variedades $(2n+1)$-dimensionales, y estructuras casi-Kahler (Kahler) en orbifolds $2n$-dimensionales. Básicamente, las primeras se obtienen como fibrados de Seifert de círculos sobre las segundas. De esta manera, el problema de encontrar variedades de K-contacto que no admitan estructuras Sasakianas es equivalente a un problema sobre hallar orbifolds simplécticos que no admitan estructuras Kahler y cumpliendo ciertos requisitos. En esta tesis explotamos este hecho para la construcción de una variedad de K-contacto que no admite una estructura Sasakiana semi-regular y con primer grupo de homología nulo.

Palabras Clave: **Orbifold, Geometría Simpléctica, Geometría Sasakiana.**

Agradecimientos.

En primer lugar, quiero agradecer a mi director Vicente Muñoz todo su trabajo y apoyo en la realización de esta tesis. No sólo ha sido un guía y orientador excelente, sino que se ha manchado las manos repetidamente. Su ayuda y trabajo han sido indispensables con las partes técnicas más difíciles de la tesis. Ha sido una suerte y un privilegio poder aprender matemáticas de él todo este tiempo.

En segundo lugar, agradezco a mis padres todo el apoyo que me han brindado durante estos cuatro años de doctorado. En general todo suele ser más fácil si tienes a tus padres cerca, y yo he tenido ese privilegio durante todo este tiempo.

Por otro lado, ha sido una gran suerte poder trabajar con Aleksy Tralle y Vicente, trabajo en cuyo fruto una sustancial parte de esta tesis está basada. Agradezco a Alesky todo su trabajo, sus ideas y contribuciones. También agradezco mucho a Marisa Fernández por recibirme en la Universidad del País Vasco, donde Aleksy y yo tuvimos la oportunidad de dar a conocer nuestro trabajo. Fue una visita muy positiva gracias a la hospitalidad de Marisa y de la UPV en general.

También quiero agradecer a Simon Chiossi por su caluroso recibimiento en la Universidad Federal Fluminense de Río de Janeiro, donde disfruté de una estancia muy agradable. No sólo tuve la oportunidad de aprender matemáticas con él, sino que pude conocer a una gran persona. Asímismo agradezco a Luis Hernández Corbato toda la ayuda y consejos sobre dicha estancia en Río de Janeiro.

Dignos de mención son también otros miembros del departamento de Geometría de la UCM con los que he tenido el gusto de colaborar en tareas docentes, como Javier Lafuente, Elena Martín, Otto Rutwig y Jesús Ruiz. Ha sido una experiencia gratificante poder impartir docencia con ellos. Además, es pertinente destacar que Jesús, Vicente y Enrique Arrondo fueron los profesores que (durante el grado y el máster) hicieron que me interesara por la rama de la geometría. Como antiguo alumno suyo, agradezco su excelente labor docente.

Agradezco mucho también la labor de mi tutor Marco Castrillón, que siempre me ha ayudado en todo lo que he necesitado, tanto en burocracias como en cuestiones académicas.

También menciono la ayuda que he recibido de otros investigadores. Destaco a Fran Presas por todo lo que me enseñó del h-principio, y a Jaume Amorós por compartir con nosotros sus conocimientos de fibraciones de Lefschetz.

Reconozco también la labor de los evaluadores externos Charles Boyer y Fran Presas, que se han leído cuidadosamente este documento y han ayudado a mejorarlo en varios aspectos. Asímismo agradezco a los miembros del tribunal su disponibilidad y su tiempo para evaluar este trabajo.

Debo mencionar también a todos mis compañeros pasados y presentes de despacho, tanto del zulo (308-H) como del 249. Éstos son Juanjo, Héctor, Eduardo, Pedro, y Tayomara. Con todos ellos el ambiente en el despacho ha sido siempre muy bueno, tanto para trabajar como para conversar cuando la situación daba pié a ello.

También agradezco a Ángel, Lucía y Giovanni el poder discutir sobre matemáticas y espinores en el seminario de la UCM, y a Giovanni doblemente por organizar dicho seminario.

Por otra parte, he tenido la suerte de conservar relación en estos años con varios de mis amigos de la carrera, entre los que se encuentran Eva, Moisés, Juanjo y Víctor. Además, a Eva y Moisés he podido recurrir en varias ocasiones en busca de ayuda con temas de matemáticas, lo cual es muy de agradecer.

Asímismo, son dignos de mención varios compañeros del ICMAT y/o de la UCM con los que he tenido la oportunidad de aprender matemáticas en numerosas ocasiones, (o simplemente de tomar un café), como Álvaro del Pino, Paco Torres, Pablo Portilla, Manuel Sheriff, Miguel Robredo, David Gómez, María Pe, Mari Ángeles García, Jorge González, David Alfaya y Luis Maire. Estoy seguro que en este apartado me olvido de gente, pero supongo que es inevitable. Quede dicho de antemano que lamento cualquier omisión relevante, en este y/o otros apartados.

Reconozco también el importante papel de mis amigos de fuera del mundo de las matemáticas por ayudarme a mantener a raya (al menos en parte) la locura durante todo este tiempo. O, quizás mejor dicho, a focalizar parte de ella en otros menesteres distintos de las matemáticas. En esto último destaco la labor de Iván, Luis y Ribot.

También agradezco a Estela por estar ahí para hablar de las miserias de la vida del doctorando, ya que esto siempre ayuda a sobrellevarlas mejor. Dignos son también de mencionar amigos con los que siempre que les puedo ver paso un buen rato, como Calero, Martín, y Diego.

Juan Ángel Rojo Carulli,

18 de Noviembre de 2018

La elaboración de esta tesis ha sido financiada por una beca predoctoral La Caixa, International PhD. Programme, Severo Ochoa 2014-2018, en el ICMAT y la UCM.

Asmismo, el autor ha sido parcialmente financiado por el proyecto MTM2010-17389 del MICINN, España.

Contents

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Introduction.

An orbifold is a space which is locally modelled on balls of $\mathbb{R}^n$ quotient by a finite group. These have been very useful in many geometrical contexts, see for instance [**50**]. In the setting of symplectic geometry, symplectic orbifolds have been introduced mainly as a way to construct symplectic manifolds satisfying some requirements. The usual strategy is to consider a group acting on a symplectic manifold by symplectomorphisms, the action being so that the quotient is a symplectic orbifold. This group is usually chosen in such a way that the quotient orbifold satisfies some desirable topological or geometrical properties. Then, a symplectic resolution is performed in the orbifold, giving a symplectic manifold. If one is able to control the change that the resolution process induces on the topology and geometry of the orbifold, then a symplectic manifold (the resolution) is obtained with the corresponding topological and geometrical properties.

On the other hand, in the setting of K-contact (Sasakian) geometry, orbifolds appear naturally as the natural structure inherited by the space of leaves of the Reeb field acting on a K-contact (Sasakian) manifold. Actually, the orbifold encompasses all the relevant information about the K-contact (Sasakian) structure, so these structures can in fact be recovered from a suitable orbifold structure.

In both cases we see a common theme, precisely that there exists a process to construct a geometric structure on a manifold starting from a suitable geometric structure on an orbifold. This applies to constructing symplectic manifolds from symplectic orbifolds via resolving their singularities. It also applies to obtaining a K-contact (Sasakian) manifold from an almost-Kahler (Kahler) orbifold, via constructing a Seifert circle bundle over the orbifold, thus creating a smooth total space of the bundle. In this thesis we explore these two methods for contructing manifolds from orbifolds.

Let us state here our convention regarding the different classes of points in an orbifold X. A point $p \in X$ is called *an isotropy point* if the isotropy group of p is non-trivial. It is called an *smooth point* if the topological underlying space of X is a topological manifold near the point p. Recall that there can be isotropy points which are also smooth. It is called a *singular point* if it is not a smooth point. An orbifold X is called *smooth* if its underlying topological space is a topological manifold, which is equivalent to all points of X being smooth points. The concept we use of smooth orbifold is not related at all with the orbifold atlas on X.

This should not cause any confusion because of the following. Whenever we employ the word *orbifold*, the orbifold atlas is assummed to be differentiable in the sense that the orbifold change of charts are $\mathcal{C}^\infty$. In the same vein, the word *manifold* in this thesis means differentiable manifold.

Problem 1: symplectic resolution of orbifolds.

First let us talk about symplectic resolution of orbifolds. One strong motivation for developing a process of resolution of symplectic orbifolds is that the construction of symplectic manifolds via resolving a symplectic orbifold has proved to be quite powerful in providing many examples of interesting manifolds. However, up to now, for each symplectic orbifold which has been resolved, it has been necessary to use a particular method of resolution by taking into account particular characteristics of the orbifold at hand. Some important results in which symplectic resolution has proved useful are the following.

- Fernández and Muñoz give in [**21**] the first example of a simply-connected symplectic 8-manifold which is non-formal, as the resolution of a suitable symplectic 8-orbifold. To resolve this orbifold, they use a method developed by themselves and Cavalcanti in [**16**], which allows to resolve symplectic orbifolds with isolated isotropy points. Recall that the manifold of [**21**] was proved to have also a complex structure by Bazzoni and Muñoz in [**6**].
- Bazzoni, Fernández and Muñoz give in [**4**] the first example of a simply-connected non-Kahler manifold which is simultaneously complex and symplectic in dimension 6. For this, they construct first an orbifold of dimension 6 with isotropy sets of dimensions 0 and 2. Then they construct a symplectic resolution of this orbifold. The manifold thus obtained as the resolution of the orbifold was used to give the sought non-Kahler, symplectic, complex and simply connected 6-manifold. However, the construction of the resolution is ad-hoc for the particular example at hand as it satisfies that the normal bundle to the 2-dimensional isotropy set is trivial.

This leads to the question of whether it is possible to construct a systematic procedure to resolve symplectic orbifolds. Let us call it from now on the *symplectic resolution question.* In order to illustrate some of the most obvious difficulties, let us point out that the analogies between the symplectic and the algebraic blow-up are tricky even in the most basic constructions like that of the proper transform of a symplectic submanifold. See Subsection 6.4 if interested in the reason why.

Now that the reader is (hopefully) convinced of the reasons why a symplectic resolution is relevant, let us review briefly some history about the symplectic resolution question. The problem of resolution of singularities and blow-up in the symplectic setting was first posed by Gromov in [**27**]. Few years later, the symplectic blow-up of a symplectic manifold along a symplectic submanifold was rigorously studied by McDuff [**40**] and it was used to construct the first example of a simply-connected symplectic manifold with no Kahler structure.

After that, McCarthy and Wolfson developed in [**37**] a symplectic resolution for isolated singularities of orbifolds in dimension 4. This was done via gluing along a suitable hypersurface.

Later on, Cavalcanti, Fernández and Muñoz gave the already mentioned method of [**16**] to resolve symplectic orbifolds with isolated singularities. This resolution is done gluing local resolutions of the isolated singular points with the rest of the orbifold. These local resolutions are performed in the complex algebraic category. Then, the original orbi-symplectic form is glued with a Kahler form defined in the complex-algebraic resolution of the neighborhoods of the singular points. The method of resolution we develop in this thesis is a generalization of this idea (gluing local resolutions) to a broader type of orbifolds.

Also, Niederkruger and Pasquotto in [**45, 46**] provide a method for resolving symplectic orbifold singularities via symplectic reduction, which can be used for some classes of symplectic singularities, including cyclic orbifold singularities, even if these are not isolated.

In [**19**], Chen has detailed a method for resolving arbitrary symplectic 4-orbifolds. He uses the fact that the singular points of the underlying space have to be isolated in dimension 4, so the resolution of [**4**] applies. Hence, his method of resolution is only a small step forward from [**4**], the only novelty being that there can be also isotropy surfaces composed of smooth points with non-trivial isotropy. In such points the orbifold is topologically a manifold, so the question only amounts to being able to deform the orbi-symplectic form into an ordinary symplectic form in a neighborhood of the isotropy surfaces. The method of resolution given in [**19**] uses techniques from symplectic reduction. Using this resolution, some restrictions are given for the existence of symplectic actions of finite groups on symplectic 4-manifolds. It is worth recalling that there is a more elementary way to deform an orbifold symplectic form into an ordinary simplectic form (see Chapter 4, Section 2), without using techniques from symplectic reduction.

Another recent result on symplectic resolution of orbifolds is the example of Bazzoni, Fernández and Muñoz in [**4**], where the first construction of a symplectic resolution of an orbifold with positive dimensional singular (non-smooth) locus is given. As remarked above, this construction is ad hoc for the particular example at hand, since the normal bundle of the isotropy locus is trivial.

Now that the symplectic resolution question is contextualized, let us talk about the results that we present here. In this thesis we give a partial result towards a positive answer of the symplectic resolution question, showing a sistematic way to construct a symplectic resolution in the case of *homogeneous isotropy orbifolds*. These are orbifolds X whose isotropy set is composed of disjoint submanifolds D_i so that each of the D_i have the same isotropy groups at all its points. We call this D_i a homogeneous isotropy set and we call such orbifold X an homogeneous isotropy orbifold, or HI-orbifold. Homogeneous isotropy orbifolds can have singular locus of *arbitrary* high dimension. In other words, the condition for an orbifold to have homogeneous isotropy allows the existence of submanifolds $D_i \subset X$ composed of singular (non-smooth) points and of arbitrarily high positive dimension.

Thus, in the context of the symplectic resolution question, we give a procedure to resolve a wider type of singularities in a symplectic orbifold X of arbitrary dimension $2n$. Since the singular points of the orbifold are not necessarily isolated here, new techniques are required in order to perform the resolution. These techniques rely on being able to endow the normal bundle of the D_i with a nice structure. Using this structure it is possible to effectively perform fiberwise the algebraic resolution of singularities of Encinas and Villamayor [**20**], and then glue these local resolutions into a global resolution $\widetilde{X}$ of X. Later, we deal with the problem of constructing a symplectic form in $\widetilde{X}$ that coincides with the original symplectic form of X away from the singular locus.

The general strategy is to endow the normal bundle ν_D of any homogeneous isotropy submanifold $D \subset X$ with the structure of an orbifold bundle with structure group $\mathrm{U}(k)$, where $2k$ is the codimension of D. The singularities of X at points of D are quotient singularities in the fibers $F = \mathbb{C}^k/\Gamma$ of ν_D, where Γ is the isotropy group of D. The usual resolution of singularities for algebraic geometry allows to resolve each of the fibers F of ν_D separately. However, we need this resolution to glue nicely when we change trivializations. For this we need an improvement of the classical theorem of resolution of singularities by Hironaka [**31**]. This improvement is the constructive resolution of singularities by Encinas and Villamayor [**20**], which is compatible with group actions. Using their result we are able to construct the resolution $\widetilde{\nu}_D$ of X near D as a smooth manifold.

The resolution $\widetilde{\nu}_D$ has the structure of a fiber bundle over D, with fiber the resolution $\widetilde{F}$ of $F = \mathbb{C}^k/\Gamma$. Both base D and fiber $\widetilde{F}$ of the total space $\widetilde{\nu}_D$ are symplectic, but this does not imply directly that $\widetilde{\nu}_D$ admits a symplectic form. First we need to prove that there is no cohomological obstruction for this, which amounts to finding a cohomology class on the total space $\widetilde{\nu}_D$ that restricts to the cohomology class of the symplectic form of the fiber. After that, we have to develop a globalization procedure for symplectic fiber bundles with non-compact symplectic fiber. The final step is to glue the symplectic form on $\widetilde{\nu}_D$ with the original symplectic form of $X \setminus D$.

The main result of symplectic resolution in this thesis, from Chapter 3, is the following.

THEOREM 0.1. *Let (X, ω) be a symplectic orbifold with isotropy set consisting of disjoint homogeneous isotropy subsets. Then there exists a symplectic manifold*

$$(\widetilde{X}, \widetilde{\omega})$$

and a smooth map

$$b : (\widetilde{X}, \widetilde{\omega}) \to (X, \omega)$$

which is a symplectomorphism outside an arbitrarily small neighborhood of the isotropy set of X.

Finally, we also provide some examples in which Theorem 0.1 applies.

Problem 2: construction of K-contact non-Sasakian manifolds.

Let us now talk about the other main concern of this thesis, i.e. the construction of K-contact non-Sasakian compact manifolds in dimension 5 with smallest possible fundamental group.

Lately, there is much interest on constructing K-contact manifolds which do not admit Sasakian structures. This is an analogous problem in odd dimension to the well-established question of finding symplectic non-Kahler manifolds. The modern monograph [**9**] is the standard reference for current research on Sasakian and K-contact geometry.

Let us put some context on this problem. In the following discussion, all manifolds will be understood to be compact. A very important tool for finding K-contact non-Sasakian compact manifolds is the fact that the odd Betti numbers up to dimension n of Sasakian $2n+1$-manifolds must be even, while this does not necessarily happen for K-contact manifolds. This property of Sasakian manifolds comes from the fact that the basic cohomology of the Reeb foliation of a Sasakian manifold has a natural Hodge structure, while this is not true for general K-contact manifolds.

The parity of b_1 was used to produce the first examples of K-contact manifolds with no Sasakian structure in [**9**, example 7.4.16]. These consist of K-contact manifolds with odd first betti numbers, hence not Sasakian.

The fundamental group can also be used to construct K-contact non-Sasakian manifolds. Fundamental groups of Sasakian manifolds are called Sasaki groups, and satisfy strong restrictions. Using this it is possible to construct (non-simply connected) compact manifolds which are K-contact but not Sasakian [**17**]. Also, it has been used to provide an example of a solvmanifold of dimension 5 which satisfies the hard Lefschetz property and which is K-contact and not Sasakian [**15**].

When one moves to the case of simply connected manifolds, the first betti number is not avalaible, so the easiest thing to consider is the third betti number. In this way, K-contact non-Sasakian examples of any dimension ≥ 9 were constructed in [**28**]. These are K-contact manifold with odd third betti number, hence non-Sasakian by the evenness of the third Betti number of a compact Sasakian manifold.

Apart from the odd betti numbers, another important fact is that the cohomology algebra of a Sasakian manifold satisfies a hard Lefschetz property [**13**]. This can be used to construct K-contact manifolds satisfying the restriction that the odd betti numbers be even, but not admitting Sasakian structures. In this vein, examples of (non-simply connected) K-contact non-Sasakian manifolds are produced in [**14**] in dimensions 5 and 7. These examples are nilmanifolds with even Betti numbers, so in particular they are not simply connected. Using the hard Lefschetz property, it is also possible to construct examples [**36**] of simply connected K-contact non-Sasakian manifolds of any dimension ≥ 9 .

In [**52**] and in [**7**] the rational homotopy type of Sasakian manifolds is studied. In [**7**] it is proved that all higher order Massey products for simply connected Sasakian manifolds vanish, although there can be non-vanishing triple Massey products. This yields examples of simply connected K-contact non-Sasakian manifolds in dimensions ≥ 17.

The problem of finding *simply connected* K-contact non-Sasakian manifolds was solved in the affirmative for dimensions ≥ 9 in [**13, 14, 28**]. Later it was also solved for dimension 7 in [**43**], where a combination of various techniques based on homotopy theory and symplectic geometry produced a K-contact manifold with odd third Betti number, hence non-Sasakian. In dimension 5 the problem appears to be much more difficult. Indeed, betti numbers, Massey products and the hard Lepschetz property are not suitable for constructing K-contact non-Sasakian manifolds in low dimensions, say ≤ 5. In dimension 3 (the lowest possible dimension), the classification of three manifolds yields that every K-contact manifold is Sasakian [**32**], so the problem is solved. In dimension 5, the only betti number that can be used is the first one, so once you require your manifold to be simply connected, betti numbers are useless. The only known obstructions to the existence of Sasakian structures in 5-manifolds that can be useful in the simply connected case are subtle properties [**34**] of the torsion of the second homology group over the integers.

In fact, the problem of the existence of simply connected K-contact non-Sasakian compact manifolds (open problem 7.4.1 in [**9**]) is still open in dimension 5. To solve this problem it appears that one has to use the arguments of [**34**] in order to extract obstructions to the existence of Sasakian structures. These obstructions are associated to the classification of Kahler complex surfaces.

A simply connected compact oriented 5-manifold is called a *Smale-Barden manifold*. These manifolds are classified topologically by $H_2(M, \mathbb{Z})$ and the second Stiefel-Whitney class. Chapter 10 of the book [**9**] by Boyer and Galicki is devoted to a description of some Smale-Barden manifolds which carry Sasakian structures. The following problem is still open (open problem 10.2.1 in [**9**]).

Do there exist Smale-Barden manifolds which carry K-contact but do not carry Sasakian structures?

In this thesis we make the first step towards a positive answer for the above question. A *homology Smale-Barden manifold* is a compact 5-dimensional manifold M with $H_1(M, \mathbb{Z}) = 0$. A Sasakian structure is *regular* if the leaves of the Reeb flow are a foliation by circles with the structure of a circle bundle over a smooth manifold. A Sasakian structure is *quasi-regular* if the Reeb foliation is a Seifert circle bundle over a (cyclic) orbifold. It is *semi-regular* if this foliation is a Seifert circle bundle over a smooth orbifold. The orbifold being smooth means that the underlying space is a topological manifold, i.e. that the orbifold has only locus of non-trivial isotropy in codimension 2. Any manifold admitting a Sasakian structure is known to also admit a quasi-regular Sasakian structure. Semi-regularity is only a small extra requirement. With this notions, our main result is the following (see Theorem 5.8).

THEOREM 0.2. *There exists a homology Smale-Barden manifold which admits a semi-regular K-contact structure but which does not carry any semi-regular Sasakian structure.*

In order to put our result into a general context, it is worth recalling Kollar's obstructions to Sasakian structures [**34**]. If a 5-dimensional manifold M has a Sasakian structure, then it has a quasi-regular Sasakian structure. Then it is a Seifert bundle structure over a Kahler orbifold X with isotropy locus a collection of complex curves. If $H_1(M) = 0$, then these curves span $H_2(X)$. Moreover, the torsion of the second homology $H_2(M, \mathbb{Z})$ allows to recover the genus of these curves if they are disjoint. In [**34**], 5-manifolds M are constructed which are Seifert bundles over 4-orbifolds X, with the property that the isotropy of X consists of surfaces not satisfying the adjunction equality, hence X cannot be Kahler, and this implies that M cannot be Sasakian. To prove this one has to use that the genus of the isotropy surfaces of X appears in $H_2(M, \mathbb{Z})$, so for any other Seifert bundle structure $M \to X'$ on M (over some 4-orbifold X'), the isotropy surfaces of X' would also contradict the adjunction equality. However, the same arguments show that for the Seifert bundles $M \to X$ constructed by Kollar, the orbifold X cannot be symplectic either (since the adjunction formula generalizes for symplectic surfaces in 4-manifolds), so M cannot be K-contact either. Hence, these examples are not useful per se for finding K-contact non-Sasakian manifolds, but the techniques used to construct them are promising because they can be adapted to exploit some other properties that set apart Kahler geometry and symplectic geometry in real dimension 4.

To produce K-contact 5-dimensional manifolds we need to produce symplectic 4-dimensional orbifolds with suitable symplectic surfaces spanning the second homology. Such a K-contact 5-manifold can be shown not to admit a Sasakian structure if we prove that such configuration of surfaces (genus, disjointness and spanning the second homology) cannot happen for complex curves inside a Kahler orbifold. We propose the following conjecture:

There does not exist a Kahler manifold or a Kahler orbifold X with $b_1 = 0$ and with disjoint complex curves spanning $H_2(X,\mathbb{Q})$, all of genus $g \geq 1$.

We give the first result in this direction in Theorem 5.17, which we reproduce below.

THEOREM 0.3. *Let S be a smooth closed Kahler surface with $H_1(S,\mathbb{Q}) = 0$ and containing $D_1, \ldots, D_b$, $b = b_2(S)$, smooth disjoint complex curves with $g(D_i) = g_i > 0$, and spanning $H_2(S,\mathbb{Q})$. Assume the following.*

- *At least two g_i are bigger than 1,*
- *All $g_i \leq 3$. In other words $g := \max\{g_i : 1 \leq i \leq 36\} \leq 3$.*

Then $b \leq 2g + 3$.

Our construction of a K-contact 5-manifold which does not admit a Sasakian structure relies on producing a symplectic 4-manifold with disjoint symplectic surfaces spanning the second homology, of genus $g \geq 1$, but also with genus $g \leq 3$, to fit with our needs in Theorem 0.3. This is the content of the delicate construction in the last chapter of this thesis.

Finally, our examples of K-contact manifolds are also of interest because they are quasi-regular but do not admit a regular K-contact structure (Remark 5.7). Most of the previous constructions of quasi-regular K-contact manifolds also admit regular K-contact structures for two reasons. Many of these manifolds are given by circle bundles over smooth symplectic manifolds (and hence they are regular). Other examples of K-contact manifolds have an initial K-contact structure which is quasi-regular and non-regular, but such manifolds also admit regular K-contact structures. One instance of this are Seifert circle bundles over weighted projective spaces $\mathbb{CP}^n_w$ whose total space is the sphere $\mathbb{S}^{2n+1}$. There are however more examples of Sasakian quasi-regular manifolds in [**10**] which do not admit regular structures. These are spin Smale-Barden manifolds with $H_2(X,\mathbb{Z})$ torsion and non-zero, and by Remark 4.39, they cannot admit regular Sasakian (or K-contact) structures

Sketch of the contents.

This thesis is organised as follows.

In Chapter 1 we give the basic tools from symplectic topology that will be used throughout this thesis. These are studied separately in different sections. In the first section we give a quick overview on linear symplectic algebra. Then, in the second section we study Darboux theorem, which allows to work locally on a symplectic manifold like if it were a symplectic vector space. The action of the group of symplectomorphisms on a symplectic manifold can be shown to be transitive with the help of Darboux theorem, and we do this in the third section. In the fourth section we give a standard model for the symplectic type of the tubular neighborhood of a symplectic submanifold inside a symplectic ambient manifold. This model will be generalized later to suborbifolds inside a symplectic ambient orbifold, and it will prove very useful in our symplectic resolution of orbifolds. In the fifth section we prove a method to substitute a pair S_1, S_2 of transversely and positively intersecting symplectic surfaces in a 4-manifold X for a single symplectic surface S so that S is smooth (no singular points) and the homology class of S is the same as the homology class of $S_1 \cup S_2$. This is done by replacing a small neighborhood in $S_1 \cup S_2$ of all the intersection points in $S_1 \cap S_2$ (such neighborhood is a cone) for a small cilinder, thus obtaining a smooth surface S representing the homology class of $S_1 \cup S_2$. In the sixth section we study the process of blowing-up in the symplectic category. We emphasise that there are two possible, equally valid methods of blowing-up, which we call *elementary* and *standard*. The standard method is the well-known method appearing in [**39**], while the elementary is maybe less well known but equally useful for our purposes. We also study the topology of the blow-up manifold, and the problems with the existence or not of proper transforms in the symplectic blow-up process. In the seventh section we sketch a method for constructing symplectic manifolds gluing two symplectic manifolds along a symplectic submanifold. This is called *symplectic sum*, and was extensively exploited by Gompf in [**27**] to construct many symplectic manifolds displaying non-Kahler behaviours. In the eighth and ninth sections we study Lefschetz and elliptic fibrations, which are important tools to construct symplectic manifolds. In the tenth and last section we give a way to transform Lagrangian submanifolds (inside an ambient symplectic manifold) into symplectic ones.

Chapter 2 gives the basics of orbifolds. We do not attempt to make a comprehensive study of orbifolds, for which we refer to the classical reference [**50**], or to [**9**] for a more modern approach. Our aims are simply to fix notation and establish some basic results of orbifolds that will be used in this thesis and maybe are not so standard in the classical literature of orbifolds. These results concern orbifold de Rham cohomology and some tensors on orbifolds, like orbifold almost-complex structures and orbifold symplectic forms. For instance, in this chapter it is proved the existence of orbifold almost-complex structures on a symplectic orbifold, as well as an orbifold version of the Darboux theorem, and also unitary local models for a simplectic orbifold.

Chapter 3 deals with one of the main results of this thesis, the problem of resolution of symplectic orbifolds. In the first section of this chapter we find a nice model for the tubular neighborhood ν_D of the isotropy set $D \subset X$ of a symplectic orbifold X. This is the previously mentioned generalization to orbifolds of the classical symplectic tubular neighborhood theorem for symplectic submanifolds (which we study in Chapter 1). In the second section we use the constructive resolution of singularities to construct a resolution $\widetilde{\nu}_D$ of the tubular neighborhood ν_D. This is done by resolving the singularities fiberwise in ν_D and then gluing these resolutions by lifting the action of the structure group of ν_D on its fibers to the resolutions of these fibers. We obtain thus the transition functions of a fiber bundle $\widetilde{\nu}_D$, whose fiber is the resolution of the fiber of ν_D. In the third and fourth sections we handle the issue of constructing symplectic forms. First we do this on the resolution $\widetilde{F}$ of the fiber F of ν_D, obtaining in this way a symplectic fiber-bundle structure on $\widetilde{\nu}_D$ in the third section. After that, in the fourth section we construct a global symplectic form in $\widetilde{\nu}_D$. For this globalization procedure we first need to overcome some cohomological restrictions. Once cohomology is handled, we need to extend to the case of non-campact fibers the well-known results to globalize a symplectic form on the fibers of a symplectic fiber bundle (with compact fibers) to a symplectic form on the total space of the bundle. Finally, in the fifth section we glue the symplectic form in $\widetilde{\nu}_D$ (constructed in the fourth section) with the original symplectic form in the orbifold X away from the isotropy locus. As a conclusion, in the sixth section we give some examples of orbifolds on which our method of resolution applies.

In Chapter 4 we explore the role of 4-orbifolds in 5-dimensional K-contact and Sasakian geometry. In the first and second sections we study the kind of orbifolds appearing in the context of K-contact and Sasakian geometry, i.e. *cyclic orbifolds* equipped with almost Kahler and Kahler orbifold structures. We show how symplectic manifolds and symplectic cyclic orbifolds are closely related, giving a method for constructing symplectic cyclic orbifolds starting from symplectic cyclic manifolds. In the third and fourth sections we study general Seifert bundles, mainly from the topological viewpoint. This type of twisted circle bundles are the link between orbifolds and K-contact or Sasakian manifolds, so its study is of crucial importance to us. The fourth section consists mainly of a careful study of the Leray spectral sequence of a general Seifert bundle. This spectral sequence relates the homology with integer coefficients of the base orbifold X and the total space M of the Seifert bundle. Crucially, the Leray sequence shows how in the second homology $H_2(M,\mathbb{Z})$ the genus of the isotropy surfaces of X can be detected. After this, in the fifth section we define Sasakian and K-contact structures, and show how both Seifert bundles and almost Kahler (Kahler) orbifolds play a central role in K-contact (Sasakian) geometry. We have decided to postpone the definition of K-contact and Sasakian structures until this point for the sake of linearity in the exposition, as our approach to these geometric satructures relies exclusively on Seifert bundles.

In Chapter 5, we conclude with the other main result of this Thesis, i.e. the construction of a semi-regular K-contact manifold not admitting any semi-regular Sasakian structure. This is done in several steps. The first step is to construct a symplectic 4-manifold X with $b = b_2(X)$ disjoint symplectic surfaces spanning $H_2(X, \mathbb{Q})$. This construction is somewhat involved, the first four sections of Chapter 5 being devoted to it. First, we make three symplectic sums of a 4-torus $\mathbb{T}^4$ with three copies of the elliptic fibration $E(1)$. We make these symplectic sums with some care, so as to being able to assure the existence of as many disjoint symplectic surfaces as possible. After that, we need to construct some additional symplectic surfaces. We do this by applying symplectic resolution of positive intersections, and symplectic blow-up, both studied in Chapter 1. We end up with the previously mentioned symplectic 4-manifold X with second homology generated by disjoint symplectic surfaces. Finally, in the fifth and last section of this Chapter, we construct Seifert bundles over some suitable 4-orbifold symplectic structure on the symplectic manifold X previously constructed. This 4-orbifold structure is constructed as follows. In each of the disjoint symplectic surfaces generating $H_2(X, \mathbb{Q})$ we put as isotropy a cyclic p-group of order p^i, with $p \in \mathbb{Z}$ a prime, and then we construct a symplectic orbifold structure on the manifold X with these surfaces as isotropy surfaces. Then we prove that these Seifert bundles (which are surely K-contact) cannot admit any Sasakian structures. This involves a theorem showing the non-existence of a Kahler complex surface S with too many disjoint complex curves of positive genus spanning the second homology.

Introducción.

Un orbifold es un espacio topológico que está localmente modelado en cocientes de bolas de $\mathbb{R}^n$ bajo la acción de un grupo finito. Los orbifolds han sido muy útiles en muchos cuestiones relacionadas con la geometría, veáse por ejemplo [**50**].

En el contexto de la geometría simpléctica, los orbifolds simplécticos se han introducido sobretodo como herramienta para construir variedades simplécticas con ciertas propiedades. La estatregia habitual es considerar la acción de un grupo en una variedad simpléctica por simplectomorfismos, de manera que el cociente es un orbifold simpléctico. Dicha acción se elige de forma que el orbifold cociente tenga propiedades geométricas o topológicas deseables, para después aplicar un proceso de resolución simpléctica a este orbifold y así obtener una variedad simpléctica con las propiedades geométricas o topológicas deseadas, suponiendo que se pueda controlar el proceso de resolución para que estas propiedades sean conservadas.

Por otra parte, en el contexto de las geometrías de K-contacto y Sasakiana, los orbifolds aparecen de manera natural como la estructura que hereda el espacio cociente de las órbitas del campo de Reeb en una variedad de K-contacto o Sasakiana. De hecho, la estructura orbifold del espacio cociente de órbitas recoge toda la información relevante de una estructura de K-contacto o Sasakiana, de tal manera que las últimas estructuras geométricas pueder ser recuperadas a partir de la estructura de orbifold casi-Kahler o Kahler del espacio cociente.

En ambos casos vemos un hilo conductor común: la posibilidad de construir una variedad con una estructura geométrica determinada a partir de un orbifold con una correspondiente estructura geométrica. Esto último es aplicable al proceso de resolución simpléctica de orbifolds, en el cual se empieza con un orbifold simpléctico y a partir de él se construye una variedad simpléctica (la resolución). Es asímismo aplicable al proceso de construccción de una estructura de K-contacto o Sasakiana en una variedad a partir de un orbifold casi-Kahler o Kahler, donde ahora el método de obtención de la variedad a partir del orbifold consiste en tomar un fibrado de Seifert de círculos (sobre dicho orbifold) de tal manera que el espacio total del fibrado sea una variedad diferenciable. En esta tesis se exploran ambos métodos para construir variedades a partir de orbifolds.

Dejemos claro aquí cuál es nuestra convención en lo que refiere a los distintos puntos que se pueden encontrar en un orbifold. Sea X un orbifold, y sea $p \in X$ un punto. Decimos que p es un *punto de isotropía* de X si su grupo de isotropía en el orbifold X es no trivial. Decimos que p es un *punto singular* de X si el espacio topológico subyacente a X no es una variedad topológica cerca de p. Nótese que puede haber puntos de isotropía que no sean puntos singulares, acorde a las definiciones anteriores.

Destaquemos también que la palabra *orbifold* en esta tesis está reservada para orbifolds tal que los cambios de cartas orbifold de su atlas asociado son $\mathcal{C}^\infty$, también llamados orbifolds diferenciables. Asímismo, la palabra *variedad* significa variedad diferenciable.

Problema 1: resolución simpléctica de orbifolds.

Una motivación de la utilidad de desarrollar un proceso sitemático de resolución de orbifolds simplécticos es el hecho de que el método de construcción de variedades simplécticas pasando primero por construir un orbifold simpléctico para luego resolverlo ha probado ser un método muy poderoso para construir variedades simplécticas, proporcionando bastantes ejemplos de variedades simplécticas con propiedades muy interesantes. Sin embargo, hasta el momento este método de construcción de variedades no es en absoluto sistemático, ya que para cada orbifold particular se ha tenido que desarrollar un proceso de resolución simpléctica específico, usando propiedades muy particulares del orbifold en cuestión. Ilustremos dos ejemplos de resultados importantes en este contexto:

- Fernández y Muñoz construyen en [**21**] el primer ejemplo de una variedad simplemente conexa, simpléctica y no-formal de dimensión 8. Dicha variedad es construída como la resolución de cierto orbifold simpléctico de dimensión 8. Para llevar a cabo la resolución simpléctica de este orbifold, usan el método de resolución simpléctica de orbifolds con singularidades aisladas, método desarrollado por ellos mismos y Cavalcanti en [**16**]. Por otro lado, Bazzoni y Muñoz han probado en [**6**] que esta misma variedad construida en [**21**] también tiene una estructura compleja, con lo que es simpléctica y compleja, pero no es Kahler (pues no es formal).
- Bazzoni, Fernández y Muñoz han construido en [**4**] el primer ejemplo de una variedad simplemente conexa de dimensión 6 que es simultáneamente compleja y simpléctica pero no es Kahler. Para construir esta variedad lo primero que hacen es construir un orbifold de dimensión 6 con estratos de isotropía de dimensión 0 y 2, para más tarde desarrollar un método de resolución simpléctica de este orbifold y obtener así la variedad de dimensión 6 buscada. Como antes mencionamos, este método de resolución simpléctica es ad-hoc para el orbifold particular, ya que en este caso el fibrado normal al estrato de puntos de isotropía de dimensión 2 resulta ser un fibrado trivial.

Esto lleva a plantearse la cuestión de si es posible desarrollar un procedimiento sistemático para resolver orbifolds simplécticos, llamemos a esto *el problema de resolución simpléctica*. Se conocen bastantes analogías entre las geometrías compleja y simpléctica, así que uno podría (si es optimista) esperar analogías en los procesos de desingularización para orbifolds simplécticos y algebraicos. Sin embargo, incluso para las construcciones más básicas como el blow-up simpléctico (explosión simpléctica) y la definición de transformada propia de una subvariedad, vemos notables diferencias entre las categorías simpléctica y algebraica. Al lector interesado referimos a la subsección 6.4 del capítulo 1.

Una vez motivado el interés, procedemos a enunciar brevemente la historia del problema de resolución simpléctica. La cuestión del blow-up (explosión) y la resolución de singularidades en la categoría simpléctica fue inicialmente planteada por Gromov en [**27**]. Al cabo de unos pocos años, la explosión simpléctica de una variedad simpléctica en una subvariedad simpléctica fue estudiada y definida rigurosamente por McDuff en [**40**], para luego usarlo en la primera construcción de una variedad simplemente conexa, simpléctica, y que no admite estructuras Kahler. Esta variedad es concretamente $\mathbb{CP}^5$ explotado en la variedad de Kodaira-Thurston.

Después de esto, McCarthy and Wolfson desarrollan en [**37**] un método de resolución simpléctica para orbifolds con singularidades aisladas en dimensión 4. Este método consiste en encontrar una hipersuperficie adecuada a través de la cual hacer un proceso de cortar y pegar para sustituir el punto singular por uno regular.

Más tarde, Cavalcanti, Fernández y Muñoz desarrollaron el antes mencionado método de [**16**] que sirve para resolver singularidades aisladas de orbifolds simplécticos en cualquier dimensión. Esta resolución consiste en construir una resolución local de los puntos singulares y después pegar estas resoluciones locales al resto del orbifold. La resolución local en un entorno de los puntos singulares se construye usando los resultados de geometría algebraica, disponibles ya que todo punto de un orbifold simpléctico tiene una estructura Kahler *local* en un pequeño entorno. Después, la forma simpléctica orbifold debe ser deformada cerca de los puntos singulares, para que coincida con la forma Kahler que la resolución local tiene como variedad Kahler. El método de resolución que se desarrolla en esta tesis es una generalización de esta idea a tipos de singularidades de orbifolds más generales.

Por otra parte, Niederkruger y Pasquotto en [**45, 46**] dan un método para resolver algunos tipos singularidades de orbifolds simplécticos usando técnicas de resolución simpléctica. Estas técnicas funcionan para ciertos tipos de singularidades, incluyendo las singularidades que presentan los orbifolds cíclicos simplécticos, incluso cuando éstas no son aisladas.

También, en [**19**], Chen detalla un método para resolver orbifolds simplécticos de dimensión 4 arbitrarios. Usa de manera esencial las restricciones que la dimensión 4 impone a las singularidades de un orbifold simpléctico, como el hecho de que en esta dimensión los puntos singulares (aquellos puntos que no son localmente euclídeos) deben ser aislados, y el hecho de que los puntos no singulares y con isotropía no trivial deben estar dispuestos en superficies que intersecan transversalmente. Su método de resolución es, por tanto, sólo un pequeño paso extra en comparación con el método de [**4**] aplicado a dimensión 4, ya que la única novedad son las superficies de isotropía formadas por puntos localmente euclídeos. Cerca de estos puntos el orbifold es ya una variedad, con lo que sólo es necesario deformar la forma simpléctica orbifold y convertirla en una forma simpléctica ordinaria de la variedad topológica subyacente al orbifold. El método de resolución propuesto en [**19**] usa técnicas de reducción simpléctica, y como aplicación a esta resolución simpléctica se dan algunas restricciones para la existencia de acciones simplécticas de grupos finitos en variedades simplécticas de dimensión 4. Merece la pena destacar que es posible dar una forma más elemental para deformar la forma simpléctica orbifold de un orbifold de dimensión 4 cerca de una superficie de isotropía, para ello referimos al lector al Capítulo 4, Sección 2.

Otro reciente resultado sobre resolución de orbifolds simpléctica se encuentra en el ya mencionado ejemplo de Bazzoni, Fernández y Muñoz ([**4**]), donde se construye por primera vez la resolución simpléctica de un orbifold simpléctio con estrato de puntos singulares de dimensión positiva, concretamente 2. Para esta resolución se usa el hecho particular de que dicho estrato de puntos singulares tiene fibrado normal trivial, facilitando esto el proceso de resolución simpléctica.

Una vez contextualizado el problema de resolución simpléctica, pasemos a hablar brevemente de los resultados que se presentan aquí. En esta tesis damos un resultado parcial al problema de resolución simpléctica, avanzando considerablemente en cuanto al tipo de singularidades de orbifolds simplécticos que se saben resolver. Se muestra una manera sistemática de construir una resolución simpléctica de un tipo de orbifolds llamados *orbifolds de isotropía homogénea*. Dichos orbifolds son aquellos orbifolds X que cumplen que sus puntos de isotropía están dispuestos un subvariedades conexas disjuntas D_i, en cada una de las cuales la isotropía es constante, es decir grupos de isotropía en puntos distintos de D_i tienen acciones equivalentes. Cada una de estas subvariedades de isotropía D_i decimos que tiene isotropía homogénea. Nótese que la dimensión de los D_i no está restringida, con lo que un orbifold de isotropía homogénea puede tener subvariedades de puntos singulares de dimensión arbitraria.

Por tanto, en el contexto del problema de resolución simpléctica, se presenta un método para resolver un tipo más general de singularidades en un orbifold simpléctico de dimensión $2n$. Como los puntos singulares de un orbifold con isotropía homogénea no son necesariamente aislados, nuevas técnicas se necesitan para abordar la resolución en este caso. Estas técnicas se basan en encontrar un modelo del fibrado normal ν_{D_i} de las surperficies D_i que sea manejable, dándole una estructura adecuada (en concreto, una estructura de fibrado orbifold con grupo de estructura unitario, y con grupo de isotropía unitario y constante). Usando esta estructura es posible utilizar en cada una de las fibras la resolución canónica de singularidades en la categoría algebraica, desarrollada por Encinas y Villamayor en [**20**], para luego pegar estas resoluciones locales (hechas en abiertos trivializantes del fibrado normal ν_{D_i}) y así obtener una resolución global $\widetilde{X}$ de X. Una vez obtenida $\widetilde{X}$ como variedad, nos encargamos de construir una forma simpléctica en $\widetilde{X}$ que coincida con la forma simpléctica original de X fuera de un entorno de los puntos de isotropía de X, dando así la deseada resolución simpléctica de X.

La estrategia para llevar a cabo todo esto es dotar al fibrado normal ν_D (de una superficie de isotropía $D \subset X$) de una reducción a $\mathrm{U}(k)$ del grupo de estructura, siendo $2k$ la codimensión de D en X. Las singularidades de X en los puntos de D son sigularidades de tipo cociente en las fibras $F = \mathbb{C}^k/\Gamma$ de ν_D, donde Γ es el grupo de isotropía de los puntos de D. La resolución estándar de singularidades en geometría algebraica permite encontrar una resolución de cada una de las fibras F de ν_D separadamente, pero necesitamos que estas resoluciones se peguen correctamente al cambiar de trivialización. Para llevar esto a cabo necesitamos una mejora del teorema clásico de resolución de singularidades de Hironaka [**31**]. Esta mejora es la resolución canónica (o constructiva) de singularidades desarrollada por Encinas y Villamayor en [**20**], la cual es compatible con acciones de grupos en el sentido de que las acciones de grupos en el espacio singular se pueden elevar a la resolución. Usando este resultado podemos construir la resolución $\widetilde{X}$ como una variedad diferenciable. Queda ahora la tarea de poner en $\widetilde{X}$ una forma simpléctica que coincida con la de X fuera de los puntos singulares.

La resolución $\widetilde{X}$ tiene, cerca de D, la estructura de un fibrado sobre D con fibra la resolución $\widetilde{F}$ de $F = \mathbb{C}^k/\Gamma$. Tanto la base como la fibra del espacio total de este fibrado $\widetilde{\nu}_D$ son simplécticas, pero esto no garantiza que el espacio total del fibrado admita una forma simpléctica, y lo primero para probar que en efecto sí lo hace es ver que la obstrucción cohomológica para que esto ocurra se satisface. Esta restricción cohomológica consiste en encontrar una clase de cohomología en el espacio total $\widetilde{\nu}_D$ tal que su restricción a cada una de las fibras $\widetilde{F}$ induzca la clase de cohomología de la forma simpléctica de la fibra. Una vez encontrada dicha clase de cohomología, tenemos que desarrollar un proceso de globalización para poder construir una forma simpléctica en $\widetilde{\nu}_D$. Estas técnicas de globalización están bien documentadas en la bibliografía en el caso en que la fibra sea compacta, pero aquí debemos adaptarlas al caso en que la fibra es no compacta. Una vez la form simpléctica en $\widetilde{\nu}_D$ está construida, el paso final es pegar fuera de un entorno de D esta forma simpléctica con la forma simpléctica orbifold inicial de X.

El resultado principal sobre resolución simpléctica de orbifolds de esta tesis está en el capítulo 3 y es el siguiente.

THEOREM 0.4. *Sea (X, ω) un orbifold simpléctico con isotropía homogénea. Entonces existe una variedad simpléctica*

$$(\widetilde{X}, \widetilde{\omega})$$

y una aplicación diferenciable

$$b : (\widetilde{X}, \widetilde{\omega}) \to (X, \omega)$$

tal que b es un simplectomorfismo fuera de un entorno de los puntos de isotropía de X. Además, dicho entorno se puede tomar arbitrariamente pequeño.

Finalmente, proporcionamos algunos ejemplos en los que el teorema anterior se puede aplicar.

Problema 2: variedades de K-contacto sin estructuras Sasakianas.

Ahora introducimos brevemente la otra cuestión principal de esta tesis: la construcción de variedades de K-contacto en dimensión 5 que no admitan estructuras Sasakianas y con el grupo fundamental más pequeño posible.

En los últimos años ha habido mucho interés en construir variedades de K-contacto que no admitan estructuras Sasakianas. Este es un problema análogo en dimensión impar al ya asentado y famoso problema de encontrar variedades simplécticas sin estructuras Kahler. El reciente libro [**9**] es el libro de referencia estándar sobre investigación en geometría Sasakiana, en el que se exponen un buen número de problemas abiertos relacionados con la geometría Sasakiana. Uno de ellos es el de encontrar variedades de K-contacto que no admitan estructuras Sasakianas, que sigue abierto a día de hoy en el caso de dimensión 5 y simplemente conexo.

Contextualizemos un poco este problema. Una herramienta importante para encontrar variedades compactas (en lo que sigue, todas las variedades se asumen compactas), de K-contacto y sin estructuras Sasakianas es el hecho de que para una variedad Sasakiana de dimensión $2n+1$, los números de Betti b_i con $1 \leq i \leq n$, y con i impar, deben ser pares, mientras que esta propiedad es falsa para variedades de K-contacto. Esta propiedad de las variedades Sasakianas se deriva del bien conocido hecho de que la cohomología básica de la foliación de Reeb de una variedad Sasakiana tiene una estructura de Hodge natural, lo cual es falso para estructuras de K-contacto generales.

La paridad de b_1 fue usada para construir los primeros ejemplos de variedades de K-contacto sin estructuras Sasakianas en [**9**, example 7.4.16], los cuales son variedades de K-contacto con primer número de betti b_1 impar, y por tanto sin estructuras Sasakianas.

El grupo fundamental se puede también usar para construir variedades de K-contacto sin estructuras Sasakianas, puesto que la existencia de una estructura Sasakiana impone fuertes restricciones al grupo fundamental de una variedad Sasakiana. Tanto es así que los grupos isomorfos al grupo fundamental de una variedad Sasakiana reciben el nombre de *grupos de Sasaki*. Usando esto, es posible construir variedades de K-contacto sin estructuras Sasakianas como en [**17**], y también es posible construir ejemplos en dimensión 5 de variedades resolubles (solvmanifolds) de K-contacto, sin estructuras Sasakianas y que satisfacen la propiedad fuerte de Lefschetz (véase [**15**]), probando así que las restricciones en el grupo fundamental impuestas por la existencia de estructuras Sasakianas son independientes de las restricciones cohomológicas.

Cuando uno se propone abordar este problema para variedades simplemente conexas, la tarea se vuelve más complicada. Éste es el problema abierto 7.4.1 en [**9**],

Existen variedades de K-contacto, simplemente conexas, y sin estructuras Sasakianas?

Puesto que en este caso el primer número de Betti b_1 no está disponible como obstrucción, el recurso más próximo es intentar usar el tercero, b_3. Usando esto, se han construido en [**28**] variedades de K-contacto simplemente conexas y sin estructuras Sasakianas (con b_3 impar) de cualquier dimensión ≥ 9.

Aparte de la paridad de los números de Betti impares y del grupo fundamental, otro hecho importante que distingue la existencia de estructuras Sasakianas es el hecho de que el álgebra de cohomología de de Rham de una variedad Sasakiana satisface la propiedad dura de Lepschetz, (aquí, es multiplicar por $\eta \wedge (d\eta)^p$ lo que induce isomorfismos en cohomología de De Rham), véase [**13**]. De esta manera se han construido ejemplos de variedades de K-contacto sin estructuras Sasakianas en dimensiones 5 y 7 (con grupo fundamental no trivial), véase [**14**]. Estos ejemplos son nil-variedades cuyos números de Betti b_i, $i \leq n$ impar, satisfacen que b_i son pares (y con $b_1 \neq 0$). Se demuestra además que la obstrucción a la existencia de estructuras Sasakianas impuesta por la paridad de los números de Betti impares es más débil que la impuesta por la propiedad dura de Lefschetz, ya que la propiedad dura de Lefschetz induce una forma bilineal no degenerada y antisimétrica en los grupos de cohomología de dimensión i con $0 \leq i \leq n$, i impar, lo que implica que los correspondientes b_i deben ser pares. Por otra parte, en [**36**] se usa la propiedad dura de Lefschetz para dar ejemplos de variedades de K-contacto, simplemente conexas y sin estructuras Sasakianas en cualquier dimensión ≥ 9.

En los artículos [**52**] y [**7**] se estudia el tipo de homotopía racional de las variedades Sasakianas. Concretamente, en [**7**] se prueba que todos los productos de Massey de orden ≥ 4 se anulan en una variedad Sasakiana simplemente conexa, pudiendo haber productos triples de Massey no nulos. Usando esta obstrucción se construyen se construyen ejemplos de variedades de K-contacto sin estructuras Sasakianas en cualquier dimensión ≥ 17.

Centremos ahora atención al problema de encontrar variedades de K-contacto sin estructuras Sasakianas y *simplemente conexas*. Este problema fue resuelto para dimensiones ≥ 9 in [**13, 14, 28**], donde tales ejemplos se construyeron. Más tarde este problema fue también resuelto para dimensión 7 en [**43**], donde, mediante una combinación de varias técnicas basadas en teoría de homotopía y geometría simpléctica, se construyó una variedad de K-contacto simplemente conexa de dimensión 7 y con tercer número de Betti impar, por tanto sin estructuras Sasakianas.

En dimensión 5, este problema parece ser considerablemente más difícil. En efecto, los números de Betti, productos de Massey, ni la propiedad dura de Lefschetz (ni por supuesto el grupo fundamental) obstruyen la existencia de variedades Sasakianas simplemente conexas en dimensión 5. Esto es debido a:

- que $5 = 2n+1$ con $n = 2$, por lo que b_1 es el único número de Betti disponible, y debe ser $b_1 = 0$;
- que en dimensión ≤ 6 toda variedad simplemente conexa es formal, véase [**41**];

Por otra parte, en dimensión 3 la clasificación de las variedades de esta dimensión ayuda a demostrar que toda variedad de K-contacto es automáticamente Sasakiana, véase [**32**], lo que resuelve el problema.

Por tanto, para variedades Sasakianas simplemente conexas de dimensión 5, las obstrucciones a estructuras Sasakianas anteriormente expuestas no son tal, y parece que las únicas obstrucciones a la existencia de estructuras Sasakianas relevantes en esta dimensión son propiedades bastante sutiles que se detectan en la parte de torsión del segundo grupo de homología sobre $\mathbb{Z}$, véase [**34**]. Esta dificultad del problema en dimensión 5 hace que la existencia de variedades de K-contacto sin estructuras Sasakianas en dimensión 5 sea todavía un problema abierto. Para abordar este problema todo indica que hay que usar los argumentos de [**34**] para extraer obstrucciones a la existencia de estructuras Sasakianas derivadas de la clasificación de superficies complejas Kahler.

Una variedad compacta y simplemente conexa de dimensión 5 se llama una *variedad de Smale-Barden*, nombre dado debido que en la clasificación de estas variedades ambos contribuyeron. Estas variedades están clasificadas topológicamente por el segundo grupo de homología sobre $\mathbb{Z}$ y la clase de Stiefel-Whitney. El capítulo 10 del libro [**9**] escrito por Boyer y Galicki describe algunas variedades de Smale-Barden que admiten estructuras Sasakianas. El siguiente problema está todavía abierto (problema abierto 10.2.1 en [**9**]):

Determinar si existen variedades de Smale-Barden que admitan estructuras de K-contacto y no admitan estructuras Sasakianas.

En esta tesis damos un primer paso en la resolución en afirmativo de este problema. Una *variedad de Smale-Barden homológica* es una variedad compacta de dimensión 5 con primer grupo de homología nulo. Una estructura de K-contacto o Sasakiana es *regular* si las órbitas del flujo de Reeb definen una foliación con estructura de fibrado de círculos sobre una variedad, es *quasi regular* si la foliación de Reeb tiene estructura de fibrado de círculos sobre un orbifold (cíclico) arbitrario, y es *semi regular* si dicha foliación de Reeb tiene estructura de fibrado de Seifert de círculos sobre un orbifold (cíclico) cuyo espacio topológico subyacente es una variedad topológica (lo que quiere decir que el orbifold tiene subvariedades de isotropía solamente en codimensión 2). Cualquier variedad que admita una estructura de K-contacto (resp. Sasakiana) también admite una estructura de K-contacto (resp. Sasakiana) quasi regular. La condición de semi regularidad es una pequeña hipótesis adicional. Con estas nociones dadas, el principal resultado de la presente tesis en cuanto al problema abierto anterior (Teorema 5.8) es el siguiente:

THEOREM 0.5. *Existe una variedad de Smale-Barden homológica que admite una estructura de K-contacto semi regular pero no admite ninguna estructura Sasakiana semi regular.*

Para contextualizar mejor este resultado, merece la pena llamar la atención sobre las obstrucciones a la existencia de estructuras Sasakianas halladas por Kollar en [**34**]. Si una variedad M de dimensión 5 admite una estructura Sasakiana, también admite una estructura Sasakiana quasi regular, con lo que es un fibrado de Seifert de círculos sobre un orbifold Kahler X, cuyas subvariedades de isotropía son una colección de curvas D_i complejas y una colección de puntos singulares, que pueden estar tanto aislados con en una de las curvas de isotropía. En ciertos casos, el género de las curvas $D_i \subset X$ se puede recuperar con información topológica de M que involucra la parte de torsión del segundo grupo de homología $H_2(M, \mathbb{Z})$. En [**34**] se dan ejemplos de variedades M de dimensión 5 que son el espacio total de un fibrado de Seifert de círculos tal que los géneros de las curvas de isotropía no satisfacen la fórmula de adjunción. Este fibrado está construido de tal modo que para cualquier otra estructura de fibrado Seifert de círculos en M, el género de las curvas de isotropía es recuperable a partir de $H_2(M, \mathbb{Z})$. Por tanto, ninguna estructura de fibrado de Seifert de círculos en M puede dar lugar a una estructura Sasakiana, ya que en tal caso la base de dicho fibrado de Seifert sería una superficie Kahler y las correspondientes curvas de isotropía serían complejas y por tanto (las no singulares) deberían satisfacer la fórmula de adjunción, lo cual no ocurre por construcción. El inconveniente de este argumento para la cuestión que nos ocupa, es que no sirve para distinguir variedades de K-contacto y Sasakianas, ya que las subvariedades simplécticas de dimensión 2 (superficies simplécticas) en una variedad simpléctica de dimensión 4 también satisfacen la fórmula de adjunción relacionando el género y la clase de homología. Aún así, el tipo de argumentos utilizados, teniendo en cuenta la posibilidad de obtener información sobre las superficies de isotropía $D_i \subset X$ a partir de la torsión de M, se pueden variar de muchas maneras para acomodarlos al problema de distinguir variedades de K-contacto y Sasakianas, puesto que el comportamiento de una configuración de superficies simplécticas en una 4-variedad simpléctica difiere en varios aspectos del comportamiento de una correspondiente configuración de curvas complejas en una superficie Kahler.

El argumento que en esta tesis se usa a este respecto sigue las siguientes líneas. Sea M una variedad Sasakiana de dimensión 5. Entonces M admite una estructura Sasakiana quasi regular, y por tanto es un fibrado de Seifert de círculos sobre un orbifold Kahler X. Asumiendo además que la estructura Sasakiana es semi regular, podemos asegurar que dicho orbifold Kahler tiene una estructura de superficie compleja, de modo que los puntos de isotropía del orbifold forman una configuración de curvas complejas sin singularidades $D_i \subset X$. Con la hipótesis de que $H_1(M, \mathbb{Z}) = 0$, una serie de argumentos topológicos permiten asegurar que las curvas D_i generan $H_2(X, \mathbb{Q})$, y si las isotropías de las curvas están bien elegidas se puede asegurar además que las curvas D_i son disjuntas, y por tanto diagonalizan la forma de intersección de X. Asímismo, la torsión de $H_2(M, \mathbb{Z})$ permite recuperar los géneros de las curvas D_i. Si somos capaces de probar que una tal configuración de curvas complejas D_i en una superficie Kahler X satisfaciendo lo anterior no puede existir para cierta elección de géneros g_i, entonces habremos encontrado una nueva obstrucción a la existencia de estructuras Sasakianas semi regulares.

De esta manera, para construir variedades de K-contacto de dimensión 5 sin estructuras Sasakianas semi regulares, necesitamos construir primero un orbifold simpléctico X de dimensión 4 con una adecuada configuración de superficies de isotropía simplécticas (deben generar $H_2(X, \mathbb{Q})$, ser disjuntas, y tener géneros positivos). Este orbifold X se construye poniendo una estructura de orbifold simpléctico en una variedad simpléctica de dimensión 4 con 36 superficies simplécticas disjuntas, siendo estas 36 superficies las superficies de isotropía del orbifold, cada una con grupo de isotropía cíclico de orden p^i, con $p \in \mathbb{Z}$ un número primo impar. Después, hay que tomar un fibrado de Seifert de círculos M sobre este orbifold X, para luego probar que M no admite una estructura Sasakiana semi-regular usando la no existencia de una configuración de curvas complejas en una superficie Kahler con las propiedades anteriores (ser disjuntas, generar la segunda homología y tener determinados géneros). En este espíritu, se propone la siguiente conjetura:

No existe una variedad Kahler, o un orbifold Kahler, con $b_1 = 0$ y con una configuración de curvas complejas disjuntas de géneros positivos que generen $H_2(X, \mathbb{Q})$.

Damos el primer resultado en esta dirección con alguna hipótesis técnica extra respecto de las hipótesis planteadas en la anterior conjetura, resultado que en todo caso nos sirve para nuestros propósitos con respecto al teorema 0.5.

THEOREM 0.6. *Sea S una superficie Kahler con $H_1(S, \mathbb{Q}) = 0$ y con una configuración de curvas complejas, no singulares y disjuntas $D_1, \ldots, D_b$, siendo $b = b_2(S)$, y con géneros $g_i = g(D_i) > 0$ satisfaciendo además:*

- *al menos dos de ellas tienen género $g_i > 1$,*
- *todas tienen $g_i \leq 3$, o de otra manera $g := \max\{g_i : 1 \leq i \leq 36\} \leq 3$.*

Entonces se cumple que $b = b_2(S) \leq 2g + 3$.

Nuestra construcción de una variedad de K-contacto en dimensión 5, con primer grupo de homología nulo y sin estructuras Sasakianas semi regulares (es decir, una variedad de Smale-Barden homológica de K-contacto sin estructuras Sasakianas semi regulares), está basada en la previa construcción de una variedad simpléctica de dimensión 4 con una configuración de superficies simplécticas disjuntas generando el segundo grupo de homología y con géneros $1 \leq g_i \leq 3$, siendo tres de los g_i iguales a 3, de modo que se satisfacen las condiciones de los géneros del Teorema 0.6. Dicha variedad simpléctica se construye en la sección 1 del capítulo 5.

Finalmente, las variedades de K-contacto que construimos en el Teorema 0.5 tienen también interés ya que dan ejemplos de variedades de K-contacto de dimensión 5, quasi regulares y que no admiten estructuras de K-contacto regulares, véase la observación 5.7. Este no es el primer ejemplo de tales variedades de K-contacto, pues en [**10**] se construyen ejemplos de variedades con estructuras Sasakianas quasi-regulares que no admiten estructuras de K-contacto regulares. La razón de la imposibilidad de dotar a estas variedades de estructuras de K-contacto regulares es la existencia de torsión no trivial en su segundo grupo de homología, por lo que la observación 4.39 descarta la existencia en ellas de una estructura de fibrado de círculos sobre una variedad de dimensión 4, y en particular descarta la existencia de estructuras de K-contacto regulares. Sin embargo, la inmensa mayoría de construcciones previas de variedades de K-contacto quasi regulares también admiten estructuras de K-contacto regulares, bien porque la estructura de K-contacto de partida es ya regular desde un inicio (las variedades están dadas como fibrados de círculos sobre variedades simplécticas), o bien porque, siendo la estructura de K-contacto de partida quasi regular y no regular, es sencillo ver que en dicha variedad también se pueden poner estructuras de K-contacto regulares. Este último es el caso de los fibrados de Seifert sobre los espacios proyectivos con pesos $\mathbb{CP}^n_w$, cuyo espacio total es la esfera $\mathbb{S}^{2n+1}$ (la cual obviamente admite una estructura Sasakiana regular ya que es un fibrado de círculos sobre $\mathbb{CP}^n$).

Esquema de los contenidos.

Esta tesis está organizada de la siguiente manera.

En el capítulo 1 se dan las herramientas y técnicas básicas de geometría y topología simpléctica que se usan por doquier en la tesis. Dichas técnicas se estudian separadamente por secciones. En la primera sección damos un breve resumen de los resultados del álgebra lineal simpléctica. En la segunda, se estudia el teorema de Darboux, el cual permite trabajar localmente en una variedad simpléctica como si fuera un espacio vectorial simpléctico, mientras que en la tercera se usa este teorema para demostrar que el grupo de simplectomorfismos actúa transitivamente en una variedad simpléctica. En la cuarta sección damos un modelo estándar de un entorno tubular de una subvariedad simpléctica dentro de una variedad simpléctica ambiente, modelo que se generalizará más adelante (en el capítulo 3) para las subvariedades de isotropía de un orbifold simpléctico. En la quinta sección se estudia un método para construir una superficie simpléctica S (conexa y sin singularidades) dentro de una variedad simpléctica X de dimensión 4 partiendo de dos subvariedades simplécticas $S_1, S_2 \subset X$ que intersequen transversal y positivamente, de modo que S represente en homología la misma clase que $S_1 \cup S_2$, y sin cambiar el espacio ambiente X. Esto se hace sustituyendo un pequeño entorno $U^p \subset S_1 \cup S_2$ de cada punto de intersección $p \in S_1 \cap S_2$ (nótese que U^p es homeomorfo a un cono) por un cilindro C_p de modo que las singularidades de $S_1 \cup S_2$ se sustituyen por pequeños cilindros. En la sexta sección se estudia el proceso de explosión en la categoría simpléctica. Enfatizamos que no hay una única manera de llevar a cabo este proceso, y damos dos maneras diferentes e igualmente válidas de construir la explosión de una variedad simpléctica en un punto, maneras que llamamos respectivamente *elemental* y *estándar*, siendo la manera elemental más sencilla de desarrollar y quizás menos conocida, mientras que la manera estándar es la que se ha adoptado clásicamente como método de explosión, y es la que aparece en el libro de McDuff y Salamon [**39**]. Ambas son igualmente útiles para nuestros propósitos. En esta sexta sección también se estudia la topología de la variedad simpléctica obtenida tras el proceso de explosión, comparándola con la variedad simpléctica inicial. En la séptima sección repasamos un bien conocido método para construir variedades simplécticas mediante un método de cortar y pegar, el cual suprime un entorno de una variedad simpléctica encajada como subvariedad dentro en dos variedades simplécticas distintas para luego pegar estas dos variedades a lo largo de su borde. Este método para construir variedades simplécticas se llama suma simpléctica y fue explotado intensivamente por Gompf en [**27**] donde la suma simpléctica se usó para construir un gran número de variedades simplécticas con numerosos comportamientos topológicos y geométricos imposibles en el mundo Kahler, por tanto probando que hay una diferencia considerable entre ambos mundos. En las secciones octava y novena se estudian las fibraciones de Lefschetz y las fibraciones elípticas respectivamente, las cuales constituyen una importante herramienta para construir variedades simplécticas. En la décima y última sección se estudia una manera para transformar subvariedades Lagrangianas (de una variedad simpléctica ambiente) en subvariedades simplécticas.

En el capítulo 2 se estudian las propiedades básicas de los orbifolds. No se pretende realizar un estudio completo de la teoría de orbifolds (para lo cual referimos al lector a la referencia clásica [**50**] o a [**9**] para un enfoque más moderno) sino que los objetivos son sencillamente fijar notaciones y enunciar (y en su caso demostrar) resultados sobre orbifolds que se usan a lo largo de esta tesis y que quizás no son resultados estándar que se encuentran habitualmente en la bibliografía sobre orbifolds. Dichos resultados abarcan principalmente la cohomología de De Rham orbifold, estructuras casi-complejas orbifold, y algunos teoremas clásicos de geometría simpléctica extendidos a formas simplécticas orbifold. Estos incluyen la existencia de estructuras casi-Kahler en orbifolds simplécticos y una versión orbifold del teorema de Darboux en la que se obtiene además un modelo local del orbifold con grupo de isotropía unitario.

En el capítulo 3 se aborda uno de los principales problemas de la tesis, precisamente el problema de resolución simpléctica de orbifolds con isotropía homogénea. En la sección primera demostramos la existencia de un modelo simpléctico adecuado de un entorno tubular suficientemente pequeño de una subvariedad $D \subset X$ de isotropía homogénea dentro de un orbifold simpléctico X. En la sección segunda usamos la resolución constructiva de singularidades de [**20**] para construir una resolución $\widetilde{\nu}_D$ del entorno tubular de una subvariedad de isotropía cualquiera $D \subset X$. Nótese que dicho entorno tubular se puede identificar mediante un simplectomorfismo con el fibrado normal ν_D, resultado que se prueba en la sección anterior. Por tanto, para construir una resolución simpléctica de un entorno tubular de D podemos tomar el fibrado normal orbifold ν_D y resolver cada una de las fibras de ν_D, las cuales todas tienen la forma $F = \mathbb{C}^k/\Gamma$, siendo $2k$ la codimensión de D en X y siendo Γ el grupo de isotrpía común de todos los puntos de D. Las resoluciones de las diferentes fibras de ν_D se llevan a cabo tomando trivializaciones del fibrado orbifold ν_D, y luego distintas resoluciones asociadas a distintas trivializaciones se deben pegar, para lo cual usamos que la acción del grupo de estructura de ν_D en la fibra $\mathbb{C}^k/\Gamma$ se puede ver como la restricción de una acción algebraica, y por tanto es posible elevarla a la resolución de singularidades constructiva de la fibra, obteniendo así unas funciones de transición que permiten construir la resolución $\widetilde{\nu}_D$ como un fibrado sobre D con fibra $\widetilde{F}$ la resolución de F. En las secciones tercera y cuarta nos encargamos de construir formas simplécticas. Primeramente, en la tercera sección hacemos esto en la resolución $\widetilde{F}$ de la fibra, obteniendo una forma simpléctica en las fibras de $\widetilde{\nu}_D$ que es invariante por las funciones de transición de dicho fibrado, y que por tanto define en él una estructura de fibrado con fibras simplécticas. Después de esto, en la cuarta sección construimos una forma simpléctica global en $\widetilde{\nu}_D$. Para realizar este proceso de globalización, primero tenemos que ver que se satisfacen ciertas obstrucciones topológicas que conciernen el segundo grupo de cohomología de $\widetilde{\nu}_D$. Una vez estas restricciones son superadas, necesitamos extender al caso en que las fibras no son compactas las técnicas usuales de globalización de formas simplécticas, y de esta manera utilizamos particiones de la unidad para construir una forma simpléctica global en $\widetilde{\nu}_D$ a partir de las formas simplécticas en las fibras $\widetilde{F}$ y en la base D. Finalmente, en la quinta sección nos encargamos de pegar la previamente construida forma simpléctica en $\widetilde{\nu}_D$ con la forma simpléctica orbifold original de X. Este pegado debe hacerse fuera de un entorno de las subvariedades de isotropía de X. Como conclusión, en la sexta sección se dan algunos ejemplos en los que este proceso de resolución de orbifolds con isotropía homogénea puede aplicarse.

En el capítulo 4 exploramos el papel de los orbifolds de dimensión 4 en el contexto de las geometrías de K-contacto y Sasakiana en dimensión 5. En las primeras dos secciones estudiamos el tipo concreto de orbifolds que aparecen de forma natural en dichas geometrías, los *orbifolds cíclicos*, (orbifolds tales que los grupos de isotropía de todos sus puntos son grupos cíclicos), que aparecen además equipados con estructuras casi-Kahler y Kahler. Estudiamos además maneras en que se pueden construir orbifolds cíclicos simplécticos a partir de variedades simplécticas, sin cambiar nada en el espacio topológico subyacente pero sí modificando el atlas de variedad para convertirlo en un atlas orbifold, y también cambiando la forma simpléctica para deformarla en una forma simpléctica orbifold. En las secciones tercera y cuarta se estudian fibrados de Seifert de círculos en un contexto topológico. Los fibrados de Seifert de círculos son el nexo de unión entre orbifolds casi-Kahler (resp. Kahler) y variedades de K-contacto (resp. Sasakianas), por lo que un estudio exhaustivo de la topología de estos fibrados se hace imprescindible para resolver problemas de existencia de variedades de K-contacto sin estructuras Sasakianas. De esta manera, la sección cuarta consiste enteramente en un estudio exhaustivo y detallado de la sucesión espectral de Leray asociada a un fibrado de Seifert de círculos. Esta sucesión espectral relaciona la homología del orbifold X (la base del fibrado) y la homología del espacio total M del fibrado de Seifert, ambas con coeficientes en $\mathbb{Z}$. De manera esencial, la sucesión espectral de Leray muestra de qué manera se pueden detectar en la torsión de $H_2(M,\mathbb{Z})$ los géneros g_i de las superficies de isotropía D_i del orbifold X. También da una serie de condiciones interesantes que caracterizan, a partir de la información del orbifold X, cuándo el primer grupo de homología del espacio total M se anula. Después de esto, en la quinta sección ya por fin damos la definición de estructura de K-contacto y Sasakiana, y mostramos cómo los orbifolds casi-Kahler y Kahler cíclicos juegan un papel fundamental en este tipo de geometrías. Se ha decidido postponer hasta este punto las definiciones de estas estructuras geométricas en aras de un desarrollo lineal de la temática, puesto que para estudiarlas desde el punto de vista en que estamos interesados se hace necesario primeramente introducir tanto orbifolds cíclicos como fibrados de Seifert.

En el capítulo 5 concluimos con el otro resultado principal de esta tesis, la construcción de una variedad de K-contacto semi-regular que no admite estructuras Sasakianas semi-regulares. Dicha construcción se realiza en varios pasos. El primer paso es construir una variedad simpléctica X de dimensión 4 que tenga $b = b_2(X)$ superficies simplécticas disjuntas S_i. Estas superficies $S_i \subset X$ claramente diagonalizan la forma de intersección de X y son por tanto una base de $H_2(X,\mathbb{Q})$. Esta construcción es bastante delicada, y a ella están dedicadas las cuatro primeras secciones del capítulo 5. Para construir esta variedad X, primero sumamos simplécticamente a un toro $\mathbb{T}^4$ tres copias de la fibración elíptica $E(1)$. Dicha suma se realiza a lo largo de un 2-toro, que en cada una de las tres copias de $E(1)$ es una fibra genérica y en $\mathbb{T}^4$ es un 2-toro distinto para cada suma. El pegado en estas sumas simplécticas debe hacerse con cuidado porque estamos interesados en construir tantas superficies simplécticas en la suma simpléctica como sea posible, para lo cual queremos pegar superficies simplécticas de cada uno de los sumandos y obtener así un número alto de superficies simplécticas en la suma simpléctica. Después de esto, todavía se hace necesario construir alguna superficie simpléctica extra, para lo cual usamos tanto el método de resolución simpléctica de intersecciones positivas como el proceso de explosión simpléctica, ambos desarrollados en el capítulo 1. Tras esto obtenemos la antes mencionada variedad simpléctica X con segundo grupo de homología $H_2(X, \mathbb{Q})$ generado por $b_2(X) = 36$ superficies simplécticas disjuntas S_i, $1 \leq i \leq 36$. Finalmente, en la quinta y última sección de este capítulo se construye un fibrado de Seifert de círculos M sobre un orbifold simpléctico de dimensión 4 construido sobre la variedad simpléctica X, dejando el espacio X intacto pero cambiando la estructura de variedad simpléctica de X por una de orbifold simpléctico cuyas superficies de isotropía sean las superficies simplécticas S_i previamente construidas, cada una de las S_i con grupo de isotropía cíclico $\mathbb{Z}_{p^i}$, siendo p un número primo cualquiera, mayor que 2. Por construcción, este fibrado de Seifert de círculos M tiene una estructura de K-contacto semi regular. Acto seguido, se prueba que M no admite estructuras Sasakianas semi regulares, lo cual se hace mediante un teorema que muestra la no existencia de una superficie Kahler S con un número demasiado elevado de curvas complejas de géneros positivos que sean una base de $H_2(S, \mathbb{Q})$.

CHAPTER 1

Symplectic topology.

In this chapter we recall some classical and very useful techniques in the study of symplectic topology. We work exclusively with symplectic manifolds here, leaving orbifolds for subsequent chapters.

1. Symplectic linear algebra.

Let us start with the basics of symplectic linear algebra. Recall that we consider only finite dimensional vector spaces here.

DEFINITION 1.1. *A* symplectic vector space *is a pair* (V, ω)*, where* V *is an* $\mathbb{R}$*-vector space, and* $\omega : V \times V \to \mathbb{R}$ *is a non-degenerate antisymmetric bilinear form.*

We call such an ω a *linear symplectic form*. The non-degeneray of ω means that given any non-zero $v \in V$ there exists some $u \in V$ with $\omega(u, v) \neq 0$. Recall that since ω is non-degenerate, the map

$$\begin{aligned} I_\omega : & V \to V^* \\ & v \mapsto \iota_v \omega \end{aligned}$$

is an isomorphism, where $\iota_v\omega : V \to \mathbb{R}$ satisfies $\iota_v\omega(u) = \omega(v, u)$ for all $u \in V$.

REMARK 1.2. *Not every vector space* V *admits a linear symplectic form. A necessary and sufficient condition for this is that* $\dim V = 2n$ *be even. In such a case, pick a basis* $B = \{v_1, u_1, \ldots, v_n, u_n\}$ *of* V*, and let* $B^* = \{v_1^*, u_1^*, \ldots, v_n^*, u_n^*\}$ *its dual basis. The form*

$$\omega = \sum_{i=1}^{n} v_j^* \wedge u_j^*$$

is clearly a linear symplectic form on V.

It can be proved via an orthonormalization procedure that any symplectic linear form ω *on a vector space* V *of dimension* $2n$ *can be expressed in this way for some basis* B *of* V*. Any such* B *is called a* symplectic basis *for* ω*, and the matrix of* ω *in this basis consists on* n *diagonal blocks* (a_{ij}) *of size* 2×2 *with* $a_{11} = 0 = a_{22}$, $a_{12} = 1 = -a_{21}$.

DEFINITION 1.3. *Let* (V, ω) *be a symplectic vector space. Let* $H < V$ *be a linear subspace.*

(1) *The* symplectic orthogonal *to* H *is defined as*

$$H^{\perp\omega} = \{v \in V : \omega(v, h) = 0 \quad \forall h \in H\} = \{v \in V : \iota_v\omega|_H = 0\} = (I_\omega)^{-1}(H^0)$$

where $H^0 < V^*$ *is the annhiliator of* H*, i.e.* $H^0 = \{f \in V^* : f|_H = 0\}$*. Clearly,* $\dim H + \dim H^{\perp\omega} = \dim V$.

(2) H *is called* symplectic *if the restriction* $\omega|_H : H \times H \to \mathbb{R}$ *is non-degenerate. It is easy to see that* H *is a symplectic subspace iff* $H \cap H^{\perp\omega} = \{0\}$.

(3) H *is called* Lagrangian *if* $H^{\perp\omega} = H$*. It is easy to see that* H *is Lagrangian iff* $\omega|_H = 0$ *and* $\dim H = \frac{1}{2} \dim V$.

Now we define symplectic manifold. Recall first that we always assume a manifold to be connected, unless otherwise specified.

DEFINITION 1.4. *A symplectic manifold (X, ω) is a smooth manifold X equipped with a non-degenerate closed 2-form $\omega \in \Omega^2(X)$.*

We call such a 2-form ω a *symplectic form* on X. For any $x \in X$ we have thus a symplectic vector space $(T_x X, \omega|_x)$. This forces that the tangent spaces be of even dimension. Call $2n = \dim T_x X$. Since all manifolds are connected to us, this yields $2n = \dim X$, so any symplectic manifold must be of even dimension. Also, recall that a symplectic manifold (X, ω) is canonically oriented since the differential form $\omega^n \in \Omega^{2n}(X)$ is non-degenerate, hence a volume form on X.

We define now a special type of submanifolds that a symplectic manifold can have.

DEFINITION 1.5. *Let (X, ω) be a symplectic manifold with $2n = \dim X$.*

(1) *A subset $D \subset X$ is called a* symplectic submanifold *if D is an embedded submanifold and $\iota^*\omega$ is a symplectic form on D, where $\iota : D \to X$ is the inclusion map.*
(2) *A subset $L \subset X$ is called a* Lagrangian submanifold *if L is an embedded submanifold, $\dim L = n$, and $\iota^*\omega = 0$, where $\iota : L \to X$ is the inclusion map.*

It is clear that $D \subset X$ is a symplectic submanifold iff it is an embedded submanifold and $T_x D < T_x X$ is a symplectic subspace for all $x \in D$. Analogously, $L \subset X$ is a Lagrangian submanifold iff it is an embedded submanifold and $T_x L < T_x X$ is a Lagrangian subspace for all $x \in L$.

On the other hand, recall that a symplectic submanifold inherits a canonical orientation, the one induced by $\iota^*\omega$.

2. Darboux theorem.

In symplectic geometry there is no analogous concept to that of curvature in Riemannian geometry. This is basically because in the definition of a symplectic form we impose flatness, i.e. $d\omega = 0$. The conclusion is that there are no local invariants, all symplectic manifolds of the same dimension look locally the same, as the following result shows.

THEOREM 1.6. *Let X be a symplectic manifold and let $Z \subset X$ be a submanifold. Suppose that ω_0, ω_1 are two symplectic forms on X such that $\omega_0|_z = \omega_1|_z$ for all $z \in Z$. Then, there is a symplectomorphism $\varphi : (U_0, \omega_0) \to (U_1, \omega_1)$ between open neighborhoods U_0, U_1 of Z in X such that $\varphi|_Z = \mathrm{Id}_Z$.*

PROOF. See [**11**], Ch. 7.3. □

This result will be generalized for orbifolds in Proposition 3.12, so the reader may also look for a proof there.

COROLLARY 1.7. *(Darboux coordinates) Let (X, ω) be a symplectic manifold. Around any point $p \in X$ there are coordinates $(x_1, y_1, \ldots, x_n, y_n)$ in which the symplectic form can be expressed as $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.*

PROOF. Fix auxiliary coordinates around p so that p gets mapped to $0 \in \mathbb{R}^{2n}$. We identify a neighborhood $U^p \subset X$ with its image by the chart in $\mathbb{R}^{2n}$. Apply Theorem 1.6 above to $Z = \{p\} \subset \mathbb{R}^{2n}$ and the forms ω and $\omega|_p$, which are both symplectic forms on $\mathbb{R}^{2n}$. Note that ω changes with the point while $\omega|_p$ is constant in $\mathbb{R}^{2n}$. We have then a symplectomorphism between two neighborhoods of p in $\mathbb{R}^{2n}$ which pulls ω back to $\omega|_p$. Since $\omega|_p$ is given by a constant matrix, we can take a symplectically orthogonal basis of $\mathbb{R}^{2n}$ so in the coordinates of this basis ω_p becomes the standard form $\omega_{std} = \sum_{i=1}^n dx_i \wedge dy_i$, and we are done. □

The category of symplectic manifolds **SMan** is defined as follows.

(1) The objects are symplectic manifolds (X, ω).
(2) The morphisms between (X_1, ω_1) and (X_2, ω_2) are smooth maps $f : (X_1, \omega_1) \to (X_2, \omega_2)$ such that $f^*\omega_2 = \omega_1$.

Since symplectic forms are non-degenerate, the inverse function Theorem implies that any morphism is an immersion. Hence, the isomorphisms in this category (called symplectomorphisms) are the bijective morphisms:

DEFINITION 1.8. *A smooth map $f : (X_1, \omega_1) \to (X_2, \omega_2)$ is called a* symplectomorphism *if f is bijective and $f^*\omega_2 = \omega_1$.*

3. The group of symplectomorphisms.

Using Darboux coordinates (and conectedness) we can prove that in a connected symplectic manifold the group of symplectomorphisms acts transitively.

LEMMA 1.9. *Let (X, ω) be a (connected) symplectic manifold, and $p, q \in X$ points. There exists a symplectomorphism $\varphi : X \to X$ so that φ is isotopic to the identity and $\varphi(p) = q$. Moreover, given a connected open set V containing p and q, the symplectomorphism φ can be taken as the identity outside V.*

PROOF. Fix $q \in X$, and let A_q be the set of points p of X so that there exists a symplectomorphism $\varphi : X \to X$ with $\varphi(p) = q$. Clearly the above defines an equivalence relation, and $A_q = [q]$ is the equivalence class of q, so $X = \sqcup_{x \in X}[x]$. If we see that $A_q = [q] \subset X$ is open, it must be $X = [q]$ by conectedness and we are done.

Take $p \in A_q$ and a Darboux chart $U = U^p \subset X$ with coordinates (x_i, y_i), $1 \leq i \leq n$ near p. Take a point $z \in U$. We can assume (maybe after a linear change of coordinates of $\mathbb{R}^{2n}$) that p is transformed by the coordinate chart in $0 \in \mathbb{R}^{2n}$ and z is transformed in $(a, 0)$ with $a \in \mathbb{R}$. The symplectic form is $\omega = \sum_i dx_i \wedge dy_i$. Take the function $f(x_i, y_i) = \rho(x_i, y_i) a y_1$, with ρ a bump function which equals 1 in $B(0, |a| + \varepsilon)$ and equals 0 in $B(0, |a| + 2\varepsilon)$, where ε is taken small enough to ensure that the balls are inside the chart. Take the 1-form df and consider a vector field V so that $\omega(V, \cdot) = df$.

Note that $df = a dy_1$ in $B(0, |a| + \varepsilon)$, so $\omega(V, \cdot) = a dy_1$ there, which implies that $V = a\partial_{x_1}$ in $B(0, |a| + \varepsilon)$. Moreover $df = 0$ outside $B(0, |a| + 2\varepsilon)$, hence $V = 0$ there.

Consider the flow φ_t of V, which satisfies $\varphi_t = \tau_v$ is the translation of vector $v = a\partial_{x_1}$ in $B(0, |a| + \varepsilon)$, and $\varphi_t = \mathrm{Id}$ outside $B(0, |a| + 2\varepsilon)$. Consider $\varphi = \varphi_t|_{t=1}$. Then $\varphi(0) = 0 + v = (a, 0)$. Moreover φ is a symplectomorphism because $\mathcal{L}_V \omega = d(\iota_V \omega) + \iota_V(d\omega) = ddf = 0$, hence the flow of V is made of symplectomorphisms. Since $\varphi = \mathrm{Id}$ away from $(a, 0)$, it can be extended as the identity to a symplectomorphism of the symplectic manifold (X, ω), as desired.

This proves that $A_q = [q]$ is open in X, so $A_q = X$. We conclude that for any other $p \in X$ the exists a symplectomorphism φ with $\varphi(p) = q$. The fact that φ can be supposed to be the identity outside an open connected set V containing p and q is straightforward, since p and q can be connected by a chain of small open sets of X contained in the domain of Darboux charts, and we can construct the symplectomorphism φ as a composition of symplectomorphisms constructed in charts, all of which are the identity outside the union of the chain of open sets. □

COROLLARY 1.10. *Let (X_1, ω_1) and (X_2, ω_2) be two symplectomorphic symplectic manifolds. Given points $p_1 \in X_1$ and $p_2 \in X_2$ there exists a symplectomorphism $\varphi : X_1 \to X_2$ with $\varphi(p_1) = p_2$.*

Moreover, given an initial symplectomorphism $f : X_1 \to X_2$, φ can be made to agree with f outside any connected open set V containing p_1 and $f^{-1}(p_2)$.

Proof. Let $f : (X_1, \omega_1) \to (X_2, \omega_2)$ be any initial symplectomorphism. If $f(p_1) = p_2$, take $\varphi = f$ and we are done. Otherwise let $q = f^{-1}(p_2) \neq p_1$. By Lemma 1.9 there exists a symplectomorphism $g : X_1 \to X_1$ with $g(p_1) = q$ and $g = \mathrm{Id}$ outside any open connected set V containing p_1 and q. The symplectomorphism $\varphi = f \circ g : X_1 \to X_2$ satisfies $\varphi(p_1) = p_2$ and $\varphi = f$ outside V as desired. □

Finally, we obtain the following:

Lemma 1.11. *Let (X_1, ω_1), (X_2, ω_2) be symplectic manifolds which are symplectomorphic. Let $A = \{p_1, \ldots, p_l\} \subset X_1$, $B = \{q_1, \ldots, q_l\} \subset X_2$ be two subsets of l different points. There exists a symplectomorphism $\varphi : X_1 \to X_2$ so that $\varphi(p_j) = q_j$ for all $j = 1, \ldots, l$.*

Moreover, given an initial symplectomorphism $f : X_1 \to X_2$, φ can be made to agree with f outside the union $V = \cup_{j=1}^{l} V_j$ of connected open sets V_j containing p_j and $f^{-1}(q_j)$.

Proof. The case $l = 1$ follows by Corollary 1.10. By induction, suppose the result is true for $l - 1 \geq 1$ and let us prove it for l. Let $f : X_1 \to X_2$ be any initial symplectomorphism. Consider a symplectomorphism $g : X_1 \to X_2$ with $g(p_l) = q_l$ (provided by Corollary 1.10) with $g = f$ outside a neighborhood V_l of p_l and $f^{-1}(q_l)$. This induces by restriction a symplectomorphism $g| : X_1 \setminus \{p_l\} \to X_2 \setminus \{q_l\}$. By induction hypothesis, there exists a symplectomorphism

$$\varphi : X_1 \setminus \{p_l\} \to X_2 \setminus \{q_l\}$$

mapping p_j to q_j for $1 \leq j \leq l - 1$ and such that $\varphi = g|$ outside a neighborhood $V \subset X_1 \setminus \{p_l\}$ of p_j and $f^{-1}(q_j)$ for $1 \leq j \leq l - 1$. Hence φ extends to the desired symplectomorphism in all X_1 sending also p_l to q_l. Clearly φ coincides with f outside $V \cup V_l$. □

4. Tubular neighborhoods of symplectic submanifolds.

Given X a symplectic manifold and $Z \subset X$ a symplectic submanifold, the *symplectic normal bundle* is the bundle $\nu_Z \to Z$ such that the fiber over any point $z \in Z$ is

$$(T_z Z)^{\perp\omega} = \{u \in T_z X : \omega(u, v) = 0 \quad \forall v \in T_z Z\}$$

the symplectic ortogononal to $T_z Z \subset T_x X$. Since $T_z Z$ is a symplectic subspace we have a decomposition

$$T_x X = T_z Z \oplus (T_z Z)^{\perp\omega}.$$

Darboux theorem can be used to give easy models (up to symplectomorphism) of tubular neighborhoods of symplectic submanifolds $Z \subset X$, as we do in the following Proposition.

Theorem 1.12. *Let X be a $2n$-symplectic manifold, and let $Z \subset X$ be a $(2n-2)$-symplectic submanifold with compact closure $\overline{Z} \subset X$. Suppose that the normal symplectic bundle ν_Z is trivializable. Then there is a tubular neighborhood $W = W^Z \subset X$ of Z and a small $r > 0$ so that (W, ω) is symplectomorphic to $(Z \times B_r^2, \Omega)$, where $\Omega = p_1^*\omega_Z + p_2^*\omega_0$, being $\omega_Z = \omega|_Z$ and $\omega_0 = dt \wedge ds$ the standard symplectic form on the ball $B_r^2 = \{(t, s) : t^2 + s^2 \leq r^2\}$.*

Proof. To ease notation we denote $\Omega = p_1^*\omega_Z + p_2^*\omega_0 = \omega_Z + \omega_0$ and omit the projections p_1, p_2 of the product $Z \times B^2$ onto its factors. Take any trivialization of ν_Z. This is given by two sections $u_i : Z \to \nu_Z$, $i = 1, 2$, giving linearly independent vectors at each $z \in Z$. By multiplying these sections by non-negative functions we can assume that $\omega(u_1, u_2) = 1$. Put a Riemannian metric on X and consider the exponential map

$$\begin{aligned} f = \exp : Z \times B_\rho^2 &\to W \\ (z, t, s) &\mapsto \exp_z(t u_1(z) + s u_2(z)) \end{aligned}$$

We claim that $f^*\omega|_{(z,0)} = \Omega|_{(z,0)}$ for all $z \in Z$. Assuming this for the moment, we can use Theorem 1.6 to get a symplectomorphism $\varphi : (U_0, \Omega) \to (U_1, f^*\omega)$, with $U_0, U_1 \subset Z \times B^2_\rho$ open subsets of $Z \times \{0\}$. We have thus a symplectomorphism

$$g = f \circ \varphi : (U_0, \Omega) \to (W, \omega).$$

If we take $r > 0$ so that $Z \times B^2_r \subset U_0$ and restrict g to $Z \times B^2_r$, we are done.

To end, let us prove for $z \in Z$ that $(f^*\omega)|_{(z,0)} = \Omega|_{(z,0)}$, with $f = \exp$ as above. Note that $d_{(z,0)}f : T_zZ \times \mathbb{R}^2 \to T_zX$ sends vectors of the form $(v, (0,0))$ to v, and vectors of the form $(0, (a,b))$ to $au_1(z) + bu_2(z)$. This yields

$$\begin{aligned} \omega|_z(df(v,(a,b)), df(v',(a',b'))) &= \omega(v + au_1(z) + bu_2(z), v' + a'u_1(z) + b'u_2(z)) \\ &= \omega_Z(v, v') + ds \wedge dt((a,b),(a',b')) \\ &= (\omega_Z + \omega_0)((v,(a,b)),(v',(a',b'))) \end{aligned}$$

and hence the claim. We have used above that $\{u_1(z), u_2(z)\}$ is a symplectic basis of $(T_zZ)^{\perp\omega}$. □

Remark 1.13. *The assumption that the symplectic submanifold $Z \subset X$ has codimension two is not necessary in Theorem 1.12. The proof given above can be easily adapted for any submanifold $Z \subset X$ of codimension $2k$. The only thing that changes a bit is that one has to construct what we call a* symplectic frame *of the normal bundle ν_Z, i.e. sections $u_1, v_1, \ldots, u_k, v_k : Z \to \nu_Z$ so that at each point $z \in Z$ the vectors $(u_j(z), v_j(z))$, $1 \le j \le k$, give a symplectic basis of $T_zZ^{\perp\omega}$. Note that in the case of codimension 2, any frame of ν_Z is a symplectic frame up to scaling.*

The construction of these symplectic sections in arbitrary codimension, say $2k$, can be done by starting with any trivialization of ν_Z given by any frame of $2k$-sections, an then carrying out a standard Gram-Schmidt procedure to transform the initial sections in symplectic ones. The rest of the proof is the same.

Theorem 1.12 above is a particular case of Theorem 1.14 below. We have included above a simpler proof for the trivializable case (and in fact this case is the only case we will need). Anyway, for completeness it is worth recalling the general theorem about tubular neighborhoods of symplectic submanifolds.

Theorem 1.14. *Let (X_0, ω_0) , (X_1, ω_1) be symplectic manifolds with symplectomorphic compact symplectic submanifolds $Z_0 \subset X_0$ and $Z_1 \subset X_1$. Let $i_0 : Z_0 \to X_0$, $i_1 : Z_1 \to X_1$ be their inclusions. Suppose there is an isomorphism $\varphi : NZ_0 \to NZ_1$ of the corresponding symplectic normal bundles covering a symplectomorphism $\phi : (Z_0, i_0^*\omega_0) \to (Z_1, i_1^*\omega_1)$. Then there exist neighborhoods $U_0 \subset X_0$ of Z_0, $U_1 \subset X_1$ of Z_1, and a symplectomorphism $\psi : U_0 \to U_1$ extending ϕ such that the restriction of $d\psi$ to the normal bundle NZ_0 is φ.*

Proof. See [**12**], Th. 1.11. □

5. Symplectic resolution of positive intersections.

In the following we focus primarily in symplectic 4-manifolds, since the 4-dimensional case is what we will need later on.

Let us make first some remarks about notation we will use in what follows.

- We shall denote sometimes smooth coordinates using smooth complex-valued functions. For example (z_1, z_2) may denote coordinates of a symplectic 4-manifold X, where $z_1 : U \to \mathbb{C}$ is a smooth $\mathbb{C}$-valued function defined in $U \subset X$ an open set. We have $z_j = x_j + iy_j$, $\overline{z_j} = x_j - iy_j$ for $j = 1, 2$. We may express the symplectic form in Darboux coordinates as

$$\omega = -\frac{i}{2}(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2})$$

in this coordinate chart. Here $dz_j = dx_j + idy_j$ and $d\overline{z_j} = dx_j - idy_j$ both lie in $\Omega^1(U) \otimes \mathbb{C}$, the $\mathbb{C}$-valued smooth 1-forms in X. A simple computation shows that

$$-\frac{i}{2}(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2}) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

so (z_1, z_2) are in fact Darboux coordinates.

- Notations as in above. We denote

$$\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$$
$$\partial_{\overline{z_j}} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$$

These are $\mathbb{C}$-valued derivations in $U \subset X$, i.e. they lie in $TX \otimes \mathbb{C}$. Recall that given a smooth $f : U \to \mathbb{C}$, if $f = u + iv$ is decomposed in real an imaginary parts, then

$$\partial_{x_j} f = \partial_{x_j} u + i\partial_{x_j} v$$
$$\partial_{y_j} f = \partial_{y_j} u + i\partial_{y_j} v$$

This also shows how ∂_{z_j} and $\partial_{\overline{z_j}}$ act on f.

- The exterior differential of $\mathbb{R}$-valued smooth k-forms $d : \Omega^k(X) \to \Omega^{k+1}(X)$ extends naturally to $\mathbb{C}$-valued forms giving

$$d : \Omega^k(X) \otimes \mathbb{C} \to \Omega^{k+1}(X) \otimes \mathbb{C}$$

Moreover we can express d in terms of the above basis as

(1) $$d = \sum_j \partial_{z_j} dz_j + \partial_{\overline{z_j}} d\overline{z_j}$$

For instance, given $f : U \to \mathbb{C}$ a function defined on $U \subset X$, we have

$$df = \sum_j \partial_{z_j} f dz_j + \partial_{\overline{z_j}} f d\overline{z_j}$$

where $\partial_{z_j} f = \frac{1}{2}(\partial_{x_j} f - i\partial_{y_j} f)$. Sometimes it is easier to compute the differential of an $\mathbb{R}$-valued smooth function by considering it as $\mathbb{C}$-valued and using formula (1).

- Formula (1) above should not be undertood as $d = \partial + \bar{\partial}$. This a famous formula for *complex* manifolds, but here X is in principle a smooth (symplectic or not) manifold. The expression $\sum_j \partial_{z_j} f dz_j$ is not independent of coordinates in a smooth manifold, so there is no operator ∂ in a general smooth manifold. The same happens with $\bar{\partial}$.
- On the other hand, symplectic manifolds are almost-complex, and in an almost-complex manifold there is indeed a way to define operators ∂ and $\bar{\partial}$ generalizing the classical operators from complex geometry. This is done using that the existence of an almost-complex structure J on a smooth manifold X gives an involution of the $\mathbb{C}$-vector space $\Omega^1(X) \otimes \mathbb{C}$, hence a decomposition in direct sum as eigenspaces

$$\Omega^1(X) \otimes \mathbb{C} = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$$

associated to $+i$ and $-i$. Then for f a smooth $\mathbb{C}$-valued function we define ∂f, and $\bar{\partial} f$ as

$$\partial f = \pi_{1,0}(df) \in \Omega^{1,0}(X)$$
$$\bar{\partial} f = \pi_{0,1}(df) \in \Omega^{0,1}(X)$$

with

$$\pi_{1,0} : \Omega^1(X) \otimes \mathbb{C} \to \Omega^{1,0}(X)$$
$$\pi_{0,1} : \Omega^1(X) \otimes \to \Omega^{0,1}(X)$$

the projections.

- However, for a smooth $\mathbb{C}$-valued function f defined in an almost complex manifold X as in the item above, the computation of ∂f cannot be made in coordinates (z_j) with the formula $\sum_j \partial_{z_j} f dz_j$. This is true only if (z_j) are *holomorphic* coordinates, and there is no notion of holomorphic coordinates in an almost-complex manifold X (unless it is complex).

Let X be a symplectic 4-manifold and let Σ_1 and Σ_2 be embedded symplectic surfaces intersecting transverseley and positively at a point $q \in X$. This yields a singular manifold $\Sigma_1 \cup \Sigma_2$. Let us see that $\Sigma_1 \cup \Sigma_2$ represents the homology class $[\Sigma_1] + [\Sigma_2] \in H_2(X, \mathbb{Z})$.

Recall that for pointed topological spaces $(X, x), (Y, y)$ the wedge sum is defined as

$$X \vee Y := \frac{X \sqcup Y}{\sim}$$

where $x \sim y$ is the only identification. The homology groups of the wedge sum satisfy $H_k(X \vee Y) \cong H_k(X) \oplus H_k(Y)$ for $k \geq 1$ in a canonical way.

Lemma 1.15. *Let X be 4-manifold and let Σ_1 and Σ_2 be embedded oriented surfaces intersecting transverseley at a point $q \in X$. Denote $i_1 : \Sigma_1 \to X$ and $i_2 : \Sigma_2 \to X$ the embeddedings.*

Then there is a canonically induced embedding $i : \Sigma_1 \vee \Sigma_2 \to X$ with image $i(\Sigma_1 \vee \Sigma_2) = i_1(\Sigma_1) \cup i_2(\Sigma_2)$ and moreover

$$i_* : H_2(\Sigma_1 \vee \Sigma_2) \to H_2(X)$$

satisfies $i_*([\Sigma_1] + [\Sigma_2]) = i_{1*}([\Sigma_1]) + i_{2*}([\Sigma_2])$.

Remark 1.16. *Note that $[\Sigma_1] + [\Sigma_2]$ is regarded in $H_2(\Sigma_1 \vee \Sigma_2) \cong H_2(\Sigma_1) \oplus H_2(\Sigma_2)$. Note also that the wedge sum $\Sigma_1 \vee \Sigma_2$ is done by identifying p_1 with p_2, being p_1 and p_2 the preimages of q in Σ_1 and Σ_2 respectively, and the embedding $i : \Sigma_1 \vee \Sigma_2 \to X$ is simply i_1 on Σ_1 and i_2 on Σ_2.*

Let $\Sigma_1, \Sigma_2 \subset X$ be embedded surfaces of a 4-manifold X intersecting transversally in a point q as above. We will express Lemma 1.15 by the formula $[\Sigma_1 \cup \Sigma_2] = [\Sigma_1] + [\Sigma_2]$.

Now suppose additionally that X is a symplectic manifold and that Σ_1, Σ_2 are two symplectic surfaces intersecting transversely and positively at a point q. Let us see how to make smooth the union $\Sigma_1 \cup \Sigma_2$ at the intersection point so as to get a symplectic surface representing the homology class $[\Sigma_1 \cup \Sigma_2] = [\Sigma_1] + [\Sigma_2]$. We need a nice local model near the intersection point $q = \Sigma_1 \cap \Sigma_2$. First let us recall the following.

Definition 1.17. *Given an oriented 4-manifold X and two oriented 2-dimensional submanifolds $S, N \subset X$ such that S, N intersect transversely, we say that S and N* intersect positively *(*negatively*) at a point $x \in S \cap N$ if there exist positive basis (u_1, u_2) of T_xS and (v_1, v_2) of T_xN so that (u_1, u_2, v_1, v_2) is a positive (negative) basis of T_xX.*

Let us make some comments about Definition 1.17 above.

(1) The definition makes sense, i.e. the choice of basis of the tangent spaces is irrelevant. This means that if S, N intersect transversely at the point x, then S and N intersect positively (negatively) iff for *any* basis (u_1, u_2) of T_xS and (v_1, v_2) of T_xN we have that (u_1, u_2, v_1, v_2) is a positive (negative) basis of T_xX.
(2) The above definition of intersecting positively and negatively at a point $x \in N \cap S$ is symmetric in S, N, i.e. S and N intersect positively (negatively) at x iff N and S do. This is true because (u_1, u_2, v_1, v_2) and (v_1, v_2, u_1, u_2) differ by an even number of permutations (four). Recall that this is in contrast with the behaviour of intersections of curves (1-cycles) in an ambient surface, which is antisymmetric.

(3) If S and N intersect transversely we define the *intersection number* of S and N as

$$S \cdot N = \sum_{x \in S \cap N} \varepsilon_x$$

where ε_x equals 1 if S and N intersect positively, and -1 if S and N intersect negatively. By item (1) we have $S \cdot N = N \cdot S$.

(4) The number $S \cdot N$ can also be defined if S and N do not intersect transversely. It is enough to make a slight perturbation of S and N to get S' and N' so that S' and N' intersect transversely. Then $S \cdot N$ is defined as $S' \cdot N'$. This definition is consistent, i.e. independet of the small perturbation chosen.

(5) The *intersection form* of a 4-manifold X can be defined using the above. It is well-known that any second homology class $[A] \in H_2(X, \mathbb{Z})$, for $A \in C_2(X, \mathbb{Z})$ a 2-chain, can be represented by a oriented surface $S_A \subset X$, so that $[A] = [S_A]$. Given two homology classes $[A], [B] \in H_2(X, \mathbb{Z})$ take oriented surfaces $S_A, S_B \subset X$ representing them, and define $[A] \cdot [B] := S_A \cdot S_B$. This gives a bilinear symmetric form over the integers

$$I : H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$$

called the *intersection form* of the 4-manifold X.

(6) Using Poincaré Duality we can see the intersection form as defined in cohomology, i.e. $I : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$. Moreover, extending coefficients we can define I over $\mathbb{R}$ and interpret I in De Rham Cohomology, giving $I([\alpha], [\beta]) = [\alpha \wedge \beta]$ for $\alpha, \beta \in \Omega^2(X)$ closed 2-forms. Note that if Vol_X is a chosen volume form on X, then it gives a generator $[\mathrm{Vol}_X] \in H^4_{DR}(X) \cong \mathbb{R}$, so $[\alpha \wedge \beta] = \lambda_{\alpha,\beta}[\mathrm{Vol}_X]$ for some $\lambda_{\alpha,\beta} \in \mathbb{R}$. In this way we have $I : H^2_{DR}(X) \times H^2_{DR}(X) \to \mathbb{R}$.

PROPOSITION 1.18. *Let (X, ω) be a symplectic manifold and let $S, N \subset X$ be symplectic 2-dimensional submanifolds of X so that S and N intersect ω-orthogonally at a point $x \in S \cap N$. Then S and N intersect positively at x.*

PROOF. Note first that X is oriented by ω, and S, N are oriented by the restriction of ω, since both are symplectic submanifolds. Take positive basis (u_1, u_2) of T_xS and (v_1, v_2) of T_xN. This means $\omega(u_1, u_2) > 0$ and $\omega(v_1, v_2) > 0$. Since T_xS and T_xN are ω-orthogonal subspaces of T_xX, we have that $(\omega \wedge \omega)(u_1, u_2, v_1, v_2)$ is a positive constant times $\omega(u_1, u_2)\omega(v_1, v_2)$, hence $(\omega \wedge \omega)(u_1, u_2, v_1, v_2) > 0$ and (u_1, u_2, v_1, v_2) is a positive basis of T_xX. We conclude that S and N intersect positively. □

The following lemma proves the existence of Darboux charts adapted to symplectic submanifolds.

LEMMA 1.19. *Let (X, ω) be a $2n$-symplectic manifold and $\Sigma \subset X$ be a symplectic $2d = 2n - 2k$ submanifold. Take $p \in \Sigma$. Then there exists a Darboux chart $\psi : U \to \psi(U) \subset \mathbb{R}^{2n}$ near p so that $\psi(\Sigma \cap U) = \{z_1 = 0, \ldots, z_{2k} = 0\}$.*

The chart ψ is called an Darboux chart adapted to Σ.

PROOF. Take $p \in \Sigma$, and take $V = V^p \subset \Sigma$ a small neighborhood of p in Σ with a symplectomorphism $\varphi : (V, \omega_V) \to (\mathbb{C}^d, \omega_d)$, with $\omega_V = \omega|_V$ and ω_d the standard symplectic form of $\mathbb{C}^d \cong \mathbb{R}^{2d}$. We have that $V \subset X$ is a symplectic submanifold with trivializable normal bundle ν_V, so using Theorem 1.12 we see that a tubular neighborhood $U = U^V \subset X$ of V in X is symplectomorphic to $(V \times B_r^{2k}, p_1^*\omega_V + p_2^*\omega_k)$, being ω_k the standard symplectic form of $\mathbb{C}^k \cong \mathbb{R}^{2k}$. Since V is a darboux chart of p in Σ, we can assume that $V \cong B_r^{2d} \subset \mathbb{C}^d$ is open and $\omega_V = \omega_d$ is the standard symplectic form, so we have a symplectomosphism

$$(U, \omega) \cong (B_r^{2d} \times B_r^{2k}, \omega_0)$$

where U is a neighborhood of V in X and $\omega_0 = p_1^*\omega_d + p_2^*\omega_k$ is the standard symplectic form on $\mathbb{C}^n$. We are done. □

The following technical result will be useful in the following. It allows to perturb symplectic (positively intersecting) surfaces on a 4-manifold to make them symplectically orthogonal at the intersection points. Hence, after this perturbation is done, we get a nice local picture of the intersection of two symplectic surfaces in a symplectic 4-manifold.

LEMMA 1.20. *Let (X, ω) be a symplectic 4-manifold, and suppose that $S, N \subset X$ are symplectic surfaces intersecting transversely and positively. Then we can perturb S to get an S' in such a way that:*

1. *The perturbed surface S' is symplectic.*
2. *The perturbation is small in the $\mathcal{C}^0$-sense and it only changes S near the points of intersection with N, but leaving these points fixed, i.e. $S \cap N = S' \cap N$.*
3. *There are Darboux coordinates (z, w) near all the intersection points of N and S' in which $N = \{z = 0\}$ and $S' = \{w = 0\}$. In particular, S' and N intersect ω-orthogonally.*

PROOF. Let us show that we can arrange that the intersection becomes orthogonal after a small symplectic isotopy around the intersection point. Take an initial Darboux chart (z, w) adapted to N. The symplectic form in this chart is $\omega = -\frac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w})$, and N is given by the equation $N = \{z = 0\}$. On the other hand, S is given as the graph of a map

$$S = \{(z, w) : w = az + b\overline{z} + g(z)\}$$

where $a, b \in \mathbb{C}$, and g a smooth function satisfying the following bounds:

$$\begin{aligned} |g(z)| &\leq C|z|^2 \\ |\partial_z g(z)| + |\partial_{\overline{z}} g(z)| &\leq C|z|. \end{aligned}$$

The point q of intersection is $(0, 0)$ in the chart. Denote $a = \alpha_1 + i\alpha_2$, $b = \beta_1 + i\beta_2$. In real coordinates (x, y, u, v) with $z = x + iy$, $w = u + iv$, S is given by

$$S = \{(\alpha_1 + \beta_1)x + (\beta_2 - \alpha_2)y - u - O(|z|^2) = 0, \quad (\alpha_2 + \beta_2)x + (\alpha_1 - b_1)y - v - O(|z|^2) = 0\}$$

hence the tangent spaces at q are

$$\begin{aligned} T_q S &= \mathrm{Span}\langle \partial_x + (\alpha_1 + \beta_1)\partial_u + (\alpha_2 + \beta_2)\partial_v, \partial_y + (\beta_2 - \alpha_2)\partial_u + (\alpha_1 - \beta_1)\partial_v \rangle \\ T_q N &= \mathrm{Span}\langle \partial_u, \partial_v \rangle \end{aligned}$$

In real coordinates, $\omega^2 = dx \wedge dy \wedge du \wedge dv$. Clearly the four vectors given above (in order of appearance) define a ω^2-positive basis of $T_q S \times T_q N$ (the determinant of the 4×4 matrix with the vectors as columns equals 1), Also, the basis for N is $\omega|_N$-positive, so S and N intersect positively iff the basis for S is $\omega|_S$-positive. In S we have

$$\begin{aligned} du &= (\alpha_1 + \beta_1)dx + (\beta_2 - \alpha_2)dy + O(|z|) \\ dv &= (\alpha_2 + \beta_2)dx + (\alpha_1 - \beta_1)dy + O(|z|) \\ \omega|_S &= (1 + (\alpha_1 + \beta_1)(\alpha_1 - \beta_1) - (\alpha_2 - \beta_2)(\alpha_2 + \beta_2))dx \wedge dy + O(|z|) \\ &= (1 + |a|^2 - |b|^2)dx \wedge dy + O(|z|). \end{aligned}$$

From here we see that if $|a|^2 - |b|^2 + 1 > 0$ then $\omega|_S$ is non-degenerate near q and the basis chosen before for $T_q S$ is $\omega|_S$-positive. Therefore, the condition for S to be symplectic and intersect positively with N is that $|a|^2 - |b|^2 + 1 > 0$.

One can deform S locally to

$$S' = \left\{ w = \rho\left(\left(\frac{|z|}{\varepsilon} \right)^{2\alpha} \right) (az + b\overline{z} + g(z)) \right\},$$

for some $\varepsilon > 0$ and $\alpha > 0$ to be determined later, where $\rho(t)$ is a bump function which is 0 on $[0, 1]$ and 1 on $[2, \infty)$, and such that $|\rho'(t)| \leq 2$ for all $t \in \mathbb{R}$.

Clearly S' intersects N at $(0,0)$ orthogonally with respect to ω. Moreover S' is symplectic in $\{|z| \leq \varepsilon\}$ (since it is $\{w=0\}$ there) and in $\{|z| \geq 2\varepsilon\}$ (since $S' = S$ there). We are going to see that it is also symplectic in $\{\varepsilon \leq |z| \leq 2\varepsilon\}$.

Denote $\rho_\varepsilon = \rho\left(\left(\frac{|z|}{\varepsilon}\right)^{2\alpha}\right)$, and $\rho'_\varepsilon = \rho'\left(\left(\frac{|z|}{\varepsilon}\right)^{2\alpha}\right)$ (considered as numbers). A calculation using $d = \partial_z dz + \partial_{\overline{z}} d\overline{z}$ gives that

$$d\left(\rho\left(\left(\frac{|z|}{\varepsilon}\right)^{2\alpha}\right)\right) = \left(\frac{|z|}{\varepsilon}\right)^{2\alpha} \alpha\rho'_\varepsilon\left(\frac{1}{z}dz + \frac{1}{\overline{z}}d\overline{z}\right)$$

From this it follows that

$$dw = \left\{\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} \alpha\rho'_\varepsilon\left(a + b\frac{\overline{z}}{z} + O(|z|)\right) + a\rho_\varepsilon + O(|z|)\right\} dz + \left\{\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} \alpha\rho'_\varepsilon\left(a\frac{z}{\overline{z}} + b + O(|z|)\right) + b\rho_\varepsilon + O(|z|)\right\} d\overline{z}$$

$$d\overline{w} = \left\{\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} \alpha\rho'_\varepsilon\left(\overline{a}\frac{\overline{z}}{z} + \overline{b} + O(|z|)\right) + \overline{b}\rho_\varepsilon + O(|z|)\right\} dz + \left\{\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} \alpha\rho'_\varepsilon\left(\overline{a} + \overline{b}\frac{z}{\overline{z}} + O(|z|)\right) + \overline{a}\rho_\varepsilon + O(|z|)\right\} d\overline{z}$$

After some cancellations coming from symmetry, we get

$$\omega|_{S'} = -\frac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w}) \tag{2}$$

$$= \left(1 + \left(\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} 2\alpha\rho_\varepsilon\rho'_\varepsilon + \rho_\varepsilon^2\right)(|a|^2 - |b|^2) + O\left(\frac{|z|^{2\alpha+1}}{\varepsilon^{2\alpha}}\right) + O(|z|)\right)\frac{-idz \wedge d\overline{z}}{2}. \tag{3}$$

Note that $-idz \wedge d\overline{z} = 2dx \wedge dy$, so we must see that the function multiplying $-idz \wedge d\overline{z}$ in the above expression is positive for adequate choices of ε, α.

Given any $\delta > 0$, if we choose α small enough we have

$$\left|\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} 2\alpha\rho_\varepsilon\rho'_\varepsilon\right| \leq \alpha C\rho_\varepsilon 2^{2\alpha+1} < \delta\rho_\varepsilon \leq \delta,$$

where we have used that $|z| \leq 2\varepsilon$, and that $|\rho'|$ is bounded by some $C \leq 2$. We have two cases:

Case i): If $|a|^2 - |b|^2 \geq 0$, clearly

$$1 + \left(\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} 2\alpha\rho_\varepsilon\rho'_\varepsilon + \rho_\varepsilon^2\right)(|a|^2 - |b|^2) \geq 1 + \rho_\varepsilon(-\delta + \rho_\varepsilon)(|a|^2 - |b|^2) > 0$$

choosing $\delta > 0$ small enough.

Case ii): If $0 > |a|^2 - |b|^2 > -1$, then $1 + |a|^2 - |b|^2 > 0$ so

$$1 + \left(\left(\frac{|z|}{\varepsilon}\right)^{2\alpha} 2\alpha\rho_\varepsilon\rho'_\varepsilon + \rho_\varepsilon^2\right)(|a|^2 - |b|^2) \geq 1 + (\delta+1)(|a|^2 - |b|^2) = (1 + |a|^2 - |b|^2) - \delta(|b|^2 - |a|^2) > 0,$$

choosing $\delta > 0$ small enough.

The error term in (2) is $O(\varepsilon)$ since we work in $\{|z| \leq 2\varepsilon\}$, so this error can be neglected for ε small enough. This completes the proof that S' is a symplectic surface. Clearly, close to $(0,0)$, S' is given by the equation $w = 0$, and outside the support □

We can also deform the second surface S in other ways. The following corollary shows infinitely many possible deformations.

COROLLARY 1.21. *Let (X, ω) be a symplectic 4-manifold, and suppose that $S, N \subset X$ are symplectic surfaces intersecting transversely and positively. For any $\lambda \in \mathbb{C}$ with $|\lambda|$ small enough, we can deform S to an $\hat{S}_\lambda$ in such a way that:*

(1) *The perturbed surface $\hat{S}_\lambda$ is symplectic.*

(2) *The deformation of S only occurs near the intersection points of N and S, and $S \cap N = \hat{S}_\lambda \cap N$.*
(3) *Near all the intersection points of N and $\hat{S}_\lambda$ there are Darboux coordinates (z, w) in which $N = \{z = 0\}$ and $\hat{S}_\lambda = \{w = \lambda z\}$.*

PROOF. Take a darboux chart (z, w) adapted to N, near some point $q \in N \cap S$. Suppose that $S = \{w = az + b\overline{z} + g(z)\}$ in this chart for $g(z) = O(|z|^2)$. Take an initial deformation S' of S as in the proof of Lemma 1.20, given by

$$S' = \left\{ w = \rho \left(\left(\frac{|z|}{\varepsilon} \right)^{2\alpha} \right) (az + b\overline{z} + g(z)) \right\},$$

where $\rho(t)$ is a bump function which is 0 on $[0, 1]$ and 1 on $[2, \infty)$, and such that $|\rho'(t)| \leq 2$ for all $t \in \mathbb{R}$. As seen in the proof of Lemma 1.20, for all $\varepsilon > 0$ and $\alpha > 0$ small enough the surface S' is symplectic.

Take

$$\hat{S} = \left\{ w = \lambda z \left(1 - \rho \left(\left(\frac{|z|}{\varepsilon} \right)^{2\alpha} \right) \right) + \rho \left(\left(\frac{|z|}{\varepsilon} \right)^{2\alpha} \right) (az + b\overline{z} + g(z)) \right\}$$

Clearly, if we take $|\lambda|$ small enough, then S' and $\hat{S}$ can be made arbitrarily $\mathcal{C}^1$-close, uniformly in all the gluing region, i.e. in $\{\varepsilon \leq |z| \leq 2\varepsilon\}$. Since S' is symplectic, then $\hat{S}$ is also symplectic for $|\lambda|$ small enough. □

Finally, we can use the above to handle intersections of more than two symplectic surfaces. Indeed, repeating the procedure of Corollary 1.21 with an arbitrary number $l \in \mathbb{N}$ of surfaces, we get the following:

COROLLARY 1.22. *Let (X, ω) be a symplectic 4-manifold, and suppose that $N, S_1, \ldots, S_l \subset X$ are symplectic surfaces intersecting transversely and positively all of them at the same point $q \in X$. For any $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ distinct complex numbers with $\max_j\{|\lambda_j|\}$ small enough, we can deform the surface S_j to an $\hat{S}_{\lambda_j}$ in such a way that:*

(1) *The perturbed surfaces $\hat{S}_{\lambda_j}$ are symplectic for $1 \leq j \leq l$.*
(2) *The deformation of the surfaces S_j only occurs near q, and $\hat{S}_{\lambda_j} \cap N = \{q\}$ for any $1 \leq j \leq l$.*
(3) *Near the point q there are Darboux coordinates (z, w) in which $N = \{z = 0\}$ and $\hat{S}_{\lambda_j} = \{w = \lambda_j z\}$ for $1 \leq j \leq l$.*

REMARK 1.23. *The assumption that the surfaces intersect positively in Lemma 1.20 (and the corollaries derived from it) is crucial.*

For instance, take Lemma 1.20, and note that since the perturbation S' of S only occurs near the intersection points $S \cap N$, we see that the intersection number at any point of $S' \cap N$ has to be the same that the intersection number at the corresponding point of $S \cap N$. So if S can be deformed to S' so that S' intersects ω-orthogonally with N, then $S \cdot N = S' \cdot N > 0$ by Proposition 1.18.

Let us note that the above results about intersection of two symplectic surfaces are folklore among symplectic topology specialists. Anyway, we have not found explicit references proving them, so it seemed a good idea to include a proof here.

Now we finally have the tools necessary to perform the resolution of a transverse and positive intersection of two symplectic surfaces of a symplectic 4-manifold.

Let (X, ω) be a symplectic 4-manifold and let $\Sigma_1, \Sigma_2 \subset X$ be two symplectic surfaces of X intersecting transversely and positively at a point q. By Lemma 1.20, after slightly perturbing Σ_1

we can take Darboux coordinates (z_1, z_2) in a 4-ball neighbourhood U of q, so that $\Sigma_1 = \{z_1 = 0\}$ and $\Sigma_2 = \{z_2 = 0\}$. Then the union $\Sigma_1 \cup \Sigma_2$ is described locally as

$$F = \{(z_1, z_2) \in U : z_1 z_2 = 0\ ,\ |z_1|^2 + |z_2|^2 \leq 1\}.$$

Note that F has a singular point at $(0,0)$. The singularity in $(0,0)$ is an ordinary double point. Topologically, a neighborhood of $(0,0)$ in F is the union of a 2-disc D_1^2 in the plane $\{z_1 = 0\}$ and another 2-disc D_2^2 in the plane $\{z_2 = 0\}$. The discs D_1^2 and D_2^2 only intersect at $(0,0)$, so F is topologically the union of a couple of 2-discs by the center, so F is homeomorphic to the cone of $\mathbb{R}^3$.

This kind of cone singularity can be resolved easily. One has to replace the vertex of the cone with a small cilinder as follows. Consider the cylinder R' given by

$$R' = \{(z_1, z_2) : z_1 z_2 = \varepsilon\ ,\ |z_1|^2 + |z_2|^2 \leq 1\}.$$

To see that R' is a cylinder, note that it can be expressed in polar coordinates as

$$\begin{aligned} R' &= \{(r_1 e^{i\theta}, r_2 e^{-i\theta}) : r_1 r_2 = \varepsilon\ ,\ r_1^2 + r_2^2 \leq 1\ ,\ \theta \in [0, 2\pi]\} \\ &= \{(re^{i\theta}, \tfrac{\varepsilon}{r} e^{-i\theta}) : r \in I_\varepsilon\ ,\ \theta \in [0, 2\pi]\} \\ &= \{(z, \tfrac{\varepsilon}{z}) : a_\varepsilon \leq |z| \leq b_\varepsilon\} \end{aligned}$$

being $I_\varepsilon = (a_\varepsilon, b_\varepsilon)$ an interval of $\mathbb{R}$, with $a_\varepsilon = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\varepsilon^2}}$ and $b_\varepsilon = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\varepsilon^2}}$.

The strategy is to replace F by a tiny perturbation of R' for ε small enough. The perturbation should be done in such a way that the boundary of the perturbed R' glues smoothly with F away from $(0,0)$. To ensure that the perturbed surface remains symplectic, we only have to make sure that the boundary of the cylinder R' is arbitrarily close to the boundary of the cone F in the $\mathcal{C}^1$-topology for ε small. This is because being a symplectic submanifold is an open condition in the $\mathcal{C}^1$-topology.

Finally let us prove the main result of this section.

THEOREM 1.24. *Let (X, ω) be a symplectic 4-manifold and let $\Sigma_1, \Sigma_2 \subset X$ be embedded symplectic surfaces intersecting transverseley and positively at a point $q \in X$. We can perturb slightly $\Sigma_1 \cup \Sigma_2$ around q to obtain a symplectic embedded (smooth) surface $\Sigma \subset X$ that satisfies:*

(1) *It represents an homology class $[\Sigma]$ with $[\Sigma] = [\Sigma_1] + [\Sigma_2] \in H_2(X)$.*
(2) *It has genus $g(\Sigma) = g(\Sigma_1) + g(\Sigma_2)$.*

PROOF. Take Darboux coordinates (z_1, z_2) in a 4-ball neighbourhood $U \cong B_r(0) \subset \mathbb{C}^2$ of q, so that $\Sigma_1 = \{z_1 = 0\}$ and $\Sigma_2 = \{z_2 = 0\}$. By the discussion above, the union $\Sigma_1 \cup \Sigma_2$ is described locally as $F = D^2 \times \{0\} \cup \{0\} \times D^2 \subset \mathbb{C}^2$, with D^2 the disc of radius r. Consider the cylinder R', parametrized as above by $R' = \{(z, \frac{\varepsilon}{z}) : a_\varepsilon \leq |z| \leq b_\varepsilon\}$. Making a Taylor series expansion of $h(\varepsilon) = \sqrt{1 - 4\varepsilon^2}$ we see that $h(\varepsilon) = 1 - 2\varepsilon^2 - 2\varepsilon^4 + O(\varepsilon^6)$ hence

$$\begin{aligned} a_\varepsilon &= \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\varepsilon^2}} = O(\varepsilon)\,, \\ b_\varepsilon &= \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\varepsilon^2}} = \sqrt{1 + O(\varepsilon^2)} \end{aligned}$$

This implies that if $|z| = a_\varepsilon$ then the point $(z, \frac{\varepsilon}{z})$ has a limit as $\varepsilon \to 0$ and this limit is on $\{0\} \times \mathbb{S}^1$. Analogously, if $|z| = b_\varepsilon$ then the point $(z, \frac{\varepsilon}{z})$ has a limit as $\varepsilon \to 0$ and this limit is on $\mathbb{S}^1 \times \{0\}$. This proves that the boundary of R' is arbitrarily close to the boundary of F in the $\mathcal{C}^0$-topology.

Let us see that the tangent spaces of R' at points of $\partial R'$ also converge to the tangent spaces of F at points of ∂F. For this we parametrize R' as a real surface of $\mathbb{R}^4$. Put $z = u + iv$, so

$$R' = \{(u, v, \tfrac{\varepsilon u}{u^2 + v^2}, \tfrac{-\varepsilon v}{u^2 + v^2}) : a_\varepsilon \leq u^2 + v^2 \leq b_\varepsilon\}$$

Call (x, y, z, t) the standard coordinates of $\mathbb{R}^4$. It follows that

$$\partial_u = \partial_x + \frac{\varepsilon(v^2 - u^2)}{(u^2+v^2)^2}\partial_z + \frac{2\varepsilon uv}{(u^2+v^2)^2}\partial_t$$
$$\partial_v = \partial_y + \frac{-2\varepsilon uv}{(u^2+v^2)^2}\partial_z + \frac{\varepsilon(v^2 - u^2)}{(u^2+v^2)^2}\partial_t$$

The tangent space at the point of R' with coordinates $z = (u, v) \neq (0, 0)$ is

$$\begin{aligned}\text{Span}\langle\partial_u, \partial_v\rangle &= \text{Span}\langle|z|^4\partial_u, |z|^4\partial_v\rangle \\ &= \text{Span}\langle|z|^4\partial_x + \varepsilon(v^2-u^2)\partial_z + 2\varepsilon uv\partial_t \ , \quad |z|^4\partial_y - 2\varepsilon uv\partial_z + \varepsilon(v^2-u^2)\partial_t\rangle\end{aligned}$$

Now we distinguish cases:

(1) If $z = (u, v)$ satisfies $|z| = a_\varepsilon = O(\varepsilon)$, divide by ε^3 and the tangent space is given by

$$\text{Span}\langle O(\varepsilon)\partial_x + O(1)\partial_z + O(1)\partial_t \ , \quad O(\varepsilon)\partial_y + O(1)\partial_z + O(1)\partial_t\rangle$$

with limit $\text{Span}\langle\partial_z, \partial_t\rangle$ as $\varepsilon \to 0$ which is the tangent space of $\{0\} \times D^2$.

(2) If $z = (u, v)$ satisfies $|z|^2 = b_\varepsilon^2 = 1 + O(\varepsilon^2)$, the tangent space is given by

$$\text{Span}\langle(1+O(\varepsilon^2))\partial_x + O(\varepsilon)\partial_z + O(\varepsilon)\partial_t \ , \quad (1+O(\varepsilon^2))\partial_y + O(\varepsilon)\partial_z + O(\varepsilon)\partial_t\rangle$$

with limit $\text{Span}\langle\partial_x, \partial_y\rangle$ as $\varepsilon \to 0$ which is the tangent space of $D^2 \times \{0\}$.

This proves that some neighborhoods of ∂F and $\partial R'$ are arbitrarily $\mathcal{C}^1$-close if ε is small. Now we have to perturb R' to make the boundaries of F and R' glue smoothly. Consider a bump function $\rho : \mathbb{R} \to \mathbb{R}$ so that $\rho(t)$ equals 1 for $|t| \leq r/3$ and $\rho(t)$ equals 0 for $|t| \geq 2r/3$, with r the radius of the Darboux chart $U = B_r(0)$. Let us define

$$R = \{(z_1, z_2) \in U : z_1 z_2 = \varepsilon\rho(|z|^2)\}$$

with $|z|^2 = |z_1|^2 + |z_2|^2$. Note that $R \cap \{|z| \leq r/3\} = R' \cap \{|z| \leq r/3\}$ and $R \cap \{|z| \geq 2r/3\} = F \cap \{|z| \geq 2r/3\}$. Clearly, R defines a smooth surface in U, diffeomorphic to a cylinder.

Moreover, if we take ε sufficiently small, R is symplectic. Indeed, in a neighborhood of $(0, 0) \in U$ it is $\mathcal{C}^1$-close to the complex curve $R' \subset U$, so it is symplectic there. Outside the above neighborhood of $(0, 0)$ it is $\mathcal{C}^1$-close to the complex curve F (note that F is smooth outside $(0, 0)$), so it is also symplectic there. The details to see the $\mathcal{C}^1$-closeness are analogous to the computations done before, so we omit them.

The conclusion is that we can construct the cylinder $R \subset U$ by perturbing R' (for $\varepsilon > 0$ sufficiently small) to achieve that R is symplectic and that $F = R$ near ∂U, so R and F glue smoothly. Now cut out the pair (U, F) and replace it with (U, R). This construction replaces $\Sigma_1 \cup \Sigma_2$ by a (smooth) symplectic surface $\Sigma = ((\Sigma_1 \cup \Sigma_2) \setminus F) \cup R$ of genus $g(\Sigma) = g(\Sigma_1) + g(\Sigma_2)$.

In order to see that Σ represents the homology class $[\Sigma_1] + [\Sigma_2]$, note that Σ is obtained by a small continuous deformation of $\Sigma_1 \cup \Sigma_2$, so clearly $[\Sigma] = [\Sigma_1 \cup \Sigma_2]$. Now use Lemma 1.15 to conclude that $[\Sigma] = [\Sigma_1 \cup \Sigma_2] = [\Sigma_1] + [\Sigma_2]$. This yields the claim. □

We call the above construction of Σ starting from $\Sigma_1 \cup \Sigma_2$ the *resolution of the positive intersection*. Sometimes it is also called *resolution of the transverse intersection*, but we think the former name is better because positivity of the intersection is crucial.

Note that this procedure of resolution is a baby case of symplectic resolution of singularities. In this case it smoothens a singular symplectic surface $\Sigma_1 \cup \Sigma_2$ with an ordinary double point embedded in a 4-manifold X. No changes to the ambient manifold X are done.

6. Symplectic blow-up.

In this section we develop the blow-up process in the symplectic category. We focus on dimension 4 for concreteness since it is what we need for later use, but in reality all the arguments presented are valid for $2n$-symplectic manifolds with exactly the same proof. We will give two methods of performing the blow-up. Each method makes the gluing in a different way. The first one is very elementary and it would be enough for our purposes, but it is somewhat ad-hoc and non-canonical. The second one is the classical symplectic blow-up, done in a more canonical way. We follow [**39**] for the second method.

Let X be a symplectic 4-manifold and $q \in X$. To perform the symplectic blow-up we need to construct $(\widetilde{X}, \widetilde{\omega})$ starting from (X, ω). The space $\widetilde{X}$ as a smooth manifold will be constructed as in the cassical blow-up from complex geometry (using diffeomorphisms instead of biholomorphisms). However, the symplectic forms ω on $U \subset X$ and ρ_λ on $\widetilde{U} \subset U \times \mathbb{CP}^1$ (coming from its Kahler structure) have to agree in $(X \setminus U) \cap \widetilde{U}$, since we want to construct a symplectic structure $\widetilde{\omega}$ on $\widetilde{X}$. Hence a process of gluing is needed here. This gluing can be done *in many ways*. In what follows we give two different methods of doing it.

The first, which we call *elementary*, constructs $\widetilde{X}$ first and then glues the symplectic forms with cohomological arguments and a bump function.

The second, called here *classical*, puts care in the construction of $\widetilde{X}$. It glues the set $\widetilde{U}$ and $X \setminus U$ via a symplectomorphism. The symplectic form on $\widetilde{X}$ then follows automatically.

6.1. Elementary Method. The *symplectic blow-up* of X at q is defined as follows. Take Darboux coordinates (z_1, z_2) in a 4-ball neighbourhood U of q, and put the standard complex structure J on U. Consider

$$\widetilde{U} = \{((z_1, z_2), [w_1, w_2]) \in U \times \mathbb{CP}^1 \ , \ z_1 w_2 = z_2 w_1\}.$$

Then there is an holomorphic map

$$\pi : \widetilde{U} \to U \ , \ ((z_1, z_1), [w_1, w_2]) \mapsto (z_1, z_2)$$

such that $\pi : \widetilde{U} \setminus E \to U \setminus \{(0,0)\}$ is a biholomorphism, where $E = \{(0,0)\} \times \mathbb{CP}^1$. Note that $\pi^{-1}(\{(0,0)\}) = E$. We cut out U from X and replace it with $\widetilde{U}$, gluing with the biholomorphism $\pi|_{\widetilde{U} \setminus E}$ to obtain the manifold

$$\widetilde{X} = (X \setminus \{(0,0)\}) \cup_\pi \widetilde{U}.$$

On the other hand, recall that there are natural Kahler structures of $\mathbb{C}^2 \times \mathbb{CP}^1$ as the product of the Kahler structures of $(\mathbb{C}^2, \omega_0)$ and $(\mathbb{CP}^1, \lambda^2 \Omega)$ respectively. Here ω_0 is the standard symplectic form on $\mathbb{C}^2 \cong \mathbb{R}^4$, and $\lambda^2 \Omega$ is the Fubini-Study form multiplied by the number $\lambda^2 > 0$. The set $U \times \mathbb{CP}^1$ inherits a Kahler structure from $(\mathbb{C}^2 \times \mathbb{CP}^1, \omega_0 + \lambda^2 \Omega)$. Hence the manifold $\widetilde{U}$ has a natural Kahler structure as a complex submanifold of the Kahler manifold $U \times \mathbb{CP}^1$. The symplectic form in $\widetilde{U}$ is $\rho_\lambda = (\omega_0 + \lambda^2 \Omega)|_{\tilde{U}}$.

Note that in the construction of $\widetilde{X}$ we glue $\widetilde{U}$ to $X \setminus U$ via the biholomorphism $\pi : \widetilde{U} \setminus E \to U \setminus \{(0,0)\}$. Since $U \setminus \{(0,0)\}$ is diffeomorphic to $(0,1) \times S^3$, its second cohomology group vanishes. It follows that $\pi^* \omega - \rho_\lambda = d\eta$ for some 1-form η in $\widetilde{U} \setminus E$. Let θ be a bump function which equals 1 in a neighborhood of $E \subset \widetilde{U}$ and vanishes outside $\widetilde{U}$. For $\varepsilon > 0$ small enough, it is easy to prove that the form $\widetilde{\omega} = \pi^* \omega - \varepsilon d(\theta \eta) \in \Omega^2(\widetilde{X})$ is symplectic (see Lemma 1.25 below). Moreover $\widetilde{\omega}$ equals ω outside $\widetilde{U}$ as we intend for the blow-up symplectic form.

LEMMA 1.25. *With notations and hypothesis as above, the form $\widetilde{\omega} = \pi^* \omega - \varepsilon d(\theta \eta)$ defines a symplectic form in $\widetilde{X}$ if we take $\varepsilon > 0$ small enough.*

Proof. We can choose the Darboux chart $U = B_r$ to be a ball of radius r centered at 0, and the bump function θ of the form $\theta = \pi^*\theta_0$ for θ_0 a bump function in $\mathbb{C}^2 = \mathbb{R}^4$ which equals 1 in B_{r_1} and equals 0 outside B_{r_2}, being $r_1 < r_2 \in (0, r)$.

In the set $\pi^{-1}(B_{r_1}) \setminus E$ we have

$$\begin{aligned}\widetilde{\omega} &= \pi^*\omega_0 - \varepsilon d\eta = \pi^*\omega_0 - \varepsilon(\pi^*\omega_0 - \rho_\lambda) \\ &= (1-\varepsilon)\pi^*\omega_0 + \varepsilon(\pi^*\omega_0 + \lambda^2\Omega) = \pi^*\omega_0 + \varepsilon\lambda^2\Omega = \rho_{\sqrt{\varepsilon}\lambda}\end{aligned}$$

so we extend $\widetilde{\omega}$ to E by declaring $\widetilde{\omega} := \rho_{\sqrt{\varepsilon}\lambda}$ in E. In this way $\widetilde{\omega}$ is symplectic in the neighborhood $\pi^{-1}(B_{r_1})$ of E.

Also, $\widetilde{\omega} = \pi^*\omega$ in $\pi^{-1}(B_r \setminus B_{r_2})$, so $\widetilde{\omega}$ extends to a global 2-form in $\widetilde{X}$ which is clearly symplectic outside B_{r_2}.

Finally, the closure of $\pi^{-1}(B_{r_2} \setminus B_{r_1}) \subset \widetilde{X}$ is a compact set K. If we make ε small enough, we can achieve that $\widetilde{\omega} = \pi^*\omega - \varepsilon d(\theta\eta)$ is also symplectic in K. This is because

(1) Any choice of ε small enough makes the term $\varepsilon d(\theta\eta)$ negligible uniformly in all K.
(2) The form $\pi^*\omega$ is symplectic in $\widetilde{U} \setminus E$ which contains K.

□

We have thus constructed a symplectic manifold $(\widetilde{X}, \widetilde{\omega})$, called the *symplectic blow-up* of X (elementary method). The set $E = \pi^{-1}(0,0) = \mathbb{CP}^1 \subset \widetilde{U}$ is called the *exceptional sphere*.

Note that near the exceptional sphere $E \subset \widetilde{U}$ the form $\widetilde{\omega}$ equals $\rho_{\sqrt{\varepsilon}\lambda}$, so we have glued ω with $\rho_{\sqrt{\varepsilon}\lambda}$ for $\varepsilon > 0$ small enough.

6.2. Standard Method. In the elementary method we first contructed the manifold $\widetilde{X}$ as in the complex category, without any care for the symplectic form. Only later did we put a symplectic form $\widetilde{\omega}$ on $\widetilde{X}$, but $\widetilde{\omega}$ was constructed ad hoc, and we had to change the symplectic form of X in the gluing region.

Now we are going to put care in the construction of $\widetilde{X}$ as a manifold. We want the gluing of $\widetilde{U}$ and U to be done via a symplectomorphism. Hence the symplectic form on $\widetilde{X}$ will be automatically induced.

Let us introduce some notations first. Call $L = \{(z, [w]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : z \in [w] \cup \{0\}\}$ the tautological line bundle over $\mathbb{CP}^1$. Let

$$\pi : L \to \mathbb{C}^2 \ , \ (z, [w]) \mapsto z$$

be the blow-up map, and let

$$p : L \to \mathbb{CP}^1 \ , \ (z, [w]) \mapsto [w]$$

be the bundle projection. For $\delta > 0$, denote $L(\delta) = \pi^{-1}(B(\delta))$, with $B(\delta) = B_\delta(0) \subset \mathbb{C}^2$ the ball of radius δ centered at 0. We also denote $L_0 = \pi^{-1}(0) \cong \mathbb{CP}^1$ the zero section of the bundle map p, also called the exceptional sphere of the blow-up map π.

Let Ω be the Fubini-Study symplectic form on $\mathbb{CP}^1$. In homogeneous coordinates $[w_0, w_1]$ it is given by

$$\begin{aligned}\Omega &= \tfrac{i}{2}\partial\overline{\partial}\log(|w_0|^2 + |w_1|^2) \\ &= \tfrac{i}{2}\frac{1}{|w|^4}[|w_1|^2 dw_0 \wedge d\overline{w_0} + |w_0|^2 dw_1 \wedge d\overline{w_1} - w_0\overline{w_1} dw_1 \wedge d\overline{w_0} - w_1\overline{w_0} dw_0 \wedge d\overline{w_1}].\end{aligned}$$

For $\lambda > 0$ let us see the expression in local coordinates of the Kahler form ρ_λ discussed previously, induced on L by restriction of $(\mathbb{C}^2 \times \mathbb{CP}^1, \omega_0 + \lambda^2\Omega)$. We have:

$$\begin{aligned}\rho_\lambda &= \pi^*\omega_0 + \lambda^2 p^*\Omega \\ &= \frac{i}{2}(dz_0 \wedge d\overline{z_0} + dz_1 \wedge d\overline{z_1}) \\ &+ \frac{i}{2}\frac{1}{|w|^4}(|w_1|^2 dw_0 \wedge d\overline{w_0} + |w_0|^2 dw_1 \wedge d\overline{w_1} - w_0\overline{w_1}dw_1 \wedge d\overline{w_0} - w_1\overline{w_0}dw_0 \wedge d\overline{w_1}).\end{aligned}$$

Consider also the Kahler form on $\mathbb{C}^2 \setminus \{0\}$ given in coordinates (z_0, z_1) by

$$\begin{aligned}\omega_\lambda &= \frac{i}{2}\partial\overline{\partial}(|z|^2 + \lambda^2 \log(|z|^2)) \\ &= \frac{i}{2}\left(\left(1 + \frac{\lambda^2}{|z|^2}\right)\sum_j dz_j \wedge d\overline{z_j} - \frac{\lambda^2}{|z|^4}\sum_{j,k}\overline{z_j}z_k dz_j \wedge d\overline{z_k}\right)\end{aligned}$$

where $j, k = 0, 1$. Finally, let us consider the diffeomorphism

$$\begin{aligned}F: \quad & \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \overline{B(\lambda)} \\ & z \mapsto F(z) = \sqrt{|z|^2 + \lambda^2}\frac{z}{|z|}.\end{aligned}$$

The map F expands the hole of $\mathbb{C}^2 \setminus \{0\}$, transforming it into a ball of radius λ. The following lemma summarises the technical facts we need for the symplectic blow-up.

LEMMA 1.26. *With notations as above, the following statements hold:*

(1) *The blow-up map $\pi : L(\delta) \setminus L_0 \to B(\delta) \setminus \{0\}$ satisfies $\pi^*\omega_\lambda = \rho_\lambda$.*
(2) *The radial map $F : B(\delta) \setminus \{0\} \to B(\sqrt{\delta^2 + \lambda^2}) \setminus B(\lambda)$ satisfies $F^*\omega_0 = \omega_\lambda$.*

PROOF. Let us see (1) first. Consider homogeneous coordinates $(z_0, z_1) \times [w_0, w_1]$ in $L(\delta) \subset L$. Denote $|z| = \sqrt{|z_0|^2 + |z_1|^2}$.

Take the dense open set $W = \{w_0 \neq 0, z_0 \neq 0\} \subset L(\delta)$, and let us see $(\pi^{-1})^*\rho_\lambda = \omega_\lambda$ in W. The set W has coordinates $(z_0, z_1) \times [1, \frac{z_1}{z_0}]$. Hence $w_0 = 1$, $dw_0 = 0 = d\overline{w_0}$, and $w_1 = \frac{z_1}{z_0}$, so

$$\begin{aligned}dw_1 &= \frac{1}{z_0}dz_1 - \frac{z_1}{z_0^2}dz_0 \\ d\overline{w_1} &= \frac{1}{\overline{z_0}}d\overline{z_1} - \frac{\overline{z_1}}{\overline{z_0}^2}d\overline{z_0}\end{aligned}$$

We plug the above in the expression of ρ_λ and we get

$$\begin{aligned}(\pi^{-1})^*\rho_\lambda &= \frac{i}{2}[dz_0 \wedge d\overline{z_0} + dz_1 \wedge d\overline{z_1}] \\ &+ \lambda^2\frac{i}{2}\left(\frac{|z_0|^2}{|z|^4}dz_1 \wedge d\overline{z_1} + \frac{|z_1|^2}{|z|^4}dz_0 \wedge d\overline{z_0}\right) \\ &- \frac{i}{2}\frac{\lambda^2}{|z|^4}\left(z_0\overline{z_1}dz_1 \wedge d\overline{z_0} + z_1\overline{z_0}dz_0 \wedge d\overline{z_1}\right)\end{aligned}$$

Substitute in the above expression $|z_0|^2 = |z|^2 - |z_1|^2$ and $|z_1|^2 = |z|^2 - |z_0|^2$, to get

$$\begin{aligned}(\pi^{-1})^*\rho_\lambda &= \frac{i}{2}(1 + \frac{\lambda^2}{|z|^2})\sum_j dz_j \wedge d\overline{z_j} \\ &- \frac{i}{2}\frac{\lambda^2}{|z|^4}\left(z_0\overline{z_0}dz_0 \wedge d\overline{z_0} + z_1\overline{z_1}dz_1 \wedge d\overline{z_1} + z_0\overline{z_1}dz_1 \wedge d\overline{z_0} + z_1\overline{z_0}dz_0 \wedge d\overline{z_1}\right) \\ &= \omega_\lambda.\end{aligned}$$

This proves that the equality $\pi^*\omega_\lambda = \rho_\lambda$ is true in $W \subset L(\delta)$. Since W is dense in $L(\delta)$ and both $\pi^*\omega_\lambda$, ρ_λ are smooth forms in $L(\delta)$, the equality $\pi^*\omega_\lambda = \rho_\lambda$ must be true in all $L(\delta)$, and this proves (1).

Now let us see (2). Put $u = F(z)$, so $u_j = \sqrt{1+\lambda^2|z|^{-2}}z_j$ for $j = 0, 1$. We compute the differentials using that $d = \partial + \overline{\partial}$ in $\mathbb{C}^2$. To simplify notation we denote $a = 1 + \lambda^2|z|^{-2}$, hence

$$\begin{aligned}
du_0 &= -\left(\frac{\lambda^2|z_0|^2}{2|z|^4}a^{-1/2} + a^{1/2}\right)dz_0 - \frac{\lambda^2 z_0^2}{2|z|^4}a^{-1/2}d\overline{z_0} \\
&\quad - \frac{\lambda^2\overline{z_0 z_1}}{2|z|^4}a^{-1/2}dz_1 - \frac{\lambda^2 z_0 z_1}{2|z|^4}a^{-1/2}d\overline{z_1}\,, \\
d\overline{u_0} &= -\frac{\lambda^2\overline{z_0}^2}{2|z|^4}a^{-1/2}dz_0 - \left(\frac{\lambda^2|z_0|^2}{2|z|^4}a^{-1/2} + a^{1/2}\right)d\overline{z_0} \\
&\quad - \frac{\lambda^2\overline{z_0 z_1}}{2|z|^4}a^{-1/2}dz_1 - \frac{\lambda^2\overline{z_0}z_1}{2|z|^4}a^{-1/2}d\overline{z_1}
\end{aligned}$$

after some straightforward computations we get

$$\begin{aligned}
du_0 \wedge d\overline{u_0} &= \left(a - \frac{\lambda^2|z_0|^2}{|z|^4}\right)\ dz_0 \wedge d\overline{z_0} - \frac{\lambda^2\overline{z_0 z_1}}{2|z|^4}dz_0 \wedge dz_1 \\
&\quad - \frac{\lambda^2\overline{z_1}z_0}{2|z|^4}dz_1 \wedge d\overline{z_0} - \frac{\lambda^2\overline{z_0}z_1}{2|z|^4}dz_0 \wedge d\overline{z_1}\,, \\
du_1 \wedge d\overline{u_1} &= \left(a - \frac{\lambda^2|z_1|^2}{|z|^4}\right)\ dz_1 \wedge d\overline{z_1} - \frac{\lambda^2\overline{z_1 z_0}}{2|z|^4}dz_1 \wedge dz_0 \\
&\quad - \frac{\lambda^2\overline{z_0}z_1}{2|z|^4}dz_0 \wedge d\overline{z_1} - \frac{\lambda^2\overline{z_1}z_0}{2|z|^4}dz_1 \wedge d\overline{z_0}
\end{aligned}$$

Note that by simmetry of the map F, $du_1 \wedge d\overline{u_1}$ is obtained interchanging the role of indexes 0 and 1. Finally, using that $a = 1 + \lambda^2|z|^{-2}$, we easily obtain that

$$\begin{aligned}
F^*\omega_0 &= \frac{i}{2}\left(du_0 \wedge d\overline{u_0} + du_1 \wedge d\overline{u_1}\right) \\
&= \frac{i}{2}\left(\left(1 + \frac{\lambda^2}{|z|^2}\right)\sum_j dz_j \wedge d\overline{z_j} - \frac{\lambda^2}{|z|^4}\sum_{j,k}\overline{z_j}z_k dz_j \wedge d\overline{z_k}\right) \\
&= \omega_\lambda
\end{aligned}$$

as claimed. □

Now let us consider the symplectomorphisms given by

$$\pi : (L(\delta) \setminus L_0, \rho_\lambda) \to (B(\delta) \setminus \{0\}, \omega_\lambda)$$
$$F : (B(\delta) \setminus \{0\}, \omega_\lambda) \to (B(\sqrt{\delta^2+\lambda^2}) \setminus B(\lambda), \omega_0)$$

The composition is a symplectomorphism

$$f = F \circ \pi : (L(\delta) \setminus L_0, \rho_\lambda) \to (B(\sqrt{\delta^2+\lambda^2}) \setminus B(\lambda), \omega_0).$$

and this f is what we were looking for.

Now, given a symplectic manifold (X, ω), take a point $p \in X$ and a neighborhood U of p so that there exists a Darboux chart ψ defined on U. Hence we have a symplectomorphism $\psi : (U, \omega) \to (B(\sqrt{\delta^2+\lambda^2}), \omega_0)$, for some $\lambda > 0$ and $\delta > 0$. The symplectomorphism we will use for gluing is

$$\Upsilon = f^{-1} \circ \psi : \left(\psi^{-1}\left(B(\sqrt{\delta^2+\lambda^2}) \setminus \overline{B(\lambda)}\right), \omega\right) \to (L(\delta) \setminus L_0, \rho_\lambda)\,.$$

We are now in a position to define the symplectic blow-up (standard method).

Let (X, ω) be a symplectic manifold and $p \in X$. Consider $R(p)$ be the *symplectic radius* of the point p, i.e. the supremum of all radii r so that there exists a symplectic embedding $\varphi : (B(r), \omega_0) \to (X, \omega)$ with $\varphi(0) = p$. In other words, $R(p)$ is the supremum of the radii r so that there is a Darboux chart to a standard symplectic ball $(B(r), \omega_0)$ of $\mathbb{R}^{2n}$ of radius r. Take $\lambda > 0$ with $\lambda < R(p)$, and let Υ be the symplectomorphism above.

DEFINITION 1.27. *We define the* symplectic blow-up *of (X, ω) at p of size $\lambda > 0$ as*

$$\widetilde{X} = (X \setminus \psi^{-1}(B(\lambda))) \bigcup_{\Upsilon} L(\delta).$$

The symplectic form $\widetilde{\omega}$ in $\widetilde{X}$ is given by $\widetilde{\omega} = \omega$ in $X \setminus B(\lambda)$ and $\widetilde{\omega} = \rho_\lambda$ in $L(\delta)$.

The form $\widetilde{\omega}$ is well defined because $\Upsilon^* \rho_\lambda = \omega$, since Υ is a symplectomorphism.

REMARK 1.28. *In the above definition, λ should be thought of as a symplectic characteristic of the blow-up process, because the size of λ is limited by the symplectic radius $R(p)$ of p. We may choose different values of λ with $0 < \lambda < R(p)$ to perform different symplectic blow-ups. On the other hand, $\delta > 0$ is just a technical device to make the gluing smooth, so it is chosen arbitrarily small.*

Note that from a symplectic point of view we are actually obtaining a symplectic manifold $\widetilde{X}$ with less volume than X, because we remove a ball of radius λ and change it by $(L(\delta), \rho_\lambda)$ which has volume δ because $L(\delta) \setminus L_0$ is symplectomorphic to the annulus $(B(\sqrt{\delta^2 + \lambda^2}) \setminus B(\lambda), \omega_0)$.

Recall that the symplectic blow-up manifold $\widetilde{X}$ as done in the standard method is diffeomorphic as a manifold to the blow-up manifold of the elementary method in which the biholomorphism $\pi : \widetilde{U} \to U$ was used as gluing map. However, the symplectic forms used in the elementary method and the second are different in $\widetilde{U}$.

6.3. Topology of the blow-up. In this subsection we focus on the blow-up manifold without taking in consideration the symplectic form. First we note that there is an alternative way to construct $\widetilde{X}$ using connected sum, shown below.

LEMMA 1.29. *Let X be a 4-manifold and $q \in X$ a point. Let $\widetilde{X}$ be the blow-up of X at q Then $\widetilde{X}$ is diffeomorphic in a natural way to $X \# \overline{\mathbb{CP}^2}$. This diffeomorphism identifies the exceptional sphere $E \subset \widetilde{X}$ with a line $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$.*

PROOF. Recall that by definition

$$\widetilde{X} = (X \setminus \{q\}) \cup_\pi \widetilde{U}$$

with $U \subset X$ an open neighborhood of q

$$\widetilde{U} = \{((z_1, z_2), [w_1, w_2]) \in U \times \mathbb{CP}^1 \ , \ z_1 w_2 = z_2 w_1\}.$$

and the gluing is done via $\pi : \widetilde{U} \setminus E \to U \setminus \{(0,0)\}$, with $E = \{(0,0)\} \times \mathbb{CP}^1$. The open set U can be supposed to be a 4-ball $B_r \subset \mathbb{C}^2$ centered at 0. We claim that $\widetilde{U} \setminus E$ is diffeomorphic to $\mathbb{CP}^2 \setminus B$ with B a ball. After we prove this claim, we are done, since $\widetilde{X}$ is obtained by removing a ball from $\mathbb{CP}^2$, another ball from X, and then gluing around the boundaries. This means that $\widetilde{X}$ is the connected sum as desired.

To see the claim, note that $\widetilde{U}$ is a neighborhood of the zero section of the tautological line bundle $\mathbb{C} \to L \to \mathbb{CP}^1$. Now, given $p = [0,0,1] \in \mathbb{CP}^2$, let us see that $\overline{\mathbb{CP}^2} \setminus \{p\}$ is diffeomorphic to the total space of L. To see this we are going to see that both are isomorphic as vector line bundles. The line bundle structure of $\overline{\mathbb{CP}^2} \setminus \{p\}$ is induced by the map

$$f : \overline{\mathbb{CP}^2} \setminus \{p\} \mapsto \mathbb{CP}^1, \ \ [w_0, w_1, w_2] \mapsto [w_0, w_1]$$

is a surjective map whose fibers are the sets $f^{-1}([w_0, w_1]) = V([w_0, w_1, 0], p) \setminus \{p\} \cong \mathbb{C}$, where $V(a, b)$ stands for the projective line generated by two points $a, b \in \mathbb{CP}^n$. In the chart $V_0 = \{w_0 \neq 0\} \subset \mathbb{CP}^1$ we have a diffeomorphism $V_0 \times \mathbb{C} \cong f^{-1}(V_0)$ given by $([1, w], \lambda) \mapsto [1, w, \lambda]$ such that every fiber $f^{-1}(\{[1, w]\})$ corresponds to $\{[1, w]\} \times \mathbb{C}$ via a $\mathbb{C}$-linear isomorphism. Doing the analogous in the chart $V_1 = \{w_1 \neq 0\}$ we get a line bundle atlas for $\overline{\mathbb{CP}^2} \setminus \{p\}$.

Let us construct two sections of L and $\overline{\mathbb{CP}^2} \setminus \{p\}$ respectively. Consider $s : \mathbb{CP}^1 \to L$ given by

$$s([w_0, w_1]) = [w_0, w_1] \times \frac{\overline{w_0}}{|w_0|^2 + |w_1|^2}(w_0, w_1)$$

Note that s is well defined, i.e. independent of the representative of $[w_0, w_1]$. Clearly s defines a section of the bundle L. This section has just one zero at the point $[0 : 1]$ on which the intersection with the zero section is negative. This shows that the Chern class of L is -1, meaning $-\,\mathrm{PD}[\mathbb{CP}^1]$, with $[\mathbb{CP}^1]$ the fundamental class.

On the other hand consider $\tilde{s} : \mathbb{CP}^1 \to \overline{\mathbb{CP}^2} \setminus \{p\}$ given by

$$\tilde{s}([w_0, w_1]) = [w_0, w_1, w_0]$$

As before, this section has just one zero at $[0 : 1]$ and intersects the zero section $\mathbb{CP}^1 \times \{0\}$ of $\overline{\mathbb{CP}^2} \setminus \{p\}$ only at $[0 : 1 : 0]$. Note that this intersection is negative precisely because we are considering $\overline{\mathbb{CP}^2} \setminus \{p\}$ with reverse orientation.

This shows that the Chern class of $\overline{\mathbb{CP}^2} \setminus \{p\}$ is also -1. Since complex line bundles are classified by the Chern class, it follows that the bundles $\overline{\mathbb{CP}^2} \setminus \{p\}$ and L are isomorphic as complex vector bundles of rank 1, so in particular they are diffeomorphic as smooth manifolds.

When doing the blow up, we cut out a small ball $B_r(0) \cong U \subset X$ with coordinates (z_0, z_1) and then glue

$$\widetilde{U} = \{(z_0, z_1) \times [w_0, w_1] \in B_r(0) \times \mathbb{CP}^1 : (z_0, z_1) \in [w_0 : w_1] \cup \{0\}\}$$

Note that $\widetilde{U}$ is a radial neighborhood of the zero section of the bundle L. Since L is isomorphic to $\overline{\mathbb{CP}^2} \setminus \{p\}$ as bundles, it follows that $\widetilde{U}$ is diffeomorphic to $\mathbb{CP}^2 \setminus B^p$ with B^p a ball around $p = [0, 0, 1]$.

The gluing of $\widetilde{U}$ and $X \setminus U$ along $\partial\widetilde{U} \cong \partial U$ corresponds under this diffeomorphism to the gluing of $\mathbb{CP}^2 \setminus B^p$ and $X \setminus U$ along $\partial(\mathbb{CP}^2 \setminus B^p) \cong \partial U$, and this yields the diffeomorphism $\widetilde{X} \cong X \# \overline{\mathbb{CP}^2}$. □

COROLLARY 1.30. *The second homology group of* $\widetilde{X}$ *satisfies*

$$H_2(\widetilde{X}, \mathbb{Z}) \cong H_2(X, \mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2}, \mathbb{Z})$$

via a natural isomorphism induced by inclusions.

Moreover, the intersection form $I : H_2(\widetilde{X}, \mathbb{Z}) \times H_2(\widetilde{X}, \mathbb{Z}) \to \mathbb{Z}$ *is orthogonal with respect to this decomposition, i.e. if the class* $c \in H_2(\widetilde{X}, \mathbb{Z})$ *corresponds to* $a + b \in H_2(X, \mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2}, \mathbb{Z})$ *then* $c \cdot c = a \cdot a + b \cdot b$.

PROOF. We use Mayer-Vietoris in $\widetilde{X} \cong X \# \overline{\mathbb{CP}^2} = U \cup V$ with $U = X \setminus B_1$, $V = \overline{\mathbb{CP}^2} \setminus B_2$, and B_1, B_2 are balls. We have that $U \cap V \cong S^3 \times (0, 1)$ has trivial first and second homology, so from the Mayer-Vietoris sequence we get

$$H_2(U \cap V, \mathbb{Z}) = 0 \to H_2(U, \mathbb{Z}) \oplus H_2(V, \mathbb{Z}) \xrightarrow{j} H_2(\widetilde{X}, \mathbb{Z}) \to 0 = H_1(U \cap V, \mathbb{Z})$$

Therefore the map $j : H_2(U, \mathbb{Z}) \oplus H_2(V, \mathbb{Z}) \to H_2(\widetilde{X}, \mathbb{Z})$ given by $j(a, b) = a - b$ is an isomorphism.

To prove the claim about the intersection form we need to understand the above isomorphism geometrically. First note that the fact that j is an isomorphism means that each 2-chain in $\widetilde{X}$ is homologous to a sum of two 2-chains contained in U and V respectively, and moreover this decomposition is unique at the homology level.

We can visualize the inverse j^{-1} of the above isomorphism as follows. Recall first that every second homology class on a 4-manifold can be represented by an embedded surface, see for instance [**25**], page 8. Take $c = [\Sigma] \in H_2(\widetilde{X}, \mathbb{Z})$ represented by the embedded surface $\Sigma \subset \widetilde{X}$. If the surface Σ representing the class c can be taken contained in either $U = X \setminus B_1$ or $V = \overline{\mathbb{CP}^2} \setminus B_2$, then $j^{-1}(c)$ equals $[\Sigma] \in H_2(U, \mathbb{Z}) \oplus H_2(V, \mathbb{Z})$. If not, this means that Σ intersects the boundaries of the balls B_1 and B_2 (note that the boundaries are identified in $\widetilde{X}$ via a diffeomorphism).

We can choose Σ so that the intersection of Σ and $\partial B_1 \equiv \partial B_2$ is transversal, so $\gamma = \Sigma \cap \partial B_1$ is a circle contained in a 3-sphere S^3. Obviously γ does not fill S^3 so we can think of it as being in $\mathbb{R}^3 \cong S^3 \setminus \{pt\}$.

From this we see that the surface Σ consists of the connected sum of two surfaces $\Sigma_1 \subset X$ and $\Sigma_2 \subset \overline{\mathbb{CP}^2}$, and the gluing of Σ_1 and Σ_2 occurs along a circle γ in $S^3 \setminus \{pt\} \cong \mathbb{R}^3$. We can bound the circle γ by small two half-spheres (discs) D_1, D_2 in $\mathbb{R}^3$ with opposite orientations, so that the homology class of Σ does not change after adding D_1 and D_2. We add these discs to undo the connected sum $\Sigma = \Sigma_1 \# \Sigma_2$ without changing the homology class $[\Sigma]$. Finally, we can perturb slightly to make Σ_1 and Σ_2 disjoint, with $\Sigma_1 \subset U = X \setminus B_1$ and $\Sigma_2 \subset V = \overline{\mathbb{CP}^2} \setminus B_2$. Summarasing,

$$[\Sigma] = [\Sigma_1 \# \Sigma_2] = [(\Sigma_1 \cup D_1) \sqcup (D_2 \cup \Sigma_2)] = [\Sigma_1] + [\Sigma_2]$$

and this shows that $j^{-1}([\Sigma]) = [\Sigma_1] + [\Sigma_2] \in H_2(U) \oplus H_2(V)$.

Now, if we want to cumpute the intersection product $[\Sigma] \cdot [S]$ of two homology classes represented by surfaces Σ and S, we can first take $\Sigma = \Sigma_1 \sqcup \Sigma_2$, $S = S_1 \sqcup S_2$ disjoint unions of surfaces contained in U and V respectively. If we make a small perturbation on Σ, it becomes a small perturbation in Σ_1 and Σ_2 separately. The same holds for S.

The conclusion is that $[\Sigma] \cdot [S] = [\Sigma_1] \cdot [S_1] + [\Sigma_2] \cdot [S_2]$, as claimed. □

Let $E \subset \widetilde{X}$ be the exceptional sphere. Its homology class $[E]$ is denoted by $e \in H_2(\widetilde{X}, \mathbb{Z}) \cong H_2(X, \mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2}, \mathbb{Z})$. It corresponds to the homology class of a negatively oriented line $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$. Using Corollary 1.30 we see that the homology class $e = [E] \in H_2(\widetilde{X}, \mathbb{Z})$ satisfies $e \cdot e = -1$, because two different negatively oriented complex lines in $\mathbb{CP}^2$ intersect transversely in one point, and positively with respect to the standard orientation of $\mathbb{CP}^2$, hence negatively in $\overline{\mathbb{CP}^2}$.

Now consider a symplectic surface $\Sigma \subset X$ and blow up X at a point $q \in \Sigma$. Let $\widetilde{X}$ be the blow-up of X at q constructed by the elementary method. Call $\pi : \widetilde{X} \to X$ the blow-up map. We define the *proper transform* of Σ as the closure in $\widetilde{X}$ of the set $\pi^{-1}(\Sigma \setminus \{q\})$. We denote the proper transform as $\widetilde{\Sigma} = \overline{\pi^{-1}(\Sigma \setminus \{q\})} \subset \widetilde{X}$.

Take coordinates (z_1, z_2) in a chart U such that $\Sigma = \{z_1 = 0\}$ and $q = (0,0)$ and let $\widetilde{U} = \{(z_1, z_2) \times [w_0, w_1] : z_1 w_2 = z_2 w_1\}$. The surface $\widetilde{\Sigma} \subset \widetilde{X}$ is defined locally in $\widetilde{U}$ by

$$\widetilde{\Sigma} = \{z_1 = 0, w_0 = 0\} = \{(0, z_2) \times [0,1] : (0, z_2) \in U\}$$

To see this, note that any point in $\pi^{-1}(\Sigma \setminus \{q\})$ has the form $(0, z_2) \times [0,1]$ with $z_2 \neq 0$, so $(0,0) \times [0,1]$ is the only limit point of $\pi^{-1}(\Sigma \setminus \{q\})$.

Note that $\widetilde{\Sigma}$ intersects the exceptional sphere E transversely in a unique point. This point has coordinates $(0,0) \times [0,1]$ in $\widetilde{U} \subset \widetilde{X}$. The intersection of $\widetilde{\Sigma}$ and E is positive in $\widetilde{X}$ therefore $[\widetilde{\Sigma}] \cdot e = 1$.

Now we study the homology class of the proper transform $[\widetilde{\Sigma}]$ of $[\Sigma]$.

LEMMA 1.31. *Let X be a 4-manifold and $\Sigma \subset X$ an embedeed surface. Let $q \in \Sigma \subset X$ be a point, and $\widetilde{X}$ the blow-up of X at q. Identifying $H_2(\widetilde{X}) \cong H_2(X) \oplus H_2(\overline{\mathbb{CP}^2})$, we have $[\widetilde{\Sigma}] = [\Sigma] - e$.*

PROOF. For the sake of clarity let us call $j : H_2(X) \oplus H_2(\overline{\mathbb{CP}^2}) \to H_2(\widetilde{X})$ the isomorphism induced by Mayer-Vietoris. We want to show that $j([\Sigma]) = [\widetilde{\Sigma}] + e$. Equivalently, $j^{-1}([\widetilde{\Sigma}]) = [\Sigma] - j^{-1}(e) = [\Sigma] - [\overline{\mathbb{CP}^1}]$.

Let us first see that $p_1\left(j^{-1}([\widetilde{\Sigma}])\right) = [\Sigma]$, being $p_1 : H_2(X) \oplus H_2(\overline{\mathbb{CP}^2}) \to H_2(X)$ the projection. This is very easy because $\widetilde{\Sigma} = (\widetilde{\Sigma} \setminus U) \cup (\widetilde{\Sigma} \cap \widetilde{U})$, with U the Darboux chart used in the definition of $\widetilde{X}$. Note that $\widetilde{\Sigma} \setminus U$ can be regarded as $\Sigma \setminus U$. Now, by the explicit description of the map $j^{-1} : H_2(\widetilde{X}) \to H_2(X) \oplus H_2(\overline{\mathbb{CP}^2})$ given in Corollary 1.30, $p_1\left(j^{-1}([\widetilde{\Sigma}])\right)$ is obtained by adding a 2-disc $D \subset \partial U \cong S^3$ to $\Sigma \setminus U$. This yields a surface $\Sigma' \subset X$. Since Σ' is a small perturbation of $\Sigma \subset X$, we get $p_1\left(j^{-1}([\widetilde{\Sigma}])\right) = [\Sigma'] = [\Sigma]$.

Therefore $j^{-1}([\widetilde{\Sigma}]) = [\Sigma] + m[\overline{\mathbb{CP}^1}]$ for some $m \in \mathbb{Z}$, being $[\overline{\mathbb{CP}^1}]$ the generator of $H_2(\overline{\mathbb{CP}^2}) \cong \mathbb{Z}$. Recall that $j([\overline{\mathbb{CP}^1}]) = e$ is the exceptional sphere.

Now, as $[\widetilde{\Sigma}] \cdot e = 1$, we have

$$1 = [\widetilde{\Sigma}] \cdot e = (j([\Sigma]) + mj([\overline{\mathbb{CP}^1}])) \cdot e = me \cdot e = -m$$

which implies that $m = -1$, so $j^{-1}([\widetilde{\Sigma}]) = [\Sigma] - [\overline{\mathbb{CP}^1}]$. If we drop the tedious notation, this is written as $[\widetilde{\Sigma}] = [\Sigma] - e$, as we wanted. □

From the Lemma above it follows that

$$[\Sigma]^2 = ([\widetilde{\Sigma}] + e)^2 = [\widetilde{\Sigma}]^2 + e^2 + 2[\widetilde{\Sigma}] \cdot e = [\widetilde{\Sigma}]^2 + 1.$$

In other words

$$[\widetilde{\Sigma}]^2 = [\Sigma]^2 - 1 \tag{4}$$

So the self intersection of a surface Σ is reduced by one each time we blow it up.

6.4. Proper Transforms of Symplectic Submanifolds. Now we focus our attention in how *symplectic* submanifolds Σ of X transform under the symplectic blow-up process. Let $\widetilde{X}$ be the blow-up manifold of X at the point $q \in X$. It would be most desirable that any symplectic submanifold Σ of X passing through the point q had a well-defined proper transform in the symplectic category. This however, is somewhat tricky.

First let us be precise about the meaning here of proper transform. The definition is the one adopted in [**38**]. Let X be a symplectic manifold and let $\Sigma \subset X$ be a symplectic closed submanifold. Let $q \in \Sigma$ be a point and $\widetilde{X}$ the blow-up of X at q. Denote by $\pi : \widetilde{X} \to X$ the blow-up map. There are neighborhoods U^E of the exceptional sphere $E \subset \widetilde{X}$, and U^q of the point $q \in X$ so that the blow-up map outside of these neighborhoods given by

$$\pi : (\widetilde{X} \setminus U^E, \widetilde{\omega}) \to (X \setminus U^q, \omega)$$

is a symplectomorphism.

DEFINITION 1.32. *Notations and hypothesis as above. We say that a symplectic closed submanifold $\widetilde{\Sigma} \subset \widetilde{X}$ is a* symplectic proper transform *of Σ if $\pi(\widetilde{\Sigma} \setminus U^E) = \Sigma \setminus B^q$.*

Note that in the above definition it is allowed in principle some freedom to construct $\widetilde{\Sigma} \cap U^E$, as long as it is symplectic and glues smoothly with $\pi^{-1}(\Sigma \setminus B^q)$ to form a smooth symplectic manifold $\widetilde{\Sigma}$.

The existence of proper transforms in the standard method (the 2th) of symplectic blow-up is conditional to the choice of the Darboux chart in the definition of $\widetilde{X}$. Indeed, if you want to make sure that some specific surface Σ has a proper transform $\widetilde{\Sigma} \subset \widetilde{X}$, you have to choose a Darboux chart adapted to Σ and use this chart for the gluing in the definition of $\widetilde{X}$.

Otherwise there can be problems. The main issue is that, if there are various symplectic submanifolds intersecting at q, one of the surfaces can be transformed into a linear complex plane in the Darboux chart, but the behaviour of the other surfaces in the chart is in principle not controlled.

It is true that for every symplectic surface $\Sigma \subset X$ it can be defined ad-hoc a blow-up $\widetilde{X}$ with a symplectic form $\widetilde{\omega}$ so that there exists a proper transform $\widetilde{\Sigma}$ of Σ. However, there is no way to define the symplectic blow-up $\widetilde{X}$ of X at q in such a way that *all* symplectic submanifolds of X passing through q have a proper transform in $\widetilde{X}$. This was proved in [**38**], where it is shown the following:

THEOREM 1.33. *Let Σ be a symplectic inmersed surface in a 4-manifold X. Suppose that Σ self-intersects transversely and negatively at a point q, and that q is a double point of Σ, meaning that the inmersion $\Sigma \to X$ has two preimages of q. Then Σ does not admit a proper transform in the symplectic blow-up $\widetilde{X}$ at q of X.*

The proof relies on the symplectic adjunction formula to show that the genus of any proper transform of Σ in $\widetilde{X}$ would strictly decrease, which is imposible since blow-up only changes a small neighborhood around the point q. Note that this covers the case of two symplectic surfaces Σ_1, Σ_2 of genus g_1 and g_2 intersecting negatively at a point q, since $\Sigma_1 \cup \Sigma_2$ is an inmersed surface of genus $g_1 + g_2$ with a negative double point at q.

This shows that there is no hope in trying to define the proper transform in the context of symplectic blow-up as a functor from symplectic surfaces in X to symplectic surfaces in $\tilde{X}$ assigning to each $\Sigma \subset X$ a proper transform $\widetilde{\Sigma} \subset \widetilde{X}$.

Anyway, we are mainly interested in the symplectic blow-up process in order to apply it to geography problems in symplectic geometry, i.e. prove or disprove the existence of symplectic manifolds satisfying some given properties. The categorical behaviour of the symplectic blow-up is not important for this at all, so we will just prove what we need and sweep this matter under the carpet.

The crucial thing we need is that the symplectic proper transform works fine if we restrict ourselves to positively intersecting symplectic surfaces, and if we allow ourselves to slightly perturb this surfaces near the intersection points where the blow-up occurs.

The following Propositions show this.

PROPOSITION 1.34. *(Proper transforms in the standard symplectic blow-up) Let (X, ω) be a symplectic 4-manifold.*

1. *Let $\Sigma \subset X$ be a symplectic surface and $q \in \Sigma$ be a point. Then we can choose a Darboux chart adapted to Σ to perform the standard symplectic blow-up $(\widetilde{X}, \widetilde{\omega})$, and there is a symplectic proper transform $\widetilde{\Sigma} \subset \widetilde{X}$ of Σ.*
2. *Let $\Sigma_1, \Sigma_2 \subset X$ be symplectic surfaces intersecting transversely and positively at $q \in \Sigma_1 \cap \Sigma_2$. Then, after a small perturbation of one of the surfaces, we can choose a Darboux chart adapted to Σ_1, Σ_2 to perform the standard symplectic blow-up $(\widetilde{X}, \widetilde{\omega})$, and there are proper symplectic transforms in $\widetilde{X}$, $\widetilde{\Sigma}_1$ of Σ_1 and $\widetilde{\Sigma}_2$ of Σ_2. Moreover $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ are disjoint surfaces in $\widetilde{X}$.*
3. *Let $\Sigma_1, \dots, \Sigma_m \subset X$ be a finite number of symplectic surfaces intersecting transversely and positively at a point $q \in X$. Suppose there exists a Darboux chart U near q so that*

$\Sigma_j \cap U$, $1 \le j \le m$, are given by complex lines (i.e. the zero set of complex polynomials of degree 1).

Then, if we use this chart U to perform the standard symplectic blow-up, there exist symplectic proper transforms $\widetilde{\Sigma}_1, \dots, \widetilde{\Sigma}_m$. The proper transforms are moreover disjoint submanifolds of the blow-up manifold $(\widetilde{X}, \widetilde{\omega})$.

PROOF. Let us see (1) first. Take a Darboux chart $U^q \cong B(\sqrt{\lambda^2+\delta^2})$ near q adapted to Σ, whose existence is proved in Lemma 1.19. Denote (z,w) the coordinates of the chart so we have $\Sigma \cap U^q \cong \{z=0\} \subset B(\sqrt{\lambda^2+\delta^2})$.

The manifold $\widetilde{X}$ is the union $(X \setminus U^q)\bigcup_\upsilon L(\delta)$, where the gluing is via a symplectomorphism $\upsilon = f^{-1}\circ\psi$, with ψ a chart near q, and

$$f^{-1} = \pi^{-1}\circ F^{-1} : (B(\sqrt{\delta^2+\lambda^2}) \setminus \overline{B(\lambda)}, \omega_0) \to (L(\delta)\setminus L_0, \rho_\lambda).$$

Recall the notations of the construction of the standard symplectic blow-up in Definition 1.27. The map $f = F\circ\pi$, with F the radial symplectomorphism

$$F : (B(\delta)\setminus\{0\}, \omega_\lambda) \to (B(\sqrt{\delta^2+\lambda^2}) \setminus B(\lambda), \omega_0)$$
$$F(z) = \sqrt{|z|^2+\lambda^2}\,\tfrac{z}{|z|}$$

and the map π is the blow-up map $\pi : (L(\delta)\setminus L_0, \rho_\lambda) \to (B(\delta)\setminus\{0\}, \omega_\lambda)$.

Now, F^{-1} sends

$$\{z=0\} \cap [B(\sqrt{\delta^2+\lambda^2}) \setminus B(\lambda)]$$

to the plane

$$\{z=0\} \cap [B(\delta)\setminus\{0\}]$$

since it is a radial map. Hence $F^{-1}(\Sigma) = \{z=0\} \subset B(\delta)\setminus\{0\}$ is a standard complex submanifold of $\mathbb{C}^2$, so the closure of $\pi^{-1}(F^{-1}(\Sigma)) \subset \mathbb{C}^2\times\mathbb{CP}^1$ is a smooth surface $\widetilde{\Sigma}$. Note that the symplectic form $\widetilde{\omega}$ on $\widetilde{X}$ equals ω in $\widetilde{X}\setminus B(\lambda)$ and equals ρ_λ in $L(\delta)$. Since $\Sigma\setminus B(\lambda)$ is symplectic and $\widetilde{\Sigma}$ is a complex submanifold of $L(\delta)$, the surface $\widetilde{\Sigma}$ is clearly symplectic. Hence $\widetilde{\Sigma}$ is a proper transform of Σ.

To see (2), take a Darboux coordinates (z,w) near q so that $\Sigma_1 = \{w=0\}$ and $\Sigma_2 = \{z=0\}$ in the chart $U^q \cong B(\sqrt{\lambda^2+\delta^2})$. This can be done after a small perturbation according to Lemma 1.20.

Let F be as above. We have $F^{-1}(\Sigma_1) = \{w=0\} \subset B(\delta)\setminus\{0\}$ and $F^{-1}(\Sigma_2) = \{z=0\} \subset B(\delta)\setminus\{0\}$ are transverse standard complex submanifold of $\mathbb{C}^2$, so the closures of $\pi^{-1}(F^{-1}(\Sigma_i)) \subset \mathbb{C}^2\times\mathbb{CP}^1$, for $i=1,2$ are smooth disjoint surfaces $\widetilde{\Sigma}_1$ and $\tilde{S}_2$. Clearly they are symplectic in $(\widetilde{X},\widetilde{\omega})$, so they are proper transforms.

The proof of (3) is completely analogous. □

REMARK 1.35. (1) *Note that for the proof above to work, we need the surfaces Σ_j locally expressed as linear subspaces. Otherwise $F^{-1}(\Sigma_j) \subset B(\delta)\setminus\{0\}$ might accumulate to the point $0\in\mathbb{C}^2$ with many different tangent directions, and the closure of $\pi^{-1}(F^{-1}(\Sigma_i))$ in $\mathbb{C}^2\times\mathbb{CP}^1$ could not be a closed manifold.*

(2) *The assumption we make in item (3) of Proposition 1.34 about the existence of a chart U in which the Σ_j are complex polynomials is always true (up to small deformations of the Σ_j), as shown in Corollary 1.22. However, note that if were to obtain the chart U in a different way (for instance, maybe without the need to deform the surfaces) then the existence of the proper transforms of the Σ_j holds anyway.*

The conclusion is that we can always make proper transforms of positively intersecting symplectic surfaces Σ_j of X, (deforming them a bit near the intersection). Note that a small deformation near the blown-up point is allowed in the definition of symplectic proper transform.

Note however that it is not clear what happens if we want to avoid the use of deformations of the Σ_j, since in that case it is not clear whether there exists a chart U in which all the Σ_j are expressed as complex linear subspaces.

By giving up the standard method of blowing-up and adopting the elementary method (where there is more freedom to change the symplectic form) we can soften a bit the requirements on the chart U above. Now it will be enough that the surfaces Σ_j be expressed in U as the zero sets of complex polynomials.

PROPOSITION 1.36. *(Proper transforms in the elementary symplectic blow-up)*

Let (X, ω) be a symplectic manifold. Let $\Sigma_1, \dots, \Sigma_m$ be a finite number of symplectic surfaces intersecting transversely and positively at a point $q \in X$. Suppose there exists a Darboux chart $\psi : U \to \psi(U) \subset \mathbb{C}^2$ near q so that $\psi(\Sigma_1 \cap U)$, ... , $\psi(\Sigma_m \cap U)$ are given by the zero set of complex polynomials.

Then, if we use this chart U to perform the elementary *symplectic blow-up, there exist symplectic proper transforms $\widetilde{\Sigma}_1$, ... , $\widetilde{\Sigma}_m$. The proper transforms are moreover disjoint (symplectic) submanifolds of the blow-up manifold $(\widetilde{X}, \widetilde{\omega})$.*

PROOF. Note that in the elementary blow-up we construct the manifold $\widetilde{X}$ by using a local Kahler structure in some chart U around q. If we take the chart U so that the surfaces Σ_j are the zero-set of complex polynomials, then the proper transforms are well-defined as smooth manifolds. We have to make sure that we can choose the symplectic form $\widetilde{\omega}$ so that the surfaces $\widetilde{\Sigma}_j \subset \widetilde{X}$ are symplectic.

But this is clearly possible if we choose $\varepsilon > 0$ small enough in the construction of $\widetilde{\omega}$ given in Lemma 1.25. Indeed, near the exceptional sphere E, the $\widetilde{\Sigma}_j$ are symplectic because they are Kahler submanifolds of $\mathbb{C}^2 \times \mathbb{CP}^1$ and the form $\widetilde{\omega}$ is the Kahler form of $\mathbb{C}^2 \times \mathbb{CP}^1$. Far away from E, they are also symplectic since $\widetilde{\omega} = \omega$ and $\widetilde{\Sigma}_j = \Sigma_j$. In the gluing region, a compact set away from E, $\widetilde{\omega} = \omega + \varepsilon d(\theta\eta)$, and making ε small enough ensures that ω dominates, so $\widetilde{\Sigma}_j$ is also symplectic there. □

REMARK 1.37. *The condition imposed in Proposition 1.36 above on the surfaces Σ_j, $1 \leq j \leq m$ that they can be locally expressed in some chart as the zero sets of complex polynomials is automatically satisfied (up to a small perturbation of some surfaces near the point q) by Corollary 1.22. However the surfaces may be expressed as complex polynomials using a different method to that used in Corollary 1.22, in which case the proper transforms of the Σ_j still exist.*

7. Symplectic sum.

There is a way of making a connected sum of two symplectic manifolds by removing tubular neighborhoods of 2-codimensional symplectic submanifolds and gluing by a symplectomorphism along the boundaries. This construction was first outlined in [**27**], and it was exploited and popularized in [**24**] to give examples of symplectic 4-manifolds displaying many non-Kahler and exotic behaviours, and for this reason it is also called the *Gompf symplectic sum.* To carry out this construction one needs two symplectic manifolds M_1, M_2 and two codimension-2 symplectic submanifolds $N_1 \subset M_1$ and $N_2 \subset M_2$ with N_1 and N_2 symplectomorphic, and such that the normal bundles ν_1 of N_1 and ν_2 of N_2 are orientation-reversing isomorphic.

We detail below how this works in the case of 4-manifolds M_j and surfaces N_j with trivial normal bundles. The connected sum in the case of non-trivial normal bundles is analogous but with some technical difficulties, see [**24**] for details.

Let (M_1, ω_1) and (M_2, ω_2) be closed symplectic 4-manifolds, and $N_1 \subset M_1$, $N_2 \subset M_2$ symplectic surfaces of the same genus and with $N_1^2 = N_2^2 = 0$. Suppose that N_1 and N_2 are symplectomorphic and fix a symplectomorphism $f : N_1 \to N_2$. We can understand this as having two symplectic embeddings $\iota_1 : (N, \omega_N) \to (M_1, \omega_1)$ and $\iota_2 : (N, \omega_N) \to (M_2, \omega_2)$ of the same symplectic manifold (N, ω_N).

Let ν_j be the normal bundle to $N_j \subset M_j$. As $N_1^2 = -N_2^2 = 0$, both surfaces have trivializable normal bundles ν_1 and ν_2, so there is a reversing-orientation bundle isomorphism $\psi : \nu_1 \to \nu_2$ covering f.

Now we identify the normal bundles ν_j with tubular neighbourhoods U^j of N_j in M_j as follows. By Theorem 1.14 we get symplectomorphisms $(U_j, \omega_j) \cong (N_j \times D^2_\varepsilon, p_1^*\omega_{N_j} + p_2^*\omega_0)$ for $j = 1, 2$, with $N_j \times D^2_\varepsilon$ a (trivialized) neighborhhod of the zero section of ν_j.

Let us consider the restriction $\psi : N_1 \times D^2_\varepsilon \to N_2 \times D^2_\varepsilon$ and can compose ψ with the map

$$g : N_2 \times D^2_\varepsilon \setminus (N_2 \times \{0\}) \to N_2 \times D^2_\varepsilon \setminus (N_2 \times \{0\})$$
$$(x, v) \mapsto \left(x, \sqrt{\varepsilon^2 - |v|^2}\,\frac{\overline{v}}{|v|}\right)$$

that turns each punctured disc $D^2_\varepsilon \setminus \{0\}$ inside out symplectomorphically with respect to $\omega_0 = dx \wedge dy = rdr \wedge d\theta$ the standard symplectic form in $D^2_\varepsilon \setminus \{0\}$. Here, $\overline{v}$ is the complex conjugate of $v \in \mathbb{R}^2 \cong \mathbb{C}$. Note that g in polar coordinates is

$$(r, \theta) \mapsto (\sqrt{\varepsilon^2 - r^2}, -\theta)$$

and a simple calculation shows that this map preserves $\omega_0 = rdr \wedge d\theta$.

In this way get an orientation-preserving $\psi_+ = g \circ \psi$ diffeomorphism

$$\psi_+ : U^1 \setminus N_1 \to U^2 \setminus N_2$$

covering the symplectomorphism $f : N_1 \to N_2$ and gluing the normal slices inside out, as required for the connected sum. The Gompf symplectic sum $M := M_1 \#_N M_2$ is the manifold obtained from $(M_1 \setminus N_1) \sqcup (M_2 \setminus N_2)$ by gluing with φ above, i.e. the manifold M given by

$$M = M_1 \#_N M_2 = (M_1 \setminus N_1) \bigcup_\varphi (M_2 \setminus N_2).$$

The manifold M clearly inherits naturally a symplectic form from those of M_1 and M_2. Note that the Euler characteristic of the Gompf symplectic sum is given by $\chi(M) = \chi(M_1) + \chi(M_2) - 2\chi(N)$, where $N = N_1 = N_2$.

The following lemma tells us that if we make the Gompf symplectic sum carefully we can glue together two symplectic surfaces $S_1 \subset M_1$ and $S_2 \subset M_2$ transverse to the manifolds N_1 and N_2 cut out.

LEMMA 1.38. *Suppose that $S_1 \subset M_1$ and $S_2 \subset M_2$ are symplectic surfaces intersecting transversely and positively with N_1, N_2, respectively, such that $S_1 \cdot N_1 = S_2 \cdot N_2 = d$. Then S_1, S_2 can be glued to a symplectic surface $S = S_1 \# S_2 \subset M_1 \#_{N_1 = N_2} M_2$ with self-intersection $S^2 = S_1^2 + S_2^2$ and genus $g(S) = g(S_1) + g(S_2) + d - 1$.*

PROOF. When doing the Gompf symplectic sum of M_1, M_2 along N_1, N_2, we arrange the symplectomorphism $f : N_1 \to N_2$ to take the intersection points $S_1 \cap N_1$ to the points $S_2 \cap N_2$, possible by Lemma 1.11. Note that we construct the normal bundles ν_j of N_j by using the symplectic orthogonal to T_pN_j at each $p \in N_j \cap S_j$. Hence, to glue smoothly S_1 and S_2 it suffices that S_j intersects ω_j-orthogonally with N_j for $j = 1, 2$.

If we assume this, then S_1 and S_2 glue nicely to give a symplectic surface S in the Gompf symplectic sum, and moreover S is the usual connected sum of symplectic surfaces.

Finally, note that for $j = 1, 2$ we can always arrange that the intersection of S_j and N_j becomes orthogonal after a small perturbation of S_j around the intersection points, as done in Lemma 1.20.

The claim about the self intersection and the genus follow because S is the connected sum of the surfaces S_1 and S_2. □

8. Lefschetz fibrations.

In the same way that an real valued Morse function $m : M \to \mathbb{R}$ on a manifold M induces a fibration of codimension one submanifolds $Z_t = m^{-1}(\{t\}) \subset M$, we have an analogous concept for generic $\mathbb{C}$-valued smooth functions $f : X \to \mathbb{C}$ defined on a manifold X, whose fibers $Y_w = f^{-1}(\{w\}) \subset X$ are of codimension two.

The fibrations induced by such maps f are called *Lefschetz fibrations*. These are the fibrations induced by *generic* complex valued maps, where generic means having only non-degenerate critical points. It is well known that a Morse function defined on a manifold tells information about the topology of the manifold, and the same is true for generic $\mathbb{C}$-valued maps and its corresponding Lefschetz fibrations.

Definition 1.39. *Let X be a 4-manifold. A* Lefschetz fibration *on X is a smooth map $f : X \to \Sigma$, with Σ a surface, such that:*

(1) *The critical points of π are isolated.*
(2) *For each critical point q of f there are neighborhoods $U = U^q \subset X$ of q , $V = V^{f(q)} \subset \Sigma$ of $f(q)$, and coordinate charts $(z_1, z_2) : U \to \mathbb{C}^2$, $w : V \to \mathbb{C}$ so that in these coordinates the map f has the expression $w = f(z_1, z_2) = z_1^2 + z_2^2$.*

Let us try to make a local picture of a regular fiber of a Lefschetz fibration f near a singular point. Any such regular fiber is locally $F_t = f^{-1}(t)$, with $t \in \mathbb{C}$, so it is given locally by the equation

$$F_t \cap U \cong \{z_1^2 + z_2^2 = t\} \subset \mathbb{C}^2.$$

We can assume $t > 0$, since if $t \in \mathbb{C}$ we make a change $z' = \lambda z$ choosing λ so that $t' := \lambda^2 t \in \mathbb{R}_+$, so if $z = (z_1, z_2) \in F_t$ we have $z' = (z_1', z_2') \in F_{t'}$ with $t' = \lambda^2 t \in \mathbb{R}_+$.

The intersection $F_t \cap \mathbb{R}^2 \subset \mathbb{C}^2$ is a circle $\{x_1^2 + x_2^2 = t\} \subset \mathbb{R}^2$, where $z_j = x_j + iy_j$. This circle is called the *vanishing cycle*. Each vanishing cycle bounds a disc D_ε in X defined by

$$D_\varepsilon = \{(z_1, z_2) \in F_t \cap \mathbb{R}^2 \mid t \in [0, \varepsilon]\}$$

which is called the *vanishing cycle* of the critical point.

Now we introduce the concept of symplectic Lefschetz fibrations.

Definition 1.40. *Let (X, ω) be a symplectic manifold. A* symplectic Lefschetz fibration *is a Lefschetz fibration $f : X \to \Sigma$ such that for all $t \in \Sigma$ the fiber $F_t = f^{-1}(t) \subset X$ is a symplectic submanifold of X.*

Recall that for those values of $t \in \Sigma$ so that the fiber F_t is singular, then being symplectic means that F_t minus the singular points is symplectic.

It is very useful the fact that in a symplectic Lefschetz fibration $f : X \to \Sigma$ we can give a special construction of the vanishing disks so that they are realised as Lagrangian discs. To see it, first we need to introduce a symplectic connection. The symplectic connection in X is given by taking the symplectic orthogonal to the fibers $f^{-1}(t) \subset X$ as horizontal spaces. This connection yields a corresponding parallel transport, (in the case of a symplectic connection, parallel transport acts by symplectomorphisms). Then we make a parallel transport of a fixed vanishing cycle contained in some fiber, and arrange that this parallel transport ends in a critical point of f, so that we have a disc formed with parallel transported vanishing cycles shrinking to a critical point. Let us be more precise.

Let $f : X \to \Sigma$ a symplectic Lefschetz fibration. Suppose X is compact, so there are a finite number of critical points in X. Denote $\mathcal{C} = \{p_j\}_{j=1}^{l} \subset \Sigma$, the critical values of f. Let $t \in \Sigma \setminus \mathcal{C}$ and consider the fiber $F_t = f^{-1}(\{t\})$. Since F_t is a symplectic surface in X, for any point $x \in F_t$ we have a decomposition $T_xX = T_xF_t \oplus (T_xF_t)^{\perp\omega}$. The differential $d_xf : T_xX \to T_t\Sigma$ is an isomorphism restricted to $(T_xF_t)^{\perp\omega}$, so given a vector $v \in T_t\Sigma$ there is a unique lifting $\tilde{v} \in (T_xF_t)^{\perp\omega}$ so that $d_xf(\tilde{v}) = v$.

Now, given a curve $\gamma : [0,1] \to \Sigma \setminus \mathcal{C}$ we can lift the vector field $\gamma'(s)$ over the curve γ to a vector field $\widetilde{\gamma'}$ defined in $f^{-1}(\gamma([0,1])) \subset X$. At a point $x \in f^{-1}(\gamma([0,1]))$, if we denote $f(x) = \gamma(s) \in \Sigma$ for some $s \in [0,1]$, then the vector field $\widetilde{\gamma'}|_x \in (T_xF_t)^{\perp\omega}$ is defined to be the lifting of $\gamma'(s) \in T_{\gamma(s)}\Sigma$ to X.

Consider the flow Ψ_s of the vector field $\widetilde{\gamma'}$ restricted to a fixed fiber $F_{\gamma(s_0)}$. Given a point $x \in F_{\gamma(s_0)}$, consider the curve $s \mapsto f(\Psi_s(x))$, whose derivative at time s is

$$\frac{d}{ds} f(\Psi_s(x)) = d_{\Psi_s(x)} f(\tilde{\gamma}'|_{\Psi_s(x)}) = \gamma'|_{f(\Psi_s(x))}$$

Hence it must be $f(\Psi_s(x)) = \gamma(s + s_0)$, since both have same initial data and solve the same first order ODE. In particular we see that $\Psi_s(x) \in F_{\gamma(s+s_0)}$ for any $x \in F_{\gamma(s_0)}$. We denote

$$P_\gamma(s_0, s_0 + s) : F_{\gamma(s_0)} \to F_{\gamma(s_0+s)}$$

the restriction of Ψ_s to $F_{\gamma(s_0)}$, and call the map $P_\gamma(s_0, s_0+s)$ the *parallel transport along* γ (from s_0 to $s_0 + s$) given by the symplectic connection of X. We denote

$$P_\gamma := P_\gamma(0,1) : F_{\gamma(0)} \to F_{\gamma(1)}$$

for the parallel transport along $\gamma : [0,1] \to \Sigma$.

Let us see first that parallel transport along γ moves the fibers by symplectomorphisms.

PROPOSITION 1.41. *Let (X,ω) be a symplectic manifold. Consider $f : X \to \Sigma$ be a symplectic Lefschetz fibration with critical values $\mathcal{C} \subset \Sigma$, and a curve $\gamma : [0,1] \to \Sigma \setminus \mathcal{C}$.*

Parallel transport along γ gives a symplectomorphism

$$P_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$$

PROOF. By definition P_γ is the restriction to a fiber of Ψ_s, where Ψ_s is the flow of the vector field $\widetilde{\gamma'}$ obtained as the horizontal lifting of γ'. Therefore to prove the claim it is enough to see that if V_Σ is a vector field in Σ, then the vector field V in X given as the horizontal lifting of V_Σ satisfies $\mathcal{L}_V\omega = 0$.

To see it, note that

$$\mathcal{L}_V\omega = d[\iota_V\omega] + \iota_V d\omega = d\alpha$$

with $\alpha = \iota_V\omega \in \Omega^1(X)$.

Take a local frame V_1, V_2 of vertical vector fields in a neighborhood of some point $x_0 \in X$ not lying in a singular fiber. We have that

$$d\alpha(V_1, V_2) = V_1(\alpha(V_2)) - V_2(\alpha(V_1)) - \alpha([V_1, V_2]) = 0.$$

This is because $\alpha(V_j) = \omega(V, V_j) = 0$ for $j = 1, 2$ (since V is horizontal and V_j is vertical), and because $[V_1, V_2]$ is vertical since the regular fibers are submanifolds of X so the Lie bracket of two vector fields tangent to a fiber remains tangent (i.e. the Lie bracket of two vertical vector fields remains vertical).

The conclusion is that $\mathcal{L}_V\omega$ vanishes on vertical vectors. This implies that for any non-singular fiber F the restriction $\mathcal{L}_V\omega|_F = 0$, and this gives that the flow of the vector field V induces by restriction a symplectomorphism on the fibers. The result follows. □

Consider a path $\gamma : [0,1] \to \Sigma$ with $\gamma(0) = p \in \mathcal{C}$ and $\gamma((0,1]) \subset \Sigma \setminus \mathcal{C}$. Let q be a critical point of f lying over p, and consider $U = U^q \subset X$ the neighborhood of the definition of Lefschetz fibration in which $f(z_1, z_2) = z_1^2 + z_2^2$. Fix some $s_0 \in (0, \varepsilon)$ and call $t_0 = \gamma(s_0) \in \Sigma$. We can assume $t_0 \in \mathbb{R}_+$ after a change of coordinates in Σ. The fiber F_{t_0} over t_0 satisfies $F_{t_0} \cap U = \{z_1^2 + z_2^2 = t_0\}$. Fix the vanishing cycle $C_{t_0} = \{x_1^2 + x_2^2 = t_0\} \subset F_{t_0}$.

For any $s \in (0, 1]$ consider the parallel transport

$$P_\gamma(s_0, s) : F_{\gamma(s_0)} \to F_{\gamma(s)}.$$

Now consider the following model of the vanishing disk, associated to the curve γ:

$$D_\gamma = \bigcup_{s\in(0,1]} P_\gamma(s_0, s)(C_{t_0}) \bigcup \{\gamma(0)\}. \tag{5}$$

Let us now prove that this model of the vanishing disk is Lagrangian and self-intersects negatively.

Proposition 1.42. *Let (X, ω) be a symplectic manifold and let $f : X \to \Sigma$ be a symplectic Lefschetz fibration. The vanishing disk $D_\gamma \subset X$ of (5) is an embedded smooth disc of self-intersection -1, and Lagrangian with respect to the symplectic structure of X.*

Proof. To see that D is Lagrangian, note that each point x of D has the form $x = P_\gamma(s_0, s)(z)$ for some $z \in C_{t_0}$. The tangent space T_xD is generated by two distinguished vectors. One is the tangent vector in the direction of the parallel transport, i.e. $v_1 = \frac{d}{d\sigma}|_{\sigma=s} P_\gamma(s_0, \sigma)(z)$. The other is the vector v_2 tangent to the circle $P_\gamma(s_0, \sigma)(C_{t_0}) \subset F_{\gamma(s)}$. Clearly v_1 is horizontal (since it is the horizontal lift of γ'), and v_2 is vertical. Therefore $\omega(v_1, v_2) = 0$, and this proves that D_γ is Lagrangian.

Now let us see that D has self-intersection -1. For this take the local model of the vanishing disk provided by coordinates near a singular point $(0, 0)$ of f in which $f(z_1, z_2) = z_1^2 + z_2^2$, so D is parametrized in these coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ as

$$D = \{z_1 = t\cos\theta\,,\ z_2 = t\sin\theta\,,\ t \in [0, 1], \theta \in [0, 2\pi]\} = \{y_1 = 0\,,\ y_2 = 0\}\,.$$

Now we deform D to a nearby disc D' as follows. Take a small positive number s and consider a complex number $\lambda = \alpha + i\beta$ such that

$$\lambda^2 = 1 - e^{2is} \tag{6}$$

and with imaginary part $\beta < 0$. To ease notation let us denote

$$a = \cos s\ ,\ b = \sin s$$

The equality (6) means that

$$\alpha^2 - \beta^2 = 1 - \cos 2s = 2b^2$$
$$2\alpha\beta = -\sin 2s = 2ab\ .$$

Now we consider the deformation D' of D parametrized by

$$D' = \begin{cases} z_1 = e^{is} t\cos\theta - \lambda t\sin\theta \\ z_2 = \lambda t\cos\theta + e^{is} t\sin\theta \end{cases} \qquad t \in [0, 1]\,, \theta \in [0, 2\pi]$$

A straigtforward computation gives that for any fixed t, the coordinates of D' satisfy $z_1^2 + z_2^2 = t^2$, so the disk D' is composed of circunferences of radius t contained in the fibers $\{z_1^2 + z_2^2 = t^2\}$ of the elliptic fibration. This means that D' is another vanishing disk of the elliptic fibration $E(1)$, so the deformation D' given above can be used to compute the self-intersection of D.

In real coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, D' is given by the equations

$$D' = \begin{cases} x_1 = at\cos\theta - \alpha t\sin\theta \\ y_1 = bt\cos\theta - \beta t\sin\theta \\ x_2 = \alpha t\cos\theta + at\sin\theta \\ y_2 = \beta t\cos\theta + bt\sin\theta \end{cases} \qquad t \in [0,1]\,,\ \theta \in [0,2\pi]\,.$$

We convert the above parametric equations into implicit ones and we get

$$D' = \begin{cases} -(\alpha\beta + ab)x_1 + (\alpha^2 + a^2)y_1 + (a\beta - b\alpha)x_2 = 0 \\ -(b^2 + \beta^2)x_1 + (\alpha\beta + ab)y_1 + (a\beta - b\alpha)y_2 = 0 \end{cases}$$

Note that the implicit equations of D are

$$D = \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases}$$

and this clearly shows that $D \cap D' = \{(0,0)\}$ is just the point $(0,0) \in \mathbb{C}^2$.

To see the sign of the self-intersection of D we choose the orientation of D given by the basis $\mathcal{B}$ of partial derivatives

$$[\mathcal{B}] = [(\partial_{x_1}, \partial_{x_2})]\,.$$

The map $\psi(x) = x'$ transforming $D \cong \{(x_1, x_2)\}$ into D' is expressed as

$$\psi : \begin{cases} x_1' = ax_1 - \alpha x_2 \\ y_1' = bx_1 - \beta x_2 \\ x_2' = \alpha x_1 + ax_2 \\ y_2' = \beta x_1 + bx_2\ . \end{cases}$$

This shows that the induced orientation in D' is given by

$$[\psi(\mathcal{B})] = [(a\partial_{x_1} + b\partial_{y_1} + \alpha\partial_{x_2} + \beta\partial_{y_2}\,,\ -\alpha\partial_{x_1} - \beta\partial_{y_1} + a\partial_{x_2} + b\partial_{y_2})]\,.$$

If we arrange both basis in a 4×4-matrix $P = (\mathcal{B}, \psi(\mathcal{B}))$ and compute its determinant we get $\det P = -(b^2 + \beta^2) < 0$. This proves that the intersection number of D and D' is -1, so the self intersection of D is $[D] \cdot [D] = -1$ as we wanted to see. □

9. Elliptic fibrations.

Here we summarise some properties of elliptic fibrations. This topic is discussed in [**25**].

Definition 1.43. *Let S be a complex surface.*

(1) *An* elliptic fibration *in S consists of an holomorphic map $f : S \to C$ onto some complex curve C such that for all $t \in C$ except for a finite set the preimages $f^{-1}(t)$ are smooth elliptic curves.*
(2) *A* genus-g fibration *in S consists of an holomorphic map $f : S \to C$ onto some complex curve C such that for all $t \in C$ except for a finite set the preimages $f^{-1}(t)$ are smooth curves of genus g.*

Note that an elliptic fibration is simply a genus-1 fibration, in accordance with the classical name *elliptic curves* given to algebraic genus-1 curves.

Let us see how genus-g fibrations can be constructed. Let $L \to S$ be an holomorphic line bundle over a complex surface S. Les us make some assumptions.

- Suppose $L = \mathcal{O}(D)$ is associated to a divisor $D \subset S$.
- Assume there are two linearly independent sections s_1, s_2 of L, so we have a Lefschetz pencil of sections of L of the form $\mathbb{P}^1 \cong \mathbb{P}(\mathrm{Span}(\langle s_1, s_2\rangle)) \subset \mathbb{P}(H^0(L))$.

- Suppose also that the zero sets of s_1, s_2 are transverse curves going through $Y \subset D$, with Y a finite set of points, so that $Y = s_1^{-1}(\{0\}) \cap s_2^{-1}(\{0\})$. More precisely, note that the cardinality of Y must be the self-intersection $[D]^2$.

Under these assumptions we have that for any point $q \in S \setminus Y$ there exists a unique point $[\lambda, \gamma] \in \mathbb{CP}^1$ so that $\lambda s_1(q) + \gamma s_2(q) = 0$, i.e. $[\lambda, \gamma] = [s_2(q), -s_1(q)]$. The holomorphic map $q \mapsto [\lambda, \gamma]$ defined in $S \setminus Y$ can be extended to the blow-up

$$\widetilde{S} = Bl_Y S$$

of S at Y as follows.

Recall that near a point $y \in Y$, when blowing-up we replace an affine neighborhood $U = U^y \subset S$ of y with coordinates (z_0, z_1) by the set

$$\widetilde{U}^y = \{(z_0, z_1) \times [w_0, w_1] \in \psi(U) \times \mathbb{CP}^1 : w_0 z_1 - w_1 z_0 = 0\}$$

being $\psi : U \to \psi(U) \subset \mathbb{C}^2$ an holomorphic chart near y. Let us call $W = W^Y = \cup_{j=1}^{m_1} U^{y_j}$ the neighborhood of Y removed when blowing-up, and $\widetilde{W} = \cup_{j=1}^{m_1} \widetilde{U}^{y_j}$.

We define

$$\pi : \widetilde{S} \to \mathbb{P}^1$$

by steps. First, if $q \in S \setminus W$ we set $\pi(q) = [\lambda, \gamma] = [s_2(q), -s_1(q)]$. Second, take any point $y \in Y$ and holomorphic coordinates (z_0, z_1) in a neighborhood $U = U^y$ around y so that y has coordinates $(0, 0)$. For $(z_0, z_1) \times [w_0, w_1] \in \widetilde{U}$ with $(z_0, z_1) \neq (0, 0)$ we define

$$\pi((z_0, z_1) \times [w_0, w_1]) = [s_2(z_0, z_1), -s_1(z_0, z_1)]$$

where $s_i(z_0, z_1) \in \mathbb{C}$ stands for $s_i(\psi^{-1}(z_0, z_1)) \in L$ seen in the trivialization of $L|_U \cong U \times \mathbb{C}$.

We have yet to define $\pi((0,0) \times [w_0, w_1])$. Since $0 = s_1(0,0) = s_2(0,0)$ we have that $s_1(z_0, z_1) = \alpha z_0 + \beta z_1 + O(|z|^2)$ and $s_2(z_0, z_1) = \delta z_0 + \gamma z_1 + O(|z|^2)$, for some constants $\alpha, \beta, \delta, \gamma \in \mathbb{C}$. By transversality, the vectors (α, β) and (δ, γ) are linearly independent. If we approach the point $(0,0) \times [w_0, w_1]$ by points in $\widetilde{U}$ we get points of the form $(tw_0, tw_1) \times [w_0, w_1]$ with $t \to 0$ in $\mathbb{C}$. Hence

$$\begin{aligned}\pi((tw_0, tw_1) \times [w_0, w_1]) &= [\delta t w_0 + \gamma t w_1 + O(|t|^2), -\alpha t w_0 - \beta t w_1 + O(|t|^2)] \\ &= [\delta w_0 + \gamma w_1 + O(|t|), -\alpha w_0 - \beta w_1 + O(|t|)]\end{aligned}$$

therefore

$$\lim_{t \to 0} \pi((tw_0, tw_1) \times [w_0, w_1]) = [\delta w_0 + \gamma w_1, -\alpha w_0 - \beta w_1]$$

so we define

$$\pi((0,0) \times [w_0, w_1]) = [\delta w_0 + \gamma w_1, -\alpha w_0 - \beta w_1].$$

If E is the exceptional divisor, clearly π is holomorphic in $\widetilde{U} \setminus E$, and continuous in $\widetilde{U}$, so it is holomorphic in $\widetilde{U}$. We repeat this procedure at each point $y \in Y$, so we have constructed an holomorphic map $\pi : \widetilde{S} \to \mathbb{P}^1$ (or algebraic if we work in the algebraic setting).

Moreover, let $g = g(D)$ the genus of the divisor D associated to the line bundle $L = \mathcal{O}(D)$. We claim that the map π is a genus-g fibration for generic choices of sections s_1 and s_2, and the fiber over $[\lambda, \gamma]$ is the proper transform of the zero set of the section $\lambda s_1 + \beta s_2 \in H^0(L)$. If we see this claim, then it follows that for generic $[\lambda, \gamma]$ the fiber over $[\lambda, \gamma]$ has the same genus as the proper transform of the smooth divisor D.

To see the claim we can assume that $D = s_1^{-1}(0)$, and we must see that $\widetilde{D} = \pi^{-1}([1, 0])$. Outside the set W this is obvious. In a neighborhood U^{y_j}, $y_j \in Y \subset D$, with coordinates (z_0, z_1), we can assume that D is given in coordinates as $\{z_0 = 0\}$ in U^{y_j}. The points $(z_0, z_1) \times [w_0, w_1] \in$

$\widetilde{U}^{y_j}$ which get mapped to $[1,0]$ are those such that $s_1(z_0,z_1)=0$ and $(z_0,z_1)\neq(0,0)$, together with those of the type $(0,0)\times[w_0,w_1]$ with $s_1([w_0,w_1])=0$. That is,

$$\{(0,z_1)\times[0,1]\}\cup\{(0,0)\times[0,1]\}=\{z_0=0\}\subset\widetilde{U}^{y_j}.$$

This is exactly $\widetilde{D}$ in $\widetilde{U}^{y_j}$, so we are done.

REMARK 1.44. *Let us remark a particular case of the discussion above to construct some genus-g fibrations in $\mathbb{CP}^2$ blown-up at d^2 points, $d\in\mathbb{N}$. Take two generic homogeneous degree-d polynomials $p_1([x,y,z]),p_2([x,y,z])$ such that their zero sets $p_1^{-1}(0)$ and $p_2^{-1}(0)$ intersect at d^2 different points. These polynomials can be seen as sections of the line bundle $\mathcal{O}(d)$, so the procedure discussed above applies, giving a map*

$$\widetilde{\pi}:\mathbb{CP}^2\#\,d^2\,\overline{\mathbb{CP}^1}\to\mathbb{CP}^1.$$

For generic choices of the polynomials p_1 and p_2, this is a genus-g fibration with

$$g=\frac{(d-1)(d-2)}{2}$$

by the degree-genus formula.

We will be interested in the following particular elliptic fibration, which is called $E(1)$. The elliptic fibration $E(1)$ is defined on $\mathbb{CP}^2$ blown-up at 9 points as follows. Take two generic cubics in $\mathbb{CP}^2$ given by polynomials $p_0([x,y,z])=0$, $p_1([x,y,z])=0$. These cubics intersect in 9 points $q_1,\ldots,q_9$. Consider the pencil of cubics $t_0p_0+t_1p_1$ parametrized by $[t_0,t_1]\in\mathbb{CP}^1$. For any point $q\in\mathbb{CP}^2\setminus\{q_1,\ldots,q_9\}$ there is only one cubic $t_0p_0+t_1p_1$ going through q. This defines an algebraic map

$$f:\mathbb{CP}^2\setminus\{q_1,\ldots,q_9\}\to\mathbb{CP}^1\quad,\quad f(q)=[t_0:t_1]=[-p_1(q):p_0(q)].$$

Blowing up $\mathbb{CP}^2$ at $q_1,\ldots,q_9$, we get a Kahler surface $E(1)\cong\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ and the map f can be extended to $E(1)$ as in the discussion above.

We denote the extension also by f so we have an algebraic map $f:E(1)\to\mathbb{CP}^1$, which is an elliptic fibration since by the choice of the cubics p_0 and p_1, the general fiber is a smooth cubic of the form $f^{-1}([t_0:t_1])=(t_0p_0+t_1p_1)^{-1}(0)$, hence it is an elliptic surface.

Let us summarize the properties of $E(1)$ which will be used later, from [**3**] and [**25**].

PROPOSITION 1.45. *The elliptic fibration $E(1)$ has the following properties.*

(1) $\pi_1(E(1))=\{e\}$, $\chi(E(1))=12$, $b_2(E(1))=10$.
(2) *Every exceptional sphere E_i of the blow-up at a point q_i is a section of the elliptic fibration $f:E(1)\to\mathbb{CP}^1$, hence there are 9 disjoint sections.*
(3) *If $h\in H_2(\mathbb{CP}^2,\mathbb{Z})$ is the homology class of the line $L\subset\mathbb{CP}^2$, and e_j are the homology classes of exceptional spheres E_j for $1\le j\le 9$, then $H_2(E(1),\mathbb{Z})=\langle h,e_1,\ldots,e_9\rangle$.*
(4) *Let F be a generic fiber of $f:E(1)\to\mathbb{CP}^1$. Then $\pi_1(E(1)\setminus F)=\{e\}$.*
(5) *The homology class of F is $[F]=3h-e_1-\ldots-e_9$. Hence a generic line $L\subset\mathbb{CP}^2$ intersects F transversely in 3 points.*
(6) *For a generic choice of cubics p_0 and p_1, $f:E(1)\to\mathbb{CP}^1$ is a symplectic Lefschetz fibration.*
(7) *For a generic choice of cubics p_0 and p_1, the singular fibers of f only have one simple nodal singularities. There are 12 such singular fibers.*
(8) *In a fiber F there are 12 vanishing cycles. They come in two packets of six 1-cycles homologous to a and six 1-cycles homologous to b, where $\{a,b\}$ is a basis of $H_1(F,\mathbb{Z})$.*

PROOF. Items (1) and (3) follow immediatly since $E(1)$ is diffeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. Item (2) is an immediate consequence of how the map f is extended to the blow-up $E(1)$, (see the discussion above).

Item (5) is because the fibers of f are the proper transforms of the cubics $t_0p_0 + t_1p_1$. Since these cubics are blown-up at 9 points, their proper transforms have homology class $[3h]-\sum_{j=1}^{9} e_j$.

Item (6) is because of the *holomorphic Morse lemma* which states that any holomorphic function can be expressed in a chart near a non-degenerate critical point as a sum of squares. Of course, for generic cubics p_0, p_1 we can make sure that $f : E(1) \to \mathbb{CP}^1$ will not have any degenerate critical point.

Item (7) is a standard result of algebraic geometry using dimensions, which says that pencils of two generic plane cubics do not intersect the variety of cubics having either cusps or points of multiplicity greater than 2. It can also be derived from Item (6), since near a singular point the fibers are $\{z_1^2+z_2^2 = t\}$, so the singular fiber in this chart is the fiber over 0, i.e. $(z_1+iz_2)(z_1-iz_2) = 0$, the union of two planes. This singularity is an ordinary node. See [**25**], Proposition 3.1.5 and Lemma 3.1.6.

Now, $\chi(E(1)) = 12$ and each singular point of f is a critical point of signature $(4, 0)$ of $|f|^2 : E(1) \to \mathbb{R}$, which is a Morse function. By Morse theory, this means that each critical point adds $+1$ to $\chi(E(1))$, so there must be exactly 12 such critical points. Since each critical point is contained in a unique fiber for generic cubics, we have 12 singular fibers.

Item (8) comes from the fact that the vanishing cycles can be transported from any (fixed) fiber near a singular point to all the fibers of f. This can be done using the parallel transport induced by the symplectic connection (discussed in the previous section for Lefschetz fibrations). The claim about the homology classes of vanishing the cycles can be found in [**25**], page 300. Alternatively, see [**42**]. □

10. Making lagrangian submanifolds symplectic.

In this section we give a result that allows to transform Lagrangian submanifolds into symplectic ones under suitable conditions. First we need an auxiliar result about tubular neighborhoods.

LEMMA 1.46. *Let X be a 4-manifold and let $\Sigma_1, \dots, \Sigma_k$ be embedded surfaces of X intersecting transversely, not three at the same point. Call $\Sigma_1 = \Sigma$. There exists a tubular neighborhood W of Σ and a (smooth) projection $p : W \to \Sigma$ such that for all $j = 2, \dots, k$, the projection p sends any connected component of the intersection $W \cap \Sigma_j$ to the corresponding point of intersection of Σ and Σ_j.*

PROOF. Take $q_j \in \Sigma_j \cap \Sigma$. Since Σ_j and Σ intersect transversely, there exists a chart $\psi_{q_j} : U^{q_j} \to V \subset \mathbb{R}^4$ around q_j with coordinates (z, w) so that $\Sigma = \{z = 0\}$ and $\Sigma_j = \{w = 0\}$.

Construct a Riemannian metric in a tubular neighborhood of $\Sigma \subset X$ as follows. Note that the U^{q_j} can be made disjoint by shrinking if necessary. Take a locally finite open cover $\Sigma \subset W = \cup_i U_i$, with $U_i \subset X$ small open sets diffeomorphic to balls of $\mathbb{R}^4$ via the chart $\psi_i : U_i \to \psi_i(U_i) \subset \mathbb{R}^4$. We can arrange that for all the points q_j in $\Sigma_j \cap \Sigma$ the sets U^{q_j} belong to the cover, and moreover that U^{q_j} be the only one of the U_i's containing q_j. Now define the Riemannian metric $g = \sum_i \rho_i \psi_i^*(g_0)$, with ρ_i a partition of unity subordinated to U_i.

The metric g is flat in some $V^{q_j} \subset U^{q_j}$ an smaller open neighborhood of q_j. Consider the normal bundle ν_Σ with respect to g. The exponential map $\exp : \nu_\Sigma \to W$ induces a diffeomorphism onto (a maybe smaller) W. The sought projection $p : W \to \Sigma$ is given by $p = \exp \circ p_\nu \circ \exp^{-1}$ with $p_\nu : \nu_\Sigma \to \Sigma$ the bundle projection. To see this, note that by construction of g, the fiber of ν_Σ above any point $q_j \in \Sigma \cap \Sigma_j$ is mapped by exp to $\Sigma_j \cap W$ so the projection p maps $\Sigma_j \cap W$ to the point q_j as desired. □

Now we can prove the result about getting symplectic submanifolds starting from Lagrangian ones. The following is a slight modification of Lemma 1.6 in [**24**].

LEMMA 1.47. *Let (M,ω) be a 4-dimensional compact symplectic manifold. Assume that $[F_1],\dots,[F_k]\in H_2(M,\mathbb{Z})$ are linearly independent homology classes represented by k orientable Lagrangian surfaces $F_1,\dots F_k$ which intersect transversely and not three of them intersect in a point. Then there is an arbitrarily small perturbation ω' of the symplectic form ω such that all $F_1,\dots,F_k$ become symplectic.*

If we further assume that the only Lagrangian surface intersecting F_j is F_{j+1} for all $1\le j\le k-1$ and that $F_j\cap F_{j+1}$ is a point, then the symplectic form ω' can be constructed so that the symplectic surfaces $F_1,\dots,F_k$ intersect positively with respect to ω'.

PROOF. Since $[F_1],\dots,[F_k]$ are linearly independent, there exists a cohomology class $[\eta]\in H^2_{DR}(X)$ such that $\int_{F_j}[\eta]=1$, for all $j=1,\dots,k$, with $\eta\in\Omega^2(X)$ a closed form.

Put any orientation on F_j and take symplectic (volume) forms ω_j on F_j such that $\int_{F_j}\omega_j=1$. Then $\int_{F_j}(\omega_j-\eta|_{F_j})=0$ so there are 1-forms α_j on F_j such that $\omega_j-\eta|_{F_j}=d\alpha_j$. We extend α_j to a tubular neighbourghood U_j of F_j by pulling-back via a projection $p_j:U_j\to F_j$. We can arrange this projection to project any surface F_i intersecting F_j to a point ($i\ne j$), by Lemma 1.46 proved above. Then we extend $p_j^*(\alpha_j)$ to the whole of M by multiplying by a cut-off function ρ_j which equals 0 off U^j and equals 1 in a smaller neighbourhood of F_j.

Set $\eta'=\eta+\sum_j d(\rho_j(p_j^*\alpha_j))$. Let us call $\iota_j:F_j\to X$ the inclusion, so the pull-back of any form by ι_j is its restriction to F_j, i.e. $\iota_j^*(\cdot)=(\cdot)|_{F_j}$. Clearly, $d\eta'=d\eta=0$ and moreover we have

$$\begin{aligned}\iota_j^*\eta'&=\iota_j^*\eta+\sum_k(\iota_j^*)d(\rho_k(p_k^*\alpha_k))\\&=\iota_j^*\eta+\sum_k d[\rho_k((p_k\circ\iota_j)^*\alpha_k)]\\&=\iota_j^*\eta+d[\rho_j((p_j\circ\iota_j)^*\alpha_j))]\\&=\iota_j^*\eta+d\alpha_j\\&=\omega_j\end{aligned}$$

for all j. The form

$$\omega'=\omega+\varepsilon\eta'$$

is symplectic for $\varepsilon>0$ small enough, and $\omega'|_{F_j}=\iota_j^*\omega'=\varepsilon\iota_j^*\eta_j'=\varepsilon\omega_j$, so all F_j are symplectic with respect to ω'.

Let us see that the F_j, $j=1,\dots,k$, can be made to intersect positively with respect to ω', under the additional assumption that for any $1\le j\le k-1$ and for any $l\ne j+1$, we have $F_j\cap F_l=\emptyset$, and $F_j\cap F_{j+1}=\{q_j\}$ is a unique point.

Take the point $q_j\in F_j\cap F_{j+1}$. Put orientations in F_j and F_{j+1} so that they intersect positively at q_j with respect to the initial symplectic form ω. This means that we choose basis $\{u_j,v_j\}$ of $T_{q_j}F_j$, $\{u_{j+1},v_{j+1}\}$ of $T_{q_j}F_{j+1}$ with

$$(\omega\wedge\omega)(u_j,v_j,u_{j+1},v_{j+1})>0.$$

Now we choose the sympletic forms ω_j and ω_{j+1} as before but compatible with these orientations and repeat the above procedure to get $\omega'=\omega+\varepsilon\eta'$. Then we have

$$\begin{aligned}&(\omega'\wedge\omega')(u_j,v_j,u_{j+1},v_{j+1})=(\omega\wedge\omega+\varepsilon^2\eta'\wedge\eta'+2\varepsilon\omega\wedge\eta')(u_j,v_j,u_{j+1},v_{j+1})\\&=(\omega\wedge\omega)(u_j,v_j,u_{j+1},v_{j+1})+O(\varepsilon)\end{aligned}$$

which is positive if ε is small, by the choice of orientations on F_j, F_{j+1}. □

CHAPTER 2

Differentiable orbifolds.

Definition 2.1. *An n-dimensional differentiable orbifold is a Hausdorff and second countable space X endowed with an atlas $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$, where $\{V_\alpha\}$ is an open covering of X, $U_\alpha \subset \mathbb{R}^n$, $\Gamma_\alpha < \mathrm{Diff}(U_\alpha)$ is a finite group acting by diffeomorphisms, and $\phi_\alpha : U_\alpha \to V_\alpha \subset X$ is a Γ_α-invariant map which induces a homeomorphism $U_\alpha/\Gamma_\alpha \cong V_\alpha$.*

There is a condition of compatibility of charts for intersections. For each point $x \in V_\alpha \cap V_\beta$ there is some $V_\delta \subset V_\alpha \cap V_\beta$ with $x \in V_\delta$ so that there are group monomorphisms $\rho_{\delta\alpha} : \Gamma_\delta \hookrightarrow \Gamma_\alpha$, $\rho_{\delta\beta} : \Gamma_\delta \hookrightarrow \Gamma_\beta$, and open differentiable embeddings $\imath_{\delta\alpha} : U_\delta \to U_\alpha$, $\imath_{\delta\beta} : U_\delta \to U_\beta$, which satisfy $\imath_{\delta\alpha}(\gamma(x)) = \rho_{\delta\alpha}(\gamma)(\imath_{\delta\alpha}(x))$ and $\imath_{\delta\beta}(\gamma(x)) = \rho_{\delta\beta}(\gamma)(\imath_{\delta\beta}(x))$, for all $\gamma \in \Gamma_\delta$.

The concept of topological orbifold is defined analogously but only requiring that the embeddings above be continuous open maps. In this thesis we will exclusively deal with differentiable orbifolds, hence the word orbifold will mean differentiable orbifold from now on (unless explicit mention of the contrary).

The concept of change of charts in orbifolds is borrowed from its analogue in manifolds, although it is a bit different.

Definition 2.2. *For an orbifold X, a* change of charts *is the map*

$$\psi^\delta_{\alpha\beta} = \imath_{\delta\beta} \circ \imath^{-1}_{\delta\alpha} : \imath_{\delta\alpha}(U_\delta) \to \imath_{\delta\beta}(U_\delta).$$

Note that $\imath_{\delta\alpha}(U_\delta) \subset U_\alpha$ and $\imath_{\delta\beta}(U_\delta) \subset U_\beta$, so $\psi^\delta_{\alpha\beta}$ is a change of charts from U_α to U_β. Clearly a change of charts between U_α and U_β depends on the inclusion of a third chart U_δ. This dependence is up to the action of an element in Γ_δ. In general this dependence is irrelevant, so we abuse notation and write $\psi_{\alpha\beta}$ for any change of chart between U_α and U_β.

Proposition 2.3. *Let X be an orbifold. For any point $x \in X$, we can arrange always a chart (U, V, ϕ, Γ) with $U \subset \mathbb{R}^n$, $U/\Gamma \cong V$, so that the preimage $\phi^{-1}(\{x\}) = \{u\}$ is only a point, and the group Γ acting on U leaves the point u fixed, i.e. $\gamma(u) = u$ for all $\gamma \in \Gamma$.*

Proof. Take an initial chart $(U_0, V_0, \phi_0, \Gamma_0)$. Take any preimage of x in U_0, say $u \in \phi_0^{-1}(\{x\})$. For any $\gamma_0 \in \Gamma_0$ so that $\gamma_0(u) \neq u$ there is a neighborhood U_{γ_0} of u so that $\gamma_0(u) \notin U_{\gamma_0}$. Take $U' = \bigcap_{\gamma_0} U_{\gamma_0}$, where the intersection is taken on all the $\gamma_0 \in \Gamma_0$ not fixing u. Now define

$$U = \bigcap_{\gamma_1 \in \Gamma_u} \gamma_1(U')$$

being $\Gamma_u < \Gamma_0$ the isotropy subgroup of u. Note that U is an open neighborhood of u, it holds that $\Gamma := \Gamma_u$ acts on U. This is because for $\gamma \in \Gamma$ we have $\gamma(U) = \bigcap_{\gamma_1 \in \Gamma_u} \gamma\gamma_1(U') = U$. □

Remark 2.4. *In the proof above, if we choose any other $u' \in \phi_0^{-1}(\{x\})$, then $u' = \gamma_0 u$ for some $\gamma_0 \in \Gamma_0$, and the group $\Gamma_{u'} = \gamma_0 \Gamma_u \gamma_0^{-1}$ is conjugate to Γ_u, (via a diffeomorphism $g_0 \in G$). Hence the conjugacy class of Γ_u only depends on the point $x \in X$.*

Given a point $x \in X$, take a chart (U, V, ϕ, Γ) as in Proposition 2.3. In this case, we call Γ the *isotropy group* at x, and we denote it by Γ_x. This group is well defined up to conjugation by a diffeomorphism of a small open set of $\mathbb{R}^n$ (by the previous comment).

COROLLARY 2.5. *Every orbifold atlas can be refined in such a way that all the groups Γ_α on the definition of orbifold atlas are the isotropy groups of some point.*

DEFINITION 2.6. (1) *We call $x \in X$ a* smooth *point if a neighbourhood of x is homeomorphic to a ball in $\mathbb{R}^n$, and* singular *otherwise.*
(2) *We call $x \in X$ a* regular *point if the isotropy group $\Gamma_x = \{\mathrm{Id}\}$ is trivial, and we call it an* isotropy *point if it is not regular.*

Clearly a regular point is smooth, but not conversely.

REMARK 2.7. *Indeed, it can happen that the quotient of a space by a group leaving some fixed points still preserves the property of being a topological manifold. The quotient $\mathbb{C}/\langle\xi\rangle$ with ξ a primitive m-th root of unity acting by complex multiplication provides an example of this, since the map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^m$ clearly induces a homeomorphism $\overline{f} : \mathbb{C}/\langle\xi\rangle \to \mathbb{C}$.*

DEFINITION 2.8. *An orbifold X is* smooth *if all its points $x \in X$ are smooth points*

REMARK 2.9. *For an orbifold X, the condition of being smooth is equivalent to the topological space X being a topological manifold. For instance, any orbifold X so that its set of isotropy points Σ consists of a single submanifold of codimension ≤ 2 can be shown to be smooth. This is because the normal slices to Σ are the quotient of a ball of $\mathbb{R}^2$ by a finite group, and it is known that such qoutients are topological manifolds, possibly with boundary.*

In the next proposition we obtain a local model where the isotropy groups act by linear isometries of $\mathbb{R}^n$.

PROPOSITION 2.10. *Every orbifold X has an atlas $\{(U_\alpha, V_\alpha, \phi_a, \Gamma_\alpha)\}$ where the isotropy groups $\Gamma_\alpha < \mathrm{O}(n)$.*

PROOF. Consider an initial small orbifold chart

$$\phi : U \to V \cong U/\Gamma$$

around a point $x \in X$, with Γ (the isotropy group of x) acting on $U \subset \mathbb{R}^n$ by diffeomorphisms. We can assume that the point $x = \phi(0)$ and that all elements of Γ fix 0. We consider the standard metric g_{std} on U and take $g = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma^* g_{std}$. Then g is a Riemannian metric on $U \subset \mathbb{R}^n$ and it is Γ-invariant. We consider now the exponential map for the metric g, $\exp_0 : T_0U = \mathbb{R}^n \to U$. Recall that any $\gamma \in \Gamma$ acts by isometries with respect to the metric g, so geodesics through 0 are transformed by γ into geodesics through 0. It follows that

$$\exp_0 \circ d_0\gamma(v) = \gamma \circ \exp_0(v)$$

for all $v \in \mathbb{R}^n$. Take $\varepsilon > 0$ small enough so that $\exp_0 : B_\varepsilon(0) \to U' = \exp_0(B_\varepsilon(0)) \subset U$ is a diffeomorphism. Then we have a chart $\phi' = \phi \circ \exp_0 : B_\varepsilon(0) \to V' = \phi(U')$ and the group Γ acts on $B_\varepsilon(0)$ via $\Gamma \hookrightarrow \mathrm{GL}(n)$, $\gamma \mapsto d_0\gamma$.

Moreover, since all $g \in G$ satisfies $\gamma^* g = g$ and $\gamma(0) = 0$, it follows that $d_0\gamma$ are isometries with respect to the metric g at the point 0, i.e. $g|_0$. Take an orthonormal basis of $\mathbb{R}^n$ with respect to $g|_0$, and put coodinates on $\mathbb{R}^n$ with respect to this basis, say P is the matrix of change of coordinates. Then the induced action of Γ in the new coordinates, is given by a representation $\Gamma \hookrightarrow \mathrm{O}(n)$. In other words, we have $\{P \circ d_0\gamma \circ P^{-1} : \gamma \in \Gamma\} < \mathrm{O}(n)$. □

Now let X be an orbifold and let $x \in X$, and take some orbifold chart (U, V, ϕ, Γ) near x with $\phi : U \to V$ inducing an homeomorphism $U/\Gamma \cong V \subset X$, and with Γ the isotropy group of x. For any diffeomorphism $\varphi : U \to U'$ of open sets of $\mathbb{R}^n$ we have an induced orbifold chart (U', V, ϕ', Γ') with $\Gamma' = \varphi \circ \Gamma \circ \varphi^{-1}$, and $\phi' = \phi \circ \varphi^{-1}$.

The relation $(U, V, \phi, \Gamma) \cong (U', V, \phi', \Gamma')$ iff there exists a diffeomorphism φ so that the above is satisfied is clearly an equivalence relation in the set of orbifold charts of $x \in X$. We denote $[\Gamma]$ the equivalence class of Γ.

Definition 2.11. *We say that the two orbifold charts* (U, V, ϕ, Γ) *and* (U', V, ϕ', Γ') *of* x *mentioned above are* equivalent.

Let us denote $[\Gamma]$ the set of all groups Γ' obtained as above by conjugation with some diffeomorphism of open sets of $\mathbb{R}^n$. We may write $\Gamma \cong \Gamma'$ if $G' \in [G]$, i.e. if Γ and Γ' are isotropy groups of two equivalent orbifold charts.

Remark 2.12. *Note that by Proposition 2.10 we can always find* $\Gamma' \in [\Gamma]$ *such that* $\Gamma' < \mathrm{O}(n)$. *In this vein, being able to find nice representatives* Γ' *of* $[\Gamma]$ *is in general very useful when working with orbifolds. We shall see later how geometric structures in orbifolds are related to the possibility of finding more refined representatives* Γ' *of* $[\Gamma]$.

As with manifolds, open subsets of orbifolds inherit naturally orbifold structures.

Proposition 2.13. *Let* X *be an orbifold and let* $V \subset X$ *be an open subset of* X. *Then* V *inherits a natural orbifold structure from the orbifold structure of* X.

Proof. Let $x \in V$ be a point in V. Take $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$, a chart around x in the orbifold structure of X such that $\Gamma_\alpha \subset \mathrm{O}(n)$ is the isotropy group of x and U_α is an open set of $\mathbb{R}^n$ containing 0. Since $V_\alpha \cap V \subset X$ is an open set containing the point x, there exists $\varepsilon > 0$ such that $W^x = \phi_\alpha(B_\varepsilon(0)) \subset V$, and since $\Gamma_\alpha < \mathrm{O}(n)$ we have an induced action on $B_\varepsilon(0)$.

We declare $(B_\varepsilon(0), W^x, \phi_\alpha, \Gamma_\alpha)$ to be a chart of the orbifold structure of V. Since $x \in V$ was arbitrary, the sets W^x constructed above cover V. It is easy to see that $\{(B_\varepsilon(0), W^x, \phi_\alpha, \Gamma_\alpha)\}_{x \in V}$ define an orbifold atlas of V. $\square$

Let us now give the notion of isomorphism in the orbifold category. We define the category **Orb** of differentiable orbifolds. The objects are differentiable orbifolds. Given two objects X_1, X_2, we define an *orbifold morphism*, also called a *smooth map between orbifolds*, as a map $f : X_1 \to X_2$ such that for any point $x_1 \in X_1$ there exists a chart $(U_1, V_1, \phi_1, \Gamma_1)$ of X_1 with $x_1 \in V_1$, and there exists a chart $(U_2, V_2, \phi_2, \Gamma_2)$ of X_2 with $f(x_1) \in V_2$ such that $f|_{V_1} : V_1 \to V_2$ admits a lifting $\tilde{f} : U_1 \to U_2$ with $\tilde{f}$ a smooth function between open subsets of Euclidean spaces. That $\tilde{f}$ is a lifting means that

$$f \circ \phi_1 = \phi_2 \circ \tilde{f} : U_1 \to U_2 .$$

Note that any lifting has to be (Γ_1, Γ_2)-equivariant, i.e. it has to satisfy that $\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x)$ for all $x \in U_1$, for all $\gamma \in \Gamma_1$, and for some homomorphism $\rho : \Gamma_1 \to \Gamma_2$. This is proved as follows. From the fact that $\varphi \circ \phi_1 = \phi_2 \circ \tilde{f}$ it follows that for any $x \in U_1$ and $\gamma \in \Gamma_1$, there exists $\gamma'_x \in \Gamma_2$ such that $\tilde{f}(\gamma x) = \gamma'_x \tilde{f}(x)$. Now, the fact that $\tilde{f}$ is continuous implies that the map $U_1 \to \Gamma_2$, $x \mapsto \gamma'_x$ must be constant, say some $\gamma' \in \Gamma_2$. Hence $\tilde{f}(\gamma x) = \gamma' \tilde{f}(x)$ for all $x \in U_1$. We define a map $\rho : \Gamma_1 \to \Gamma_2$ as $\rho(\gamma) = \gamma'$, with γ' the unique element of Γ_2 such that $\tilde{f}(\gamma x) = \gamma' \tilde{f}(x)$ for all $x \in U_1$. It is straightforward to chech that ρ is a homomorphism of groups.

The isomorphisms in this category are also called *orbi-diffeomorphisms* or *orbifold diffeomorphisms*. Recall that a morphism (a smooth orbifold map) $f : X_1 \to X_2$ is an isomorphism (an orbi-diffeomorphism) iff f is bijective and its inverse $f^{-1} : X_2 \to X_1$ is a morphism (a smooth orbifold map).

Now let us define the concept of suborbifold. First recall that if a group G acts on a manifold U by diffeomorphisms, we say that a submanifold $\Sigma \subset U$ is a G-invariant submanifold if G induces by restriction an action on Σ. Equivalently, if for each $g \in G$, g maps Σ into Σ, i.e. $g|_\Sigma : \Sigma \to \Sigma$.

Definition 2.14. *Let* X *be an orbifold and let* $\Sigma \subset X$ *be a subset. We say that* $\Sigma \subset X$ *is a* suborbifold *of* X *if for each* $x \in \Sigma$ *there exists a chart* $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$ *around* x *such that* $\phi_\alpha^{-1}(V_\alpha \cap \Sigma) \subset U_\alpha$ *is a* Γ_α*-invariant submanifold of* $(U_\alpha, \Gamma_\alpha)$.

Let X be an orbifold and let $\Sigma \subset X$ be a suborbifold. It follows readily from Definition 2.14 above that Σ inherits from X in a natural way an orbifold structure. The following Proposition shows that the isotropy sets of an orbifold are suborbifolds. Recall the norion of equivalence between orbifold charts discussed in Definition 2.11.

PROPOSITION 2.15. *Let X be an orbifold, and let Σ be its isotropy subset. For every finite subgroup $H < \mathrm{O}(n)$, we define the set*

$$\Sigma_H = \{x \in X : \Gamma_x \cong H\}.$$

The closure $\overline{\Sigma}_H$ is a suborbifold of X, and $\Sigma_H = \overline{\Sigma}_H \setminus \bigcup_{H<H'} \Sigma_{H'}$ is a submanifold of X.

PROOF. Recall first that Σ_H consits of the points of X whose isotropy group can be transformed into H after conjugation with a diffeomorphism of open subsets of $\mathbb{R}^n$.

Now, let $x_0 \in \Sigma$ be an isotropy point and take a local chart (U, V, ϕ, Γ) near x_0 with $\Gamma < \mathrm{O}(n)$. Let $\Gamma = \{\gamma_1 = \mathrm{Id}, \gamma_2, \ldots, \gamma_N\}$ and consider the linear subspaces $L_i = \ker(\gamma_i - \mathrm{Id}) \subset \mathbb{R}^n$, for $1 \leq i \leq N$. For every subgroup $H < \Gamma$, we define

$$L_H = \bigcap_{\gamma_i \in H} L_i \subset \mathbb{R}^n.$$

This gives a finite collection of subspaces, which are stratified, in the sense that $H' < H$ implies that $L_H \subset L_{H'}$. Given $H < \Gamma$, let

$$L_H^0 = L_H \setminus \bigcup_{H'>H} L_{H'}.$$

If L_H^0 is not empty, then a point $x \in L_H^0$ satisfies that its isotropy is exactly H. So $\Sigma_H \cap V = \phi(L_H^0 \cap U)$. Clearly $\overline{\Sigma}_H \cap V = \phi(L_H \cap U)$. Let us see that the map

$$\phi| : L_H \cap U \to \overline{\Sigma}_H \cap V$$

is an orbifold chart. Take the minimal normal subgroup $\langle H \rangle$ containing H. Then $\Gamma/\langle H \rangle$ acts on $L_H \cap U$ and $\phi|$ acts equivariantly with respect to $\Gamma/\langle H \rangle$. Moreover, $\phi|$ induces a homeomorphism in the quotient $(L_H \cap U)/\Gamma/\langle H \rangle$. This gives $\overline{\Sigma}_H$ its orbifold structure with orbifold charts given by $(U \cap L_H, V \cap \overline{\Sigma}_H, \phi|, \Gamma/\langle H \rangle)$.

Note that for any subgroup of Γ conjugate to H, say $\widehat{H} = \gamma H \gamma^{-1}$ for some $\gamma \in \Gamma$, it happens that

$$L_{\widehat{H}} = \gamma L_H$$
$$\phi(L_H \cap U) = \phi(L_{\widehat{H}} \cap U) = V \cap \overline{\Sigma}_H.$$

This gives another orbifold chart for $\overline{\Sigma}_H$ given by $(U \cap L_{\widehat{H}}, V \cap \overline{\Sigma}_H, \phi|, \Gamma/\langle \widehat{H} \rangle)$. Note that $\langle \widehat{H} \rangle = \langle H \rangle$ since both are conjugate subgroups, and note that the orbifold chart obtained using $\widehat{H}$ is clearly equivalent to the one we obtained using H. □

Let us now define the notion of smooth function in orbifolds. In an analogous manner to manifolds, a function is smooth iff its local expressions in orbifold charts are smooth between open Euclidean sets.

DEFINITION 2.16. *Let X be an orbifold and let $V \subset X$ be an open set. An* orbifold smooth function *on V consists of $f : V \to \mathbb{R}$ a continuous function such that $f \circ \phi_\alpha : U_\alpha \cap \phi_\alpha^{-1}(V) \to \mathbb{R}$ is smooth for every α, being $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$ the orbifold charts in the orbifold atlas of X.*

Note that this is equivalent to giving smooth functions f_α on U_α which are Γ_α-equivariant and which agree under the changes of charts. Let us denote $\mathcal{C}^\infty_{orb}(V)$ for the set of orbifold (differentiable) functions on V. It is easy to see that $\mathcal{C}^\infty_{orb}(V) \subset \mathcal{C}^0(V)$, with $\mathcal{C}^0(V)$ the continuous functions. As usual, the concept of orbifold smooth function gives raise to a sheaf. We shall dwell on this a bit later.

DEFINITION 2.17. *An orbifold partition of unity subordinated to the open cover* $\{V_\alpha\}$ *of* X *consists of orbifold smooth functions* $\rho_\alpha : X \to [0,1]$ *such that the support of* ρ_α *lies inside* V_α *and the sum* $\sum_\alpha \rho_\alpha \equiv 1$ *on* X.

Let X be a topological space. Recall the following:

(1) An open cover $X = \bigcup_\alpha U_\alpha$ is *locally finite* if each $x \in X$ has a neighbourhood $V = V^x$ such that V intersects only a finite number of the sets U_α.
(2) A *refinement* of an open cover $X = \bigcup_\alpha U_\alpha$ is another cover $X = \bigcup_\beta W_\beta$ such that for each W_β there exists some U_α such that $W_\beta \subset U_\alpha$.

Recall also the definition of a *paracompact* space X. This means that for each open cover $X = \bigcup_\alpha U_\alpha$ there exists a refinement $X = \bigcup_\beta W_\beta$ such that W_β is a locally finite open cover of X. We have the following folklore result:

THEOREM 2.18. *If a topological space* X *is locally compact, second countable and Hausdorff then it is paracompact.*

The conclusion is that as orbifolds are locally compact, they are also paracompact. This allows to construct orbifolds partitions of unity as we do below in Proposition 2.20.

REMARK 2.19. *There are two possible definitions of an orbifold* X*, imposing two different topological requirements:*

(1) *That the topological space* X *be Hausdorff and second countable.*
(2) *That* X *be a paracompact space.*

Requirement (1) implies requirement (2) for locally compact spaces, so it a a stronger requirement to impose in the definition of orbifold.

We now construct orbifold partitions of unity.

PROPOSITION 2.20. *Let* X *be an* n*-orbifold. For any open cover* $\{V_\alpha\}$ *of* X *there exists an orbifold partition of unity subordinated to* $\{V_\alpha\}$.

PROOF. Suppose that the open cover $\{V_\alpha\}$ of X has already been refined so that it is formed by coordinate patches $V_\alpha \cong B_{\varepsilon_\alpha}(0)/\Gamma_\alpha$ with $\Gamma_\alpha < \mathrm{O}(n)$ and so that $V'_\alpha \cong B_{\delta_\alpha}(0)/\Gamma_\alpha$ is also an open cover of X, with $\delta_\alpha < \varepsilon_\alpha$ for all α. We can assume moreover that both V_α and V'_α are locally finite open covers since X is paracompact. Take $\tilde{f}_\alpha : \mathbb{R}^n \to \mathbb{R}$ be a radial bump function so that $\tilde{f}_\alpha \equiv 0$ on $B_{\varepsilon_\alpha}(0) \setminus B_{\delta_\alpha}(0)$ and $\tilde{f} \equiv 1$ on $B_{\delta_\alpha}(0)$. Since $\tilde{f}_\alpha$ is a radial function and $\Gamma_\alpha < \mathrm{O}(n)$, it descends to the quotient and gives a continuous function $f_\alpha : V_\alpha \to \mathbb{R}$ which can be extended by zero to all X so we write $f_\alpha : X \to \mathbb{R}$. The sum

$$S(x) = \sum_\alpha f_\alpha(x) > 0$$

is a positive function at all points x of X because the sets V'_α form a cover of X. We define

$$\rho_\alpha = \frac{f_\alpha}{S} : X \to \mathbb{R}$$

and thus $\sum_\alpha \rho_\alpha \equiv 1$ on X, so this is the desired partition of unity. □

Orbifold partitions of unity are central in the study of (smooth) orbifolds for the same reasons as they are for (smooth) manifolds. In many cases they allow to construct orbi-tensors locally and then gluing the local constructions together to form a globally defined orbi-tensor. We will see several instances of this in the future.

1. Orbifold cohomology.

There are various ways of defining cohomologies associated with orbifolds. Some, like orbifold cohomology in the grupoid category, or in the stack category, are quite intricated and not very useful to our purposes, so we shall not go into that. We refer the reader interested in a treatment of orbifolds as groupoids to [**1**].

For our purposes, we will be interested in emulatating the theory of De Rham cohomology of smooth manifolds in the orbifold category. As in the manifold case, De Rham cohomology groups of orbifolds will turn out to give isomorphic groups to the groups given by the singular cohomology of X as a topological space. Nevertheless, the perspective of differential forms in cohomology will be very useful when dealing with orbifold tensors.

Let $k \in \mathbb{N}$. Our aim here is to define the sheaf of k-forms on an orbifold X. First let us note that if $V \subset X$ is an open set, by Proposition 2.13, V inherits an orbifold structure from X. Hence in order to define the sheaf of orbifold k-forms it will be enough to define k-forms in a general orbifold, which we will call X as usual.

Definition 2.21. *An orbifold smooth form η of degree k in X is given by smooth k-forms η_α defined in the orbifolds charts $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$ which moreover satisfy:*

(1) *They are Γ_α-invariant, i.e. $f^*\eta_\alpha = \eta_\alpha$ for all $f \in \Gamma_\alpha$.*
(2) *They match under every the orbifold change of charts.*

We may use the term *orbifold k-form* or *orbi k-form*, and denote the set of orbifold k-forms as $\Omega^k_{orb}(X)$.

Remark 2.22. *Let us be more explicit as to what we mean by the second item in Definition 2.21, i.e. the meaning of matching under change of charts for an orbifold k-form. Let $\iota_{\gamma\alpha} : U_\gamma \to U_\alpha$ and $\iota_{\gamma\beta} : U_\gamma \to U_\beta$ be two inclusions of an open set $V_\gamma \subset V_\alpha \cap V_\beta$ as in Definition 2.1. The condition that η_α and η_β match means that for every $V_\gamma \subset V_\alpha \cap V_\beta$ we have*

$$\iota^*_{\gamma\alpha}\eta_\alpha = \eta_\gamma = \iota^*_{\gamma\beta}\eta_\beta$$

The above requirements are not specific to orbifold k-forms. In fact, all covariant orbitensors are required to behave like this under change of charts.

The usual constructions with k-forms work analogously for orbifold k-forms. Let us sketch some instances of this.

(1) The exterior derivative. If $\eta \in \Omega^k_{orb}(X)$, take η_α its representative in U_α, and define $d\eta$ as the orbifold form such that its representative in U_α is by definition $d\eta_\alpha$. Since the exterior differential commutes with pull-backs, we see that the forms $d\eta_\alpha$ are Γ_α-invariant and match under any change of charts, hence we can define a new orbifold form $d\eta \in \Omega^{k+1}_{orb}(X)$ by the local representatives $d\eta_\alpha$. Therefore we have
$$d = d_k : \Omega^k_{orb}(X) \to \Omega^{k+1}_{orb}(X)$$
a well defined operator. It is straightforward that $d \circ d = 0$, since this can be checked locally at charts.
(2) The wedge product. Given $\eta \in \Omega^k_{orb}(X)$ and $\nu \in \Omega^l_{orb}(X)$ we can define $\eta \wedge \nu \in \Omega^{k+l}_{orb}(X)$ as the orbi-form whose local representatives in the orbi-charts U_α are given by $\eta_\alpha \wedge \nu_\alpha$, with η_α and ν_α the local representatives of η and ν respectively. Since the pull-back of the wedge equals the wedge of the pull-backs, we see that $\eta_\alpha \wedge \nu_\alpha$ are both Γ_α-invariant and match under any change of charts.

The conclusion is that we can define a cochain complex $(\Omega^*_{orb}(X), d)$ of differential graded algebras.

Definition 2.23. *Let X be an orbifold. The k-th* orbifold De Rham cohomology group *of X is defined as*

$$H^k_{orb}(X) = \frac{\ker d_k}{\operatorname{im} d_{k-1}} \tag{7}$$

Let $\eta \in \ker d_k \subset \Omega^k_{orb}(X)$ be a closed orbifold k-form. As usual, we will denote $[\eta] \in H^k_{orb}(X)$ its cohomology class. Recall that the sets $H^k_{orb}(X)$, called groups by tradition, have actually more structure. They are differential graded algebras, just as the De Rham cohomology groups for smooth manifolds.

Now we prove the *Poincaré Lemma* for orbi-forms, saying the closed orbi-forms in simply connected charts have a primitive, i.e. they are exact.

Proposition 2.24. *Let X be an orbifold, and $V \subset X$ open such that there is a chart (U, ϕ, V, Γ) with $V \cong U/\Gamma$ and $U \subset \mathbb{R}^n$ simply connected.*

Suppose that $\beta \in \Omega^{k+1}_{orb}(V)$ satisfies $d\beta = 0$ for $k \geq 0$. Then there exists $\eta \in \Omega^k_{orb}(V)$ such that $d\eta = \beta$.

Proof. The form $\beta \in \Omega^{k+1}_{orb}(V)$ has a Γ-invariant representative in $U \subset \mathbb{R}^n$. We call again $\beta \in \Omega^{k+1}(U)$ this representative, which satisfies $d\beta = 0$ on $U \subset \mathbb{R}^n$ and $\gamma^*\beta = \beta$ for all $\gamma \in \Gamma$. By the Poincaré Lemma (of open subsets of $\mathbb{R}^n$) there exists $\eta' \in \Omega^k(U)$ with $d\eta' = \beta$. The form η' might not be Γ-invariant, by we can average it to get

$$\eta = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* \eta'.$$

This form η satisfies $d\eta = d\eta' = \beta$ and it is clearly Γ-invariant, so η is the representative of an orbiform $\eta \in \Omega^k_{orb}(V)$, giving the result. □

Orbi-forms can also be adressed from a sheaf-theoretic perspective. From the definition of orbi-forms and the existence of orbifold partitions of unity, two facts are immediate:

- The orbifold p-forms define a sheaf Ω^p_{orb} on X. When $p = 0$ we have $\Omega^0_{orb} = \mathcal{C}^\infty_{orb}$ is the sheaf of orbifold (differentiable) functions.
- For any integer $p \geq 0$, the sheaf of orbifold p-forms Ω^p_{orb} is a fine sheaf on X. In particular, the cohomology of X with respect to the sheaf Ω^p_{orb} vanishes.

Let us now prove the equivalence between all cohomologies for orbifolds. Recall that we can define the following three cohomologies in X:

- Singular cohomology of X as a topological space, denoted $H^*(X, \mathbb{R})$.
- Sheaf cohomology of X using the locally constant real-valued sheaf $\underline{\mathbb{R}}$, denoted $H^*(X, \underline{\mathbb{R}})$.
- De Rham cohomology of X using the cochain complex $(\Omega^*(X), d)$, denoted $H^*_{orb}(X)$.

These cohomologies give isomorphic groups:

Proposition 2.25. *Let X be an orbifold. The three cohomologies above yield isomorphisc groups, i.e.*

$$H^*_{orb}(X) \cong H^*(X, \mathbb{R}) \cong H^*(X, \underline{\mathbb{R}}). \tag{8}$$

Proof. By the Poincaré Lemma for orbi-forms, we have a resolution of the sheaf $\underline{\mathbb{R}}$

$$0 \to \underline{\mathbb{R}} \to \Omega^0_{orb} \to \Omega^1_{orb} \to \dots$$

given by the fine sheaf of orbi-forms Ω^*_{orb} and the exterior differential d. Taking global sections this gives $H^*(X, \underline{\mathbb{R}}) \cong H^*_{orb}(X)$.

On the other hand, it is a standard result that for any paracompact and locally contractible space, the Cech and singular cohomology coincide (using a good cover for the Cech cohomology). Also, Cech cohomology with real coefficients (with respect to a good cover) is isomorphic to sheaf cohomology with respect to the sheaf $\underline{\mathbb{R}}$. Any orbifold is clearly paracompact, locally contractible, and admits good covers, so we have an isomorphism $H^*(X,\underline{\mathbb{R}}) \cong H^*(X,\mathbb{R})$. □

2. Tensors on orbifolds.

Let X be an orbifold. Let us define what we understand by tangent space of X at a point $x \in X$.

Take a local chart (U, V, ϕ, Γ) with $\Gamma = \Gamma_x$ the isotropy group of x, and $\phi : U \to V$ inducing a homeomorphism $\tilde{\phi} : U/\Gamma \to V$. Clearly, ϕ maps two points of U to the same image if and only if they belong to the same orbit of the Γ-action. In particular, the point $x \in X$, being Γ its isotropy group, has only one preimage in U. Let us call $a \in U$ such preimage, i.e. $a \in U$ is the unique point such that $\phi(a) = x$. Note that in the tanget space T_aU there is a natural action of $\Gamma_x < \mathrm{GL}(T_xU)$ induced by $d_x\gamma$, for $\gamma \in \Gamma$ acting on U.

Definition 2.26. *With the above hypothesis and notations, we define the* orbifold tangent space *of X at the point x as $T_xX = T_aU$, equipped with the action of the group $d_x\Gamma = \{d_x\gamma : \gamma \in \Gamma\}$*

Let us denote $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$ the atlas of the orbifold X. In general, an orbifold tensor on X is a collection of tensors T_α on each U_α which are Γ_α-invariant, and which match under the orbifold changes of charts. We already gave an explicit description of the meaning of *matching* for a covariant orbi-tensor in Remark 2.22. In particular, we have already described in definition 2.21 the set of orbifold differential forms $\Omega^p_{orb}(X)$. In an analogous manner we can define orbifold Riemannian metrics g, orbi-hermitian metrics h, orbifold almost complex structures J, and so on.

We can also define orbi-tensors constructed with derivatives of other tensors, e.g. the Nijenhuis orbi-tensor, the curvature orbi-tensor, etc. Also, in the same vein as we defined the exterior differential for orbi-forms, we can define covariant derivatives, Lie bracket, and other kinds of differential operators involving orbi-tensors.

Proposition 2.27. *Let X be an orbifold. There exists an orbifold Riemannian metric g on X.*

Proof. Let us consider a locally finite atlas $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$ where the isotropy groups $\Gamma_\alpha \subset \mathrm{O}(n)$, whose existence is proved in Proposition 2.10. Consider the standard metric g_α on U_α which is in particular Γ_α-invariant. Take a differentiable orbi-partition of unity ρ_α subordinated to $\{V_\alpha\}$, given by Proposition 2.20. Define $g = \sum_\alpha \rho_\alpha g_\alpha$. This is an orbifold tensor on X, since g_α are orbifold tensors and ρ_α orbifold functions. It is an orbifold Riemannian metric by the usual convexity argument. □

Definition 2.28. *An orbifold X is orientable if all Γ_α act by orientation preserving diffeomorphisms and all embeddings $\imath_{\delta\alpha}$ in Definition 2.1 preserve orientation.*

In this case we have an atlas of X with all $\Gamma_\alpha < \mathrm{SO}(n)$ and all changes of charts preserving orientation. This is equivalent to the existence of a globally non-zero orbi-form of maximal degree $n = \dim X$.

Proposition 2.29. *An n-orbifold X is orientable if and only if there exists a never vanishing n-form $\nu \in \Omega^n_{orb}(X)$.*

This gives raise to the following definition.

Definition 2.30. *Let X be an orbifold of dimension n. A never vanishing n-form is called a* volume orbi-form

Hence Proposition 2.29 above can be rephrased saying that an orbifold X is orientable if and only if it admits a volume orbi-form.

3. Symplectic and Kahler orbifolds.

This kind of orbifolds will be our main objects of interest.

Definition 2.31. *A* symplectic orbifold (X, ω) *is an orbifold X equipped with an orbifold 2-form $\omega \in \Omega^2_{orb}(X)$ such that $d\omega = 0$ and $\omega^n > 0$, where $2n = \dim X$.*

In particular, it is oriented by Proposition 2.29.

Definition 2.32. *An* almost Kahler orbifold (X, J, ω) *consists of an orbifold X, an orbifold almost complex structure J and an orbifold symplectic form ω such that $g(u, v) = \omega(u, Jv)$ defines an orbifold Riemannian metric with $g(Ju, Jv) = g(u, v)$.*

Definition 2.33. *A* Kahler orbifold *is an almost Kahler orbifold (X, J, ω) satisfying the integrability condition that the Nijenhuis tensor $N_J = 0$.*

Using the Newlander-Nirenberg Theorem, it is immediate that the Nijenhuis tensor vanishing is equivalent to being able to refine the orbifold atlas in such a way that the orbifold changes of charts are biholomorphisms of open sets of $\mathbb{C}^n$, with $2n = \dim X$, hence giving the orbifold X the structure of a *complex orbifold*.

Analogously as for manifolds, every symplectic orbifold is almost Kahler:

Proposition 2.34. *Let (X, ω) be a symplectic orbifold. Then (X, ω) admits an almost Kahler orbifold structure (X, ω, J, g).*

Proof. Consider an auxiliary orbifold Riemannian metric g_0 on X. We define the orbifold endomorphism $A \in \mathrm{End}(TX)$ by the requirement $g_0(u, Av) = \omega(u, v)$. The adjoint of A with respect to g is the orbifold endomorphism $A^* \in \mathrm{End}(TX)$ such that $g_0(u, A^*v) = g_0(Au, v)$. We have that $A^* = -A$ since

$$g_0(u, A^*v) = g_0(Au, v) = g_0(v, Au) = \omega(v, u) = -\omega(u, v) = -g_0(u, Av) = g_0(u, -Av).$$

The orbifold endomorphism $B = AA^* = -A^2$ is symmetric and positive. Indeed $g_0(u, Bu) = g_0(A^*u, A^*u) > 0$ for $u \neq 0$, and $g_0(u, Bv) = g_0(A^*u, A^*v) = g_0(A^*v, A^*u) = g_0(v, Bu)$.

Let us see that B admits a square root $\sqrt{B} \in \mathrm{End}(TX)$, which is an orbifold endomorphism. On every chart $\phi : U \to V = U/\Gamma$, the endomorphism B is given by a matrix valued function $B(x)$ on U which is Γ-equivariant. At every $x \in U$, it has positive eigenvalues and diagonalises, so we can define $\sqrt{B}$ locally as the matrix which has the same eigenvectors as B with eigenvalues the (positive) square root of the eigenvalues of B. We have to see that $\sqrt{B}$ is Γ-equivariant. We take a real constant $\mu > 0$ so that $\|\mu B - \mathrm{Id}\| < 1$, in some operator norm, so we have

$$\sqrt{\mu}\sqrt{B} = \sqrt{\mu B} = \mathrm{Id} + \frac{1}{2}\mu B - \frac{1}{8}\mu^2 B^2 + \frac{1}{16}\mu^3 B^3 + \dots$$

by the usual power series expansion of the square root. This yields the formula

$$\sqrt{B} = \frac{1}{\sqrt{\mu}}(\mathrm{Id} + \frac{1}{2}\mu B + \dots)$$

As Γ commutes with B, we have that it also commutes with $\sqrt{B}$.

Now define $J = -(\sqrt{B})^{-1}A$, which is an orbifold endomorphism. As $\sqrt{B} = \sqrt{-A^2}$ commutes with A by the power series expansion, its inverse $(\sqrt{B})^{-1}$ also commutes with A, and hence J

commutes with both $\sqrt{B}$ and A. Also $J^2 = B^{-1}A^2 = (-A^2)^{-1}A^2 = -\operatorname{Id}$, so J is an orbifold almost complex structure. As $J^* = A^*\sqrt{B^*} = -A\sqrt{B} = -J$, we have that $g(u,v) = \omega(u, Jv)$ is a symmetric bilinear orbifold tensor. Indeed we have

$$\begin{aligned} g(u,v) &= \omega(u,Jv) = -\omega(Jv,u) = -g_0(Jv,Au) = -g_0(v,J^*Au) \\ &= g_0(v,JAu) = g_0(v,AJu) = \omega(v,Ju) = g(v,u) \end{aligned}$$

Moreover

$$g(u,v) = \omega(u,Jv) = g_0(u,AJv) = g_0(u,(\sqrt{AA^*})^{-1}AA^*v) = g_0(u,\sqrt{AA^*}v)$$

which implies that g is positive definite, and hence an orbifold Riemannian metric. Finally, J is compatible with ω since $\omega(Ju,Jv) = g(Ju,AJv) = g(J^*Ju,Av) = g(u,Av) = \omega(u,v)$. So (X,ω,g,J) is an almost Kahler orbifold. □

A consequence of the above is that symplectic orbifolds are locally quotients of open sets of $\mathbb{C}^n$ by unitary subgroups.

COROLLARY 2.35. *Let (X,ω) be a symplectic $2n$-orbifold. Around any point $x \in X$ there exists an orbifold chart (U,V,ϕ,Γ) with $0 \in U$, $\phi(0) = x$ and $\Gamma < \mathrm{U}(n)$.*

PROOF. Put any almost Kahler structure (ω, J, g) on X as provided by Proposition 2.34. Fix a chart (U,V,ϕ,Γ) near a point $x \in X$ with $\Gamma < \mathrm{GL}(2n,\mathbb{R})$ the isotropy group of x, $U \subset \mathbb{R}^{2n}$ a neighborhood of 0 and $\phi(0) = x$. Denote $h = g - i\omega$ the orbi-hermitian metric induced by the almost Kahler structure. Consider h in the chart U, and let $h|_0 = g|_0 - i\omega|_0$ be the evaluation at $0 \in U$ of h. Taking coordinates on $\mathbb{C}^n$ with respect to a unitary basis of $h|_0$ we can assume that $h|_0$ is the standard hermitian metric $h_0 = g_0 - i\omega_0$ of $\mathbb{C}^n$, so $J|_0 = J_0$ is standard complex structure of $\mathbb{C}^n$ and $g|_0 = g_0$ is the standard metric of $\mathbb{R}^{2n}$.

As J is an orbifold almost complex structure we have $\gamma_* \circ J = J \circ \gamma_*$ for all $\gamma \in \Gamma$. Evaluating at 0 we get $d_0\gamma \circ J_0 = J_0 \circ d_0\gamma$ for all $\gamma \in \Gamma$. As γ is linear, we have that $d_0\gamma = \gamma$, hence γ preserves the complex standard structure of $\mathbb{C}^n = (\mathbb{R}^{2n}, J_0)$. This means that $\Gamma < \mathrm{GL}(n,\mathbb{C})$.

In the same vein, as g is an orbi-metric we have $\gamma^* \circ g = g$ for all $\gamma \in \Gamma$. In particular, evaluating at 0 we get $d_0\gamma^* \circ g_0 = g_0$ for all $\gamma \in \Gamma$. Since $d_0\gamma = \gamma$, γ preserves the standard metric g_0 of $\mathbb{R}^{2n}$, so $\Gamma < \mathrm{O}(2n)$.

The conclusion is that $\Gamma < \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$. □

Another consequence is that the structure of the isotropy set given in Proposition 2.15 can be improved for symplectic orbifolds.

COROLLARY 2.36. *Notations as in Proposition 2.15. Let (X,ω) be a symplectic $2n$-orbifold. The isotropy set Σ of (X,ω) consists of inmersed symplectic suborbifolds $\overline{\Sigma}_H$. Moreover, if we endow X with an almost Kahler orbifold structure (ω, J, g), then the sets $\overline{\Sigma}_H$ are almost Kahler suborbifolds.*

PROOF. Put any almost Kahler structure (ω, J, g) on X as provided by Proposition 2.34. Take a point $x \in \Sigma$. We can arrange a chart (U,V,ϕ,Γ) so that $0 \in U$, $\phi(0) = x$, and $(\omega|_0, g|_0, J|_0) = (\omega_0, g_0, J_0)$ is the standard almost-Kahler structure of $\mathbb{R}^{2n}$. As proved in Corollary 2.35, this implies that $\Gamma < \mathrm{U}(n)$.

As proved in Proposition 2.15, the isotropy set $\Sigma \cap V$ is the union of $\overline{\Sigma}_H \cap V = \phi(U \cap L_H)$, for some subgroups $H < \Gamma$. Hence $x \in \overline{\Sigma}_H$ for some subgroup H. We have $L_H = \bigcap_{\gamma \in H} L_\gamma$. Since $\gamma : U \to U$ are induced by complex endomorphisms, the fixed points of γ are $L_\gamma = \ker(\gamma - \operatorname{Id})$ so L_H is a complex linear subspace of $\mathbb{C}^n$. This proves that J_0 leaves invariant $T_0\overline{\Sigma}_H = L_H$, which means that $J|_x$ leaves invariant the (orbifold) tangent space $T_x\overline{\Sigma}_H$ of $\overline{\Sigma}_H$ at x. By compatibility of $\omega|_x$ and $J|_x$ this automatically yields that $\omega|_x$ is non-degenerate in $T_x\overline{\Sigma}_H$.

This happens at every point $x \in \overline{\Sigma}_H$, hence $\overline{\Sigma}_H$ is an almost Kahler orbifold. In particular, it is a symplectic suborbifold of (X, ω). $\square$

The following result is a Darboux theorem adapted for symplectic orbifolds. Note that in the orbifold version of these local Theorems we can also keep track of the isotropy group Γ so as to get a representative of the conjugacy class $[\Gamma]$ as nice as possible in the sense of Remark 2.12. This may make these local results trickier in the orbifold case than the corresponding Theorems for manifolds.

PROPOSITION 2.37. *Let (X, ω) be a symplectic orbifold and $x_0 \in X$. There exists an orbifold chart (U, V, ϕ, Γ) around x_0 with local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ such that the symplectic form has the expression $\omega = \sum dx_i \wedge dy_i$ and $\Gamma < \mathrm{U}(n)$ is a subgroup of the unitary group.*

PROOF. Take an initial orbifold chart (U, V, ψ, Γ) with $\Gamma < \mathrm{U}(n)$, $0 \in U$, $\phi(0) = x$ and so that the evaluation $\omega|_0$ is the standard symplectic form ω_0, i.e. $\omega|_0 = \sum dx_i \wedge dy_i = \omega_0$. Since U is contractible we have that $\omega - \omega_0 = d\mu$, for some $\mu \in \Omega^1(U)$. We can assume that μ is Γ-invariant, since otherwise we put $\tilde{\mu} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* \mu$ and $\tilde{\mu}$ also satisfies

$$d\tilde{\mu} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* d\mu = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* (\omega - \omega_0) = \omega - \omega_0 \, .$$

We can further suppose that $\mu|_0 = 0$ (i.e. it vanishes as a 1-form at the point $0 \in U$), since otherwise we put $\tilde{\mu} = \mu - \mu|_0$ which also satisfies $d\tilde{\mu} = \omega - \omega_0$ and $\tilde{\mu}$ is Γ-equivariant.

Now we apply Moser's trick. Consider $\omega_t = t\omega + (1-t)\omega_0 = \omega_0 + t\, d\mu$. Since $\omega_t|_0 = \omega_0$ is non-degenerate evaluated at 0, by shrinking the chart we can assume that ω_t is a symplectic form for all $t \in [0, 1]$.

Since ω_t is non-degenerate, there exists a unique a vector field X_t such that $\iota_{X_t}\omega_t = -\mu$. Recall that $X_t|_0 = 0$ because $\mu|_0 = 0$. Let us call φ_t the flow of the vector field X_t at time t which satisfies $\frac{d}{dt}\varphi_t(x) = X_t|_{\varphi_t(x)}$ for each $x \in U$. Recall that as $X_t|_0 = 0$ for all $t \in [0, 1]$, all the maps φ_t fix the point 0, hence we can assume they are defined in the same small neighborhood of $0 \in U$.

Then for each $s \in [0, 1]$ we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=s} \varphi_t^* \omega_t &= \frac{d}{dt}\Big|_{t=s} \varphi_t^* \omega_s + \varphi_s^* \left(\frac{d}{dt}\Big|_{t=s} \omega_t \right) = \varphi_s^* (\mathcal{L}_{X_s} \omega_s) + \varphi_s^* (d\mu) \\ &= \varphi_s^* \left(d(\iota_{X_s} \omega_s) + \iota_{X_s} d\omega_s \right) + \varphi_s^* (d\mu) = -\varphi_s^* (d\mu) + \varphi_s^* (d\mu) = 0, \end{aligned}$$

using Cartan formula for the Lie derivative $\mathcal{L}_X = d\iota_X + \iota_X d$. This implies that $\omega_0 = \varphi_0^* \omega_0 = \varphi_1^* \omega_1 = \varphi_1^* \omega$. Consider the diffeomorphism $\varphi := \varphi_1$ which is defined in some neighborhood of $0 \in U$.

As said before, we have $\varphi_t(0) = 0$ for all t, so $\varphi(0) = 0$. Finally, as μ and ω_t are Γ-equivariant, and $\iota_{X_t}\omega_t = -\mu$, we have that the vector fields X_t are Γ-equivariant, i.e. $\gamma_* X_t = X_t \circ \gamma$ since for a point $p \in U$ and a tangent vector $V \in \mathbb{R}^{2n} = T_pU$ we have

$$\begin{aligned} \omega_t|_{\gamma(p)}(d_p\gamma(X_t|_p), d_p\gamma(V)) &= \omega_t|_p(X_t|_p, V) \\ &= -\mu|_p(V) = -\mu_{\gamma(p)}(d_p\gamma(V)) = \omega_t|_{\gamma(p)}(X_t|_{\gamma(p)}, d_p\gamma(V)) \end{aligned}$$

which yields $d_p\gamma(X_t|_p) = X_t|_{\gamma(p)}$.

This implies that the flow φ_t are Γ-equivariant diffeomorphisms. Indeed, $\gamma^{-1} \circ \varphi_t \circ \gamma$ is the flow of a vector field which evaluated at a point $p \in U$ is $d_p\gamma^{-1}(X_t|_{\gamma(p)}) = X_t|_p$. Therefore $\gamma^{-1} \circ \varphi_t \circ \gamma = \varphi_t$, and so $\varphi = \varphi_1$ is Γ-equivariant.

Summarising, we have a diffeomorphism $\varphi : U \to U'$ between two neighborhoods of 0 and $\varphi^* \omega = \omega_0$ is a constant symplectic form on U'. Moreover, since $\varphi \circ \gamma \circ \varphi^{-1} = \gamma$ for all $\gamma \in \Gamma$, the Γ-action induced by φ on U' is the same as on U. The sought orbifold chart is $(U', V, \varphi \circ \psi, \Gamma)$. $\square$

Corollary 2.38. *Let (X,ω) be a symplectic orbifold. Then (X,ω) admits a* Darboux orbifold atlas*, i.e. an atlas $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$ where all the isotropy groups $\Gamma_\alpha < \mathrm{U}(n)$ and the expression in coordinates of ω on each $U_\alpha \subset \mathbb{R}^{2n}$ is the canonical symplectic form of $\mathbb{R}^{2n}$, i.e.*

$$\omega|_{U_\alpha} = \sum dx_j \wedge dy_j = -\frac{i}{2}\sum dz_j \wedge d\overline{z_j}.$$

Moreover, if $\overline{\Sigma}_H \subset X$ is an isotropy suborbifold of codimension $2k$, we can arrange that for each open set V_α which intersects $\overline{\Sigma}_H$, the intersection $\overline{\Sigma}_H \cap V_\alpha$ is given by $\{z_1 = 0, \dots, z_k = 0\} \subset U_\alpha$.

Proof. By Proposition 2.37, there is a Darboux atlas as required. Let us see that it can be adapted to the submanifold $\overline{\Sigma}_H$. For each chart $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$ intersecting $\overline{\Sigma}_H$ it holds $\overline{\Sigma}_H \cap V_\alpha = \phi_\alpha(L_H \cap U_\alpha)$, where $L_H \subset \mathbb{C}^n$, the fixed subset of Γ_α, is a complex linear subspace. We can then take a unitary basis of $\mathbb{C}^n$ so that $L_H = \{z_1 = 0, \dots, z_k = 0\}$ in the coordinates z_i induced by this basis, and clearly the symplectic form in the new coordinates is again ω_0, since $\mathrm{U}(n) < \mathrm{Sp}(2n, \mathbb{R})$. ∎

CHAPTER 3

Symplectic resolution of HI-orbifolds.

In the setting of symplectic geometry, symplectic orbifolds have been introduced mainly as a way to construct symplectic manifolds by resolving their singularities. Let us put context on this.

The problem of resolution of singularities and blow-up in the symplectic setting was posed by Gromov in [**27**]. Few years later, the symplectic blow-up was rigorously defined by McDuff [**40**] and it was used to construct a simply-connected symplectic manifold with no Kahler structure. The construction consisted on two steps. Firstly, the Kodaira-Thurston manifold was symplectically embedded in $\mathbb{CP}^n$. Secondly, a blow-up was carried out in $\mathbb{CP}^n$ along the Kodaira-Thurston manifold to get a simply conected symplectic non-Kahler manifold.

McCarthy and Wolfson developed in [**37**] a method of symplectic resolution for isolated singularities of an orbifold X in dimension 4. This was done finding a suitable three manifold Y embedded near an isolated singular point p so that for some neighborhood U^p of p it holds $Y = \partial U^p$. The resolution is then carried out by symplectically gluing $X \setminus U^p$ to a suitable symplectic 4-manifold Γ_Y (constructed from Y) such that $\partial\Gamma_Y = Y$.

Later on, Cavalcanti, Fernández and Muñoz gave a method of performing symplectic resolution of orbifold isolated singularities in all dimensions [**16**]. Since the singularities are isolated, they used a complex local model for the singularities and then invoked the well-known Theorems of resolution of complex algebraic singularities. The symplectic form was then constructed by gluing the symplectic forms of the orbifold and the resolution.

This resolution of isolated singularities was used in [**21**] to give the first example of a simply-connected symplectic 8-manifold which is non-formal, as the resolution of a suitable symplectic 8-orbifold. The starting point was a non-formal nilmanifold (hence not simply connected), on which they defined an action of a finite group by symplectomorphisms. The action was defined in such a way that the fundamental group was killed in the quotient space. The result was a simply connected symplectic orbifold with isolated singularities, on which the resolution of singularities was performed, giving the desired manifold. Later on, this manifold was proved to have also a complex structure in [**6**].

Niederkruger and Pasquotto [**45, 46**] provided a method for resolving symplectic orbifold singularities via symplectic reduction, which can be used for some classes of symplectic singularities, including cyclic orbifold singularities, even if these are not isolated. Recently, Chen [**19**] has detailed a method of resolving arbitrary symplectic 4-orbifolds, using the fact that the singular points of the underlying space have to be isolated in dimension 4. The novelty is that there can be also surfaces of non-trivial isotropy, and the symplectic orbifold form has to be modified on these surfaces also.

In this dimension, the work of the author with Muñoz and Tralle in [**44**] (included in this thesis in Proposition 4.8) also serves to resolve symplectic 4-orbifolds whose isotropy set is of codimension 2. In such case the orbifold is topologically a manifold (the isotropy points are non-singular), so the question only amounts to change the orbifold symplectic form into a smooth symplectic form.

Bazzoni, Fernández and Muñoz [**4**] have given the first construction of a symplectic resolution of an orbifold of dimension 6 with isotropy sets of dimension 0 and 2, although the construction is ad hoc for the particular example at hand, since it satisfies that the normal bundle to the 2-dimensional isotropy set is trivial. This was used to give the first example of a simply-connected non-Kahler 6-manifold which is simultaneously complex and symplectic.

In this section we give a procedure to resolve a wider type of singularities in a symplectic orbifold X of arbitrary dimension $2n$. We are able to develop such resolution for orbifolds X whose isotropy set is composed of disjoint submanifolds D_i so that each of the D_i have the same isotropy groups at all its points. We call this D_i a homogeneous isotropy set and such orbifold X an homogeneous isotropy orbifold (HI-orbifold). This allows for the existence of positive dimensional submanifolds composed of singular points, whose normal bundle may or may not be trivialized. The singular points of the topological underlying space are not isolated, hence new techniques are required in order to perform the resolution. We give a method to endow the normal bundle to D_i with a nice structure in which to effectively perform fiberwise the constructuve algebraic resolution of singularities of [**20**], and then glue these local resolutions into a resolution $\widetilde{X}$ of X.

Let us briefly sketch the steps in our construction. The strategy is to endow the normal bundle ν_D of any homogeneous isotropy submanifold $D \subset X$ with the structure of a special kind of orbibundle with structure group $\mathrm{U}(k)$, where $2k$ is the codimension of D.

The singularities of X at the points of D are quotient singularities in the fibers $F = \mathbb{C}^k/\Gamma$ of ν_D, where Γ is the isotropy group of D. The usual resolution of singularities for algebraic geometry allows to resolve each of the fibers F of ν_D separately. However, we need this resolution to glue nicely when we change trivializations. For this we need an improvement of the classical theorem of resolution of singularities by Hironaka [**31**]. This improvement is the *constructive resolution of singularities* by Encinas and Villamayor [**20**], which is compatible with algebraic group actions. Using their result we are able to construct the resolution $\widetilde{\nu}_D$ of X near D as a smooth manifold.

The resolution $\widetilde{\nu}_D$ has the structure of a fiber bundle over D, with fiber the resolution $\widetilde{F}$ of $F = \mathbb{C}^k/\Gamma$. Both base D and fiber $\widetilde{F}$ of the total space $\widetilde{\nu}_D$ are symplectic, but this does not imply directly that $\widetilde{\nu}_D$ admits a symplectic form. First we need to prove that there is no cohomological obstruction for this, which amounts to finding a cohomology class on the total space $\widetilde{\nu}_D$ that restricts to the cohomology class of the symplectic form of the fiber. Secondly, we have to develop a globalization procedure for symplectic fiber bundles with non-compact symplectic fiber. The final step consists of gluing the symplectic form on $\widetilde{\nu}_D$ with the original symplectic form of $X \setminus D$.

1. Tubular neighbourhood of the isotropy set.

Let us point out that for our method of resolution of singularities to work, we need to restrict to the case where the isotropy locus Σ of our orbifold X is a smooth submanifold with all its points having the same isotropy group. In other words, we require that the singular part of the orbifold X is concentrated in the slices orthogonal to Σ, being the isotropy set Σ a smooth submanifold of X. The following two definitions summarise this concept.

Definition 3.1. *Notations as in Proposition 10. We say that an isotropy subset $\overline{\Sigma}_H$ is* homogeneous *if $\overline{\Sigma}_H = \Sigma_H$. That is, all the points of $\overline{\Sigma}_H$ have isotropy equal to H.*

By Proposition 2.15, if $\overline{\Sigma}_H$ is homogeneous, then it is a submanifold of X.

Definition 3.2. *We say that an orbifold X is HI (abbreviation for homogeneous isotropy) if all its isotropy subsets are homogeneous, i.e. for every isotropy subset $\overline{\Sigma}_H$ we have $\overline{\Sigma}_H = \Sigma_H$.*

In this chapter we shall work exclusively with an HI orbifold X. Thus, in this chapter the word orbifold will mean HI orbifold.

The following results analizes the local picture near an homogenous isotropy set, first for a general orbifold and later for a symplectic orbifold.

LEMMA 3.3. *Let X be an orbifold of dimension n. Suppose $\overline{\Sigma}_H \subset X$ is an homogeneous isotropy set and let $d = \dim \overline{\Sigma}_H$.*

Then $\overline{\Sigma}_H$ is an isolated isotropy submanifold, that is, no other isotropy set intersects it. Moreover, around any point $x_0 \in \overline{\Sigma}_H$ we have a chart (U, V, ϕ, H) such that:

(1) *The open set $U \subset \mathbb{R}^n$ is a product of the type $U = U' \times U''$, $U' \subset \mathbb{R}^d$, $U'' \subset \mathbb{R}^{n-d}$, and $\overline{\Sigma}_H$ corresponds to $U' \times \{0\}$.*
(2) *The group $H < \mathrm{O}(n-d)$ acts only on the second factor U'', hence $V \cong U/H = U' \times (U''/H)$.*

PROOF. Take an orbifold chart (U, V, ϕ, Γ) around x_0 with $\Gamma < \mathrm{O}(n)$. We have $\overline{\Sigma}_H \cap V = \phi(L_H \cap U)$, with

$$L_H = \bigcap_{\gamma \in \Gamma} \ker(\gamma - \mathrm{Id}).$$

The linear subspace L_H is d-dimensional, and we can write $\mathbb{R}^n = L_H \oplus (L_H)^\perp \cong \mathbb{R}^d \times \mathbb{R}^{n-d}$. Note that Γ fixes L_H, so it acts on $(L_H)^\perp \cong \mathbb{R}^{n-d}$. Moreover $\Gamma = H$ since $x_0 \in \overline{\Sigma}_H = \Sigma_H$ by homogeneity. The result follows. □

We have an analoguous result for symplectic orbifolds.

LEMMA 3.4. *Let (X, ω) be a symplectic orbifold of dimension $2n$ and let $\overline{\Sigma}_H$ be an homogeneous isotropy set of dimension $2d$. For every $x \in \overline{\Sigma}_H$ there is a Darboux chart (U, V, ϕ, H) around x such that:*

(1) *The open set U is a product of the type $U = U' \times U''$, $U' \subset \mathbb{C}^d$, $U'' \subset \mathbb{C}^{n-d}$, and $\overline{\Sigma}_H$ corresponds to $U' \times \{0\}$.*
(2) *The isotropy group $H < \mathrm{U}(n-d)$ acts only on the second factor, hence $V \cong U/H = U' \times (U''/H)$,*

PROOF. This follows analogously to the previous Lemma, this time using Corollary 2.38 to obtain an orbifold chart (U, V, ϕ, Γ) with $\Gamma < \mathrm{U}(n)$ and

$$\phi(U \cap L_H) = \overline{\Sigma}_H \cap V$$

being $L_H = \{z_1 = 0, \dots, z_{2n-2d} = 0\} \cap U$. □

To understand the global structure of X around an homogeneous isotropy subset $\overline{\Sigma}_H$, let us introduce the notion of *bundle with orbi-fibers*. This is basically a bundle of orbifolds (as fibers) over a manifold. This definition differs from the usual definition of orbibundle in that the base space is left untouched by the action of the local groups. See [**23**] for a broader definition of the concept of orbibundle.

DEFINITION 3.5. *A* space with a geometric structure (M, G) *is a smooth manifold M with a Lie group G acting on M. We call G the automorphism group of the structure and write $G = \mathrm{Aut}(M)$.*

Examples of spaces with geometric structures (M, G) of interest to us consist of M being a vector space (think of the fiber of a vector bundle) and $G < \mathrm{GL}(n, \mathbb{R})$ some subgroup of the general linear group (think of the structure group of a vector bundle). For instance:

(1) $M = \mathbb{R}^n$ (or $M = \mathbb{C}^n$) and $G = \mathrm{GL}(n, \mathbb{R})$ (or $G = \mathrm{GL}(n, \mathbb{C})$), the fiber and structure group of a real (or complex) vector bundle.
(2) $M = \mathbb{R}^{2n}$ and $G = \mathrm{Sp}(2n)$ the linear symplectic group, the fiber and structure group of a symplectic vector bundle.
(3) $M = \mathbb{C}^n$ and $G = \mathrm{U}(n)$ the unitary group, the fiber and structure group of an hermitian vector bundle.

Let us now define the concept of bundle with orbi-fibers $M/\Gamma \to E \to B$. Let (M, G) be a space with some geometric structure, let $\Gamma < G = \mathrm{Aut}(M)$ be a finite subgroup of automorphisms of M, and let B be a smooth manifold.

Definition 3.6. *A* bundle with orbi-fibers *E with fiber $F = M/\Gamma$ and base space B, denoted $M/\Gamma \to E \to B$, consists of an orbifold E endowed with a map $\pi : E \to B$ and with a special type of orbifold atlas. The atlas consists on an open cover $\{V_\alpha\}$ of E and orbifold charts $\phi_\alpha : U_\alpha \times M \to V_\alpha$, with $U_\alpha \subset B$ open, so that:*

(1) The groups $\Gamma_\alpha < \mathrm{Aut}(M)$ act on $U_\alpha \times M$ as $\gamma(x, m) = (x, \gamma m)$, $\gamma \in \Gamma_\alpha$. Hence we have induced homeomorphisms $\widetilde{\phi_\alpha} : U_\alpha \times M/\Gamma \to V_\alpha$.
(2) The map π has the local expression $\pi|_{V_\alpha} : V_\alpha \xrightarrow{p_1 \circ \widetilde{\phi_\alpha}^{-1}} U_\alpha \subset B$.
(3) Every inclusion $\iota_{\delta\alpha} : U_\delta \times M \to U_\alpha \times M$ of the orbifold atlas of E has the form $\iota_{\delta\alpha}(x, m) = (\psi_{\delta\alpha}(x), A_{\delta\alpha}(x)m)$ with

$$A_{\alpha\beta} : \iota_{\delta\alpha}(U_\delta) \to \mathrm{Aut}(M)$$

a smooth map taking values in the group of automorphisms of M, and $\psi_{\delta\alpha} : U_\delta \to U_\alpha$ a smooth embedding.
(4) The monomorphisms $\rho_{\delta\alpha}$ associated to the inclusions $\iota_{\delta\alpha}$ are in fact isomorphisms $\rho_{\delta\alpha} : \Gamma_\delta \to \Gamma_\alpha$. We denote Γ the isomorphim class of the isotropy groups.

Let us make some comments about the definition above.

(1) Recall Definition 2.1. Note that the maps ϕ_α above are not orbifold charts in a strict sense, since $U_\alpha \times M$ are not open sets of $\mathbb{R}^n$. So the atlas of a bundle with orbi-fibers is not an orbifold atlas strictly speaking, but a space locally modelled by the quotient of a smooth manifold by a finite group.
(2) Obviously, an orbifold atlas can be constructed from the atlas above by taking smaller open subsets of V_α and using the atlases of B and M, both of which are smooth manifolds.
(3) According to the definition of an orbifold (Definition 2.1), the inclusions satisfy $\iota_{\delta\alpha}(\gamma(x, m)) = \rho_{\delta\alpha}(\gamma)\iota_{\delta\alpha}(x, m)$ which here means $A_{\delta\alpha}(x)(\gamma m) = \rho_{\delta\alpha}(\gamma)(A_{\delta\alpha}(x)m)$ for all $(x, m) \in U_\delta \times M$. This implies that for any $x \in U_\delta$, $g \in \Gamma_\alpha$ we have the relation

$$A_{\delta\alpha}(x) \circ \gamma = \rho_{\delta\alpha}(\gamma) \circ A_{\delta\alpha}(x)$$

where $\circ$ denotes the operation in the group $G = \mathrm{Aut}(M)$.
(4) The changes of charts $\varphi_{\alpha\beta}$ of this orbifold atlas of E are obtained from the inclusions according to Definition 2.1, i.e.

$$\begin{aligned} \varphi_{\alpha\beta} = \iota_{\delta\beta} \circ \iota_{\delta\alpha}^{-1} : \psi_{\delta\alpha}(U_\delta) \times M &\to \psi_{\delta\beta}(U_\delta) \times M \\ (x, m) &\mapsto (\psi_{\alpha\beta}(x), A_{\alpha\beta}(x)m) \end{aligned}$$

with $\psi_{\alpha\beta}(x) = \psi_{\delta\beta} \circ \psi_{\delta\alpha}^{-1}$, and $A_{\alpha\beta}(x) = A_{\delta\beta}(x) \left(A_{\delta\alpha}(x)\right)^{-1}$. In particular we see that

$$A_{\alpha\beta} : \iota_{\delta\alpha}(U_\delta) \to \mathrm{Aut}(M)$$

is a smooth map taking values in the group of automorphisms of M. These $A_{\alpha\beta}$ are the *transition functions* of a bundle with orbi-fibers.

(5) For a change of charts $\varphi_{\alpha\beta}$ we conclude that $A_{\alpha\beta}(x)\gamma m = \rho_{\alpha\beta}(\gamma)A_{\alpha\beta}(x)m$ for all $\gamma \in \Gamma_\alpha$, being
$$\rho_{\alpha\beta} = \rho_{\delta\beta} \circ \rho_{\delta\alpha}^{-1} : \Gamma_\alpha \to \Gamma_\beta$$
where
$$\rho_{\alpha\beta}(\gamma) = A_{\alpha\beta}(x)\gamma A_{\alpha\beta}(x)^{-1}.$$
So the isomorphisms $\rho_{\alpha\beta}$ are given by conjugations by $A_{\alpha\beta}(x) \in G$. Note that there are many conjugations that give raise to $\rho_{\alpha\beta}$, as the conjugation by any of the $A_{\alpha\beta}(x)$ for $x \in \psi_{\delta\alpha}(U_\delta)$ is valid.

(6) A bundle with orbi-fibers satisfies that E is a topological fiber bundle of the form $F = M/\Gamma \to E \to B$. The transition functions of the fiber bundle $F \to E \to B$ are given by taking quotients in the transition functions of E:
$$[\varphi_{\alpha\beta}] : \psi_{\delta\alpha}(U_\delta) \times M/\Gamma \to \psi_{\delta\beta}(U_\delta) \times M/\Gamma$$
$$(x, [m]) \longrightarrow (\psi_{\alpha\beta}(x), [A_{\alpha\beta}(x)m])$$
where $[m]$ denotes the equivalence class of $m \in M$ on M/Γ. Note that this map is well defined in M/Γ by Item (5).

(7) As noted before, a *vector bundle with orbi-fibers* corresponds to the case where M is a (real or complex) vector space and $\mathrm{Aut}(M)$ is a subgroup of the group of linear maps of M.

Now let (X, ω) be a HI symplectic orbifold, and let $D = \overline{\Sigma}_H = \Sigma_H \subset X$ be an homogeneous isotropy set of dimension $2d$. To maintain the previous notation, denote $\Gamma = H$ the isotropy of D. Let $2k = 2n - 2d$ be the codimension of D in X. We are going to define the orbifold normal bundle of D and then show that it satisfies the definition of a vector bundle with orbi-fibers, with fiber $\mathbb{C}^k/\Gamma$ and structure group $G = \mathrm{U}(k)$.

Recall first from Definition 2.26 that for $x \in X$ the orbifold tangent space T_xX is given as follows. Take a local chart (U, V, ϕ, Γ) with $\Gamma = \Gamma_x$ the isotropy group of x. Let $a \in U$ be the unique point such that $\phi(a) = x$. We may abuse notation an call such a point $a \in U$ simply $x \in U$. The tangent space at $x \in X$ is T_xU endowed with the linearized action of $\Gamma_x < \mathrm{GL}(T_xU)$, given by $d_x\gamma : T_xU \to T_xU$, for $\gamma \in \Gamma$.

In particular if $x \in D$, then T_xU is a symplectic vector space of dimension $2n$ and T_xD is the fix set of Γ_x, a symplectic subspace of T_xU of dimension $2d$. The symplectic orthogonal $(T_xD)^{\perp\omega}$ is also a symplectic subspace of dimension $2k$. Note that since X is an HI orbifold, the action induced by Γ_x on $T_xU = T_xD \times (T_xD)^{\perp\omega}$ occurs only in the second factor because D is left untouched by the action of Γ_x.

Definition 3.7. *Let X be an HI symplectic orbifold, and let D be one of the isotropy submanifolds of X. We define the* orbifold normal space *of D at the point $x \in D$ as*
$$\nu_{D,x} = (T_xD)^{\perp\omega}/\Gamma_x$$
where Γ_x is the isotropy subgroup of $x \in D$. The normal bundle ν_D is the union of all $\nu_{D,x}$, for $x \in D$.

Now let (X, ω) be a symplectic orbifold with an homogeneous isotropy submanifold $D \subset X$. Let $2d$ be the dimension of D and $2k = 2n - 2d$ its codimension. Then we take (ω, g, J) any orbifold almost Kahler structure for (X, ω). For $x_0 \in D$, using Corollary 2.38, we take an orbifold Darboux chart (U, V, ϕ, Γ) adapted to D, with $\Gamma < \mathrm{U}(k)$. So the lifting of D to U, which we may call D again, is given by
$$D = \{z_{d+1} = 0, \ldots, z_n = 0\}$$
By compatibility of g and ω, we have that $(T_{x_0}D)^{\perp\omega} = (T_{x_0}D)^{\perp g}$, and it has the structure of a J-complex subspace of $T_{x_0}U = \mathbb{C}^n$. Therefore in the chart U it must be
$$(T_{x_0}D)^{\perp\omega} = \{z_1 = 0, \ldots, z_d = 0\}$$

The action of Γ on U is given by $\gamma(x,y) = (x, \gamma y)$ for $x = (z_1, \dots, z_d) \in \mathbb{C}^d$ and $y = (z_{d+1}, \dots, z_n) \in \mathbb{C}^k$.

The orbifold atlas given by charts like the above induces in the normal bundle ν_D a structure of a symplectic vector bundle with orbi-fibers, as the following shows.

Proposition 3.8. *Let (X, ω) be an HI symplectic orbifold, and let $D \subset X$ be an isotropy submanifold. Then the normal bundle ν_D admits the structure of a symplectic vector bundle with orbi-fibers over D.*

The fiber of ν_D is $\mathbb{C}^k$, its structure group is $\mathrm{Sp}(2k, \mathbb{R})$ and the local isotropy group is $\Gamma < \mathrm{U}(k) < \mathrm{Sp}(2k, \mathbb{R})$.

Proof. We take a collection of Darboux symplectic charts $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$ adapted to D, given by Corollary 2.38. Denote $2d = \dim D$ and let $2k = 2n - 2d$ be the codimension of D. By Lemma 3.4 we can assume that $U_\alpha = U'_\alpha \times U''_\alpha$, where $U'_\alpha \subset \mathbb{C}^d$, $U''_\alpha \subset \mathbb{C}^k$, $\Gamma_\alpha < \mathrm{U}(k)$, and $V_\alpha \cong U'_\alpha \times (U''_\alpha / \Gamma_\alpha)$. Here, $D \cap U_\alpha$ corresponds to $U'_\alpha \times \{0\}$.

For any $p \in U'_\alpha \subset D$, the tangent space $T_pD = \mathbb{C}^d \times \{0\}$ and $(T_pD)^{\perp\omega} = \{0\} \times \mathbb{C}^k$. The action of Γ_α in the chart $U_\alpha = U'_\alpha \times U''_\alpha$ is done in the slices $\{x\} \times U''_\alpha$, with $x \in D$. Note that $\{x\} \times U''_\alpha$ represents the ω-orthogonal slice to D at the point $x \in D$.

Therefore we have $\nu_D|_{U'_\alpha} \cong U'_\alpha \times (\mathbb{C}^k/\Gamma_\alpha)$, where $\nu_D|_{U'_\alpha}$ denotes the collection of normal spaces to points $p \in U'_\alpha$. Then there is an orbifold chart

$$U'_\alpha \times \mathbb{C}^k \to \nu_D|_{U'_\alpha}\,,$$

where Γ_α acts on $\mathbb{C}^k$ by the inclusion $\Gamma_\alpha < \mathrm{U}(k)$. The fiber is $M = \mathbb{C}^k$ with $\mathrm{Aut}(M) = \mathrm{U}(k)$. Let us see that the orbifold changes of charts satisfy (3) in Definiton 3.6. By Definition 2.1, the change of charts for U_α and U_β is given by a map

$$\psi_{\alpha\beta} : \imath_{\delta\alpha}(U'_\delta \times U''_\delta) \to \imath_{\delta\beta}(U'_\delta \times U''_\delta), \quad \psi_{\alpha\beta}(x,y) = (\psi'_{\alpha\beta}(x,y), \psi''_{\alpha\beta}(x,y)).$$

The group homomorphisms $\rho_{\delta\alpha} : \Gamma_\delta \hookrightarrow \Gamma_\alpha$ and $\rho_{\delta\beta} : \Gamma_\delta \hookrightarrow \Gamma_\beta$ are isomorphisms (since all points have the same isotropy), so the map $\rho_{\alpha\beta} = \rho_{\delta\beta} \circ \rho_{\delta\alpha}^{-1} : \Gamma_\alpha \to \Gamma_\beta$ is an isomorphism. The map $\psi_{\alpha\beta}$ satisties $\psi_{\alpha\beta}(x, \gamma y) = \rho_{\alpha\beta}(\gamma)(\psi_{\alpha\beta}(x,y))$, i.e.

$$\psi''_{\alpha\beta}(x, \gamma y) = \rho_{\alpha\beta}(\gamma)\psi''_{\alpha\beta}(x,y), \tag{9}$$

for $\gamma \in \Gamma_\alpha$. Take a point $x = (x, 0) \in U'_\alpha \subset U_\alpha$. The map at the tangent space T_xX is given by $(d\psi_{\alpha\beta})_{(x,0)}$. Therefore the induced map on $(T_xD)^{\perp\omega} = \{0\} \times \mathbb{C}^k$ is given by the differential in the direction of y, which is

$$A_{\alpha\beta}(x) = \frac{\partial \psi''_{\alpha\beta}}{\partial y}\Big|_{(x,0)}.$$

By differentiating (9), we have $A_{\alpha\beta}(x)\gamma m = \rho_{\alpha\beta}(\gamma) A_{\alpha\beta}(x) m$, for $m \in \mathbb{C}^k$. Note that $A_{\alpha\beta}(x) \in \mathrm{Sp}(2k, \mathbb{R})$, since $\psi_{\alpha\beta}$ are symplectomorphisms. We consider the geometric space $M = \mathbb{C}^k$ with group $\mathrm{Aut}(M) = \mathrm{Sp}(2k, \mathbb{R})$. This completes the proof. □

With the following result we start the preparations to get a nice model of the normal bundle of an HI isotropy submanifold D of an orbifold X

Proposition 3.9 (Tubular neighbourhood for orbifolds). *Let X be an orbifold and $D \subset X$ an HI isotropy submanifold. Then there exists a neighborhood $\mathcal{U}$ of the zero section $D \times \{0\}$ of the orbifold normal bundle ν_D which is orbi-diffeomorphic to a tubular neighborhood $\mathcal{V}$ of D in X (via some orbi-deffeomorphism F), and moreover the orbi-diffeomorphism*

$$F : \mathcal{U} \to \mathcal{V}$$

maps $D \times \{0\} \subset \mathcal{U}$ to $D \subset \mathcal{V}$ as the identity map, and $d_{(x,0)}F : T_{(x,0)}\nu_D \to T_xX$ is the identity map identifying $T_{(x,0)}\nu_D \cong T_xD \times (T_xD)^{\perp\omega} \cong T_xX$.

PROOF. Consider an orbifold Riemannian metric g for X. We use the exponential map associated to the metric to find the desired diffeomorphism. Take the normal bundle $\nu_D = \{(x,u)|u \in (T_{(x,0)}D)^\perp\}$ and let $D = D \times \{0\} \subset \nu_D$ be the zero section. Define

$$\exp : \nu_D \longrightarrow U/\Gamma \subset X$$
$$(x,[u]) \mapsto [\alpha_{((x,0),u)}(1)]$$

where $\alpha_{((x,0),u)}$ is the geodesic from $(x,0) \in U$ with direction u. The brackets stand for the equivalence classes modulo the local isotropy groups.

We have to check that the expression of the map exp in each orbifold chart is Γ-equivariant. Consider a chart of ν_D as above, so we have $\exp : \nu_D|_{U'} \to U/\Gamma = U' \times (U''/\Gamma)$. The isotropy groups Γ act by isometries on the orbifold charts and hence commute with the exponential map, so $\exp(x,\gamma u) = \gamma(\exp(x,u))$ for $\gamma \in \Gamma$. This gives that exp is equivariant, so it is an orbifold smooth map.

There are open sets $\mathcal{U}$ with $D \times \{0\} \subset \mathcal{U} \subset \nu_D$, and $\mathcal{V}$ with $D \subset \mathcal{V} \subset M$, so that $F = \exp : \mathcal{U} \to \mathcal{V}$ is defined. As exp is the identity on D, it yields an orbifold diffeomorphism $F : \mathcal{U} \to \mathcal{V}$ for small open sets.

The fact the the differential of F is the identity at points of the zero-section can be checked locally in a chart, and hence it follows in a straightforward manner from the properties of the exponential map. □

Consider the orbi-diffeomorphism $F := \exp : \mathcal{U} \to \mathcal{V}$ provided by Proposition 3.9, where $\mathcal{U}$ is a neighbourhood of the zero section $D \times \{0\} \subset \nu_D$ and $\mathcal{V}$ is a neighbourhood of $D \subset (X,\omega)$. We can consider the pull-back $F^*\omega$ of ω to $\mathcal{U}$, which we may call $F^*\omega := \omega$ again. So $\omega \in \Omega^2_{orb}(\mathcal{U})$ is a symplectic orbifold form in a niegborhood of $D \times \{0\} \subset \nu_D$.

REMARK 3.10. *A warning about abuse of notation.*

(1) *No confusion should arise from the abuse of notation of identifying $F^*\omega$ in ν_D with ω in X, since we fix from now on an orbi-diffeomorphism*
$$F : \mathcal{U} \to \mathcal{V}$$
as in Proposition 3.9, identifying neighborhoods $\mathcal{U} \subset \nu_D$ and $\mathcal{V} \subset X$ of D. Therefore ω may be viewed both in $\mathcal{U}$ and in $\mathcal{V}$.

(2) *Recall that an isotropy submanifold D can be canonically viewed as a subset of its normal bundle ν_D as the zero section, and moreover at a point $x = (x,0) \in D = D \times \{0\} \subset \nu_D$, there is a canonical identification of $T_x\nu_D \cong T_xX$. Indeed, $T_x\nu_D = T_xD \oplus (T_xD)^\perp$ is decomposed as the tangent space to the zero section and the fiber, which is precisely the decomposition of T_xX. Being that said, the subsequent abuse of notation may follow.*

In the following proposition we deform the symplectic form $\omega \in \Omega^2(\mathcal{U})$, with $\mathcal{U} \subset \nu_D$, to get another symplectic form ω' so that ω' is *constant along the fibers* of ν_D. More precisely:

PROPOSITION 3.11. *Let (X,ω) be a symplectic orbifold and D a homogeneous isotropy submanifold. Consider $\omega := F^*\omega$ the form ω seen in an open subset of ν_D. Then the normal bundle ν_D admits a closed orbifold 2-form ω' such that:*

- *The forms ω' and ω coincide along the zero section $D \times \{0\} \subset \nu_D$, in particular ω' is symplectic on an open set $\mathcal{U}$ with $D \times \{0\} \subset \mathcal{U} \subset \nu_D$.*
- *Restricted to any fiber $F_x = \nu_{D,x} = (T_xD)^{\perp\omega}/\Gamma_x$, the form $\omega'|_{F_x}$ is constant on the vector space $(T_xD)^{\perp\omega}$.*

PROOF. We consider a local trivialization of ν_D, given by a chart $\phi : U_\alpha \times \mathbb{C}^k \to \nu_D|_{U_\alpha}$, with group $\Gamma_\alpha < \mathrm{U}(k)$. Denote $(x,y) \in U_\alpha \times \mathbb{C}^k$ the coordinates. Let us consider the form

$\omega_x := \omega_{(x,0)}|_{(T_xD)^{\perp\omega}}$, which is a Γ_α-equivariant symplectic form on the vector space $(T_xD)^{\perp\omega}$. We have that

$$\omega_{(x,0)}|_{(T_xD)^{\perp\omega}} = \omega_x = \sum b_{ij}(x) dy_i \wedge dy_j$$

for some smooth functions $b_{ij} : U_\alpha \to \mathbb{R}$. Now define

$$\beta_\alpha = d\left(\sum_{\gamma\in\Gamma} \gamma^* \left(\sum_{i,j} b_{ij}(x) y_i dy_j\right)\right) \in \Omega^2(U_\alpha \times \mathbb{C}^k)$$

Clearly β_α is a Γ_α-invariant closed 2-form defined in the chart $U_\alpha \times \mathbb{C}^k$, and satisfies $(\beta_\alpha)|_{F_x} = \omega_x$ for every $x \in D$. Now consider

$$\begin{aligned}\omega'_\alpha &= \pi^*(\omega|_D) + \beta_\alpha \\ &= \pi^*(\omega|_D) + \sum_{\gamma\in\Gamma} \gamma^* \left(\sum_{i,j,k} (\partial_{x_k} b_{ij}) y_i dx_k \wedge dy_j + \sum_{i,j} b_{ij}(x) dy_i \wedge dy_j\right).\end{aligned}$$

This form ω'_α is Γ_α-invariant, satisfies $(\omega'_\alpha)_{(x,0)} = \omega_{(x,0)}$ for all $x \in U_\alpha$, and it is constant restricted to fibers, since $\pi^*(\omega|_D)$ and $dx_k \wedge dy_j$ vanish when restricted to fibers. Actually, note that ω'_α equals ω_x when restricted to the fiber $(T_xD)^{\perp\omega}$.

On the other hand we have $\beta_\alpha = d\eta_\alpha$ on $U_\alpha \times \mathbb{C}^k$, with η_α the Γ_α-invariant 1-form given by

$$\eta_\alpha = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \gamma^* \left(\sum_{i,j} b_{ij}(x) y_i dy_j\right) \in \Omega^1(U_\alpha \times \mathbb{C}^k).$$

Therefore $\omega'_\alpha = \pi^*(\omega|_D) + d\eta_\alpha$, for some $\eta_\alpha \in \Omega^1_{orb}(\nu_D|_{U_\alpha})$. Note that all the 2-forms $\beta_\alpha = d\eta_\alpha$ restrict to 0 on $U_\alpha \times \{0\}$ and restrict to ω_x on every fiber F_x over a point $x \in U_\alpha$.

Now we glue all these locally defined 2-forms ω'_α as follows. Take any smooth partition of unity ρ_α subordinated to the cover U_α of D. Consider the form

$$\omega' = \pi^*(\omega|_D) + \sum_\alpha d\left((\pi^*\rho_\alpha)\eta_\alpha\right) \tag{10}$$

where $\pi : \nu_D \to D$ is the bundle projection. Note that ω' is invariant by the local groups since all objects involved in its definition are. Restricting to a fiber F_x, we have

$$\omega'|_{F_x} = \sum_\alpha d(\rho_\alpha(x)\eta_\alpha) = \sum_\alpha \rho_\alpha(x)(d\eta_\alpha)|_{F_x} = \sum_\alpha \rho_\alpha(x)\omega_x = \omega_x.$$

For $(x, 0) \in \nu_D$, we have from the expression

$$\omega' = \pi^*(\omega|_D) + \sum_\alpha d(\pi^*\rho_\alpha) \wedge \eta_\alpha + \sum_\alpha (\pi^*\rho_\alpha) d\eta_\alpha$$

and the fact that η_α vanishes at $(x, 0)$, that

$$\omega'_{(x,0)} = \pi^*(\omega|_D) + \sum_\alpha \rho_\alpha(x) d\eta_\alpha$$

which equals $\omega_{(x,0)}$, since both $\omega'_{(x,0)}$ and $\omega_{(x,0)}$ coincide in T_xD and in $(T_xD)^{\perp\omega}$.

In particular, ω' is non-degenerate at every point $(x, 0)$ in the zero section, which implies that ω' is also non-degenerate in some open neighborhood $\mathcal{U}$ of the zero section in ν_D. Since ω' is closed, it is symplectic on $\mathcal{U}$. □

The next result is the orbifold version of the tubular neighbourhood theorem for symplectic submanifolds. It gives a symplectomorphism $(\mathcal{U}, \omega') \cong (\mathcal{V}, \omega)$.

More precisely, it says that the open set $\mathcal{U} \subset \nu_D$ with the symplectic form ω' constant along the fibers (constructed in Proposition 3.11 above) is symplectomorphic to a tubular neighborhood $\mathcal{V}$ of D in X with the initial symplectic form ω.

PROPOSITION 3.12 (Symplectic tubular neighborhood for orbifolds). *Let (X, ω) be a symplectic orbifold and let $D \subset X$ be an HI isotropy submanifold. Let $\mathcal{U} \subset \nu_D$ be a neighborhood of $D \times \{0\}$ in the orbifold normal bundle ν_D and suppose that $(\mathcal{U}, \omega')$ is a symplectic manifold such that the symplectic form ω' satisfies that ω'_x and ω_x coincide on T_xX for all points $x \in D$. Then there are open sets $\mathcal{U}', \mathcal{V}'$ with $D \subset \mathcal{U}' \subset \mathcal{U} \subset \nu_D$ and $D \subset \mathcal{V}' \subset X$ and an orbifold symplectomorphism $\varphi : (\mathcal{U}', \omega') \to (\mathcal{V}', \omega)$ so that $\varphi|_D = \mathrm{Id}_D$ and $d_x f = \mathrm{Id}_{T_xX}$ for all $x \in D$.*

PROOF. The proof is similar to the equivariant Darboux theorem (Proposition 2.37). Take first any orbifold diffeomorphism $F : \mathcal{U} \subset \nu_D \to \mathcal{V} \subset X$ such that $F|_D = \mathrm{Id}_D$, and $\mathrm{Id} = d_{(x,0)}F : T_xX \to T_xX$. We saw the existence of F in Proposition 3.9 (maybe reducing $\mathcal{U}$). Denote $\omega_0 = \omega'$, $\omega_1 = F^*(\omega)$.

Let us call $\iota_D : D \to \nu_D$ the inclusion of D as the zero section. The hypothesis that $\omega|_x = \omega'|_x$ for all $x \in D$ implies that $\omega_0|_x = \omega_1|_x$ for all $x \in D \cong D \times \{0\}$. So in particular they satisfy $\iota_D^*(\omega_1 - \omega_0) = 0$.

As for manifolds, (see the map Q below) it is proved that the inclusion $\iota_D : D \to \mathcal{U}$ induces an isomorphism $\iota_D^* : H^2_{orb}(\mathcal{U}) \to H^2(D)$. So there exists an orbifold 1-form $\mu \in \Omega^1_{orb}(\mathcal{U})$ such that $d\mu = \omega_1 - \omega_0$. Since $(\omega_1 - \omega_0)|_x = 0$ for all $x \in D \cong D \times \{0\} \subset \nu_D$, we can also arrange that $\mu|_x = 0$ for all $x \in D$. This comes from the explicit construction of the primitive given by the chain homotopy between the maps $\mathrm{Id}_{\mathcal{U}}$ and $\iota_D \circ \pi$, with $\pi : \nu_D \to D$ the bundle projection. It is easily shown that if $\rho_t : \mathcal{U} \to \mathcal{U}$ is given by $\rho_t(x, u) = (x, tu)$ and $v_t = \frac{d}{dt}\rho_t$ is the vector field generating ρ_t, then the map

$$
\begin{aligned}
Q : \Omega^k_{orb}(\mathcal{U}) &\to \Omega^{k-1}_{orb}(\mathcal{U}) \\
\alpha &\longrightarrow \int_0^1 \rho_t^*(\iota_{v_t}\alpha)
\end{aligned}
$$

satisfies

$$
\mathrm{Id} - (\iota_D \circ \pi)^* = d \circ Q + Q \circ d
$$

i.e. it is chain homotopy in $\Omega^*_{orb}(\mathcal{U})$ between Id and $(\iota_D \circ \pi)^*$. The primitive of $\omega_1 - \omega_0$ given by this chain homotopy is

$$
\mu = Q(\omega_1 - \omega_0)
$$

and for $(x, 0) \in D \times \{0\}$ we clearly have $\mu|_{(x,0)} = 0$.

Consider the form $\omega_t = t\omega_1 + (1-t)\omega_0 = \omega_0 + t\, d\mu$, for $0 \le t \le 1$. For any $x \in D$ we have that $\omega_t|_x = \omega_0|_x = \omega_1|_x$ is symplectic on T_xX, so we can assume, reducing $\mathcal{U}$ if necessary, that ω_t is symplectic on some neighborhood, which we call $\mathcal{U}$ again, of the zero section D of ν_D. The equation

$$
\iota_{X_t}\omega_t = -\mu
$$

admits a unique solution X_t which is a vector field on $\mathcal{U}$. Since μ vanishes at $D \times \{0\}$, the same happens for X_t. Consider the flow $\varphi_t = \varphi_s^{X_t}|_{s=t}$ of the family of vector fields X_t, which satisfies $\varphi_t|_D = \mathrm{Id}_D$. There is some $\mathcal{U}' \subset \mathcal{U}$ such that $\varphi_t : \mathcal{U}' \to \varphi_t(\mathcal{U}') \subset \mathcal{U}$ for all $t \in [0, 1]$. Moreover $\varphi_0 = \mathrm{Id}_{\mathcal{U}'}$. Now compute

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=s}\varphi_t^*\omega_t &= \varphi_s^*\left(\mathcal{L}_{X_s}\omega_s\right) + \varphi_s^*(d\mu) \\
&= \varphi_s^*\left(d(\iota_{X_s}\omega_s) + \iota_{X_s}d\omega_s\right) + \varphi_s^* d\mu = -\varphi_s^*(d\mu) + \varphi_s^*(d\mu) = 0.
\end{aligned}
$$

This implies that $\omega_0 = \varphi_0^*\omega_0 = \varphi_1^*\omega_1$. Recall that $\omega_0 = \omega'$ and $\omega_1 = F^*\omega$, so we conclude that $\varphi := \varphi_1 : (\mathcal{U}', \omega') \to (\mathcal{U}, F^*(\omega))$ is a symplectomorphism. It remains to see that φ is Γ_α-equivariant by all the local isotropy groups Γ_α. Fix a chart of ν_D and suppose that the group

Γ_α acts on this chart. As ω_t and μ are Γ_α-equivariant, we have that the vector fields X_t are Γ-equivariant. This implies that the diffeomorphisms φ_t are Γ_α-equivariant (i.e. commute with Γ). In particular, $\varphi = \varphi_1$ is Γ_α-equivariant.

Finally, take the composition

$$\psi = F \circ \varphi : (\mathcal{U}', \omega') \to (\mathcal{V}, \omega)$$

and this is our desired orbifold symplectomorphism of $\mathcal{U}'$ onto $\mathcal{V}' = \psi(\mathcal{U}') \subset \mathcal{V}$. □

We have seen so far that the form $F^*\omega$ in ν_D can be deformed to a symplectic form ω' in $\mathcal{U} \subset \nu_D$ so that ω' is constant along the normal fibers $(T_xD)^{\perp\omega'}$. Moreover, this form ω' is symplectomorphic to the form $F^*\omega$ in some open neighborhood $\mathcal{U} \subset \nu_D$ of the zero section, so in order the make the symplectic resolution we can forget ω and work with the form ω'.

The fact that ω' is constant along fibers means that the normal slices to D in the symplectic manifold $(\mathcal{U}, \omega')$ behave like symplectic vector spaces. This will prove to be very useful to apply the algebraic resolution of singularities later on.

In the following proposition we use that the isotropy groups can be chosen in $\mathrm{U}(n)$ to obtain *local Kahler models* for $(\mathcal{U}, \omega')$. This entails first putting an arbitrary almost-Kahler structure, and then deforming this structure to get another almost Kahler-structure constant along the fibers. Therefore the normal slices in $\mathcal{U}$ inherit the structure of a Kahler manifold. Using this Kahler model we can perform the algebraic resolution of singularities fiberwise in $\mathcal{U}$, and then (after some work) match them together to obtain the resolution. The following is the first step in this direction.

PROPOSITION 3.13. *Let (X, ω, g, J) be an almost Kahler orbifold and D a homogeneous isotropy submanifold. An open neighborhood $\mathcal{U} \subset \nu_D$ of the zero section $D = D \times \{0\} \subset \nu_D$ admits an orbifold almost Kahler structure (ω', g', J') such that:*

- *For a point $(x, 0)$ in the zero-section we have that, under the natural splitting $T_{(x,0)}\nu_D = T_xD \times (T_xD)^\perp \cong T_xX$, the restriction of (ω', g', J') to T_xX coincides with (ω, g, J).*
- *The tensors (ω', g', J') are constant along the fibers $F_x = \nu_{D,x}$, for $x \in D$.*

PROOF. First note that in the statement of the Proposition, $(T_xD)^\perp$ can be taken with respect to either ω or g, because for a compatible metric, the symplectic orthogonal of a symplectic linear space coincides with the metric orthogonal.

Now, take the symplectic structure ω' in ν_D provided by Proposition 3.11. Let us define first an auxiliar metric g^* on $\mathcal{U} \subset \nu_D$. We define g^* so that it coincides with g on T_xD and on $(T_xD)^\perp$ for $x \in D$. On the fiber $F_x = \nu_{D,x} = (T_xD)^\perp/\Gamma_x$, the tensor $g_x|_{(T_xD)^\perp}$ is Γ_x-equivariant, so we can define a constant tensor on F_x which varies smoothly for $x \in D$: at any point $y \in F_x$ we define $g^*_y|$ as $g_x|_{(T_xD)^\perp}$. We have thus defined the restriction of g^* to the fibers, so it remains to define g^* at horizontal directions.

Now we extend g^* to a Riemannian metric on $\mathcal{U} \subset \nu_D$. This is done as follows. For $(x, u) \in \mathcal{U} \subset \nu_D$, with $u \neq 0$, we consider the splitting $T_{(x,u)}\nu_D = T_{(x,u)}F_x \oplus (T_{(x,u)}F_x)^{\perp\omega'}$. We define g^* by making these subspaces orthogonal so that g^* restricted to $(T_{(x,u)}F_x)^{\perp\omega'}$ is $\pi^*(g|_{T_xD})$ under the isomorphism $\pi_* : (T_{(x,u)}F_x)^{\perp\omega'} \to T_xD$. The metric g^* may not be equivariant, so we make it equivariant by averaging and then we use the method of the proof of Proposition 2.34 to modify g^* into an orbifold Riemannian metric g' such that $g'(u, v) = \omega'(u, J'v)$ defines an orbifold almost-Kahler structure J'. Note that the tensor A defined by $g^*(u, Av) = \omega'(u, v)$ satisfies that $A = J$ at the points of $D \subset \nu_D$, as desired. For $(x, u) \in F_x$, the definition $g^*(u, Av) = \omega'(u, v)$ and the fact that $T_{(x,u)}F_x \oplus (T_{(x,u)}F_x)^{\perp\widetilde{\omega}}$ is at the same time the Riemannian orthogonal decomposition, implies that A equals $J_x|_{T_xD^\perp}$ restricted to $T_{(x,u)}F_x = F_x$. So J' is constant along F_x. This concludes the proof. □

Now we need to see that the transition functions on the normal orbi-bundle ν_D can be made to be $\mathrm{U}(k)$-valued, i.e. the structure group of the bundle with orbi-fibers ν_D *reduces* to $\mathrm{U}(k)$

This is well known in the case of normal bundles of symplectic submanifolds. In this case the reduction of the structure group consists basically on choosing local frames which are orthonormal for a hermitian metric in the fibers of the normal bundle.

In the orbifold case, however, we want to reduce the structure group and at the same time maintain the local isotropy groups Γ_α as subgroups of $\mathrm{U}(k)$. Choosing orthonormal frames is no longer enough, since the Gram-Schmidt orthonormalization procedure gives a matrix of change of coordinates C, and this matrix does not satisfy necessarily that $C \circ \Gamma_\alpha \circ C^{-1}$ is equal to Γ_α, as we would desire. In order to fix this, we will use the following retraction.

Proposition 3.14. *There is a natural retraction.*

$$r : \mathrm{Sp}(2k, \mathbb{R}) \to \mathrm{U}(k), \qquad r(A) = A(A^t A)^{-1/2}$$

This retraction satisfies the following:

(1) *If there is a group $\Gamma < \mathrm{U}(k)$ and an isomorphism $\rho : \Gamma \to \Gamma' < \mathrm{U}(k)$ such that a matrix $A \in \mathrm{Sp}(2k, \mathbb{R})$ is (ρ, Γ)-equivariant in the sense that $A \circ \gamma = \rho(\gamma) \circ A$ for all $\gamma \in \Gamma$, then $r(A)$ is also (ρ, Γ)-equivariant.*

Proof. See [**39**, Prop. 2.2.4] □

The following is a technical result we will use later on. It says that if two unitary matrices are conjugate in $\mathrm{Sp}(2k, \mathbb{R})$ then they are also conjugate in $\mathrm{U}(k)$.

Lemma 3.15. *Let $A, C \in \mathrm{U}(k)$ and $B \in \mathrm{Sp}(2k, \mathbb{R})$ such that $A = B^{-1}CB$. Then $A = r(B)^{-1} C\, r(B)$.*

Proof. The fact that $B \in \mathrm{Sp}(2k, \mathbb{R})$ means that $B^t J_0 B = J_0$, where J_0 is the matrix of the standard complex structure of $\mathbb{R}^{2n}$. So $B^t = -J_0 B^{-1} J_0$, $A^t A = C^t C = \mathrm{Id}$, $AJ_0 = J_0 A$ and $CJ_0 = J_0 C$. Then

$$\begin{aligned}(B^t B)A &= -J_0 B^{-1} J_0 B A = -J_0 B^{-1} J_0 C B = -J_0 B^{-1} C J_0 B \\ &= -J_0 A B^{-1} J_0 B = -A J_0 B^{-1} J_0 B = A(B^t B).\end{aligned}$$

This means that A commutes with $B^t B$. Therefore A commutes with $(B^t B)^{1/2}$ as well. Hence

$$r(B)^{-1} C r(B) = (B^t B)^{1/2} B^{-1} C B (B^t B)^{-1/2} = (B^t B)^{1/2} A (B^t B)^{-1/2} = A$$

as required. □

Now we are ready to see that the structure group of the orbi-bundle ν_D reduces to $\mathrm{U}(k)$.

Proposition 3.16. *The normal bundle with orbi-fibers ν_D admits an atlas such that the transition functions $A_{\alpha\beta}$ are $\mathrm{U}(k)$-valued. In the terminology of Definition 3.6, the structure group of ν_D reduces to $\mathrm{U}(k)$.*

Proof. By Propositions 3.8 and 3.13, the normal bundle with orbi-fibers ν_D admits:

- An almost Kahler structure which is constant along the fibers (let us change notation and denote it (ω, J, g) this time).
- The structure of a $\mathrm{Sp}(2k, \mathbb{R})$-bundle with orbi-fibers.

Take an atlas $\{(U_\alpha \times \mathbb{C}^k, \Gamma_\alpha\ , \omega_0)\}$ of ν_D so that $\Gamma_\alpha < \mathrm{U}(k)$, ω_0 is the standard symplectic form in $\mathbb{C}^k$, and the transition functions are $A_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Sp}(2k, \mathbb{R})$. The orbifold almost-Kahler structure in each trivialization $(U_\alpha \times \mathbb{C}^k, \Gamma_\alpha\ , \omega_0)$ is given by tensors (ω_0, J_x, g_x) which are Γ_α-equivariant.

Fix a chart $U_\alpha \times \mathbb{C}^k$ and call (x,y) the corresponding coordinates. Call h the hermitian metric associated with (ω, J, g). The hermitian metric h induces a linear hermitian metric h_x on each fiber $\{x\} \times \mathbb{C}^k$ varying smoothly with $x \in U_\alpha$. In this trivilization we can make Gram-Schmidt to get an h_x-unitary frame of each fiber $\{x\} \times \mathbb{C}^k$. Note that this h_x-unitary frame is determined by a matrix $C_\alpha(x) \in \mathrm{Sp}(2k, \mathbb{R})$, the change of coordinates between the initial (symplectic) frame and the h_x-unitary frame.

If we introduce new coordinates $(x, \tilde{y}) = (x, C_\alpha(x)y)$ then the orbifold almost Kahler structure is given by the standard tensors (ω_0, J_0, g_0) defining the complex structure and metric in $\mathbb{C}^k$, but the action of the isotropy group is given by the varying group $\Gamma_\alpha^x = C_\alpha(x)\Gamma_\alpha C_\alpha(x)^{-1}$. Clearly $\Gamma_\alpha^x < \mathrm{U}(k)$ because it preserves the hermitian structure (ω_0, g_0, J_0). The group Γ_α^x acts on the fiber $\{x\} \times \mathbb{C}^k$ and vary with the point $x \in U_\alpha$, so the action of the isotropy is not linear on the chart $U_\alpha \times \mathbb{C}^k$. On the other hand, in the coordinates $(x, \tilde{y})$ the transition functions of the bundle are $\mathrm{U}(k)$-valued as we want. Now we have to get the isotropy to act linearly, i.e. we need to eliminate the dependance on x of the groups Γ_α^x.

Now define new coordinates $(x, y') = (x, r(C_\alpha(x))^{-1}\tilde{y})$ where r is the retraction (3.14). The hermitian metric in the new coordinates is the standard metric of $\mathbb{C}^k$ because it was so in the coordinates $(x, \tilde{y})$ and $r(C_\alpha(x))^{-1} \in \mathrm{U}(k)$. So the orbifold almost Kahler structure in the coordinates (x, y') is given by (ω_0, J_0, g_0). However, the isotropy group is the group $\Gamma_\alpha < \mathrm{U}(k)$ that we began with. Indeed, $\Gamma_\alpha = C_\alpha(x)^{-1}\Gamma_\alpha^x C_\alpha(x)$ implies, by Lemma 3.15, that $\Gamma_\alpha = r(C_\alpha(x))^{-1}\Gamma_\alpha^x\, r(C_\alpha(x))$. Carrying out this procedure for each coodinate patch, the corresponding transition functions are in $\mathrm{U}(k)$, whereas the isotropy is given by the (constant) groups $\Gamma_\alpha < \mathrm{U}(k)$. □

COROLLARY 3.17. *If $D \subset X$ is a connected homogeneous isotropy submanifold, then the normal bundle ν_D admits an atlas $\{U_\alpha \times \mathbb{C}^k\}$ with the transition functions $A_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(k)$ and with the isotropy group Γ fixed, i.e. all trivializations $U_\alpha \times \mathbb{C}^k$ have the same isotropy group $\Gamma_\alpha = \Gamma$. Actually, the image of $A_{\alpha\beta}$ lies in the normalizer of $\Gamma < \mathrm{U}(k)$, i.e. in the subgroup of $\mathrm{U}(k)$ given by $N_{\mathrm{U}(k)}(\Gamma) = \{A \in \mathrm{U}(k) : A\Gamma A^{-1} = \Gamma\}$.*

REMARK 3.18. *Therefore, if an homogeneous isotropy submanifold $D \subset X$ has an isotropy group $\Gamma < \mathrm{U}(k)$ so that its normalizer $N_{\mathrm{U}(k)}(\Gamma)$ is finite, then its normal bundle ν_D has constant transition functions $A_{\alpha\beta}$ and hence the Chern classes $c_k(\nu_D) = 0$ for all $k \geq 1$. This gives a somewhat unexpected connection between the isotropy group of D and the Chern classes of its normal bundle ν_D.*

2. Resolution of the normal bundle.

In this section we will use the previously obtained structure of the normal bundle ν_D of an HI-submanifold $D \subset X$ of a symplectic orbifold X, to construct a resolution of ν_D.

By Corollary 3.17, we fix an atlas $\{U_\alpha \times \mathbb{C}^k\}$ with $\Gamma < \mathrm{U}(k)$ acting on the fiber, and with the transition functions $A_{\alpha\beta} : U_\alpha \cap U_\beta \to N_{\mathrm{U}(k)}(\Gamma)$. The group $G = N_{U(k)}(\Gamma)$ is a closed Lie subgroup of $\mathrm{U}(k)$ since Γ is finite. In particular G is compact, and acts on $\mathbb{C}^k/\Gamma$ by matrix multiplication.

Recall that $F_x \cong \mathbb{C}^k/\Gamma$ is a singular complex variety, hence we can use the *constructive algebraic resolution* of [**20**] and [**53**]. The method to obtain the constructive resolution has the property that any algebraic action on the singular variety admits a unique lifting to the resolution. In other words, the constructive resolution can be permorfed in an equivariant way.

THEOREM 3.19 ([**53**, Prop. 7.6.2]). *Let $X \subset W$ be a subscheme of finite type of a smooth scheme W, with X reduced, and $\theta \in \mathrm{Aut}(W)$ an algebraic automorphism of X. Let $b : \widetilde{X} \to X$ be the constructive resolution of singularities. Then $\theta : X \to X$ lifts uniquely to an isomorphism*

$$\tilde{\theta} : \widetilde{X} \to \widetilde{X}$$

of the constructive resolution of singularities $\widetilde{X}$ *of* X *such that* $b \circ \tilde{\theta} = \theta \circ b$.

Note that the uniqueness of the lifting follows immediately from the existence because any two liftings have to coincide in the Zariski open set where $b : \widetilde{X} \to X$ is an isomorphism.

Now, recall that $\mathrm{U}(k)$ is not a complex algebraic Lie group (actually it is a real one) since in the equations defining the conditions to be unitary $A\bar{A}^t = \mathrm{Id}$ appear non-holomorphic terms, such as the complex conjugation.

Recall that the structure group of ν_D is

$$G := N_{\mathrm{U}(k)}\Gamma < \mathrm{U}(k)$$

by Corollary 3.17, being Γ the isotropy group. Again, G is a real-algebraic Lie group (being a closed subgroup of $\mathrm{U}(k)$), but it is not a complex algebraic group, as we would need to use Theorem 3.19.

This can be solved like this. The group G is a subgroup of the closed and complex Lie subgroup $L < \mathrm{GL}(k, \mathbb{C})$ given by

$$L := N_{\mathrm{GL}(k,\mathbb{C})}(\Gamma) < \mathrm{GL}(k, \mathbb{C}).$$

Actually, $G = L \cap \mathrm{U}(k)$, so G is a closed Lie subgroup of L. As we show below, L is a complex algebraic Lie group.

Let us first state here once and for all that *affine variety* for us means a complex algebraic affine variety i.e. a subset of some $\mathbb{C}^N$ given as the zero set of some polynomials in $\mathbb{C}[x_1, \ldots, x_N]$. In the same line, *projective variety* means complex algebraic projective variety i.e. a subset of some $\mathbb{CP}^N$ given as the zero set of some homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_N]$. In other words, closed Zariski subsets of $\mathbb{C}^N$ and $\mathbb{CP}^N$ respectively.

Now let us prove that $L = N_{\mathrm{GL}(k,\mathbb{C})}(\Gamma) < \mathrm{GL}(k, \mathbb{C})$ is a complex-algebraic Lie subgroup.

Lemma 3.20. *Let* $\Gamma < \mathrm{U}(k)$ *be a finite subgroup of unitary matrices. The normalizer* $L = N_{\mathrm{GL}(k,\mathbb{C})}(\Gamma)$ *in* $\mathrm{GL}(k, \mathbb{C})$ *of* Γ *is a complex-algebraic Lie subgroup of* $\mathrm{GL}(k, \mathbb{C})$.

Proof. Note that a non-singular complex matrix $A \in \mathrm{GL}(k, \mathbb{C})$ belongs to L if and only if $A \circ \Gamma \circ A^{-1} = \Gamma$. This means that for every $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that $A \circ \gamma \circ A^{-1} = \gamma'$.

For $\gamma, \gamma' \in \Gamma$ we consider the sets

$$A_{\gamma\gamma'} := \{A \in \mathrm{GL}(k, \mathbb{C}) : A \circ \gamma \circ A^{-1} = \gamma'\}, \quad A_\gamma := \bigcup_{\gamma' \in \Gamma} A_{\gamma\gamma'}$$

We claim that $A_{\gamma\gamma'}$ is an affine variety in $\mathrm{GL}(k, \mathbb{C}) \cong \mathbb{C}^{k^2}$, since it is given as the zero locus of a finite set of polynomials over $\mathbb{C}$. To see this, let us multiply by the complex determinant $\det(A)$ in the relation $A \circ \gamma \circ A^{-1} = \gamma'$ to get $A \circ \gamma \circ \mathrm{Adj}(A) = \det(A)\gamma'$, being $\mathrm{Adj}(A)$ the adjoint matrix of A. This last equation is equivalent to k^2 polynomial (of degree k) in the entries of the matrix A.

We conclude that A_γ is also an affine variety (Zariski-closed), being a finite union of variaties (Zariski-closed subsets). Finally, note that $L = \cap_{\gamma \in \Gamma} A_\gamma$ is the intersection of Zariski-closed subsets, hence closed, i.e. a variety. □

On the other hand, note that both groups L and G act naturally on $F = \mathbb{C}^k/\Gamma$ by matrix multiplication, i.e. $A[u] := [Au]$ for $A \in L$ or $A \in G$, $u \in \mathbb{C}^k$. Here the bracket stands for the equivalence class of $u \in \mathbb{C}^k$ in the quotient $\mathbb{C}^k/\Gamma$. The actions of L and G are both well defined: let $A \in L$ or $A \in G$. If $[u] = [u']$ then there exists $\gamma \in \Gamma$ with $u = \gamma u'$ and hence $Au = A\gamma u' = \gamma' Au'$ for some $\gamma' \in \Gamma$. Therefore $[Au] = [Au']$ as desired.

We have so far proved that:

(1) The group L is an algebraic group containing the group G as a subgroup.

(2) There is a well defined action of L in the space $F = \mathbb{C}^k/\Gamma$. This action restricted to $G < L$ yields the action of G.

In order to see that the action of L in F is algebraic, first we have to see that $F = \mathbb{C}^k/\Gamma$ has a natural structure of an algebraic variety. To see this we are going to embed F in some $\mathbb{C}^N$ with N large enough.

EXAMPLE 3.21. *Let us illustrate the strategy to embed F into $\mathbb{C}^N$ with an easy example.*

Consider the group $\Gamma \cong \mathbb{Z}_3$ acting on $\mathbb{C}^2$ by $\xi(z_1, z_2) = (\xi z_1, \xi^2 z_2)$, with $\xi = e^{\frac{2\pi i}{3}}$. Clearly $\Gamma < U(2)$ acts on $\mathbb{C}^2$ by matrix multiplication.

Let us constract explicitly an embedding of $F = \mathbb{C}^2/\Gamma$ in $\mathbb{C}^3$. Note that the polinomials $f_1, f_2, f_3 \in \mathbb{C}[x, y]$ given by $f_1(x, y) = x^3$, $f_2(x, y) = xy$, $f_3(x, y) = y^3$ are polynomials invariant by the action of Γ, hence they descend to functions in the quotient space $F = \mathbb{C}^2/\Gamma$. We have a map

$$\varphi : F \to \mathbb{C}^3 : [(x, y)] \mapsto (x^3, xy, y^3)$$

induced by $f = (f_1, f_2, f_3) : \mathbb{C}^2 \to \mathbb{C}^3$. Let us see that φ is a homeomorphism between F and the affine variety $Z = \{(u, v, w) \in \mathbb{C}^3 : uw = v^3\} \subset \mathbb{C}^3$.

If $f(x, y) = f(x', y')$ then $x^3 = x'^3$, $xy = x'y'$, $y^3 = y'^3$. The first and third conditions imply that $x' = \xi^j x$, $y' = \xi^l y$ for $j, l \in \{0, 1, 2\}$. Now the second condition reads $xy = x'y' = \xi^{l+j} xy$ which implies that $l + j$ is either 0 or 3. If $l + j = 0$ then $l = j = 0$, hence $(x, y) = (x', y')$. The case $l + j = 3$ is subdivided in two cases. If $(j, l) = (1, 2)$ then $(x', y') = \xi(x, y)$. If $(j, l) = (2, 1)$ then $(x', y') = \xi^2(x, y)$.

The conclusion is that $f(x, y) = f(x', y')$ if and only if $[(x, y)] = [(x', y')] \in F = \mathbb{C}^2/\Gamma$. By the properties of the quotient topology, this implies that $\varphi : F \to \varphi(F)$ is a homeomorphism.

Finally let us see that the image of φ is Z. Let $(u, v, w) \in Z$, so $uw = v^3$. Suppose that $uvw \neq 0$ (when it is zero, easier). The complex numbers u, w have three cube roots. Choose $x_0, y_0 \in \mathbb{C}$ one of such roots, so that $x_0^3 = u$ and $y_0^3 = w$. Then (x_0, x_1, x_2) and (y_0, y_1, y_2) are the three roots of u and v respectively, with $x_1 = \xi x_0$, $x_2 = \xi^2 x_0$, $y_1 = \xi y_0$, $y_2 = \xi^2 y_0$. Note that any combination $x_i y_j$ is a 3-th root of uw. Since $uw = v^3$, we see that v must be either $x_0 y_0$, $\xi x_0 y_0$, or $\xi^2 x_0 y_0$. In each case respectively, choose $(x, y) = (x_0, y_0)$, $(x, y) = (x_1, y_0)$ and $(x, y) = (x_2, y_0)$ (more choices are possible). Then $\varphi([(x, y)]) = (u, v, w)$.

Now we aim to generalize the above procedure to the action of any finite group of complex matrices. Let us establish some notations.

(1) Let $\mathbb{K}$ be a field of characteristic zero.
(2) Given a group of matrices $\Gamma < \mathrm{GL}(n, \mathbb{K})$ acting on $\mathbb{K}^n$ by matrix multiplication, we denote

$$\mathbb{K}[x_1, \dots, x_n]^\Gamma = \{f \in \mathbb{K}[x_1, \dots, x_n] : f(Ax) = f(x) \quad \forall A \in \Gamma\}$$

for the $\mathbb{K}$-algebra of polynomials invariant by the action of Γ.
(3) The symmetric group S_n acts on $\mathbb{K}^n$ by permutation of the coordinates of points. This corresponds to the linear action of the subgroup of $\mathrm{GL}(n, \mathbb{K})$ of permutation matrices, i.e. matrices obtained from the identity permuting rows. In this case, the invariant polynomials $\mathbb{K}[x_1, \dots, x_n]^{S_n}$ of this action are called the symmetric polynomials.
(4) Recall the definition of the elementary symmetric polynomials $\sigma_1, \dots, \sigma_n \in \mathbb{K}[x_1, \dots, x_n]$ given by the expression

$$(x - x_1) \cdot \ldots \cdot (x - x_n) = x^n - \sigma_1 x^{n-1} + \dots + (-1)^k \sigma_n \tag{11}$$

(5) We define the k-th power sum as $s_k = x_1^k + \dots + x_n^k \in \mathbb{K}[x_1, \dots, x_n]$, $k \in \mathbb{N}$.
(6) The well-known Newton-Girard formulas say that for $1 \leq k \leq n$

$$0 = s_k - s_{k-1}\sigma_1 + \dots + (-1)^{k-1} s_1 \sigma_{k-1} + (-1)^k k \sigma_k.$$

This gives an important relation between the k-th powers and the elementary symmetric polynomials.

LEMMA 3.22. *We have the following expressions regarding $\mathbb{K}[x_1,\dots,x_n]^{S_n}$, the $\mathbb{K}$-algebra of symmetric polynomials:*

(1) $\mathbb{K}[x_1,\dots,x_n]^{S_n} = \mathbb{K}[\sigma_1,\dots,\sigma_n]$
(2) $\mathbb{K}[x_1,\dots,x_n]^{S_n} = \mathbb{K}[s_1,\dots,s_n]$

PROOF. The first assertion is taught in any standard algebra course.

The second assertion follows from the first and the fact that the elementary symmetric polynomials $\sigma_1\dots,\sigma_n$ can all be expressed as some polynomial in $s_1,\dots,s_n$, as we show below.

We use the Newton-Girard formulas

$$0 = s_k - s_{k-1}\sigma_1 + \cdots + k(-1)^k\sigma_k, \quad 1 \le k \le n$$

With this formula, we proceed by induction on $k \ge 1$. For $k = 1$ we have $\sigma_1 = s_1$. Suppose that $k \ge 2$ and $\sigma_1,\dots,\sigma_{k-1}$ can all be expressed as some polynomial in $(s_1,\dots,s_n)$. Then we can put $k(-1)^k\sigma_k = s_{k-1}\sigma_1 - \cdots - \sigma_k$, so σ_k can also be expressed in terms on $s_1,\dots,s_n$. □

LEMMA 3.23. *Let $\Gamma < \mathrm{GL}(n,\mathbb{K})$ be a finite group acting on $\mathbb{K}^n$ by matrix multiplication. The $\mathbb{K}$-algebra $\mathbb{K}[x_1,\dots,x_n]^\Gamma \subset \mathbb{K}[x_1,\dots,x_n]$ of polynomials invariant by Γ is finitely generated. In other words, there exist a finite number of polynomials $f_1,\dots,f_N$ such that $\mathbb{K}[x_1,\dots,x_k]^\Gamma = \mathbb{K}[f_1,\cdots,f_N]$ as sets.*

PROOF. This is a standard result of Invariant Theory. The proof is given for self-completeness and lack of a concise reference for this result.

Define the average operator

$$P_\Gamma : \mathbb{K}[x_1,\dots,x_n] \to \mathbb{K}[x_1,\dots,x_n]$$

$$f = f(x_1,\dots,x_n) \mapsto P_\Gamma(f) = (P_\Gamma(f))(x_1,\dots,x_n) = \tfrac{1}{|\Gamma|}\textstyle\sum_{A\in\Gamma} f(Ax)$$

where in the expression Ax, x stands for $(x_1,\dots,x_n)$ in column form, as usual in matrix multiplication. It is easy to see that P_Γ is a projection onto the subspace of Γ-invariants $\mathbb{K}[x_1,\dots,x_n]^\Gamma \subset \mathbb{C}[x_1,\dots,x_n]$.

We claim the following: $\mathbb{K}[x_1,\dots,x_n]^\Gamma = \mathbb{K}[P_\Gamma(x^\beta) : |\beta| \le |\Gamma|]$, where $\beta = (\beta_1,\dots\beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ is its degree, and $|\Gamma|$ is the order of Γ.

Once the claim is proved, just note that there are a finite number of multi-indices β satisfying the above condition, so we are done. So let us prove the claim. Consider $f = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x_1,\dots,x_n]^\Gamma$. We have $f = P_\Gamma(f) = \sum_\alpha a_\alpha P_\Gamma(x^\alpha)$, so it is enough to see that for every monomial x^α, its projection $P_\Gamma(x^\alpha) \in \mathbb{C}[P_\Gamma(x^\beta) : |\beta| \le |\Gamma|]$.

The trick is to introduce new variables $u_1,\dots,u_n$, and consider the l-th power for an arbitrary integer $l \ge 1$

$$\sum_{A\in\Gamma}(u\cdot Ax)^l = \sum_{|\alpha|=l} c_\alpha u^\alpha \sum_{A\in\Gamma}(Ax)^\alpha = \sum_{|\alpha|=l} |\Gamma| c_\alpha u^\alpha P_\Gamma(x^\alpha) \tag{12}$$

where c_α are the binomial coefficients for multi-indices.

On the other hand, call $v_A = u\cdot Ax$, and put $\Gamma = \{A_j\}_{j=1}^m$, with $m = |\Gamma|$. Denote $v_A = v_j$ if $A = A_j \in \Gamma$.

Then the left hand side of (12) becomes $s_l(v) = s_l(v_1,\dots,v_m) = \sum_{j=1}^m v_j^l$. Now, by Lemma 3.22, any symmetric polynomial in the variables $v_1,\dots,v_m$ can be expressed as a polynomial in the polinomials $s_1,\dots,s_m$, therefore $s_l(v) = F(s_1(v),\dots,s_m(v))$ for some polynomial F in m variables.

The computation above is valid for $l \in \mathbb{N}$ arbitrary, hence $s_j(v) = \sum_{|\alpha|=j} |\Gamma| c_\alpha u^\alpha P_\Gamma(x^\alpha)$. The conclusion is that for $l \geq 1$ and $m = |\Gamma|$,

$$\sum_{|\alpha|=l} |\Gamma| c_\alpha u^\alpha P_\Gamma(x^\alpha) = \sum_{A \in \Gamma} (u \cdot Ax)^l = F\left(\sum_{|\alpha|=1} |\Gamma| c_\alpha u^\alpha P_\Gamma(x^\alpha), \ldots, \sum_{|\alpha|=m} |\Gamma| c_\alpha u^\alpha P_\Gamma(x^\alpha) \right)$$

so if we equate coefficients of the variables u^α of both sides we get the claim. □

LEMMA 3.24. *Let Γ be a finite subgroup of $\mathrm{GL}(n, \mathbb{K})$. Suppose that for two points $y, z \in \mathbb{K}^n$ we have $f(y) = f(z)$ for all $f \in \mathbb{K}[x_1, \ldots, x_n]^\Gamma$. Then there exists $\gamma \in \Gamma$ so that $\gamma y = z$, i.e. $[z] = [y]$ in $\mathbb{K}^n/\Gamma$.*

PROOF. Suppose $[z] \neq [y]$. Then the orbits Γz and Γy are disjoint. Let us write the following

(1) $\Gamma = \{\mathrm{Id} = \gamma_1, \gamma_2, \ldots, \gamma_m\}$, where $m = |\Gamma|$.
(2) $\Gamma y = \{\gamma y : \gamma \in \Gamma\} = \{y_1 = y, \ldots, y_l\}$, where $l = \frac{m}{|\Gamma_y|}$.
(3) $\Gamma z = \{\gamma z : \gamma \in \Gamma\} = \{z_1 = z, \ldots, z_r\}$, where $r = \frac{m}{|\Gamma_z|}$.

where Γ_y and Γ_z are the isotropy groups of y and z respectively. Take a function $f \in \mathbb{K}[x_1, \ldots, x_n]$ such that $f(y_i) = 0$ for all $1 \leq i \leq l$ and $f(z_j) = 0$ for all $2 \leq j \leq r$, and $f(z) \neq 0$.

Consider the average $P_\Gamma(f) \in \mathbb{K}[x_1, \ldots, x_n]^\Gamma$. We have that

$$P_\Gamma(f)(z) = \frac{1}{m} \sum_{\gamma \in \Gamma} f(\gamma z) = \frac{|\Gamma_z|}{m} f(z) = \frac{1}{r} f(z) \neq 0$$

whereas $P_\Gamma(f)(y) = 0$, proving the result. □

Now we are ready to prove the embedding of $F = \mathbb{C}^k/\Gamma$ into $\mathbb{C}^N$.

PROPOSITION 3.25. *The fiber $F = \mathbb{C}^k/\Gamma$ is an affine variety. Therefore, its constructive resolution $\widetilde{F}$ is a quasi-projective variety.*

PROOF. Let us construct an embeding $\imath : F \to \mathbb{C}^N$ for some $N \in \mathbb{N}$ large enough. Indeed by Lemma 3.23, $\mathbb{C}[x_1, \ldots, x_k]^\Gamma \subset \mathbb{C}[x_1, \ldots, x_k]$ is a finitely generated $\mathbb{C}$-algebra, say $\mathbb{C}[x_1, \ldots, x_k]^\Gamma = \mathbb{C}[f_1, \ldots, f_N]$ for some polynomials $f_j \in \mathbb{C}[x_1, \ldots, x_k]$, $1 \leq j \leq N$.

Let us define $\imath : \mathbb{C}^k/\Gamma \to \mathbb{C}^N$, $\imath([(x_1, \ldots, x_k)]) = (f_1(x), \ldots, f_N(x))$. We claim that ι is an embedding. Indeed, if $\iota([z]) = \iota([y])$, then we see that $f_j(z) = f_j(y)$ for all $1 \leq j \leq N$. Since the f_j generate the Γ-invariant functions, this implies that $f(z) = f(y)$ for all $f \in \mathbb{C}[x_1, \ldots, x_k]^\Gamma$. By Lemma 3.24, there exists $\gamma \in \Gamma$ so that $z = \gamma y$, hence $[z] = [y] \in Z$ and ι is injective.

Moreover, the image of ι equals $\varphi(\mathbb{C}^k)$, being $\varphi : \mathbb{C}^k \to \mathbb{C}^N, x \mapsto (f_1(x), \ldots, f_N(x))$. Clearly φ is an algebraic morphism, so $\varphi(\mathbb{C}^k) = Z \subset \mathbb{C}^N$ is an affine variety as desired. Finally, by the universal property of the quotient, ι is an homeomorphism from F to its image Z. □

This proves that F can be thought of as an affine variety, hence it is quasi-projective. We can use the model $\imath(F) \subset \mathbb{C}^N$ to perform the resolution of singularities. The resolution $\widetilde{F}$ of $\imath(F)$ is obtained via a finite number of blow-ups starting from $\mathbb{C}^N$ so $\widetilde{F}$ is also quasi-projective. We have an algebraic morphism $b : \widetilde{F} \to F$ which is bijective in $\widetilde{F} \setminus Z$, being $Z = \beta^{-1}([0])$ the exceptional locus.

Let $L = N_{\mathrm{GL}(k,\mathbb{C})}\Gamma$, $G = N_{\mathrm{U}(k)}\Gamma$ as before.

PROPOSITION 3.26. *The action*

$$\begin{aligned} \varphi : L \times F &\to F \\ (A, y) &\mapsto Ay \end{aligned}$$

is algebraic. In particular, for each matrix $A \in L$ the map

$$\begin{aligned}\varphi_A : F &\to F\\ y &\mapsto Ay\end{aligned}$$

is a biholomorphism.

PROOF. It follows immediatly from the fact that the action in $F = \mathbb{C}^k/\Gamma$ is induced from matrix multiplication by $L < \mathrm{GL}(k, \mathbb{C})$ in $\mathbb{C}^k$. □

By Theorem 3.19, this action lifts to the constructive resolution to give an algebraic action $\tilde{\varphi} : L \times \widetilde{F} \to \widetilde{F}$ so there is a well-defined map

$$\begin{aligned}(\mathrm{Id}, \tilde{\varphi}) : L \times \widetilde{F} &\to L \times \widetilde{F}\\ (A, y) &\mapsto (A, Ay)\end{aligned}$$

The map $(\mathrm{Id}, \tilde{\varphi})$ is a bijection between smooth algebraic varieties, and it is algebraic on the Zariski dense open subset $(L \times \widetilde{F}) \setminus (L \times Z)$, where Z is the exceptional locus. So it is an algebraic automorphism of $L \times \widetilde{F}$. This implies that the map $L \to \mathrm{Aut}(\widetilde{F})$ corresponding to the action $\tilde{\varphi}$ is algebraic, where we denote $\mathrm{Aut}(\widetilde{F})$ the algebraic automorphisms of $\widetilde{F}$, i.e. the biholomorphisms of $\tilde{F}$.

Consider the restriction of the map $L \to \mathrm{Aut}(\widetilde{F})$ to G, given by

$$\Lambda : G \to \mathrm{Aut}(\widetilde{F}).$$

Recall that $G < L$ is a Lie subgroup so it is a submanifold. Since Λ is induced by restriction of an algebraic action, Λ is a smooth map.

Now let X be an orbifold, and let $V \subset X$ an open set covered by a chart (U, V, ϕ, Γ). If we have a smooth map $A : U \to G$, $x \mapsto A(x)$, then the induced map $\Lambda \circ A : U \to \mathrm{Aut}(\widetilde{F})$ is smooth, therefore the map

$$\begin{aligned}U \times \tilde{F} &\to \tilde{F}\\ (x, y) &\mapsto A(x)y\end{aligned}$$

is smooth by the previous discussion. Now we can take as the map A any transition function of the normal bundle ν_D of $D \subset X$ an HI-isotropy submanifold. We conclude the following.

COROLLARY 3.27. *Let $\Gamma < \mathrm{U}(k)$ be a finite subgroup of the unitary group. Let $L = N_{\mathrm{GL}(k,\mathbb{C})}(\Gamma)$, and $G = N_{\mathrm{U}(k)}(\Gamma) < L$. Consider the space $F = \mathbb{C}^k/\Gamma$ with the structure of an algebraic variety inherited from Proposition 3.25. Let $\widetilde{F}$ be its constructive resolution and $b : \widetilde{F} \to F$ be the resolution map.*

Then we have algebraic actions

$$\begin{aligned}\varphi : L \times F \to F, &\quad (A, [u]) \mapsto A \star [u] = [Au]\\ \tilde{\varphi} : L \times \widetilde{F} \to \widetilde{F}, &\quad (A, y) \mapsto A \cdot y\end{aligned}$$

so that $b(A \cdot y) = A \star b(y)$ for all $y \in \widetilde{F}$, $A \in L$. (i.e. the actions commute with the resolution map).

Moreover, when restricted to $G \times F \subset L \times F$, the actions φ and $\tilde{\varphi}$ give smooth actions of the group G in F and $\widetilde{F}$ respectively (which also commute with b).

Finally, the action of G in F given by φ coincides with the action of the structure group of ν_D given in Proposition 3.12.

In particular, note the following. Let $b : \widetilde{F} \to F$ be the blow-up map, and denote by $Z = b^{-1}([0])$ the exceptional divisor. For the bundle ν_D, each transition matrix $A_{\alpha\beta}(x) \in G < \mathrm{U}(k)$ has a corresponding unique lifting

$$B_{\alpha\beta}(x) : \widetilde{F} \to \widetilde{F}$$

with $B_{\alpha\beta}(x) \in \mathrm{Aut}(\widetilde{F})$, and satisfying $b(B_{\alpha\beta}(x)y) = A_{\alpha\beta}(x)(b(y))$, for each $y \in \widetilde{F}$, i.e.

$$b \circ B_{\alpha\beta}(x) = A_{\alpha\beta}(x) \circ b : \widetilde{F} \to F.$$

The maps $B_{\alpha\beta}(x)$ depend smoothly on x, as mentioned in Corollary 3.27.

PROPOSITION 3.28. *The maps $B_{\alpha\beta}(x) : \widetilde{F} \to \widetilde{F}$ for $x \in U_\alpha \cap U_\beta$ are the transition functions of a smooth fiber bundle $\widetilde{\nu}_D$ over D with $\widetilde{F}$ as fiber, i.e.*

$$\tilde{F} \to \widetilde{\nu}_D \to D.$$

Moreover, for each $x \in U_\alpha \cap U_\beta$ the map $B_{\alpha\beta}(x)$ is a biholomorphism of $\widetilde{F}$.

PROOF. We only need to check the cocycle condition. In a triple intersection we know that $A_{\alpha\beta} \circ A_{\delta\alpha} \circ A_{\beta\delta} = \mathrm{Id}_F$. Since lifting respects composition and the identity lifts to the identity, we have that $B_{\alpha\beta} \circ B_{\delta\alpha} \circ B_{\beta\delta} = \mathrm{Id}_{\widetilde{F}}$, as required.

The fact that $B_{\alpha\beta}(x) : \widetilde{F} \to \widetilde{F}$ is a biholomorphism follows from the fact that $A_{\alpha\beta}(x) : F \to F$ is a biholomorphism (since it acts by matrix multiplication and $A_{\alpha\beta}(x) \in \mathrm{U}(k)$, recall Proposition 3.26). From the properties of the constructive resolution of Theorem 3.19, the lifting $B_{\alpha\beta}(x)$ is also a biholomorphism of $\widetilde{F}$. □

Therefore we have finally constructed our fiber bundle $\widetilde{F} \to \widetilde{\nu}_D \to D$ such that each fiber of $\widetilde{\nu}_D$ is the constructive resolution of the corresponding fiber $F = \mathbb{C}^k/\Gamma$ of ν_D.

We call $b : \widetilde{\nu}_D \to \nu_D$ the blow-up map (or resolution map), because it is induced on each fiber by the blow-up map $b : \widetilde{F} \to F$. Note that $b : \widetilde{\nu}_D \to \nu_D$ is a diffeomorphism outside the subbundle $Z \to E \to D$ whose fiber is the exceptional locus $Z \subset \widetilde{F}$. We call $E \subset \widetilde{\nu}_D$ the *exceptional subbundle* of $\widetilde{\nu}_D$.

3. Symplectic form on the resolution of the fiber.

In this section our aim is to construct a symplectic form in the resolution $\widetilde{\nu}_D$ of the normal bundle ν_D.

The first step towards this end consists on constructing a symplectic form on the resolution $\widetilde{F}$ of the complex variety $F = \mathbb{C}^k/\Gamma$, with $\Gamma < \mathrm{U}(k)$ as above. Here, $F \cong F_x$ is diffeomorphic to the orbifold normal space $T_xD^{\perp\omega}/\Gamma$ of the HI-submanifold $D \subset X$. Since D does not intersect any other isotropy submanifold of the orbifold X, we see that $0 \in \mathbb{C}^k$ is the only fixed point of the action of the group $\Gamma < \mathrm{U}(k)$. Hence the singular locus of F reduces to the point $[0] \in F = \mathbb{C}^k/\Gamma$. The exceptional locus is $Z = b^{-1}([0]) \subset \widetilde{F}$, and consists of a finite union of irreducible components Z_j which are divisors intersecting transversally.

PROPOSITION 3.29. *The resolution $\widetilde{F}$ of $F = \mathbb{C}^k/\Gamma$ admits a Kahler structure $(\omega_{\widetilde{F}}, J_{\widetilde{F}}, g_{\widetilde{F}})$ which is invariant by the action of $G = N_{\mathrm{U}(k)}(\Gamma)$ on $\widetilde{F}$.*

PROOF. By Proposition 3.25, $\widetilde{F}$ is a quasi-projective variety, so it is a complex submanifold of some $\mathbb{CP}^N$ for N high enough. Consider $(\mathbb{CP}^N, \omega_{FS}, J, g_{FS})$ the standard Kahler structure on $\mathbb{CP}^N$, where ω_{FS} is the Fubini-Study Kahler form. The restriction of (ω_{FS}, J, g_{FS}) to $\widetilde{F}$ defines a Kahler structure $(\omega_1, J_{\widetilde{F}}, g_1)$ on $\widetilde{F}$, where $J_{\widetilde{F}}$ is the given complex structure on $\widetilde{F}$.

The complex structure $J_{\widetilde{F}}$ is preserved by the transition functions $B_{\alpha\beta}(x)$ because they act on $\widetilde{F}$ as biholomorphisms. But the symplectic structure ω_1 may not be preserved, so we need to make an average. As G is compact, we put on G any right-invariant Riemannian metric and call μ the measure induced by this metric. Let

$$\omega_{\widetilde{F}} = \frac{1}{\mu(G)} \int_G h^* \omega_1 d\mu(h) \in \Omega^2(\widetilde{F}).$$

We claim that $\omega_{\widetilde{F}}$ is a symplectic form invariant by the action of G on $\widetilde{F}$. The invariance follows easily by the usual arguments of averages. Indeed, take $g \in G$ and compute

$$\begin{aligned} g^* \omega_{\widetilde{F}} &= \frac{1}{\mu(G)} \int_G g^*(h^* \omega_1) d\mu(h) \\ &= \frac{1}{\mu(G)} \int_G (hg)^*(\omega_1) d\mu(h) = \frac{1}{\mu(G)} \int_G k^* \omega_1 d\mu(k) = \omega_{\widetilde{F}}, \end{aligned}$$

where we have made the change of variables $hg = k$, and $d\mu(h) = d\mu(k)$ since translations are isometries. The closeness is clear as

$$d\omega_{\widetilde{F}} = \frac{1}{\mu(G)} \int_G d(h^* \omega_1) d\mu(h) = \frac{1}{\mu(G)} \int_G (h^* d\omega_1) d\mu(h) = 0$$

because ω_1 is closed.

Finally, let us see that $\omega_{\widetilde{F}}$ is a Kahler form. As $\omega_1(u, v) = g_1(u, -Jv)$, we have that $h^* \omega_1(u, v) = h^* g_1(u, -Jv)$. It follows that

$$\omega_{\widetilde{F}}(u, v) = \frac{1}{\mu(G)} \int_G h^* g_1(u, -Jv) d\mu(h) = g_{\widetilde{F}}(u, -Jv)$$

where

$$g_{\widetilde{F}} = \frac{1}{\mu(G)} \int_G h^* g_1 d\mu(h)$$

is a G-invariant Riemannian metric, since the set of Riemannian metrics on a manifold is a cone, therefore closed by averages. Moreover $g_{\widetilde{F}}(Ju, Jv) = g_{\widetilde{F}}(u, v)$. This gives a Kahler structure $(\omega_{\widetilde{F}}, J_{\widetilde{F}}, g_{\widetilde{F}})$ on $\widetilde{F}$ which is invariant by the action of the group G, as desired. Note that we have not changed $J_{\widetilde{F}}$, so it is the complex structure inherited from $\mathbb{CP}^N$. □

Let $b : \widetilde{F} \to F$ be blow-up map, $Z = b^{-1}([0]) \subset \widetilde{F}$ the exceptional divisor. So $b : \widetilde{F} \setminus Z \to F \setminus \{[0]\}$ is a biholomorphism. We now modify the Kahler form on $\widetilde{F}$ by a cut and paste process so as to make it agree with the original Kahler form ω_F on F, in the complement of a neighbourhood of Z.

Proposition 3.30. *The resolution $\widetilde{F}$ admits a symplectic form $\Omega_{\widetilde{F}}$ which satisfies:*

- *It coincides with the form $b^*(\omega_F)$ outside an arbritarily small neighborhood U^Z of $Z \subset \widetilde{F}$.*
- *It coincides with the Kahler form $\omega_{\widetilde{F}}$ in some neighborhood V^Z of Z so that $V^Z \subset U^Z$.*
- *It is invariant by the action of the group $G = N_{\mathrm{U}(k)}(\Gamma)$ in $\widetilde{F}$.*

Proof. For any choice of $\varepsilon > 0$ we consider $U = U^Z = b^{-1}(B_{4\varepsilon}(0)/\Gamma)$ and $V = V^Z = b^{-1}(B_\varepsilon(0)/\Gamma)$, where the balls are taken with respect to the metric g_F on $\mathbb{C}^k$. Consider also $W = b^{-1}(\{\frac{1}{2}\varepsilon < |z| < \frac{9}{2}\varepsilon\}/\Gamma)$ so that $U^Z \setminus V^Z \subset \overline{W}$.

As the map $b : \widetilde{F} \to F$ is a diffeomorphism outside $Z \subset \widetilde{F}$, we see that W is homotopy equivalent to a lens-space $S^{2k-1}/\Gamma \subset F = \mathbb{C}^k/\Gamma$. In particular $H^2(W, \mathbb{R}) = 0$, so we have

$$\omega_{\widetilde{F}} - b^*(\omega_F) = d\eta, \tag{13}$$

for some $\eta \in \Omega^1(W)$. Take $\rho : \widetilde{F} \to \mathbb{R}$ a bump-function so that $\rho = 1$ on V and $\rho = 0$ on $\widetilde{F} \setminus U$. Define the form

$$\Omega_\delta = b^*(\omega_F) + \delta\, d(\rho\eta),$$

for $\delta > 0$. Note that $\Omega_\delta = (1-\delta)b^*(\omega_F) + \delta\,\omega_{\widetilde{F}}$ in $W \cap V$, and $\Omega_\delta = b^*(\omega_F)$ in $W \setminus U$. This shows that Ω_δ can be extended to all $\widetilde{F}$ so that Ω_δ equals $(1-\delta)b^*(\omega_F) + \delta\,\omega_{\widetilde{F}}$ on V and $b^*(\omega_F)$ on $\widetilde{F} \setminus U$. Moreover Ω_δ is obviously closed.

We need to show that Ω_δ is non-degenerate for an adequate choice of δ. We already know that Ω_δ is non-degenerate except for the set $U \setminus V$, on which Ω_δ has the form $\Omega_\delta = b^*(\omega_F) + \delta\, d(\rho\eta)$. Since $b^*(\omega_F)$ is non-degenerate on $U \setminus V$, by choosing δ small enough the form Ω_δ will be non-degenerate on $U \setminus V$. Note that $\Omega_\delta = (1-\delta)b^*(\omega_F) + \delta\,\omega_{\widetilde{F}}$ on V, both (F, g_F, J_F, ω_F) and $(\widetilde{F}, g_{\widetilde{F}}, J_{\widetilde{F}}, \omega_{\widetilde{F}})$ are Kahler, and b is a biholomorphism. From this we see that for a tangent vector u at a point in V we have $\Omega_\delta(u, -J_{\widetilde{F}}u) = (1-\delta)(b^*g_F)(u,u) + \delta g_{\widetilde{F}}(u,u) > 0$ if $u \neq 0$. This also shows that Ω_δ is $J_{\widetilde{F}}$ -compatible in V, hence $(V, \Omega_\delta, J_{\widetilde{F}}, (1-\delta)b^*g_F + \delta g_{\widetilde{F}})$ defines a Kahler structure on V.

It remains to see the invariance of the symplectic form

$$\Omega_\delta = b^*(\omega_F) + \delta\, d(\rho\eta),$$

under the action of group $G = N_{\mathrm{U}(k)}(\Gamma)$. Recall that the average over the compact Lie group G of a form α is given by $\frac{1}{\mu(\alpha)}\int_G h^*\alpha d\mu(h)$, where μ is the measure induced by any right-invariant metric on G. The average operator is a linear projection onto the vector subspace of G-invariant forms. The form η can be chosen to be G-invariant by averaging over G in the equation (13). The bump function ρ can also be chosen G-invariant. It suffices to take $\rho = b^*\rho_0$ with ρ_0 a bump-function on $\mathbb{C}^k$ which is radial with respect to the metric g_F. Indeed, since G acts on F by unitary matrices and ρ_0 is radial, ρ_0 is a G-invariant function on F. Since the actions of G on $\widetilde{F}$ and F commute with b, for $h \in G$ we have $h^*b^*\rho_0 = (b \circ h)^*\rho_0 = b^*\rho_0$, proving the invariance of $\rho = b^*\rho_0$.

The conclusion is that Ω_δ is also G-invariant for these choices of ρ and η, and we are done. □

The proposition above shows that we can construct a symplectic form $\Omega_{\widetilde{F}}$ on the fiber $\widetilde{F}$ of $\widetilde{\nu}_D$ which is preserved by the transition functions $B_{\alpha\beta}(x)$ of the bundle $\widetilde{\nu}_D$. This gives the bundle $\widetilde{\nu}_D$ the structure of a fiber bundle with symplectic fibers, also called a symplectic fiber bundle. But we want an honest symplectic form defined in $\widetilde{\nu}_D$, both in vertical and horizontal directions.

Note that we have a blow-up map

$$b : \widetilde{\nu}_D \to \nu_D,$$

such that $b^{-1}(F_x) = \widetilde{F}_x$ for all $x \in D$. Let 0_x be the origin of the fiber $F_x \cong \mathbb{C}^k/\Gamma$, and let $Z_x = b^{-1}(0_x) \cong Z$ the exceptional divisor. The base space of the bundle ν_D is identified with $D \subset \nu_D$ the zero section, and $E = b^{-1}(D)$ is the subbundle whose fibers are the exceptional locus of the resolution $\tilde{F}_x \to F_x$.

Definition 3.31. *Let us call E the* exceptional subbundle *of $\widetilde{\nu}_D$.*

Note that $b : E \to D$ is a fibre bundle whose fiber is Z_x at every $x \in D$. In short notation we will write $Z \to E \to D$.

As we will see, it is not be necessary to construct a symplectic form on all of $\widetilde{\nu}_D$, but it will be enough for this symplectic form to be defined in some open set of $\widetilde{\nu}_D$ containing the exceptional subbundle E. So now our objective will be to obtain a symplectic form in some small neighborhood of the exceptional subbundle E of $\widetilde{\nu}_D$. This will take us some time because first of all we have to overcome some topological obstructions.

4. Symplectic form in the resolution of the normal bundle.

Let us begin with an example showing that the existence of the required symplectic form is not automatic and it is in general obstructed by topology.

REMARK 3.32. *The following is a simple example of a bundle with symplectic fibers over a symplectic base space which does not admit a symplectic form defined on the total space of the bundle. More about these topological obstructions can be found* [**26**].

Consider the Hopf fibration $S^1 \to S^3 \to S^2$ and multiply by S^1 to get a bundle

$$S^1 \times S^1 \to S^3 \times S^1 \to S^2.$$

Both base and fiber are symplectic, however the total space $S^3 \times S^1$ is compact and has trivial second cohomology so it does not admit any symplectic form.

4.1. Cohomological Obstruction. The first thing that we need is to find a cohomology class $[\eta]$ on the manifold $\widetilde{\nu}_D$ that restricts to the cohomology class $[\Omega_{\widetilde{F}}]$ of the symplectic form of the fiber $\widetilde{F}$, constructed in Proposition 3.30. If we do this, the cohomological obstructions of Remark 3.32 vanish and we will be able to construct a symplectic form on an open set $\mathcal{U}$ of $\widetilde{\nu}_D$ containing the exceptional subbundle E (cf. Proposition 3.38).

PROPOSITION 3.33. *The homology $H_{2k-2}(\widetilde{F})$ of $\widetilde{F}$ is freely generated by the exceptional divisors Z_j, $j = 1, \dots, l$ (the irreducible components of $Z \subset \widetilde{F}$). In other words $H_{2k-2}(\widetilde{F}) = \bigoplus_{j=1}^{l} \mathbb{Z}\langle Z_j\rangle$.*

PROOF. The exceptional locus Z of the constructive resolution of singularities of [**20**] is a tree of exceptional divisors Z_j with normal crossings. These are smooth complex submanifolds of dimension $k-1$, hence $(2k-2)$-dimensional smooth real manifolds, so $H_{2k-2}(Z_j) = \mathbb{Z}\langle Z_j\rangle$. Now, $Z_i \cap Z_j$ for $i \neq j$ is of complex dimension $\leq (k-2)$, hence of real dimension $\leq (2k-4)$. So

$$\begin{aligned} H_{2k-2}(Z) = H_{2k-2}\left(Z/(\cup_{i\neq j}(Z_i \cap Z_j))\right) &= H_{2k-2}\left(\bigvee\nolimits_{j=1}^{l} Z_j/(\cup_{i\neq j}(Z_i \cap Z_j))\right) \\ \cong \bigoplus_{j=1}^{l} H_{2k-2}\left(Z_j/(\cup_{i\neq j}(Z_i \cap Z_j))\right) &= \bigoplus_{j=1}^{l} H_{2k-2}(Z_j) = \bigoplus_{j=1}^{l} \mathbb{Z}\langle Z_j\rangle. \end{aligned}$$

There is a deformation retraction from $\widetilde{F}$ to Z induced by lifting the radial vector field $r\frac{\partial}{\partial r}$ from $F = \mathbb{C}^k/\Gamma$ to $b : \widetilde{F} \to F$. Therefore $H_{2k-2}(\widetilde{F}) = H_{2k-2}(Z) = \bigoplus_{j=1}^{l} \mathbb{Z}\langle Z_j\rangle$, as required. □

REMARK 3.34. (1) *Proposition 3.33 above implies that in the bundle $\widetilde{F} \to \widetilde{\nu}_D \to D$ there is a canonical unordered basis for $H_{2k-2}(\widetilde{F})$ at the level of chains, namely the set of exceptional divisors. Note that for each ordering of the exceptional divisors Z_j, we have a basis of $H_{2k-2}(\widetilde{F})$, but the transition functions $B_{\alpha\beta}(x) : \widetilde{F} \to \widetilde{F}$ induce a permutation on this basis, so it is the (unordered) set $\{Z_1, \dots, Z_l\}$ what is preserved. This property of the bundle $\widetilde{\nu}_D$ will be crucial to construct a symplectic form on the total space $\widetilde{\nu}_D$.*

(2) *Poincaré duality for $\widetilde{F}$ gives an isomorphism*

$$PD : H^2_c(\widetilde{F}, \mathbb{R}) \xrightarrow{\cong} H_{2k-2}(F, \mathbb{R}).$$

We claim in addition that $H^2_c(\widetilde{F}, \mathbb{R}) \cong H^2(\widetilde{F}, \mathbb{R})$. To see it, consider the radial function $r : \widetilde{F} \to [0, \infty)$, and introduce the sets $A_R = \{y \in \widetilde{F} : r(y) \leq R\} \subset \widetilde{F}$, for each $R > 0$. Then

$$H^2_c(\widetilde{F}, \mathbb{R}) \cong H^2_c(A_R, \mathbb{R}) \cong H^2(\overline{A}_R, \partial A_R, \mathbb{R}) \cong H^2(\overline{A}_R, \mathbb{R}) \cong H^2(\widetilde{F}, \mathbb{R}),$$

since $\partial A_R \cong S^{2k-1}/\Gamma$ has $H^2(\partial A_R, \mathbb{R}) = 0$.

Now we move on to the final step, namely constructing a global symplectic form on a neighborhood $\mathcal{U} \subset \widetilde{\nu}_D$ of the exceptional subbundle $E \subset \widetilde{\nu}_D$. This symplectic form is also required to coincide with $\Omega_{\widetilde{F}}$ restricted to every fiber.

Note that this construction will provide a symplectic form on a sufficiently small neighbourhood $\mathcal{U}$ of the exceptional locus $E \subset \widetilde{\nu}_D$, and not on all of $\widetilde{\nu}_D$. First we deal with the cohomological obstruction mentioned in Remark 3.32.

PROPOSITION 3.35. *Let $\widetilde{F} \to \widetilde{\nu}_D \to D$ be as before, with $(\widetilde{F}, \Omega_{\widetilde{F}})$ the symplectic structure on $\widetilde{F}$ constructed in Proposition 3.30. There exists a cohomology class $a \in H^2(\widetilde{\nu}_D, \mathbb{R})$ whose restriction to each fiber is $[\Omega_{\widetilde{F}}]$.*

PROOF. Consider the atlas of the bundle $\widetilde{\nu}_D$ consisting of charts $\phi_\alpha : U_\alpha \times \widetilde{F} \to V_\alpha \subset \widetilde{\nu}_D$, $U_\alpha \subset D$, and with changes of trivializations $B_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Sympl}(\widetilde{F}, \Omega_{\widetilde{F}})$. We refine the open cover given by the $U_\alpha \subset D$ in such a way that there exists a smooth map $T_\alpha : [0,1]^{2n-2k} \to U_\alpha$ with image $Q_\alpha \subset U_\alpha$, so that the simplices Q_α form a triangulation of D. As D is compact and symplectic, it is an oriented manifold of dimension $2n-2k$. Let $[D] \in H_{2n-2k}(D)$ denote its fundamental class, represented by the chain $\sum_\alpha Q_\alpha \in C_{2n-2k}(D)$.

On the other hand, consider the cohomology class $[\Omega_{\widetilde{F}}] \in H^2(\widetilde{F}, \mathbb{R})$. By Poincaré duality and Remark 3.34, we have $H^2(\widetilde{F}, \mathbb{R}) \cong H^2_c(\widetilde{F}, \mathbb{R}) \cong H_{2k-2}(\widetilde{F}, \mathbb{R})$. Choose a basis $\{Z_1, \dots, Z_l\}$ of exceptional divisors of $H_{2k-2}(\widetilde{F})$. There exists unique real numbers $a_i \in \mathbb{R}$ so that

$$\mathrm{PD}[\Omega_{\widetilde{F}}] = \sum_{i=1}^{l} a_i [Z_i].$$

For each trivialization $\phi_\alpha : U_\alpha \times \widetilde{F} \to V_\alpha \subset \widetilde{\nu}_D$, consider the chain

$$A_\alpha = \sum_{i=1}^{l} a_i \phi_\alpha (Q_\alpha \times Z_i) \in C_{2n-2}(\widetilde{\nu}_D).$$

We claim that the chain $A = \sum_\alpha A_\alpha$ is closed, so it defines a homology class $[A] \in H_{2n-2}(\widetilde{\nu}_D)$. Certainly,

$$\partial A = \sum \partial A_\alpha = \sum_\alpha \sum_i a_i \phi_\alpha (\partial Q_\alpha \times Z_i). \tag{14}$$

If $x \in \partial Q_\alpha \cap \partial Q_\beta \subset U_\alpha \cap U_\beta$, the transition function $g = B_{\alpha\beta}(x) : \widetilde{F} \to \widetilde{F}$ is a symplectomorphism of $(\widetilde{F}, \Omega_{\widetilde{F}})$, hence it preserves the homology class $\mathrm{PD}([\Omega_{\widetilde{F}}]) = \sum_{i=1}^{l} a_i [Z_i]$. On the other hand, g permutes the exceptional divisors Z_i. But if $g(Z_{i_1}) = Z_{i_2}$ then the corresponding coefficients in $[\Omega_{\widetilde{F}}]$ are the same, i.e. $a_{i_1} = a_{i_2}$. This follows from the equality

$$\mathrm{PD}[\Omega_{\widetilde{F}}] = \sum_{i=1}^{l} a_i [Z_i] = (g)_* (\mathrm{PD}[\Omega_{\widetilde{F}}]) = \sum_{i=1}^{l} a_i [g(Z_i)]$$

by looking on both sides at the coefficient of $[Z_{i_2}]$. Therefore, if $g(Z_{i_1}) = Z_{i_2}$ then

$$a_{i_1} \phi_\alpha (T \times Z_{i_1}) + a_{i_2} \phi_\beta (T \times Z_{i_2}) = 0 \in C_{2n-3}(\widetilde{\nu}_D), \tag{15}$$

where $T \subset \partial Q_\alpha \cap \partial Q_\beta$ is a $(2n-3)$-simplex that is common to the boundary of both Q_α and Q_β. Note that we are taking into account that the orientations of T induced by Q_α and Q_β are opposite. Plugging (15) into (14), we get that $\partial A = 0$.

Hence $A \in H_{2n-2}(\widetilde{\nu}_D)$ determines via Poincaré duality a unique $a = [\eta] \in H^2(\widetilde{\nu}_D, \mathbb{R})$ so that $\mathrm{PD}(a) = A$. The relation between $a = [\eta]$ and A is given by the equality $\int_{\widetilde{\nu}_D} \eta \wedge \beta = \int_A \beta$, for all $[\beta] \in H^{2n-2}(\widetilde{\nu}_D)$. To see that the cohomology class $[\eta]$ restricts to $[\Omega_{\widetilde{F}}]$ over each fiber $\widetilde{F}$, we need to check that

$$\int_{\widetilde{F}} \eta \wedge \gamma = \int_{\widetilde{F}} \Omega_{\widetilde{F}} \wedge \gamma$$

for all $[\gamma] \in H^{2k-2}(\widetilde{F})$. For this, take any $x \in D$ with fiber $\widetilde{F}_x \subset \widetilde{\nu}_D$, and some Q_α containing x. Take any $[\gamma] \in H^{2k-2}(\widetilde{F}_x)$. Consider a bump $2(n-k)$-form $\nu \in \Omega^{2n-2k}(D)$ with support contained in Q_α and $\int_D \nu = 1$. Then $\pi^*\nu$ has support in $Q_\alpha \times \widetilde{F}$ and so

$$\int_{\widetilde{F}_x} \eta \wedge \gamma = \int_{Q_\alpha \times \widetilde{F}_x} \eta \wedge \gamma \wedge \pi^*\nu = \int_{\widetilde{\nu}_D} \eta \wedge \gamma \wedge \pi^*\nu = \int_A \gamma \wedge \pi^*\nu$$
$$= \int_{A \cap (Q_\alpha \times \widetilde{F})} \gamma \wedge \pi^*\nu = \sum_i a_i \int_{Q_\alpha \times Z_i} \gamma \wedge \pi^*\nu = \sum_i a_i \int_{Z_i} \gamma = \int_{\widetilde{F}_x} \Omega_{\widetilde{F}} \wedge \gamma.$$

This completes the proof. □

4.2. Symplectic forms on proper symplectic bundles. In the paper [**51**] it is outlined a construction of a symplectic form on the total space of a fiber bundle with symplectic base and *compact* symplectic fibers, once we know the existence of a cohomology class that restricts to the cohomology class of the sympletic form on the fibers. We have to do a slight extension to a case with non-compact symplectic fiber. We start with a lemma.

LEMMA 3.36. *Let (B, g) be a compact Riemannian manifold and let ω be a symplectic form in B. There exists a constant $m > 0$ which satisfies the following. For each $x \in B$ and $u \in T_xB$, there exists $v \in T_xB$ so that $\omega(u, v) \geq m|u||v|$.*

PROOF. Let $S(TB)$ be the unit sphere bundle of B, and consider the function $s : S(TB) \to \mathbb{R}$ defined by

$$s(x, u) = \max_{v \in S(T_xB)} \omega(u, v)$$

This is a continuous function, which is strictly positive since ω is symplectic. It follows that s attains a minimun m on the compact set $S(TB)$, so for all $x \in B$ and for all $u \in T_xB$ with $|u| = 1$ there exists $v \in T_xB$ with $|v| = 1$ so that $\omega(u, v) \geq m$. This implies the required assertion. □

DEFINITION 3.37. *Let B be a compact manifold, and (N, ω_N) a (possibly non-compact) symplectic manifold with a proper* height *function $H : N \to [0, \infty)$. A* proper symplectic bundle *is a fiber bundle $N \to M \to B$ such that the transition functions take values in* $\mathrm{Sympl}(N, \omega_N, H) = \{f : N \to N : f^*\omega_N = \omega_N, H \circ f = H\}$.

A couple of comments.

1. If $N \to M \to B$ is a proper symplectic bundle, then the height function H defines a smooth proper function $H_M : M \to [0, \infty)$. Note that H_M can be defined in the obvious way taking any trivialization because the transition functions of the bundle M preserve the height function H by hypothesis.
2. For $R > 0$, we introduce the sets $M_R = H_M^{-1}([0, R]) \subset M$ and $N_R = H^{-1}([0, R]) \subset N$. Then N_R and M_R are compact and $N_R \to M_R \to B$ is a fibre bundle. If $R > 0$ is a regular value of H, then (N_R, ω_R) is a symplectic manifold with boundary, so $N_R \to M_R \to B$ is a compact symplectic bundle.

PROPOSITION 3.38. *Let $N \to M \xrightarrow{\pi} B$ be a proper symplectic bundle. Suppose the base space (B, ω_B) is a compact symplectic manifold, and denote (N, ω_N, H) the symplectic structure of the fiber and the height function $H : N \to [0, \infty)$.*

Suppose that there exists a cohomology class $e \in H^2(M, \mathbb{R})$ which restricts to $[\omega_N] \in H^2_{DR}(N)$ on every fiber. Fix $R > 0$. Then there exists a closed 2-form $\omega_M \in \Omega^2(M)$ which is non-degenerate on $M_R \subset M$, so that ω_M restricts to ω_N on every fiber $N_x = \pi^{-1}(x) \subset M$.

In particular (M_R, ω_M) is a symplectic manifold.

PROOF. Take $e = [\eta]$ with $\eta \in \Omega^2(E)$ a representative of the class e. Take U_α a good cover of B so that $\phi_\alpha : U_\alpha \times N \to V_\alpha \subset M$ are trivialisations of the bundle M, and the transition functions

$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Sympl}(N, \omega_N, H)$. On each trivialisation the (locally defined) vertical projection $q_\alpha : U_\alpha \times N \to N$ induces an isomorphism in cohomology, hence $(\phi_\alpha^{-1})^* q_\alpha^* \omega_N - \eta|_{V_\alpha} = d\theta_\alpha$ for some 1-form $\theta_\alpha \in \Omega^1(V_\alpha)$. Take a partition of unity ρ_α subordinated to the open cover U_α of B and define

$$\omega_M = K\pi^*(\omega_B) + \eta + \sum_\alpha d((\pi^*\rho_\alpha)\theta_\alpha), \tag{16}$$

for a real number $K > 0$ to be chosen later. We claim that ω_M is symplectic in $M_R \subset M$ if $K > 0$ is large enough. The form ω_M is clearly closed. We rewrite it as

$$\begin{aligned}\omega_M &= K\pi^*\omega_B + \eta + \sum_\alpha (\pi^* d\rho_\alpha) \wedge \theta_\alpha + \sum_\alpha (\pi^*\rho_\alpha) \wedge ((\phi_\alpha^{-1})^* q_\alpha^* \omega_N - \eta) \\ &= K\pi^*\omega_B + \sum_\alpha (\pi^* d\rho_\alpha) \wedge \theta_\alpha + \sum_\alpha (\pi^*\rho_\alpha)(\phi_\alpha^{-1})^* q_\alpha^* \omega_N \,.\end{aligned}$$

On a fiber $N_x = \pi^{-1}(x)$, we have

$$(\omega_M)|_{N_x} = \sum_\alpha \rho_\alpha(x)(\phi_\alpha^{-1})^* q_\alpha^* \omega_N = \sum_\alpha \rho_\alpha(x)\omega_N = \omega_N,$$

since all $\phi_\alpha : \{x\} \times N \to N_x$ are symplectomorphims. We are using here that the transition functions of the bundle are symplectomorphisms of (N, ω_N).

To see that ω_M is non-degenerate on M_R, take a vector $u \in T_yM$ and let us see that there exists another vector v such that $\omega_M(u, v) \neq 0$. We fix some (any) background metrics on M and D and assume that $|u| = 1$. If $u \in T_y N_{\pi(y)}$ lies in the tangent space to the fiber, then it is clear since $\omega_M|_{N_{\pi(y)}} = \omega_N$ is symplectic. Since being non-degenerate is an open condition, there is an open set $W \subset S(TM_R)$ containing all unitary tangent vectors to fibers $T_yN_{\pi(y)}$, for $y \in M_R$, with the property that ω_M is non-degenerate on W. The set W can be taken of the form

$$W = \{(y, w) \in S(TM_R) : \mathrm{dist}(w, T_yN_{\pi(y)}) < \delta\}$$

for some $\delta > 0$. As M_R is compact, we can take a uniform δ for all points $y \in M_R$.

Take now $u \in S(T_yM_R) \setminus W$. Since u is well away from the tangent spaces to the fibers and $|u| = 1$, there exists a constant $\delta_1 > 0$, valid for all $u \in S(T_yM_R) \setminus W$, so that $|\pi_*(u)| > \delta_1$.

On the other hand, by Lemma 3.36, there exists a constant $m_1 > 0$ (independent of u) and a vector $w \in T_yM_\varepsilon$ (depending on u) so that $\pi^*\omega_B(u, w) \geq m_1|\pi_*(u)||\pi_*(w)|$.

By compactness, there are constants $C_1, C_2 > 0$ so that the map $\pi_* : (T_yN_{\pi(y)})^\perp \to T_{\pi(y)}B$ satisfies that

$$C_1|v| \leq |\pi_*(v)| \leq C_2|v|, \quad \text{for all } v \in (T_yN_{\pi(y)})^\perp \,, \tag{17}$$

and for all $y \in M_R$.

Choosing $w \in (T_yN_{\pi(y)})^\perp$ and unitary (which can be done since $u \notin W$), we have that $|\pi_*(w)| \geq C_1$ and $\pi^*\omega_B(u, w) \geq m_1|\pi_*(u)||\pi_*(w)|$.

Finally, recall the second term μ in (16) with $\omega_M = K\pi^*\omega_B + \mu$, which is given by

$$\mu = \eta + \sum_\alpha d((\pi^*\rho_\alpha)\theta_\alpha)$$

We have that

$$\begin{aligned}\omega_M|_y(u, w) &= K\pi^*\omega_B(u, w) + \mu(u, w) \\ &\geq Km_1|\pi_*(u)||\pi_*(w)| - m_2|u||w| \geq Km_1\delta_1C_1 - m_2,\end{aligned}$$

where m_2 is a constant which bounds μ on M_R. The above constants are valid for all $y \in M_R$ and for all $u \notin W$ with $|u| = 1$. It is enough to take $K \geq \frac{m_2}{m_1\delta_1C_1} + 1$ to get that the form ω_M is non-degenerate on M_R. □

Let us denote $|\cdot|$ the Euclidean metric of $\mathbb{C}^k \cong \mathbb{R}^{2k}$.

Proposition 3.39. *Consider the bundle $\widetilde{F} \to \widetilde{\nu}_D \to D$ with symplectic fiber $(\widetilde{F}, \Omega_{\widetilde{F}})$ and height function $H : \widetilde{F} \to [0, \infty)$ given by $H(y) = |b(y)|$.*

We have that $\widetilde{\nu}_D$ is a proper symplectic bundle.

Proof. First note that H is well defined. This is because the Euclidean metric $|\cdot|$ is well-defined in equivalence classes of $F = \mathbb{C}^k/\Gamma$, since $\Gamma < \mathrm{U}(k) < \mathrm{SO}(2k)$. The function H is clearly proper.

We only have to see that the transition functions of the bundle belong to $\mathrm{Sympl}(\widetilde{F}, \Omega_{\widetilde{F}}, H)$ in the sense of Definition 3.37. Let $B_{\alpha\beta}(x)$ such a transition function. We already know by Proposition 3.29 that $B_{\alpha\beta}(x)$ acts on $\tilde{F}$ by symplectomorphisms, so it only remains to see that the height function H is preserved.

Indeed, we have $H(B_{\alpha\beta}(x)y) = |b(B_{\alpha\beta}(x)y)| = |A_{\alpha\beta}(x)b(y)| = |b(y)| = H(y)$, where we have used that $A_{\alpha\beta}(x) \in \mathrm{U}(k)$ preserves the euclidean metric. We are done. □

Therefore we can apply Proposition 3.38 above to the proper symplectic bundle $\widetilde{\nu}_D$, and we have the following.

Theorem 3.40. *The bundle $\widetilde{F} \to \widetilde{\nu}_D \to D$ admits closed 2-form $\omega_{\widetilde{\nu}_D}$ so that:*

- *The restriction of $\omega_{\widetilde{\nu}_D}$ to each fiber $\widetilde{F}_x$ coincides with $\Omega_{\widetilde{F}}$.*
- *If $E \subset \widetilde{\nu}_D$ is the exceptional locus, then the form $\omega_{\widetilde{\nu}_D}$ is non-degenerate on a neighborhood $\mathcal{U}^E = \mathcal{U}$ of E in $\widetilde{\nu}_D$.*

The form $\omega_{\widetilde{\nu}_D}$ has the local expression

$$\omega_{\widetilde{\nu}_D} = K\pi^*\omega_D + \sum_\alpha d(\pi^*\rho_\alpha) \wedge \eta_\alpha + \sum_\alpha (\pi^*\rho_\alpha)(\phi_\alpha^{-1})^*\Omega_{\widetilde{F}}, \tag{18}$$

for some $K > 0$ large enough, a finite atlas of symplectic-bundle charts $\phi_\alpha : U_\alpha \times \widetilde{F} \to V_\alpha \subset \widetilde{\nu}_D$, some 1-forms η_α, and a partition of unity ρ_α subordinated to the cover U_α of D.

5. Gluing the symplectic form.

Finally, we glue the symplectic form $\omega_{\widetilde{\nu}_D}$ constructed in Theorem 3.40 with the symplectic form of the initial symplectic orbifold (X, ω). Recall some notations of the previous sections. We have a symplectic fiber bundle $\pi : \widetilde{\nu}_D \to D$ with fiber $\widetilde{F}$, the exceptional subbundle $E \subset \widetilde{\nu}_D$ is a fiber sub-bundle $Z \to E \to D$ with fiber Z the exceptional locus of $\widetilde{F}$, and the blow-up map is denoted $b : \widetilde{\nu}_D \to \nu_D$. Recall that by Proposition 3.11, the space ν_D admits a closed orbi-form $\omega' \in \Omega^2_{orb}(\nu_D)$, which is symplectic on a neighbourhood of the zero section and constant along fibers.

By Proposition 3.39 we know that $\widetilde{\nu}_D$ is a proper symplectic bundle. In particular the height funtion $H : \tilde{F} \to [0, \infty)$ is preserved by the transition functions on $\widetilde{\nu}_D$, so the height function can be extended to the total space of the bundle to give a function $H_{\widetilde{\nu}_D} = \tilde{H} : \widetilde{\nu}_D \to \widetilde{\nu}_D$ given by $\widetilde{H}(y) = |b(y)|$, for $y \in \widetilde{\nu}_D$, where $b(y) \in F_{\pi(y)} \cong \mathbb{C}^k/\Gamma$ and $|b(y)|$ is its euclidean norm in $\mathbb{C}^k$.

We denote $U_R = \{y \in \widetilde{\nu}_D : \widetilde{H}(y) < R\}$ for $R > 0$. We fix a neighbourhood

$$W = U_{R_0} \subset \widetilde{\nu}_D$$

of the exceptional locus such that $\omega_{\widetilde{\nu}_D}$ is symplectic on W, as provided by Theorem 3.40.

Proposition 3.41. *For $\varepsilon > 0$ small enough there exists a symplectic form Ω_W on W so that $\Omega_W = (1-\varepsilon)b^*(\widetilde{\omega}) + \varepsilon\frac{1}{K}\omega_{\widetilde{\nu}_D}$ on some small neighborhood $U_\delta \subset W$ of E, and $\Omega_W = b^*(\omega')$ outside of some larger neighborhood $U_{\delta'} \subset W$, $0 < \delta < \delta' < R_0$.*

Proof. By construction,

$$\omega_{\widetilde{\nu}_D} = K\pi^*(\omega_D) + \eta + \sum_\alpha d((\pi^*\rho_\alpha)\theta_\alpha)$$

where the form η is a representative of the Poincaré dual of the homology class given by the cycle $A = \sum_\alpha \sum_i a_i Q_\alpha \times Z_i$. In particular we can take η to be very close to a Dirac delta around the cycle A, hence we can assume that the support of η is contained in a small neighborhood of E, say U_δ, for $0 < 2\delta < R_0$. By the construction of ω' in Proposition 3.11, we have

$$b^*(\omega') = \pi^*(\omega_D) + d\left(b^*\left(\sum_\alpha (\pi^*\rho_\alpha)\eta_\alpha\right)\right)$$

for some 1-forms η_α. On the other hand, outside of the support of η, we have

$$\frac{1}{K}\omega_{\widetilde{\nu}_D} = \pi^*(\omega_D) + d\left(\frac{1}{K}\sum_\alpha (\pi^*\rho_\alpha)\theta_\alpha\right).$$

This implies that $b^*(\omega')$ and $\frac{1}{K}\omega_{\widetilde{\nu}_D}$ define the same cohomology class outside U_δ. So there exists a 1-form $\gamma \in \Omega^1(W \setminus U_\delta)$ such that

$$\frac{1}{K}\omega_{\widetilde{\nu}_D} - b^*(\omega') = d\gamma$$

on $W \setminus U_\delta$. Now define

$$\Omega_W = b^*(\omega') + \varepsilon\, d(\rho\gamma)$$

with $\rho : E \to [0,1]$ a bump function so that $\rho \equiv 1$ on U_δ and $\rho \equiv 0$ outside some $U_{\delta'}$ with $\delta < \delta' < R_0$. In the set U_δ the form Ω_W satisfies

$$\Omega_W = b^*(\omega') + \varepsilon\, d\gamma = b^*(\omega') + \varepsilon\left(\frac{1}{K}\omega_{\widetilde{\nu}_D} - b^*(\omega')\right) = (1-\varepsilon)b^*(\omega') + \varepsilon\frac{1}{K}\omega_{\widetilde{\nu}_D} \tag{19}$$

We extend Ω_W with the same formula to all of U_δ. Also $\Omega_W = b^*(\omega')$ on $W \setminus U_{\delta'}$.

It remains to see that Ω_W is symplectic on W if we choose $\varepsilon > 0$ small enough. This is clear on $W \setminus U_{\delta'}$ because $\Omega_W = b^*(\omega')$ there. It is also strightforward in $U_{\delta'} \setminus U_\delta$, since there we have $\Omega_W = b^*(\omega') + \varepsilon\, d(\rho\gamma)$, where $b^*(\omega')$ is non-degenerate. As this is a compact set, making $\varepsilon > 0$ small we can assure that Ω_W is symplectic there.

It only remains to see that $\Omega_W|_{U_\delta}$ given by (19) is symplectic in U_δ. Take $y \in U_\delta$, then $T_y\widetilde{\nu}_D \cong T_y\widetilde{F}_{\pi(y)} \times T_{\pi(y)}D$ by splitting (non-canonically) into vertical directions and projecting onto D. The form $b^*(\omega')$ vanish on the vertical directions, whereas $\omega_{\widetilde{\nu}_D}$ is symplectic over $T_y\widetilde{F}_{\pi(y)}$, hence for $u \in T_y\widetilde{F}_{\pi(y)}$ there is some v such that $\Omega_W(u,v) \neq 0$. The same happens for vectors in a neighbourhood of $S(T_y\widetilde{F}_{\pi(y)})$ in the sphere tangent bundle $S(TW)$.

Finally, for unitary vectors $u \in T_y\widetilde{\nu}_D$ such that $|\pi_*(u)| \geq \delta_1$ (using some background metrics), we have that there exists a suitable unitary vector w and there exist constants $m_1 > 0$ (provided by Lemma 3.36), and $C_1 > 0$ (provided by the arguments used in (17)), so that $|b^*(\omega')(u,w)| \geq m_1\delta_1 C_1$.

We can bound $|\omega_{\widetilde{\nu}_D}(u,w)| \leq m_2$, so for $\varepsilon > 0$ small enough, we have that the expression (19) implies that $|\Omega_W(u,w)| > 0$. This completes the proof. □

Take the form Ω_W constructed in the Proposition 3.41. It is symplectic on some neighborhood W of $E \subset \widetilde{\nu}_D$. By Proposition 3.12, there are neighborhoods $\mathcal{U} \subset \nu_D$ and $\mathcal{V} \subset X$ of D and a symplectomorphism $\varphi : (\mathcal{U}, \omega') \to (\mathcal{V}, \omega)$. By shrinking we can arrange that φ be defined on larger open sets. Consider the open set $\widetilde{\mathcal{U}} = b^{-1}(\mathcal{U}) \subset \widetilde{\nu}_D$, which we assume contained in W. We define

$$\widetilde{X} = W \cup_f (X \setminus \bar{U}_\varepsilon),$$

where $\widehat{U}_\varepsilon = \varphi(b(U_\varepsilon))$ is a tubular neighborhood of $D \subset X$ of radius $\varepsilon > 0$. This is chosen with $\varepsilon > \delta'$, given in Proposition 3.41. The gluing map is $f = \varphi \circ b : W \backslash U_\varepsilon \to V \subset X$, whose image is some open set $V \subset \mathcal{V}$. Note that $V \subset X$ is the result of removing a tubular neighborhood of $D \subset X$ from a larger tubular neighborhood, i.e. V is a fiber bundle over D with fiber $(\varepsilon, R_0) \times S^{2k-1}/\Gamma$. Since $f^*(\omega) = b^*\varphi^*\omega = b^*\omega' = \Omega_W$, we see that f is a symplectomorphism. Hence $\widetilde{X}$ is a symplectic manifold. We have proved the following.

THEOREM 3.42. *Let (X, ω) be a symplectic orbifold such that all its isotropy set consists of homogeneous disjoint embedded submanifolds in the sense of definition 3.2. There exists a symplectic manifold $(\widetilde{X}, \widetilde{\omega})$ and a smooth map*

$$b : (\widetilde{X}, \widetilde{\omega}) \to (X, \omega)$$

which is a symplectomorphism outside an arbitrarily small neighborhood of the isotropy points.

6. Examples.

In this section, we want to give some examples where we can apply Theorem 3.42.

Example 1. A symplectic divisor. Let (X, ω) be a symplectic orbifold of dimension $2n$ such that the isotropy locus $D \subset X$ is a divisor, that is, $\dim D = 2n - 2$, and the isotropy is given by $\Gamma = \mathbb{Z}_k = \langle g \rangle$ acting on the normal space $\mathbb{C}$ by $g(z) = e^{2\pi i/k} z$. Then X is topologically a manifold since $\mathbb{C}/\mathbb{Z}_k$ is homeomorphic to $\mathbb{C}$. The algebraic resolution of $F = \mathbb{C}/\mathbb{Z}_k$ is given by $\widetilde{F} = \mathbb{C}$, with map $b : \widetilde{F} \to F$, $b(w) = w^k$. Note that b is the homeomorphism mentioned above. Theorem 3.42 applies to get a smooth symplectic manifold $(\widetilde{X}, \widetilde{\omega})$ with a map $b : \widetilde{X} \to X$ which is a symplectomorphism outside a small neighbourhood of D.

Note that b is bijective, hence a homeomorphism. Then we can identify $\widetilde{X} \cong X$, and hence Theorem 3.42 in this case means that we can change the orbifold atlas of X by a smooth atlas, and the orbifold symplectic form ω by a smooth symplectic form $\widetilde{\omega}$. This process is the reverse process to that of Proposition 4.6, where we start with a smooth symplectic manifold and a set a prescribed divisors, and produce from it an orbifold atlas (with the divisors as isotropy sets) and an orbifold symplectic form. Observe that in Proposition 4.6 the result is stated for dimension 4, but the proof is valid in any dimension.

Example 2. A product. Let (M, ω_1) be a symplectic orbifold with isolated orbifold singularities. By [**16**], we have a symplectic resolution $b : (\tilde{M}, \widetilde{\omega}_1) \to (M, \omega_1)$. Let (N, ω_2) be a smooth symplectic manifold. Then $(X = M \times N, \omega_1 + \omega_2)$ is a symplectic orbifold with homogeneous isotropy sets. Actually, if $x \in M$ is a singular point of M, then $D = \{x\} \times N$ is an isotropy submanifold of X. The map $b : (\tilde{M} \times N, \widetilde{\omega}_1 + \omega_2) \to (M \times N, \omega_1 + \omega_2)$ is a symplectic resolution, agreeing with Theorem 3.42. In this case, the symplectic normal bundle to D is trivial.

Example 3. Symplectic bundle over an orbifold. Let (F, ω_F) be a symplectic manifold, (B, ω_B) a symplectic orbifold with isolated singularities, and let $F \to M \xrightarrow{\pi} B$ be a smooth bundle, where (M, ω) is a symplectic orbifold such that $(F_x, \omega|_{F_x})$ is symplectomorphic to (F, ω_F), for all fibers $F_x = \pi^{-1}(x)$, $x \in B$. That is, M is a symplectic orbifold with a compatible symplectic bundle structure over an orbifold symplectic base. For a small orbifold chart (U, V, φ, Γ) of B, we have $\pi^{-1}(V) \cong V \times F \cong (U/\Gamma) \times F = (U \times F)/\Gamma$, where Γ acts on the first factor. As we are assuming that B has isolated singularities, the isotropy sets are F_x, where $x \in B$ is a singularity of B. Hence M is an HI symplectic orbifold. Theorem 3.42 guarantees the existence of a symplectic resolution of M.

Actually, the resolution can be explicitly given as follows. Take a resolution

$$b : (\widetilde{B}, \widetilde{\omega}_B) \to (B, \omega_B)$$

provided by [**16**], and take the pull-back by b of the bundle $F \to M \xrightarrow{\pi} B$. Denote the pull-back bundle by

$$F \to \widetilde{M} \xrightarrow{\widetilde{\pi}} \widetilde{B}.$$

Then for every singular point $x \in B$ with orbifold chart (U, V, φ, Γ), we call $\widetilde{V} := b^{-1}(V) \subset \widetilde{B}$. We glue the symplectic form $\widetilde{\omega}_B \times \omega_F$ on $\widetilde{\pi}^{-1}(\widetilde{V}) \cong \widetilde{V} \times F$ to ω_M along the complement of a neighbourhood of F_x. Anyway, Theorem 3.42 does the job without having to care about the details.

Example 4. Mapping torus. Let (M, ω_M) be a compact symplectic orbifold with isolated singularities. Let $f : M \to M$ be an orbifold symplectomorphism and consider the mapping torus $M_f = (M \times [0,1])/\sim$ with $(x,0) \sim (f(x),1)$. Let t be the coordinate of $[0,1]$ and consider a circle S^1 with coordinate θ. Then $X = M_f \times S^1$ is a symplectic orbifold with symplectic form $\omega = \omega_M + dt \wedge d\theta$.

The isotropy sets are 2-tori. To see it, take a singular point $x \in M$ and let $x_0 = x, x_1 = f(x_0), x_2 = f^2(x_0), \ldots$ be the orbit of x. As all of them are singular points and there are finitely many of them in M, there is some $n > 0$ such that $x_n = x_0$, and we take the minimum of such n. Consider the circle C_x given by the image of $\{x_0, \ldots, x_{n-1}\} \times [0,1]$ in M_f, which is a $n:1$ covering of $[0,1]/\sim \cong S^1$. The tori $D = C_x \times S^1 \subset X$, for $x \in M$ the singular points, are the only isotropy sets of $X = M_f \times S^1$, and moreover they are HI-isotropy submanifolds of X whose codimension in X equals $\dim M$.

Theorem 3.42 gives a symplectic resolution of X. This can be constructed alternatively by taking the symplectic resolution $b : \widetilde{M} \to M$ of M given by [**16**]. If we arrange to do it in an equivariant way around the singular points, then we may lift f to a symplectomorphism $\widetilde{f} : \widetilde{M} \to \widetilde{M}$ of the resolved manifold, and then

$$\widetilde{X} = \widetilde{M}_{\widetilde{f}} \times S^1$$

is a symplectic resolution of X.

Example 5. An example with non-trivial normal bundle. Take a standard 6-torus $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$ with the standard symplectic form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$, and consider the maps

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_2, -x_3, -x_4, -x_5, -x_6),$$
$$g(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + \tfrac{1}{2}, x_2, -x_3, -x_4, x_5, x_6).$$

Then the quotient

$$X = \frac{\mathbb{T}^6}{\langle f, g \rangle}$$

is a symplectic orbifold. Note that f and g commute, and $g^i \circ f^j$ has no fixed points if $i \neq 0$ because the first coordinate is moved by g and not by f. Hence the only fixed points by the action of $\langle f, g \rangle$ in $\mathbb{T}^6$ are those of f. This implies that the isotropy locus are the image by the quotient map of the subsets

$$S_{\mathbf{a}} = \{(x_1, x_2, a_3, a_4, a_5, a_6) | (x_1, x_2) \in \mathbb{R}^2\}, \quad \text{for} \quad \mathbf{a} = (a_3, a_4, a_5, a_6) \in \{0, 1/2\}^4.$$

We denote

$$D_{\mathbf{a}} = \frac{S_{\mathbf{a}}}{\langle \mathbb{Z}^6, f, g \rangle} \subset X$$

for the quotient of $S_{\mathbf{a}}$ seen in X. Note that $D_{\mathbf{a}}$ is isomorphic to a 2-torus, since

$$D_{\mathbf{a}} := [S_{\mathbf{a}}] \cong \frac{\mathbb{R}^2}{\langle (\frac{1}{2}, 0), (0,1) \rangle} \subset \mathbb{T}^6/\langle f, g \rangle = X$$

The fiber of the normal bundle $\nu_{D_{\mathbf{a}}}$ of $D_{\mathbf{a}}$ in X is $F = \mathbb{C}^2/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ in $\mathbb{C}^2$ is given by $(z_1, z_2) \sim (-z_1, -z_2)$. Denote $\mathbb{T}^2_{\mathbf{a}} = S_{\mathbf{a}}/\mathbb{Z}^6 \subset \mathbb{T}^6$. It is easy to see that $\nu_{D_{\mathbf{a}}}$ is the quotient of the trivial bundle $\mathbb{T}^2_{\mathbf{a}} \times F$, by the map g. It follows that $\nu_{D_{\mathbf{a}}}$ is non-trivial, since g moves points in the base $\mathbb{T}^2_{\mathbf{a}}$ and identifies the corresponding fibers by a map different from the identity. However, $\nu_{D_{\mathbf{a}}}$ is trivializable.

Example 6. Resolving the quotient of a symplectic nilmanifold. To give an explicit example of a resolution, we shall take a symplectic 6-nilmanifold from [**5**] and perform a suitable quotient to get a symplectic 6-orbifold with homogeneous isotropy. For instance we take the nilmanifold corresponding to the Lie algebra $L_{6,10}$ of Table 2 in [**5**], which is symplectic since it appears in Table 3 of [**5**]. Take the group G of (7×7)-matrices given by

$$\begin{pmatrix} 1 & x_2 & x_1 & x_4 & x_1x_2 & x_5 & x_6 \\ 0 & 1 & 0 & -x_1 & x_1 & x_1^2/2 & x_3 \\ 0 & 0 & 1 & 0 & x_2 & -x_4 & x_2^2/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_i \in \mathbb{R}$, for any $i = 1, \ldots, 6$. Then, a global system of coordinate functions $\{x_1, \ldots, x_6\}$ for G is given by $x_i(X) = x_i$, $i = 1, \ldots, 6$, $X \in G$. Note that if a matrix $A \in G$ has coordinates a_i, then the change of coordinates of a matrix $X \in G$ by the left translation $L_A : G \to G$, $X \mapsto AX$ are given by

$$\begin{aligned} &L_A^*(x_1) = x_1 + a_1, && L_A^*(x_2) = x_2 + a_2, \\ &L_A^*(x_3) = x_3 + a_1x_2 + a_3, && L_A^*(x_4) = x_4 - a_2x_1 + a_4, \\ &L_A^*(x_5) = x_5 + \tfrac{1}{2}a_2x_1^2 - a_1x_4 + a_1a_2x_1 + a_5, \\ &L_A^*(x_6) = x_6 + \tfrac{1}{2}a_1x_2^2 + a_2x_3 + a_1a_2x_2 + a_6. \end{aligned}$$

A standard calculation shows that a basis for the left invariant 1-forms on G consists of

$$\{dx_1, dx_2, dx_3 - x_1dx_2, dx_4 + x_2dx_1, dx_5 + x_1dx_4, dx_6 - x_2dx_3\}.$$

Let Γ be the discrete subgroup of G consisting of matrices with entries $(x_1, x_2, \ldots, x_6) \in (2\mathbb{Z})^2 \times \mathbb{Z}^4$, that is x_i are integer numbers and x_1, x_2 are even. It is easy to see that Γ is a subgroup of G. So the quotient space of right cosets $M = \Gamma\backslash G$ is a compact 6-manifold. Hence the 1-forms

$$\begin{aligned} e_1 &= dx_1, \\ e_2 &= -dx_2, \\ e_3 &= dx_3 - x_1dx_2 - dx_4 - x_2dx_1 = d(x_3 - x_4 - x_1x_2), \\ e_4 &= dx_4 + x_2dx_1, \\ e_5 &= dx_5 + x_1dx_4, \\ e_6 &= dx_6 - x_2dx_3 \end{aligned}$$

satisfy

$$\begin{aligned} de_1 &= de_2 = de_3 = 0, \\ de_4 &= e_1e_2, \\ de_5 &= e_1e_4, \\ de_6 &= e_2e_3 + e_2e_4\,. \end{aligned}$$

This coincides with $L_{6,10}$ in Table 2 in [**5**]. The symplectic form of M is $\omega = e_1e_6 + e_2e_5 - e_3e_4$ (see Table 3 in [**5**]).

Now we consider the map $\varphi(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, -x_3, -x_4, -x_5, x_6)$. This is given in terms of the matrices as $\varphi(A) = PAP$, where P is the diagonal matrix

$$P = \operatorname{diag}(1, -1, 1, -1, -1, -1, 1)\,.$$

Note that for $N \in \Gamma$, $PNAP = (PNP)(PAP)$. As $\varphi(\Gamma) = \Gamma$, we see that φ descends to $M = \Gamma\backslash G$. This is clearly a symplectomorphism with $\varphi^2 = \mathrm{Id}$, hence

$$X = M/\langle\varphi\rangle$$

is a symplectic orbifold. The isotropy locus is formed by the sets

$$S_{\mathbf{b}} = \{(x_1, b_2, b_3, b_4 - b_2 x_1, b_5 + \frac{1}{2} b_2 x_1^2, x_6) : (x_1, x_6) \in \mathbb{R}^2\},$$

for $\mathbf{b} = (b_2, b_3, b_4, b_5) \in \{0,1\} \times \{0, 1/2\}^3$. This is computed solving the equation $\varphi(x) = Ax$ for some $A \in \Gamma$, which translates to $x_1 = L_A^*(x_1)$, $-x_i = L_A^*(x_i)$ for $2 \leq i \leq 5$ and $x_6 = L_A^*(x_6)$. Therefore the isotropy consits of a collection of 16 tori, each of them of homogeneous isotropy $\mathbb{C}^2/\mathbb{Z}_2$.

Recall that the above manifold M is a circle bundle (with coordinate x_6) over a mapping torus (with coordinate x_1) of a 4-torus (with coordinates x_2, x_3, x_4, x_5). Then we take a quotient of T^4 by $\mathbb{Z}_2$ acting as $\pm\,\mathrm{Id}$. So this fits with Example 4 above.

Let us compute the Betti numbers of the resolution $\widetilde{X}$ of X. The Betti numbers of M appear in Table 2 of [**5**] and are $b_1(M) = 3, b_2(M) = 5, b_3(M) = 6$. Easily we get that $H^1(M) = \langle e_1, e_2, e_3\rangle$ and $H^2(M) = \langle e_2e_3, e_1e_5, e_1e_3, e_2e_6, e_3e_6 + e_4e_6\rangle$. Taking the invariant part by the action of φ, we have

$$H^1(X) = \langle e_1\rangle, \quad H^2(X) = \langle e_2e_3\rangle,$$

so $b_1(X) = 1$ and $b_2(X) = 1$. By Poincaré duality, $b_4(X) = b_5(X) = 1$. Now $\chi(X) = 0$ since $\chi(M) = 0$ and the ramification locus are T^2 which have $\chi(T^2) = 0$. Therefore $b_3(X) = 2$.

The resolution process changes $F = \mathbb{C}^2/\mathbb{Z}_2$ by the single blow-up at the origin $\widetilde{F}$, which has exceptional divisor $Z = \mathbb{CP}^1$ with $Z^2 = -2$. Then each exceptional locus increases by 1 the second Betti number b_2 (cf. the computations of cohomology in [**22**]). Therefore $b_1(\widetilde{X}) = 1, b_2(\widetilde{X}) = 1 + 16 = 17$. By Poincaré duality, $b_4(\widetilde{X}) = 17, b_5(\widetilde{X}) = 1$. Again $\chi(\widetilde{X}) = 0$, since the exceptional divisors are $\mathbb{CP}^1$-bundles over T^2 and hence they have $\chi(E) = 0$. So $b_3(\widetilde{X}) = 34$.

CHAPTER 4

Orbifolds in Sasakian and K-contact geometry.

Sasakian geometry has become an important and active subject, especially after the appearance of the fundamental treatise of Boyer and Galicki [**9**]. Chapter 7 of this book contains an extended discussion of the topological problems in the theory of Sasakian, and, more generally, K-contact manifolds. These are odd-dimensional analogues to Kahler and symplectic manifolds, respectively.

The key feature that characterizes K-contact (Sasakian) manifolds (i.e. manifolds admitting a K-contact (Sasakian) structure, see Section 5) is the fact that they can be expressed as the total space of a circle orbi-bundle, (that we will call *Seifert bundle*), over some base space X with an orbifold symplectic (Kahler) structure. In this chapter we study the tools necessary to study the geography of Sasakian and K-contact manifolds in dimension 5.

In the Section 1 we study cyclic orbifolds in dimension 4 (i.e. 4-orbifolds whose isotropy at every point consists of a cyclic group). Cyclic orbifolds are important since they appear as the base space of Seifert bundles, hence K-contact and Sasakian manifolds are constructed from cyclic 4-orbifolds by putting orbifold geometric structures (symplectic or Kahler) and then taking a circle orbi-bundle over these (symplectic or Kahler) orbifolds.

After studying 4-orbifolds, in Section 2 we pass to the definition of Seifert bundles, and then study the basic properties of them. We go on to analize the cohomology of a Seifert bundle, i.e. how the cohomology of the total space is computed from that of the base. This is given by the Leray spectral sequence, which is studied in detail in Section 3.

In Section 4 we give an introduction to K-contact and Sasakian geometry, with the basic definitions and the (so called) Structure Theorems, which characterize the existence of K-contact and Sasakian structures on a manifold M in terms of being able to construct M as a Seifert bundle over a symplectic (Kahler) orbifold. These Structure Theorems allow us to study geography problems of K-contact and Sasakian manifolds by studying the corresponding geography problems of symplectic and Kahler orbifolds. We use the correspondence between K-contact vs Sasakian and symplectic vs Kahler in Chapter 5 to construct a K-contact manifold with first homology zero and not admitting any semi-regular Sasakian structure.

1. 4-dimensional cyclic orbifolds.

In this section we analize more in detail orbifolds for the particular case of dimension 4. This dimension is small enough to display particular behaviours. The isotropy set of a 4-orbifold is considerably easier to handle than the isotropy set of a general orbifold. For instance, the singular points of a 4-orbifold are a discrete subset, as we will see later.

Recall that given an orbifold X we denote an orbifold atlas by $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$, where $U_\alpha \subset \mathbb{R}^n$, $\Gamma_\alpha < \mathrm{GL}(\mathbb{R}^n)$ is a finite group acting linearly, and $\phi_\alpha : U_\alpha \to V_\alpha \subset X$ is a Γ_α-invariant map inducing a homeomorphism $U_\alpha/\Gamma_\alpha \cong V_\alpha$ onto an open set V_α of X.

Moreover, if we fix a point $x \in X$, we can arrange always a chart $\phi : U \to V$ with $U \subset \mathbb{R}^n$ is a ball centered at 0, $\phi(0) = x$, and $U/\Gamma \cong V$, with Γ the isotropy group of the point x. As the groups Γ_α are finite, we can arrange (after a suitable conjugation) that $\Gamma_\alpha < \mathrm{O}(n)$, as we saw in

Proposition 2.10. If the orbifold X is orientable then we can further refine the atlas so that all isotropy groups $\Gamma_\alpha < \mathrm{SO}(n)$ preserve orientation.

DEFINITION 4.1. *An orientable orbifold X is called a* cyclic *orbifold if the isotropy group of every point $x \in X$ is a cyclic group, i.e. if for every $x \in X$ there exists $m = m(x) \in \mathbb{N}$ so that $\mathbb{Z}_m \cong \Gamma_x < \mathrm{SO}(n)$, with Γ_x being the isotropy group of x.*

REMARK 4.2. *In our definition of cyclic orbifold we impose orientability also. We exclude non-orientable orbifolds with cyclic isotropy groups because these are not interesting to us. But note that the existence of these orbifolds is certainly possible. An example is $\mathbb{C}^2/\Gamma$ with the action $\gamma(z_1, z_2) = (z_1, \overline{z_2})$ for γ the generator of $\Gamma \cong \mathbb{Z}_2$.*

From now on our main interest will lie in cyclic orbifolds, since this is the kind of orbifolds that appear as the space of leaves of K-contact and Sasakian structures. Moreover we will focus specifically on 4-dimensional cyclic orbifolds, since we are interested in 5-dimensional Sasakian and K-contact manifolds.

Let X a cyclic 4-orbifold. Take $x \in X$ and a chart $\phi : U \to V$ around x. Let $\Gamma = \mathbb{Z}_m < \mathrm{SO}(4)$ be the isotropy group. Then V is homeomorhic to an open neighbourhood of $0 \in \mathbb{R}^4/\mathbb{Z}_m$. A matrix of finite order in SO(4) is conjugate to a diagonal matrix in U(2) of the type $(\exp(2\pi i j_1/m), \exp(2\pi i j_2/m)) = (\xi^{j_1}, \xi^{j_2})$, where $\xi = e^{2\pi i/m}$. Therefore we can suppose that $U \subset \mathbb{C}^2$ and $\Gamma = \mathbb{Z}_m = \langle \xi \rangle \subset \mathrm{U}(2)$ acts on U as

$$\xi \cdot (z_1, z_2) := (\xi^{j_1} z_1, \xi^{j_2} z_2). \tag{20}$$

Here j_1, j_2 are defined modulo m. As the action is effective, we have $\gcd(j_1, j_2, m) = 1$. Let us list the possible local models for an action given by the formula (20).

Let us recall the following from chapter one.

(1) We call $x \in X$ a regular point if $m(x) = 1$, otherwise we call it a (non-trivial) isotropy point.
(2) We call $x \in X$ a smooth point if there exists a neighbourhood of x in X is homeomorphic to a ball in $\mathbb{R}^4$, and a singular point otherwise.
(3) We say that $D \subset X$ is an isotropy surface of multiplicity m if D is closed and there is a dense open subset $D^\circ \subset D$ which is a surface and $m(x) = m$, for $x \in D^\circ$.

Clearly every regular point is smooth, but the converse does not hold in general (it does in dimension 2 and 3). Dimension 4 is the first when the converse fails (see example below).

Recall the following terminology. For an orbifold X the *link* of a point $x \in X$ is the sphere S_ε/Γ of a neighborhood of x modelled in some small chart by a ball $B_{2\varepsilon}/\Gamma$, with Γ acting by linear isometries. It is easy to check that:

(1) The homeomorphism type of the link is independent of the radius of the small sphere chosen around x.
(2) For any two small enough charts around x, the quotient of a sphere centered in x by the isotropy group has the same homeomorphism class.

So there is a well defined homeomorphism type of link around a point $x \in X$. Alternatively, the link around x can be seen as a genuine sphere aound x with respect to any orbi-riemannian metric on the orbifold X. Anyway, it is clear that a smooth point always has a link homeomorphic to a sphere.

EXAMPLE 4.3. (1) *Let $m \geq 2$ be an integer. Consider the action of $\mathbb{Z}_m$ in $\mathbb{C}^2$ given by $\xi(z_1, z_2) = (\xi z_1, \xi z_2)$, where ξ is a primitive m-th root of unity (e.g. take $\xi = e^{\frac{2\pi i}{m}}$). Then any sphere $S^3_\varepsilon(0)$ centered at 0 of radius ε transforms into a lens space $S^3_\varepsilon(0)/\mathbb{Z}_m \subset \mathbb{C}^2/\mathbb{Z}_m$. In particular it has fundamental group isomorphic to $\mathbb{Z}_m$, by covering theory. Hence $0 \in \mathbb{C}^2/\mathbb{Z}_m$ is both an isotropy point and a singular point.*

(2) *Consider now the action of* $\mathbb{Z}_6$ *in* $\mathbb{C}^2$ *given by* $\xi(z_1, z_2) = (\xi^2 z_1, \xi^3 z_2)$*, with* $\xi = e^{\frac{2\pi i}{6}}$*. Note that* $\xi^2(z_1, z_2) = (\xi^4 z_1, z_2)$ *acts only in the first factor, leaving the second fixed. Analogously,* $\xi^3(z_1, z_2) = (z_1, \xi^3 z_2)$ *leaves fixed the first factor and acts on the second. Since* $\langle\xi\rangle = \langle\xi^2, \xi^3\rangle$*, the action of* $\langle\xi\rangle \cong \mathbb{Z}_6$ *despomposes as* $\mathbb{C}^2/\langle\xi\rangle = \mathbb{C}/\langle\xi^2\rangle \times \mathbb{C}/\langle\xi^3\rangle \cong \mathbb{C}/\mathbb{Z}_3 \times \mathbb{C}/\mathbb{Z}_2$ *is a product of cones, hence a topological manifold. Hence the point* $0 = (0,0) \in \mathbb{C}^2/\mathbb{Z}_6$ *provides an example of a non-trivial isotropy point which is smooth.*

The next proposition studies in full generality the possible local models for a cyclic 4-orbifold.

PROPOSITION 4.4. *Let* X *be a cyclic, 4-dimensional orbifold and* $x \in X$ *with local model* $\mathbb{C}^2/\mathbb{Z}_m$*. Then there are at most two isotropy surfaces* D_i*, with multiplicity* $m_i|m$*, through* x*. If there are two such surfaces* D_i, D_j*, then they intersect transversely and* $\gcd(m_i, m_j) = 1$*. The fundamental group of the link of* x *has order* d *with* $(\prod m_i)d = m$*, the product over all* m_i *such that* $x \in D_i$*. So the point is smooth if and only if* $\prod m_i = m$*.*

PROOF. For an action of $\mathbb{Z}_m$ in $\mathbb{C}^2$ given as in (20) by

$$\xi(z_1, z_2) = (\xi^{j_1} z_1, \xi^{j_2} z_2)$$

we set $m_1 := \gcd(j_1, m)$, $m_2 := \gcd(j_2, m)$. Note that $\gcd(m_1, m_2) = 1$, so we can write $m_1 m_2 d = m$, for some integer d. Put $j_1 = m_1 e_1$, $j_2 = m_2 e_2$, $m = m_1 c_1 = m_2 c_2$. Clearly $c_1 = m_2 d$ and $c_2 = m_1 d$ and $d = \gcd(c_1, c_2)$.

We have five cases:

(a) x is an isolated singular point. This corresponds to $m_1 = m_2 = 1$. As $\gcd(j_1, m) = \gcd(j_2, m) = 1$, the only fixed point is $(0,0)$ since any power of ξ rotates both copies of $\mathbb{C}$ non trivially. In this case the quotient space is singular, and the singularity is a cone over a lens space $S^3/\mathbb{Z}_m$, which is the link of the origin. Note that $d = m$.

(b) Two isotropy surfaces and x is a smooth point, $m_1, m_2 > 1$, $d = 1$. Let us see that the action is equivalent to the product of one action on each factor $\mathbb{C}$. In this case $c_2 = m_1$ and $c_1 = m_2$. So $\gcd(c_1, c_2) = 1$ and $m = c_1 c_2$. The action is given by $\xi \cdot (z_1, z_2) := (\exp(2\pi i e_1/c_1) z_1, \exp(2\pi i e_2/c_2) z_2)$. We see that

$$\xi^{c_1} \cdot (z_1, z_2) = (z_1, \exp(2\pi i c_1 e_2/c_2) z_2),$$
$$\xi^{c_2} \cdot (z_1, z_2) = (\exp(2\pi i c_2 e_1/c_1) z_1, z_2),$$

so the surfaces $D_1 = \{(z_1, 0)\}$ and $D_2 = \{(0, z_2)\}$ have isotropy groups $\langle\xi^{c_1}\rangle = \mathbb{Z}_{m_1}$ and $\langle\xi^{c_2}\rangle = \mathbb{Z}_{m_2}$, respectively. In this case $m = m_1 m_2$, $d = 1$.

Note that $\mathbb{Z}_m = \langle\xi^{c_1}\rangle \times \langle\xi^{c_2}\rangle$ if and only if $d = \gcd(c_1, c_2) = 1$. In this case the action of $\mathbb{Z}_m$ decomposes as the product of the actions of $\mathbb{Z}m_2$ and $\mathbb{Z}_{m_1}$ on each of the factors $\mathbb{C}$. The quotient space is $\mathbb{C}^2/\mathbb{Z}_m \cong \mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1}$, which is homeomorphic to $\mathbb{C} \times \mathbb{C}$, and hence x is a smooth point (its link is S^3).

(c) Two isotropy surfaces intersect at x and x is a singular point. In this case $d = \gcd(c_1, c_2) > 1$ and $m_1, m_2 > 1$. Now $\langle\xi^{c_1}, \xi^{c_2}\rangle = \langle\xi^d\rangle = \mathbb{Z}_{m'}$ with $d\,m' = m$. As $m' = m_1 m_2$, case (b) applies to the action of ξ^d and the quotient space is $\mathbb{C}^2/\mathbb{Z}_{m'} \cong \mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1}$, which is homeomorphic to a ball in $\mathbb{C}^2$ via the map $(z_1, z_2) \mapsto (w_1, w_2) = (z_1^{m_2}, z_2^{m_1})$. The points of $D_1 = \{(w_1, 0)\}$ and $D_2 = \{(0, w_2)\}$ define two surfaces intersecting transversely, and with multiplicities m_1, m_2, respectively.

Now ξ acts on $\mathbb{C}^2/\mathbb{Z}_{m'}$ by the formula

$$\xi \cdot (w_1, w_2) = (\xi^{m_2 j_1} w_1, \xi^{m_1 j_2} w_2) = (\exp(2\pi i e_1/d) w_1, \exp(2\pi i e_2/d) w_2)$$

where $\gcd(e_1, d) = \gcd(e_2, d) = 1$. Therefore this action falls into case (a). The quotient is therefore $\mathbb{C}^2/\langle\xi\rangle \cong (\mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1})/\mathbb{Z}_d$, the point x has as link a lens space $S^3/\mathbb{Z}_d$, and the images of D_1 and D_2 are the points with non-trivial isotropy, with multiplicities m_1, m_2, respectively.

(d) One isotropy surface and x is a smooth point. In this case $m_2 = 1$ and $m_1 = m$. As $d = 1$, this is basically as case (b). The action is $\xi \cdot (z_1, z_2) = (z_1, \exp(2\pi j_2/m) z_2)$. There is only one surface $D_1 = \{(z_1, 0)\}$ with non-trivial isotropy m, and all its points have the same isotropy. The quotient $\mathbb{C}^2/\mathbb{Z}_m = \mathbb{C} \times (\mathbb{C}/\mathbb{Z}_m)$ is topologically smooth.

(e) One isotropy surface and x is a singular point. In this case $m_2 = 1$, $m_1 d = m$ and $d > 1$. This is basically as case (c). Now $c_2 = m$ and $c_1 = d$. Let $dm' = m$ so $m' = m_1$. The quotient space $\mathbb{C}^2/\mathbb{Z}_{m'} \cong \mathbb{C} \times \mathbb{C}/\mathbb{Z}_{m_1}$ is homemorphic to a ball in $\mathbb{C}^2$ and the points of $D_1 = \{(z_1, 0)\}$ define a surface with isotropy m_1. Now for the quotient $\mathbb{C}^2/\mathbb{Z}_m = (\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_{m_1}))/\mathbb{Z}_d$, the image of D_1 consists of points with isotropy m_1, except for the origin which has isotropy $m = m_1 d$. The link around x is the lens space $S^3/\mathbb{Z}_d$, hence it is singular. The rest of the points of D_1 are smooth.

□

Recall our definition of *smooth* orbifold. We say that a 4-orbifold X is *smooth* if all its points are smooth. That is, all points of X fall into cases (b) or (d) in Proposition 4.4. This is equivalent to the topological space X being a topological manifold.

REMARK 4.5. *Let us give a standard model for the action of the isotropy of a* smooth *point $x \in X$.*

(1) *In case the smooth point lies in the intersection of two isotropy surfaces as in (b) of Proposition 4.4, we can change the generator $\xi = e^{2\pi i/m}$ of $\mathbb{Z}_m$ to $\xi' = \xi^k$ for k such that $ke_i \equiv 1 \pmod{m_i}$, $i = 1, 2$. This system of congruences has a solution because m_1 and m_2 are coprime. The action of ξ' and ξ give the same quotient space, so if we denote $\xi' := \xi$ we have that $\xi' \cdot (z_1, z_2) = (\exp(\frac{2\pi i}{m_2}) z_1, \exp(\frac{2\pi i}{m_1}) z_2)$. With this new generator, the action has model $\mathbb{C}^2$ and moreover is given by $\xi \cdot (z_1, z_2) = (\xi^{m_1} z_1, \xi^{m_2} z_2)$, $\xi = e^{2\pi i/m}$.*

(2) *In case the smooth point lies in a unique isotropy surface D of isotropy m as in (d) of Proposition 4.4, we can change the generator of the group so that the action is given by $\xi(z_1, z_2) = (z_1, \xi z_2)$ with $\xi = e^{2\pi i/m}$, and $D = \{z_2 = 0\}$ in the chart.*

Recall that we say that an orbifold is *smooth* if all its points are smooth points, i.e. if the underlaying topological space associated to the orbifold is a topological manifold. Note that a smooth orbifold is not the same as a smooth manifold, since a smooth orbifold is not equipped with a manifold atlas. However, there is a mechanism to produce a smooth orbifold from a smooth manifold, as the following proposition shows.

PROPOSITION 4.6. *Let X be an oriented 4-manifold with embedded surfaces D_i intersecting transversely, and coefficients $m_i > 1$ such that $\gcd(m_i, m_j) = 1$ if D_i, D_j intersect. Then there is a smooth orbifold X with isotropy surfaces D_i of multiplicities m_i.*

PROOF. We consider X with its atlas as smooth manifold. We start by fixing a Riemannian metric such that in a neighbourhood of the (finitely many) points which are in the intersection of two of the D_i's, it is standard, that is, for $x \in D_i \cap D_j$ there is a chart $f : B^2_\varepsilon(0) \times B^2_\varepsilon(0) \to V$, with $f(0,0) = x$, $D_i \cap V = f(B^2_\varepsilon(0) \times \{0\})$, $D_j \cap V = f(\{0\} \times B^2_\varepsilon(0))$, and g is the standard metric on V.

Now let $x \in X$ be a point. If x does not lie in any D_i, take a smooth chart $f : B^4_\varepsilon(0) \to V$ to a neighbourhood V of x not touching any D_i. Then we consider the orbifold chart $(B^4_\varepsilon(0), f, \{1\})$.

If x lies in only one $D = D_i$ with $m = m_i$, take a chart as follows. Take a small neighbourhood $W \subset D$ of x, and by using coordinates we identify $W \subset \mathbb{R}^2$. Consider the exponential map from the normal bundle (on W) ν_D to X, $\exp : \nu_D \to X$. For small $\varepsilon > 0$,

$$\exp : \nu^\varepsilon_D = \{(x, v) : x \in W, v \in (T_x D)^\perp, |v| < \varepsilon\} \to X$$

is a diffeomorphism onto its image. Trivialize the normal bundle, so that $\nu_D^\varepsilon \cong W \times B_\varepsilon(0)$. This gives a smooth chart $f : W \times B_\varepsilon(0) \to U$, $f(w_1, w_2) = \exp_{w_1}(w_2)$, with coordinates (w_1, w_2). We define the following orbifold chart. Consider $U = W \times B_\varepsilon(0)$ and define

$$\phi : U = W \times B_\varepsilon(0) \to V \subset X$$
$$\phi(z_1, z_2) = f(z_1, re^{2\pi m i\theta}) = \exp_{z_1}(re^{2\pi m i\theta})$$

with $z_2 = re^{2\pi i\theta}$. The action of $\mathbb{Z}_m$ is given by $\xi \cdot (z_1, z_2) = (z_1, \xi z_2)$, $\xi = e^{2\pi i/m}$. This defines a chart $(U, V, \phi, \mathbb{Z}_m)$ at x.

If x lies in the intersection of two surfaces, say D_1, D_2, with coefficients m_1, m_2, then $\gcd(m_1, m_2) = 1$, by assumption. Consider a smooth chart $f : B_\varepsilon^2(0) \times B_\varepsilon^2(0) \to V$, with $f(0,0) = x$, $D_1 \cap V = f(B_\varepsilon^2(0) \times \{0\})$, $D_2 \cap V = f(\{0\} \times B_\varepsilon^2(0))$, and g is the standard metric on $V \subset X$. We define the orbifold chart as follows: consider $U = B_\varepsilon^2(0) \times B_\varepsilon^2(0) \subset \mathbb{C}^2$ and

$$\phi : U \to V \subset X$$
$$\phi(z_1, z_2) = \phi(r_1 e^{2\pi i\theta_1}, r_2 e^{2\pi i\theta_2}) = f(r_1 e^{2\pi i m_2 \theta_1}, r_2 e^{2\pi i m_1 \theta_2}) .$$

The action of $\mathbb{Z}_m = \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_1}$, $m = m_1 m_2$, is given by $\xi \cdot (z_1, z_2) = (\xi^{m_1} z_1, \xi^{m_2} z_2)$, where $\xi = e^{2\pi i/m}$. Then $(U, V, \phi, \mathbb{Z}_m)$ is a chart at x.

We have to see that these orbifold charts are compatible with the $\mathbb{Z}_m$-actions, and that we can define orbifold change of charts satisfying the conditions of being smooth and equivariant. We only need to check this near the points of an isotropy surface D not lying in any other isotropy surface. Near these points we can assume that the changes of charts of the manifold structure are those of the respective normal bundle ν_D. Therefore, suppose that the changes of charts for the manifold structure of X near any such point of D have the form $(w_1, w_2) \mapsto (w_1', w_2') = (\varphi(w_1), h(w_1) w_2)$, for some smooth maps $\varphi : W_\alpha \subset \mathbb{C} \to W_\beta \subset \mathbb{C}$, $h : V_\alpha \to S^1$, and with $D = \{w_2 = 0\}$. Then the changes of charts for the orbifold structure of X are of the form

$$U_\alpha = W_\alpha \times B_\varepsilon(0) \to U_\beta = W_\beta \times B_\varepsilon(0)$$
$$(z_1, z_2) \mapsto (z_1', z_2') = (\varphi(z_1), h(z_1)^{1/m} z_2)$$

for some smooth m-th root of h. Therefore they are smooth and $\mathbb{Z}_m$-equivariant with respect to the $\langle \xi \rangle \cong \mathbb{Z}_m$-action $\xi(z_1, z_2) = (z_1, \xi z_2)$. It is easy to check that $\phi_\alpha(z_1, z_2) = \phi_\beta(z_1', z_2') \in X$, where $\phi_\alpha : U_\alpha \to V_\alpha \subset X$ and $\phi_\beta : U_\beta \to V_\beta \subset X$ are the orbifold charts defined above. □

In the above proof, the orbifold constructed is the same topological space X as the smooth manifold, but the manifold atlas has been transformed into an orbifold atlas. So actually the procedure above gives a method for constructing and orbifold atlas starting with a manifold atlas of X.

Note that we have not introduced the freedom of choosing coefficients j_i for each D_i. As mentioned in Remark 4.5, near a smooth point one can always arrange so that the local actions are as above.

Remark 4.7. *Proposition 4.6 can be seen as a way to add singularities on a smooth manifold via an orbifold chart, i.e. to put on a smooth manifold an orbifold chart with isotropy points, such that these isotropy points can be distinguished from the generic points (with trivial isotropy) via the orbifold atlas, so we can say that singularities have been added (in some sense).*

Note that the added singularities can only be distinguished via the orbifold atlas, not through topology since we do not change the underlying topological space. This process of singularization is similar to the process of pinching a sphere, adding a corner but not changing the topology.

We have in this way a reverse of the process to desingularize orbifolds given in the previous chapter, though applying only to a specific case. Note that the proof of 4.6 is written for dimension 4*, but it actually applies to any smooth n-manifold with codimension* 2 *submanifolds D_i, yielding* *a cyclic smooth n-orbifold with the same underlying topological space and the D_i as isotropy submanifolds (divisors).*

Also a smooth cyclic 4-orbifold X can be converted into a smooth manifold with the same underlying space such that the isotropy surfaces are embedded submanifolds intersecting transversely.

PROPOSITION 4.8. *Let X be a smooth cyclic 4-orbifold. Then we can endow X with a smooth atlas in a natural way, thus X has a natural structure of a smooth manifold.*

The explicit proof of this result consists of going in the opposite direction in Proposition 4.6. We do not give the details since this result will not be used later. Note that this is a baby case of desingularization of orbifolds, in which no topological changes are made, but only the orbifold atlas is transformed into a smooth manifold atlas. Also note that if the orbifold X is smooth and has only HI-isotropy submanifolds (which happens iff all isotropy points of X fall into case d) of Proposition 4.4) then Theorem 3.42 of Chapter 4 automatically proves Proposition 4.8.

We can say more about the correspondence between smooth orbifolds and manifolds, this time in respect to De Rham cohomology. We already know by Proposition 2.25 that if a topological space X has at the same time an orbifold structure and a manifold structure, then the orbifold De Rham cohomology of the orbifold X is isomorphic to the De Rham cohomology of the manifold X. In Proposition 2.25, the isomorphism is constructed through sheaf cohomology so it is quite an indirect argument. The next proposition shows how this isomorphism can be explicitly constructed in the special case of the orbifold constructed from a manifold as in Proposition 4.6.

PROPOSITION 4.9. *Let X be an manifold, and endow X with the orbifold structure of Proposition 4.6. We have a natural isomorphism $H^*_{orb}(X) \cong H^*_{DR}(X)$.*

PROOF. We adapt the proof given in [**16**, p. 8] of an analogous result for orbifolds with isolated isotropy points. Consider a smooth map $\varphi : X \to X$ such that

(1) It is the identity off a neighbourhood of $\cup_i D_i$.
(2) Maps a tubular neighborhood of each D_i to D_i.
(3) For each point $p_{ij} \in D_i \cap D_j$ lying in the intersection between two isotropy surfaces, it contracts a neighborhood of p_{ij} into p_{ij}.

If we construct such φ then the map $\varphi^* : \Omega^*_{orb}(X) \to \Omega^*(X)$ will induce our desired isomorphism $\varphi^* : H^*_{orb}(X) \to H^*_{DR}(X)$ at cohomology level.

Let us construct the map φ. As seen in the proof of Lemma 1.46, we can arrange a metric so that for each pair D_i, D_j of intersecting surfaces, the normal bundle ν_{D_i} with respect to this metric satisfy that given $p_{ij} \in D_i \cap D_j$ and ε small, the fiber of

$$\nu^{\varepsilon}_{D_i} = \{(x, u) : x \in D_i, u \in T_x D_i^{\perp}, |u| \leq \varepsilon\}$$

over p_{ij} corresponds to D_j via the exponential map

$$f_i : \nu^{\varepsilon}_{D_i} \to U^{D_i}, \quad (x, u) \mapsto \exp_x(u).$$

Consider a bump function ρ so that $\rho(t) = 1$ for $t \geq \frac{2\varepsilon}{3}$ and $\rho(t) = 0$ for $t \leq \frac{\varepsilon}{3}$, and define the map

$$h_i : U^{D_i} \to X, \quad \exp_x(u) \mapsto \exp_x(\rho(|u|)u).$$

The map h_i projects a tubular neighborhood of D_i onto D_i and extends to all X as the identity. Also, $D_j \cap U^{D_i}$ is mapped to the intersection points of D_j and D_i. Now consider $\varphi = h_1 \circ \cdots \circ h_l$ (with l the number of isotropy surfaces). This map φ clearly satisfies the three items above. Note that all the maps $h_1, \ldots, h_l$ are homotopic to the identity so φ is also homotopic to the identity. Moreover, φ is a smooth map for the manifold structure of X, and an orbifold map for the orbifold structure of X, so it induces in a functorial way maps in the both the ordinary and the orbifold smooth forms.

More precisely, φ induces two maps of cochain complexes

$$\varphi^{\#} : \Omega^*_{orb}(X) \to \Omega^*(X)$$
$$\varphi^{\#} : \Omega^*(X) \to \Omega^*_{orb}(X)$$

since the pull-back of an orbi-form (or an ordinary form) by φ can be seen both as an orbi-form and as an ordinary form. The composition of the above maps (in any order) is a cochain map inducing the identity in cohomology (because φ is homotopic to Id_X). Hence the map φ induces the desired isomorphism $\varphi^* : H^*_{orb}(X) \to H^*(X)$ as we wanted to see. □

2. 4-dimensional cyclic symplectic and Kahler orbifolds.

Let us study now how to obtain a symplectic cyclic orbifold from a symplectic manifold. This means that, apart from transforming a manifold atlas into an orbifold one, we have to construct a symplectic orbi-form from a symplectic form.

PROPOSITION 4.10. *Let (X, ω) be a symplectic smooth 4-manifold with symplectic surfaces D_i intersecting transversely and positively, and choose coefficients $m_i > 1$ such that $\gcd(m_i, m_j) = 1$ if D_i, D_j intersect. Then there is a smooth symplectic cyclic orbifold $(X, \hat{\omega})$ with isotropy surfaces D_i of multiplicities m_i.*

PROOF. By Lemma 1.20, we can assume that the surfaces D_i intersect orthogonally. As in the proof of Proposition 4.6, we start by fixing a metric. We do this as follows. First at any point at an intersection $D_i \cap D_j$, fix a Darboux chart

$$f : B_\varepsilon(0) \times B_\varepsilon(0) \to U$$

with $D_i \cap U = f(B_\varepsilon(0) \times \{0\})$ and $D_j \cap U = f(\{0\} \times B_\varepsilon(0))$.

Take a standard metric on U, and the corresponding almost complex structure J_U on U. Fix now compatible almost complex structures J_i on each D_i (that is, $J_i : T_x D_i \to T_x D_i$ at each $x \in D_i$), which agree on U with J_U. The normal bundle ν_{D_i} over D_i is a symplectic bundle. Take a Riemannian metric on ν_{D_i} compatible with its symplectic structure, and define a Riemannian metric on each $T_x X = T_x D_i \oplus \nu_{D_i,x}$, $x \in D_i$, by declaring the direct sum orthogonal. We extend this metric g on $\bigcup D_i$ to a Riemannian metric on the whole of X compatible with the symplectic form. This produces an almost Kahler structure on the whole of X for which each D_i is a J-invariant surface.

Now we use this metric g for producing the atlas of Proposition 4.6 that gives X the structure of a smooth orbifold. Let us now construct the orbifold symplectic form. We need first to modify ω to a nearby ω' as follows.

Let $D = D_i$ be one of the isotropy surfaces. On ν_D^ε we have a radial coordinate r, and an angular coordinate θ, well-defined in every chart up to addition of a function on D. By construction, we have $\omega = \omega|_D + r\, dr \wedge d\theta$ along D. For the bundle $\nu_D \to D$, consider a connection 1-form $\eta \in \Omega^1(\nu_D - D)$, and let $F = d\eta \in \Omega^2(\nu_D)$ be its curvarture. Thus $\Omega = r dr \wedge \eta - \frac{1}{2} r^2 F + \omega|_D$, is a closed form on ν_D that coincides with ω along D. In the last expression, $\omega|_D$ stands for the pull-back of $\omega|_D$ by the bundle projection. Now $|\Omega - \omega| \leq Cr$, where C is a constant independent of r. On ν_D^ε, $\Omega - \omega$ is closed so (being zero on D) it is exact, say $\Omega - \omega = d\beta$.

We can choose the 1-form β so that it satisfies $|\beta| \leq Cr^2$, by the usual standard procedure to produce a primitive of an exact form. Indeed, if $\Omega - \omega = \alpha_0 \wedge dr + \alpha_1$, one takes $\beta = \int_0^r \alpha_0 dr$, which is smooth (see [**24**, p. 542]).

We also arrange the 1-form η to be equal to $d\theta$ on $U \cap \nu_D^\varepsilon$, so that $F = 0$ on $U \cap \nu_D^\varepsilon$ and so $\Omega = \omega$ on $U \cap \nu_D^\varepsilon$.These forms Ω's for the different D's paste to a globally defined Ω on a neighbourhood of $\bigcup D_i$.

Take a cut-off function $\rho : [0, \varepsilon] \to [0, 1]$ with $\rho(r) \equiv 1$ for $r \in [0, \frac{1}{3}\varepsilon]$, and $\rho(r) \equiv 0$ for $r \in [\frac{2}{3}\varepsilon, \varepsilon)$, and $|\rho'| \leq C/\varepsilon$. Hence $\omega' = \omega + d(\rho\beta)$ satisfies that it is equal to Ω for $|r| \leq \frac{\varepsilon}{3}$, equal to ω for $|r| \geq \frac{2\varepsilon}{3}$, and $|\omega' - \omega| = |d(\rho\beta)| = |d\rho \wedge \beta + \rho \wedge d\beta| \leq C\varepsilon$. This produces a globally defined 2-form ω' on X. For ε small enough, ω' is symplectic.

Now let us define our orbi-symplectic form. Take first a point x in some $D = D_i$ and not in U. We have smooth coordinates (w_1, w_2), $w_2 = re^{2\pi i\theta}$, and orbifold coordinates (z_1, z_2), $z_1 = w_1$ and $z_2 = re^{2\pi i\vartheta}$, $\theta = m\vartheta$. The action is $\xi \cdot (z_1, z_2) = (z_1, \xi z_2)$. Here $\omega' = \Omega = \alpha + r\, dr \wedge d\theta + r\, dr \wedge \gamma$, where α is a 2-form and γ is a 1-form, and both α and γ are invariant in the fiber direction, in particular SO(2)-equivariant (recall that the connection 1-form is $\eta = d\theta + \gamma$).

We set, in the orbifold coordinates (z_1, r, ϑ),

$$\hat{\omega} = \alpha + m\, r\, dr \wedge d\vartheta + r\, dr \wedge \gamma.$$

This is closed, smooth, symplectic and $\mathbb{Z}_m$-invariant. Moreover, $\hat{\omega}$ agrees with the pull-back of ω' via the orbifold chart $(z_1, z_2) \mapsto (w_1, w_2)$, and this implies that $\hat{\omega}$ is invariant by the orbifold change of charts.

Finally, on U, we take smooth coordinates (w_1, w_2), $w_1 = r_1 e^{2\pi i\theta_1}$, $w_2 = r_2 e^{2\pi i\theta_2}$, and orbifold coordinates are $z_1 = r_1 e^{2\pi i\vartheta_1}$, $z_2 = r_2 e^{2\pi i\vartheta_2}$, with $\theta_1 = m_2\vartheta_1$, $\theta_2 = m_1\vartheta_2$. Here $\omega' = r_1\, dr_1 \wedge d\theta_1 + r_2\, dr_2 \wedge d\theta_2$. We set

$$\hat{\omega} = m_2 r_1\, dr_1 \wedge d\vartheta_1 + m_1 r_2\, dr_2 \wedge d\vartheta_2\,,$$

which defines an orbifold symplectic form on U. □

Remark 4.11. *We observe that in the proof of Proposition 4.10, the perturbed symplectic form $\omega' \in \Omega^2(X)$ satisfies*

$$[\omega'] = [\omega] \in H^2_{DR}(X).$$

This is checked by integrating along any oriented surface $S \subset X$. Take S to intersect transversely all D_i. Let S_δ be S minus small δ-balls around the intersections $S \cap D_i$. Then

$$\langle [\omega'] - [\omega], [S] \rangle = \int_S d(\rho\beta) = \lim \int_{S_\delta} d(\rho\beta) = \lim \int_{\partial S_\delta} \rho\beta = \lim \int_{\partial S_\delta} \beta = 0$$

since $|\beta| \leq Cr^2$. Note that β may not be $\mathcal{C}^\infty$ at $r = 0$ when expressed in the polar coordinates of the normal bundle (that is the reason for this indirect argument).

Let X the orbifold constructed in Proposition 4.10. Consider the orbifold forms

$$(\Omega_{orb}(X), d)$$

and their cohomology, denoted by $H^*_{orb}(X)$. We already know that the orbifold De Rham cohomology of the cyclic orbifold X is isomorphic to the De Rham cohomology of the manifold X, (since both cohomologies are isomorphic to the singular cohomology of the topological space X by Proposition 2.25). So there exists an isomorphism

$$H^*_{orb}(X) \cong H^*_{DR}(X)$$

and moreover in this case the isomorphism can be constructed explicitly in a geometric way using the map φ constructed in Proposition 4.9.

The orbifold symplectic form $\hat{\omega}$ constructed in the proof of Proposition 4.10 defines a class in

$$[\hat{\omega}] \in H^2_{orb}(X)$$

and we have the correspondence

$$[\hat{\omega}] = [\omega'] = [\omega]$$

under the isomorphism $H^*_{orb}(X) \cong H^*_{DR}(X)$. This is because $\varphi^*\omega' = \varphi^*\hat{\omega}$ as forms (seen either in $\Omega^2(X)$ or in $\Omega^2_{orb}(X)$).

The next proposition shows that the correspondence between smooth cyclic orbifolds and smooth manifolds is compatible with complex structures.

PROPOSITION 4.12. *If (X, J) is a smooth complex cyclic orbifold, then X inherits naturally the structure of a smooth complex manifold and the isotropy surfaces $D_i \subset X$ are complex curves intersecting transversely.*

PROOF. As the almost-complex structure J is integrable, we can refine the orbifold smooth atlas of X to get an holomorphic orbifold atlas. Now take holomorphic orbifold charts

$$\phi : (U, J_0) \to (V, J) \subset X$$

where $U \subset \mathbb{C}^2$ a neighborhhod of $0 \in \mathbb{C}^2$. Being holomorphic means that $\phi_* \circ J_0 = J \circ \phi_*$, with J_0 the standard complex structure on $\mathbb{C}^2$.

The group $\Gamma = \mathbb{Z}_m$ acts on U by a biholomorphism $f : \widetilde{U} \to \widetilde{U}$. The map

$$\varphi(z) = \frac{1}{m} \sum_{k=0}^{m-1} d_0 f^{-k}(f^k(z))$$

satisfies $d_0\varphi = \mathrm{Id}$ and $\varphi(0) = 0$, so it is a diffeomorphism between small neighborhoods of 0.

Define now a new orbifold chart by $\phi' = \phi \circ \varphi^{-1}$, defined in a maybe smaller neighborhood U_0 of 0. A straightforward computation gives $d_0 f(\varphi(z)) = \varphi(f(z))$. Hence $d_0 f = \varphi \circ f \circ \varphi^{-1}$, so the action of Γ in the chart ϕ' is linear.

The conclusion is that we can assume that $\Gamma < \mathrm{GL}(2, \mathbb{C})$ acts by complex transformations in an holomorphic orbifold chart $\phi : (U, J_0) \to (V, J) \subset X$. This defines an holomorphic orbifold atlas (U, V, ϕ, Γ) of X with $\Gamma < \mathrm{GL}(2, \mathbb{C})$ acting by linear complex maps.

Now, the quotient U/Γ has a natural complex structure. This means that the complex structure on the complement of the isotropy locus $\bigcup D_i$ extends naturally to $\bigcup D_i$. Also, the induced map $\bar{\phi} : U/\Gamma \to V$ is holomorphic, and thus biholomorphic since it is bijective. These maps $\bar{\phi}$ define a complex manifold atlas of the space X as follows. Take an orbifold chart (U, V, ϕ, Γ). Since $\Gamma \cong \mathbb{Z}_m$ is cyclic we must have $\Gamma = \langle \xi \rangle$ with $\xi = e^{2\pi i/m}$ acting in U by $\xi(z_1, z_2) = (\xi^{j_1} z_1, \xi^{j_2} z_2)$. Note that if (z_1, z_2) are the coordinates for an orbifold holomophic chart, with action $\xi \cdot (z_1, z_2) = (\xi^{m_2} z_1, \xi^{m_1} z_2)$, $\xi = e^{2\pi i/m}$, $m = m_1 m_2$, then $w_1 = z_1^{m_1}$, $w_2 = z_2^{m_2}$ define holomorphic coordinates for the quotient $U/\Gamma \cong V$, so (w_1, w_2) are holomorphic coordinates in the coordinate patch $V \subset X$.

Now take any of the surfaces D_i, let us call it D. We have that D is expressed in U as the fixed points of $\Gamma < \mathrm{GL}(2, \mathbb{C})$. Hence the surface D is given as $D = \{z_1 = 0\}$ or $D = \{z_2 = 0\}$ in U. In the induced holomorphic coordinates (w_1, w_2) of V, D is given by $D = \{w_1 = 0\}$ or $D = \{w_2 = 0\}$, so D is a complex curve of the complex manifold X, as we wanted to see. □

3. Seifert bundles.

A Seifert bundle is a space fibered by circles over an orbifold. We give a precise definition.

DEFINITION 4.13. *Let X be a cyclic n-dimensional orbifold. A* Seifert bundle *over X is an oriented $(n+1)$-dimensional manifold M equipped with a smooth S^1-action and a continuous surjective map $f : M \to X$ such that for any $x \in X$ there is an orbifold chart $(U, V, \phi, \mathbb{Z}_m)$, with $x \in V$ and there is a S^1-equivariant diffeomorphism φ such that the following is a commutative* *diagram:*

$$\begin{array}{ccc} (S^1 \times U)/\mathbb{Z}_m & \xrightarrow{\varphi} & f^{-1}(V) \\ \downarrow p_2 & & \downarrow f \\ U/\mathbb{Z}_m & \xrightarrow{\bar{\phi}} & V \end{array}$$

Where the following holds:

(1) *The map $\bar{\phi}$ is the homeomorphsim induced by the orbifold chart $\phi : U \to V$.*
(2) *The $\mathbb{Z}_m$-action on $S^1 \times U$ is the diagonal action given by $\xi(s, u) = (\xi s, \xi \cdot u)$.*
(3) *The diffeomorphism φ of the diagram is S^1-equivariant with respect to the action of S^1 on $(S^1 \times U)/\mathbb{Z}_m$ given by $s \cdot [(t, u)] = [(st, u)]$.*
(4) *The map p_2 maps $[(s, u)] \in (S^1 \times U)/\mathbb{Z}_m$ to $[u] \in U/\mathbb{Z}_m$.*

The orbit $O_e := (S^1 \times \{0\})/\mathbb{Z}_m \subset (S^1 \times U)/\mathbb{Z}_m$ is called a *exceptional orbit*, and the natural identification of O_e with nearby orbits over non-isotropy points has degree m. The other fibers will be called *generic* fibers. Note that points on exceptional fibers have non trivial stabilizer. The existence of exceptional fibers will be the reason why we cannot avoid local coefficients in the Leray spectral sequence.

Remark 4.14. *Let us be more explicit in the case that X is a cyclic 4-orbifold. In this case $U \subset \mathbb{C}^2$.*

(1) *The action of $\mathbb{Z}_m$ is given in $S^1 \times U$ by $\xi(s, z_1, z_2) = (\xi s, \xi^{j_1} z_1, \xi^{j_2} z_2)$, and in U by $\xi(z_1, z_2) = (\xi^{j_1} z_1, \xi^{j_2} z_2)$.*
(2) *The action of S^1 in $(S^1 \times U)/\mathbb{Z}_m$ is given by $t[(s, z_1, z_2)] = [(ts, z_1, z_2)]$ for $t \in S^1$. Here a bracket denotes the equivalence class in the corresponding space. Obviously this S^1-action is well defined on the quotient since S^1 is abelian.*

Note also that a Seifert bundle is not *a fiber bundle in the usual sense. In particular it does not have the homotopy lifting property. Although it is true that all the fibers are homeomorphic, the inclusions of distinct fibers on the total space M may not be homotopy-equivalent.*

For instance, suppose that $j_1 = j_2 = 1$ in the action of $\mathbb{Z}_m$ in item (1), so

$$\xi(s, z_1, z_2) = (\xi s, \xi z_1, \xi z_2).$$

Consider the fiber over the point $[(0, 0)] \in U/\mathbb{Z}_m$. When $t \in S^1$ acts on $[(s, 0, 0)]$ we go m times round the fiber as t traverses S^1, since $[(s, 0, 0)] = [\xi(s, 0, 0)] = [(\xi s, 0, 0)]$ for $\xi = e^{2\pi i/m}$. On the other hand when $t \in S^1$ acts on (s, z_1, z_2) with $(z_1, z_2) \neq (0, 0)$, the action only goes round one time the fiber when t traverses S^1. This is because $(ts, z_1, z_2) = (\xi^j s, \xi^j z_1, \xi^j z_2)$ only when $j = 0$ and $t = 1$.

In this vein we may call an orbit $O_z := (S^1 \times \{(z_1, z_2)\})/\mathbb{Z}_m$ with $z = (z_1, z_2) \neq (0, 0)$ a generic orbit*, and the fiber $O_e := (S^1 \times \{(0, 0)\})/\mathbb{Z}_m$ a* exceptional orbit*. The above shows that generic orbits O_z are m-times longer that the exceptional orbit O_e over $[(0, 0)]$. In other words, the orbit O_e closes m-times sooner than the generic fiber O_z. This implies that the map $O_z \to O_e$ given by $[(s, z_1, z_2)] \mapsto [(s, 0, 0)]$ has degree m. The preimages of $[(s, 0, 0)]$ by this map are the points $[(\xi^j s, z_1, z_2)]$ for $0 \leq j \leq m - 1$.*

The conclusion is that for a Seifert bundle, inclusions of nearby fibers are not in general homotopy equivalent.

Although the definition of Seifert bundle can appear quite intricate, we are going to see shortly that it is a quite natural structure to consider. Indeed, a Seifert bundle structure on M is equivalent to a circle action in which no point of M is fixed by all elements of the circle. Let us prove this step by step.

DEFINITION 4.15. *Let G be a group and M a set. We say that an action of G on M is* fixed point free *if for every point $y \in M$ the isotropy subgroup $G_y = \{g \in G : gy = y\}$ is a proper subgroup of G.*

LEMMA 4.16. *The only closed proper subgroups of S^1 are the finite cyclic subgroups generated by some m-th complex root of 1.*

PROOF. Let us prove that any infinite subgroup of S^1 must be dense. Let $H < S^1$ be an infinite subgroup. Select an arbitrary natural number N, and choose $h_1, \dots, h_N$ distinct elements of H. There must be at least two of the h_i whose angle difference is less than $2\pi/N$. Call them h_k and h_j, and let $h = h_k h_j^{-1} \in H$. By construction $h = e^{2\pi i\theta}$ for some $0 < \theta < 2\pi/N$. There exists a natural number M so that $M\theta \geq 1$. Consider the set $\{h, h^2, \dots, h^M\}$. It is easy to check that for any $g \in S^1$ there exists some $1 \leq l \leq M$ so that $|g - h^l| \leq \pi/N$. Since $N \in \mathbb{N}$ was arbitrary, this proves H is dense. The conclusion is that any closed and proper subgroup of S^1 must be finite.

Let us see that any finite subgroup H of S^1 is cyclic. Let $h_0 \in H$, $h_0 \neq 1$ and $h_0 = e^{i\theta_0}$ the point of H with smallest angle θ_0 in $(0, 2\pi)$. Let us see that h_0 generates H. Given $h \in H$, $h \neq 1$, put $h = e^{i\theta}$ for $\theta \in (0, 2\pi)$. We have $0 < \theta_0 < \theta$. Put $\theta = \theta_0 q + r$ for some $q > 0$ and some $r \in [0, \theta_0)$. Then $h = e^{i\theta} = h_0^q h'$, being $h' = e^{ir} = h h_0^{-q} \in H$. Since h' has angle r, it must be $r = 0$ by construction of θ_0 so $h' = 1$ and $h = h_0^m$. This concludes the proof. □

Let us point out that if a Lie group G acts smoothly on a manifold M, for any $y \in M$ the isotropy group $G_y = \{g \in G : gy = y\}$ is a closed subgroup of G since it is the preimage $(m_y)^{-1}(\{y\})$ of the closed set $\{y\}$ by the smooth function $m_y : G \to M$, $g \mapsto gy$.

We conclude the following.

COROLLARY 4.17. *Suppose that we have a fixed point free smooth action of $G = S^1$ in a manifold M. Then all isotropy groups G_y for $y \in M$ are finite and cyclic subgroups of S^1, so for any $y \in M$ there exists $m = m(y) \in \mathbb{N}$ so that $G_y \cong \mathbb{Z}_m$.*

PROOF. Apply Lemma 4.16. □

The following result says that any smooth S^1 action on an oriented manifold M acts by orientation-preserving diffeomorphisms.

PROPOSITION 4.18. *Let G be a connected Lie group. If M is an oriented manifold with a smooth action of G, the action preserves orientation.*

PROOF. Let $g \in G$ be arbitrary and let us see that $\varphi_g : M \to M$, $y \mapsto gy$ preserves orientation. Since G is connected and locally path-connected, it is path connected. Choose a path $g_t \in G$ so that $g_0 = 1$ and $g_1 = g$. For an arbitrary point $p \in M$ choose a positive chart around p and (using this chart) consider the Jacobian determinant $\det(d_p\varphi_{g_t})$, which is a continuous and never vanishing function on $t \in [0, 1]$, positive at $t = 0$ since $\varphi_0 = \mathrm{Id}_M$. We see that $\det(d_p\varphi_g)$ is positive, so φ_g preserves orientation. □

PROPOSITION 4.19. *An oriented $(n+1)$-manifold M endowed with a smooth and fixed point free action of S^1 is a Seifert bundle over a cyclic (oriented) n-orbifold X.*

PROOF. Let M be a manifold endowed with a fixed point free action of S^1. Then X will be the space of leaves of the S^1-action. The orbifold structure on X is obtained as follows. Take an auxiliary Riemannian metric g on M and average it over S^1 to make it S^1-invariant. Then the group S^1 acts on M by isometries with respect to this metric.

For a point $p \in M$, let O_p be the orbit of p. Let $I_p = \mathbb{Z}_m = \langle \xi \rangle$, $\xi = e^{2\pi i/m}$ be the isotropy of p. Then the action of ξ, say $f : M \to M$, fixes p and the tangent direction R_p to the orbit

O_p. Hence the differential of d_pf fixes the orthogonal hyperplane $H_p = R_p^\perp \subset T_pM$, inducing an action of $\mathbb{Z}_m$ on it. Since M is oriented, d_pf preserves orientation by Proposition 4.18, so $\mathbb{Z}_m = \langle d_pf \rangle < \mathrm{SO}(n)$.

For a small $U \subset H_p$, the exponential map and the S^1-action give a local diffeomorphism

$$\varphi : S^1 \times U \to M, \quad \varphi(s,u) = s \cdot \exp_p(u).$$

On $S^1 \times \{0\}$ the isotropy of the S^1-action on M is $\mathbb{Z}_m$, hence there is an induced homeomorphism $\bar{\varphi} : (S^1 \times U)/\mathbb{Z}_m \cong W$, being $W \subset M$ a neighbourhood of $O_p = \varphi(S^1 \times \{0\}) \subset M$.

Hence, a neighborhood of O_p in M is modelled on $(S^1 \times U)/\mathbb{Z}_m$. The $\mathbb{Z}_m$-action on $S^1 \times U$ is by multiplication by ξ on the S^1-factor, and by the action of d_pf on U. We conclude that the space of leaves X of the S^1-action on M is locally modelled in $U/\mathbb{Z}_m$, and $(U, V, \phi, \mathbb{Z}_m)$ gives the desired orbifold chart, with

$$\phi : U \to V \subset X, \quad u \mapsto O_{\exp_p(u)}$$

mapping each $u \in U$ to the orbit $O_{\exp_p(u)} = \{s \cdot \exp_p(u) : s \in S^1\}$ of the corresponding point $\exp_p(u) \in M$. Recall that $V = \{O_{\exp_p(u)} : u \in U\} \subset X$. Two such orbits $\phi(u)$ and $\phi(u')$ are equal iff $u = (d_pf)^j(u')$ for some $0 \le j \le m-1$, so this yields an homeomorphism $\bar{\phi} : U/\mathbb{Z}_m \to V$. This completes the proof. □

Suppose that X is a 4-dimensional orbifold and $f : M \to X$ is a Seifert bundle over X. According to the normal form of the $\mathbb{Z}_m$-action given in (20), the open subset $\pi^{-1}(U) \cong (S^1 \times \widetilde{U})/\mathbb{Z}_m$ is parametrized by $(u, z_1, z_2) \in S^1 \times \mathbb{C}^2$, modulo the $\mathbb{Z}_m$-action $\xi \cdot (u, z_1, z_2) = (\xi u, \xi^{j_1} z_1, \xi^{j_2} z_2)$, for some integers j_1, j_2, where $\xi = e^{2\pi i/m}$. The S^1-action is given by $s \cdot (u, z_1, z_2) = (su, z_1, z_2)$, so $\mathbb{Z}_m \subset S^1$ is the isotropy group of the orbit $O_p \subset M$, and the exponents j_1, j_2 are determined by the S^1-action.

Now we want to show that a Seifert bundle can be determined by some data on the base space X (more concretely, the orbit invariants and Chern class of the Seifert bundle, which we define below). This is specially useful to us when X is a smooth orbifold.

DEFINITION 4.20. *Let $f : M \to X$ be a Seifert bundle.*

We say that $\{(D_i, m_i, j_i)\}$ are the orbit invariants *of the Seifert bundle if $D_i \subset X$ are the isotropy surfaces, with multiplicities m_i, and for every i the local model around any point $p \in D_i^o = D_i \setminus \bigcup_{k \neq i}(D_i \cap D_k)$ is of the form $(S^1 \times U)/\mathbb{Z}_{m_i}$ with action $\xi \cdot (u, z_1, z_2) = (\xi u, z_1, \xi^{j_i} z_2)$, being $D_i = \{z_2 = 0\}$ and $\langle \xi \rangle \cong \mathbb{Z}_{m_i}$, $\xi = e^{2\pi i/m_i}$, and $(m_i, j_i) = 1$.*

REMARK 4.21. *If the orbifold X is smooth, then the orbit invariants determine the orbifold structure of X. Indeed, for a point $p \in D_i \cap D_k$, the local model is of the form $(S^1 \times U)/\mathbb{Z}_m$, $m = m_i m_k$, with action of $\mathbb{Z}_m = \langle \xi \rangle$ given by $\xi \cdot (u, z_1, z_2) = (\xi u, \xi^{j_k} z_1, \xi^{j_i} z_2)$, and with $D_i = \{z_2 = 0\}$, $D_k = \{z_1 = 0\}$.*

Let us now define the *Chern class* of a Seifert bundle. Note that a Seifert bundle does not fit into the definition of a fiber bundle, so this has to be defined ad-hoc.

Associated to a Seifert bundle $f : M \to X$ there is an (ordinary, locally trivial) circle bundle $f/\mu : M/\mu \to X$, obtained as a quotient of M by all the stabilizers. More precisely, let $\mu \subset S^1$ be the group generated by all the stabilizers of points of M. Let us see that μ is a finite group of S^1 if M is compact. If we call μ_y the stabilizer (or isotropy subgroup) of $y \in M$, then $\mu = \bigcup_{y \in M} \mu_y$. We already know that μ_y is finite. On the other hand, for an open set of M of the kind $W = f^{-1}(V) \cong (S^1 \times U)/\mathbb{Z}_m$ as in Definition 4.13, we see that $\bigcup_{y \in W} \mu_y \cong \mathbb{Z}_m$ is finite. If we cover $M = \bigcup_{i=1}^{l} W_i$ by a finite number of open sets W_i of this kind, each with isotropy $\mathbb{Z}_{m_i}$, then $\mu \cong \mathbb{Z}_{m(X)}$ with $m(X) = \mathrm{lcm}\{m_i, 1 \le i \le l\}$ the least common multiple.

Obviously, the S^1-action on M descends to a $S^1/\mu \cong S^1$ action on the space M/μ, and by construction this action has no fixed points. Note that M/μ has a natural orbifold structure as a global quotient of a manifold by a finite group. Therefore we can consider the map

$$f/\mu : M/\mu \to X$$

which is an ordinary (locally trivial) circle bundle. We have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\;q\;} & M/\mu \\ & \searrow^{f} & \downarrow{\scriptstyle f/\mu} \\ & & X \end{array}$$

The association $f \to f/\mu$ allows us to define the Chern class of a Seifert bundle.

Definition 4.22. *For a Seifert bundle $f : M \to X$, we define its* Chern class*, denoted $c_1(M/X)$, as follows. Let $\mu = \mathbb{Z}_{m(X)}$, where $m(X) = \operatorname{lcm}\{m(x) : x \in X\}$. Consider the circle fiber bundle $M/\mu \to X$ and its Chern class $c_1(M/\mu) \in H^2(X, \mathbb{Z})$. We define*

$$c_1(M/X) = \frac{1}{m(X)} c_1(M/\mu) \in H^2(X, \mathbb{Q}).$$

The next proposition shows that the orbit invariants $\{(D_i, m_i, j_i)\}$ and the Chern class $c_1(M/X)$ determine the Seifert bundle globally when X is smooth. It also shows the existence of a Seifert bundle over a base space equipped with a given set of data (orbit invariants and Chern class).

Proposition 4.23. *Let X be an oriented 4-manifold and $D_i \subset X$ oriented surfaces of X which intersect transversely. Let $m_i > 1$ such that $\gcd(m_i, m_k) = 1$ if D_i and D_k intersect. Let $0 < j_i < m_i$ with $\gcd(j_i, m_i) = 1$ for every i. Let $0 < b_i < m_i$ such that $j_i b_i \equiv 1 \pmod{m_i}$. Finally, let B be a complex line bundle on X. Then there is a Seifert bundle $f : M \to X$ with orbit invariants $\{(D_i, m_i, j_i)\}$ and first Chern class*

$$c_1(M/X) = c_1(B) + \sum_i \frac{b_i}{m_i}[D_i]. \tag{21}$$

The set of all such Seifert bundles forms a principal homogeneous space under $H^2(X, \mathbb{Z})$, where the action corresponds to changing B.

Proof. As mentioned above, recall that (since X is an smooth orbifold) the orbit invariants determine uniquely the orbifold structure of X. Let $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$ be an orbifold atlas of X. The orbit invariants determine the local model of any possible Seifert bundle with base X, so the possible Seifert bundles with these orbifold invariants are given by the gluing of the local models. This is defined by transition orbi-functions

$$g_{\alpha\beta} : V_\alpha \cap V_\beta \to S^1$$

which are Γ_γ-invariant for $V_\gamma \subset V_\alpha \cap V_\beta$, and such that $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\delta}$. Therefore it is defined by a 1-cocycle in $\mathcal{C}^\infty_{orb}(S^1)$, the orbifold functions with values in S^1. Using the exponential short exact sequence of sheaves $0 \to \mathbb{Z} \to \mathcal{C}^\infty_{orb} \to \mathcal{C}^\infty_{orb}(S^1) \to 0$, where the sheaf of orbifold differentiable (real) functions $\mathcal{C}^\infty_{orb}$ is a fine sheaf, we have that the possible Seifert bundles are parametrized by $H^1(X, \mathcal{C}^\infty_{orb}(S^1)) \cong H^2(X, \mathbb{Z})$. We can consider the tensor product $M \otimes B$, for a line bundle $B \to X$, by multiplying the transition functions. Therefore the set of Seibert bundles forms a homogeneous space under $H^2(X, \mathbb{Z})$.

For $M \otimes B$, we have that $(M \otimes B)/\mu = (M/\mu) \otimes B^{\otimes m}$, since the quotient is given locally by $(u, z_1, z_2) \mapsto (u^m, z_1, z_2)$, where $m = m(X)$. So

$$c_1((M \otimes B)/X) = \frac{1}{m} c_1((M \otimes B)/\mu) = \frac{1}{m}(c_1(M/\mu) + m\, c_1(B)) = c_1(M/X) + c_1(B).$$

To prove (21) is equivalent to prove that

$$c_1(M/\mu) = m\, c_1(M/X) \equiv \sum_i b_i \frac{m}{m_i}[D_i] \pmod{m}.$$

For this we take a 2-cycle $S \subset X$ intersecting transversely the D_j's, and compute $\langle c_1(M/\mu), S\rangle$. To compute $c_1(M/\mu)$, we fix a transverse section s of the line bundle associated to $M/\mu \to X$.

In $\langle c_1(M/\mu), S\rangle$ there is a contribution coming from balls $B_p \subset S$ around each intersection point $p \in S \cap (\bigcup D_i)$ and a contribution from $S^o = S \setminus \bigcup B_p$. The second one is $\langle c_1(M/\mu), S^o\rangle = \langle m\, c_1(M/X), S^o\rangle \in m\,\mathbb{Z}$, since $M \to X$ is an honest circle bundle over the locus S^o. For this equality we choose s to be the image of a section of the line bundle associated to the circle bundle $M|_{S^o} \to S^o$.

Now we look at the circle bundle $M/\mu \to X$ at a point $p \in S \cap D_i$. We can arrange orbifold coordinates (z_1, z_2) in a chart U with $U/\mathbb{Z}_m = B_p \subset X$ such that $D_i = \{z_2 = 0\}$ and $S = \{z_1 = 0\}$ in U. The Seifert bundle is given by coordinates (u, z_1, z_2) modulo $\xi \cdot (u, z_1, z_2) = (\xi u, z_1, \xi^{j_i} z_2)$, $\xi = e^{2\pi i/m_i}$. Equivalently, modulo $(u, z_1, z_2) \mapsto (\xi^{b_i} u, z_1, \xi z_2)$. The circle bundle M/μ is parametrized by $(v = u^m, z_1, z_2)$ modulo $(v, z_1, z_2) \mapsto (v, z_1, \xi z_2)$. The section s of M/μ lifts to a section $\hat{s}$ of M over ∂B_p. In orbifold coordinates of X, $\hat{s}$ is of the form $\hat{s}(z_1, z_2) = (u(z_1, z_2), z_1, z_2)$, with $u(z_1, \xi z_2) = \xi^{b_i} u(z_1, z_2)$. This means that we can choose $u(z_1, z_2) = z_2^{b_i}$.

Therefore, the section s is given in the chart U as $s(z_1, z_2) = (v, z_1, z_2)$ with $v = z_2^{b_i m}$. Going back to smooth coordinates (w_1, w_2) of $B_p \subset X$, we have $w_1 = z_1$, $w_2 = z_2^{m_i}$, so the section s is written as

$$s(w_1, w_2) = (w_2^{b_i m/m_i}, w_1, w_2).$$

Therefore the zero set of s in $B_p \subset X$ along $S = \{w_1 = 0\}$ is the point $p = (0, 0)$ with multiplicity $b_i m/m_i$. This multiplicity contibutes positively or negatively in $\langle c_1(M/\mu), S\rangle$ depending on whether S and D_i intersect positively or not at p. Adding the contributions of all points $p \in S \cap D_i$, we get the contribution $\frac{b_i m}{m_i}\langle [D_i], S\rangle$ to the pairing $\langle c_1(M/\mu), S\rangle$. Summing in all i, we get

$$\langle c_1(M/\mu), S\rangle \equiv \sum_i b_i \frac{m}{m_i}\langle [D_i], S\rangle \pmod{m}$$

This proves the sought formula for $c_1(M/X)$. □

Let $f : M \to X$ be a Seifert bundle, $p \in M$ and $x = f(p) \in X$. We call fiber over x to the the orbit O_p, which is of the form $S^1/\mathbb{Z}_m$, where $m = m(x) = m(p)$ is both the isotropy of x (as orbifold point) and the isotropy of p (for the S^1-action on M).

DEFINITION 4.24. *Let $p \in M$, and $f : M \to X$ a Seifert bundle. We call the orbit O_p* semi-regular *if the orbifold point $x = f(p) \in X$ is a smooth orbifold point.*

This means that the local model in Proposition 4.4 is of type (b) or (d). In the case (d), the orbit O_p has nearby orbits $O_{p'}$ with multiplicity $m(p') = m(p)$. In case (b) we have $m = m_1 m_2$ with $\gcd(m_1, m_2) = 1$, and O_p has nearby orbits O_{p_1} and O_{p_2} of multiplicities m_1, m_2, respectively.

DEFINITION 4.25. *We say that a Seifert bundle $f : M \to X$ is* semi-regular *if the base orbifold X is smooth, that is all orbits are semi-regular.*

Semi-regularity of a Seifert bundle is a property we will need for some results in the subsequent Chapters.

4. Topology of Seifert bundles.

Now we want to relate the homology of M with that of X for a Seifert bundle $f : M \to X$. Following [**34**], we analize the topology of a Seifert bundle structure $f : M \to X$ via the Leray spectral sequence. This entails homology and cohomology groups, which will be our main concern here.

Let us sumarise first some results we obtained before.

PROPOSITION 4.26. *Let $f : M \to X$ be a Seifert bundle. Suppose that M is closed, orientable and connected. Then the following holds*

(1) *The orbifold X is closed, connected and orientable.*
(2) *For any choice of orientation, the smooth S^1-action on M is orientation preserving and this induces an orientation on X.*

PROOF. The fact that X is orientable was proved in Proposition 4.19. The rest of the claims are obvious. □

Let us state some assumptions for this subsection.

(1) We shall assume that $f : M \to X$ is a Seifert bundle with M a closed 5-manifold and X a 4-orbifold. This hypothesis on the dimension is only for concreteness, since it is the case in which we are interested.
(2) We will suppose that M is oriented and connected, so the smooth S^1-action on M preserves the orientation. This induces an orientation on the space of leaves X. Under these hypothesis on M, the orbifold X has to be closed, orientable and connected.
(3) We will try to follow the notation s_i for singular points of X and p_i for the smooth points of X. When we want to refer to any kind of isotropy point of X, we will use q_i. Moreover we will denote D_i for the surfaces of orbifold points of X.
(4) Let $\{s_i\}$ be the singular points of X. Note that this set in discrete if X has dimension 4. Let us call $X^0 = X \setminus \{s_i\}$ and $X^s := \{s_i\}$, the subsets of smooth and singular points of X. Note that the set X^0 of smooth points is comprised of both smooth isotropy points and regular (non-isotropy) points.

In addition, let us clarify that when the coefficients of homology and cohomology are not especified, they are the integers $\mathbb{Z}$. For instance $H_*(X)$ denotes $H_*(X, \mathbb{Z})$, and analogously for cohomology groups.

Now we want to see how the topologies of X and M influence each other. First let us go to cohomology and homology. The cohomology groups $H^i(M, \mathbb{Z}) = H^i(M)$ are computed by the Leray spectral sequence whose E_2 term is given by

$$E_2^{i,j} = H^i(X, R^j f_* \mathbb{Z}_M) \Rightarrow H^{i+j}(M, \mathbb{Z})$$

Here $R^j f_* \mathbb{Z}_M$ is the j-th right derived functor of the pushforward sheaf of $\mathbb{Z}_M$, given by the sheafification of the presheaf $R^j f_* \mathbb{Z}_M(U) = H^j(f^{-1}(U), \mathbb{Z}_X))$.

Since the fibers of M are homeomorphic to S^1, we see that $R^j f_* \mathbb{Z}_M = 0$ for $j \geq 2$. Also, there is a canonical isomorphism $R^0 f_* \mathbb{Z}_M \cong \mathbb{Z}_X$.

The only non-trivial sheaf is $R^1 f_* \mathbb{Z}_M$. Therefore this sheaf will be our enemy from now on. The next proposition is the first step to defeat our enemy.

PROPOSITION 4.27. *Let $f : M \to X$ be a Seifert bundle.*
(1) There is a natural injection

$$\tau_M : R^1 f_* \mathbb{Z}_M \hookrightarrow \mathbb{Z}_X$$

which is an isomorphism over regular points of X (points with trivial stabilizer).
(2) If $U \subset X$ is connected, then

$$\tau_M(H^0(U, R^1 f_* \mathbb{Z}_M)) = m(U) H^0(U, \mathbb{Z}) \cong m(U)\mathbb{Z}$$

being $m(U)$ the least common multiple of the orders of points of U.

(3) The previous injection τ_M induces a natural isomorphism

$$\tau_M : R^1 f_* \mathbb{Q}_M \hookrightarrow \mathbb{Q}_X$$

which we denote in the same way.

Proof. The main point of the argument is that if $U^x \subset X$ is an small open subset with isotropy $\mathbb{Z}_{m(U^x)}$, then the exceptional fiber $f^{-1}(x)$ generates $H^1(f^{-1}(U^x), \mathbb{Z}_M)$, and any generic fiber $f^{-1}(y)$ is $m(U^x)$ times the exceptional fiber in $H^1(f^{-1}(U^x), \mathbb{Z}_M)$.

More concretely, as $R^1 f_* \mathbb{Z}_X(U^x)$ consists of *constant* sections for U^x small, any section s sends every $y \in U^x$ to $af^{-1}(x) \in H^1(f^{-1}(U^x), \mathbb{Z}_M)$ for some $a \in \mathbb{Z}$. If we take now $V^y \subset U^x$ open and not containing isotropy points, then the restriction of s to V^y must send each $z \in V^y$ to $b\, f^{-1}(z) = b\, f^{-1}(y)$ in $H^1(f^{-1}(V^y), \mathbb{Z}_M)$. Hence we must have that $af^{-1}(x)$ equals to $b\, f^{-1}(y) = b\, m(U^x) f^{-1}(x)$ in $H^1(f^{-1}(U^x), \mathbb{Z}_M)$, so $a = b\, m(U^x)$ for some $b \in \mathbb{Z}$.

Hence we see that any $s \in R^1 f_* \mathbb{Z}_X(U^x)$ is an integer multiple of the section $m(U^x) f^{-1}(x)$, so we see that $R^1 f_* \mathbb{Z}_X(U^x)$ identifies with $m(U^x)\mathbb{Z}\langle f^{-1}(x)\rangle = m(U^x)\mathbb{Z}_X$. This proves (1) and (2) for small U^x, and it is easy to extend this for general connected U.

To see (3) note that the previous injection is also surjective with $\mathbb{Q}$ coefficients, since by the previous discussion $R^1 f_* \mathbb{Q}_X(U^x)$ identifies with $m(U^x)\mathbb{Q}\langle f^{-1}(x)\rangle \cong m(U^x)\mathbb{Q}_X = \mathbb{Q}_X$. □

Recall that the Chern class of a Seifert bundle is defined as

$$c_1(M/X) := \frac{1}{|\mu|} c_1((M/\mu)/X) \in H^2(X, \mathbb{Q})$$

with μ the finite subgroup of S^1 generated by all the stabilizers of points of M, and we denote $|\mu| = m(X)$ the least common multiple of all the stabilizers, and $q : M \to M/\mu$ the quotient map, which has degree $m(X)$.

Proposition 4.28. *For a Seifert bundle $f : M \to X$, the edge homomorphisms of the Leray spectral sequence*

$$d_2^{i,1} : H^i(X, R^1 f_* \mathbb{Q}_M) \to H^{i+2}(X, \mathbb{Q})$$

are identified with cup product with $c_1(M/X) \in H^2(X, \mathbb{Q})$.

Proof. With notation from Proposition 4.27, we have sheaf isomorphisms

$$\mathbb{Q}_X \xrightarrow{\tau_{M/\mu}^{-1}} R^1(f/\mu)_* \mathbb{Q}_{M/\mu} \xrightarrow{q^*} R^1 f_* \mathbb{Q}_M \xrightarrow{\tau_M} \mathbb{Q}_X$$

whose composition $\mathbb{Q}_X \to \mathbb{Q}_X$ evaluated at $U \subset X$ is given by multiplication by $m(U)$, the least common multiple of stabilizers on U. Here $q : M \to M/\mu$ is the quotient map. Hence we can think of q^* as multiplication by the isotropy. We see that q induces isomorphisms between the following spectral sequences

$$S/\mu : H^i(X, R^j f_* \mathbb{Q}_{M/\mu}) \Rightarrow H^{i+j}(M/\mu, \mathbb{Q})$$
$$S : H^i(X, R^j f_* \mathbb{Q}_M) \Rightarrow H^{i+j}(M, \mathbb{Q})$$

Now, since f/μ is a genuine circle bundle, the edge homomorphisms

$$H^i(X, R^1(f/\mu)\mathbb{Q}_{M/\mu}) \xrightarrow{d_2/\mu} H^{i+2}(X, \mathbb{Q})$$

are identified with cup product with the chern class $c_1(M/\mu) \in H^2(X, \mathbb{Z})$. So the edge homomorphisms

$$H^i(X, R^1 f_* \mathbb{Q}_M) \xrightarrow{d_2} H^{i+2}(X, \mathbb{Q})$$

are given by the following commutative diagram

$$\begin{array}{ccc} H^i(X, R^1(f/\mu)_*\mathbb{Q}_{M/\mu}) & \xrightarrow{d_2^{i,1}/\mu} & H^{i+2}(X,\mathbb{Q}) \\ \downarrow q^* & & \downarrow \| \\ H^i(X, R^1 f_*\mathbb{Q}_M) & \xrightarrow{d_2^{i,1}} & H^{i+2}(X,\mathbb{Q}) \end{array}$$

From this we see that

$$(d_2/\mu)(\alpha) = c_1(M/\mu) \cup \alpha = d_2(q^*\alpha) = d_2(m(X)\alpha) = m(X)d_2(\alpha)$$

so $d_2\alpha = c_1(M/X) \cup \alpha$, and we see that for a Seifert bundle the edge homomorphisms are also given by cup product with its Chern class. □

By regarding the torsion-free part of $H^i(X, R^1 f_*\mathbb{Z}_M)$ embedded in $H^i(X, R^1 f_*\mathbb{Q}_M)$, we conclude

COROLLARY 4.29. *In the non-torsion part, the edge homomorphisms*

$$d_2^{i,1} : H^i(X, R^1 f_*\mathbb{Z}_M) \to H^{i+2}(X,\mathbb{Z})$$

are identified with cup product with $c_1(M/X) \in H^2(X,\mathbb{Q})$. Moreover, the image of

$$d_2^{0,1} : H^0(X, R^1 f_*\mathbb{Z}_M) \to H^2(X,\mathbb{Z})$$

is generated by $c_1(M/\mu) \in H^2(X,\mathbb{Z})$.

PROOF. Note that q^* induces an isomorphism $H^0(X, R^1(f/\mu)_*\mathbb{Z}_{M/\mu}) \xrightarrow{q^*} H^0(X, R^1 f_*\mathbb{Z}_M)$, so the image of $d_2^{0,1}$ and $d_2^{0,1}/\mu$ coincide. □

Let us continue exploring the spectral sequence. Up to now, we know that E_2 term of the spectral sequence is like this

$$\begin{array}{ccccc} H^0(X, R^1 f_*\mathbb{Z}_M) & H^1(X, R^1 f_*\mathbb{Z}_M) & H^2(X, R^1 f_*\mathbb{Z}_M) & H^3(X, R^1 f_*\mathbb{Z}_M) & H^4(X, R^1 f_*\mathbb{Z}_M) \\ H^0(X,\mathbb{Z}) & H^1(X,\mathbb{Z}) & H^2(X,\mathbb{Z}) & H^3(X,\mathbb{Z}) & H^4(X,\mathbb{Z}) \end{array}$$

The following gives the first piece of useful information.

PROPOSITION 4.30. *Let $f : M \to X$ be a Seifert bundle.*

(1) $H^1(M,\mathbb{Q}) = 0 \iff H^1(X,\mathbb{Q}) = 0$ *and* $c_1(M/X) \neq 0$.
(2) *If* $H^1(M,\mathbb{Q}) = 0$ *then* $\dim H^2(M,\mathbb{Q}) = \dim H^2(X,\mathbb{Q}) - 1$

PROOF. To see (1), recall that the edge homomorphism

$$H^0(X, R^1 f_*\mathbb{Q}_M) \xrightarrow{d_2^{0,1}} H^2(X,\mathbb{Q})$$

is injective if and only if $c_1(M/X) \neq 0$. Looking at the spectral sequence, we see that $H^1(M,\mathbb{Q}) = 0$ if and only if $H^1(X,\mathbb{Q}) = 0$ and $d_2^{0,1}$ is injective. To see (2) recall that if $H^1(M,\mathbb{Q}) = 0$, then $H^1(X,\mathbb{Q}) = 0$, so also $H^1(X, R^1 f_*\mathbb{Q}_M) = 0$ since $\mathbb{Q}_X \cong R^1 f_*\mathbb{Q}_M$ as sheaves. Hence

$$H^2(M,\mathbb{Q}) \cong \frac{H^2(X,\mathbb{Q})}{\operatorname{im} d_2^{0,1}} = \frac{H^2(X,\mathbb{Q})}{\langle c_1(M/\mu)\rangle}$$

so $\dim H^2(M,\mathbb{Q}) = \dim H^2(X,\mathbb{Q}) - 1$. □

We need a couple of additional tools to relate homologies with constant and local coefficients.

PROPOSITION 4.31. *Let $f : M \to X$ be a Seifert bundle with notations as above. Consider the homomorphism of sheaves given by restriction and quotient*

$$r : \mathbb{Z}_X \longrightarrow \bigoplus_i (\mathbb{Z}_{m_i})_{D_i}$$
$$\mathbb{Z}_X(U) \mapsto \bigoplus_i (\mathbb{Z}_{m_i})_{D_i}(U \cap D_i).$$

Let us call $K = \ker(r)$ the sheaf given by the kernel of r.

(1) *The injection $\tau_M : R^1 f_* \mathbb{Z}_M \hookrightarrow \mathbb{Z}_X$ has image contained in $K = \ker r$. We call*
$$t : R^1 f_* \mathbb{Z}_M \hookrightarrow K$$
the induced injection. We also call $Q = K/\operatorname{im} t$ the quotient sheaf.

(2) *The support of the sheaf Q, denoted $\operatorname{Supp}(Q)$, is contained in the set of singular points of X, hence it is (at most) a discrete set of points.*

PROOF. Recall that for $U \subset X$ connected we have $\tau_M(R^1 f_* \mathbb{Z}_M(U)) = m(U)\mathbb{Z}_X$, being $m(U)$ the least common multiple of the isotropies of points in U. In particular, for all D_i intersecting U, m_i divides $m(U)$, so $\tau_M(R^1 f_* \mathbb{Z}_M) \subset \ker(r)$ and this gives (1).

To see (2), recall that for $(U, V, \phi, \mathbb{Z}_m)$ an orbifold chart with $0 \in U \subset \mathbb{C}^2$ open, we have that $V \cong U/\mathbb{Z}_m$ is a smooth orbifold (i.e. a topological manifold) if and only if $m = m_1 m_2$, being m_i the isotropies of the surfaces D_i through 0 in U. Therefore

$$Q(U) \cong \frac{(m_1 m_2)\mathbb{Z}}{(m)\mathbb{Z}} \cong \frac{\mathbb{Z}}{(l)\mathbb{Z}} = \mathbb{Z}_l$$

with $l = \frac{m}{m_1 m_2}$, and U is a manifold iff $l = 1$ iff $Q(U) = 0$. ☐

So we have the exact sequence of sheaves $0 \to R^1 f_* \mathbb{Z}_M \to K \to Q \to 0$, which gives a long exact sequence

$$\cdots \to H^{j-1}(X, Q) \to H^j(X, R^1 f_* \mathbb{Z}_M) \to H^j(X, K) \to H^j(X, Q) \to \dots \tag{22}$$

Since $\operatorname{Supp}(Q)$ is a discrete set of points (dimension 0), it follows $H^j(X, Q) = 0$ for $j \geq 1$, and this implies that

$$H^k(X, R^1 f_* \mathbb{Z}_M) \cong H^k(X, K), \quad \text{for} \quad k \geq 2. \tag{23}$$

On the other hand, since the homomorphism r of Proposition 4.31 above is surjective, we also have the exact sequence

$$0 \to K \to \mathbb{Z}_X \to \bigoplus_i (\mathbb{Z}_{m_i})_{D_i} \to 0$$

which gives the long exact sequence

$$\cdots \to \bigoplus_i H^{j-1}(D_i, \mathbb{Z}_{m_i}) \to H^j(X, K) \to H^j(X, \mathbb{Z}) \to \bigoplus_i H^j(D_i, \mathbb{Z}_{m_i}) \to \dots \tag{24}$$

where we used that $H^j(X, (\mathbb{Z}_{m_i})_{D_i}) \cong H^j(D_i, \mathbb{Z}_{m_i})$. This gives

$$H^4(X, R^1 f_* \mathbb{Z}_M) \cong H^4(X, K) \cong H^4(X, \mathbb{Z}).$$

The most interesting part of the exact sequence (24) is obtained using (23) to get

$$\begin{aligned} H^1(X, \mathbb{Z}) \quad & \to \textstyle\bigoplus_i H^1(D_i, \mathbb{Z}_{m_i}) = \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i} \to \\ & \to H^2(X, R^1 f_* \mathbb{Z}_M) \to H^2(X, \mathbb{Z}) \to \\ & \to \textstyle\bigoplus_i H^2(D_i, \mathbb{Z}_{m_i}) = \bigoplus_i \mathbb{Z}_{m_i} \to H^3(X, R^1 f_* \mathbb{Z}_M) \to H^3(X, \mathbb{Z}) \end{aligned} \tag{25}$$

Let us give now an auxiliary result which will help us later. It is a Poincaré duality result for a 4-orbifold (possibly with singular points).

Proposition 4.32. *Let $f : M \to X$ be a Seifert bundle. Call $X^s = \{s_i\}$ the singular points of X, and call $X^0 = X \setminus X^s$ the smooth points of X. For $k \geq 2$ we have*

$$H^k(X, \mathbb{Z}) \cong H_{4-k}(X^0, \mathbb{Z})$$
$$H_k(X, \mathbb{Z}) \cong H^{4-k}(X^0, \mathbb{Z})$$

Proof. Since X^s is a discrete set, for $k \geq 2$ we have $H_k(X, \mathbb{Z}) \cong H_k(X, X^s, \mathbb{Z})$. Call $U := \cup_i U_i$ the union of small neighborhoods U_i of the singular points s_i. We take the U_i disjoint and contractible, which obviously can be done by taking U_i the quotient of a small ball in an orbifold chart $U_i \cong B/\mathbb{Z}_{m_i}$. Since U deformation retracts onto X^s, we have $H_k(X, X^s, \mathbb{Z}) \cong H_k(X, U, \mathbb{Z})$.

We want to use excision, so define $U' = \bigcup_i U'_i$ with U'_i an smaller neighborhood of s_i, with $U'_i \cong B'/\mathbb{Z}_{m_i}$ for a smaller ball $B' \subset B$. Let $N := X \setminus U'$ be the manifold with boundary obtained by substracting open neighborhoods of the singular points. Note that the closure of U' is contained in U, and $U \setminus U'$ deformation retracts onto ∂N, which is a union of lens spaces. Finally we use excision, Lefschetz duality, and the fact that X^0 deformation retracts onto N to get the result. The entire process is below

$$\begin{aligned} H_k(X, \mathbb{Z}) &\cong H_k(X, X^s, \mathbb{Z}) \cong H_k(X, U, \mathbb{Z}) \cong H_k(X \setminus U', U \setminus U', \mathbb{Z}) \\ &\cong H_k(N, \partial N, \mathbb{Z}) \cong H^{4-k}(N, \mathbb{Z}) \cong H^{4-k}(X^0, \mathbb{Z}) \end{aligned}$$

With the same technique we obtain

$$\begin{aligned} H^k(X, \mathbb{Z}) &\cong H^k(X, X^s, \mathbb{Z}) \cong H^k(X, U, \mathbb{Z}) \cong H^k(X \setminus U', U \setminus U', \mathbb{Z}) \\ &\cong H^k(N, \partial N, \mathbb{Z}) \cong H_{4-k}(N, \mathbb{Z}) \cong H_{4-k}(X^0, \mathbb{Z}) \end{aligned}$$

and this gives the result we wanted. □

We are interested mainly in the following topological questions regarding a Seifert bundle $f : M \to X$:

(1) How the hypothesis that $H_1(M) = 0$ affects the topology of both M and X.
(2) What (as easy as possible) conditions can be verified in the base space X in order to ensure that $H_1(M)$ vanishes.

So let us start by the first item.

4.1. Case 1: $H_1(M) = 0$. Let $f : M \to X$ be a Seifert bundle. Throughout this subsection it will be assumed by default that $H_1(M) = 0$. Note however that in the statement of Propositions, etc, of this subsection it will be stated exactly the hypothesis needed in their proofs, so fewer assumptions may be made.

That said, note that by Poincaré Duality the condition $H_1(M) = 0$ is equivalent to $H^4(M) = 0$. We know the following up to now. Let k be the rank of $H_2(M)$. By Proposition 4.30 we know that the rank of $H_2(X)$ is $k+1$. Moreover the rank of $H_1(X)$ is zero. Since $H^1(X)$ is torsion free, it must be 0. We use *tor* to denote any unspecified torsion group. We know up to now the following:

$$\begin{aligned} &H_0(X) \cong \mathbb{Z}\,, \\ &H_1(X) = tor\,, \\ &H_2(X) \cong \mathbb{Z}^{k+1} + tor\,, \\ &H_4(X) \cong H^0(X^0) \cong \mathbb{Z}\,. \end{aligned}$$

Let us prove a couple of results that will allow us to determine these groups much better. We start with the proof that $H_1(X)$ equals 0.

Proposition 4.33. *Let $f : M \to X$ be a Seifert bundle and suppose that $H_1(M) = 0$. Then $H_1(X) = 0$.*

Proof. We only have to see that $H_1(X)$ has no torsion. This will prove that it is zero because we saw before that $H_1(M,\mathbb{Q})=0$ implies $H_1(X,\mathbb{Q})=0$. Note that the torsion of $H_1(X,\mathbb{Z})$ equals the torsion of $H^2(X,\mathbb{Z})$. On the other hand, the spectral sequence clearly stops in the third page. Therefore the E^3-term is the E^∞-term, and by the convergence of the spectral sequence we get the following

$$0\to F^2H^2(M)\to F^1H^2(M)\to E_\infty^{1,1}=\ker(d_2^{1,1})\to 0$$
$$0\to F^3H^2(M)\to F^2H^2(M)\to E_\infty^{2,0}\cong H^2(X)/\operatorname{im}(d_2^{0,1})\to 0$$
$$0\to F^{1-k}H^2(M)\to F^{-k}H^2(M)\to E_\infty^{-k,2+k}=0\,,\quad \text{for } k\geq 0$$
$$0\to F^{4+k}H^2(M)\to F^{3+k}H^2(M)\to E_\infty^{3+k,-1-k}=0\,,\quad \text{for } k\geq 0.$$

Since the filtration is decreasing, we conclude that

$$F^{3+k}H^2(M)=0\,,\quad \text{for } k\geq 0$$
$$F^{1-k}H^2(M)\cong H^2(M)\,,\quad \text{for } k\geq 0$$

This implies that $F^2H^2(M,\mathbb{Z})\cong E_\infty^{2,0}\cong H^2(X)/\operatorname{im}(d_2^{0,1})$ and the first exact sequence becomes

$$0\to H^2(X,\mathbb{Z})/\operatorname{im}(d_2^{0,1})\to H^2(M)\to\ker(d_2^{1,1})\to 0$$

Now recall that $d_2^{0,1}:\mathbb{Z}\cong H^0(X,R^1f_*\mathbb{Z}_M)\to H^2(X)\cong\mathbb{Z}^{k+1}$ is given by cup product with the Chern class $c_1(M/X)\in H^2(X,\mathbb{Q})$ and its image is generated by $m(X)c_1(M/X)=c_1(M/\mu)\in H^2(X,\mathbb{Z})$, so

$$\operatorname{im}(d_2^{0,1})=\langle c_1(M/\mu)\rangle.$$

Note that $c_1(M/\mu)\in H^2(X,\mathbb{Z})$ must be non-torsion because $c_1(M/X)\neq 0$ in $H^2(X,\mathbb{Q})$ by Proposition 4.30. Also, $c_1(M/\mu)$ must be primitive because $H^2(M,\mathbb{Z})$ is torsion free (the torsion of $H^2(M,\mathbb{Z})$ equals the torsion of $H_1(M,\mathbb{Z})=0$).

The conclusion is that if $H^2(X,\mathbb{Z})$ had torsion, this torsion would survive in the quotient $H^2(X,\mathbb{Z})/\operatorname{im}(d_2^{0,1})$ and hence would be embedded in $H^2(M,\mathbb{Z})$. But we know that $H^2(M,\mathbb{Z})$ has no torsion, so $H^2(X,\mathbb{Z})$ cannot have any torsion either. This finally yields $H_1(X,\mathbb{Z})=0$. □

We need another result to go on.

Proposition 4.34. *Let $f:M\to X$ be a Seifert bundle over a 4-orbifold X. Let $X^0=X\backslash\{s_i\}$ be the smooth points of X.*

If $H_1(M)=0$ then $H_1(X^0)=0$.

Proof. Let $M^s=\bigcup_i f^{-1}(\{s_i\})$ be the fibers over the singular points $\{s_i\}$ of X. Let us call $M^0=M\setminus M^s$, so the restriction of f to M^0 gives a Seifert bundle

$$f^0:=f|_{M^0}:M^0\to X^0.$$

Now, the inclusion $\iota:M^0\to M$ induces an isomorphism

$$\iota_*:\pi_1(M^0)\to\pi_1(M)$$

by transversality reasons, since $M^0=M\setminus M^s$ with M^s a disjoint union of closed 1-submanifolds (circunferences). This implies that any loop (any homotopy of loops) in M can be made transversal to M^s, so it can be deformed to a loop (a homotopy of loops) not touching M^s, hence contained in M^0.

Taking the abelianization functor we have an isomorphism

$$\iota_*:H_1(M^0)\to H_1(M)$$

so $H_1(M^0)=0$. Now, since $f^0:M^0\to X^0$ is a Seifert bundle with total space M^0 satisfying $H_1(M^0)=0$, using Proposition 4.33 we get $H_1(X^0)=0$ as desired. □

Under the hypothesis of this subsection, (i.e. $H_1(M) = 0$), and calling k to the rank of $H_2(M)$ as before, we claim that this yields the following:

$$\begin{aligned} H_0(X) &\cong \mathbb{Z}\,, \\ H_1(X) &= 0\,, \\ H_2(X) &\cong \mathbb{Z}^{k+1}\,, \\ H_3(X) &\cong H^1(X^0) = 0\,, \\ H_4(X) &\cong H^0(X^0) = \mathbb{Z}\,. \end{aligned}$$

To see that $H_1(X) = 0$ we have used Proposition 4.33. To see that $H_2(X)$ has no torsion note the following. First, the torsion of $H_2(X)$ is isomorphic to the torsion of $H^3(X)$. But $H^3(X) \cong H_1(X^0)$ by Proposition 4.32. Since $H_1(X^0) = 0$ by Proposition 4.34, we conclude that $H_2(X)$ is torsion free.

The cohomology of X are computed by the Universal Coefficient Theorem (U.C.T.), which gives

$$\begin{aligned} &H^0(X) \cong \mathbb{Z}\,, \qquad H^1(X) = 0\,, \qquad H^2(X) \cong \mathbb{Z}^{k+1}\,, \\ &H^3(X) = 0\,, \qquad H^4(X) \cong \mathbb{Z} \end{aligned}$$

On the other hand, using that $H_1(M) = 0$, the U.C.T., and Poincaré duality we get

$$\begin{aligned} &H_0(M) \cong \mathbb{Z}\,, \qquad H_1(M) = 0\,, \qquad H_2(M) \cong \mathbb{Z}^k + A_{tors}\,, \\ &H_3(M) \cong \mathbb{Z}^k\,, \qquad H_4(M) = 0\,, \qquad H_5(M) \cong \mathbb{Z}\,, \end{aligned}$$

where A_{tors} is a torsion group to be determined later, and for cohomology

$$\begin{aligned} &H^0(M) \cong \mathbb{Z}\,, \qquad H^1(M) = 0\,, \qquad H^2(M) \cong \mathbb{Z}^k\,, \\ &H^3(M) \cong \mathbb{Z}^k + A_{tors}\,, \qquad H^4(M) = 0\,, \qquad H^5(M) \cong \mathbb{Z}\,. \end{aligned}$$

Now we come back to the exact sequence (25)

$$\begin{aligned} 0 &\to \textstyle\bigoplus_i H^1(D_i, \mathbb{Z}_{m_i}) = \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i} \to H^2(X, R^1 f_* \mathbb{Z}_M) \to \\ &\to H^2(X, \mathbb{Z}) \to \textstyle\bigoplus_i H^2(D_i, \mathbb{Z}_{m_i}) = \bigoplus_i \mathbb{Z}_{m_i} \to H^3(X, R^1 f_* \mathbb{Z}_M) \to 0 \end{aligned} \tag{26}$$

Let us write it again more briefly as

$$\begin{aligned} 0 &\to \textstyle\bigoplus_i (\mathbb{Z}_{m_i})^{2g_i} \xrightarrow{\alpha_0} H^2(X, R^1 f_* \mathbb{Z}_M) \xrightarrow{\alpha_1} H^2(X, \mathbb{Z}) \to \\ &\xrightarrow{\alpha_2} \textstyle\bigoplus_i \mathbb{Z}_{m_i} \xrightarrow{\alpha_3} H^3(X, R^1 f_* \mathbb{Z}_M) \to 0 \end{aligned} \tag{27}$$

where g_i is the genus of D_i, so in the notation introduced earlier $2g_i = b_1(D_i)$ is the first Betti number of D_i. Since $H^2(X, \mathbb{Z})$ is torsion free, $\mathrm{im}(\alpha_1) = \alpha_1(\text{free part}) = \ker(\alpha_2)$, and $\ker(\alpha_2)$ is a subgroup of $H^2(X, \mathbb{Z})$ of maximal rank, i.e. of rank equal to $k + 1$.

Therefore $\alpha_1(\text{free part})$ has rank $k + 1$. By exactness, α_1 restricted to the free part of $H^2(X, R^1 f_* \mathbb{Z}_M)$ is injective. It follows that the free part of $H^2(X, R^1 f_* \mathbb{Z}_M)$ has rank $k + 1$. Also, all the torsion part of $H^2(X, R^1 f_* \mathbb{Z}_M)$ is contained in $\ker(\alpha_1) = \mathrm{im}(\alpha_0)$, since $H^2(X, \mathbb{Z})$ is torsion free. We conclude that

$$H^2(X, R^1 f_* \mathbb{Z}_M) \cong \mathbb{Z}^{k+1} \oplus \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i}.$$

Note also that $H^3(X, R^1 f_* \mathbb{Z}_M) \cong \mathrm{coker}(\alpha_2)$ so it is a torsion group.

We claim in addition that $H^1(X, R^1 f_* \mathbb{Z}_M)$ is a torsion group. The easiest way to see this is considering the short exact sequence of sheaves

$$0 \to R^1 f_* \mathbb{Z}_M \xrightarrow{\tau_M} \mathbb{Z}_X \to P \to 0$$

where $P = \frac{\mathbb{Z}_X}{R^1 f_* \mathbb{Z}_M}$ is the quotient sheaf and τ_M is the injection previously considered. This gives a long exact sequence

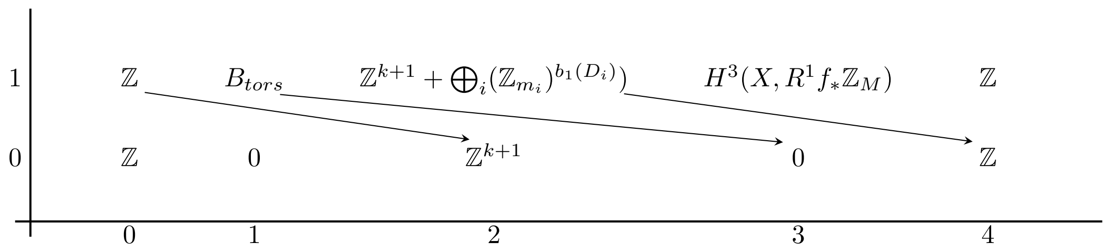
$$0 \to H^0(X, R^1 f_* \mathbb{Z}_M) \xrightarrow{\tau_M} H^0(X, \mathbb{Z}_X) \to H^0(X, P) \to H^1(X, R^1 f_* \mathbb{Z}_M) \to 0\,.$$

It is easy to see that $H^0(X, P)$ is a torsion group. Indeed, for any $s \in H^0(X, P)$ it is clear that $m(X)s = 0$ since it is zero on each stalk, i.e. for $U \subset X$ a small open set $P(U) \cong \mathbb{Z}/m(U)\mathbb{Z}$ so $m(X)s|_U = 0$.

We conclude that $H^0(X, P)$ is a torsion group, and hence so it is $H^1(X, R^1 f_* \mathbb{Z}_M)$.

Now we come back to analyze the spectral sequence, and recall that the E_2-term is $E_2^{i,j} = H^i(X, R^j f_* \mathbb{Z}_M)$, and the differentials are $d_2^{i,j} : E_2^{i,j} \to H^{i+j}(X, \mathbb{Z})$. The i-coordinate stands for the column, and the j-coordinate for the row in the diagram below.

The E_2-term is this:



The E_3 term is the E_∞ term, and it is the following:

1	$\ker(d_2^{0,1})$	B_{tors}	$\ker(d_2^{0,1})$	$H^3(X, R^1 f_* \mathbb{Z}_M)$	$\mathbb{Z}$
0	$\mathbb{Z}$	0	$\frac{\mathbb{Z}^{k+1}}{\mathrm{im}(d_2^{0,1})}$	0	$\frac{\mathbb{Z}}{\mathrm{im}(d_2^{2,1})}$
	0	1	2	3	4

By the convergence of the Leray spectral sequence we know that

$$E_\infty^{p,q} \cong \frac{F^p H^{p+q}(M, \mathbb{Z})}{F^{p+1} H^{p+q}(M, \mathbb{Z})}$$

which is equivalent to the exact sequence

$$0 \to F^{p+1} H^{p+q}(M) \to F^p H^{p+q}(M) \to E_\infty^{p,q} \to 0$$

Now we analyze case by case

Dimension $p + q = 4$:

We have the following short exact sequences

$$0 \to F^4H^4(M) \to F^3H^4(M) \to E_\infty^{3,1} = H^3(X, R^1f_*\mathbb{Z}_M) \to 0$$
$$0 \to F^5H^4(M) \to F^4H^4(M) \to E_\infty^{4,0} = \frac{\mathbb{Z}}{\operatorname{im}(d_2^{2,1})} \to 0$$
$$0 \to F^{6+k}H^4(M) \to F^{5+k}H^4(M) \to E_\infty^{5+k,-1-k} = 0 \quad \text{for } k \geq 0$$
$$0 \to F^{3-k}H^4(M) \to F^{2-k}H^4(M) \to E_\infty^{2-k,2+k} = 0 \quad \text{for } k \geq 0$$

Since the filtration is decreasing, we conclude that $F^{5+k}H^4(M) = 0$ and $F^{3-k}H^4(M) \cong H^4(M)$ for all $k \geq 0$. Then we have the exact sequence

$$0 \to F^4H^4(M) \to E_\infty^{4,0} \to 0$$

and this gives the sequence given by

$$0 \to \frac{\mathbb{Z}}{\operatorname{im} d_2^{2,1}} \to H^4(M, \mathbb{Z}) = 0 \to H^3(X, R^1f_*\mathbb{Z}_M) \to 0$$

so we conclude that $d_2^{2,1} : \mathbb{Z}^{k+1} + \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i} \to \mathbb{Z}$ is surjective and $H^3(X, R^1f_*\mathbb{Z}_M) = 0$.

Dimension $p + q = 3$**:**

We have the following short exact sequences

$$0 \to F^3H^3(M) \to F^2H^3(M) \to E_\infty^{2,1} = \ker(d_2^{2,1}) \to 0$$
$$0 \to F^{4+k}H^3(M) \to F^{3+k}H^3(M) \to E_\infty^{3+k,-k} = 0 \quad \text{for } k \geq 0$$
$$0 \to F^{2-k}H^3(M) \to F^{1-k}H^3(M) \to E_\infty^{1-k,2+k} = 0 \quad \text{for } k \geq 0$$

Since the filtration is decreasing, we conclude that $F^{3+k}H^3(M) = 0$ and $F^{2-k}H^3(M) \cong H^3(M)$ for all $k \geq 0$. If we substitute this in the first exact sequence above we get

$$0 \to H^3(M) \to E_\infty^{2,1} = \ker(d_2^{2,1}) \to 0$$

hence $H^3(M, \mathbb{Z}) \cong \ker(d_2^{2,1})$. Since $d_2^{2,1}$ is surjective (seen in the previous case $p+q = 4$), we get that $\ker(d_2^{2,1}) \cong \mathbb{Z}^k + \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i}$, so it follows

$$H_2(M, \mathbb{Z}) \cong H^3(M, \mathbb{Z}) \cong \mathbb{Z}^k + \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i}.$$

We have discovered now that $A_{tors} \cong \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i}$.

Dimension $p + q = 2$**:**

We have the following short exact sequences

$$0 \to F^2H^2(M) \to F^1H^2(M) \to E_\infty^{1,1} = B_{tors} \to 0$$
$$0 \to F^3H^2(M) \to F^2H^2(M) \to E_\infty^{2,0} \cong H^2(X)/\operatorname{im}(d_2^{0,1}) \to 0$$
$$0 \to F^{1-k}H^2(M) \to F^{-k}H^2(M) \to E_\infty^{-k,2-k} = 0 \quad \text{for } k \geq 0$$
$$0 \to F^{4+k}H^2(M) \to F^{3+k}H^2(M) \to E_\infty^{3+k,-1-k} = 0 \quad \text{for } k \geq 0$$

Since the filtration is decreasing, we conclude that $F^{3+k}H^2(M) = 0$ and $F^{1-k}H^2(M) \cong H^2(M)$ for all $k \geq 0$. This implies that

$$F^2H^2(M, \mathbb{Z}) \cong E_\infty^{2,0} \cong \frac{H^2(X)}{\operatorname{im}(d_2^{0,1})}$$

and the short exact sequence

$$0 \to \frac{H^2(X)}{\operatorname{im}(d_2^{0,1})} \to H^2(M) \to B_{tors} \to 0$$

Now recall that $d_2^{0,1} : \mathbb{Z} \cong H^0(X, R^1 f_* \mathbb{Z}_M) \to H^2(X) \cong \mathbb{Z}^{k+1}$ is given by cup product with the Chern class $c_1(M/X) \in H^2(X, \mathbb{Q})$. We know by Corollary 4.29 that the image of $d_2^{0,1}$ is generated by $m(X)c_1(M/X) = c_1(M/\mu) \in H^2(X, \mathbb{Z})$. The conclusion is that $c_1(M/\mu)$ must be a primitive element of $H^2(X, \mathbb{Z})$, since $H^2(M)$ is torsion free.

Dimension $p + q = 1$:

Since the only term in the diagonal $p+q = 1$ is $E_\infty^{0,1} = \ker(d_2^{0,1})$, in this case we only conclude that $H^1(M) \cong \ker(d_2^{0,1}) = 0$, in other words

$$d_2^{0,1} : \mathbb{Z} \cong H^0(X, R^1 f_* \mathbb{Z}_M) \to H^2(X) \cong \mathbb{Z}^{k+1}$$

is injective, fact that we already knew since it is cup product with $c_1(M/X) \neq 0$.

Conclusion

With these calculations we have determined completely the homology groups of M and X in the case that $H_1(M) = 0$.

It remains to see an interesting surjection which will be important in the future. To see it we come back to the exact sequence (27) to see that

$$\begin{aligned} 0 \to \textstyle\bigoplus_i H^1(D_i, \mathbb{Z}_{m_i}) &\xrightarrow{\alpha_0} H^2(X, R^1 f_* \mathbb{Z}_M) \xrightarrow{\alpha_1} \\ &\xrightarrow{\alpha_1} H^2(X, \mathbb{Z}) \xrightarrow{\alpha_2} \textstyle\bigoplus_i H^2(D_i, \mathbb{Z}_{m_i}) \xrightarrow{\alpha_3} H^3(X, R^1 f_* \mathbb{Z}_M) \to 0 \end{aligned}$$

We saw before that $H^3(X, R^1 f_* \mathbb{Z}_M) = 0$ so

$$\alpha_2 : H^2(X, \mathbb{Z}) \to \bigoplus_i H^2(D_i, \mathbb{Z}_{m_i})$$

is the so-called interesting surjection.

Summing up what we have seen until now, we obtain the following Theorem.

THEOREM 4.35. *Let $f : M \to X$ be a Seifert bundle with M a closed 5-manifold so that $H_1(M) = 0$. Let k be the rank of $H^2(M, \mathbb{Z})$ and let $b_1(D_i)$ be the first Betti numbers of the surfaces $D_i \subset X$.*

Then the homology of X is:

$$\begin{aligned} &H_0(X) \cong \mathbb{Z}\,, \quad H_1(X) = 0\,, \quad H_2(X) \cong \mathbb{Z}^{k+1}\,, \quad H_3(X) = 0\,, \quad H_4(X) = \mathbb{Z} \\ &H^0(X) \cong \mathbb{Z}\,, \quad H^1(X) = 0\,, \quad H^2(X) \cong \mathbb{Z}^{k+1}\,, \quad H^3(X) = 0\,, \quad H^4(X) = \mathbb{Z} \end{aligned}$$

The homology of X with coefficients in the non-friendly sheaf is:

$$\begin{aligned} &H^0(X, R^1 f_* \mathbb{Z}_M) \cong \mathbb{Z}\,, \qquad H^1(X, R^1 f_* \mathbb{Z}_M) = B_{tors}\,, \quad H^2(X, R^1 f_* \mathbb{Z}_M) \cong \mathbb{Z}^{k+1} + \textstyle\bigoplus_i (\mathbb{Z}/m_i)^{b_1(D_i)}\,, \\ &H^3(X, R^1 f_* \mathbb{Z}_M) = 0; \qquad H^4(X, R^1 f_* \mathbb{Z}_M) = \mathbb{Z} \end{aligned}$$

The homology and cohomology of M are:

$$\begin{aligned} &H_0(M) \cong \mathbb{Z}\,, \quad H_1(M) = 0\,, \quad H_2(M) \cong \mathbb{Z}^k + \textstyle\sum_i (\mathbb{Z}/m_i)^{b_1(D_i)}\,, \\ &H_4(M) \cong \mathbb{Z}\,, \quad H_5(M) \cong \mathbb{Z}\,, \quad H_3(M) \cong \mathbb{Z}^k\,, \\ &H^0(M) \cong \mathbb{Z}\,, \quad H^1(M) = 0\,, \quad H^2(M) \cong \mathbb{Z}^k\,, \\ &H^4(M) = 0\,, \quad H^5(M) \cong \mathbb{Z}\,, \quad H^3(M) \cong \mathbb{Z}^k + \textstyle\sum_i (\mathbb{Z}/m_i)^{b_1(D_i)} \end{aligned}$$

Moreover, more is satisfied:

(1) *The map given by*

$$\alpha_2 : H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{k+1} \to \bigoplus_i H^2(D_i, \mathbb{Z}_{m_i}) \cong \bigoplus_i \mathbb{Z}_{m_i} \tag{28}$$

is surjective.

(2) *The first Chern class $c_1(M/\mu) \in H^2(X,\mathbb{Z})$ of the circle bundle $M/\mu \to X$ is a primitive element in $H^2(X,\mathbb{Z})$.*

As a consequence of the previous discussion we also get the following.

Corollary 4.36. *We can construct a basis of the torsion of $H_2(M,\mathbb{Z})$ by choosing $\gamma_{ij} \subset D_i$, $1 \le j \le 2g_i$ a basis for $H_1(D_i,\mathbb{Z}_{m_i})$ and making the pull-back $\Gamma_{ij} = f^{-1}(\gamma_{ij}) \subset M$, which is an m_i-torsion 2-cycle, so the torsion part of the second homology of M is given by*

$$H_{2,tor}(M,\mathbb{Z}) \cong \bigoplus_{i,j} \mathbb{Z}_{m_i}[\Gamma_{ij}].$$

4.2. Case 2: Conditions on X to kill the first homology of M. Let us see which conditions on the space X have to be imposed to ensure that $H_1(M)$ vanishes. Let us suppose the following for this subsection:

(1) The first homology group of X vanishes, i.e. $H_1(X) = 0$.

(2) The first chern class $c_1(M/\mu)$ of the circle bundle $M/\mu \to X$ is primitive, i.e.

$$c_1(M/\mu) = m(X)c_1(M/X) \in H^2(X,\mathbb{Z})$$

is primitive.

(3) The map $\alpha_2 : H^2(X,\mathbb{Z}) \to \bigoplus_i H^2(D_i,\mathbb{Z}_{m_i})$ is surjective.

(4) The orbifold X is smooth. In the notation previously introduced this means that $X^0 = X$.

These will be the hypothesis by default in this subsection, although in the statement of Propositions fewer assumptions may be assumed if possible. We saw in Theorem 4.35 that conditions (1), (2) and (3) are all necessary to ensure the vanishing of $H_1(M)$. We are going to see now that these conditions are also sufficient if we additionally suppose condition (4), i.e. that X has no singular points (so it is a topological manifold).

First of all, if we call k the rank of $H_2(M)$, the homology of X with $\mathbb{Z}$ coefficients satisfies:

$$\begin{array}{lllll} H_0(X) \cong \mathbb{Z}\,, & H_1(X) = 0\,, & H_2(X) \cong \mathbb{Z}^{k+1}\,, & H_3(X) = 0\,, & H_4(X) = \mathbb{Z}\,, \\ H^0(X) \cong \mathbb{Z}\,, & H^1(X) = 0\,, & H^2(X) \cong \mathbb{Z}^{k+1}\,, & H^3(X) = 0\,, & H^4(X) = \mathbb{Z}\,. \end{array}$$

This is proved as follows. From the assumption that $H_1(X) = 0$ it follows that $H^1(X) = 0$ and that $H^2(X)$ has no torsion, so $H_2(X)$ has no torsion either by Poincaré duality. Poincaré duality also implies $H^3(X) = 0$ and $H_3(X) = 0$. By Proposition 4.30 we see that $H^1(M,\mathbb{Q}) = 0$, hence if $k = b_2(M)$ is the second Betti number of M, we have that $b_2(X) = k+1$.

The exact sequences of sheaves we used before are valid also here, only taking care of not using hypothesis outside the four stated above. Let us recall that we denote

$$K = \ker[\mathbb{Z}_X \to \bigoplus_i (\mathbb{Z}_{m_i})_{D_i}].$$

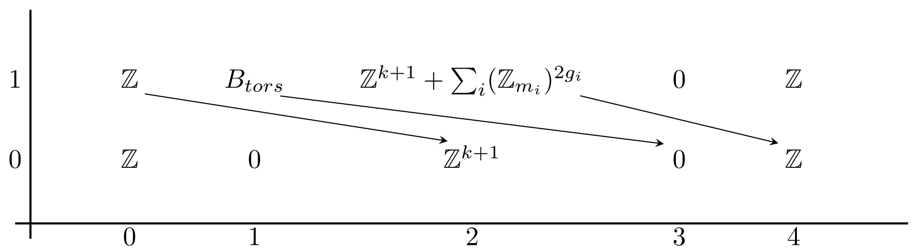
Hence, as before, we have the exact sequences of sheaves

$$\begin{aligned} &0 \to R^1 f_* \mathbb{Z}_M \to K \to Q \to 0\,, \\ &0 \to K \to \mathbb{Z}_X \to \textstyle\bigoplus_i (\mathbb{Z}_{m_i})_{D_i} \to 0 \end{aligned} \tag{29}$$

Since the support of Q is 0-dimensional and $H^1(X,\mathbb{Z}) = 0$, this yields

$$\begin{aligned} 0 \to \textstyle\sum_i H^1(D_i,\mathbb{Z}_{m_i}) = \sum_i (\mathbb{Z}_{m_i})^{2g_i} \to H^2(X, R^1 f_* \mathbb{Z}_M) \to \\ \to H^2(X,\mathbb{Z}) \xrightarrow{\alpha_2} \textstyle\sum_i H^2(D_i,\mathbb{Z}_{m_i}) = \sum_i \mathbb{Z}_{m_i} \to H^3(X, R^1 f_* \mathbb{Z}_M) \to 0 \end{aligned} \tag{30}$$

Since $H^2(X,\mathbb{Z})$ is torsion free, we conclude that $H^2(X,R^1f_*\mathbb{Z}_M)\cong\mathbb{Z}^{k+1}+\sum_i(\mathbb{Z}_{m_i})^{2g_i}$. Moreover $H^3(X,R^1f_*\mathbb{Z}_M)=0$ since α_2 is surjective by assumption. With this information we come back to the spectral sequence, with E_2 term



and with $E_3=E_\infty$ term given by

	0	1	2	3	4
1	$\ker(d_2^{0,1})$	B_{tors}	$\ker(d_2^{2,1})$	0	$\mathbb{Z}$
0	$\mathbb{Z}$	0	$\frac{\mathbb{Z}^{k+1}}{\mathrm{im}(d_2^{0,1})}$	0	$\frac{\mathbb{Z}}{\mathrm{im}(d_2^{2,1})}$

The spectral sequence gives

$$0\to\frac{H^4(X,\mathbb{Z})}{\mathrm{im}(d_2^{2,1})}\to H^4(M,\mathbb{Z})\to 0=H^3(X,R^1f_*\mathbb{Z}_M).$$

Note that $d_2^{2,1}:H^2(X,R^1f_*\mathbb{Z}_M)\to H^4(X,\mathbb{Z})$ is cup product with $c_1(M/X)$. Let us see that $\mathrm{im}(d_2^{2,1})$ equals $\langle c_1(M/\mu)\rangle$ under assumptions (1), (2), (3) and (4).

PROPOSITION 4.37. *Let $f:M\to X$ be a Seifert bundle, with M an oriented closed 5-manifold. Suppose that the four hypothesis of this subsection given in 4.2 are satisfied.*

Then the map

$$d_2^{2,1}:H^2(X,R^1f_*\mathbb{Z}_M)\to H^4(X,\mathbb{Z})$$

is surjective.

PROOF. First note that by Proposition 4.28, the differential $d_2^{2,1}:H^2(X,R^1f_*\mathbb{Z}_M)\to H^4(X,\mathbb{Z})$ is given by cup product with $c_1(M/X)$.

Now, as X is a topological manifold, the pairing

$$H^2(X,\mathbb{Z})\times H^2(X,\mathbb{Z})\to H^4(X,\mathbb{Z})$$

is perfect. Since $c_1(M/\mu)=m(X)c_1(M/X)\in H^2(X,\mathbb{Z})$ is primitive, there exists an $[\alpha]\in H^2(X,\mathbb{Z})$ so that

$$1_{H^4(X)}=c_1(M/\mu)\cup[\alpha]=m(X)c_1(M/X)\cup[\alpha]$$

where $1_{H^4(X)}$ denotes the positive generator of $H^4(X)\cong\mathbb{Z}$. On the other hand note that

$$m(X)[\alpha]\in m(X)H^2(X,\mathbb{Z})\subset\ker[\alpha_2:H^2(X,\mathbb{Z})\to\sum_i H^2(D_i,\mathbb{Z}_{m_i})]$$

the inclusion above is due to the fact that $m(X)$ is by definition multiple of all the isotropies m_i. Recall the exact sequence (30) which says

$$0 \to \bigoplus_i (\mathbb{Z}_{m_i})^{2g_i} \to H^2(X, R^1 f_* \mathbb{Z}_M) \xrightarrow{\alpha_1} H^2(X, \mathbb{Z}) \xrightarrow{\alpha_2} \sum_i H^2(D_i, \mathbb{Z}_{m_i}) \to 0 \tag{31}$$

Note that by exactness, the map

$$\alpha_1 : H^2(X, R^1 f_* \mathbb{Z}_M) \to \mathrm{im}(\alpha_1) = \ker(\alpha_2) \subset H^2(X, \mathbb{Z})$$

has to be a bijection from the free part of $H^2(X, R^1 f_* \mathbb{Z}_M)$ to $\ker(\alpha_2) \subset H^2(X, \mathbb{Z})$, and $\ker(\alpha_2)$ is a subgroup of maximal rank of $H^2(X, \mathbb{Z})$.

This means that we can identify $m(X)[\alpha] \in \ker(\alpha_2)$ as an element in $H^2(X, R^1 f_* \mathbb{Z}_M)$. We conclude that

$$1_{H^4(X)} = c_1(M/X) \cup m(X)[\alpha] = d_2^{2,1}(m(X)[\alpha]) \in \mathrm{im}(d_2^{2,1})$$

which proves that $d_2^{2,1}$ is surjective as desired. □

We are now in position to see that $H_1(M) = 0$ under the assumptions of this subsection. Looking at the spectral sequence it follows easily that

$$H^4(M, \mathbb{Z}) \cong \frac{H^4(X, \mathbb{Z})}{\mathrm{im}(d_2^{2,1})}.$$

Hence, under the hypothesis (1), (2), (3), (4) we get by Proposition 4.37 that $H^4(M, \mathbb{Z}) = 0$, and by Poincaré Duality we get $H_1(M, \mathbb{Z}) \cong H^4(M, \mathbb{Z}) = 0$ as desired.

Finally, the spectral sequence also gives

$$0 \to H^3(M, \mathbb{Z}) \to \ker(d_2^{2,1}) \to 0$$

so $H_2(M, \mathbb{Z}) \cong H^3(M, \mathbb{Z}) \cong \ker(d_2^{2,1}) \cong \mathbb{Z}^k + \sum_i (\mathbb{Z}_{m_i})^{2g_i}$. Let us sumarise the main conclusion we have got from the previous discussion.

Recall that a *Semi-regular Seifert bundle* is a Seifert bundle $f : M \to X$ in which the base space X is a smooth orbifold (an orbifold whose underlying space is a topological manifold).

THEOREM 4.38. *Let M a closed 5-manifold, and $f : M \to X$ a semi-regular Seifert bundle with isotropy surfaces D_i of multiplicities m_i. Then the following are equivalent:*

a) The total space M satisfies $H_1(M, \mathbb{Z}) = 0$

b) The base space X satisfies the following three conditions:

(1) $H_1(X, \mathbb{Z}) = 0$
(2) $H^2(X, \mathbb{Z}) \to \sum H^2(D_i, \mathbb{Z}_{m_i})$ *is surjective,*
(3) $c_1(M/\mu) \in H^2(X, \mathbb{Z})$ *is primitive.*

Moreover, any of the conditions a) or b) implies $H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus (\mathbb{Z}_{m_i})^{2g_i}$, with g_i =genus of D_i, $k = b_2(M)$ and $k + 1 = b_2(X)$.

PROOF. First note that the claim about $H_2(M)$ was already proved above.

To see that $a) \implies b)$ note that if $H_1(M) = 0$ we proved that the three conditions of $b)$ hold (along with many more) in Theorem 4.35. Recall that for $a) \implies b)$ we do not need X to be a smooth orbifold, so X can have singularities.

The proof that $b) \implies a)$ is exactly the discussion above (and it does need X to be a smooth orbifold). □

REMARK 4.39. *If X is a smooth 4-manifold and $f : M \to X$ is a circle bundle, then Theorem 4.38 also applies, just taking empty isotropy locus. Then $H_1(M, \mathbb{Z}) = 0$ if and only if $H_1(X, \mathbb{Z}) = 0$ and $c_1(M/\mu)$ is primitive, and in that case $H_2(M, \mathbb{Z}) \cong \mathbb{Z}^k$ with $k = b_2(X) - 1$. Note that $H_2(M, \mathbb{Z})$ is torsion-free in this case.*

4.3. Consequences. Here we want to extract the most important consequences to us of our analysis of the topology of Seifert bundles. These will be key tools that we will use for the construction of the K-contact non-Sasakian manifold that we give later on.

Recall first that for $m \in \mathbb{Z}$ the symbols $\mathbb{Z}_m$, $\mathbb{Z}/(m)$, $\mathbb{Z}/m$ and $\mathbb{Z}/m\mathbb{Z}$ all denote the same, i.e. the cyclic group of order m. We will use all these symbols, trying in each context to ease notation as much as possible.

LEMMA 4.40. *Let G be a torsion-free abelian group and let $T = \bigoplus_{i=1}^r \mathbb{Z}/(d_i)$ with d_i a divisor of d_{i+1} for $1 \leq i \leq r-1$. If there exists a surjective group homomorphism*

$$h : G \to T = \bigoplus_{i=1}^{r} \mathbb{Z}/(d_i)$$

then the rank of G is at least r.

PROOF. Let us call 1_{d_i} for the canonical generator of $\mathbb{Z}/(d_i) \subset T$. Since h is surjective, there exists elements $g_i \in G$ such that $h(g_i) = 1_{d_i}$ for $1 \leq i \leq r$.

Let us see that $\{g_1, \ldots, g_r\}$ is a linearly independent subset of G. Suppose that for some inetegers $n_1, \ldots, n_r$ we have $n_1 g_1 + \cdots + n_r g_r = 0$. Since G is torsion free, we can assume that the n_i have no common factor, i.e. $lcd(n_i : 1 \leq i \leq r) = 1$.

Suppose that the n_i are not all zero, and select n_l the first of the n_i which is non-equal to 0 i.e. such that $0 = n_1 = \cdots = n_{l-1}$ and $n_l \neq 0$.

Then $n_l g_l = -n_{l+1} g_{l+1} - \cdots - n_r g_r$. Apply the homomorphism h to both sides to get

$$[n_l]_{d_l} 1_{d_l} = [-n_{l+1}]_{d_{l+1}} 1_{d_{l+1}} + \cdots + [-n_r]_{d_r} 1_{d_r}$$

where $[n]_{d_i}$ stands for the reduction of $n \in \mathbb{Z}$ modulo d_i.

Now, since T is a direct sum we see that

$$n_l \equiv 0 \pmod{d_l} \quad , \quad \ldots \quad , \quad n_r \equiv 0 \pmod{d_r}$$

which means that d_j divides n_j for $l \leq j \leq r$.

The conclusion is that d_l divides n_i for $1 \leq i \leq r$, which is a contradiction with the choice of the n_i. □

The following Proposition is a consequence of much importance to our study of Sasakian and K-contact manifolds. Starting from basic topological data of M, it allows to infer quite an amount of information about the base space X of *any* semi-regular Seifert bundle $M \to X$.

PROPOSITION 4.41. *Let M be a closed 5-manifold with $H_1(M,\mathbb{Z}) = 0$ and $H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus_{i=1}^{k+1} (\mathbb{Z}/(p^i))^{2g_i}$, for some $k \geq 0$, p a prime, and $g_i \geq 1$.*

If $f : M \to X$ is a semi-regular Seifert bundle, then $H_1(X,\mathbb{Z}) = 0$, $H_2(X,\mathbb{Z}) = \mathbb{Z}^{k+1}$, and the isotropy locus of the orbifold X contains $k+1$ disjoint surfaces D_i linearly independent in rational homology and of genus $g(D_i) = g_i$.

PROOF. Let $D_1, \ldots, D_q$ be the isotropy surfaces of X with positive genus, and coefficients m_r, $1 \leq r \leq q$. Let $S_{q+1}, \ldots, S_l$ be the isotropy spheres of X with coefficients n_j, $q+1 \leq j \leq l$.

By Theorem 4.35 we know that

$$\begin{aligned} &H_1(X,\mathbb{Z}) = 0 \\ &H_2(X,\mathbb{Z}) \cong \mathbb{Z}^{k+1} \\ &H_2(M,\mathbb{Z}) \cong \mathbb{Z}^k \oplus \bigoplus_{r=1}^{q} (\mathbb{Z}/(m_r))^{2g(D_r)} \end{aligned}$$

On the other hand, we know by hypothesis that

$$H_2(M,\mathbb{Z}) \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{k+1} (\mathbb{Z}/(p^i))^{2g_i}.$$

Equating the torsion parts we see that

$$\bigoplus_{r=1}^{q} (\mathbb{Z}/(m_r))^{2g(D_r)} \cong \bigoplus_{i=1}^{k+1} (\mathbb{Z}/(p^i))^{2g_i}$$

Hence the invariant factors of the group $\bigoplus_{r=1}^{q} (\mathbb{Z}/(m_r))^{2g(D_r)}$ are p^i, $1 \le i \le k+1$, where each p^i is repeated $2g_i$ times. From here it is easy to see the following:

(1) Each of the m_r with $1 \le r \le q$ is one of the p^i.
(2) For each p^i with $1 \le i \le k+1$ there is at least one m_r so that $m_r = p^i$.

But in principle the above still allows that many of the m_r were the same p^i as long as the sum $\sum_r g(D_r)$ of the genus of the corresponding surfaces D_r equals g_i. Let us see that this cannot happen. Put

$$\{m_r : 1 \le r \le q\} = \{p^i : 1 \le i \le k+1\} \cup \{p^{i_j} : j \in J\}$$

with $|J| = q-(k+1) \ge 0$. To see that $J = \emptyset$ we need to use that the map

$$\alpha_2 : H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{k+1} \to \sum_{i=1}^{k+1} H^2(D_i, \mathbb{Z}/p^i) + \sum_{j=k+2}^{q} H^2(D_j, \mathbb{Z}/p^{i_j}) + \sum_{j=q+1}^{l} H^2(S_j, \mathbb{Z}/m_j)$$

is surjective. The group of the right hand side has at least q distinct invariant factors. Since the map is surjective it cannot be $q > k+1$ by Lemma 4.40 above. This implies that $q = k+1$, so there are exactly $k+1$ surfaces D_i with positive genus, and they have each a different isotropy coefficient p^i. It follows that $g(D_i) = g_i$. Moreover D_i and D_j are disjoint for $i \ne j$ because the isotropies of D_i and D_j are p^i and p^j, in particular not coprime.

Let us see that the classes $[D_i]$ are independent. Consider the projection map

$$p : \bigoplus_{i=1}^{k+1} H^2(D_i, \mathbb{Z}/p^i) \oplus \bigoplus_{j=k+2}^{l} H^2(S_i, \mathbb{Z}/(n_j)) \to \bigoplus_{i=1}^{k+1} H^2(D_i, \mathbb{Z}/p^i)$$

The explicit expression of the map $p \circ \alpha_2$ is given by

$$\begin{aligned} p \circ \alpha_2 : H^2(X,\mathbb{Z}) = \mathbb{Z}^{k+1} \quad &\to \quad \sum_{i=1}^{k+1} H^2(D_i, \mathbb{Z}/(p^i)) = \sum_{i=1}^{k+1} \mathbb{Z}/(p^i) \\ [\Sigma] \quad &\mapsto \quad \big([\Sigma]\cdot[D_i] \pmod{p^i}\big) \end{aligned}$$

As this map is surjective, for each $[D_i]$ there exists an element $[\Sigma_i] \in H^2(X,\mathbb{Z})$ so that

- $[\Sigma_i]\cdot[D_i] \equiv 1 \pmod{p^i}$
- $[\Sigma_i]\cdot[D_j] \equiv 0 \pmod{p^j}$ for $j \ne i$.

In particular it follows that

$$\begin{aligned} &[\Sigma_i]\cdot[D_i] \equiv 1 \pmod{p} \\ &[\Sigma_i]\cdot[D_j] \equiv 0 \pmod{p} \quad \text{for } j \ne i\,. \end{aligned}$$

Note that the classes $[D_i]$ are non-torsion, (this follows from $[\Sigma_i]\cdot[D_i] \ne 0$, or simply noting that $H^2(X)$ and $H_2(X)$ have no torsion). Now, if the $[D_i]$ were not linearly independent (over $\mathbb{Z}$ or $\mathbb{Q}$) then there would exist integers b_i so that $\sum_i b_i[D_i] = 0$. We can choose b_i that are coprime. Multiplying by $[\Sigma_j]$ we get $\sum_i b_i[D_i]\cdot[\Sigma_j] = 0$, for $1 \le j \le k+1$. Reducing modulo p, we have $b_j \equiv 0 \pmod{p}$, and this is a contradiction.

The conclusion is that the classes $[D_i]$ are independent both in $H_2(X,\mathbb{Z})$ and in $H_2(X,\mathbb{Q})$. This concludes the Theorem. □

5. Sasakian and K-contact manifolds.

First of all we need to introduce the concept of a contact, K-contact, and Sasakian manifold.

Definition 4.42. *A* contact manifold *is a pair (M,η) of a differentiable manifold M of dimension $\dim M = 2n+1$ and a 1-form $\eta \in \Omega^1(M)$ such that $\eta \wedge (d\eta)^n > 0$ everywhere on M.*

Note that a contact manifold is automatically oriented with volume form $\eta \wedge (d\eta)^n$. It is a classical result that if (M,η) is a contact manifold then the $2n$-dimensional distribution $\mathcal{D} = \ker\eta \subset TM$ is non-integrable, i.e. there does not exist a submanifold $N \subset M$ with $\dim N = 2n$ such that $T_xN = \mathcal{D}|_x$ for all $x \in N$. It is also well-known that the distribution $(\mathcal{D}, d\eta)$ is a symplectic distribution in the sense that $d\eta|_{\mathcal{D}}$ is non-degenerate.

Definition 4.43. *Let (M,η) be a contact manifold. The* Reeb vector field ξ *is the only vector field on M such that $\iota_\xi\eta = 1$ and $\iota_\xi d\eta = 0$.*

Clearly, ξ is transverse to the distribution $\mathcal{D} = \ker\eta$.

Definition 4.44. *We say that a contact manifold (M,η) is* K-contact *if there exists an endomorphism Φ of TM such that:*

- *$\Phi^2 = -\operatorname{Id} + \xi\otimes\eta$, where ξ is the Reeb vector field of η*
- *the contact form η is compatible with Φ in the sense that $d\eta(\Phi X, \Phi Y) = d\eta(X,Y)$, for all vector fields X, Y,*
- *$d\eta(\Phi X, X) > 0$ for all nonzero $X \in \ker\eta$, and*
- *the Reeb field ξ is Killing with respect to the Riemannian metric defined by the formula $g(X,Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.*

We denote (M,η,Φ,g) for the K-contact structure on M.

In other words, $d\eta$ defines a non-degenerate 2-form on the distribution $\mathcal{D} = \ker\eta \subset TM$, and the endomorphism Φ defines a complex structure on the distribution $\mathcal{D}$ compatible with $d\eta$, hence Φ is orthogonal with respect to the metric $g|_{\mathcal{D}}$. By definition, the Reeb vector field ξ is orthogonal to $\mathcal{D}$, and it is a Killing vector field for the metric g, so the flow of ξ acts on (M,g) by isometries. Also note that for any vector fields X, Y on M we have

$$g(X, \Phi Y) = d\eta(\Phi X, \Phi Y) + \eta(X)\eta(\Phi Y) = d\eta(X,Y)$$

since the image of Φ is $\mathcal{D} = \ker\eta$.

Now we define Sasakian structure, for which we need to introduce the cone of a manifold. Let (M,η,g,Φ) be a K-contact manifold. Let $C(M)$ be the *Riemannian cone* of M, i.e. the Riemannian manifold $C(M) = (M\times\mathbb{R}^+, \widetilde{g} = t^2g + dt^2)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. One defines an almost complex structure I on $C(M)$ by:

- $I(X) = \Phi(X)$ on $\ker\eta$,
- $I(\xi) = t\frac{\partial}{\partial t}$, $I(t\frac{\partial}{\partial t}) = -\xi$.

It is easy to see that $I \in \operatorname{End}(TM)$ is an almost complex structure on M acting by isometries, i.e. $\widetilde{g}(IX,IY) = \widetilde{g}(X,Y)$. The corresponding 2-form is

$$\omega(X,Y) = \widetilde{g}(X, IY)$$

An easy computation gives that

$$\omega = t^2p_1^*(d\eta) + 2t\,p_2^*(dt\wedge\eta) = t^2d\eta + 2t\,dt\wedge\eta = d(t^2\eta)\,.$$

This is clearly closed and non-degenerate so $(C(M), I, \widetilde{g}, \omega)$ is an almost-Kahler manifold. In the formula for ω we have denoted $p_1 : C(M) \to M$ and $p_2 : C(M) \to \mathbb{R}$ the projections onto the first and second factors, but we may omit these projections in the future.

DEFINITION 4.45. *Let (M, η, Φ, g) be a K-contact manifold. We say that (M, η, Φ, g) is* Sasakian *if the almost-complex structure I on $C(M)$ defined above is integrable.*

Hence a Sasakian manifold is a K-contact manifold such that the almost-Kahler manifold $(C(M), I, \widetilde{g}, \omega)$ is actually Kahler.

Let us introduce the concept of quasi-regularity and regularity of a contact structure. For a reference on this see [**9**, p. 188]).

Lat (M, η) be a contact manifold with Reeb vector field ξ. Denote by φ the flow of the Reeb vector field ξ. For a point $x \in M$ we call $\{\varphi_t(x) : t \in \mathbb{R}\}$ the leaf of ξ through x. The leafs determine a foliation $\mathcal{F}_\xi$ on M, called the *Reeb foliation* or the *characteristic foliation*.

DEFINITION 4.46. *We say that the contact manifold (M, η) is* quasi-regular *if there exists some positive integer k such that for every $x \in M$ there exists a neighborhood U^x such that each leaf on the foliation $\mathcal{F}_\xi$ passes through U^x at most k times.*

If k can be taken as 1 then the contact manifold (M, η) is called regular.

If M is compact, it is easy to see that a contact form η in M is quasi-regular if and only if all the leafs of the Reeb foliation $\mathcal{F}_\xi$ are compact, hence diffeomorphic to the circle $\mathbb{S}^1$.

The above definition gives rise to the concept of quasi-regular (regular) K-contact and Sasakian structures. Note that in a K-contact manifold (M, η, Φ, g), the flow φ_t of the Reeb vector field ξ acts on M by isometries. If M is compact the flow is defined on $\mathbb{R}$, and $H = \{\varphi_t : t \in \mathbb{R}\}$ is an abelian Lie subgroup of the compact Lie group $\mathrm{Isom}(M, g)$ of isometries of M. The closure of H in $\mathrm{Isom}(M, g)$ can be seen to be a torus, and when this torus is simply a circle then the K-contact manifold is quasi-regular. See [**9**] and the references therein for a thorough discussion of this.

A very important result is that, in order to study the *geography* of Sasakian and K-contact manifolds, i.e. questions regarding which smooth manifolds admit or not K-contact or Sasakian structures, we can restrict ourselves to quasi-regular structures. The following result of [**9**] says that if M admits a K-contact structure, then it also admits a quasi-regular K-contact structure.

THEOREM 4.47. *Let (M, η_0, Φ_0, g_0) be a K-contact manifold. Then there exists a deformation (η, Φ, g) (close in the $\mathcal{C}^\infty$-topology) of the inicial K-contact structure (η_0, Φ_0, g_0) such that (M, η, Φ, g) is a quasi-regular K-contact structure.*

PROOF. See [**9**] Theorem 7.1.10, or [**43**] for a different proof in the compact case. □

The following is the analogous Theorem regarding Sasakian structures, saying that any such structure can be approximated by a quasi-regular one.

THEOREM 4.48. *Let (M, η_0, Φ_0, g_0) be a Sasakian manifold. Then there exists a deformation (η, Φ, g) (close in the $\mathcal{C}^\infty$-topology) of the initial Sasakian structure (η_0, Φ_0, g_0) such that (M, η, Φ, g) is a quasi-regular Sasakian structure.*

PROOF. See [**47**], Prop 6.2. □

Now, another important fact is that a quasi-regular K-contact (Sasakian) manifold is the same thing as a Seifert bundle over an almost-Kahler (Kahler) cyclic orbifold. The following two results stated in [**9**] show this.

THEOREM 4.49. *Let (M, η, Φ, ξ, g) be a quasi-regular K-contact manifold. Then the space of leaves X has a natural structure of an almost-Kahler cyclic orbifold inherited from the projection $f : M \to X$, which is a Seifert bundle.*

Furthermore, if (M, η, Φ, ξ, g) is Sasakian, then X is a Kahler orbifold.

PROOF. Take a point $p \in M$, and let O_p be the orbit through p. Since O_p interesects finitely many times every small neighbourhood, then O_p must be a circle. Let ϕ_t be the Reeb flow, and consider t_p the period of $\phi_t(p)$. Let $F = \phi_{t_p}$, $H_p = \langle \xi_p \rangle^\perp = \ker \eta|_p$, and $d_pF| : H_p \to H_p$. For $\varepsilon > 0$ small, take $B_\varepsilon(0) \subset H_p$. Then

$$\begin{aligned} \varphi : \mathbb{R} \times B_\varepsilon(0) &\to M\,, \\ (t, w) &\mapsto \phi_t(\exp_p(w)) \end{aligned}$$

is an open embedding whose image

$$W = \varphi(\mathbb{R} \times B_\varepsilon(0)) \subset M$$

is a neighbourhood of O_p consisting of orbits of the Reeb flow. Recall that the Reeb flow is by isometries, so it preserves the distances to O_p. Take $q = \varphi(0, w) = \exp_p(w) \in S_p$. Since F is an isometry we have that

$$F(q) = \phi_{t_p}(q) = \phi_{t_p}(\exp_p(w)) = \exp_p(d_pF(w)).$$

As $d_pF| : H_p \to H_p$, this implies that $F|_{S_p} : S_p \to S_p$. On the other hand as ξ is a quasi-regular vector field, the orbits intersect the slice $S_p = \varphi(\{0\} \times B_\varepsilon(0))$ at finitely many points. Given $q \in S_p$, the orbit O_q intersects S_p at the points

$$F^k(q) = \exp_p((d_pF)^k(w)) \quad \text{for } k \in \mathbb{Z}$$

Since $d_pF| : H_p \to H_p$ is a linear isometry, this implies that it is of finite order. Let m be its order, so $(d_pF|_{H_p})^m = \mathrm{Id}_{H_p}$, and hence $F^m|_{S_p} = \mathrm{Id}_{S_p}$. Since $F^m = \phi_{mt_p}$, d_pF also leaves invariant the direction tangent to the orbits, so $(d_pF)^m = \mathrm{Id}_{T_pM}$, so $F^m|_W = \mathrm{Id}_W$. Therefore ϕ_t gives an S^1-action on W, with period $m\,t_p$ for some $m \in \mathbb{N}$. Using Proposition 4.19, we have a Seifert bundle

$$f : M \to X\,, \quad p \mapsto f(p) = \bar{p} = O_p \in X$$

over the space of leaves X, which is a cyclic orbifold.

Let us see that X has the structure of an almost Kahler orbifold if M is K-contact. The open set $W \cong (S^1 \times B_\varepsilon(0))/\mathbb{Z}_m$, and the orbifold chart is $(U = B_\varepsilon(0), V, \phi, \mathbb{Z}_m)$, where $\phi : B_\varepsilon(0) \to X$ is given by $\phi(w) = f(\exp_p(w))$. Then for $p \in M$, the orbifold tangent space at $\bar{p} = f(p)$ is identified with $T_0U \cong H_p$. We put at $\bar{p} \in X$ the complex and symplectic structures (J, ω) on T_0U given by $\Phi, d\eta$ on H_p, respectively. These are well defined independently of the point in the orbit, since the Reeb flow acts by isometries, preserving Φ and η. Finally, these complex and symplectic structures are $\mathbb{Z}_m$-invariant (since the action is given by d_pF, the isometry defined by the Reeb flow $F = \phi_{t_p}$).

Now suppose that M is Sasakian. Then, by definition, there is an integrable complex structure I on the cone $C(M) = M \times \mathbb{R}^+$, given by $I(V) = \Phi(V)$ for a vector field V in $\ker \eta$, and $I(\xi) = t\frac{\partial}{\partial t}$. This means that the Nijenhuis tensor vanishes, i.e. for any vector fields V_1, V_2 on $C(M)$ we have

$$N_I(V_1, V_2) = -[V_1, V_2] + I[IV_1, V_2] + I[V_1, IV_2] - [IV_1, IV_2] = 0. \tag{32}$$

Take an orbifold chart $(U, V, \phi, \mathbb{Z}_m)$ as above with $W = (S^1 \times U)/\mathbb{Z}_m$, $\bar{p} = \phi(0) \in V \subset X$. Take V_1, V_2 two $\mathbb{Z}_m$-equivariant vector fields on U. Let us see that $N_J(V_1, V_2)_{\bar{p}}$ vanishes. The vector fields V_1, V_2 on U define vector fields (that we denote V_1, V_2 again) on

$$W \times \mathbb{R}^+ = ((S^1 \times U)/\mathbb{Z}_m) \times \mathbb{R}^+ = (S^1 \times U \times \mathbb{R}^+)/\mathbb{Z}_m\,.$$

We write $V_1 = V_1' + a\xi$, $V_2 = V_2' + b\xi$, where $V_1'|_x, V_2'|_x \in H_x$, for all points $x \in W$, and a, b smooth functions with $a(p) = b(p) = 0$. We expand $N_I(V_1, V_2)$ given in (32), substitute at p, and discard the components with ξ, ∂_t (that is, project down to H_p). We get

$$-[V_1', V_2'] + \Phi[\Phi V_1', V_2'] + \Phi[V_1', \Phi V_2'] - [\Phi V_1', \Phi V_2'] = 0,$$

at p. Using the projection $h : S^1 \times U \times \mathbb{R}^+ \to U$, and that the Lie bracket is preserved ($h_*[V_1', V_2'] = [V_1, V_2]$), and the formula $h_*(\Phi V_1') = JV_1$, we get

$$N_J(V_1, V_2) = -[V_1, V_2] + J[JV_1, V_2] + J[V_1, JV_2] - [JV_1, JV_2] = 0$$

at $\bar{p}$, as required. □

As for ordinary circle bundles, for Seifert bundles we can also see the Chern class as the curvature of a connection 1-form on the total space. Let us see this more in detail.

Let $f : M \to X$ be a Seifert bundle. We construct a connection 1-form on $M \to X$ as follows. Take an orbifold cover

$$X = \bigcup_\alpha V_\alpha$$

formed by coordinate patches $V_\alpha \cong U_\alpha / \mathbb{Z}_{m_\alpha}$ and take an orbifold partition of unity $\{\rho_\alpha\}$ subordinated to the cover $\{V_\alpha\}$. For each $V_\alpha \cong U_\alpha/\mathbb{Z}_{m_\alpha}$ we have $f^{-1}(V_\alpha) = (S^1 \times U_\alpha)/\mathbb{Z}_{m_\alpha}$. Let $\eta_\alpha = u_\alpha^{-1} du_\alpha$, where u_α is the S^1-coordinate. Define

$$\eta = \sum_\alpha (f^* \rho_\alpha) \eta_\alpha. \tag{33}$$

This is an orbifold 1-form in the sense that in its definition one needs to consider local lifting to the orbifold charts U_α, but actually $\eta \in \Omega^1(M)$ defines an honest 1-form in the manifold M. We have also that

$$F := d\eta = \sum_\alpha f^*(d\rho_\alpha) \wedge \eta_\alpha$$

is the (orbifold) curvature 2-form of $f : M \to X$.

To see that $[F] = c_1(M/X)$, note that the orbi-form η descends to an orbifold 1-form $\bar{\eta}$ defined on M/μ, the total space of the circle fiber bundle $f/\mu : M/\mu \to X$. The fiber of M/μ is parametrized by $\bar{u}_\alpha = u_\alpha^m$, with $m = m(X)$. So the connection 1-form $\bar{\eta}$ on M/μ satisfies $\bar{\eta} = m\eta$ under the pull-back by the quotient

$$\pi : M \to M/\mu\,.$$

The curvature of $\bar{\eta}$ is $d\bar{\eta} = mF$ and thus

$$c_1(M/\mu) = [mF] \in H^2(X, \mathbb{Z}).$$

This implies that

$$c_1(M/X) = \frac{1}{m} c_1(M/\mu) = [F] \in H^2(M, \mathbb{Q})\,.$$

We have proved then that the curvature of the connection form $\eta \in \Omega^1(M)$ represents the Chern class of a Seifert bundle, as we wanted to see.

The following result is a reciprocal of Theorem 4.49. It appears in [**9**, p. 211], where it is referred to [**30**]. However the proof in [**30**] does not cover the orbifold case, so we have included a proof for completeness.

THEOREM 4.50. *Let (X, ω, J, g) be an almost Kahler cyclic orbifold with $[\omega] \in H^2(X, \mathbb{Q})$, and let $f : M \to X$ be a Seifert bundle with $c_1(M/X) = [\omega]$. Then M admits a K-contact structure (ξ, η, Φ, g) such that $f^*(\omega) = d\eta$.*

Furthermore, if (X, ω, J, g) is Kahler, then (ξ, η, Φ, g) is a Sasakian structure on M

PROOF. Take the connection 1-form η constructed in (33) above, and let $F = d\eta$ be its curvature. As $[F] = c_1(M/X) = [\omega]$, we have that $F - \omega = d\beta$, for some 1-form β. Then we can change η to $\eta' = \eta - \beta$, so that its curvature is $F' = F - d\beta = \omega$. Hence we can assume that the connection 1-form η satisfies $F = d\eta = \omega$.

Now the 1-form η is a smooth form on the total space M. On each $V \subset X$ small open set with $f^{-1}(V) \cong (S^1 \times U)/\mathbb{Z}_m$, we have that $d\eta = \omega$ is the 2-form coming from U. So $\eta \wedge (d\eta)^2 > 0$, and η is a contact form. Now define the Reeb vector field ξ as the one given by the S^1-action, which clearly preserves η. Define

$$H_p = \ker \eta_p \subset T_pM$$

and define

$$\Phi : T_pM \to T_pM$$

by $\Phi(\xi) = 0$ and $\Phi : H_p \to H_p$ as the almost complex structure $J_x : T_xU \to T_xU$ of the orbifold X, with $x = f(p)$, under the isomorphism $H_p \cong T_xU$. This is well-defined since the S^1-flow preserves the horizontal subspaces H_p. Clearly the Reeb flow preserves Φ.

Finally define the metric g by declaring H_p and ξ_p orthogonal, ξ_p unitary and such that g is the metric on H_p given by Φ and ω. Then the Reeb flow preserves g, i.e. it acts by isometries. This means that (M, ξ, η, Φ, g) is a K-contact manifold.

Now suppose that X is a Kahler orbifold and let us see that the K-contact structure constructed above is Sasakian. Let $(C(M), I, d(r^2\eta), dr^2 + r^2 g)$ the metric cone of M. We have to prove that the Nijenhuis tensor of I vanishes, i.e. we must show that for any vector fields V_1, V_2 on $C(M)$

$$0 = N_I(V_1, V_2) = -[V_1, V_2] + I[IV_1, V_2] + I[V_1, IV_2] - [IV_1, IV_2]\,. \tag{34}$$

Take an open set of $C(M)$ of the form $W \times \mathbb{R}^+$, being $W = (S^1 \times U)/\mathbb{Z}_m \subset M$. Since N_I is a tensor, it is enough to prove that it vanishes in a basis of $T_{(p,t)}C(M) = T_t\mathbb{R} \times T_pM$, for $(t, p) \in C(M)$ a point. We take as a basis the vectors $t\partial_t, \xi, V_i$, where V_i are a basis of $\ker \eta_p$.

Case 1: We take $V_1 = t\partial_t$, and $V_2 \in \ker \eta_p \subset T_pM$. We extend V_2 to a vector field in $C(M) = M \times \mathbb{R}^+$ invariant in the t-direction and tangent to the distribution $\ker \eta$. We can also suppose that the extension of V_2 is invariant under the Reeb flow, as we can simply extend V_2 via the push-forward by the Reeb flow. Then we have

$$N_I(t\partial_t, V_2) = -0 + I[\xi, V_2] - 0 - [\xi, IV_2] = 0\,.$$

Recall that $[\xi, V_2] = 0$ by the choice of V_2, and $[\xi, IV_2] = 0$ since both the endomorphism I and V_2 are invariant under the Reeb flow.

Case 2: Take $V_1 = \xi$, $V_2 \in \ker \eta_p$. As before, extend V_2 via the Reeb flow, so it is invariant by the flows of ξ and $t\partial_t$, and remains tangent to $\ker \eta$. We have

$$N_I(\xi, V_2) = -0 + 0 - I[\xi, IV_2] - 0 = 0\,.$$

We have used again that $[\xi, V_2] = 0 = [t\partial_t, V_2]$, and also that $[\xi, IV_2] = \mathcal{L}_\xi(IV_2) = 0$ bacause both $\mathcal{L}_\xi I$ and $\mathcal{L}_\xi V_2$ vanish.

Case 3: Take $V_1, V_2 \in \ker \eta_p$. Now extend both V_1, V_2 via the Reeb flow. Note that the vector fields obtained in this way remain in the distribution $\ker \eta$. We have

$$N_I(V_1, V_2) = -[V_1, V_2] + I[IV_1, V_2] + I[V_1, IV_2] - [IV_1, IV_2] = 0$$

because the complex structure I equals the complex structure J of the orbifold X when restricted to the distribution $\ker \eta$, and by hypothesis the complex structure J is integrable. The computations of this case are analogous of the ones carried out in the proof of Theorem 4.49. □

Theorem 4.50 corrects an statement of [**33**], where it is claimed that a K-contact structure can be constructed from an orbifold where the isotropy locus is not a collection of symplectic surfaces.

From these results we see that if any smooth manifold M admits a K-contact or Sasakian structure, then M also admits the structure of a Seifert bundle over a symplectic or Kahler orbifold. As we shall see, this gives topological restrictions for a smooth manifold to admit K-contact or Sasakian structures.

Now let us give an existence result for 5-dimensional K-contact manifolds. It produces a K-contact 5-manifold starting from a symplectic 4-manifold and a suitable configuration of symplectic surfaces. In other words, it produces a Seifert bundle whose base space is a cyclic symplectic orbifold, and this orbifold is constructed from a symplectic 4-manifold and a configuration of surfaces as in Proposition 4.6. We already saw an existence result for Seifert bundles in Proposition 4.23, and the following is an adaptation of this result to the case that the base space is symplectic.

LEMMA 4.51. *Let (X,ω) be a closed symplectic 4-manifold with a collection of embedded symplectic surfaces D_i intersecting transversely and positively, and integer numbers $m_i > 1$, with $\gcd(m_i, m_j) = 1$ whenever $D_i \cap D_j \neq \emptyset$. Let $0 < j_i < m_i$ with $\gcd(j_i, m_i) = 1$, and take $0 < b_i < m_i$ so that $j_i b_i \equiv 1 \pmod{m_i}$. Denote $m = \operatorname{lcm}\{m_i\}$. Then there is a Seifert bundle $f : M \to X$ such that:*

1. *It has orbit invariants (D_i, m_i, j_i).*
2. *It has Chern class $c_1(M/X) = [\hat\omega]$ for some orbifold symplectic form $\hat\omega$ on X.*
3. *If $\sum_i \frac{b_i m}{m_i}[D_i] \in H^2(X,\mathbb{Z})$ is primitive and the second Betti number $b_2(X) \geq 3$, then then we can further have that $c_1(M/\mu) \in H^2(X,\mathbb{Z})$ is primitive.*

PROOF. Consider the Seifert bundle $f : \widetilde{M} \to X$ with orbit invariants $\{(D_i, m_i, j_i)\}$, $m = m(X)$, and so that

$$c_1(\widetilde{M}/\mu) = m \sum \frac{b_i}{m_i}[D_i] \in H^2(X,\mathbb{Z})\,.$$

This is possible by Proposition 4.23. We fix this Seifert bundle $\widetilde{M}$. Now, the set of elements

$$\left\{ \frac{1}{mk+1} a + \frac{1}{mk+1} c_1(\widetilde{M}/X) \,:\, a \in H^2(X,\mathbb{Z}), k \geq 1 \right\} \subset H^2(X,\mathbb{R}) \tag{35}$$

is clearly dense.

Note that C^1-small perturbations of symplectic forms are still symplectic, so for any $\alpha \in \Omega^2(M)$ closed, there exists $\varepsilon(\alpha) > 0$ so that for $0 < \varepsilon < \varepsilon(\alpha)$, the closed form $\omega_{\varepsilon,\alpha} = \omega + \varepsilon\alpha$ is symplectic. We call the forms $\omega_{\varepsilon,\alpha}$ *perturbations of* ω.

Using that the set in (35) is dense, we can perturb ω slightly so that

$$[\omega] = \frac{1}{mk+1} a + \frac{1}{mk+1} c_1(\widetilde{M}/X)$$

for some $a \in H^2(X,\mathbb{Z})$ and $k \geq 1$.

Then the symplectic form $\widetilde{\omega} = (mk+1)\omega$ satisfies that $[\widetilde{\omega}] = a + c_1(\widetilde{M}/X)$. Choosing a line bundle B with $c_1(B) = a$, we have a Seifert bundle

$$M = \widetilde{M} \otimes B \quad \text{with} \quad c_1(M/X) = [\widetilde{\omega}] = a + c_1(\widetilde{M}/X)\,. \tag{36}$$

Now the process of Proposition 4.10 gives an orbifold symplectic form $\hat\omega$ on the orbifold X with isotropy surfaces D_i of multiplicities m_i. This has $[\hat\omega] = [\widetilde{\omega}] = c_1(M/X)$. This proves (1) and (2).

Now let us see (3) assuming $b_2(X) \geq 3$. Take primitive classes as follows:

$$\begin{aligned} &b_1 \in H^2(X,\mathbb{Z})\,, \quad \text{with} \quad c_1(\widetilde{M}/\mu)\cdot b_1 = 0\,,\\ &a_0 \in H^2(X,\mathbb{Z})\,, \quad \text{with} \quad a_0 \cdot b_1 = 1\,,\\ &b_2 \in H^2(X,\mathbb{Z})\,, \quad \text{with} \quad c_1(\widetilde{M}/\mu)\cdot b_2 = 0 \ \text{ and } \ a_0\cdot b_2 = 0\,.\end{aligned}$$

The above choices are possible because $b_2(X) \geq 3$. Let us see that the elements of (35) with $\gcd(a\cdot b_1, a\cdot b_2) = 1$ are dense. To ease notation, call $c_1 = c_1(\widetilde{M}/X)$. Take any element

$$x = \frac{1}{mk+1}a + \frac{1}{mk+1}c_1 \tag{37}$$

in (35) given by some $a \in H^2(X,\mathbb{Z})$ and $k \in \mathbb{N}$. Consider the integers

$$k_1 = a\cdot b_1\,,\ \ k_2 = a \cdot b_2\,.$$

Let k_0 be a large integer containing all prime factors of both k_1, k_2. Then take the element x' determined by $a' \in H^2(X,\mathbb{Z})$ and $k' \in \mathbb{N}$ as follows

$$x' = \frac{1}{mk'+1}a' + \frac{1}{mk'+1}c_1\,, \quad \text{with} \quad a' = k_0 a + a_0\,, \quad k' = k_0 k \tag{38}$$

We have

$$\gcd(a'\cdot b_1, a'\cdot b_2) = \gcd(k_0k_1+1, k_0k_2) = 1$$

by the choice of k_0. We also have that

$$\begin{aligned} |x-x'| &= \left| \frac{1-k_0}{(mk+1)(mk_0k+1)}a - \frac{1}{mk_0k+1}a_0 + \frac{mk(k_0-1)}{(mk_0k+1)(mk+1)}c_1 \right| \\ &= \left| \frac{1-k_0}{(mk_0k+1)}\left(\frac{1}{(mk+1)}a + \frac{1}{(mk+1)}c_1 - c_1\right) + \frac{1}{mk_0k+1}a_0 \right| \\ &= \left| \frac{1-k_0}{(mk_0k+1)}x + \frac{k_0-1}{(mk_0k+1)}c_1 - \frac{1}{mk_0k+1}a_0 \right| \\ &\leq \frac{2}{mk}|x| + \frac{2}{mk}|c_1| + \frac{1}{mk}|a_0| \\ &\leq C_1\frac{|x|}{k} + C_2\frac{1}{k}\,. \end{aligned}$$

Given $\varepsilon > 0$ and a cohomology class $[\alpha] \in H^2(X,\mathbb{R})$ there exists $x = x(k,a)$ as in (37) with $|[\alpha]-x| < \varepsilon$ and with $1/k < \varepsilon$. Take the element x' of (38), so we have

$$|[\alpha]-x'| \leq |[\alpha]-x| + |x-x'| \leq \varepsilon + \frac{C_1(|a|+\varepsilon)}{k} + \frac{C_2}{k} \leq C_3|\alpha|\varepsilon$$

and this shows that the elements of (35) with $\gcd(a\cdot b_1, a\cdot b_2) = 1$ are dense.

So consider an element $a \in H^2(X,\mathbb{Z})$ with $\gcd(a\cdot b_1, a\cdot b_2) = 1$ and a Seifert bundle M as in (36) with

$$c_1(M/\mu) = ma + c_1(\widetilde{M}/\mu) = m[\widetilde{\omega}] \in H^2(X,\mathbb{Z})\,.$$

Then $c_1(M/\mu)\cdot b_j = m\, a\cdot b_j$, for $j = 1,2$. Therefore if $c_1(M/\mu)$ is divisible by some ℓ, then $\ell | m$. So $c_1(\widetilde{M}/\mu) = c_1(M/\mu) - ma$ is divisible by ℓ, and hence it is not a primitive class, contrary to hypothesis. This completes the proof of (3). □

CHAPTER 5

Construction of a K-contact non-Sasakian 5-manifold.

In this section we detail the construction of a 5-dimensional K-contact manifold with first homology group $H_1 = 0$ and which does not admit any semi-regular Sasakian structure. This is Theorem 5.8.

The first thing we need is a symplectic 4-manifold X with many disjoint symplectic surfaces generating the second homology. This manifold X will be used to construct a Seifert bundle and this will produce a K-contact 5-manifold M.

To see that this K-contact manifold M cannot admit a Sasakian structure, we will have to use the fact that Kahler 4-manifolds with many disjoint algebraic surfaces (divisors) generating the homology as in Proposition 4.41 are a rare occurrence. This is the content of the last subsection.

1. A symplectic 4-manifold with many disjoint symplectic surfaces.

In this section we construct a simply connected symplectic 4-manifold X with $b_2 = b_2(X)$ disjoint symplectic curves generating the second homology $H_2(X)$. Let us start with the precise statement.

THEOREM 5.1. *There exists a simply connected symplectic* 4*-manifold* X *with* $b_2(X) = 36$ *and with* 36 *disjoint surfaces* $S_1, \ldots, S_{36}$ *such that*

(1) $g(S_1) = \ldots = g(S_9) = 1$, $g(S_{11}) = \ldots = g(S_{19}) = 1$, $g(S_{21}) = \ldots = g(S_{29}) = 1$, *and* $S_i \cdot S_i = -1$, *for* $i = 1, \ldots, 9, 11, \ldots, 19, 21, \ldots, 29$;
(2) $g(S_{10}) = 3$, $g(S_{20}) = 3$, $g(S_{30}) = 3$, *and* $S_j \cdot S_j = 1$, $j = 10, 20, 30$;
(3) $g(S_{31}) = 1$, $g(S_{32}) = 1$, $g(S_{33}) = 2$, *and* $S_{31} \cdot S_{31} = -1$, $S_{32} \cdot S_{32} = -1$, $S_{33} \cdot S_{33} = 1$;
(4) $g(S_{34}) = 1$, $g(S_{35}) = 1$, $g(S_{36}) = 2$, *and* $S_{34} \cdot S_{34} = -1$, $S_{35} \cdot S_{35} = -1$, $S_{36} \cdot S_{36} = 1$.

The homology classes $[S_j]$, $j = 1, \ldots, 36$, *generate* $H_2(X, \mathbb{Z})$.

In the subsequent subsections we will construct such X. Our basic tools are Gompf symplectic sum, symplectic blow-up, elliptic and Lefschetz fibrations, and symplectic resolution of positive intersections. These we studied in Chapter 1.

2. First step: a configuration of tori in $\mathbb{T}^4$.

Let $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$, with coordinates $x_1, \ldots, x_4$. We denote $\partial_i = \frac{\partial}{\partial x_i} \in T(\mathbb{T}^4)$ for the partial derivative in the x_i-direction.

Consider the symplectic form

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_2 \wedge dx_3 + \delta dx_1 \wedge dx_4 + dx_2 \wedge dx_4 - \delta dx_1 \wedge dx_3,$$

where $\delta > 0$ is small (to see that ω is symplectic we compute $\omega \wedge \omega$ below).

We claim that $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is a positive basis for $(\mathbb{T}^4, \omega)$. To see it, we compute the volume form of $\mathbb{T}^4$:

$$\begin{aligned} \omega \wedge \omega &= 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &\quad +2\delta dx_2 \wedge dx_3 \wedge dx_1 \wedge dx_4 - 2\delta dx_2 \wedge dx_4 \wedge dx_1 \wedge dx_3 \\ &= 2(1+2\delta) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

which proves the claim.

Consider the following six embedded tori in $\mathbb{T}^4$:

$$\begin{aligned} T_{12} &= \{(x_1, x_2, \alpha_3, \alpha_4) : x_1, x_2 \in \mathbb{R}/\mathbb{Z}\} \ , \\ T_{34} &= \{(\alpha_1, \alpha_2, x_3, x_4) : x_3, x_4 \in \mathbb{R}/\mathbb{Z}\} \ , \\ T_{23} &= \{(\beta_1, x_2, x_3, \beta_4) : x_2, x_3 \in \mathbb{R}/\mathbb{Z}\} \ , \\ T_{14} &= \{(x_1, \beta_2, \beta_3, x_4) : x_1, x_4 \in \mathbb{R}/\mathbb{Z}\} \ , \\ T_{13} &= \{(x_1, \gamma_2, x_3, \gamma_4) : x_1, x_3 \in \mathbb{R}/\mathbb{Z}\} \ , \\ T_{24} &= \{(\gamma_1, x_2, \gamma_3, x_4) : x_2, x_4 \in \mathbb{R}/\mathbb{Z}\} \ , \end{aligned}$$

where $\alpha_i, \beta_i, \gamma_i$ are generic numbers which may be fixed at convenience when necessary. We get in this way a configuration of six tori intersecting transversely in pairs $T_{12} \cap T_{34}$, $T_{23} \cap T_{14}$ and $T_{13} \cap T_{24}$. Note that each pair intersects in a single point. Moreover, different choices of the generic numbers $\alpha_i, \beta_i, \gamma_i$ give parallel disjoint copies T'_{ij} of T_{ij}.

LEMMA 5.2. *The tori T_{ij} with $i, j \in \{1,2,3,4\}$, $i < j$, are all symplectic surfaces of $\mathbb{T}^4$. All of them have the standard orientation $[\{\partial_i, \partial_j\}]$, except T_{13} which has orientation $[\{\partial_3, \partial_1\}]$.*

PROOF. Take first $(i,j) \neq (1,3)$. It is clear that the torus T_{ij} is symplectic with respect to ω. Moreover ω restricted to T_{ij} equals:

(1) For $(i,j) \neq (1,4)$, the standard symplectic structure $dx_i \wedge dx_j$ on T_{ij}.
(2) For $(i,j) = (1,4)$, a positive multiple of the standard $\delta dx_1 \wedge dx_4$ on T_{14}.

in any case, T_{ij} is symplectic and the orientation induced by ω in T_{ij} has positive basis $\{\partial_i, \partial_j\}$ i.e. the standard orientation on T_{ij}.

On the other hand T_{13} is also symplectic with respect to ω, and ω restricted to T_{13} is

$$\omega|_{T_{13}} = -\delta dx_1 \wedge dx_3 \ ,$$

so the orientation induced by ω in T_{13} is the opposite to the standard. Note also that T_{13} has area δ. □

LEMMA 5.3. *With notations as above, we have the following intersection numbers:*

(1) $[T_{12}] \cdot [T_{34}] = 1$
(2) $[T_{13}] \cdot [T_{24}] = 1$
(3) $[T_{14}] \cdot [T_{23}] = 1$

PROOF. Clearly all of them intersect in just one point. Let us see that the intersections are positive. Since $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is a positive basis for $(\mathbb{T}^4, \omega)$, it follows that $[T_{12}] \cdot [T_{34}] > 0$.

On the other hand note that T_{13} is oriented reversely, with $\{\partial_3, \partial_1\}$ as positive basis. The permutation (3124) has the same signature that (1234), so $[T_{13}] \cdot [T_{24}] > 0$. Analogously $[T_{14}] \cdot [T_{23}] > 0$ since (1423) has the same signature that (1234). □

Consider now the following specific collection of three disjoint 2-tori, which are symplectic in $(\mathbb{T}^4, \omega)$:

$$\begin{aligned} T_{12} &= \{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{R}/\mathbb{Z}\} \subset \mathbb{T}^4\,, \\ T_{13} &= \{(x_1, 0, x_3, \tfrac{1}{2}) : x_1, x_3 \in \mathbb{R}/\mathbb{Z})\} \subset \mathbb{T}^4\,, \\ T_{14} &= \{(x_1, \tfrac{1}{2}, \tfrac{1}{2}, x_4) : x_1, x_4 \in \mathbb{R}/\mathbb{Z}\} \subset \mathbb{T}^4\,. \end{aligned}$$

We want to do a Gompf symplectic sum along each of T_{12}, T_{13} and T_{14}. For this, we cut out tubular neighbourhoods $\nu(T_{ij})$ of T_{12}, T_{13} and T_{14} of some small radius $\varepsilon > 0$. Let us call

$$\begin{aligned} Y &= \mathbb{T}^4 \setminus (\nu(T_{12}) \cup \nu(T_{13}) \cup \nu(T_{14})) \\ &= \{(x_1, x_2, x_3, x_4) \in \mathbb{T}^4 : |(x_3, x_4)| \geq \varepsilon, |(x_2, x_4 - \tfrac{1}{2})| \geq \varepsilon,\ |(x_2 - \tfrac{1}{2}, x_3 - \tfrac{1}{2})| \geq \varepsilon\}. \end{aligned}$$

We shall denote $\partial_{1j} Y = \partial\nu(T_{1j})$, $j = 2, 3, 4$, the three connected components of the boundary ∂Y. Recall that

$$\begin{aligned} \partial_{12} Y &= \{(x_1, x_2, x_3, x_4) : x_3^2 + x_4^2 = \varepsilon^2\} \\ \partial_{13} Y &= \{(x_1, x_2, x_3, x_4) : x_2^2 + (x_4 - \tfrac{1}{2})^2 = \varepsilon^2\} \\ \partial_{14} Y &= \{(x_1, x_2, x_3, x_4) : (x_2 - \tfrac{1}{2})^2 + (x_3 - \tfrac{1}{2})^2 = \varepsilon^2\} \end{aligned}$$

Let us describe a configuration of certain Lagrangian tori T_1, T_2, and cylinders C_1, C_2 in Y to be used later.

$$\begin{aligned} C_1 &= \{(x_1, -\delta(\tfrac{1}{2} - 2\varepsilon)(t-1), 0, \varepsilon + (\tfrac{1}{2} - 2\varepsilon)t), t \in [0,1]\}\,, \\ C_2 &= \{(x_1, \tfrac{1}{2} + \delta(\tfrac{1}{2} - 2\varepsilon)(t-1), \varepsilon + (\tfrac{1}{2} - 2\varepsilon)t, 0), t \in [0,1]\}\,, \\ T_1 &= \{(\tfrac{1}{2} - \tfrac{\varepsilon}{\delta}(\sin\theta - \cos\theta), \varepsilon\cos\theta, x_3, \tfrac{1}{2} + \varepsilon\sin\theta), \theta \in [0, 2\pi]\}\,, \\ T_2 &= \{(\tfrac{1}{2} - \tfrac{\varepsilon}{\delta}(\sin\theta + \cos\theta), \tfrac{1}{2} + \varepsilon\cos\theta, \tfrac{1}{2} + \varepsilon\sin\theta, x_4), \theta \in [0, 2\pi]\}\,. \end{aligned}$$

PROPOSITION 5.4. *If we choose δ and ε positive and small enough, the cylinders C_1, C_2 and the tori T_1, T_2 satisfy the following:*

(1) $C_1, C_2 \subset Y$, $T_1 \subset \partial_{13} Y$, $T_2 \subset \partial_{14} Y$.
(2) $C_1 \cap C_2 = \emptyset$, $C_1 \cap T_2 = \emptyset$, $C_2 \cap T_1 = \emptyset$, $T_1 \cap T_2 = \emptyset$.
(3) C_1 *and* T_1 *intersect transversely in a unique point.*
(4) C_2 *and* T_2 *intersect transversely in a unique point.*
(5) C_1 , C_2 , T_1 , T_2 *are Lagrangian.*
(6) $\partial C_1 \subset \partial Y$ *consists of two circles, one contained* $\partial_{12} Y$ *and another in* $\partial_{13} Y$.
(7) $\partial C_2 \subset \partial Y$ *consists of two circles, one contained* $\partial_{12} Y$ *and another in* $\partial_{14} Y$.

PROOF. First let us see (1). Recall that Y has implicit equations

$$Y = \{x_3^2 + x_4^2 \geq \varepsilon^2 \ \ ,\ \ x_2^2 + (x_4 - \tfrac{1}{2})^2 \geq \varepsilon^2 \ \ ,\ \ (x_2 - \tfrac{1}{2})^2 + (x_3 + \tfrac{1}{2})^2 \geq \varepsilon^2\}.$$

To see that C_1 satisfies the above, we express it in parametric equations as

$$C_1 : x_1 = x_1\ ,\quad x_2 = -\delta(\tfrac{1}{2} - 2\varepsilon)(t-1)\ ,\quad x_3 = 0\ ,\quad x_4 = \varepsilon + (\tfrac{1}{2} - 2\varepsilon)t\ ,$$

for $t \in [0,1]$, $x_1 \in \mathbb{R}/\mathbb{Z}$. The first inequality of Y is satisfied by C_1 if $\varepsilon < \frac{1}{4}$. Same for the third. Proving that the second inequality of Y holds in C_1 causes more trouble. Let us find the roots of the second equation as a polynomial in t

$$\begin{aligned} 0 \quad &= x_2^2 + (x_4 - \tfrac{1}{2})^2 - \varepsilon^2 \\ &= \delta^2(\tfrac{1}{2} - 2\varepsilon)^2(t-1)^2 + (\varepsilon - \tfrac{1}{2} + (\tfrac{1}{2} - 2\varepsilon)t)^2 - \varepsilon^2 \\ &= (\tfrac{1}{2} - 2\varepsilon)\left\{(\tfrac{1}{2} - 2\varepsilon)(\delta^2 + 1)t^2 + (-2\delta^2(\tfrac{1}{2} - 2\varepsilon) + 2(\varepsilon - \tfrac{1}{2}))t + \delta^2(\tfrac{1}{2} - 2\varepsilon) + \tfrac{1}{2}\right\} \end{aligned}$$

If we call $\eta = \frac{1}{2} - 2\varepsilon$ and ignore the common factor, the above is equivalent to $at^2 + bt + c = 0$, with $a = \eta(\delta^2 + 1)$, $b = -(2\delta^2\eta + \eta + \frac{1}{2})$ and $c = \delta^2\eta + \frac{1}{2}$. The discriminant is

$$\Delta = (2\delta^2\eta + \eta + \tfrac{1}{2})^2 - 4\eta(\delta^2+1)[\delta^2\eta + \frac{1}{2}] = (\eta - \tfrac{1}{2})^2 = 4\varepsilon^2 > 0$$

This shows that there are two roots, namely

$$t_1 = \tfrac{b-\sqrt{\Delta}}{2a} = \frac{2\eta(\delta^2+1)}{2\eta(\delta^2+1)} = 1$$
$$t_2 = \tfrac{b+\sqrt{\Delta}}{2a} = \frac{2\eta(\delta^2+1) + 4\varepsilon}{2\eta(\delta^2+1)} = 1 + \frac{2\varepsilon}{\eta(\delta^2+1)}$$

For any ε and δ positive and sufficiently small, both roots t_1, t_2 are ≥ 1. This shows that the equation $x_2^2 + (x_4 - \frac{1}{2})^2 - \varepsilon^2 \geq 0$ holds in C_1, because $t \in [0,1]$ in the parametrization of C_1. Note that the inequality is strict for $t \in [0,1)$ and for $t = 1$ it is an equality.

The proof that $C_2 \subset Y$ is totally analogous. In this case, the first and second inequalities are trivially satisfies if $\varepsilon < \frac{1}{4}$. The third inequality for C_2 becomes the same as the second inequality for C_1 that was proved to be true above, so it also holds.

Recall that

$$\partial_{12}Y = \{x_3^2 + x_4^2 = \varepsilon^2\}$$
$$\partial_{13}Y = \{x_2^2 + (x_4 - \tfrac{1}{2})^2 = \varepsilon^2\}$$
$$\partial_{14}Y = \{(x_2 - \tfrac{1}{2})^2 + (x_3 - \tfrac{1}{2})^2 = \varepsilon^2\}$$

In the parametrization of T_1 given above we have $x_2 = \varepsilon\cos\theta$, $x_4 = \frac{1}{2} + \varepsilon\sin\theta$ so clearly $T_1 \subset \partial_{13}Y$. For T_2 we have $x_2 = \frac{1}{2} + \varepsilon\cos\theta$, $x_3 = \frac{1}{2} + \varepsilon\sin\theta$, so $T_2 \subset \partial_{14}Y$.

Now let us see (2). To see that $C_1 \cap C_2 = \emptyset$ note that in C_1 the second coordinate $x_2 = \delta(\frac{1}{2} - 2\varepsilon)(1-t)$ is close to 0 as $\delta \to 0$, while in C_2 the second coordinate is $x_2 = \frac{1}{2} - \delta(\frac{1}{2} - 2\varepsilon)(1-t)$ is close to $\frac{1}{2}$ as $\delta \to 0$. To prove $C_1 \cap T_2 = \emptyset$ look at the coordinate x_3, which stisfies $x_3 = 0$ in C_1 and $x_3 = \frac{1}{2} + \varepsilon\sin\theta$ in T_2. If $\varepsilon \to 0$ then $x_3 \to \frac{1}{2}$ in T_2 so we can take $\varepsilon > 0$ small to ensure that $C_1 \cap T_2 = \emptyset$. For $C_2 \cap T_1 = \emptyset$, we make an analogous argument looking at the coordinate x_4, which stisfies $x_4 = 0$ in C_2 and $x_4 = \frac{1}{2} + \varepsilon\sin\theta$ in T_1. The fact that $T_1 \cap T_2 = \emptyset$ follows by looking at the second coordinate. We have that $x_2 = \varepsilon\cos\theta$ in T_1 and $x_2 = \frac{1}{2} + \varepsilon\cos\theta$ in T_2.

We now prove (3). If $x \in C_1 \cap T_1$ then putting $x = (x_1, x_2, x_3, x_4)$ in coordinates we must have $x_2^2 + (x_4 - \frac{1}{2})^2 = \varepsilon^2$ since $x \in T_1$, and also $x_2 = -\delta(\frac{1}{2} - 2\varepsilon)(t-1)$, $x_4 = \varepsilon + (\frac{1}{2} - 2\varepsilon)t$ because $x \in C_1$. This is exactly the equation we solved before in the proof of (1). Hence the only solution for $t \in [0,1]$ is $t_1 = 1$. This yields $x_2 = 0$ and $x_4 = \frac{1}{2} - \varepsilon = \frac{1}{2} + \varepsilon\sin\theta$ from which it follows $\theta = \frac{3\pi}{2}$. Hence the only point x in $C_1 \cap T_1$ has coordinates $x_1 = \frac{1}{2} + \frac{\varepsilon}{\delta}$, $x_2 = 0$, $x_3 = 0$, $x_4 = \frac{1}{2} - \varepsilon$. To see that C_1 and T_1 intersect transversely, we compute the tangent spaces at the point of intersection x. Let us call $\eta = \frac{1}{2} - 2\varepsilon$ as before. We have

$$\begin{aligned} T_xC_1 &= \langle \partial_{x_1}, -\delta\eta\partial_{x_2} + \eta\partial_{x_4} \rangle \\ T_xT_1 &= \langle -\tfrac{\varepsilon}{\delta}(\cos\theta + \sin\theta)\partial_{x_1} - \varepsilon\sin\theta\partial_{x_2} + \varepsilon\cos\theta\partial_{x_4}, \partial_{x_3} \rangle|_{\theta=3\pi/2} \\ &= \langle \tfrac{\varepsilon}{\delta}\partial_{x_1} + \varepsilon\partial_{x_2}, \partial_{x_3} \rangle \end{aligned}$$

The four vectors generating the tangent spaces are linearly independent, because if we assemble their coordinates as the columns of a 4×4-matrix P we compute its determinant and get $\det P = \eta\varepsilon \neq 0$. This proves that the intersection is transverse.

The proof of (4) is analogous to the proof of (3) done above. Now the intersection $C_2 \cap T_2$ is computed by subtituting the parametric equations of C_2 given by $x_2 = \frac{1}{2} + \delta\eta(t-1)$, $x_3 = \varepsilon + \eta t$, into the implicit equation $(x_2 - \frac{1}{2})^2 + (x_3 - \frac{1}{2})^2 = \varepsilon^2$ of T_1. This equation was seen before to have $t = 1$ as its only solution in $[0,1]$, hence $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{2} - \varepsilon$. If follows that $\theta = 3\pi/2$, which implies that $x_1 = \frac{1}{2} + \frac{\varepsilon}{\delta}$. Thus the unique point $x \in C_2 \cap T_2$ is $x = (\frac{1}{2} + \frac{\varepsilon}{\delta}, \frac{1}{2}, \frac{1}{2} - \varepsilon, 0)$. The intersection at x is transverse since

$$\begin{aligned} T_x C_2 &= \langle \partial_{x_1}, \delta\eta\partial_{x_2} + \eta\partial_{x_3} \rangle \\ T_x T_1 &= \langle -\tfrac{\varepsilon}{\delta}(\cos\theta - \sin\theta)\partial_{x_1} - \varepsilon \sin\theta \partial_{x_2} + \varepsilon \cos\theta \partial_{x_3}, \partial_{x_4} \rangle|_{\theta=3\pi/2} = \langle -\tfrac{\varepsilon}{\delta}\partial_{x_1} + \varepsilon\partial_{x_2}, \partial_{x_4} \rangle \end{aligned}$$

with $\eta = \frac{1}{2} - 2\varepsilon$ as usual. If we assemble this four vectors as the columns of a matrix P, we have $\det P = -\varepsilon\eta \neq 0$, proving transversality.

Let us see (5). This is a very straighforward computation, so we will prove it for C_1 and T_1. We take differentials in the parametric equations of C_1 to obtain $dx_1 = dx_1$, $dx_2 = -\delta\eta dt$, $dx_3 = 0$, $dx_4 = \eta dt$, so the pull-back of the symplectic form ω to C_1 is

$$\omega|_{C_1} = dx_1 \wedge (-\delta\eta dt) + \delta dx_1 \wedge (\eta dt) = 0$$

and this proves that C_1 is Lagrangian.

Now take differentials in the parametric equations of T_2 to obtain $dx_1 = -\frac{\varepsilon}{\delta}(\cos\theta + \sin\theta)d\theta$, $dx_2 = -\varepsilon \sin\theta d\theta$, $dx_3 = dx_3$, $dx_4 = \varepsilon\cos\theta d\theta$ so the pull-back of ω becomes

$$\omega|_{T_1} = dx_3 \wedge (\varepsilon\cos\theta d\theta) + (-\varepsilon\sin\theta d\theta) \wedge dx_3 - \delta(-\tfrac{\varepsilon}{\delta}(\cos\theta + \sin\theta)d\theta) \wedge dx_3 = 0$$

proving that T_1 is Lagrangian.

We now prove (6) and (7). The boundaries of the Lagrangian cilinders C_1 and C_2 are obtained by plugging $t = 0$ and $t = 1$ in the parametric equations of the cylinders. Hence we have

$$\begin{aligned} \partial C_1 &= \{(x_1, \delta\eta, 0, \varepsilon) : x_1 \in \mathbb{R}/\mathbb{Z}\} \bigsqcup \{(x_1, 0, 0, \tfrac{1}{2} - \varepsilon) : x_1 \in \mathbb{R}/\mathbb{Z}\} \subset \partial_{12}Y \bigcup \partial_{13}Y \\ \partial C_2 &= \{(x_1, \tfrac{1}{2} - \delta\eta, \varepsilon, 0) : x_1 \in \mathbb{R}/\mathbb{Z}\} \bigsqcup \{(x_1, \tfrac{1}{2}, \tfrac{1}{2} - \varepsilon, 0) : x_1 \in \mathbb{R}/\mathbb{Z}\} \subset \partial_{12}Y \bigcup \partial_{14}Y \end{aligned}$$

and this proves (6) and (7), so we are done. □

3. Second step: the symplectic manifold Z.

Clearly, the normal bundles of $T_{1j} \subset \mathbb{T}^4$ are trivial. By Proposition 1.45, the normal bundle of the generic fiber F of the elliptic surface $E(1)$ is also trivial. We can make a Gompf sum identifying F and T_{1j} as follows. Take three copies of the elliptic surface $E(1)$, call them $E(1)_2$, $E(1)_3$ and $E(1)_4$, with generic fibers F_2, F_3, F_4, respectively, and form the Gompf symplectic sum

$$Z = \mathbb{T}^4 \#_{T_{12}=F_2} E(1)_2 \#_{T_{13}=F_3} E(1)_3 \#_{T_{14}=F_4} E(1)_4\,. \tag{39}$$

To compute the fundamental group of Z we have the following result.

LEMMA 5.5. *Let X be a 4-manifold with an embedded symplectic surface $T \subset X$ of self-intersection zero and genus 1. Denote $\iota : T \to X$ the inclusion map. Then the Gompf connected sum $X' = X \#_{T=F} E(1)$ has fundamental group $\pi_1(X') = \pi_1(X)/H$, where H is the normal subgroup generated by the image of $\iota_* : \pi_1(T) \to \pi_1(X)$.*

PROOF. By definition $X' = (X \setminus \nu(T)) \cup_B (E(1) \setminus \nu(F))$, where

$$B = \partial(X \setminus \nu(T)) \cong \partial(E(1) \setminus \nu(F)) \cong \mathbb{T}^3 = (\mathbb{S}^1)^3.$$

Applying Seifert-Van Kampen theorem, $\pi_1(X')$ is isomorphic to the amalgamated product $\pi_1(X \setminus \nu(T)) *_{\pi_1(B)} \pi_1(E(1) \setminus \nu(F))$. On the other hand recall that by Proposition 1.45 we have $\pi_1(E(1) \setminus \nu(F)) = \{1\}$. It follows that $\pi_1(X')$ is isomorphic to the quotient of $\pi_1(X \setminus \nu(T))$ by the image of $\pi_1(B)$.

On the other hand, using Seifert-Van Kampen theorem for the decomposition $X = (X \setminus \nu(T)) \cup_B \nu(T)$, it follows that $\pi_1(X) \cong \pi_1(X \setminus \nu(T)) *_{\pi_1(B)} \pi_1(\nu(T))$. Hence the quotient of $\pi_1(X)$ by the image of $\pi_1(T)$ is isomorphic to the quotient of $\pi_1(X \setminus \nu(T))$ by the image of $\pi_1(B)$. The result follows. □

Using Lemma 5.5 three times, we have that $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(\mathbb{T}^4)$ by the images of $\pi_1(T_{12}), \pi_1(T_{13}), \pi_1(T_{14})$, hence Z is simply-connected. In particular $b_1(Z) = 0$, and by Poincaré Duality $b_3(Z) = 0$. Using the formula for the Euler characteristic of the Gompf symplectic sum and the fact that $\chi(\mathbb{T}^n) = 0$, $\chi(E(1)) = 12$, one obtains

$$\begin{aligned}\chi(Z) &= \chi(\mathbb{T}^4) + 3\chi(E(1)) - 3\chi(F) = 36 \\ &= b_0(Z) - b_1(Z) + b_2(Z) - b_3(Z) + b_4(Z) = 2 + b_2(Z)\end{aligned}$$

hence we conclude that $b_2(Z) = 34$.

Now we are going to construct 34 symplectic surfaces in Z. This will be done in several steps. First, let us focus on the first Gompf symplectic sum $\mathbb{T}^4 \#_{T_{12}=F_2} E(1)_2$. To ease notation let us call $E(1) = E(1)_2$, $F = F_2$, $T = T_{12}$. By Proposition 1.45 there are 9 sections $E_1, \ldots, E_9$ of $E(1)$ which are disjoint symplectic (-1)-spheres intersecting the fiber F transversely and positively at one point. The intersection is positive since $E_i, F \subset E(1)$ are complex curves. By Lemma 1.38, we can glue each of the sections E_i to a different parallel copy of T_{34}, to get nine symplectic surfaces

$$S_1 = E_1 \# T_{34} \, , \ldots , \; S_9 = E_9 \# T_{34} \subset Z \, .$$

Note that $S_1, \ldots, S_9$ are disjoint symplectic tori of self-intersection -1.

Now take a generic line $L \subset E(1)$, which intersects F tranversely and positively in three points by Proposition 1.45, and does not intersect any of the exceptional spheres E_i. The surface L is a symplectic sphere of self-intersection 1, and when doing the Gompf sum, it can be glued by Lemma 1.38 to three parallel copies of T_{34}, resulting in a symplectic surface $S_{10} = L \# 3T_{34}$ of genus 3 and self-intersection 1, which is moreover disjoint from the previous ones $S_1, \ldots, S_9$.

When doing the second and third Gompf symplectic sums in (39), we construct analogous collections $S_{11}, \ldots, S_{19}, S_{20}$ and $S_{21}, \ldots, S_{29}, S_{30}$ of symplectic surfaces in Z. Sumarasing, we have 30 symplectic surfaces $S_1, \ldots, S_{30}$ of Z with genus and self-intersections as follows.

- $g(S_1) = \ldots = g(S_9) = 1$, $g(S_{11}) = \ldots = g(S_{19}) = 1$, $g(S_{21}) = \ldots = g(S_{29}) = 1$, $g(S_{10}) = g(S_{20}) = g(S_{30}) = 3$.
- $S_k \cdot S_k = -1$, $1 \leq k \leq 9$, $S_{10} \cdot S_{10} = 1$, $S_{10+k} \cdot S_{10+k} = -1$, $1 \leq k \leq 9$, $S_{20} \cdot S_{20} = 1$, $S_{20+k} \cdot S_{20+k} = -1$, $1 \leq k \leq 9$, $S_{30} \cdot S_{30} = 1$.

All of them are disjoint if we make the construction with some care. Indeed, note that

- For constructing S_k , $k = 1, \ldots, 10$, we glue with parallel copies of T_{34}.
- For constructing S_{10+k} , $k = 1, \ldots, 10$, we glue with parallel copies of T_{24}.
- For constructing S_{20+k} , $k = 1, \ldots, 10$, we glue with parallel copies of T_{23}.

We can arrange as many copies as we wish of T_{34}, T_{24}, T_{23} which do not intersect, so the surfaces S_i with $1 \leq i \leq 30$ are disjoint.

The four remaining surfaces are constructed as follows. Consider the Lagrangian cylinders C_1 , C_2 and tori T_1 , T_2 from Proposition 5.4. Recall that they are contained in

$$Y = \mathbb{T}^4 \setminus (\nu(T_{12}) \cup \nu(T_{13}) \cup \nu(T_{14})) \, ,$$

so they are disjoint with the tori T_{12}, T_{13}, T_{14}. Moreover, we can take collections of parallel copies of T_{34}, T_{24}, T_{23} which do not intersect any of C_1, C_2, T_1, T_2. Therefore we can assume that C_i and T_i, $i = 1, 2$, are disjoint from $S_1, \ldots, S_{30}$ in Z.

We use the cylinders C_1, C_2 to construct Lagrangian spheres in Z as follows. We detail the construction for C_1, being the construction for C_2 completely analogous. The boundary $\partial C_1 \subset \partial_{12} Y \sqcup \partial_{13} Y$ consists of two circles γ_{12}, γ_{13}. We can arrange the identifications

$$F_2 \times S^1 \cong \partial(E(1)_2 \setminus \nu(F_2)) \xrightarrow{\cong} \partial_{12} Y \cong T_{12} \times S^1$$

$$F_3 \times S^1 \cong \partial(E(1)_3 \setminus \nu(F_3)) \xrightarrow{\cong} \partial_{13} Y \cong T_{13} \times S^1$$

to match both circles γ_{12} and γ_{13} with vanishing cycles c_{12} and c_{13} of the elliptic fibrations $E(1)_2$ and $E(1)_3$. Let us see that this is possible. First note that, as seen before, we have

$$\partial C_1 = \{(x_1, \delta\eta, 0, \varepsilon) : x_1 \in \mathbb{R}/\mathbb{Z}\} \bigsqcup \{(x_1, 0, 0, \tfrac{1}{2} - \varepsilon) : x_1 \in \mathbb{R}/\mathbb{Z}\} \subset \partial_{12}Y \bigcup \partial_{13}Y$$

so in particular we have the following.

- The circle $\gamma_{12} \subset T_{12} \times \{(0, \varepsilon)\}$ lies inside a torus. Moreover $[\gamma_{12}]$ is a generator of $\pi_1(T_{12})$. Also, the vanishing cycle $c_{12} \subset F_2$ represents a generator of $\pi_1(F_2)$, so we can build a model of the vanishing disk of F_2 starting from the image of γ_{12} under the identification of $T_{12} \xrightarrow{\cong} F_2$. See the construction of the vanishing disk done in Proposition 1.42.
- Analogously, the circle γ_{13} lies inside $T_{13} \times \{(0, 1/2 - \varepsilon)\}$ and $[\gamma_{12}]$ is a generator of $\pi_1(T_{13})$. The vanishing cycle $c_{13} \subset F_3$ also represents a generator of $\pi_1(F_3)$, so we can build a model of the vanishing disk of F_3 starting from the image of γ_{13} under the identification of $T_{13} \xrightarrow{\cong} F_3$.

Let V_2, V_3 be vanishing disks in $E(1)_2 \setminus \nu(F_2)$ and $E(1)_3 \setminus \nu(F_3)$ such that $\partial V_2 = c_{12}$, $\partial V_3 = c_{13}$. Note that V_2 and V_3 are Lagrangian (-1)-disks by Proposition 1.42.

They can be glued to C_1 to obtain a Lagrangian sphere

$$L_1 = V_2 \bigcup_{c_{12} \cong \gamma_{12}} C_1 \bigcup_{\gamma_{13} \cong c_{13}} V_3$$

of self-intersection -2, since C_1 has self-intersection 0 and both discs have self-intersection -1.

To make the gluing smooth in the definition of L_1, we may need to change the gluing in the Gompf connected sum as follows. First note that the gluing region in $E(1)$ is a neighbourhood of $F \times S^1$ of the form $F \times S^1 \times (-\varepsilon, \varepsilon)$, with θ the coordinate of S^1 and $t \in (-\varepsilon, \varepsilon)$. The symplectic form can be written as $\omega_F + d\theta \wedge dt$ in this neighborhood $F \times S^1 \times (-\varepsilon, \varepsilon)$. This is a consequence of the symplectic tubular neighborhood, Theorem 1.12.

Let V be any of the Lagrangian discs V_1, V_2. The tangent spaces to V are spanned by the derivative γ' of the curve γ, and by some vector of the form $a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial t}$, for some $a, b \in \mathbb{R}$, $b \neq 0$. A diffeomorphism of the form $(\theta, s) \mapsto (\theta + g(s), s)$ can serve to arrange $a = 0$, so that the Lagrangian enters the gluing region in the radial direction and thus can be glued without corner.

Hence we have constructed an smooth lagrangian (-2)-sphere $L_1 \subset Z$. This intersects the Lagrangian torus T_1 transversely at one point by Proposition 5.4.

The construction for the cylinder C_2 works analogously. In this case we obtain another Lagrangian (-2)-sphere L_2 as

$$L_2 = \widetilde{V}_2 \bigcup_{\widetilde{c}_{12} \cong \widetilde{\gamma}_{12}} C_2 \bigcup_{\widetilde{\gamma}_{14} \cong \widetilde{c}_{14}} \widetilde{V}_4$$

where

$$\partial C_2 = \widetilde{\gamma}_{12} \sqcup \widetilde{\gamma}_{14} \subset \partial_{12}Y \sqcup \partial_{14}Y\,,$$

the circles $\widetilde{\gamma}_{12}$ and $\widetilde{\gamma}_{13}$ are identified with vanishing cycles $\widetilde{c}_{12}$ and $\widetilde{c}_{14}$ of $E(1)_2$ and $E(1)_4$, and $\widetilde{V}_2$, $\widetilde{V}_4$ are vanishing disks bounding $\widetilde{c}_{12}$ and $\widetilde{c}_{14}$ respectively.

In this way we obtain another pair L_2, T_2 of a Lagrangian (-2)-sphere L_2 and a Lagrangian torus T_2 of self-intersection 0, both intersecting transversely at one point.

Moreover, we claim that we can arrange L_1, L_2 to be disjoint. To see it, note that by Proposition 1.45 we can choose different vanishing cycles c_{12} and $\widetilde{c}_{12}$ in $E(1)_2$ to match the two boundary components $\gamma_{12} \subset \partial C_1$ and $\widetilde{\gamma}_{12} \subset \partial C_2$. Hence we can arrange the vanishing disks V_2 and $\widetilde{V}_2$ to be disjoint, which yields that L_1 and L_2 are disjoint in the manifold Z.

Looking at the intersection form, we see that the 34 surfaces $S_1, \ldots, S_{30}$ and L_1, L_2, T_1, T_2 are independent in homology, and since $b_2(Z) = 34$ they span $H_2(Z, \mathbb{Q})$. Finally, we apply Lemma 1.47 to change slightly the symplectic form of Z so that all the symplectic manifolds S_j remain symplectic and the Lagrangian surfaces become symplectic. Moreover, since L_i only intersects T_i for $i = 1, 2$, Lemma 1.47 shows that we arrange that both pairs of symplectic surfaces (L_1, T_1) and (L_2, T_2) intersect positively, so we assume this. This shows that $L_1 \cdot T_1 = 1, L_2 \cdot T_2 = 1$.

4. Making all symplectic surfaces disjoint.

To make the surfaces in Z disjoint we have to do the following process with both pairs L_1, T_1 and L_2, T_2.

Suppose we have any symplectic manifold Z and $L, T \subset Z$ a pair of a symplectic sphere L and a symplectic torus T with

$$[L] \cdot [L] = -2\,, \quad [L] \cdot [T] = 1\,, \quad [T] \cdot [T] = 0\,.$$

We are going to show how after making a couple of resolutions of positive intersections and one blow-up, we get a manifold $\widetilde{Z}$ so that the part of the homology concerning T and L is generated by symplectic and *disjoint* surfaces.

Take a parallel copy of T, call it T', displacing via the normal bundle, so $[T] = [T'] \in H_2(Z)$. We can resolve the intersection point $T' \cap L$ by applying the process of symplectic resolution of positive intersections of Chapter 1 to get a torus T'' satisfying

$$\begin{aligned} [T''] &= [T] + [L] \\ [T''] \cdot [T''] &= ([T] + [L]) \cdot ([T] + [L]) = [T]^2 + [L]^2 + 2[T][L] = -2 + 2 = 0 \\ [T''] \cdot [T] &= ([T] + [L]) \cdot [T] = 1 \end{aligned}$$

Therefore T and T'' intersect at exactly one point, say $\{p\} = T \cap T''$.

After perturbing slightly T'' if necessary, we get a Darboux chart U near p so that the model in this chart is given by $T \cup T'' = \{z \cdot w = 0\}$, where $T = \{z = 0\}$, and $T'' = \{w = 0\}$. Resolve again symplectically the intersection $T \cap T''$, producing a symplectic genus 2 surface Σ with $[\Sigma] = [T] + [T'']$. We can move Σ slightly in the Darboux chart so that it intersects T and T'' in the same point p. This is possible by the following. The process of resolution of positive intersections gives that in the darboux chart U near the point p, the surface Σ can be written as $\Sigma \cap U = \{zw = \varepsilon^2\}$, for ε small enough. We can make a small translation $\tau : (z, w) \mapsto (z+\varepsilon, w+\varepsilon)$, so that the translated Σ has equation $\tau(\Sigma \cap U) = \{(z-\varepsilon)(w-\varepsilon) = \varepsilon^2\}$. If we change $\Sigma \cap U$ by $\tau(\Sigma \cap U)$ and then glue $\tau(\Sigma \cap U)$ to the rest of Σ, we obtain our desired surface (which we call Σ again) of genus 2, homologous to $[T] + [T'']$ and passing throught p.

Now, the equalities

$$[\Sigma] \cdot [T] = ([T] + [T'']) \cdot [T] = 1, [\Sigma] \cdot [T''] = ([T] + [T'']) \cdot [T''] = 1$$

show that p is the only intersection point of the three surfaces T, T'', Σ, and that they intersect transversely. Moreover, $\Sigma^2 = (T + T'')^2 = 2$. Blowing up symplectically at p, we get a symplectic manifold $\tilde{Z} \cong Z \# \overline{\mathbb{CP}}^2$. Since there is a chart in which T, T'' and Σ are given by the zero set of complex polinomials, by Proposition 1.36 the proper transforms

$$\widetilde{T}, \widetilde{T}'', \widetilde{\Sigma} \subset \tilde{Z}$$

are disjoint symplectic surfaces of $\tilde{Z}$ of genus $1, 1, 2$ respectively, and its self-intersection numbers are

$$[\widetilde{T}]^2 = [T]^2 - 1 = -1$$
$$[\widetilde{T}'']^2 = [T'']^2 - 1 = -1$$
$$[\widetilde{\Sigma}]^2 = [\Sigma]^2 - 1 = 1\,.$$

We claim that the proper transforms above generate the same three-dimensional space in $H_2(\tilde{Z})$ as $[T], [L]$ and e, where $e = [E] \in H_2(\tilde{Z})$ is the homology class of the exceptional sphere $E \subset \tilde{Z}$. We have to see that

$$\mathrm{Span}\langle[\widetilde{T}], [\widetilde{T}''], [\widetilde{\Sigma}]\rangle = \mathrm{Span}\langle[T], [L], e\rangle \subset H_2(\tilde{Z}).$$

First note that by definition

$$[\widetilde{T}] = [T] - e$$
$$[\widetilde{T}''] = [T''] - e = [T] + [L] - e$$
$$[\widetilde{\Sigma}] = [\Sigma] - e = [T] + [T''] - e = 2[T] + [L] - e$$

which yields

$$[T] = [\widetilde{\Sigma}] - [\widetilde{T}'']$$
$$e = -[\widetilde{T}] - [\widetilde{T}''] + [\widetilde{\Sigma}]$$
$$[L] = -[\widetilde{T}] + [\widetilde{T}'']\,.$$

Moreover recall that both $\widetilde{T}$ and $\widetilde{T}''$ have genus 1 and self-intersection -1, and $\widetilde{\Sigma}$ has genus 2 and self-intersection 1.

Now come back to our specific symplectic manifold Z constructed in the previous subsection. Use this method for both pairs L_1, T_1 and L_2, T_2, and after two symplectic blow-ups of Z we end up with a symplectic manifold $X \cong Z \# 2\,\overline{\mathbb{CP}^2}$ such that

$$H_2(X) = \mathrm{Span}\langle S_i, T_1, L_1, e_1, T_2, L_2, e_2\rangle$$
$$= \mathrm{Span}\langle S_i, \widetilde{T}_1, \widetilde{T}_1'', \widetilde{\Sigma}_1, \widetilde{T}_2, \widetilde{T}_2'', \widetilde{\Sigma}_2\rangle\,,$$

with $1 \le i \le 30$. We see that $b_2(X) = 36$, the 36 disjoint symplectic surfaces

$$S_1\,,\,\ldots,\,S_{30}\,,\widetilde{T}_1\,,\,\widetilde{T}_1''\,,\,\widetilde{\Sigma}_1\,,\,\widetilde{T}_2\,,\,\widetilde{T}_2''\,,\widetilde{\Sigma}_2$$

are a basis of $H_2(X)$, and moreover the intersection form of X is diagonal with respect to this basis. The genus and self-intersections of the surfaces are those stated in Theorem 5.1. This finishes the proof of Theorem 5.1.

COROLLARY 5.6. *Take a prime p, and let $g_i = g(S_i)$, $1 \le i \le 36$, as given in Theorem 5.1. Then there is a 5-dimensional K-contact manifold M with $H_1(M, \mathbb{Z}) = 0$ and*

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}.$$

PROOF. Consider the symplectic manifold (X, ω) provided by Theorem 5.1, and let

$$S_i\,,\ 1 \le i \le 36$$

be the collection of disjoint symplectic surfaces. Put coefficients $m_i = p^i$ for S_i. Using Proposition 4.10, we give X the structure of a symplectic orbifold with isotropy surfaces S_i of multiplicities m_i. By Proposition 2.34, X admits an almost Kahler orbifold structure. Now, Lemma 4.51 implies that, after a small perturbation of ω, there exists a Seifert bundle $M \to X$ such that

$$c_1(M/\mu) = l[\omega] \in H^2(X, \mathbb{Z})$$

for some integer l. Finally, by Theorem 4.50 M admits a K-contact structure.

We compute the homology of M using Theorem 4.38. As X is simply connected, $H_1(X,\mathbb{Z}) = 0$. By Lemma 4.51, we can arrange that $c_1(M/\mu) \in H^2(X,\mathbb{Z})$ is primitive. Now $k+1 = b_2(X) = 36$, so $k = b_2(M) = 35$ and

$$H_2(M,\mathbb{Z}) \cong \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}\,.$$

The map $H^2(X,\mathbb{Z}) \to H^2(S_i,\mathbb{Z})$ sends $[S_j]$ to $[S_j]\cdot[S_i] = 0$ for $j \neq i$ since S_i and S_j are disjoint, and it sends $[S_i]$ to $[S_i]^2$. Taking the quotient modulo p^i we have that

$$H^2(X,\mathbb{Z}) \to H^2(S_i,\mathbb{Z}/p^{e_i})$$

sends $[S_i]$ to $[S_i]^2 \pmod{p^i}$. Given the self-intersection numbers in Theorem 5.1, this is $+1$ or -1, since $p > 2$.

The conclusion is that

$$H^2(X,\mathbb{Z}) \to \bigoplus_{i=1}^{36} H^2(S_i,\mathbb{Z}/p^i)$$

is surjective. Hence $H_1(M,\mathbb{Z}) = 0$ by Theorem 4.38. The result follows. □

REMARK 5.7. *The manifold M of Corollary 5.6 does not admit a regular K-contact structure. This follows from Remark 4.39 since $H_2(M,\mathbb{Z})$ has torsion.*

5. Kahler surfaces with many disjoint complex curves.

Now we want to find obstructions for the existence of Sasakian structures on 5-dimensional manifolds. In particular, we are going to prove that the 5-manifold M constructed in Corollary 5.6, which admits a K-contact structure, cannot admit a semi-regular Sasakian structure.

THEOREM 5.8. *Let M be a 5-dimensional manifold with $H_1(M,\mathbb{Z}) = 0$ and*

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}$$

where $g_i = g(S_i)$ are the numbers given in Theorem 5.1, and $p > 2$ is a prime number. Then M does not admit a semi-regular Sasakian structure.

PROOF. Let M be a 5-dimensional manifold with $H_1(M,\mathbb{Z}) = 0$ which admits a Sasakian structure. Then it also admits a quasi-regular Sasakian structure by Theorem 4.48. This means that M is a Seifert bundle $\pi : M \to X$ over a Kahler orbifold X, by Theorem 4.49. By Proposition 4.41, $H_1(X,\mathbb{Z}) = 0$, $H_2(X,\mathbb{Z}) = \mathbb{Z}^{36}$ and the ramification locus contains a collection of 36 disjoint surfaces D_i with $g(D_i) = g_i$.

If the Sasakian structure is semi-regular, then X is a smooth Kahler orbifold. By Proposition 4.12, the smooth Kahler orbifold X inherits naturally a structure of a Kahler manifold, and the ramification surfaces D_i are smooth curves of the Kahler manifold X, spanning the second homology (again, see Proposition 4.41). We see in Theorem 5.17 below that this is not possible. □

A smooth Kahler manifold with disjoint complex curves spanning its second homology is a rare phenomenom. We have the first result in this direction in Theorem 5.17.

Before going to the proof, let us first recall some preliminary results from Kahler geometry. Let (Z,h) be a Kahler manifold with $h = g - i\omega$ its Kahler metric. Denote

$$L : H^k(Z,\mathbb{R}) \to H^{k+2}(Z,\mathbb{R})$$
$$[\alpha] \mapsto [\alpha] \wedge [\omega]$$

the Lefschetz operator given by wedge product with the Kahler class $[\omega] \in H^2(Z,\mathbb{R})$. The *Hard Lefschetz Theorem* states that for $0 \leq k \leq n$ the $(n-k)$-th power of the Lefschetz map given by

$$L^{n-k} : H^k(Z,\mathbb{R}) \to H^{2n-k}(Z,\mathbb{R})$$
$$[\alpha] \mapsto [\omega]^{n-k} \wedge [\alpha]$$

is an isomorphism.

Denote $H^k(Z,\mathbb{R})_{pr} = \ker L^{n-k+1}$, the *primitive* cohomology. It can be proved that $H^k(Z,\mathbb{R})_{pr} \subset H^k(Z,\mathbb{R})$ inherits a Hodge structure, so the primitive Dolbeaut groups

$$H^{p,q}(Z)_{pr} = H^{p,q}(Z) \cap H^k(Z,\mathbb{C})_{pr}$$

are defined and we have a decomposition

$$H^k(Z,\mathbb{C})_{pr} = \oplus_{p+q=k} H^{p,q}(Z)_{pr}\,.$$

We define also the map

$$Q_\omega : H^k(Z,\mathbb{R}) \times H^k(Z,\mathbb{R}) \to \mathbb{R}$$
$$Q_\omega([\alpha],[\beta]) = \int_Z \alpha \wedge \beta \wedge \omega^{n-k}\,.$$

This is a non-degenerate bilinear form (by Poincaré Duality and the Hard Lefschetz Theorem), and it is symmetric or antisymmetric depending on the parity of k.

We can extend Ψ_ω to the complexification $H^k(Z,\mathbb{C}) = H^k(Z,\mathbb{R}) \otimes \mathbb{C}$, and define an Hermitian pairing

$$H_\omega : H^k(Z,\mathbb{C}) \times H^k(Z,\mathbb{C}) \to \mathbb{C}$$

by the formula

$$H_\omega(\alpha,\beta) = i^k Q_\omega(\alpha,\bar\beta) = i^k \int_Z \alpha \wedge \bar\beta \wedge \omega^{n-k}\,.$$

The *Riemann-Hodge relations* state the following.

(1) The Hodge decomposition $H^k(Z,\mathbb{C}) = \sum_{p+q=k} H^{p,q}(Z)$ is orthogonal with respect to the maps H_ω and Q_ω defined above.
(2) The Hermitian map H_ω is definite on the subspaces

$$L^r H^{p,q}(Z)_{pr} \subset H^{p+r,q+r}(Z) \subset H^k(Z)$$

with sign $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k}$, $k = p+q+2r$.

In particular, if $Z = S$ is a complex Kahler surface, $n = 2$, so the intersection form

$$I : H^2(S,\mathbb{R}) \times H^2(S,\mathbb{R}) \to \mathbb{R}$$

coincides with Q_ω. The Hermitian pairing H_ω is definite in $H^{1,1}(S)_{pr} = L^0 H^{1,1}(S)_{pr}$, $k = 2$, with sign -1.

Recall that if $[\alpha],[\beta]$ are classes in $H^{1,1}(S) \cap H^2(S,\mathbb{R})$ then $[\bar\beta] = -[\beta]$, so $H_\omega([\alpha],[\beta]) = \int_S \alpha \wedge \beta$. In particular if $[\alpha] = \mathrm{PD}[D_1]$ and $[\beta] = \mathrm{PD}[D_2]$ are Poincaré duals of divisors in S, then

$$H_\omega([D_1],[D_2]) = [D_1] \cdot [D_2]$$

is the intersection product.

On the other hand, the subspace $H^{1,1}(S)_{pr}$ is by definition

$$H^{1,1}(S)_{pr} = \ker[L : H^{1,1}(S) \to H^{2,2}(S) \cong \mathbb{C}].$$

In other words $H^{1,1}(S)_{pr} = \{\alpha \in H^{1,1}(S) : \alpha \wedge \omega = 0\} = \langle \omega \rangle^\perp$ is the hiperplane annihilator of ω under wedge product.

Finally, note that $H_\omega([\omega],[\omega]) = \int_S \omega \wedge \omega = \mathrm{Vol}(S) > 0$.

The conclusion is that the intersection form on $H^{1,1}(S) \cap H^2(S, \mathbb{R})$ is non-degenerate and has signature $(1, b^{1,1} - 1)$.

Corollary 5.9. *(Hodge Index Theorem) Let S be a Kahler smooth complex surface. The signature of the intersection form $I : H^2(S) \times H^2(S) \to \mathbb{Z}$ restricted to $H^{1,1}(S)$ is $(1, b^{1,1} - 1)$.*

For more details see [**29**], Ch. V.1. for the projective case, and Ch. 7.2.1 in [**54**] for the Kahler case.

Now let us recall some facts about line bundles.

Definition 5.10. *Let $L \to X$ be an holomorphic (algebraic) line bundle over a complex (algebraic) manifold X. We say that L is* very ample *if there are sections $s_0, \ldots, s_m \in H^0(L)$ such that*

(1) *The sections generate $H^0(L)$ as a vector space, i.e. $\mathbb{C}\langle s_0, \ldots, s_m\rangle = H^0(L)$.*
(2) *The map*
$$\iota : X \to \mathbb{P}^m \ , \ x \mapsto [s_0(x) : \ldots : s_m(x)] \, .$$
is an embedding.

Note that from the definition above it follows that $\iota^*(\mathcal{O}(1)) = h^0(L)$, being $\mathcal{O}(1)$ the sheaf given by the homogeneous degree-1 polynomials on $\mathbb{P}^m$ and $h^0(L)$ the sheaf of sections of the line bundle L.

Now we give a classical result about sections of line bundles over complex (algebraic) curves.

Proposition 5.11. *Let $L \to C$ be an holomorphic line bundle over a complex smooth algebraic curve C of genus g. Denote $d = \deg(L)$.*

(1) *If $d \geq 2g - 1$. Then $\dim(H^0(L)) = d - g + 1 \geq g$.*
(2) *If $d \geq 2g + 1$ then L is very ample.*

Proof. See [**2**], Proposition 11.9. Alternatively, see [**29**]. □

The following is the main result about transversality of sections that we need.

Theorem 5.12. *(Bertini's Theorem) For a smooth quasi-projective variety $X \subset \mathbb{P}^m$, a generic hiperplane $H \subset \mathbb{P}^m$ intersects X transversally.*

Proof. See Chapter III.10. of [**29**]. □

Bertini's Theorem can be rephrased in several ways. A common one is by saying that a generic hyperplane section of X is transversal to the zero section, meaning that a generic section of the bundle $\mathcal{O}(1)|_X$ is transverse to the zero section X.

As a corollary, we get the existence of transverse sections of very ample line bundles as follows.

Corollary 5.13. *Let X be a complex (algebraic) manifold. If $L \to X$ is a very ample line bundle, then a generic section $s \in H^0(L)$ is transverse to the zero section.*

In particular, $H^0(L)$ is generated as a vector space by sections $s_0, \ldots, s_m$ which are transverse to the zero section.

Proof. Take $\iota : X \to \mathbb{P}^m$ any embedding induced by a basis of $H^0(L)$. We have that $\iota^*(\mathcal{O}(1)) = h^0(L)$ as sheaves. Then the pull-back by ι of an hyperplane $H \in H^0(\mathcal{O}(1))$ is a section $s = \iota^*(H) \in H^0(L)$ corresponding to the hyperplane section $H \cap X$ under the isomorphism $L \cong \mathcal{O}(1)|_X$. Hence a generic section s of L will correspond to a generic hyperplane section $H \cap X$, so it is transversal to the zero section by Bertini's Theorem. □

Now we give some results about numerical invariants of algebraic fibrations of surfaces over curves.

DEFINITION 5.14. *A genus-g fibration $f : S \to C$ of a complex surface S onto a complex curve C is an holomorphic map with connected fibers such that the general fiber has genus g. This means that:*

(1) *For all $t \in C$ the fiber $f^{-1}(t) \subset S$ is a connected curve.*
(2) *There is a Zariski-open subset $C^0 \subset C$ such that for all $t \in C^0$ the fiber $f^{-1}(t)$ is a smooth genus-g curve.*

The following definitions appear in [**55**] and [**17**].

DEFINITION 5.15. *Let $f : S \to C$ be a genus-g fibration of a complex smooth surface S onto a complex smooth curve C. Denote $b = g(C)$ the genus of C.*

(1) *We define the following relative numerical invariants:*

$$K^2_{S/C} = K^2_S - 8(g-1)(b-1)\,.$$
$$\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1)\,.$$

(2) *We say that f is* locally non-trivial *if $\chi_f \neq 0$.*
(3) *For a locally non-trivial fibration $f : S \to C$ we define its* slope *as*

$$\lambda_f = \frac{K^2_{S/C}}{\chi_f}\,.$$

The following appears in [**55**], Theorem 2. It is the key result we will need to bound the slope of a relatively minimal fibration.

THEOREM 5.16. *Let $f : S \to C$ be a relatively minimal genus-g fibration of a smooth projective surface S onto a smooth curve C. Suppose that f is not locally trivial and that $g \geq 2$. Then*

$$4 - \frac{4}{g} \leq \lambda_f \leq 12\,.$$

We are ready now to prove the main Theorem of this section.

THEOREM 5.17. *Let S be a smooth closed Kahler surface with $H_1(S, \mathbb{Q}) = 0$ and containing $D_1, \dots, D_b$, $b = b_2(S)$, smooth disjoint complex curves with $g(D_i) = g_i > 0$, and spanning $H_2(S, \mathbb{Q})$. Assume the following.*

- *At least two g_i are bigger than 1.*
- *All $g_i \leq 3$. In other words $g := \max\{g_i : 1 \leq i \leq b\} \leq 3$.*

Then $b \leq 2g + 3$.

PROOF. First, it is clear that the Poincaré duals of $[D_1], \dots, [D_b]$ are a basis of $H^2(S, \mathbb{Q})$, since they are generators (by hypothesis), and b is the dimension of $H^2(S, \mathbb{Q})$. Being the D_i complex submanifolds, $[D_i]$ are classes of type $(1,1)$. It follows that $h^{1,1} = b = b_2(S)$, so $H^2(S, \mathbb{C}) = H^{1,1}(S)$. Also, the geometric genus is $p_g = h^{2,0} = 0$, and the irregularity is $q = h^{1,0} = 0$ since $b_1 = 0$. From these data we see that the complex surface S is an algebraic projective surface (by the Enriques-Kodaira classification).

By Noether's formula (see [**3**]) we see that

$$\frac{1}{12}(K^2_S + c_2(S)) = \chi(\mathcal{O}_S) = 1 - q + p_g = 1.$$

Also, $c_2(S) = \chi(S) = 2 + b$, since $b = b_2$ and $b_1 = b_3 = 0$. Therefore $K^2_S = 10 - b$, where K_S is the canonical divisor of S.

By the Riemann-Hodge relations, the signature of $H^{1,1}(S)$ is $(1, b-1)$. Therefore, we can assume $D_1^2 = m_1$, $D_i^2 = -m_i$, $i = 2, \ldots, b$, where all m_i are positive integer numbers. By the adjunction formula, we have

$$K_S \cdot D_i + D_i^2 = 2g_i - 2,$$

so $K_S \cdot D_i = 2g_i - 2 - D_i^2$, and hence

$$K_S = \sum_{i=1}^{b} \frac{2g_i - 2 - D_i^2}{D_i^2} D_i$$

and

$$K_S^2 = \sum_{i=1}^{b} \frac{(2g_i - 2 - D_i^2)^2}{D_i^2}.$$

For $i \geq 2$ we have

$$\frac{(2g_i - 2 - D_i^2)^2}{D_i^2} = -\frac{(2g_i - 2 + m_i)}{m_i}(2g_i - 2 + m_i) \leq -(2g_i - 2 + m_i) \leq -1,$$

since $2g_i - 2 \geq 0$. Then

$$10 - b = K_S^2 \leq \frac{(2g_1 - 2 - D_1^2)^2}{D_1^2} - (b - 1).$$

With the hypothesis that at least one g_i, $i \geq 2$, satisfies that $g_i > 1$, we have an strict inequality. So

$$\frac{(2g_1 - 2 - m_1)^2}{m_1} \geq 10.$$

This is rewritten as $m_1^2 - (4g_1 + 6)m_1 + 4(g_1 - 1)^2 \geq 0$. Hence

$$m_1 \geq 2g_1 + 3 + \sqrt{20g_1 + 5} \qquad \text{or} \qquad m_1 \leq 2g_1 + 3 - \sqrt{20g_1 + 5}.$$

By the hypothesis that $g_1 \leq 3$, the second inequality is impossible (since $m_1 \geq 1$). Hence it must be $m_1 \geq 2g_1 + 3$.

Now we have that there is a curve D_1 of genus g_1 with self-intersection $m_1 = D_1^2 \geq 2g_1 + 3$. Take the line bundle $L = \mathcal{O}(D_1)$. This has $m_1 = \deg(L|_{D_1}) \geq 2g_1 + 3$, so $L|_{D_1}$ has sections by Proposition 5.11, precisely $\dim(H^0(L|_{D_1})) = m_1 - g_1 - 1 \geq g_1 + 4 \geq 5$, and moreover Proposition 5.11 also gives that $L|_{D_1}$ is very ample. In particular, there is a section $s \in H^0(L|_{D_1})$ transversal to the zero section, so s vanishes at exactly m_1 different points. Call $Y = s^{-1}(0) = \{y_1, \ldots, y_{m_1}\} \subset D_1$.

On the other hand, the long exact sequence in cohomology associated to the short exact sequence of sheaves $0 \to \mathcal{O} \to L \to L|_{D_1} \to 0$, together with the fact that $H^1(\mathcal{O}) = H^{0,1}(S) = 0$, gives a short exact sequence

$$0 \to \mathbb{C} \to H^0(L) \to H^0(L|_{D_1}) \to 0.$$

Take the preimage of the section $s \in H^0(L|_{D_1})$. This is a 2-dimensional subspace of $H^0(L)$ generated by two sections $s_1, s_2 \in H^0(L)$ such that $s_1^{-1}(0) \cap D_1 = s_2^{-1}(0) \cap D_1 = Y$.

As both s_1 and s_2 are linearly independent sections of $L = \mathcal{O}(D_1)$, their zero sets are distinct divisors linearly equivalent to D_1, i.e. $[s_1^{-1}(0)] = [s_2^{-1}(0)] = [D_1]$, so it follows that $s_1^{-1}(0) \cap s_2^{-1}(0) = Y$. In particular, it follows that the zero-sets of s_1 and s_2 are transversal, since Y consists of $m_1 = [D]^2$ distinct points.

In this way we have a Lefschetz pencil

$$\mathbb{P}^1 \cong \mathbb{P}(\mathrm{Span}(\langle s_1, s_2 \rangle)) \subset \mathbb{P}(H^0(L))$$

of sections whose zero sets are curves going through Y. Blow-up Y to get a smooth complex surface $\tilde{S}$ and a Lefschetz fibration

$$\pi : \widetilde{S} \to \mathbb{P}^1.$$

The construction of π was done in Chapter 1, in the discussion previous to Remark 1.44. Now, the proper transform of D_1, say $C_1 = \widetilde{D}_1$ is a smooth fiber of π, so the general fiber of π has genus g_1. The other D_j, $2 \leq j \leq b$, are not touched by the blow-up loci, so we do not change their names and denote $D_j \subset \widetilde{S}$, $j = 2, \dots, b$.

Now let E_i, $i = 1, \dots, m_1$, be the exceptional divisors of the blow-up $\widetilde{S} \to S$. These are sections of $\pi : \widetilde{S} \to \mathbb{P}^1$, as proved in Chapter 1. Note that $C_1, E_1, \dots, E_{m_1}, D_2, \dots, D_b$ are a basis of

$$H_2(\widetilde{S}, \mathbb{Q}) \cong H_2(S, \mathbb{Q}) \oplus \bigoplus_{k=1}^{m_1} \mathbb{Q}\langle E_k \rangle$$

and $[C_1] = [D_1] - [E_1] - \cdots - [E_{m_1}]$ under the above isomorphism.

On the other hand, for any $2 \leq j \leq b$ we have that $D_j \cdot E_i = 0$ for all $i = 1, \dots, m_1$, so $D_j \subset \widetilde{S}$ is contained in a fiber of π. This can be seen as follows. Since $\pi : \widetilde{S} \to \mathbb{P}^1$ is an isomorphism outside the blow-up loci, π transforms the Zariski closed set $D_j \subset \widetilde{S}$ to a Zariski closed set $\pi(D_j) \subset \mathbb{P}^1$. Since D_j is connected, $\pi(D_j)$ is either a point or $\mathbb{P}^1$. If $\pi(D_j)$ is $\mathbb{P}^1$, then D_j would intersect the fiber given by

$$[C_1] = [D_1] - \sum_{i=1}^{m_1} [E_i] \, .$$

Clearly for $j \geq 2$ it holds $[D_j] \cdot [C_1] = 0$, giving a contradiction. The conclusion is that $\pi(D_j)$ is a point, which means that D_j is contained in a fiber.

From this it follows that $g_j \leq g_1$ for $j \geq 2$, since the genus of a component of a singular fiber cannot be bigger than the genus of the generic fiber. So the maximum of the genus of the D_i is g_1, i.e.

$$g = \max\{g_i : 1 \leq i \leq b\} = g_1 \, .$$

Moreover, we can assume that not all the curves $D_2, \dots, D_b$ are contained in the same fiber of $\pi : \widetilde{S} \to \mathbb{CP}^1$. Indeed, if that were the case, all the D_j for $j \geq 2$ would be components of a singular fiber of π, whose general fiber has genus g_1. Since the sum of the genus of the components of a singular fiber is less or equal than the genus of the general fiber we would have

$$\sum_{i=2}^{b} g_i \leq g \leq 3$$

so $b \leq 4$ and the inequality $b \leq 2g + 3$ holds trivially in this case.

Hence we suppose from now on that not all the curves $D_2, \dots, D_b$ are contained in the same fiber of π. Let us see moreover that π is a relatively minimal fibration, i.e. that there are no (-1)-rational curves contained in a fiber. Suppose that B is such a curve.

Case 1: If B intersects some section say E_1, then $B \cdot E_1 = 1$. The set $B_1 = B \cup E_1$ is a rational nodal curve with homology class $[B_1] = [B \cup E_1] = [B] + [E_1]$ and self-intersection $[B_1] \cdot [B_1] = [B + E_1] \cdot [B + E_1] = -1 - 1 + 2 = 0$. It follows that there is a linear system of rational curves of self-intersection zero and hence $\widetilde{S}$ is ruled. To see it, consider the bundle $L_1 = \mathcal{O}(B_1)|_{B_1}$, which has degree $\deg(L_1) = [B_1] \cdot [B_1] = 0 \geq -1 = 2g(B_1) - 1$, so $\dim H^0(L_1) = \deg(L_1) - g(B_1) + 1 = 1$ and there exists a non-zero section of L_1, call it σ.

The short exact sequence of sheaves $0 \to \mathcal{O} \to \mathcal{O}(B_1) \to \mathcal{O}(B_1)|_{B_1} \to 0$ and the fact that $H^1(S, \mathcal{O}) = 0$ yields a short exact sequence

$$0 \to \mathbb{C} \to H^0(\mathcal{O}(B_1)) \to H^0(L_1) \to 0.$$

The preimage of the section $\sigma \in H^0(L_1)$ gives two linearly independent sections $\sigma_1, \sigma_2 \in H^0(\mathcal{O}(B_1))$, whose zero sets are two rational curves $B_1 = \sigma_1^{-1}(0)$ and $B_2 = \sigma_2^{-1}(0)$ intersecting in $[B_1] \cdot [B_2] = [B_1] \cdot [B_1] = 0$ points, i.e. disjoint. So the Lefschetz pencil $\mathbb{P}^1 \cong \mathbb{P}(\mathrm{Span}(\langle \sigma_1, \sigma_2 \rangle))$

spanned by σ_1 and σ_2 gives a ruling of S. In particular for every point of S there is a rational curve in the pencil passing throught the point.

Recall that, by hypothesis, not all the curves $D_2, \dots, D_b$ are contained in the same fiber of π. In particular there is some D_j with $j \geq 2$ contained in a different fiber than our curve B, suppose it is D_2, so $[B_1] \cdot [D_2] = ([B] + [E_1]) \cdot [D_2] = [B] \cdot [D_2] = 0$. But if we take any point $q \in D_2$ there exists $\lambda s_1 + \beta s_2$ in the Lefschetz pencil so that $\lambda s_1(q) + \beta s_2(q) = 0$, so D_2 has to intersect some class of the form $[(\lambda s_1 + \beta s_2)^{-1}(0)] = [B_1]$. This is a contradiction.

Case 2: Suppose that B does not intersect any section E_k with $1 \leq k \leq m_1$. Under this assumption, if B does not intersect any D_j, $j \geq 2$, then B is homologically trivial. To see it note that in such a case $[B]$ could be expressed in the basis of $H_2(\widetilde{X})$ given by D_1, D_j with $j \geq 2$ and E_k, so $[B]$ should be equal to $a[D_1] = a([C_1] + \sum [E_j])$ for some $a \in \mathbb{Q}$, but this is not possible since $[B] \cdot [B] = -1$ while $a[D_1] \cdot a[D_1] = a^2 m_1 > 0$. Therefore B has to intersect some D_j, say D_2, in some fiber F. Let $F_1, \dots, F_l$ be the irreducible components of the fiber F. By [**3**, (III.8.2)], the span of $\langle [F_1], \dots, [F_l] \rangle$ has dimension l, and subject to the only relation

$$[C_1] = [F] = \sum a_i [F_i] \,,$$

for some $a_i \in \mathbb{Q}$. Removing the components of F that do intersect the exceptional divisors E_j, the rest of the components, together with the D_i and the E_j, should be independent. Indeed, call $F_1, \dots F_r$ the components of F distinct of the D_i and not intersecting any of the exceptional divisors E_i. Let us see that $F_1, \dots, F_r, D_2, \dots, D_b, E_1, \dots, E_{m_1}$ are linearly independent in homology. Suppose a linear combination is zero, so

$$0 = \sum_k \lambda_k [F_k] + \sum_j \lambda_j [D_j] + \sum_i \lambda_i [E_i] \,.$$

Multipliying by $[E_{i_0}]$ we get $0 = -\lambda_{i_0}$, so we have

$$0 = \sum_k \lambda_k [F_k] + \sum_j \lambda_j [D_j] \,.$$

Now, if $[D_{j_0}]$ is not one of the components of the fiber F, D_{j_0} is contained in another fiber so $[D_{j_0}] \cdot [F_k] = 0$ for all k, and multipliying by $[D_{j_0}]$ we obtain $0 = -\lambda_{j_0} m_{j_0}$, i.e. $\lambda_{j_0} = 0$. We are left with

$$0 = \sum_k \lambda_k [F_k] + \sum_{D_j \subset F} \lambda_j [D_j]$$

and this is a linear combination of the irreducible components $F_1, \dots F_l$ of F, which are linearly independent by [**3**, (III.8.2)], so $\lambda_k = 0$ and $\lambda_j = 0$. This proves the desired linear independence. Coming back to our rational (-1)-curve $B \subset F$ not intersecting the E_i, if B existed then $B, D_2, \dots, D_b, E_i$ would be independent in homology, hence a basis. But then $B = a[D_1]$ for some $a \in \mathbb{Q}$, and this contradicts that $[D_1] \cdot [D_1] = m_1 \geq 1$.

The conclusion is that such a curve B does not appear, so $\pi : \widetilde{S} \to \mathbb{CP}^1$ is a relatively minimal fibration.

Now, for the genus-g fibration $\pi : \widetilde{S} \to \mathbb{P}^1$ we recall the invariants introduced in Definition 5.15, given by

$$\begin{aligned} K^2_{\widetilde{S}/\mathbb{P}^1} &= K^2_{\widetilde{S}} - 8(g-1)(-1) = 10 - b - m_1 + 8g - 8, \\ \chi_\pi &= \chi(\mathcal{O}_{\widetilde{S}}) - (g-1)(-1) = 1 + g - 1 = g, \\ \lambda_\pi &= K^2_{\widetilde{S}/\mathbb{P}^1} / \chi_\pi = (2 - b - m_1 + 8g)/g. \end{aligned}$$

By Theorem 5.16, since $\pi : \widetilde{S} \to \mathbb{P}^1$ is a relatively minimal fibration of genus $g \geq 2$, we have

$$4 - 4/g \leq \lambda_\pi \leq 12 \,.$$

The first inequality implies that

$$4g - 4 \leq 2 - b - m_1 + 8g \leq 2 - b - (2g + 3) + 8g\,,$$

so it follows that $b \leq 2g + 3$. This concludes the proof. □

REMARK 5.18. *The proof of Theorem 5.17 also works when we have all the complex curves D_i spanning the second homology of genus $g_i = 1$. We only have to note that in this case it follows automatically that $m_1 \geq 1 = 2g_1 - 1$, and this is enough to construct a Lefschetz fibration.*

To extend these arguments to quasi-regular Sasakian manifolds (and hence to all Sasakian manifolds), we need a version of Theorem 5.17 that covers the case that S is a cyclic Kahler orbifold. The argument should run as follows: desingularize each orbifold point (this is a Hirzebruch-Jung desingularisation [**3**]), creating a tree of rational curves of negative self-intersection, and bound $K^2_{\widetilde{S}}$ for the desingularisation $\widetilde{S} \to S$. If the bound for $K^2_{\widetilde{S}}$ permits, we could construct (in an analogous manner to that of the proof of Theorem 5.17) a pencil of curves on $\widetilde{S}$. After blowing-up $\widetilde{S}$ in the points of the basis of this pencil, we end up with an elliptic fibration

$$\widetilde{\widetilde{S}} \to \mathbb{P}^1$$

on which to apply the bounds for the slope.

We have only managed to make this argument work for the case where all complex curves are of genus $g_i = 1$. Unfortunately, we have not been able to construct a symplectic manifold X with $H_1(X, \mathbb{Z}) = 0$ and $b = b_2(X)$ disjoint symplectic tori in X spanning $H_2(X, \mathbb{Q})$.

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