Darboux transformation for the Manin-Radul supersymmetric KdV equation

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Abstract

In this paper we present a vectorial Darboux transformation, in terms of ordinary determinants, for the supersymmetric extension of the Korteweg-de Vries equation proposed by Manin and Radul. It is shown how this transformation reduces to the Korteweg-de Vries equation. Soliton type solutions are constructed by dressing the vacuum and we present some relevant plots.

1. Introduction

The Korteweg-de Vries (KdV) equation was embedded in a supersymmetric framework for the first time by Manin and Radul in [14]. Since then a number of integrable equations have been extended in this way. The role of the KdV equation and its Virasoro constraints in two dimensional quantum gravity [4,6,7] lead the group of Alvarez-Gaumé to search for analogous structures for supersymmetric two dimensional quantum gravity [1,2]. In turn, this motivated the study of Virasoro constraints for the supersymmetric Kadomtsev-Petviashvilii (KP) hierarchies available [13], which is connected with the study of additional symmetries of these hierarchies [13,5,17]. These results indicated that the supersymmetric extensions of the KdV equation, in particular the Manin-

Radul super KdV (MRSKdV), might be relevant in the study of susy 2d quantum gravity.

The search of solutions of the Manin-Radul super KP started with the work of of Radul [16] on algebro-geometric type solutions. Then in [18], from a Sato Grassmannian approach, the construction of solutions was outlined, nevertheless one can not find explicit examples in this paper. Recently in [9] some explicit solutions were obtained.

The MRSKdV system is defined in terms of three independent variables ϑ, x, t, where $\vartheta \in \mathbb{C}_a$ is an odd supernumber, and $x, t \in \mathbb{C}_c$ are even supernumbers, and two dependent variables $\alpha(\vartheta, x, t), u(\vartheta, x, t)$, where α is an odd function taking values in $\mathbb{C}_a$ and u is even function with values in $\mathbb{C}_c$. A basic ingredient is a superderivation defined by $D := \partial_\vartheta + \vartheta\partial_x$. The system is

$$\alpha_t = \frac{1}{4}(\alpha_{xxx} + 3(\alpha D\alpha)_x + 6(\alpha u)_x),$$

$$u_t = \frac{1}{4}(u_{xxx} + 6uu_x + 3\alpha_x Du + 3\alpha(Du_x)), \tag{1}$$

where we use the notation $f_x := \partial f / \partial x$ and $f_t := \partial f / \partial t$.

The following linear system for the wave function $\psi(\vartheta, x, t)$, that takes values in the Grassmann algebra $\Lambda = \mathbb{C}_c \oplus \mathbb{C}_a$,

$$\psi_{xx} + \alpha D\psi + u\psi - \lambda\psi = 0,$$

$$\psi_t - \frac{1}{2}\alpha(D\psi)_x - \lambda\psi_x - \frac{1}{2}u\psi_x + \frac{1}{4}\alpha_x D\psi + \frac{1}{4}u_x\psi = 0,$$
$$\tag{2}$$

where the spectral parameter $\lambda \in \mathbb{C}_c$ is an even super-number, has as its compatibility condition Eqs. (1), and therefore it can be considered as a Lax pair for it.

Our aim in this paper is to extend a well known tool, Darboux transformations, in integrable system theory to the supersymmetric case. This tool is a well established scheme in dealing with integrable equations and its solutions [15]. Given an integrable equation and its Lax pair the Darboux technique consists of transforming simultaneously both fields and wave functions. For the KdV equation the Lax pair is essentially the Schrödinger equation, and this was precisely the equation where Darboux developed his technique. On the one hand, recently one of the authors extended the standard Darboux transformations to the supersymmetric KdV, [10]. On the other hand, the other author has been involved recently in generalizing the standard Darboux techniques to a vectorial Darboux transformation, [8,12].

In this paper we present a vectorial Darboux transformation for the MRSKdV, this transformation is represented in terms of ordinary determinants of an even operator. Nevertheless, we do obtain essentially supersymmetric solutions. Indeed any arbitrary solution can be used as seed solution to dress, and obtain therefore large families of new solutions having the seed solution as background.

The layout of the paper is as follows. In §2 we include the main results of the paper, namely the vectorial Darboux transformation for the MRSKdV equation (1). There, we also consider the reduction to the KdV equation. Next, in §3 we study some explicit solutions selected among the large classes of explicit solutions offered by this method. In particular, we dress the vacuum solution to obtain soliton type solutions. Here we also give some plots showing the behaviour of the field α (as the ones for the u are similar we

do not include them here). We end with some conclusions and remarks in Section 4.

2. Vectorial Darboux transformation

The linear system (2) is of a scalar nature, $\lambda \in \mathbb{C}_c$, $\psi(\vartheta, x, t) \in \Lambda$. Nevertheless, it is possible to give a vector extension of these linear problem. Indeed, we may replace Λ by an arbitrary linear Grassmann space $\mathcal{E}$ over Λ and take b as an $\mathcal{E}$-valued eigenfunction, then the spectral parameter can be taken as $L \in L(\mathcal{E}_{\underline{0}}) \oplus L(\mathcal{E}_{\underline{1}})$, an even operator.

Namely, the linear system

$$b_{xx} + \alpha Db + ub - Lb = 0,$$

$$b_t - \frac{1}{2}\alpha(Db_x) - Lb_x - \frac{1}{2}ub_x + \frac{1}{4}\alpha_x Db + \frac{1}{4}u_x b = 0,$$
$$\tag{3}$$

has as its compatibility condition the MRSKdV system (1).

Notice that Eqs. (1) is also the compatibility condition of adjoint linear system:

$$\beta_{xx} + D(\alpha\beta) + u\beta - \beta M = 0,$$

$$\beta_t + \frac{1}{2}\alpha D\beta_x - \beta_x M - \frac{1}{2}(u + D\alpha)\beta_x$$

$$\qquad + \frac{1}{4}D(\alpha_x\beta) + \frac{1}{4}u_x\beta = 0,$$
$$\tag{4}$$

where $\beta(\vartheta, x, t) \in \mathcal{E}^*$ is a linear function on the supervector space $\mathcal{E}$, and $M \in L(\mathcal{E}_{\underline{0}}) \oplus L(\mathcal{E}_{\underline{1}})$.

In order to construct Darboux transformation for these linear systems we need to introduce an even operator, say V, to this end we assume that $b(\vartheta, x, t) \in \mathcal{E}_{\underline{0}}$ is an even vector and $\beta \in \mathcal{E}_{\underline{1}}^*$ an odd functional.

Proposition 1. Let $b(\vartheta, x, t)$ and $\beta(\vartheta, x, t)$ satisfy Eqs. (3) and (4), respectively. Then, there exists a potential operator $V(\vartheta, x, t) \in L(\mathcal{E}_{\underline{0}}) \oplus L(\mathcal{E}_{\underline{1}})$ given by

$$DV = b \otimes \beta,$$

$$V_t = LV_x + V_x M - D(b_x \otimes \beta_x + \frac{1}{2}uDV) - \frac{1}{4}\alpha_x DV$$

$$-\frac{1}{2}(Db) \otimes ((D\alpha)\beta - \alpha(D\beta))$$

$$+\frac{1}{2}\alpha(b \otimes \beta_x - b_x \otimes \beta) \qquad (5)$$

such that

$$LV - VM = D(b_x \otimes \beta - b \otimes \beta_x) - \alpha b \otimes \beta. \qquad (6)$$

Proof: A direct calculation shows that $DV_t = (DV)_t$ holds. We proceed by checking that the identity

$$D\left(LV - VM - D(b_x \otimes \beta - b \otimes \beta_x) + \alpha b \otimes \beta\right)$$
$$= 0$$

holds. A tedious but straightforward calculation shows

$$\partial_t\left(LV - VM - D(b_x \otimes \beta - b \otimes \beta_x) + \alpha b \otimes \beta\right)$$
$$= 0.\Box$$

Now, we state the main result of the paper.

Theorem 1. Let $b(\vartheta, x, t) \in \mathcal{E}_{\underline{0}}$ be an even vector satisfying Eq. (3), $\beta(\vartheta, x, t) \in \mathcal{E}_{\underline{0}}^*$ an odd functional solving Eq. (4) and $V \in L(\mathcal{E}_{\underline{0}}) \oplus L(\mathcal{E}_{\underline{1}})$ a non singular even operator, $\det V_{\text{body}} \neq 0$, defined in terms of the compatible Eqs. (5) and (6). Then, the objects

$$\hat{b} := V^{-1}b, \quad \hat{\beta} := \beta V^{-1}, \quad \hat{L} := M, \quad \hat{M} := L$$

$$\hat{\alpha} = \alpha - 2D^3 \ln \det V,$$

$$\hat{u} = u + 2\hat{\alpha}D \ln \det V + 2\left(\frac{\sum_j D\beta_j \det V_j}{\det V}\right)_x,$$

where V_j is an operator with associated supermatrix obtained from the corresponding one of V by replacing the j-th column by b, satisfy the Eqs. (3) and (4) whenever the unhatted variables do. Thus, $\hat{\alpha}$ and $\hat{u}$ are new solutions of the MRSKdV (1).

Proof: Is a tedious but straightforward calculation to check that

$$\hat{b} = V^{-1}b, \quad \hat{\beta} = \beta V^{-1}, \quad \hat{L} = M, \quad \hat{M} = L$$

$$\hat{\alpha} = \alpha - 2\langle \beta, V^{-1}b\rangle_x,$$

$$\hat{u} = u + 2\langle D\beta, V^{-1}b\rangle_x + 2\alpha\langle \beta, V^{-1}b\rangle$$
$$+ 2\langle \beta, V^{-1}b\rangle\langle \beta, V^{-1}b\rangle_x,$$

satisfy Eqs. (3) and (4). Observe that form $DV = b \otimes \beta$ one has the relation $\text{Tr}(DV \cdot V^{-1}) = \langle \beta, V^{-1}b\rangle$, and therefore

$$\langle \beta, V^{-1}b\rangle = D \ln \det V,$$

where we are using standard traces and determinants. Notice also that, using Cramer's rule, we have

$$\langle D\beta, V^{-1}b\rangle = \frac{\sum_j D\beta_j \det V_j}{\det V}.$$

These three remarks lead to the desired result. $\Box$

Reduction to KdV. The MRSKdV system (1) reduces to the KdV equation when $\alpha = 0$. Our Darboux transformation is in fact compatible with this reduction, giving in this manner Darboux transformation for the KdV equation. To see this let us first note that with the splitting $b(\vartheta, x, t) = b_0(x, t) + \vartheta b_1(x, t)$, $\beta(\vartheta, x, t) = \beta_1(x, t) + \vartheta \beta_0(x, t)$ and $V(\vartheta, x, t) = V_0(x, t) + \vartheta V_1(x, t)$, Eq. (5) reads

$$V_1 = b_0 \otimes \beta_1, \quad V_{0x} = b_1 \otimes \beta_1 + b_0 \otimes \beta_0$$

The linear systems (3) and (4) with $\alpha = 0$ are

$$b_{xx} + ub = Lb, \quad b_t = Lb_x + \frac{1}{2}ub_x - \frac{1}{4}u_x b$$

and

$$\beta_{xx} + u\beta = \beta M, \quad \beta_t = \beta_x M + \frac{1}{2}u\beta_x - \frac{1}{4}u_x\beta$$

To proceed further, we assume $\beta_1 = b_1 = 0$ so that $V_1 = 0$. Then, our potential satisfies

$$V_{0,x} = b_0 \otimes \beta_0,$$

In this case, our Darboux transformation tells us

$$\hat{\alpha} = -2\vartheta\langle \beta_0, V_0^{-1}b_0\rangle_x,$$

$$\hat{u} = u + 2\langle \beta_0, V_0^{-1}b_0\rangle_x$$

and since $\hat{\alpha}D\hat{b} = 0$ we conclude that $\hat{u}$ satisfies the equations

$$\hat{b}_{xx} + \hat{u}\hat{b} - \hat{L}\hat{b} = 0,$$

$$\hat{b}_t - \hat{L}\hat{b}_x - \frac{1}{2}\hat{u}\hat{b}_x + \frac{1}{4}\hat{u}_x\hat{b} = 0,$$

and therefore is a new solution of the KdV equation. Notice that here we have a subtle point, observe that $\hat{\alpha}$

is not zero but in turn does not appear in the evolution equations because its particular structure.

Observe also that using the formula $\mathrm{Tr}(V_{0,x} \cdot V_0^{-1}) = (\ln \det V_0)_x$, the transformation for field u can be rewritten neatly as

$$\hat{u} = u + 2(\ln \det V_0)_{xx},$$

which a standard form for the solutions of the KdV equation [3].

3. Exact solutions of the MRSKdV

Among the large classes of solutions provided by the just presented vectorial Darboux transformation, in this section we select some relevant example by dressing the vacuum solution $\alpha = 0$, $u = 0$. In doing so we obtain solutions, that for simplicity we denote by α, u erasing the hat, of the MRSKdV equation (1) which can be considered as a superextension of the soliton solutions of KdV.

Inserting $\alpha = u = 0$ in the linear systems one gets the equations for b

$$b_{xx} = Lb, \quad b_t = Lb_x,$$

$$\beta_{xx} = \beta M, \quad \beta_t = \beta_x M.$$

For simplicity we take L, M as diagonal even matrices, $L = \mathrm{diag}(\ell_1^2, \cdots, \ell_n^2)$ and $M = \mathrm{diag}(m_1^2, \cdots, m_n^2)$, $\ell_j, m_j \in \mathbb{C}_c$, $j = 1, \ldots, n$. Then, the functions b and β have the following form:

$$b_i = c_{i,+} \exp(\eta_i) + c_{i,-} \exp(-\eta_i),$$
$$\beta_j = \kappa_{j,+} \exp(\xi_j) + \kappa_{j,-} \exp(-\xi_j)$$

where $\eta_i(x,t) := \ell_i(x + \ell_i^2 t)$ and $\xi_j(x,t) := m_j(x + m_j^2 t)$.

The operator V, whenever $(\ell_i^2 - m_j^2)_{\mathrm{body}} \neq 0$ for all i, j, is determined by the constraint (6), namely

$$V_{ij} = \frac{1}{\ell_i^2 - m_j^2} D \varphi_{ij}$$

where

$$\varphi_{ij} = -\ell_i(c_{i,-} \exp(-\eta_i)$$
$$- c_{i,+} \exp(\eta_i))(\kappa_{j,-} \exp(-\xi_j) + \kappa_{j,+} \exp(\xi_j))$$

$$+ m_j(c_{i,-} \exp(\eta_i)$$
$$+ c_{i,+} \exp(\eta_i))(\kappa_{j,-} \exp(\xi_j) - \kappa_{j,+} \exp(\xi_j))$$

and $c_{i,\pm} = c_{i,\pm}(\vartheta)$ and $\kappa_{j,\pm} = \kappa_{j,\pm}(\vartheta)$.

The expression can be made explicitly by means of substitutions

$$c_{i,\pm} = c_{i,\pm}^0 + \vartheta c_{i,\pm}^1, \quad \kappa_{j,\pm} = \kappa_{j,\pm}^1 + \vartheta \kappa_{j,\pm}^0,$$

where the superfix indicate the parities of the variables.

Indeed, we have

$$V_{ij} = V_{ij}^0 + \vartheta V_{ij}^1$$

with

$$V_{ij}^0 = \frac{1}{\ell_i + m_j} \Big((c_{i,+}^0 \kappa_{j,+}^0 + c_{i,+}^1 \kappa_{j,+}^1) \exp(\eta_i + \xi_j)$$
$$- (c_{i,-}^1 \kappa_{j,-}^1 + c_{i,-}^0 \kappa_{j,-}^0) \exp(-\eta_i - \xi_j) \Big)$$
$$+ \frac{1}{\ell_i - m_j} \Big((c_{i,+}^1 \kappa_{j,-}^1 + c_{i,+}^0 \kappa_{j,-}^0) \exp(\eta_i - \xi_j)$$
$$- (c_{i,-}^1 \kappa_{j,+}^1 + c_{i,-}^0 \kappa_{j,+}^0) \exp(-(\eta_i - \xi_j)) \Big)$$

and

$$V_{ij}^1 = c_{i,+}^0 \kappa_{j,+}^1 \exp((\eta_i + \xi_j)) + c_{i,-}^0 \kappa_{j,-}^1 \exp(-\eta_i - \xi_j)$$
$$+ c_{i,-}^0 \kappa_{j,+}^1 \exp(-(\eta_i - \xi_j)) + c_{i,+}^0 \kappa_{j,-}^1 \exp(\eta_i - \xi_j)$$

As a particular example one can pick b_i as above and $\beta_j = \kappa_j^1 + \vartheta \kappa_j^0$, which requires $m_j = 0$. In this case, the potential is

$$V_{ij} = \frac{1}{\ell_i} (\kappa_j^0 c_{i,+}^0 - \kappa_j^1 c_{i,+}^1) \exp(\eta_j)$$
$$+ (\kappa_j^1 c_{i,-}^1 - \kappa_j^0 c_{i,-}^0) \exp(-\eta_i)$$
$$+ \vartheta \ell_i \kappa_j^1 (c_{i,+}^0 \exp(\eta_i) + c_{i,-}^0 \exp(-\eta_i))$$

To illustrate this family of solutions we shall consider the simplest case of a one dimensional space $\mathcal{E}$ (for simplicity we omit the index 1 next). Hence, we have

$$V = \frac{1}{\ell} ((\kappa^0 c_+^0 - \kappa^1 c_+^1) \exp(\eta)$$
$$+ (\kappa^1 c_-^1 - \kappa^0 c_-^0) \exp(-\eta)$$
$$+ \vartheta \ell \kappa^1 (c_+^0 \exp(\eta) + c_-^0 \exp(-\eta))$$

To invert the operator V, we have to separate its body from its soul. We choose the supernumbers $\kappa^0, \ell, c_\pm^0 \in$

$\mathbb{C}$ to have vanishing soul, so that the potential operator V takes the following form:

$$V = \frac{1}{\ell}(V_{\text{body}} + V_{\text{soul}})$$

where $V_{\text{body}} := \kappa^0(c_+^0 \exp(\eta) - c_-^0 \exp(-\eta))$ and $V_{\text{soul}} := \kappa^1(-c_+^1 \exp(\eta) + c_-^1 \exp(-\eta) - \vartheta\ell b_0)$, we notice that $b = b_0 + \vartheta b_1$ with $b_0 = c_+^0 \exp(\eta) + c_-^0 \exp(-\eta)$ and $b_1 = c_+^1 \exp(\eta) + c_-^1 \exp(-\eta)$.

The constitutive elements for our solution are now

$$\langle \beta, V^{-1}b \rangle = \ell\beta\frac{b_0}{V_{\text{body}}} - 2\vartheta\frac{\ell\kappa^0\kappa^1(c_+^0 c_-^1 - c_-^0 c_+^1)}{(V_{\text{body}})^2},$$

$$\langle D\beta, V^{-1}b \rangle = \kappa^0\ell\frac{b}{V_{\text{body}}}$$

$$- \frac{\ell\kappa^0\kappa^1}{(V_{\text{body}})^2}(-c_+^1 c_+^0 \exp(2\eta) + c_-^1 c_-^0 \exp(-2\eta)$$

$$- c_+^1 c_-^0 + c_-^1 c_+^0) + \frac{\vartheta\ell\kappa^0\kappa^1}{(V_{\text{body}})^2}(2c_+^1 c_-^1 - \ell b_0^2)$$

It is easy to see that this solution can be considered as a supersymmetric extension of the soliton solution of the KdV equation. In fact, the constants can be chosen in such a way that the functions depending solely on x, t appearing as multiplicative coefficients are exponentially localized in x and travel with constant speed. Moreover, the KdV soliton solution appears as a particular coefficient, $(b_0/V_{\text{body}})_x$. Thus, our solution can be considered as a supersymmetric deformation of the standard KdV soliton.

We write $\alpha(\vartheta, x, t) = \beta(\vartheta)f(x, t) - \vartheta\kappa^1\kappa^0 g(x, t)$ and we plot the functions f and g. The function f, which is plotted in Fig. 1, is just a KdV 1-soliton solution while g, plotted in Fig. 2, is a exponentially localized regular solution that travels with constant speed, now its shape is more involved that in the KdV soliton.

4. Conclusions and remarks

We have constructed a Darboux transformation of a vector nature for the Manin-Radul supersymmetric KdV system. We have further shown that our Darboux transformation can be reduced to the KdV equation and is very effective when exact solutions are needed. The vectorial Darboux transformation is given in terms
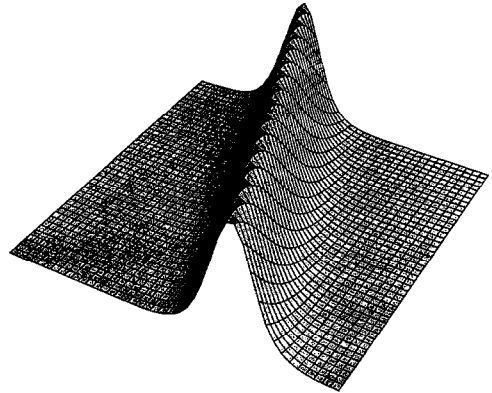

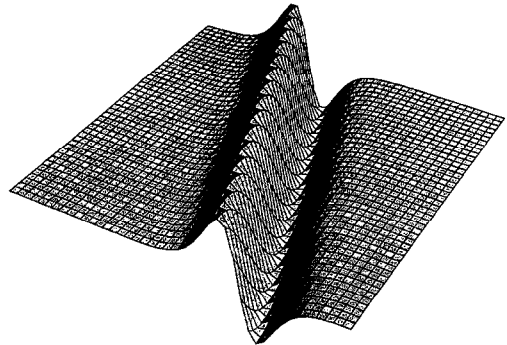Fig. 1. The function $f(x, t)$ plotted in the x, t plane, is just the standard 1-soliton of the KdV equation.

Fig. 2. The function $g(x, t)$ plotted in the x, t plane, it represents a exponentially localized regular solution in x that travels with constant speed.

of solutions of the Lax pair and its adjoint which have a well defined and opposite parities. This implies that the Darboux operator is even and that the new solution can be expressed in terms of ordinary determinants. However, this absence of superdeterminants is not a drawback because the proposed technique gives an efficient method of construction of genuine supersymmetric solutions.

We further remark that the basic Darboux transformations considered in [10] can be iterated so that the Crum type transformations may be obtained. In this case, the transformations will be represented in terms of superdeterminants of certain super Wronski matrices. This and other related results will be presented elsewhere, [11].

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