Uniqueness and collapse of solution for a mathematical model with nonlocal terms arising in glaciology

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Summary

In this paper we study a nonlinear system of differential equations which arises from a stationary 2-dimensional Ice-Sheet Model describing the *ice-streaming phenomenon*. The system consists of a multivalued nonlinear PDE of parabolic type coupled with a first order PDE and an ODE involving a nonlocal term. We study the uniqueness of weak solution under suitable assumptions (physically reasonable). We also establish that the ice thickness collapses at a finite distance (by employing a comparison principle).

1 Introduction

Nowadays, climate topics belong to the research issues which have not only found significant interest within the scientific community, but also among the general public. A great quantity of human and economical resources are devoted to climate research, partly due to the Global Warming debate. The Scientific Community, aware of the fact that ice-sheet behavior is closely related to environmental changes, has shown a renewed interest in modelling ice-sheet dynamics. So, the development and subsequent validation of Ice-Sheet models (I.S.M.) becomes essential for a better understanding of the mechanisms involved in their dynamics and their behavior.

It is well-known that ice-sheet dynamics requires an approach in terms of continuum mechanics in the sense that a set of equations (conservation equations and constitutive laws) are established which describe the operating mechanisms and deformable properties of the material, i.e., the ice. Needless to say that ice, in theoretical Glaciology, is typically modelled as a non-newtonian highly viscous fluid.

The physically based model considered here and proposed in Munõz et al[17], describes the phenomenon of the ice-stream flow over soft and deformable beds (see Fowler

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and Johnson[13],[14], Munõz et al[17], Muñoz[16], Díaz et al[8]). The ice-streaming phenomenon consists in the occurrence of lateral (transversal direction to the main flow one) oscillations in the ice flow regime. This phenomenon has been widely observed and studied in the geophysical scenario of the Siple Coast ice-streams (West Antarctica). There, slow flow areas (observed velocities of the order of ten km/year) alternate with atypically fast flow areas (velocities of the order of hundreds of km/year). The presence of such unusual oscillatory behavior is mainly attributed to the dynamical nature of the relation between basal sliding and basal drainage system configuration.

In the next section, we briefly describe the model, which mainly consists in a reformulation in terms of multivalued operators of a previous ice streaming model due to Fowler and Johnson[13]. Afterwards, we present the main results of the paper, regarding the uniqueness of bounded weak solution and the qualitative behavior of the solution. Before the proofs of these results, for reader′s convenience we comment the ideas of the proof of the existence. The paper finishes with the conclusions.

2 Ice streaming models

2.1 From the Siple Coast model to a multivalued equation

The Siple Coast model was proposed by Fowler and Johnson [13] (see also Fowler[11] and Fowler and Johnson[14]) following conservation principles and some constitutive laws.

In order to obtain a parameterization of the heat flux Fowler considered the limit thermal boundary layer theory. He proposed a generalization of the Walder and Fowler's drainage theory (Fowler and Walder[19]), incorporating pressure gradients that made possible to distinguish regions of fast and slow ice flow. Drawing on Fowler's work [11], we are going to consider the following set of variables, equations and parameters

Variables:

- x, the coordinate in the longitudinal down stream direction (main flow direction) $x \in (0, X)$;
- y, the coordinate in the lateral cross stream direction $y \in (0, L)$;
- $Q = Q(x, y)$, the scalar water flux associated to the sliding;
- $h = h(x)$, the ice thickness;
- $u = u(x, y)$, the basal ice velocity in the down stream direction;
- $\tau = \tau(x)$, the shear stress;
- $N = N(x, y)$, the effective pressure, defined as $N = p_i - p_w$, i.e. ice pressure minus water pressure;
- $q = q(x, y)$, the cooling term;
- $\xi = \xi(x, y)$ the *accumulated velocity* defined by $\xi(x, y) = \int_0^x u(\tilde{x}, y)\, d\tilde{x}$.

Equations:

- the *water flux conservation equation*:

$$\frac{\partial Q}{\partial x} = k\frac{\partial}{\partial y}\left[(Q+\bar{Q})^{-k}\frac{\partial Q}{\partial y}\right] + (\gamma + \tau u - q);$$

- the *mass conservation equation*:

$$h\int_0^L u\,dy = M;$$

- the *momentum conservation equation:* $\tau = -h\,\dfrac{\partial h}{\partial x}$;
- the *sliding law:* $\tau = u^r N^s$;
- the *drainage law:* $Q + \bar{Q} = \dfrac{1}{N^{\frac{1}{k}}}$;
- the *parameterized cooling* term: $q = \dfrac{u}{\xi^{\frac{1}{2}}} + \dfrac{\delta}{h}$.

Parameters

- k, the inverse of the exponent in Glen's non-linear flow law;
- r and s, the exponents related to Boulton Hindmarsh's rheology and to Glen's exponent;
- M, the dimensionless initial ice flux prescribed in the ice divide;
- $\bar{Q}$, the dimensionless residual basal water flux associated with the creeping component of the ice flow;
- γ represents the geothermal heat flux;
- δ measures the importance of the conductive cooling.

The model reproduces the ice streaming phenomenon, and assumes $h = h(x)$ (i.e. h is independent of y), neglecting the effect of the lateral components of velocity and stress fields. Following Fowler[11], the system for the unknown Q, h and ξ is given by:

$$\begin{cases} \dfrac{\partial Q}{\partial x} - k\dfrac{\partial}{\partial y}\left((Q+\bar{Q})^{-k}\dfrac{\partial Q}{\partial y}\right) = \tilde{f}(\xi, h, \dfrac{\partial h}{\partial x}, Q), & \text{in } \Omega^+, \\ \dfrac{\partial h}{\partial x} = -M^r h^{-1-r}\left(\displaystyle\int_0^L (Q+\bar{Q})^S dy\right)^{-r}, & \text{in } \Omega^+, \\ \dfrac{\partial \xi}{\partial x} = h^2\left|\dfrac{\partial h}{\partial x}\right|^2 (Q+\bar{Q})^S, & \text{in } \Omega^+, \\ \dfrac{\partial Q}{\partial y} = 0, & y = 0, L, \\ Q(0,y) = Q_0(y), \quad h(0) = h_0, \quad \xi(0,y) = \xi_0, & y \in (0, L). \end{cases} \tag{2.1}$$

where f, Ω^+ and S are defined by

$$\tilde{f}\left(\xi, h, \frac{\partial h}{\partial x}, Q\right) = h^2 \left|\frac{\partial h}{\partial x}\right|^2 \left(h\left|\frac{\partial h}{\partial x}\right| - \xi^{-\frac{1}{2}}\right)(Q+\bar{Q})^S + \gamma - \delta h^{-1},$$

$$\Omega^+ := \{(x,y) \in (0,X)\times(0,L), \quad Q(x,y) > 0\}, \qquad S = \frac{sk}{r}.$$

The domain Ω^+ is a moving domain and the problem is a $f$ree boundary problem that may be formulated as a variational problem. Variational formulations are useful in many free boundary problems, but of course, other different approaches are used by many authors in this context.

We introduce the variable

$$\sigma = \int_\Omega (Q+\bar{Q})^S dy \tag{2.2}$$

in order to deal with the non local integral term. The system is given by

$$\begin{cases} Q \geq 0, & \text{in } \Omega, \\ \dfrac{\partial Q}{\partial x} - k\dfrac{\partial}{\partial y}\left((Q+\bar{Q})^{-k}\dfrac{\partial Q}{\partial y}\right) - f(\xi,h,Q,\sigma) \geq 0, & \text{in } \Omega, \\ \left[\dfrac{\partial Q}{\partial x} - k\dfrac{\partial}{\partial y}\left((Q+\bar{Q})^{-k}\dfrac{\partial Q}{\partial y}\right) - f(\xi,h,Q,\sigma)\right] Q = 0, & \text{in } \Omega, \\ \dfrac{\partial h}{\partial x} = -M^r h^{-1-r}\sigma^{-r}, & \text{in } \Omega, \\ \dfrac{\partial \xi}{\partial x} = M h^{-1}\sigma^{-1}(Q+\bar{Q})^S, & \text{in } \Omega, \\ \dfrac{\partial Q}{\partial y} = 0, & y = 0, L, \\ Q(0,y) = Q_0(y), \quad h(0) = h_0, \quad \xi(0,y) = \xi_0, & y \in (0,L), \end{cases} \tag{2.3}$$

where $\Omega = (0,X)\times(0,L)$ and

$$f(\xi,h,Q,\sigma) = \frac{M}{h\sigma}\left(Q+\bar{Q}\right)^S\left(M^r h^{-r}\sigma^{-r} - \xi^{-\frac{1}{2}}\right) + \gamma - \delta h^{-1}. \tag{2.4}$$

The solution Q may present a loss of regularity at the free boundary, as recent numerical experiments shows (see Muñoz et el[17] and Calvo et al [5]).

Classical solution exists under the assumption

$$Q_0 > 0 \tag{2.5}$$

for X small enough. Since the initial datum does not satisfy (2.5) we introduce, as in Díaz et al[8], the formulation in terms of weak solutions.

Definition 2.1 *Let δ, γ, r, S and M be positive constants satisfying (2.10) and Q_0, h_0 and ξ_0 satisfying (2.16)-(2.18), then (Q, h, ξ, σ) is a* **bounded weak solution** *(b.w.s) to (2.3), if*

$$\iint_\Omega \frac{\partial Q}{\partial x} \nu dxdy + \iint_\Omega [Q - Q_0] \frac{\partial \nu}{\partial x} dydx = 0, \tag{2.6}$$

$$\forall \nu \in L^2(0, X; H^1(0, L)) \cap H^1(0, X; L^2(0, L)), \text{ such that } \nu(X, \cdot) = 0,$$

$$\iint_\Omega \frac{\partial Q}{\partial x}(\eta - Q)dydx + \iint_\Omega k(Q + \bar{Q})^{-k} \frac{\partial Q}{\partial y} \frac{\partial}{\partial y}(\eta - Q)dydx \geq$$

$$\iint_\Omega f(\xi, h, Q, \sigma) \cdot (\eta - Q)dydx, \; \forall \eta \in L^2(0, X; H^1(0, L)), \tag{2.7}$$

$$h(x) = \left[h_0^{r+2} - (r + 2)M^r \int_0^x \sigma^{-r} ds\right]^{\frac{1}{r+2}}, \; \forall \, y \in [0, L], \tag{2.8}$$

$$\xi(x, y) = \xi_0 + M \int_0^y (\sigma h)^{-1} (Q + \bar{Q})^S ds, \quad a.e. \, x \in (0, X), \quad a.e. \, y \in [0, L]. \tag{2.9}$$

Typical values of the parameters are derived from scaling processes and dimensionless analysis. As in Fowler and Johnson[13] we assume that

$$k = \frac{1}{3}, \; r, S \in (0, 1), \;\; M > 0, \;\; 0 < \bar{Q} \ll 1, \;\; \gamma \sim 0.2 \;\; \text{and } \delta \sim 0.4. \tag{2.10}$$

In order to obtain the estimates that we need in order to prove the uniqueness of solution, we introduce an equivalent formulation presented in terms of maximal monotone graph. Equations with multivalued graph are frequent in many problems evolving free boundaries. Thus, given L, X, $\bar{Q}$, γ, δ and M, the coupled system of equations for the variables Q, h and ξ, that governs the multivalued model is:

$$\frac{\partial Q}{\partial x} - k \frac{\partial}{\partial y}\left[(Q + \bar{Q})^{-k} \frac{\partial Q}{\partial y}\right] + \beta(Q) \ni f(\xi, h, Q, \sigma), \qquad \text{in } \Omega, \tag{2.11}$$

$$\frac{\partial h}{\partial x} = -M^r h^{-1-r} \sigma^{-r}, \qquad \text{in } (0, X), \tag{2.12}$$

$$\frac{\partial \xi}{\partial x} = M h^{-1} \sigma^{-1} (Q + \bar{Q})^S, \qquad \text{in } \Omega, \tag{2.13}$$

$$\left.\frac{\partial Q}{\partial y}\right|_{(x,0)} = \left.\frac{\partial Q}{\partial y}\right|_{(x,L)} = 0, \qquad x \in (0, X), \tag{2.14}$$

$$Q(0, y) = Q_0(y), \qquad h(0) = h_0, \quad \text{and} \quad \xi(0, y) = \xi_0(y), \tag{2.15}$$

where f is defined by (2.4), σ by (2.2) and β is given by

$$\beta(s) = \begin{cases} 0, & s \leq 0, \\ (-\infty, 0], & s = 0, \\ \emptyset, & s < 0. \end{cases}$$

We assume that the initial data satisfy

$$Q_0 \geq 0, \quad h_0 > 0, \quad \xi_0 > 0, \tag{2.16}$$

$$\xi_0 \in H^1(0,L), \qquad h_0 \in L^\infty(0,L), \tag{2.17}$$

$$Q_0 \in H^1(0,L) \quad \text{and} \quad \frac{\partial Q_0}{\partial y} = 0 \quad \text{at} \quad y = 0, L. \tag{2.18}$$

The choice of β is standard in free boundary problems and suggested by the physics of the problem.

Remark 2.1 *It is significant that f may become negative and it produces $\beta(Q) < 0$ keeping $Q \geq 0$ (the multivalued equation is just defined for $Q \geq 0$). Notice that, any solution of the variational problem, satisfies (2.11)-(2.15) for*

$$\beta(Q) := -\frac{\partial Q}{\partial x} + k\,\frac{\partial}{\partial y}\left[(Q+\bar{Q})^{-k}\frac{\partial Q}{\partial y}\right] + f(\xi, h, Q, \sigma),$$

and it is a solution to the multivalued model. Then, proving the existence of solutions for the variational problem, the existence for the multivalued model is done.

The variables Q, h and ξ satisfy:

$$h > 0, \quad Q \geq 0 \quad \text{and} \quad \xi > 0, \tag{2.19}$$

a hypothesis which selects only physically meaningful solutions of (2.11)-(2.15).

Remark 2.2 *The assumption $Q \geq 0$ is one of the request that Q has to satisfy to be a solution of the variational problem. Since the solution of the variational problem is also a solution to the multivalued equation we obtain that the unique solution to the multivalued equation satisfies $Q \geq 0$.*

2.2 Review of the existence

For the reader′s convenience, we comment the result of the existence of b.w.s., proven by Díaz et al[8]. Due to technical reasons related to the fixed point argument, they consider first the existence in $[0, X)$ for X satisfying

$$X < \frac{1}{2+r}\,L^r \bar{Q}^{rS} M^{-r} h_0^{2+r} \tag{2.20}$$

and the solution is extended as long as $h > 0$.

Remark 2.3 *The extension may be done by using an iterative method: first, the problem is solved in $(0, X_0)$, for X_0 defined by (2.20), then we construct a sequence X_n defined by*

$$X_n = \frac{1}{2+r}\,L^r \bar{Q}^{rS} M^{-r} h^{2+r}(X_{n-1}),$$

and we solve the problem in (X_{n-1}, X_n) in the same fashion than in $(0, X_0)$ if $h(X_{n-1}) > 0$ (if $h(X_{n-1}) = 0$, the solution exists in the interval $(0, X_{n-1})$). Notice that the sequence X_n is increasing and as we will see in Section 4 it is bounded.

Considering first the new unknown $w = \frac{k}{k+1}(Q+\bar{Q})^{1-k}$, the function $b(w) = \left(\frac{k+1}{k}w\right)^{\frac{1}{1-k}}$, and assuming $S = 1/3$ and $r = 1/2$, the system becomes

$$w \geq \Phi, \quad \text{in } \Omega, \tag{2.21}$$

$$\frac{\partial}{\partial x}b(w) - \frac{\partial^2 w}{\partial y^2} - f(\xi, h, \sigma, w) \geq 0, \quad \text{in } \Omega, \tag{2.22}$$

$$\left[\frac{\partial}{\partial x}b(w) - \frac{\partial^2 w}{\partial y^2} - f(\xi, h, \sigma, w)\right](w - \Phi) = 0, \quad \text{in } \Omega, \tag{2.23}$$

$$h^{'} = -M^r h^{-1-r}\sigma^{-r}, \quad \text{in } \Omega, \tag{2.24}$$

$$\frac{\partial \xi}{\partial x} = Mh^{-1}\sigma^{-1}\left(\frac{kw}{1+k}\right)^{\frac{S}{1-k}}, \quad \text{in } \Omega, \tag{2.25}$$

$$\frac{\partial w}{\partial y} = 0, \qquad y = 0, L, \tag{2.26}$$

$$w(0,y) = w_0(y),\ h(0,y) = h_0,\ \xi(0,y) = \xi_0, \quad y \in (0,L), \tag{2.27}$$

where f is defined by (2.4).

The definition of b.w.s. to (2.21)-(2.27) in terms of w derives from definition 2.1.

The proof of the existence follows the work [7], where an *iterative method* was applied. Consider the functional space $V = V_w \times V_h \times V_\xi$ for

$$V_w := \{\eta :\ \eta \in L^2(0, X; I\!\!K) \cap L^\infty(\Omega),\ \frac{\partial}{\partial x}b(\eta) \in L^2(\Omega)\},$$

$$V_h := \{\phi :\ \phi \in C([0, X]),\ \phi^{'} \in L^\infty(0, X)\}$$

$$\textit{and} \qquad V_\xi := \{\psi :\ \psi \in W^{1,\infty}(0, X; L^\infty(0, L)) \cap L^2(0, X; H^1(0, L))\},$$

where $I\!\!K := \{v(x) \in H^1(\Omega), \quad \text{such that} \quad v(x) \geq \Phi,\ a.e.\, x \in \Omega\}$.

Then, for any $h \in V_h$, $\xi \in V_\xi$ we solve the problem (2.21)-(2.23), (2.26) with the initial datum (2.27) by using Yosida approximations of the nonlinearity appearing in the source term. In the regularized problems, the authors follow Alt et al[1].

After obtaining the necessary estimates for w, problems (2.24) and (2.25) with initial data (2.27) are solved for w satisfying the estimates obtained before.

The authors obtain a sequence of solutions $\{(w_j, h_j, \xi_j)\}_j$ where a subsequence converges in V (in weak sense) to (w, h, ξ) that is the weak solution of the problem. The existence in obtained for X given by (2.20), starting the process from $h_0 = h(X)$ we obtain the existence as long as h is positive.

Notice that the general cases $r, S \in (0,1)$, can be analyzed with similar arguments.

2.3 Main results

Next, we state the main results of the paper are thus: Theorem 2.1 concerns the uniqueness of b.w.s. and Theorem 2.2 the collapse of the solution at finite longitudinal distance.

Theorem 2.1 *Under assumptions (2.10) - (2.18) there exists at most a unique bounded weak solution* (Q, h, ξ) *to (2.11) - (2.15) satisfying (2.19).*

The next result asserts under a certain hypothesis that in accordance with the physics of the problem the ice-sheet extends longitudinally only a finite distance.

Theorem 2.2 *Under assumptions (2.10) - (2.18) there exists a positive constant* $X_c < \infty$*, defined by*

$$X_c = ((h_0^{2+r} A^{-1} + B^{1-r})^{\frac{1}{1-r}} - B)(\gamma L)^{-1},$$

for

$$A := \frac{(2+r)(1-r)}{\gamma L} \frac{M^r \bar{Q}^{r-Sr}}{L^r} \quad \textit{and} \quad B := Q_0 + Mh_0 + \bar{Q}L,$$

such that the solution exists in $(0, X)$ *and*

$$\lim_{x \to X} h(x) = 0$$

for some $X \leq X_c$.

Notice that, in particular, Theorem 2.2 proves that the solution exists for a finite distance $(0, X)$, and it can not be extended after $X \leq X_c$.

3 Proof of Theorem 2.1: uniqueness

The proof of the Theorem 2.1 will be carried out, for clearness' sake, in several steps. We will resort to approximating problems which have regular solutions in order to derive a suitable regularity properties for the solutions to (2.11) - (2.15). Afterwards, we employ these regularity properties to establish several suitable estimates which we need for the uniqueness result. For technical reasons we consider first the uniqueness for small X and then we extend the result to all intervals where $h > 0$.

To begin with the proof we establish the following result concerning the regularity of the solutions. The major difficulty of the proof appears in the proof of the estimate (3.1), which in particular proves Lemma 3.2, a necessary step to obtain (3.30). From (3.30) we deduce the uniqueness.

Proposition 3.1 *Let* (Q,h,ξ) *be a solution to (2.11) - (2.15), satisfying (2.18)-(2.19). Then, there exists a positive constant* C *such that*

$$\iint_\Omega \left[\frac{\partial^2}{\partial y^2}(Q+\bar{Q})^{\frac{2}{3}}\right]^2 dydx \leq CX, \tag{3.1}$$

for X *small enough.*

Proof. Let us consider a parameter $\epsilon > 0$, doomed to tend to zero and $F(x,y) = f(\xi,h,Q,\sigma)$ where (ξ,h,Q,σ) is a solution to (2.11) - (2.15) and f is given by (2.4). We define the following ϵ-dependent problems

$$\frac{\partial Q_\epsilon}{\partial x} - k\frac{\partial}{\partial y}\left[(Q_\epsilon+\bar{Q})^{-k}\frac{\partial Q_\epsilon}{\partial y}\right] + \beta_\epsilon(Q_\epsilon) = F(x,y), \quad \text{in } \Omega, \tag{3.2}$$

$$Q_\epsilon(0,\cdot) = Q_0 \quad \text{and} \quad \frac{\partial Q_\epsilon}{\partial y} = 0 \quad \text{at} \quad y = 0, L. \tag{3.3}$$

The functions β_ϵ are defined by

$$\beta_\epsilon(s) = \begin{cases} 0, & s \geq 0, \\ \epsilon^{-1}s, & s < 0, \end{cases}$$

which approximates β when $\epsilon \to 0$ in the sense of graphs. Notice that β_ϵ is a maximal monotone and Lipschitz function. The existence of at least one weak solution $Q_\epsilon \in L^2(0,X : H^2(0,L)) \cap W^{1,2}(0,X : L^2(0,L))$ to (3.2), (3.3) can be proven as in Muñoz[16] or by employing sub- and super- solutions.

Now, we decompose the rest of the proof of the proposition into several steps.

Step 1: The solution Q_ϵ is bounded.
Let us consider the positive constant

$$c_f = \max_{(x,y)\in\Omega}\{f(\xi(x,y),h(x),Q(x,y),\sigma(x))\} = \max_{(x,y)\in\Omega}\{F(x,y)\}, \tag{3.4}$$

then $\bar{v} = \max Q_0 + c_f x$ is a super-solution to (3.2), (3.3) and $\underline{v} = -c_f x$ is a sub-solution. We are going to derive that $-c_f x \leq Q_\epsilon \leq max\{Q_0\} + c_f x$. In order to keep the diffusion coefficient bounded we consider

$$X \leq \frac{\bar{Q}}{2c_f}, \tag{3.5}$$

which guarantees

$$Q_\epsilon \geq -\frac{\bar{Q}}{2} \quad \text{and} \quad Q_\epsilon \leq \max\{Q_0\} + \frac{\bar{Q}}{2}. \tag{3.6}$$

Step 2 : $L^2(0,X : H^2(0,L))$ estimates.
Let us consider $\bar{b}(s)$ defined by $\bar{b}(s) = (s+\bar{Q})^{\frac{2}{3}}$ then we have the following lemma:

Lemma 3.1

$$\iint_{\Omega} \left[\frac{\partial^2}{\partial y^2}\bar{b}(Q_\epsilon)\right]^2 dydx \leq k_0. \tag{3.7}$$

Proof. Multiplying the identity (3.2) by $(Q_\epsilon + \bar{Q})^{-\frac{1}{3}}$, one obtains

$$\begin{aligned} &\frac{3}{2}\frac{\partial}{\partial x}(Q_\epsilon + \bar{Q})^{\frac{2}{3}} - \frac{1}{3}(Q_\epsilon + \bar{Q})^{-\frac{1}{3}}\frac{\partial}{\partial y}\left[(Q_\epsilon + \bar{Q})^{-\frac{1}{3}}\frac{\partial Q_\epsilon}{\partial y}\right] + \\ &\beta_\epsilon(Q_\epsilon)(Q_\epsilon + \bar{Q})^{-\frac{1}{3}} = f(\xi, h, Q, \sigma)(Q_\epsilon + \bar{Q})^{-\frac{1}{3}}, \qquad \text{in} \quad \Omega. \end{aligned} \tag{3.8}$$

Next, taking $-\frac{\partial^2}{\partial y^2}\bar{b}(Q_\epsilon)$ as test function in (3.8), it follows

$$\begin{aligned} &\frac{3}{4}\int_0^L \left[\frac{\partial}{\partial y}\bar{b}(Q_\epsilon)\right]^2 dy\Bigg|_0^X + \frac{1}{3}\iint_\Omega (Q_\epsilon + \bar{Q})^{-\frac{1}{3}}[\frac{\partial^2}{\partial y^2}(\bar{b}(Q_\epsilon))]^2 dydx \\ -&\iint_\Omega \beta_\epsilon(Q_\epsilon)(Q_\epsilon + \bar{Q})^{-\frac{1}{3}}\frac{\partial}{\partial y}\bar{b}(Q_\epsilon)dydx = -\iint_\Omega (Q_\epsilon + \bar{Q})^{-\frac{1}{3}} f \frac{\partial}{\partial y}\bar{b}(Q_\epsilon)dydx. \end{aligned} \tag{3.9}$$

Since

$$\int\int_\Omega -\frac{\beta_\epsilon(Q_\epsilon)}{(Q_\epsilon + \bar{Q})^{\frac{1}{3}}}\frac{\partial}{\partial y}\bar{b}(Q_\epsilon)dydx =$$

$$\frac{3}{2}\int\int_\Omega \left[\frac{\beta_\epsilon(Q_\epsilon)}{(Q_\epsilon + \bar{Q})^{\frac{1}{3}}}\right]' (Q_\epsilon + \bar{Q})^{\frac{1}{3}} \left[\frac{\partial^2}{\partial y^2}\bar{b}(Q_\epsilon)\right]^2 dydx,$$

$$\left[\beta_\epsilon(Q_\epsilon)(Q_\epsilon + \bar{Q})^{-\frac{1}{3}}\right]' = \beta_\epsilon'(Q_\epsilon)(Q_\epsilon + \bar{Q})^{-\frac{1}{3}} - \frac{1}{3}\beta_\epsilon(Q_\epsilon)(Q_\epsilon + \bar{Q})^{-\frac{4}{3}} \geq 0$$

and

$$-\iint_\Omega \frac{2f}{(Q_\epsilon + \bar{Q})^{\frac{1}{3}}}\frac{\partial}{\partial y}\bar{b}(Q_\epsilon)dydx \leq \iint_\Omega \frac{f^2}{(Q_\epsilon + \bar{Q})^{\frac{2}{3}}}dydx + \iint_\Omega \left[\frac{\partial^2}{\partial y^2}\bar{b}(Q_\epsilon)\right]^2 dydx,$$

we get

$$\int_0^L \left[\frac{\partial}{\partial y}\bar{b}(Q_\epsilon)\right]^2 dy + \iint_\Omega \left[\frac{\partial^2}{\partial y^2}\bar{b}(Q_\epsilon)\right]^2 dydx \leq k_0, \tag{3.10}$$

which implies (3.7). □

Remark 3.1 *Notice that, since*

$$\frac{\partial^2 Q_\epsilon}{\partial y^2} = (Q_\epsilon + \bar{Q})^{\frac{1}{3}}\frac{\partial}{\partial y}\left[(Q_\epsilon + \bar{Q})^{-\frac{1}{3}}\frac{\partial Q_\epsilon}{\partial y}\right] + \frac{1}{3}(Q_\epsilon + \bar{Q})^{-1}\left|\frac{\partial Q_\epsilon}{\partial y}\right|^2$$

and $H^2(0, L) \subset L^4(0, L)$, in view of (3.7) and (3.6) we assert

$$\iint_\Omega \left|\frac{\partial^2 Q_\epsilon}{\partial y^2}\right|^2 dydx \leq k_1. \tag{3.11}$$

Step 3: $L^2(\Omega)$ **estimates for** $\beta_\epsilon(Q_\epsilon)$**.**
We claim that there exists a positive constant k_1, such that

$$\iint_\Omega \beta_\epsilon^2(Q_\epsilon)dydx \leq k_1. \tag{3.12}$$

Proof. Let us take $\beta_\epsilon(Q_\epsilon)$ as test function in (3.2) to obtain

$$\begin{aligned}\iint_\Omega \beta_\epsilon(Q_\epsilon)\frac{\partial Q_\epsilon}{\partial x}dydx - \frac{1}{3}\iint_\Omega \beta_\epsilon(Q_\epsilon)\frac{\partial}{\partial y}\left[(Q_\epsilon+\bar{Q})^{-\frac{1}{3}}\frac{\partial Q_\epsilon}{\partial y}\right]dydx \\ + \iint_\Omega \beta_\epsilon^2(Q_\epsilon)dydx = \iint_\Omega f(\xi,h,Q,\sigma)\beta_\epsilon(Q_\epsilon)dydx.\end{aligned} \tag{3.13}$$

Next, we estimate each of the terms. We begin with

$$\iint_\Omega \beta_\epsilon(Q_\epsilon)\frac{\partial Q_\epsilon}{\partial x}dydx.$$

Notice that $\beta_\epsilon(Q_\epsilon)\frac{\partial Q_\epsilon}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left(\beta_\epsilon(Q_\epsilon)Q_\epsilon\right)$ and we have that

$$\iint_\Omega \beta_\epsilon(Q_\epsilon)\frac{\partial Q_\epsilon}{\partial x}dydx = \frac{1}{2}\int_0^L Q_\epsilon\beta_\epsilon(Q_\epsilon)dy\Big|_0^X =$$

$$\frac{1}{2}\int_0^L Q_\epsilon\beta_\epsilon(Q_\epsilon)dy\Big|_X - \frac{1}{2}\int_0^L Q_0\beta_\epsilon(Q_0)dy.$$

Since $Q_0 \geq 0$, one gets that $Q_0\beta_\epsilon(Q_0) = 0$ and then

$$\iint_\Omega \beta_\epsilon(Q_\epsilon)\frac{\partial Q_\epsilon}{\partial x}dydx = \frac{1}{2}\int_0^L Q_\epsilon\beta_\epsilon(Q_\epsilon)dy\Big|_X \geq 0.$$

We consider now the term

$$-\iint_\Omega \frac{\partial}{\partial y}\left[(Q_\epsilon+\bar{Q})^{-\frac{1}{3}}\frac{\partial Q_\epsilon}{\partial y}\right]\beta_\epsilon(Q_\epsilon)dydx,$$

from (2.14), the integration-by-parts formula yields to

$$-\iint_\Omega \frac{\partial}{\partial y}\left[(Q_\epsilon+\bar{Q})^{-\frac{1}{3}}\frac{\partial Q_\epsilon}{\partial y}\right]\beta_\epsilon(Q_\epsilon)dydx =$$

$$\iint_\Omega (Q_\epsilon+\bar{Q})^{-\frac{1}{3}}\left|\frac{\partial Q_\epsilon}{\partial y}\right|^2\beta'_\epsilon(Q_\epsilon)dydx \geq 0.$$

Finally, we deal with

$$\iint_\Omega f(\xi,h,Q,\sigma)\beta_\epsilon(Q_\epsilon)dydx.$$

In this case we employ Young's inequality and (3.4) to obtain

$$\iint_\Omega f(\xi,h,Q,\sigma)\beta_\epsilon(Q_\epsilon)dydx \leq \frac{c_f LX}{2} + \frac{1}{2}\iint_\Omega \beta_\epsilon^2(Q_\epsilon)dydx. \tag{3.14}$$

Then, summing up,

$$\int_0^L Q_\epsilon \beta_\epsilon(Q_\epsilon) dy \Big|_X + \frac{1}{3} \iint_\Omega (Q_\epsilon + \bar{Q})^{-\frac{1}{3}} |\nabla Q_\epsilon|^2 \beta'_\epsilon(Q_\epsilon) dy dx$$

$$+ \iint_\Omega \beta_\epsilon^2(Q_\epsilon) dy dx \leq \frac{c_f L X}{2} + \frac{1}{2} \iint_\Omega \beta_\epsilon^2(Q_\epsilon) dy dx,$$

and therefore, due to the fact that $s\beta_\epsilon(s) \geq 0$ and $\beta'_\epsilon \geq 0$, we have

$$\frac{1}{2} \iint_\Omega \beta_\epsilon^2(Q_\epsilon) dy dx \leq \frac{c_f L X}{2}.$$

which finishes the proof. □

Step 4: $H^1(0, X : L^2(0, L))$ **estimates.**
We claim that there exists a positive constant k_2, such that

$$\iint_\Omega \left| \frac{\partial Q_\epsilon}{\partial x} \right|^2 dy dx \leq k_0 \tag{3.15}$$

Proof. Multiplying by $\frac{\partial Q_\epsilon}{\partial x}$ in (2.11) and integrating over Ω, we get by Young's inequality that

$$\frac{1}{3} \left\| \frac{\partial Q_\epsilon}{\partial x} \right\|^2_{L^2(\Omega)} \leq \|A(Q_\epsilon)\|^2_{L^2(\Omega)} + \|\beta_\epsilon(Q_\epsilon)\|^2_{L^2(\Omega)} + \|f(\xi, h, Q, \sigma)\|^2_{L^2(\Omega)},$$

where A is defined by

$$A(v) := -\frac{1}{3} \frac{\partial}{\partial y} \left\{ (Q + \bar{Q})^{-\frac{1}{3}} \frac{\partial Q}{\partial y} \right\}. \tag{3.16}$$

So, the conclusion follows from (3.7), (3.12) and (3.4). □

Step 5: The convergence.
Since $H^2(0, L) \hookrightarrow W^{1,\infty}(0, L)$ is a compact embedding, considering the estimates given in (3.11) and (3.15), we can assert the existence of $Q^* \in L^2(0, X : H^2(0, L))$ such that $Q_{\epsilon i} \to Q^*$ in $L^2(0, T : W^{1,\infty}(0, L))$. By (3.12) there exists a subsequence $\{\beta_{\epsilon ij}(Q_{\epsilon ij})\}$ such that $\beta_{\epsilon ij}(Q_{\epsilon ij}) \rightharpoonup u$ weakly in $L^2(\Omega)$. Applying Lemma G in Benilan et al[2], we get that $u \in \beta(Q^*)$. Next, by taking limits in the weak formulation we obtain that Q^* satisfies

$$\frac{\partial Q^*}{\partial x} + A(Q^*) + \beta(Q^*) \ni F(x, y). \tag{3.17}$$

Step 6. Uniqueness of (3.17)
We present first a technical lemma, which will also be employed in the proof of the uniqueness for (2.11) - (2.15).

Lemma 3.2 *Let* $Q_1 \in L^2(0, X : H^1(0, L))$, $Q_2 \in L^2(0, X : W^{1,\infty}(0, L))$ *satisfy (2.14) and (3.6) for* X *given by (3.5). Then there exists a function* $\mu \in L^1(0, X)$ *such that*

$$\int_0^L (A(Q_1) - A(Q_2))(Q_1 - Q_2)dy \geq \mu(x) \int_0^L (Q_1 - Q_2)^2 dy. \tag{3.18}$$

Proof. Integrating the right hand side of (3.18) over $(0, L)$, applying the integration-by-parts formula and taking (2.14) into consideration, we get

$$\int_0^L (A(Q_1) - A(Q_2))(Q_1 - Q_2)dy =$$

$$\frac{1}{3}\int_0^L \left[(Q_1 + \bar{Q})^{-\frac{1}{3}} \frac{\partial Q_1}{\partial y} - (Q_2 + \bar{Q})^{-\frac{1}{3}} \frac{\partial Q_2}{\partial y}\right] \left[\frac{\partial Q_1}{\partial y} - \frac{\partial Q_2}{\partial y}\right] dy.$$

Since

$$\int_0^L \left[(Q_1 + \bar{Q})^{-\frac{1}{3}} \frac{\partial Q_1}{\partial y} - (Q_2 + \bar{Q})^{-\frac{1}{3}} \frac{\partial Q_2}{\partial y}\right] \left[\frac{\partial Q_1}{\partial y} - \frac{\partial Q_2}{\partial y}\right] dy =$$

$$\int_0^L (Q_1 + \bar{Q})^{-\frac{1}{3}} \left[\frac{\partial Q_1}{\partial y} - \frac{\partial Q_2}{\partial y}\right]^2 + \left[(Q_1 + \bar{Q})^{-\frac{1}{3}} - (Q_2 + \bar{Q})^{-\frac{1}{3}}\right] \frac{\partial Q_2}{\partial y} \left[\frac{\partial Q_1}{\partial y} - \frac{\partial Q_2}{\partial y}\right] dy$$

$$\geq \int_0^L \left[(Q_1 + \bar{Q})^{-\frac{1}{3}} - \epsilon\right] \left[\frac{\partial Q_1}{\partial y} - \frac{\partial Q_2}{\partial y}\right]^2 dy -$$

$$C_\epsilon \int_0^L \left[(Q_1 + \bar{Q})^{-\frac{1}{3}} - (Q_2 + \bar{Q})^{-\frac{1}{3}}\right]^2 \left|\frac{\partial Q_2}{\partial y}\right|^2 dy,$$

we have, for $\epsilon \ll 1$,

$$\int_0^L (A(Q_1) - A(Q_2))(Q_1 - Q_2)dy \geq -\frac{C_\epsilon}{27} \int_0^L (\hat{Q} + \bar{Q})^{-\frac{8}{3}} (Q_1 - Q_2)^2 \left|\frac{\partial Q_2}{\partial y}\right|^2 dy$$

in view of

$$[(Q_1 + \bar{Q})^{-\frac{1}{3}} - (Q_2 + \bar{Q})^{-\frac{1}{3}}] = -\frac{1}{3}(\hat{Q} + \bar{Q})^{-\frac{4}{3}} (Q_1 - Q_2),$$

for $\hat{Q} = \lambda Q_1^* + (1 - \lambda)Q_2^*$ and $\lambda = \lambda(x, t) \in (0, 1)$. Taking into account that $Q_2 \in L^2(0, X : W^{1,\infty}(0, L)$ and that $(\hat{Q}+\bar{Q})^{\frac{-8}{3}} \leq \bar{Q}^{\frac{-8}{3}}$, we conclude that there exist a positive constant k_1 such that

$$\int_0^L (A(Q_1) - A(Q_2))(Q_1 - Q_2)dy \geq -k_1 \|Q_2\|^2_{W^{1,\infty}(\Omega)} \int_0^L (Q_1 - Q_2)^2 dy.$$

So, choosing $\mu(x) = -k_1 \|Q_2\|^2_{W^{1,\infty}(0,L)}$ the lemma is proven. □

In order to establish the uniqueness of (3.17) we argue by contradiction. Let us assume there exist two solutions Q_1^* and Q_2^* to (3.17) satisfying (2.14) and (2.15). We define $Q_* = Q_1^* - Q_2^*$ and then

$$\frac{\partial Q_*}{\partial x} + A(Q_1^*) - A(Q_2^*) + \beta(Q_1^*) - \beta(Q_2^*) \ni 0.$$

Taking Q_* as test function, by (3.18), the monotonicity of β and the fact that Q_1^* and Q_2^* verify (2.15) and hence $Q_*(0) = 0$, we obtain that

$$\frac{1}{2}\int_0^L Q_*^2 dx \bigg|_{x=X} \leq \int_0^X \mu(x) \int_0^L Q_*^2 dy dx.$$

Applying Hale[15] (Lemma 3.1 p. 15) it follows that $Q_* = 0$ and the uniqueness is then proven.

Having proved $Q_\epsilon \rightharpoonup Q$ weak in $L^2(0, X : H^2(0, L))$ (where Q is the solution to (2.11) - (2.14)) by (3.7), we have established the Proposition 3.1. □

Proof of the Theorem 2.1

As before, we argue by contradiction. Assume there exist two solutions to (2.11) - (2.15): $(Q_1, h_1, \xi_1, \sigma_1)$ and $(Q_2, h_2, \xi_2, \sigma_2)$ and consider

$$Q := Q_1 - Q_2, \quad h := h_1 - h_2, \quad \xi := \xi_1 - \xi_2 \text{ and } \sigma := \sigma_1 - \sigma_2. \tag{3.19}$$

Then σ satisfies

$$\sigma = \int_0^L (Q_1 + \bar{Q})^S dy - \int_0^L (Q_2 + \bar{Q})^S dy = S \int_\Omega (\hat{Q} + \bar{Q})^{S-1} (Q_1 - Q_2) dy, \tag{3.20}$$

where $\hat{Q} = \lambda_Q Q_1 + (1 - \lambda_Q) Q_2$, for $\lambda_Q = \lambda_Q(x, y)$ and $0 \leq \lambda_Q \leq 1$, i.e.

$$\sigma = \int_0^L g_1 Q dx, \tag{3.21}$$

with $g_1 = S(\hat{Q} + \bar{Q})^{S-1}$.

We consider the equation $\frac{\partial h_i}{\partial x} = -M^r h_i^{-1-r} \sigma_i^{-r}$, for $i = 1, 2$, and multiply by h_i^{r+1} to get

$$h_i^{r+1} \frac{\partial h_i}{\partial x} = -M^r \sigma_i^{-r} \quad \text{and} \quad \frac{\partial h_i^{2+r}}{\partial x} = -(2 + r) M^r \sigma_i^{-r}.$$

Then

$$\frac{\partial h_1^{2+r}}{\partial x} - \frac{\partial h_2^{2+r}}{\partial x} = -(2 + r) M^r (\sigma_1^{-r} - \sigma_2^{-r}) = (2 + r) r M^r \tilde{\sigma}^{-1-r} \sigma$$

for some $\tilde{\sigma} = \lambda_\sigma \sigma_1 + (1 - \lambda_\sigma) \sigma_2$ where $\lambda_\sigma = \lambda_\sigma(x)$ satisfying $0 \leq \lambda_\sigma \leq 1$. Since

$$h_1^{2+r} - h_2^{2+r} = (2 + r) \tilde{h}^{1+r} h$$

with $\tilde{h} = \lambda_h h_1 + (1 - \lambda_h) h_2$, where $\lambda_h = \lambda_h(x)$ and $0 \leq \lambda_h \leq 1$. We have

$$h = g_7(x) \int_0^x g_6(\tilde{x}) \sigma d\tilde{x}, \tag{3.22}$$

for

$$g_6 = \tilde{\sigma}^{-1-r} \quad \text{and} \quad g_7 = -M^r r \tilde{h}^{-1-r}.$$

By definition, ξ satisfies

$$\frac{\partial \xi}{\partial x} = M(Q_1+\bar{Q})^S(h_1\sigma_1)^{-1} - M(Q_2+\bar{Q})^S(h_2\sigma_2)^{-1}. \tag{3.23}$$

Adding and subtracting to (3.23) the term $(Q_2+\bar{Q})^S(h_1\sigma_1)^{-1}$, we have that

$$M\left[(Q_1+\bar{Q})^S - (Q_2+\bar{Q})^S\right](h_1\sigma_1)^{-1} + M(Q_2+\bar{Q})^S\left[(h_1\sigma_1)^{-1} - (h_2\sigma_2)^{-1}\right] =$$

$$MS(\hat{Q}+\bar{Q})^{S-1}(h_1\sigma_1)^{-1}(Q_1-Q_2) + M(Q_2+\bar{Q})^S\left[h_2\sigma_2 - h_1\sigma_1\right]\left[h_1\sigma_1 h_2\sigma_2\right]^{-1},$$

where $\hat{Q} = \lambda_Q Q_1 + (1-\lambda_Q)Q_2$, as before (see (3.20)). Since

$$h_2\sigma_2 - h_1\sigma_1 = (h_2-h_1)\sigma_2 + h_1(\sigma_2-\sigma_1),$$

we obtain

$$\frac{\partial \xi}{\partial x} = MS(\hat{Q}+\bar{Q})^{S-1}(h_1\sigma_1)^{-1}Q + M(Q_2+\bar{Q})^S(h_1\sigma_1 h_2\sigma_2)^{-1}\left[h\sigma_2 + h_1\sigma\right].$$

Therefore

$$\frac{\partial \xi}{\partial x} = g_2(x,t)Q + g_3(x,t)h + g_4(x,t)\sigma \tag{3.24}$$

and

$$\xi = \int_0^x \left[g_2(\tilde{x},y)Q + g_3(\tilde{x},y)h + g_4(\tilde{x},y)\sigma\right] d\tilde{x}, \tag{3.25}$$

where

$$g_2 = MS(\hat{Q}+\bar{Q})^{S-1}(h_1\sigma_1)^{-1}, \qquad g_3 = M\sigma_2(Q_2+\bar{Q})^S(h_1\sigma_1 h_2\sigma_2)^{-1}$$

and

$$g_4 = M(Q_2+\bar{Q})^S h_1(h_1\sigma_1 h_2\sigma_2)^{-1}.$$

Let us consider the following multivalued expression:

$$\frac{\partial Q}{\partial x} + A(Q_1) - A(Q_2) - f_1 + f_2 \in -(\beta(Q_1) - \beta(Q_2)), \tag{3.26}$$

where $f_i = f(\xi_i, h_i, Q_i, \sigma_i)$ for $i = 1, 2$.

We employ the fact that

$$f_1 - f_2 = \nabla f|_{\tilde{\xi},\tilde{h},\hat{Q},\tilde{\sigma}} \cdot (\xi, h, Q, \sigma)^t, \tag{3.27}$$

where $(\tilde{\xi}, \tilde{h}, \hat{Q}, \tilde{\sigma}) = \lambda_3(\xi_1, h_1, Q_1, \sigma_1) + (1-\lambda_3)(\xi_2, h_2, Q_2, \sigma_2)$ for some $\lambda_3 \in [0,1]$ provided that $Q_i \geq 0$, $\xi_i > 0$, $h_i > 0$, $\sigma_i > 0$.

Multiply (3.26) by Q and obtain

$$Q\frac{\partial Q}{\partial x} + \left[A(Q_1) - A(Q_2)\right]Q - \left[f_1 - f_2\right]Q \in -\left[\beta(Q_1) - \beta(Q_2)\right]Q.$$

Since β is a maximal monotone graph, we get

$$-\left[\beta(Q_1) - \beta(Q_2)\right] Q \leq 0.$$

Then, integrating over Ω, it follows that

$$\frac{1}{2}\int_0^L Q^2 dy\Big|_0^X + \iint_\Omega \left[A(Q_1) - A(Q_2)\right] Q dydx - \iint_\Omega (f_1 - f_2) Q dydx \leq 0.$$

In view of $Q_1(x,0) = Q_2(x,0)$ and (3.18), one has

$$\frac{1}{2}\int_0^L Q^2 dy\Big|_{x=X} \leq \int_0^X \mu(x)\int_0^L Q^2 dydx + \iint_\Omega (f_1 - f_2) Q dydx,$$

and therefore

$$\frac{1}{2}\int_0^L Q^2 dy\Big|_X \leq \int_0^X \mu(x)\int_0^L Q^2 dydx + \iint_\Omega Q(\nabla f|_{\tilde{\xi},\tilde{h},\hat{Q},\tilde{\sigma}} \cdot (\xi, h, Q, \sigma)^t) dydx \leq$$

$$\int_0^X \mu(x)\int_\Omega Q^2 dydx + \iint_\Omega k\left(|\xi| + |h| + |Q| + |\sigma|\right)|Q| dydx, \tag{3.28}$$

where $k = \max\{|\nabla f|\}$, for $(\xi, h, Q, \sigma) = \lambda(\xi_1, h_1, Q_1, \sigma_1) + (1-\lambda)(\xi_2, h_2, Q_2, \sigma_2)$ and $\lambda \in (0,1)$. In view of (3.25), (3.22) and (3.21) we get that

$$\iint_\Omega k\left(|\xi| + |h| + |Q| + |\sigma|\right)|Q| dydx \leq k_1 \iint_\Omega Q^2 dydx. \tag{3.29}$$

Then, by means of substituting (3.29) in (3.28) we get

$$\frac{1}{2}\int_0^L Q^2 dy\Big|_X \leq \int_0^X \mu(x)\int_0^L Q^2 dydx + k_1 \iint_\Omega Q^2 dydx, \tag{3.30}$$

and by and by Hale[15] (Lemma 3.1 p. 15), $\frac{1}{2}\int_0^L Q^2 dy \leq 0$ which proves that $Q_1 = Q_2$. Consequently $\sigma_1 = \sigma_2$ which implies $h_1 = h_2$ and substituting in (3.25) the proof concludes for X small enough. Repeating the process, as in the Remark 2.3, starting from X we obtain the uniqueness of solutions as large as the solution exists. □

4 Proof of Theorem 2.2: the ice thickness collapse

In order to prove Theorem 2.2, we consider first the equation (2.12). Then, under assumption (2.16) we deduce that h is a decreasing function. Consider now h^{2+r} which satisfies

$$\frac{\partial h^{2+r}}{\partial x} = -\frac{(2+r)M^r}{L^r \sigma^r}.$$

It is then possible to obtain the following explicit expression for the ice thickness:

$$h(x) = \left[h_0^{2+r} - \frac{(2+r)M^r}{L^r}\int_0^x \sigma^{-r} d\tilde{x}\right]^{\frac{1}{2+r}}. \tag{4.1}$$

In view of (2.13) and (2.16) we assert

$$\xi \geq \xi_0 > 0. \tag{4.2}$$

Let us consider the functions ϕ_ϵ and Φ_ϵ defined by

$$\phi_\epsilon(s) = \begin{cases} 1, & s > \epsilon, \\ \dfrac{s}{\epsilon}, & 0 \leq s \leq \epsilon, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \Phi_\epsilon(s) = \begin{cases} s - \dfrac{\epsilon}{2}, & s > \epsilon, \\ \dfrac{s^2}{2\epsilon}, & 0 \leq s \leq \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\Phi'_\epsilon = \phi_\epsilon$. Taking $\phi_\epsilon(Q)$ as test function in (2.11) we get, thanks to (2.14),

$$\frac{\partial}{\partial x}\int_0^L \Phi_\epsilon(Q)dy \leq \int_0^L f(\xi, h, Q, \sigma)\phi_\epsilon(Q)dy.$$

In view of (4.2), we have

$$f(\xi, h, Q, \sigma) \leq \frac{M^{1+r}}{(h\sigma)^{1+r}}(Q + \bar{Q})^S + \gamma,$$

and, consequently,

$$f(\xi, h, Q, \sigma)\phi_\epsilon(Q) \leq \frac{M^{1+r}}{(h\sigma)^{1+r}}(Q + \bar{Q})^S + \gamma.$$

Next, integrating over $(0, L)$ the above expression, it follows

$$\int_0^L f(\xi, h, Q, \sigma)\phi_\epsilon(Q)dy \leq \int_0^L \frac{M^{1+r}}{(h\sigma)^{1+r}}(Q + \bar{Q})^S dy + \gamma L =$$

$$\frac{M^{1+r}}{(h\sigma)^{1+r}}\int_0^L (Q + \bar{Q})^S dy + \gamma L = \frac{M^{1+r}}{h^{1+r}\sigma^r} + \gamma L$$

and then

$$\frac{\partial}{\partial x}\int_0^L \Phi_\epsilon(Q)dy \leq \frac{M^{1+r}}{h^{1+r}\sigma^r} + \gamma L.$$

Let us define $Q_\epsilon = \int_0^L \Phi_\epsilon(Q)dy$, which satisfies

$$\frac{\partial Q_\epsilon}{\partial x} \leq \frac{M^{1+r}}{h^{1+r}\sigma^r} + \gamma L. \tag{4.3}$$

Consider now $\bar{Q}_\epsilon = Q_\epsilon(0) + M(h_0 - h) + \gamma L t$, then we have

$$\frac{\partial \bar{Q}_\epsilon}{\partial x} = -M\frac{\partial h}{\partial x} + \gamma L = \frac{M^{1+r}}{h^{1+r}\sigma^r} + \gamma L \geq \frac{\partial Q_\epsilon}{\partial x}.$$

Consequently, since $\bar{Q}_\epsilon(0) = Q_\epsilon(0)$, we obtain $\bar{Q}_\epsilon \geq Q_\epsilon$. Taking limits as $\epsilon \to 0$, we conclude

$$\int_0^L Q dy \leq Q_0 + M(h_0 - h) + \gamma L x,$$

and then

$$\int_0^L (Q + \bar{Q}) dy \leq Q_0 + M(h_0 - h) + \bar{Q}L + \gamma L x \leq Q_0 + M h_0 + \bar{Q}L + \gamma L x.$$

Since $0 < S \leq 1$, $0 < \bar{Q}$ and $Q \geq 0$ we get

$$(Q + \bar{Q})^S = (Q + \bar{Q})(Q + \bar{Q})^{S-1} \leq \bar{Q}^{S-1}(Q + \bar{Q}),$$

and consequently

$$\sigma = \int_0^L (Q + \bar{Q})^S dy \leq \bar{Q}^{S-1} \int_0^L (Q + \bar{Q}) dy \leq \bar{Q}^{S-1}(Q_0 + M h_0 + \bar{Q}L + \gamma L x). \quad (4.4)$$

Substituting (4.4) into (4.1) shows

$$h(x) \leq \left[h_0^{2+r} - (2+r)\frac{M^r}{L^r} \int_0^x \bar{Q}^{(1-S)r}(Q_0 + M h_0 + \bar{Q}L + \gamma L \tilde{x})^{-r} d\tilde{x} \right]^{\frac{1}{2+r}}.$$

For simplicity, let us introduce the positive constants

$$A := \frac{(2+r)(1-r)}{\gamma L} \frac{M^r \bar{Q}^{(1-S)r}}{L^r}, \quad B := Q_0 + M h_0 + \bar{Q}L \quad \text{and} \quad C := \gamma L,$$

(notice that $0 < r < 1$), then

$$h(x) \leq \left[h_0^{2+r} - A\left((B + Cx)^{1-r} - B^{1-r} \right) \right]^{\frac{1}{2+r}}. \quad (4.5)$$

Now, choosing $X_c = ((h_0^{2+r} A^{-1} + B^{1-r})^{\frac{1}{1-r}} - B) C^{-1}$ we get the desired result. □

Remark 4.1 *The special case where the variables do not depend on x, i.e.,*

$$Q(x, y) = Q(x), \quad \xi(x, y) = \xi(x), \qquad y \in \Omega = (0, L), \quad (4.6)$$

is considerable simpler. In fact, the system (2.11)-(2.14) reduces to the following coupled system of ODE:

$$\frac{\partial Q}{\partial x} = \frac{M}{h\sigma}(Q + \bar{Q})^S \left(M^r h^{-r} \sigma^{-r} - \xi^{-\frac{1}{2}} \right) + \gamma - \delta h^{-1}, \quad \textit{in } (0, X), \quad (4.7)$$

$$\frac{\partial h}{\partial x} = -M^r h^{-1-r} \sigma^{-r}, \quad \textit{in } (0, X), \quad (4.8)$$

$$\frac{\partial \xi}{\partial x} = \frac{M}{h\sigma}(Q + \bar{Q})^S, \quad \textit{in } (0, X), \quad (4.9)$$

$$Q(0) = Q_0, \quad h(0) = h_0, \quad \textit{and} \quad \xi(0) = \xi_0. \quad (4.10)$$

Notice that in this case $\sigma = L(Q+\bar{Q})^S$*. So, if we substitute this new expression of* σ *into the above system of equations, it follows*

$$\frac{\partial Q}{\partial x} = \frac{M}{hL}\left(M^r h^{-r} L^{-r}(Q+\bar{Q})^{-Sr} - \xi^{-\frac{1}{2}}\right) + \gamma - \delta h^{-1}, \quad \textit{in } (0,X), \tag{4.11}$$

$$\frac{\partial h}{\partial x} = -M^r L^{-r} h^{-1-r}(Q+\bar{Q})^{-Sr}, \quad \textit{in } (0,X), \tag{4.12}$$

$$\frac{\partial \xi}{\partial x} = \frac{M}{hL}, \quad \textit{in } (0,X). \tag{4.13}$$

Now, we will consider the function $\bar{q}$*, defined by*

$$\bar{q}(x) = ML^{-1}(h_0 - h(x)) + Q_0 + \gamma x.$$

By construction $\frac{\partial \bar{q}}{\partial x} \geq \frac{\partial Q}{\partial x}$ *and we obtain that*

$$Q \leq \bar{q} \leq ML^{-1}h_0 + Q_0 + \gamma x.$$

Substituting this into (4.1) we get (4.5) for

$$A := \frac{(2+r)(1-r)}{\gamma}\frac{M^r}{L^r}, \quad B := Q_0 + Mh_0 + \bar{Q} \quad \textit{and} \quad C := \gamma.$$

5 Conclusion

In this paper we consider an ice-streaming model related to a model originally proposed by Fowler et al[13] and later modified by Muñoz[16] and Díaz et al[8]. We have proved the collapse of the ice thickness in the main flow direction, i.e. for any initial bounded water flow and any initial ice thickness there exists a $X < \infty$ such that the ice becomes extinct before X. We also consider the case where the initial data are constant in the cross stream direction, and the equations reduce to a system of ODE.

The existence of bounded solutions was proven in Muñoz[16] and Díaz et al[8], but the question of uniqueness was left unresolved. In this work we have also established the uniqueness of solutions to the model. Different techniques have been employed in order to get suitable regularity properties that lead to assert the uniqueness of solution.

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