School Choice with Transferable Student Characteristics

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1 Introduction

The school choice problem studies the mechanisms employed by many school districts to assign students to public schools (see Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). It considers a set of students, a set of schools, and the schools' quotas, which represent the capacity of each school. Each student submits a list of preferences to a central placement authority such as a school district, and each school has a priority ranking that determines who receives a seat in the event that a school is over demanded. The school district decides which students attend each school using an algorithm that selects a matching of students to schools considering the students' reported preferences and the schools' priorities. A major concern regarding the design of school choice programs has been the ability to fairly match students to schools. A matching is fair if all students who obtain a seat at a given school have a higher priority at that school than the students who preferred that school rather than the one they are matched to, and therefore no student has justified envy. In recent years, a vast majority of school districts have implemented school choice algorithms based on Gale and Shapley's deferred acceptance algorithm (DA) (see Gale and Shapley, 1962; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005; Pathak, 2016). The application of the student-proposing DA to prospective students always results in a stable matching, that is, a fair, individually rational, and nonwasteful matching.[1] However, the matching can be Pareto dominated by another matching that does not respect school priorities.

In this paper, we add structure to the definition of school priorities. In the canonical school choice problem, priorities are a primitive aspect of the model, but school districts use several criteria to determine priority orders for schools, such as different characteristics of potential students or tie-breaker lotteries (see Abdulkadiroğlu et al., 2005). Our model relies on student characteristics as primitives. Students are endowed with characteristics specific to individual schools.[2] Each school ranks students according to priorities defined over individual characteristics of each student and the specific transferable characteristic of the student at that school. Students can exchange the characteristics of different schools and thus affect their positions in the priority rankings of those schools. In this context, a matching that Pareto improves the initial matching but may not respect fairness may become fair after an exchange of the relevant characteristics among students. We explore the efficiency gains with respect to arbitrary initial assignments of students to schools that can be justified under schools' priority rankings after the exchanges of characteristics

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[1] A matching is individually rational if no student is assigned to a school that she would rather not attend. A matching is nonwasteful if every school that a student prefers to the school she is assigned to has filled all its available seats.

[2] A similar formulation was independently proposed by Duddy (2019) that discusses the informational shortcomings of the current priority-based model and proposes a formulation based on a "priority matrix".

among the students involved in the exchange of seats (fair Pareto improvements).

We propose a class of school choice algorithms, namely, the student exchange with transferable characteristics (SETC) algorithms. Each algorithm in this class proposes a sequence of fair Pareto improvements of an initial extended matching, that is, a matching and an allocation of transferable characteristics. If the initial extended matching is individually rational and nonwasteful, then each algorithm in the class stops at an extended matching such that no further Pareto improvement can be obtained without generating some instance of justified envy (Theorem 1). If the initial matching is stable, then every SETC algorithm obtains a constrained efficient extended matching. Under a neutrality condition on the structure of school priorities, for every extended matching that can be obtained as a sequence of fair Pareto improvements of an initially stable extended matching, there is an instance of an SETC that selects essentially the same extended matching at some step of the application of the algorithm (Theorem 2).

We can think of several situations where the transferability of student characteristics can improve an allocation by solving a market design problem. For example, the lottery number can be a natural transferable characteristic when schools use different tie-breaking lotteries.[3] In fact, this issue was at the heart of the reform of the high school assignment procedure in Amsterdam in 2014. Under this reform, a system based on immediate acceptance with multiple tie-breaking and the possibility of trading assignments was replaced with a system based on DA with multiple tie-breaking without the possibility of trading assignments. In 2015, the new allocation procedure was unsuccessfully challenged in court by families who wished to exchange school seats. Our framework allows us to design a procedure with the lottery number as the unique exchangeable characteristic. This allows Pareto improvements respecting all but the priorities based on the tie-breaking lottery.[4]

Additionally, our model can be useful in situations of walk zone redistricting (see Dur et al., 2018; Casalmiglia et al., 2020). For example, in Madrid where the walk zone priority was abolished, generating winners and losers, we could design an allocation procedure that respects the walk zone priority but allows the exchange of the walk zone characteristic generating only Pareto improvements (see Górtazar et al., 2020).

In general, our approach can be useful for improving efficiency in situations where we can differentiate between allocative criteria (such as tie-breaking lotteries) and fairness constraints (such as the need for siblings to attend the same school) in the formation of

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[3] The use of multiple tie-breaking criteria can be justified, since it reduces the chances that over demanded schools will systematically reject a student who has a bad lottery draw (see Arnosti, 2016).

[4] For further reference, see Ashlagi et al. (2019); Ruijs and Oosterbeek (2019), and https://www.nemokennislink.nl/publicaties/schoolstrijd-in-amsterdam/(Schoolstrijd in Amsterdam) (Arnout Jaspers, Kennislink, July 1, 2015, accessed July 31, 2022).

priorities. An example could be the integration of transplantation programs.[5] Different programs might have different priorities or different criteria on the uses of desensitization or resorting to cadaveric donation. Therefore, a patient might receive a different treatment in different programs depending on her characteristics. The interaction between different programs might mean that a member will lose in favor of an outsider. A within-program assignment followed by an exchange of characteristics where the transplantation program is the only transferable characteristic allows for Pareto-improving reassignment of patients seeking treatment.

Finally, the algorithms in the SETC class allow Pareto-improving exchanges in any extended matching that is individually rational and nonwasteful. As the initial allocation does not have to be stable, a SETC algorithm can be used as post-allocation scramble reducing instances of justified envy.

1.1 Related Literature

The school choice problem was first presented by Balinski and Sönmez (1999). This paper introduces the idea of fairness into the context of allocating school seats to students. Abdulkadiroğlu and Sönmez (2003) analyzes this problem from a mechanism design perspective. These authors show that a student-proposing DA algorithm always selects a stable matches and is strategy-proof.[6] They also study an adaptation of Gale's trop trading cycle mechanism (TTCM) by Shapley and Scarf (1974) and show that it always selects Pareto efficient matchings and is strategy-proof. Unfortunately, stable matchings are not efficient and can have severe levels of inefficiency (see Abdulkadiroğlu et al., 2009; Kesten, 2010; Dur and Morrill, 2017).

There have been attempts to alleviate the tradeoff between stability and efficiency by weakening the notion of fairness. Kesten (2010) proposes the efficiency adjusted deferred acceptance algorithm (EADA) that finds a constrained efficient matching by incorporating the possibility that students may consent to renounce their priorities in relation to schools where they cannot obtain a seat according to the student-proposing DA algorithm.[7] Alcalde and Romero-Medina (2017) proposes an alternative weakening of fairness dubbed α-equitability. Ehlers and Morrill (2020) relaxes the fairness constraint and proposes a stable set of legal matchings that are not dominated in terms of fairness by any other legal matching. In the same spirit, Alva and Manjunath (2019) presents the concept of stable domination, Troyan et al. (2020) proposes the concept of essentially stable, and Tang and Zhang (2021) considers the concept of weak stability.

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[5] See Van der Spiegel et al. (2020) for details on the integration of national transplant programs in the European Union.

[6] A mechanism is strategy-proof if students have incentives to report their true preferences.

[7] See also Tang and Yu (2014).

A different approach is comparing the instances of justified envy generated by different mechanisms. In that sense Hakimov and Kesten (2018) proposes a Pareto efficient and strategy-proof mechanism that eliminates justified envy due to pairwise exchanges. Abdulkadiroğlu et al. (2020)) shows that TTC minimizes justified envy among all Pareto efficient and strategy-proof mechanisms in one-to-one matching. Doğan and Ehlers (2021) investigates efficient and minimally unstable Pareto improvements over the DA mechanism. Finally, Doğan and Ehlers (2022) formulates methods to compare assignments in terms of their stability in the context of priority-based allocation of objects.

With the same objective of alleviating the tradeoff between stability and efficiency, other papers explore the interaction between agents' characteristics and the solution concept of the allocation problem. For example, Klaus and Klijn (2021) presents a classical school choice problem with access rights. In general, the minimal-access rights as (siblings walk zone, etc.) are incorporated into priorities giving the students with minimal access higher priority over non-minimal-access students. Klaus and Klijn (2021) weakens stability to minimal-access stability, a concept that guarantees access to at most one school that guarantees the student minimal-access right. Additionally, Combe (2022) studies the idea of matching with ownership in situations where ownership of an object restricts the objections of agents who are not owners. In this setting, Combe (2022) defines a notion of stability with two different ownership structures and shows that stable matchings exist in both cases.

The closest paper to ours is Dur et al. (2019), which proposes an alternative weakening of stability called partial stability. Under partial stability, certain priorities of certain students at certain schools are ignored. Then, the welfare gains can be captured by applying the improvement cycles approach proposed by Erdil and Ergin (2008) for school choice problems with weak priorities and arbitrary tie-breakers. Kitahara and Okumura (2021) uses a modification of the stable improvement cycles algorithm introduced by Erdil and Ergin (2008) and considers a school choice problem where the student priorities for schools are represented by partial orders.

Similar to Dur et al. (2019) our paper uses improvement cycles. However, beyond this point, the two papers have considerable differences. First, the primitives in our model are not school priorities but the individual student characteristics on which those priorities are based. Second, in our case, the resulting extended matching is an allocation of both school seats and student characteristics. Third, the possible welfare gains that we capture are derived from exchanges of characteristics. That is, the resulting extended matching of our model is justified by the final allocation of transferable characteristics. Fourth, the SETC algorithms consider exchanges of characteristics, and contrary to the stable improvement cycle algorithm in Erdil and Ergin (2008), some of the students who participate in these cycles only exchange characteristics and facilitate other exchanges, and they are weakly better off. Finally, there is a technical difference. Our framework does not require additional conditions on the set of priorities that may be ignored as in Dur et al. (2019).[8] Our results only require that school priorities are complete and neutral (monotonic) in terms of student characteristics.

The remainder of the paper is organized as follows. In Section 2, we introduce the model and notation utilized. In Section 3, we present our main results. In Section 4, we consider incentive issues and relate our framework of transferable characteristics to that of school choice with consent proposed by Kesten (2010). In Section 5, we conclude this paper. In Section 6, we provide the proofs of our work.

2 Notation and Definitions

We present the elements of the canonical school choice problem and introduce a school choice problem with school priorities depending on transferable characteristics.

Let I be a finite set of students and S be a finite set of schools to which the students must be allocated. Each student i has a strict preference P_i over $S \cup \{\varnothing\}$, where a strict preference is a complete, transitive, and antisymmetric binary relation, and $\{\varnothing\}$ refers to the option of being unassigned. We use R_i to signify the weak preference relation associated with P_i, which is defined in the standard way. Each school s has a quota q_s of available seats ($q_s \in \mathbb{N}$).

A ***matching*** is a function $\mu : I \rightarrow S \cup \{\varnothing\}$ such that

i) for each $i \in I$, $\mu(i) \in S \cup \{\varnothing\}$.

ii) for each $s \in S$, $\#\{i \in I : \mu(i) = s\} \leq q_s$.[9]

Abusing notation, for each $s \in S$, we write $\mu^{-1}(s) = \{i \in I : \mu(i) = s\}$ and represent arbitrary matchings by the list of student-school pairs:

$$\mu = [(i, \mu(i)]_{i \in I} \, .$$

A matching μ' ***Pareto dominates*** the matching μ if for each $i \in I$, $\mu'(i) \; R_i \; \mu(i)$; and for some $j \in I$, $\mu'(j) \; P_j \; \mu(j)$. The matching μ' ***weakly Pareto dominates*** the matching μ if for each $i \in I$, $\mu'(i) \; R_i \; \mu(i)$

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[8] See Assumption 1 in Dur et al. (2019).

[9] For any set A, $\#A$ stands for the cardinality of the set A.

The last component of the canonical school choice problem is a profile of school priorities. Each school ranks its prospective students according to a priority ranking. Our contribution is to explore the structures of such priority rankings. We consider that school priorities may depend on different student characteristics. Some of these characteristics are intrinsic to individual students, but others can be exchanged among students. The relevant priorities for schools depend on the allocation of such characteristics.

For each student i, let $\omega(i) = (\omega^s(i))_{s\in S}$ be the initial endowment vector of the transferable characteristics that influence the position of student i at each school s. For each school s, let $\Omega^s = \cup_{i\in I}\omega^s(i)$. A permutation of the transferable characteristics for school s, $\lambda^s : I \to \Omega^s$, is a bijection from I to Ω^s,[10] and $\lambda^s(i)$ is the transferable characteristic of i at school s. We call $\lambda = (\lambda^s)_{s\in S}$ an ***allocation of transferable characteristics***. For each student i and each allocation λ, $\lambda(i) = (\lambda^s(i))_{s\in S}$. We denote by ω the initial allocation of transferable characteristics. Finally, for each allocation of transferable characteristics λ and each set of students $N \subseteq I$, $\lambda\,|_N$ is the restriction of λ to the students in N.

When characteristics are transferable, their allocation is relevant to define school priorities. An ***extended matching*** is a pair (μ, λ) where μ is a matching and λ is an allocation of transferable characteristics. We say that the extended matching (μ, λ) Pareto dominates the extended matching (μ', λ') if μ Pareto dominates μ'.

In a school choice problem with transferable characteristics, school priorities rank combinations of students and transferable characteristics that students present to the school choice process. Hence, school s's priority is a complete, transitive, and antisymmetric binary relation $\succ_s$ over $I \times \Omega^s$. We use the notation $\succsim_s$ to refer to the weak priority relation associated with $\succ_s$ defined in the usual way.

Neutral Priorities For each $i, j \in I$, $s \in S$, and each $l, l' \in \Omega^s$, $(i, l) \succ_s (i, l')$ if and only if $(j, l) \succ_s (j, l')$.

Under neutral priorities, for each s, the set Ω^s is naturally ordered; for each $L \subseteq \Omega^s$, we can define

$$\max\{L\} = \{l \in L : \text{ for each } i \in I, \text{ for each } l' \in L, (i, l) \succsim_s (i, l')\}.$$

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[10] For each $i \in I$ and $s \in S$, there is $j \in I$ with $\lambda^s(i) = \omega^s(j)$, and for each $j, j' \in I$ such that $j \neq j'$, $\lambda^s(j) \neq \lambda^s(j')$.

Throughout this paper, we assume that for each school s the set Ω^s is an ordered set and transferable characteristics affect all students in a neutral way. This assumption seems natural in most applications, for instance, if we assume that the transferable characteristic corresponds to tie-breaker lotteries.

A school choice problem with transferable characteristics is a 6-tuple defined by the set of students, the set of schools, the number of seats available at each school, student preferences over schools, the initial allocation of transferable characteristics, and school priorities:

$$(I, S, (q_s)_{s\in S}, (R_i)_{i\in I}, \omega, (\succ_s)_{s\in S}) .$$

We now present a stability notion for extended matchings. This notion reflects the idea that stable extended matchings should not generate rightful complaints according to school priorities from students who would like to change the school they are assigned to.

Given an extended matching (μ, λ), student i has ***justified envy of student*** j if $\mu(j)\ P_i\ \mu(i)$ and $(i, \lambda^{\mu(j)}(i)) \succ_{\mu(j)} (j, \lambda^{\mu(j)}(j))$.

An extended matching (μ, λ) is ***(ex-post) stable*** if it is

i) ***fair***: no student has justified envy of any other student,

ii) ***individually rational***: for each $i \in I$, $\mu(i)\ R_i\ \{\varnothing\}$,

iii) ***nonwasteful***: for no $i \in N$ or $s \in S$, $s\ P_i\ \mu(i)$ and $\#\{i \in I :\ \mu(i) = s\} = \mu^{-1}(s) < q_s$.

If an extended matching is fair, no student has more rights (according to school priorities) to attend a specific school that she prefers to the school she is assigned to than some other student assigned to that specific school. If an extended matching is individually rational, no student has incentives to leave the enrollment system since she prefers the remaining unassigned option. Finally, if an extended matching is nonwasteful, no student prefers to be reassigned to a school that has not fulfilled its quota of available seats.

We analyze the possibility of finding improvements of an initial extended matching both in terms of Pareto efficiency and ex-post stability. Starting from an arbitrary initial extended matching, (μ, ω), we seek new extended matchings that Pareto dominate the initial extended matching and that solve and do not generate new instances of justified envy by exchanging transferable characteristics among the students assigned to a new school under the new extended matching.

The extended matching $(\mu', \bar{\lambda})$ is a ***fair Pareto improvement of*** (μ, λ) if

i) μ' Pareto dominates μ,

ii) for each i such that $(\mu'(i), \bar{\lambda}(i)) \neq (\mu(i), \lambda(i))$ there is no j with justified envy of i at $(\mu', \bar{\lambda})$,

iii) for each $i \in I$ and for each $s \notin \{\mu(i), \mu'(i)\}$, $\lambda^s(i) = \bar{\lambda}^s(i)$.

By items i) and ii), we require a fair Pareto improvement to generate a new matching that all students consider at least as good as the initial matching (with some strict preference) such that all the changes in the matching and/or the allocation of transferable characteristics do not generate instances of justified envy. By item iii), we consider new extended matchings such that the reallocation of transferable characteristics is restricted to students and schools involved in the new assignment of students to schools. Specifically, we exclude the possibility of a student abandoning her transferable characteristic at some school she is not assigned to foster the priority of another student at that school.

In the following definition, we sequentially apply the notion of fair Pareto improvement and consider the possibility of obtaining extended matchings resulting from a finite sequence of fair Pareto improvements.

The extended matching (μ', λ') is a ***justifiable Pareto improvement*** of (μ, λ) if there is a sequence of extended matchings $\{(\mu_0, \lambda_0), (\mu_1, \lambda_1), \ldots, (\mu_n, \lambda_n)\}$ with $(\mu_0, \lambda_0) = (\mu, \lambda)$, $(\mu_n, \lambda_n) = (\mu', \lambda')$, such that for each $t \in \{1, \ldots, n\}$, (μ_t, λ_t) is a fair Pareto improvement of $(\mu_{t-1}, \lambda_{t-1})$.

When the initial transferable characteristics cannot be reallocated, the student-proposing DA algorithm selects a matching that together with the initial allocation of transferable characteristics is (ex-post) stable. For each allocation of transferable characteristics λ, we define μ_λ^{SO} as the matching obtained by the student-proposing DA algorithm for λ. We call $(\mu_\omega^{SO}, \omega)$ the ***student optimal stable extended matching (SOSEM)***. Gale and Shapley (1962) proves that the matching selected by the student-proposing DA Pareto dominates all other fair, individually rational, and nonwasteful matchings under the initial allocation of transferable characteristics ω. However, it is possible to find alternative matchings that Pareto dominate the SOSEM by generating instances of justified envy.

Our main interest is to explore the possibility of exhausting the generation of fair Pareto improvements. That is, given an initial extended matching, we seek extended matching that improve upon the initial extended matching both in terms of efficiency and stability and are justified by bidirectional exchanges of positions and transferable characteristics and such that further Pareto improvements necessarily generate justified envy. We focus on finding fair Pareto improvements of individually rational and nonwasteful extended matchings and specifically for (ex-post) stable extended matchings and the *SOSEM*, that is, (ex-post) stable extended matchings that are not Pareto dominated by other (ex-post) stable extended matchings, and therefore any Pareto improvements will imply a violation of fairness. This notion is captured with the following definition.

An extended matching (μ, λ) is ***constrained efficient*** if it is (ex-post) stable and there is no fair Pareto improvement (μ', λ') of (μ, λ).

2.1 Examples

We start the analysis by providing some examples that show the possibility of finding fair Pareto improvements for the solutions to the classical school choice problem.

Example 1 provides an instance of a fair Pareto improvement above the SOSEM (and a constrained efficient extended matching) by the direct swap of transferable characteristics at different schools between two students.

Example 1. *Let* $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, *and* $q_{s_x} = 1$ *for* $x = 1, 2, 3$. *Student preferences over schools are:*

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3
$\{\varnothing\}$	$\{\varnothing\}$	$\{\varnothing\}$

School priorities regard nontransferable characteristics intrinsic to each student and students' tie-breaker lotteries as transferable characteristics. Schools' s_1 and s_2 priorities are completely determined by the tie-breaking lottery. Student i_1 has the highest priority for s_1, student i_2 has the highest priority for s_2, while student i_3 has the second-highest priority in both schools. Hence, the initial allocation of transferable characteristics and the relevant school priorities are:[11]

$\omega^s(i)$	i_1	i_2	i_3
s_1	$\mathbf{2}^{s_1}$	$\mathbf{0}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{0}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$
s_3	$\mathbf{0}^{s_3}$	$\mathbf{1}^{s_3}$	$\mathbf{2}^{s_3}$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, \mathbf{2}^{s_1})$	$(i_2, \mathbf{2}^{s_2})$	$(i_3, \cdot)$
$(i_2, \mathbf{2}^{s_1})$	$(i_1, \mathbf{2}^{s_2})$	$\dots$
$(i_3, \mathbf{1}^{s_1})$	$(i_3, \mathbf{1}^{s_2})$	
$(i_2, \mathbf{0}^{s_1})$	$(i_1, \mathbf{0}^{s_2})$	

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[11] We do not present the complete school priority orders over students and transferable characteristics pairs but only the relevant comparisons.

Note that μ_ω^{SO} is defined by

$$\mu_\omega^{SO} = \left[(i_1, s_1), (i_2, s_2), (i_3, s_3)\right].$$

The extended matching $(\mu_\omega^{SO}, \omega)$ is (ex-post) stable but μ_ω^{SO} is Pareto dominated by the matching μ' such that:

$$\mu' = \left[(i_1, s_2), (i_2, s_1), (i_3, s_3)\right].$$

However, since $(i_3, \mathbf{1}^{s_1}) \succ_{s_1} (i_2, \mathbf{0}^{s_1})$, i_3 has justified envy of i_1, and (μ', ω) is not a fair extended matching.

When students i_1 and i_2 swap their transferable characteristics at schools s_1 and s_2, we obtain an allocation of transferable characteristics λ such that : $\lambda^{s_1}(i_1) = \omega^{s_1}(i_2)$, $\lambda^{s_1}(i_2) = \omega^{s_1}(i_1)$, $\lambda^{s_2}(i_1) = \omega^{s_2}(i_2)$, $\lambda^{s_2}(i_2) = \omega^{s_2}(i_1)$, and $\lambda^{s_3} = \omega^{s_3}$.

$\lambda^s(i)$	i_1	i_2	i_3
s_1	$\mathbf{0}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{2}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{1}^{s_2}$
s_3	$\mathbf{0}^{s_3}$	$\mathbf{1}^{s_3}$	$\mathbf{2}^{s_3}$

Note that $\mu' = \mu_\lambda^{SO}$. The extended matching (μ', λ) is (ex-post) stable and a fair Pareto improvement for the SOSEM $(\mu_\omega^{SO}, \omega)$. Since μ_λ^{SO} Pareto dominates μ_ω^{SO}, $(\mu_\omega^{SO}, \omega)$ is not constrained efficient. Since there is no matching μ'' that Pareto dominates μ', and (μ', λ) is (ex-post) stable, (μ', λ) is constrained efficient.

Example 2 presents the constraint that our focus on fair Pareto improvements introduces into our framework.

Example 2. *Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $q_{s_x} = 1$ for $x = 1, 2, 3, 4$. Student preferences over schools that are at least as good as the remaining unassigned option are:*

P_{i_1}	P_{i_2}	P_{i_3}	P_4
s_2	s_1	s_1	s_4
s_1	s_2	s_2	$\{\varnothing\}$
s_3	s_3	s_3	
$\{\varnothing\}$	$\{\varnothing\}$	$\{\varnothing\}$	

The relevant initial allocation of transferable characteristics for schools ω and the relevant school priorities are:

$\omega^s(i)$	i_1	i_2	i_3	i_4
s_1	$\mathbf{0}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{3}^{s_1}$
s_2	$\mathbf{1}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{3}^{s_2}$
s_3	$\ldots$	$\ldots$	$\ldots$	$\ldots$
s_4	$\ldots$	$\ldots$	$\ldots$	$\ldots$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
$(i_1, \mathbf{0}^{s_1})$	$(i_2, \mathbf{0}^{s_2})$	$(i_3, \cdot)$	$(i_4, \cdot)$
$(i_4, \mathbf{3}^{s_1})$	$(i_4, \mathbf{3}^{s_2})$	$\ldots$	$\ldots$
$(i_2, \mathbf{3}^{s_1})$	$(i_1, \mathbf{3}^{s_2})$		
$(i_3, \mathbf{2}^{s_1})$	$(i_3, \mathbf{2}^{s_2})$		
$(i_2, \mathbf{1}^{s_1})$	$(i_1, \mathbf{1}^{s_2})$		

This school choice problem modifies the problem in Example 1 by adding a student who has the highest lottery tickets for schools s_1 and s_2, and school priorities in such a way that i_1 and i_2 initially have the highest priorities at schools s_2 and s_1 respectively, but they do not have valuable lottery tickets to exchange.

The matching μ_ω^{SO} is defined by

$$\mu_\omega^{SO} = [(i_1, s_1,), (i_2, s_2), (i_3, s_3), (i_4, s_4)] .$$

Note that μ_ω^{SO} is Pareto dominated by

$$\mu' = [(i_1, s_2), (i_2, s_1)(i_3, s_3), (i_4, s_4)] .$$

Consider the allocation of transferable characteristics $\bar{\lambda}$ such that $\bar{\lambda}^s = \omega^s$ for $s \in \{s_3, s_4\}$ and

$\bar{\lambda}^s(i)$	i_1	i_2	i_3	i_4
s_1	$\mathbf{0}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$
s_3	$\ldots$	$\ldots$	$\ldots$	$\ldots$
s_4	$\ldots$	$\ldots$	$\ldots$	$\ldots$

According to the allocation $\bar{\lambda}$ students i_1 and i_2 obtain i_4's transferable characteristics at schools s_1 and s_2 respectively, although student i_4 is assigned to a position in s_4. Hence, $(\mu', \bar{\lambda})$ is not a fair Pareto improvement of $(\mu_\omega^{SO}, \omega)$. It turns out that $(\mu_\omega^{SO}, \omega)$ is constrained efficient.

Example 3 shows the possibility of different (incompatible) fair Pareto improvements over an initial extended matching, as well as the possibility of Pareto improvements where students willing to exchange positions need the transferable characteristic of a third student.

Example 3. *Let $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, for all $s \in S \setminus \{s_2\}$, $q_s = 1$, and $q_{s_2} = 2$. Student preferences over schools that are at least as good as the remaining unassigned option are:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_1	s_1	s_1	s_5	s_5
s_4	s_2	s_2	s_4	s_2	s_2
s_1	s_3	s_4	$\{\varnothing\}$	$\{\varnothing\}$	$\{\varnothing\}$
s_3	$\{\varnothing\}$	s_3			
$\{\varnothing\}$		$\{\varnothing\}$			

The relevant initial allocation of transferable characteristics and the relevant school priorities are:[12]

$\omega^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{5}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{0}^{s_1}$
s_2	$\mathbf{2}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{5}^{s_2}$
s_3	...	...	...	...	...	...
s_4	$\mathbf{5}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{1}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	...	...	...	...	...	...

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succsim_{s_5}$
$(i_1, \mathbf{5}^{s_1})$	$(i_5, \mathbf{0}^{s_2})$	$(i_3, \cdot)$	$(i_4, \mathbf{3}^{s_4})$	$(i_5, \cdot)$
$(i_2, \mathbf{5}^{s_1})$	$(i_6, \mathbf{5}^{s_2})$		$(i_1, \mathbf{5}^{s_4})$	$(i_6, \cdot)$
$(i_4, \mathbf{5}^{s_1})$	$(i_6, \mathbf{3}^{s_2})$		$(i_3, \mathbf{4}^{s_4})$	
$(i_3, \mathbf{4}^{s_1})$	$(i_2, \mathbf{3}^{s_2})$		$(i_1, \mathbf{3}^{s_4})$	
$(i_2, \mathbf{3}^{s_1})$	$(i_1, \mathbf{5}^{s_2})$			
$(i_4, \mathbf{2}^{s_1})$	$(i_3, \mathbf{4}^{s_2})$			
	$(i_1, \mathbf{3}^{s_2})$			

The SOSEM is defined by the matching:

$$\mu_\omega^{SO} = [(i_1, s_1), ((i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_5), (i_6, s_2)],$$

and the extended matching $(\mu_\omega^{SO}, \omega)$ *is (ex-post) stable. The matching* μ_ω^{SO} *is Pareto dominated by two alternative matchings* μ' *and* μ''*:*

$$\mu' = [(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_5), (i_6, s_2)],$$
$$\mu'' = [(i_1, s_4), (i_2, s_2), (i_3, s_3), (i_4, s_1), (i_5, s_5), (i_6, s_2)].$$

However, the extended matchings (μ', ω) *and* (μ'', ω) *are not (ex-post) stable. Note that at* (μ', ω)*,* i_2 *has justified envy of* i_1 *at school* s_2*. At* (μ'', ω)*,* i_2 *has justified envy of* i_4 *at school* s_1*.*

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[12] It is worth to note that school priorities in Example 3 are consistent with the interpretation of priorities based on weak orders over students and transferable characteristics as tie-breaking lotteries. Actually, we can interpret that student i_1 always has the highest priority at school s_1, and the s_1 ranking of students i_2, i_3, and i_4 depends on the respective transferable characteristics (with an irrelevant arbitrary criterion to define complete and strict priorities over all pairs of students and transferable characteristics). Similarly, student i_5 always has the highest priority at school s_2, student i_6 always has the second-highest priority, and i_2 the third-highest priority at school s_2, and the transferable characteristic determines the priority of students i_1 and i_3 at school s_2.

Considering the matching μ'; every fair Pareto improvement involving a swap of transferable characteristics involving only students i_1 and i_2 does not generate an (ex-post) stable extended matching because $\omega^{s_2}(i_2) = \mathbf{2}^{s_2}$ and $(i_3, \mathbf{4}^{s_2}) \succ_{s_2} (i_1, \mathbf{2}^{s_2})$. However, if student i_6 participates in the swap of transferable characteristics, we can define the allocation of transferable characteristics λ such that for each $s \in \{s_3, s_4, s_5\}$ $\lambda^s = \omega^s$ and the allocation of transferable characteristics at schools s_1 and s_2 is:

$\lambda^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{3}^{s_1}$	$\mathbf{5}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{0}^{s_1}$
s_2	$\mathbf{5}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{3}^{s_2}$
s_3	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
s_4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
s_5	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

The extended matching (μ', λ) is (ex-post) stable, and (μ', λ) is a fair Pareto improvement over (μ, ω).

On the other hand, for the matching μ'', under the initial allocation of transferable characteristics, student i_1 cannot obtain a position at school s_4 because i_4 has a higher priority at that school (despite that i_1's transferable characteristic at s_4 is higher than i_4's). In this case, student i_4 needs i_1's transferable characteristic at school s_1 to avoid generating justified envy by i_2 but i_1 does not need i_4's at s_4. Hence, we can define the allocation of transferable characteristics $\bar{\lambda}$ such that for each $s \in \{s_2, s_3, s_4, s_5\}$, $\bar{\lambda}^s = \omega^s$, and the allocation of transferable characteristics at schools s_1 and s_4 is:

$\bar{\lambda}^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{2}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{5}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{0}^{s_1}$
s_2	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
s_3	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
s_4	$\mathbf{5}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{1}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

The extended matching $(\mu'', \bar{\lambda})$ is (ex-post) stable, and $(\mu'', \bar{\lambda})$ is a fair Pareto improvement over (μ, ω). In fact both (μ', λ) and $(\mu'', \bar{\lambda})$ are constrained efficient extended matchings.

We conclude this section with a variation of Example 3.

Example 4. *Consider a variation of the school choice problem defined in Example 3 with an alternative profile of student preferences. Let P' such that for each $i \neq i_5$, $P_i = P'_i$, and $s_2 \; P'_{i_5} \; s_5 \; P'_{i_5} \; \{\varnothing\}$. The matching associated with SOSEM for the problem with the new profile of preferences is:*

$$\hat{\mu} = [(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_2), (i_6, s_5)],$$

that together with the initial allocation of transferable characteristics forms an (ex-post) stable extended matching $(\hat{\mu}, \omega)$. *Consider the matching* $\hat{\mu}'$:

$$\hat{\mu}' = [(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_2), (i_6, s_5)] \,.$$

Note that $\hat{\mu}'$ *Pareto dominates* $\hat{\mu}$, *but there is no allocation of transferable characteristics* λ *such that* $(\hat{\mu}, \lambda)$ *is a fair Pareto improvement for* $(\hat{\mu}, \omega)$ *because*

$$(i_3, \omega^{s_2}(i_3)) \succ_{s_2} (i_1, \max\{\omega^{s_2}(i_1), \omega^{s_2}(i_2), \omega^{s_2}(i_5)\}).$$

We have seen that the possibility of constructing fair Pareto improvements involving some students at different schools may depend on the remaining set of students assigned to those schools. Specifically, the transferable characteristics of the students assigned to each school determine the possibility of finding fair Pareto improvements. This characteristic of our framework contrasts with the notion of partial fairness proposed by Dur et al. (2019). In Dur et al. (2019), there is an initial set of priority violations (instances of justified envy) that can be admitted. In their framework, the possibility of a swap of positions between students that generates admissible instances of justified envy does not depend on the sets of students with their transferable characteristics assigned to the same school.

3 Improvement Cycles for Extended Matchings

In this section, we present a systematic method to obtain fair Pareto improvements starting from individually rational and nonwasteful extended matchings. We study the possible improvements in terms of efficiency and fairness by allowing exchanges of transferable characteristics. If the initial extended matching is (ex-post) stable, then the objective is to obtain constrained efficient extended matchings. Our approach follows Erdil and Ergin (2008) and Dur et al. (2019), who propose a method for finding fair Pareto-improving matchings through exchange cycles based on the outcome of the student-proposing DA algorithm for priorities with indifferences and arbitrary tie-breaker; and partially unenforceable priorities, respectively. The logic behind fair Pareto improvement cycles in both papers is related to the idea of vacancy chains introduced by Blum et al. (1997). Given an initial matching, if a student quits the position she has been assigned to, then the seat and the transferable characteristics of this student may be used by another student. To obtain a Pareto improvement over the initial matching, the candidates to fill the free position are the students who either are assigned to the same school or the students who prefer the school with the free position to the school to which they are assigned in the initial matching. To obtain a final (ex-post) stable extended matching, the seat could be assigned to a candidate that has a higher priority at this specific school than the remaining students who prefer this school to their initial match. This higher priority can be the product of either using the transferable characteristic of the leaving candidate, or keeping her own characteristics at that specific school. In this paper, instead of introducing vacancy chains that are generated by the exogenous creation of new available seats, the vacancy chains may be endogenously generated by the construction of cycles of students exchanging seats at schools and their respective transferable characteristics.

The following concepts extend the graph-theoretical approach presented by Dur et al. (2019) to the school choice with transferable characteristics framework. Unlike Dur et al. (2019), in our model, students may be willing to move to a position at a desirable school but, as in Examples 3 and 4, an instance of justified envy may appear depending on the student who exchanges the transferable characteristics. Moreover, fair Pareto improvements involving two students may require the participation of additional students who exchange transferable characteristics but do not change the school to which they are assigned.

We introduce notation with respect to the students who may be interested in occupying another student position and the students who may finally be assigned to a position without generating justified envy.

Given an individually rational and nonwasteful extended matching (μ, λ), for each student $j \in I$, let the set $D_{(\mu,\lambda)}(j)$ consist of the students who consider $\mu(j)$ at least as good as their own matches. Formally,

$$D_{(\mu,\lambda)}(j) = \{i \in I \setminus \{j\} : \mu(j) \ R_i \ \mu(i)\},$$

Next, the set $\tilde{D}_{(\mu,\lambda)}(j)$ contains all the students who strictly prefer the match of student j over their own matches. That is,

$$\tilde{D}_{(\mu,\lambda)}(j) = \{i \in I : \mu(j) \ P_i \ \mu(i)\}.$$

Clearly, $\tilde{D}_{(\mu,\lambda)}(j) \subseteq D_{(\mu,\lambda)}(j)$. The students in $D_{(\mu,\lambda)}(j)$ are willing to occupy a seat at $\mu(j)$.

The set $Y_{(\mu,\lambda)}(j)$ contains every student i assigned to $\mu(j)$ such that no student in $\tilde{D}_{(\mu,\lambda)}(j)$ would have justified envy of i if i stays at the same school but with the transferable characteristic of j at $\mu(j)$.

$$Y_{(\mu,\lambda)}(j) = \left\{ i \in D_{(\mu,\lambda)}(j) \setminus \tilde{D}_{(\mu,\lambda)}(j) : \text{ for each } k \in \tilde{D}_{(\mu,\lambda)}(j), (i, \lambda^{\mu(j)}(j)) \succsim_{\mu(j)} (k, \lambda^{\mu(j)}(k)) \right\}.$$

Analogously, the set $\tilde{Y}_{(\mu,\lambda)}(j)$ contains all the students who would enjoy an improvement by being by assigned to $\mu(j)$ and that would have the highest priority among the students in $\tilde{D}_{(\mu,\lambda)}(j)$ either with her own or with j's transferable characteristic at $\mu(j)$.

$$\tilde{Y}_{(\mu,\lambda)}(j) = \left\{ i \in \tilde{D}_{(\mu,\lambda)}(j) : \text{ for each } k \in \tilde{D}_{(\mu,\lambda)}(j) \setminus \{i\}, (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (k, \lambda^{\mu(j)}(k)) \right\}.$$

Finally, the set $X_{(\mu,\lambda)}(j)$ consists of all the students who would be willing to occupy j's position at $\mu(j)$ without generating any additional instance of justified envy because after a possible exchange of the transferable characteristic at $\mu(j)$, they have higher priority than the remaining students in $\tilde{D}_{(\mu,\lambda)}(j)$ for j's position.

$$X_{(\mu,\lambda)}(j) = Y_{(\mu,\lambda)}(j) \cup \tilde{Y}_{(\mu,\lambda)}(j).$$

Let $G = (V; E)$ be a directed application graph with the set of vertices V and the set of directed edges E, which is a set of ordered pairs of elements of V. We consider graphs with $V = I$ and directed edges consisting of ordered pairs of distinct students $ij \in I \times I$ with $i \neq j$. With slight abuse of notation, since the set of edges completely defines the graph, we write $ij \in G$ when edge ij belongs to the set of edges of G. For any directed application graph G, a set of edges $\{i_1i_2, i_2i_3, \ldots, i_ni_{n+1}\}$ is a path if the related edges $i_1i_2, i_2i_3, \ldots, i_ni_{n+1}$ are distinct, and it is a cycle if the edges $i_1i_2, i_2i_3, \ldots, i_ni_{n+1}$ are distinct and $i_1 = i_{n+1}$. We generically denote an arbitrary cycle in a graph by ϕ. Student i is involved in the cycle ϕ if there is a student j such that $ij \in \phi$. For each cycle ϕ, $N(\phi)$ denotes the set of students involved in ϕ.

For each extended matching (μ, λ), $G(\mu, \lambda)$ is the ***directed application graph associated with*** (μ, λ) where I is the set of vertices, and the set of directed edges is defined by $ij \in G(\mu, \lambda)$ if and only $i \in X_{(\mu,\lambda)}(j)$.

For an arbitrary (μ, λ), let ϕ be an arbitrary cycle of $G(\mu, \lambda)$, $N(\phi) \subseteq I$ be the students involved in the cycle ϕ, and $\hat{\lambda}$ be an arbitrary allocation of transferable characteristics. A pair formed by a cycle ϕ of $G(\mu, \lambda)$ and an allocation of transferable characteristics for the students involved in the cycle $\hat{\lambda}\mid_{N(\phi)}$, $\gamma = (\phi, \hat{\lambda}\mid_{N(\phi)})$, is an ***improvement cycle*** for $G(\mu, \lambda)$ if:

i) for some $ij \in \phi$, $\mu(i) \neq \mu(j)$, and

ii) for each i involved in a link $ij \in \phi$,

- for each $s \notin \{\mu(i), \mu(j)\}$, $\hat{\lambda}^s(i) = \lambda^s(i)$,
- if $\mu(i) = \mu(j)$, then $\hat{\lambda}^{\mu(j)}(i) = \lambda^{\mu(j)}(j)$,
- if $\mu(i) \neq \mu(j)$ then:

$$\hat{\lambda}^{\mu(j)}(i) \in \{l \in \{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\} : \text{ for each } k \in \tilde{D}_{(\mu,\lambda)}(j)\backslash\{i\},\ (i, l) \succsim_{\mu(j)} (k, \lambda^{\mu(j)}(k))\}.$$

An improvement cycle $\gamma = (\phi, \hat{\lambda}\mid_{N_\phi})$ is solved when for each $ij \in \phi$, student i is assigned to $\mu(j)$ reassigning transferable characteristics according to $\hat{\lambda}\mid_{N(\phi)}$ to obtain a new extended matching. Formally, we denote the solution of a cycle by the operator $\circ$, that is, $(\nu, \bar{\lambda}) = \gamma \circ (\mu, \lambda)$ if and only if for each $i \in N(\phi)$ and $ij \in \phi$, $\nu(i) = \mu(j)$ and $\bar{\lambda}(i) = \hat{\lambda}(i)$, and for each $i' \notin N(\phi)$, $\nu(i') = \mu(i')$ and $\bar{\lambda}(i') = \lambda(i')$.

Note that for each cycle ϕ of $G(\mu, \lambda)$ that involves students initially assigned to different schools it is possible to define at least one improvement cycle of $G(\mu, \lambda)$. If a cycle of $G(\mu, \lambda)$ generates different improvement cycles, solving any of those improvement cycles leads to extended matchings with the same matchings but different allocations of transferable characteristics.

The following algorithm is built on an extended matching and is defined by solving improvement cycles and proposing new allocations of transferable characteristics iteratively. We focus our analysis on individually rational and nonwasteful extended matching as starting points of the algorithm. Note that it would be trivial to solve for individually rational and nonwasteful Pareto improvements just by assigning students to their outside option, and assigning empty seats to the students willing to join a school according to the priority.

Student Exchange with Transferable Characteristics (SETC) Algorithm:

Step 0: Let (μ_0, λ_0) be an individually rational and nonwasteful extended matching.

Step $t \geq 1$: Given the extended matching $(\mu_{t-1}, \lambda_{t-1})$,

- if there is an improvement cycle in $G(\mu_{t-1}, \lambda_{t-1})$, solve any one of such cycles, for example, γ_t, and let $(\mu_t, \lambda_t) = \gamma_t \circ (\mu_{t-1}, \lambda_{t-1})$. Next, move to Step $t + 1$.
- if there is no improvement cycle in $G(\mu_{t-1}, \lambda_{t-1})$, then the algorithm stops and $(\mu_{t-1}, \lambda_{t-1})$ is the obtained extended matching.

Note that the definition of the SETC algorithm entails a class of algorithms, as there may be several incompatible improvement cycles and the order in which improvement cycles are solved may lead to different final outcomes.

Regarding the computational efficiency of SETC algorithms, since the sets of schools and students are finite, the algorithm stops after a finite number of steps. Using the same arguments as Erdil and Ergin (2008) to compute the running time for finding stable Pareto improvements, the running time to find a fair Pareto improvement cycle is $O(\#I\#S)$, and at most, there are $\frac{1}{2}\#I(\#I-1)$ possible Pareto improvements. Thus, the running time to solve this problem entirely is $O\left(\frac{1}{2}\#S\#I^3\right)$. Hence, the SETC algorithms are computationally efficient.

Next, we return to Example 3 to illustrate the workings of the algorithm. Example 3 shows the relevance of constructing improvement cycles for students who do not strictly benefit from exchanging their transferable characteristics.

Example 5. ***(Example 3 continued)****. Consider the school choice problem with transferable characteristics introduced in Example 3 and the corresponding SOSEM* $(\mu_\omega^{SO}, \omega)$ *and the extended matchings* (μ', λ) *and* $(\mu'', \bar{\lambda})$ *defined there to clarify the workings of the SETC. Let us construct the direct application graph associated with* $(\mu_\omega^{SO}, \omega)$*. In Figure 1(a), we represent the possibilities of improvement for the different students. Each student points to all the students that occupy a position at a school at least as good as the school prescribed to them by* μ_ω^{SO}*.*

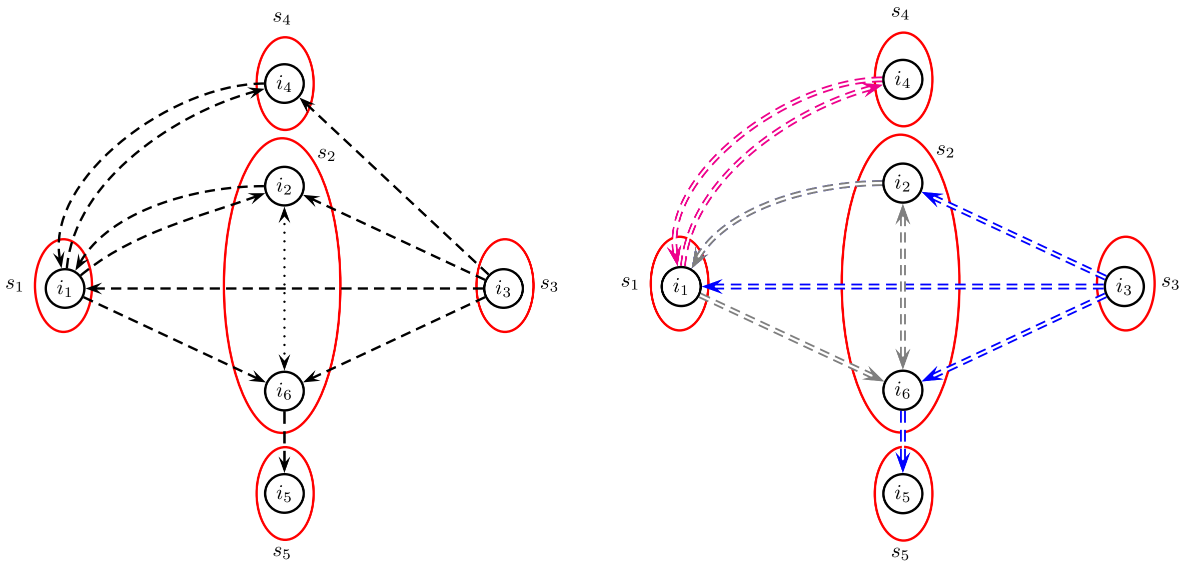
In Figure 1(b), we represent the associated graph $G(\mu_\omega^{SO}, \omega)$*. We observe that there are two cycles,* $\phi = \{i_1 i_6, i_6 i_2, i_2 i_1\}$ *and* $\phi' = \{i_1 i_4, i_4 i_1\}$*, that generate two associated improvement cycles,* $\gamma = (\phi, \lambda \mid_{\{1,2,6\}})$ *and* $\gamma' = (\phi', \bar{\lambda} \mid_{\{1,4\}})$*. Student* i_1 *is involved in both cycles and only one of the associated improvement cycles can be solved.*

The extended matching (μ', λ) *is the outcome of solving improvement cycle* γ*,* $(\mu', \lambda) = \gamma \circ (\mu_\omega^{SO}, \omega)$*. In Figure 1(c), we present the graph* $G(\mu', \lambda)$*. We observe that no student points to the students assigned to schools* s_3 *and* s_4*, and the students in* s_2 *are only pointed to by the student in* s_3*, whereas the students in* s_1 *and* s_5 *do not point to any student. Hence, the graph* $G(\mu', \lambda)$ *has no cycle. In fact,* (μ', λ) *is constrained efficient. Analogously,* $(\mu'', \bar{\lambda}) = \gamma' \circ (\mu_\omega^{SO}, \omega)$*. In Figure 1(d), we present the graph for* $G(\mu'', \bar{\lambda})$ *that does not present any additional cycle.*

Example 5 illustrates how the algorithms in the SETC class obtain fair Pareto improvements by seeking cycles in the directed application graph associated with an initial individually rational and nonwasteful extended matching. Our first result shows that by iteratively applying the same logic, the final outcome of the algorithm is a justifiable Pareto improvement of the initial extended matching. Moreover, the algorithms in the SETC class exhaust the possibilities of finding additional fair Pareto improvements.

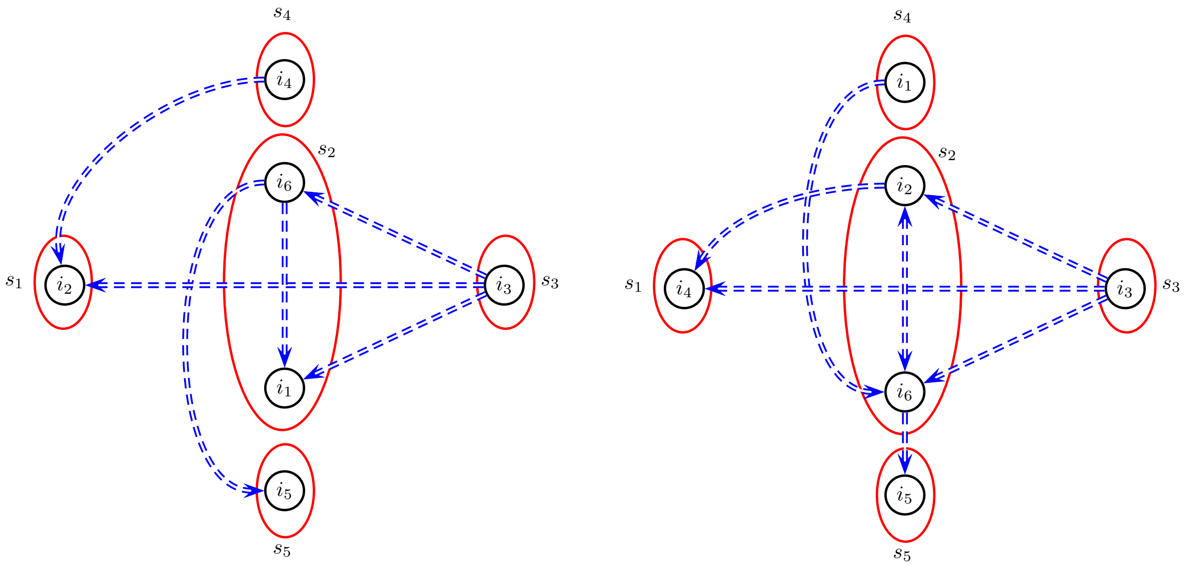
Theorem 1. *Let* (μ, ω) *be an individually rational and nonwasteful extended matching and* (μ', λ) *be the outcome obtained by an algorithm in the SETC class starting with* (μ, ω)*. Then, the extended matching* (μ', λ) *is a justifiable Pareto improvement of* (μ, ω) *and does not admit any fair Pareto improvement.*

The proof of Theorem 1 relies on showing that if an individually rational and nonwasteful extended matching admits a fair Pareto improvement, then the directed application graph has (at least) one improvement cycle. The arguments in the proof are similar to Dur et al. (2019, Theorem 1) but we need to keep track of important details that are



(a) Start. Student i_x points to student i_y if $i_x \in D_{(\mu,\omega)}(i_y)$. Dashed lines: i_x points to i_y if $i_x \in \tilde{D}_{(\mu,\omega)}(i_y)$ Dotted Lines: i_x points to i_y if $i_x \in D_{(\mu,\omega)}(i_y)$ and $\mu(i_x) = \mu(i_y)$.

(b) $G(\mu, \omega)$. Student i_x points to student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$. Two improvement cycles with cycles $\phi = \{i_1 i_6, i_6 i_2, i_2 i_1\}$ and $\phi' = \{i_1 i_4, i_4 i_1\}$



(c) $G(\mu', \lambda)$ with $(\mu', \lambda) = \gamma \circ (\mu, \omega)$. Student i_x points to student i_y if $i_x \in X_{(\mu',\lambda)}(i_y)$.

(d) $G(\mu'', \bar{\lambda})$ with $(\mu'', \bar{\lambda}) = \gamma' \circ (\mu, \omega)$. Student i_x points to student i_y if $i_x \in X_{(\mu'',\bar{\lambda})}(i_y)$.

Figure 1: Example 5. Construction of $G(\mu, \omega)$ and the application of the SETC algorithms.

absent in Dur et al. (2019)'s framework. Specifically, transferable characteristics differ between students, and only exchanges involving specific students at a school may be mutually viable. Moreover, improvement cycles may need to involve students who do not strictly benefit from these exchanges but are needed to facilitate reassignment through transferable characteristic trades. We would like to highlight that our assumption on neutral priorities is not crucial to obtain the result. For non-neutral priorities, we can construct the direct application graph $G(\mu, \lambda)$ and run the SETC algorithms, once we account for the fact that the transferable characteristics that imply a higher priority at each school may be different for different students.[13]

Next, we focus on analyzing the extended matchings that can be obtained by the application of the SETC starting at (ex-post) stable extended matchings. Since the application of the SETC obtains a fair Pareto improvement at each step of the algorithm, the final outcome of the algorithm is a justifiable Pareto improvement of the initial extended matching. If the initial extended matching is (ex-post) stable, then any outcome of the SETC algorithm is also (ex-post) stable. From Theorem 1, we obtain interesting implications. When we start with an (ex-post) stable extended matching, any extended matching that can be the result of the SETC algorithm is constrained efficient. Moreover, the resulting extended matching is the SOSEM associated with the final allocation of transferable characteristics. We formalize both implications in the following Corollaries 1 and 2.

Corollary 1. *Let (μ, ω) be an (ex-post) stable extended matching and (μ', λ) be the outcome obtained by an algorithm in the SETC class starting with (μ, ω). Then, the extended matching (μ', λ) is constrained efficient.*

Corollary 2. *For each problem, each (ex-post) stable matching (μ_0, ω), and each SETC algorithm, if the extended matching (μ, λ) is an outcome of an SETC algorithm then $(\mu, \lambda) = (\mu^{SO}_{\lambda}, \lambda)$.*

In light of Corollary 1, a natural question to consider is whether the algorithms in the SETC class are able to obtain all the constrained efficient extended matching that Pareto dominate an initial (ex-post) stable extended matching, such as the SOSEM. Note that any SETC algorithm is restricted to select justifiable Pareto improvements from an initial extended matching. Hence, there are school choice problems with a constrained efficient extended matching that Pareto dominates the initial extended matching (μ, ω) that cannot be obtained by applying an instance of a SETC algorithm (see Example 2).

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[13] Specifically, we should define a student-specific operator max to order the transferable characteristics at each school to provide a consistent definition of the set $\tilde{Y}_{(\mu,\lambda)}(j)$.

Even if we restrict our attention to constrained efficient justifiable Pareto improvements over the initial extended matching, there is an additional issue to address. At each step of the application of any instance of a SETC algorithm, the algorithm proposes an extended matching with a specific allocation of transferable characteristics among the students involved in the selected cycle. However, an extended matching with the same matching and a different allocation of transferable characteristics could also be a fair Pareto improvement over the extended matching proposed in the previous step. This situation can occur when the students who initially may have justified envy of other students occupying a position at a given school no longer desire that position after being involved in cycles solved in several steps in an application of the SETC algorithm. In that case, the transferable characteristic of the student at that position becomes irrelevant. This fact implies that at some points in the application of an SETC algorithm, the matching proposed by the SETC could be fair under an alternative allocation of transferable characteristics. We define a characteristic-wise extended matching to compare different extended matchings in terms of the fairness binding constraints.

Let (μ, λ) and $(\mu, \bar{\lambda})$ be extended matching such that (μ, λ) is (ex-post) stable. The extended matching $(\mu, \bar{\lambda})$ is ***characteristic-wise equivalent to*** (μ, λ) if for each $i \in N$, for each $s \in S$ such that $s \ R_i \ \mu(i)$, if there is a $j \in I$ with $s \ R_j \ \mu(j)$ $(j, \lambda^s(j)) \succsim_s (i, \lambda^s(i))$ implies $(j, \bar{\lambda}^s(j)) \succsim_s (i, \bar{\lambda}^s(i))$.

Remark 1. *Let* (μ, λ) *be an (ex-post) stable extended matching, if* $\bar{\lambda}$ *is such that for each* $i \in I$, $\bar{\lambda}^{\mu(i)}(i) \geq \lambda^{\mu(i)}(i)$, *and for each* $s \in S$ *with* $s \ P_i \ \mu(i)$, $\bar{\lambda}^s(i) = \lambda^s(i)$, *then* $(\mu . \bar{\lambda})$ *is charateristic-wise equivalent to* (μ, λ).

Example 6 illustrates the concept of characteristic-wise equivalent extended matchings.

Example 6. *Let* $I = \{i_1, i_2, i_3, i_4, i_5\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, *and* $q_{s_x} = 1$ *for* $x = 1, 2, 3, 4, 5$. *The relevant student preferences are as follows:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
s_2	s_3	s_1	s_2	s_4
s_1	s_5	s_3	s_3	s_5
$\{\varnothing\}$	s_1	$\{\varnothing\}$	s_5	$\{\varnothing\}$
	s_2		s_4	
	$\{\varnothing\}$		$\{\varnothing\}$	

The relevant initial allocation of transferable characteristics ω *and the relevant school priorities are:*

$\omega^s(i)$	i_1	i_2	i_3	i_4	i_5
s_1	$\mathbf{0}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{2}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{2}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{1}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	$\mathbf{1}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{2}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{0}^{s_5}$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
$(i_1, \mathbf{0}^{s_1})$	$(i_2, \mathbf{4}^{s_2})$	$(i_3, \mathbf{4}^{s_3})$	$(i_4, \mathbf{0}^{s_4})$	$(i_5, \mathbf{0}^{s_5})$
$(i_2, \mathbf{4}^{s_1})$	$(i_1, \mathbf{4}^{s_2})$	$(i_2, \mathbf{4}^{s_3})$	$(i_5, \mathbf{4}^{s_4})$	$(i_2, \mathbf{4}^{s_5})$
$(i_3, \mathbf{3}^{s_1})$	$(i_4, \mathbf{3}^{s_2})$	$(i_4, \mathbf{4}^{s_3})$		$(i_4, \mathbf{3}^{s_5})$
	$(i_1, \mathbf{2}^{s_2})$	$(i_4, \mathbf{3}^{s_3})$		
		$(i_2, \mathbf{2}^{s_3})$		

Note that for each $j \in \{1, \ldots, 5\}$, $\mu^{SO}_\omega(i_j) = s_j$. *That is,*

$$\mu^{SO}_\omega = [(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_5)] .$$

Consider now the extended matching (μ', λ) *with*

$$\mu' = [(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_5), (i_5, s_4)] .$$

and λ *such that:*

$\lambda^s(i)$	i_1	i_2	i_3	i_4	i_5
s_1	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{0}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{4}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{2}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{1}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{0}^{s_4}$	$\mathbf{4}^{s_4}$
s_5	$\mathbf{1}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{2}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{0}^{s_5}$

The extended matching (μ', λ) *is a fair Pareto improvement of* $(\mu^{SO}_\omega, \omega)$, *and it is a constrained efficient extended matching. However,* (μ', λ) *cannot be obtained by the application of the SETC algorithm. Note that in the first stage of the SETC there is only one improvement cycle* $\{i_1 i_2, i_2 i_1\}$ *with allocation of transferable characteristics* $\bar{\lambda}$ *such that* $\bar{\lambda}^{s_1}(i_2) = \omega^{s_1}(i_2) = \mathbf{4}^{s_1}$ *and* $\bar{\lambda}^{s_2}(i_1) = \omega^{s_2}(i_2) = \mathbf{4}^{s_2}$ *After two additional stages of the SETC, the SETC ends up in the constrained efficient extended matching* $(\mu', \hat{\lambda})$ *with* $\hat{\lambda}$ *defined by:*

$\hat{\lambda}^s(i)$	i_1	i_2	i_3	i_4	i_5
s_1	$\mathbf{0}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{4}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{2}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{1}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{0}^{s_4}$	$\mathbf{4}^{s_4}$
s_5	$\mathbf{1}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{2}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{0}^{s_5}$

Note that according to extended matching (μ', λ), $\tilde{D}_{(\mu', \lambda)}(i_3) = \varnothing$. *Hence, once all the students are assigned to a school according to* μ', *the allocation of the transferable characteristic to* i_3 *at* s_1 *is irrelevant, since no student may have justified envy of* i_3. *Hence, we can obtain a fair Pareto improvement of the initial extended matching* $(\mu^{SO}_\omega, \omega)$

that cannot be obtained by the SETC. This fact notwithstanding, the SETC obtains an extended matching that is characteristic-wise equivalent to (μ', λ).[14]

Example 6 shows that it is not possible to characterize the set of constrained efficient extended matching that are justifiable Pareto improvements of an initial extended matching as the set of outcomes of the SETC starting at that initial extended matching. However, Theorem 2 shows that for every extended matching that can be obtained by a sequence of fair Pareto improvements from an initial (ex-post) stable extended matching, an instance of the SETC obtains an extended matching that (weakly) Pareto dominates the initial extended matching.

Theorem 2. *Let* (μ, ω) *be an (ex-post) stable extended matching if* (μ', λ) *is a justifiable Pareto improvement of* (μ, ω)*; then there is an extended matching* $(\nu, \bar{\lambda})$ *obtained with an algorithm of the SETC class starting at* (μ, ω) *such that for each* $i \in I$, $\nu(i)\ R_i\ \mu'(i)$.

Theorem 2 follows from an intermediate key result that we prove in Section 4.2 (Proposition 4). If an extended matching (μ', λ) is a justifiable Pareto improvement of (μ, ω), then the outcome of an application of the SETC after t steps yields an extended matching (μ_t, λ_t) such that $\mu_t = \mu'$ and (μ_t, λ_t) is characteristic-wise equivalent to (μ', λ). To prove this result, we have to check that the matching involved in any fair Pareto improvement of an (ex-post) stable extended matching can be obtained by the SETC after a finite number of steps (Lemma 9), but an allocation of transferable characteristics such that improving students may obtain higher transferable characteristics at the school to which they are assigned than that prescribed by (μ', λ). Thus, it may be the case that (μ', λ) is constrained efficient, but (μ_t, λ_t) is not constrained efficient and admits further fair Pareto improvement, and the final outcome of the SETC may (strictly) Pareto dominate (μ', λ).[15]

Note that Theorem 2 applies to an (ex-post) stable extended matching and not to any initial nonwasteful extended matching as in Theorem 1. In this case, the restriction to fair Pareto improvements implies that no student may generate justified envy. This is an unnecessarily strong requirement. It is possible to define a weaker notion of fair Pareto improvement such as requiring that the set of students who have justified envy of the

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[14] There is another constrained efficient extended matching obtained by selecting an alternative improvement cycle in the second stage of the SETC with the cycle $\{i_2 i_5, i_5 i_4, i_4 i_3, i_3 i_2\}$ for the graph $G(\mu_1, \lambda_1)$.

[15] To obtain an example of a constrained efficient extended matching that is Pareto dominated by the outcome of the SETC, it suffices to consider a school choice problem with two independent replicas of the school choice problem in Example 6 and an additional student, where the students in the role of i_3 would like to exchange their positions at the replicas of s_1 but the additional student may have higher priority and generate justified envy unless they exchange the relevant characteristic $\mathbf{4}^{s_1}$.

student occupying a specific position at a school be included in the initial set of students who have justified envy of her. With this alternative notion of fair Pareto improvements in mind, we can define an alternative class of algorithms that will be analogous to those in the SETC class and would uncover more justifiable Pareto improvements than the SETC algorithm. Finally, our assumption of neutral priorities plays an implicitly relevant role in the proof of Theorem 2. Under neutral priorities, we can construct the allocations of transferable characteristics that allow every fair Pareto improvement to be generated as a sequence of solved improvement cycles by an algorithm in the SETC class.

4 Discussion

In this section, we focus on issues related to the possibility of finding a justifiable Pareto improvement for the SOSEM. Specifically, we consider the incentives of the students to reveal their true preferences to a centralized planner and a particular class of school priorities that allows us to precisely compare our framework with previous works.

4.1 Incentives and Student Transferable Characteristics

We first analyze the incentives of students to reveal their true preferences when an allocation of school seats is determined by an SETC algorithm. For that purpose, we need to introduce further notation.

Let $\mathcal{P}$ denote the complete set of student preference profiles and $\mathcal{M}$ be a set of all the extended matchings. A mechanism is a mapping $\Psi : \mathcal{P} \to \mathcal{M}$.

The application of an SETC algorithm starting with the SOSEM that corresponds to each preference profile defines a mechanism that always selects an (ex-post) stable and constrained efficient extended matching. We call this class of mechanisms the ***student optimal transferable characteristics (SOTC)*** class of mechanisms.

Strategy-proofness A mechanism Ψ satisfies ***strategy-proofness*** if for each $i \in N$, each $P, P' \in \mathcal{P}$, such that for each $j \neq i$, $P_j = P'_j$, $\Psi(P) = (\mu, \lambda)$ and $\Psi(P') = (\mu', \lambda')$, $\mu(i)\ R_i\ \mu'(i)$.

Strategy-proofness implies that no student has the capacity and the incentives to manipulate the outcome of a mechanism by misreporting her preferences regarding schools. It is well-known that the mechanism that selects the SOSEM for each student preference profile satisfies *strategy-proofness*, but it may select a Pareto dominated extended matching. According to the results in the work of Abdulkadiroğlu et al. (2009); Kesten (2010);

Alva and Manjunath (2019); Kesten and Kurino (2019), since the matching selected by any SETC algorithm that starts with the SOSEM Pareto dominates the SOSEM for the initial allocation of characteristics ω and is not Pareto dominated by any other extended matching, each mechanism in the SOTC class is manipulable for some profile of student preferences.

Proposition 1. *There is no mechanism in the SOTC class that satisfies* strategy-proofness*.*

4.2 Fully Transferable Priorities

In the previous sections, we analyzed a new setting where tradeoffs between stability and efficiency can be attenuated. In a school choice problem with transferable characteristics, some violations of initial priorities can be justified after exchanges of transferable characteristics. This new component of the canonical school choice problem does not allow us to make an immediate comparison to previous works that consider dropping stability constraints when some students do not benefit from exercising their priority rights.

In particular, the concepts of α-stability in Alcalde and Romero-Medina (2017) and of students consenting to drop their initial priorities in Kesten (2010) imply that the proposed matchings are met with no objections, although the initial priorities of some students are not respected by these matchings. In both cases, the exertion of some priorities by some students at some schools that block the assignment of seats to other students may not ultimately lead to a placement improvement for the blocking student. Therefore, students are either not allowed to exert their priority rights as in Alcalde and Romero-Medina (2017), or encouraged not to claim a seat if doing so would be ineffective as in Kesten (2010).

The proposals of Alcalde and Romero-Medina (2017) and Kesten (2010) are presented in terms of the canonical school choice problem. This prevents an immediate comparison with our results. However, there is an extreme class of school priorities that allows us to view both proposals as particular cases of extended matchings obtained by SETC algorithms. This is the case in the domain of *fully transferable extended priorities*, where we can address both concepts of priority renouncement.

Fully Transferable Priorities. For each $i, i', j, j' \in I$, $s \in S$, and for each $\lambda^s, \bar{\lambda}^s \in \mathcal{L}^s$, $(i, \lambda^s) \succ_s (i', \bar{\lambda}^s)$ if and only if $(j, \lambda^s) \succ_s (j', \bar{\lambda}^s)$.

In cases where transferable characteristics entirely determine school priorities, if a student participates in an improvement cycle and receives the transferable characteristic that initially secured her seat, then the initial priority that another student may have had for that seat is no longer relevant.

In the context of fully transferable priorities, the analysis of the SETC algorithms is simpler. This is because any student who desires the position of another student can obtain it with an exchange of transferable characteristics.

Lemma 1. *Let school priorities be fully transferable, let* (μ, λ) *be an (ex-post) stable extended matching and let* $G(\mu, \lambda)$ *be the directed application graph associated with* (μ, λ)*. If* $\mu(j)\ P_i\ \mu(i)$*, then* $ij \in G(\mu, \lambda)$*.*

Lemma 1 implies that under fully transferable priorities, students who exchange their characteristics but remain assigned to the same school do not need to participate in improvement cycles. Moreover, the possibility of justifying an exchange of position that involves a violation of school priorities under the initial allocation of transferable characteristics does not depend on the students (and their transferable characteristics) initially assigned to each school. Hence, the framework under fully transferable characteristics is equivalent to the framework of Dur et al. (2019) when all potential exchange cycles are admitted under partial stability.[16] In this context, since any fair Pareto improvement of a matching can be achieved by forming disjoint cycles among students and because such cycles correspond to an improvement cycle in directed application graph $G(\mu_\omega^{SO}, \omega)$, we immediately derive the following result.

Proposition 2. *Let school priorities be fully transferable. If* μ *is a matching that Pareto dominates* μ_ω^{SO} *and* μ *is not Pareto dominated by any matching* ν*, then there is an allocation of transferable characteristics* λ *such that* (μ, λ) *is the result of the application of an SETC algorithm that starts with the SOSEM.*

Alcalde and Romero-Medina (2017, Theorem 1) proves that the set of Pareto efficient matchings that are Pareto improvements over the initial optimal student matching coincides with an ideal set of matchings such that under the initial priorities, no student can pose an admissible objection. That is, whenever a student proposes (objects) an alternative matching where she would obtain a preferred seat for which she has a priority right, another student could rightfully object to that alternative matching (α-*fair* matching). Hence, the set of matchings produced by an SETC algorithm coincides with the set of α-*fair* matchings produced under the assumption of fully transferable priorities.

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[16] In the terms of Dur et al. (2019) this corresponds to the case where the correspondence that defines the admitted priority violations satisfies the all-or-nothing property, specifically, item *i*) of the all-or-nothing property for all schools.

Corollary 3. *Let school priorities be fully transferable. A matching μ is an α-*fair *matching if and only if there is an allocation of transferable characteristics λ such that the extended matching (μ, λ) is the result of the application of an SETC algorithm that starts with the SOSEM*

Kesten (2010) occupies a central position in the analysis of Pareto efficient matching in the context of school choice and introduces the idea of consent. Students can consent to withdraw their claims to seats that they will not accept. This idea leads to a modification of the student-proposing DA algorithm that yields a Pareto efficient matching with "minimal" violations of initial priorities, the ***efficiency adjusted deferred acceptance algorithm (EADA)***. Tang and Yu (2014) presents a simpler algorithm with the same outcome.[17] Under fully transferable priority, the matching obtained by the EADA can be obtained by a specific algorithm in the SETC class.

Proposition 3. *Let school priorities be fully transferable. There is an algorithm in the SETC class that for each school choice with transferable characteristics problem, the outcome of the algorithm starting in the SOSEM selects an extended matching (μ, λ) such that μ coincides with the EADA matching.*

5 Conclusions

In this paper, we generalize the school choice problem by defining school priorities in terms of (possibly transferable) student characteristics. We define a class of algorithms, the student exchange with transferable characteristics (SETC) class of algorithms. Each algorithm in this class begins with an individually rational and nonwasteful extended matching and produces an extended matching such that any extended matching that Pareto dominates it generates additional instances of justified envy. Moreover, for each constrained efficient extended matching obtained by a sequence of fair Pareto improvements from an initial (ex-post) stable extended matching, an algorithm in the SETC class obtains an extended matching that either is characteristic-wise equivalent to or weakly Pareto dominates the constrained efficient extended matching.

We motivate our analysis of the allocation of objects under priorities based on individual characteristics in the school choice problem. In this framework, the tie-breaking lottery is a natural transferable characteristic when schools use multiple tie-breaking lotteries. However, the algorithms in the SETC class can be used for improving efficiency in

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[17] The resulting matching is a unique Pareto efficient matching μ^* such that there is no other matching ν that can improve the situation of any student whose priority is violated in μ unless ν violates the priority of a student whose situation is worsened (see Reny, 2022).

situations where we can differentiate between allocative criteria and fairness constraints in the characteristics that define the priorities. For instance, we can avoid welfare losses in the integration of separate markets (walk zones). We can study changes in the priority structure due to the redefinition of the characteristics or because of different valuations of the existing characteristics. Finally, we can use the SETC algorithms to propose an ex-post assignment scramble in mechanisms that generate instances of justified envy.

Our analysis is based on characteristics that are specific to individual schools. This is the situation in the case of the multiple tie-breaking lotteries, the priorities for siblings attending the school, or legacy awarded priorities. However, there are other characteristics that are not, in general, school specific, such as the walk-zone priority, or all priorities associated with family circumstances, such as income or the total number of siblings. In these cases, we could adjust the algorithms in the SETC class to allow characteristics to be valid in several schools, but this adjustment must be precisely defined, and it will complicate our results. The general case for nonspecific characteristics is, therefore, left for further research.

6 Proofs

6.1 Proof of Theorem 1

Let $(I, S, (q_s)_{s\in S}, (R_i)_{i\in I}, \omega, (\succ_s)_{s\in S},)$ be a school choice problem with transferable characteristics and let (μ, ω) be an individually rational and nonwasteful extended matching. Consider an instance of an SETC algorithm with initial extended matching $(\mu_0, \lambda_0) = (\mu, \omega)$. Let T be the last step of the SETC algorithm starting by (μ_0, λ_0). For each $t \in \{1, \ldots, T\}$, let $\gamma_t = (\phi_t, \lambda_t\mid_{N(\phi_t)})$ be the improvement cycle solved at step t of the algorithm, and let $(\mu_t, \lambda_t) = \gamma_t \circ (\mu_{t-1}, \lambda_{t-1})$ be the extended matching selected at step t. Note that, the students involved in the improvement cycle are better off, some of them are strictly better off, and the students not involved in the cycle are not worse off at the new extended matching obtained by solving the improvement cycle γ_t. Thus, for each for each $t \in \{1, \ldots, T\}$, μ_t *Pareto dominates* μ_{t-1}.

Remark 2. *Let $\gamma_t = (\phi_t, \lambda_t\mid_{N(\phi_t)})$ be the improvement cycle solved at step t of the application of SETC algorithm, and let $i, j \notin N(\phi_t)$.*

i) $\tilde{D}_{(\mu_t,\lambda_t)}(j) \subseteq \tilde{D}_{(\mu_{t-1},\lambda_{t-1})}(j)$.

ii) If $ij \in G(\mu_{t-1}, \lambda_{t-1})$ then i points to $ij \in G(\mu_t, \lambda_t)$.

Lemma 2. *The extended matching (μ_T, λ_T) is individually rational and nonwasteful.*

Proof. Let $t \in \{0, \dots, T-1\}$ and let (μ_t, λ_t) be the extended matching obtained at step t of the algorithm. We prove the result by induction on t. The initial extended matching (μ_0, λ_0) is individually rational and nonwasteful.

First, we check that (μ_T, λ_T) is an individual rationality extended matching. Since (μ_0, λ_0) is individually rational, and each student is never worse off after each step of the algorithm, then (μ_T, λ_T) is individually rational.

We conclude by checking that (μ_T, λ_T) is nonwasteful. The initial match (μ_0, λ_0) is nonwasteful. At each step, students are assigned to better schools swapping their positions at schools, hence $\#\mu_t^{-1}(s)$ remains constant at each step of the algorithm. Therefore, if school s has an empty slot at step t, then school s has an empty slot at step 0. Since μ_0 is individually rational and nonwasteful, for each student i with $\mu_0(i) \neq s$, $\mu_0(i)\ P_i\ s$. Since for each i, $\mu_t(i)\ R_i\ \mu_0(i)$, we have that $\mu_t(i)\ R_i\ s$. Thus, (μ_t, λ_t) is nonwasteful. $\square$

Our next result, Lemma 3 shows that the outcome of the SETC reduces the instances of justified envy.

Lemma 3. *For each each $t \in \{1, \dots, T\}$, if student i does not have justified envy of j at $(\mu_{t-1}, \lambda_{t-1})$, then student i does not have justified envy of j at (μ_t, λ_t).*

Proof. Let $i, j \in I$ such that i does not have justified envy of j at $(\mu_{t-1}, \lambda_{t-1})$. If $\mu_t(i)\ R_i\ \mu_t(j)$, then i has not justified envy of j at (μ_t, λ_t). Hence, we assume that $\mu_t(j)\ P_i\ \mu_t(i)$. Let $\gamma_t = (\phi_t, \lambda_t\ |_{N(\phi)})$ be the improvement cycle of $G(\mu_{t-1}, \lambda_{t-1})$ solved at step t. Since (μ_t, λ_t) is the result of solving the cycle γ_t, for each student k, $\mu_t(k)\ R_k\ \mu_{t-1}(k)$, and we obtain $\mu_t(j)\ P_i\ \mu_{t-1}(i)$. Since $\mu_t(j)\ P_i\ \mu_{t-1}(i)$ and $\mu_t(j)\ P_i\ \mu_t(i)$, $\mu_t(i) \neq \mu_t(j)$ and $\mu_{t-1}(i) \neq \mu_t(j)$ imply that $\lambda_t^{\mu_t(j)}(i) = \lambda_{t-1}^{\mu_t(j)}(i)$. We consider two cases. Assume first that $j \notin N(\phi_t)$. In this case, $\mu_t(j) = \mu_{t-1}(j)$ and $\lambda_t^{\mu_t(j)}(j) = \lambda_{t-1}^{\mu_t(j)}(j)$. Since i has not justified envy of j at $(\mu_{t-1}, \lambda_{t-1})$, $(j, \lambda_{t-1}^{\mu_{t-1}(j)}(j)) \succsim_{\mu_{t-1}(j)} (i, \lambda_{t-1}^{\mu_{t-1}(j)}(i))$. Consider the second case, $j \in N(\phi)$. Let k be the student such that $jk \in \phi_t$. Note that $\mu_t(j) = \mu_{t-1}(k)$, $j \in X_{(\mu_{t-1}, \lambda_{t-1})}(k)$, and $i \in \tilde{D}_{(\mu_{t-1}, \lambda_{t-1})}(k)$. Since $j \in X_{(\mu_{t-1}, \lambda_{t-1})}(k)$ and $\lambda_t^{\mu(j)}(i) = \lambda_{t-1}^{\mu(j)}(i)$, we have $(j, \lambda_t^{\mu_t(j)}(j)) \succsim_{\mu_t(j)} (i, \lambda_{t-1}^{\mu_t(j)}(i)) = (i, \lambda_t^{\mu(j)}(i))$. Since the two cases are exhaustive and imply that $(j, \lambda_t^{\mu_t(j)}(j)) \succsim_{\mu_t(j)} (i, \lambda_t^{\mu_t(j)}(i))$, we conclude that i has not justified envy of j at (μ_t, λ_t). $\square$

From Lemma 3 and noting that students may only improve at each step of the algorithm, we immediately obtain Corollaries 4 and 5.

Corollary 4. *For each $i \in I$:*

i) If for some each $t \in \{1, \dots, T\}$, $i \in N(\phi_t)$; then no student has justified envy of i at (μ_T, λ_T).

ii) If i is not involved in any improvement cycle solved to obtain (μ_T, λ_T) and there is a student j with justified envy of i at (μ_T, λ_T), then j has justified envy of i at (μ_0, λ_0).

Corollary 5. *For each $t \in \{1, \dots, T\}$, (μ_t, λ_t) is a fair Pareto improvement of $(\mu_{t-1}, \lambda_{t-1})$.*

Lemma 4. *For each individually rational and nonwasteful extended matching (μ, λ) and $j \in I$, $X_{(\mu,\lambda)}(j) \cap \tilde{D}_{(\mu,\lambda)}(j) = \varnothing$ if and only if $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$.*

Proof. If $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$, then $D_{(\mu,\lambda)}(j) = \{i \in I : \mu(i) = \mu(j)\}$. Since $X_{(\mu,\lambda)}(j) \subseteq D_{(\mu,\lambda)}(j)$, the result is immediate. If $\tilde{D}_{(\mu,\lambda)}(j) \neq \varnothing$, then by completeness and transitivity of school priorities, there is $i \in \tilde{D}_{(\mu,\lambda)}(j)$ such that for each $i' \in \tilde{D}_{(\mu,\lambda)}(j)$, $(i, \lambda^{\mu(j)}(i)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$. Finally, since priorities are neutral, $(i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i, \lambda^{\mu(j)}(i))$. Therefore, $i \in X_{(\mu,\lambda)}(j)$. □

From Lemma 4 and the definition of the set $\tilde{D}_{(\mu,\lambda)}(j)$ we obtain Remark 3.

Remark 3. *For each individually rational and nonwasteful extended matching (μ, λ) and $j \in I$, if $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$, then for each $j' \in I$ with $\mu(j) = \mu(j')$, $\tilde{D}_{(\mu,\lambda)}(j') = \varnothing$.*

Lemma 5. *Let (μ, λ) and (ν, λ') be individually rational and nonwasteful extended matchings such that (ν, λ') Pareto dominates (μ, λ). For each $s \in S$, $\#\mu^{-1}(s) = \#\nu^{-1}(s)$.*

Proof. Let $N = \{i \in I : \nu(i) \; P_i \; \mu(i)\}$. Since (ν, λ') Pareto dominates (μ, λ) and student preferences are strict, for each $j \in I \setminus N$, $\mu(j) = \nu(j)$. Consider an arbitrary school $s \in S$ and assume to the contrary that $\#(N \cap \nu^{-1}(s)) > \#(N \cap \mu^{-1}(s))$. This implies that $\#\mu^{-1}(s) < q_s$. For each $i \in (N \cap \nu^{-1}(s))$, $\nu(i) = s \; P_i \; \mu(i)$, which contradicts that (μ, λ) is nonwasteful. Hence, $\#(N \cap \nu^{-1}(s)) \leq \#(N \cap \mu^{-1}(s))$. Finally, assume to the contrary there is s such that the strict inequality holds. Summing up the inequalities across schools, the number of students in N who are assigned to some school in matching μ is larger than the number of students in N that are assigned to some school in matching ν. Thus, there is a student $i \in N$ such that $\mu(i) \in S$, and $\nu(i) = \{\varnothing\}$. Since μ is an individually rational matching, we have that $\mu(i) \; P_i \; \nu(i)$, which contradicts the definition of N. □

Lemma 6. *Let (μ, λ) be an individually rational and nonwasteful extended matching, if (ν, λ') Pareto dominates (μ, λ) then (ν, λ') is individually rational and nonwasteful.*

Proof. Since (μ, λ) is individually rational and for each $i \in I$, $\nu(i) \; R_i \; \mu(i)$, we know that (ν, λ') is an individually rational extended matching. Let $i \in I$ such that $\nu(i) \; P_i \; \mu(i)$. Since (μ, λ) is nonwasteful, there is $j \in I$ such that $\nu(j) \neq \mu(j) = \nu(i)$. Since (ν, λ') Pareto dominates (μ, λ) and $\nu(j) \neq \mu(j)$, we have $\nu(j) \; P_j \; \mu(j)$. By Lemma 5, there is $k \in I$ such that $\nu(k) \neq \mu(k) = \nu(j)$. As S is finite, for each i with $\nu(i) \; P_i \; \mu(i)$ there is a finite sequence of students $i_1, i_2, i_3, \dots, i_n$ such that $\mu(i_i) = \nu(i_{i+1})$ and $i_1 = i_n$.

Since (μ, λ) is nonwasteful, for each $i \in I$ for each $s \in S$ such that $s\ P_i\ \mu(i)$, we have $\#\mu^{-1}(s) = \#\nu^{-1}(s) = q_s$. Finally, as $s\ P_i\ \nu(i)$ implies $s\ P_i\ \mu(i)$, and for each s such that $s\ P_i\ \mu(i)$, $\#\mu^{-1}(s) = q_s$, we have that for each s such that $s\ P_i\ \nu(i)$, $\#\nu^{-1}(s) = q_s$, which suffices to prove that (ν, λ') is nonwasteful. □

Lemma 7 provides the final step in the proof of Theorem 1.

Lemma 7. *The extended matching (μ_T, λ_T) does not admit any additional fair Pareto improvement.*

Proof. Let $(\mu, \lambda) = (\mu_T, \lambda_T)$ and assume to the contrary, that (ν, λ') is a fair Pareto improvement of (μ, λ). By Lemma 6, (ν, λ') is individually rational and nonwasteful. By the definition of the SETC algorithms, there is no improvement cycle in the graph $G(\mu, \lambda)$. There are two cases:

Case 1. For each $i \in I$, $\tilde{D}_{(\mu,\lambda)}(i) = \varnothing$. Then, by Lemma 4 and Remark 3 for each $i \in I$, $X_{(\mu,\lambda)}(i) \subseteq \{i' \in I : \mu(i) = \mu(i')\}$. This implies that each student is assigned to her best school at μ, there is no improvement cycle and ν does not Pareto dominate μ.

Case 2. There are paths in $G(\mu, \lambda)$ involving students who would like to change her assigned school, but there is no improvement cycle. This implies that there are students who are only pointed to by students assigned to the same school.

Assume we are in Case 2. Since there is no improvement cycle, there is a set of students who are not pointed to by any other student in $G(\mu, \lambda)$. Let $I_1 = \{i \in I : \tilde{D}_{(\mu,\lambda)}(i) = \varnothing\}$. Let $i_1 \in I_1$ and $s_1 = \mu(i_1)$. By Remark 3, for each j with $\mu(j) = s_1$, $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$ and $j \in I_1$. Since ν Pareto dominates μ, there does not exist any $j' \in I$, such that $\mu(j') \neq s_1$ and $\nu(j') = s_1$. Thus $\nu^{-1}(s_1) \subseteq \mu^{-1}(s_1)$. By Lemma 5, $\#\nu^{-1}(s_1) = \#\mu^{-1}(s_1)$ and we get $\nu^{-1}(s_1) = \mu^{-1}(s_1)$. Since i_1 was arbitrary, this holds for each s such that $\mu^{-1}(s) \cap I_1 \neq \varnothing$.

Next, since there is no improvement cycle in $G(\mu, \lambda)$, there is at least a student in $I \setminus I_1$ such that she is only pointed by students in I_1. Otherwise, there would be an improvement cycle or no path (Case 1). Let $I_2 = \{i \in I : \tilde{D}_{(\mu,\lambda)}(i) \subseteq I_1\} \setminus I_1$ Let $i_2 \in I_2$ and $s_2 = \mu(i_2)$. We first show that there is no j with $\mu(j) \neq s_2$ and $\nu(j) = s_2$. Assume to the contrary and since ν Pareto dominates μ, $s_2\ P_j\ \mu(j)$ and thus, $j \in \tilde{D}_{(\mu,\lambda)}(i_2)$. Nevertheless, by definition i_2 is only pointed by students in I_1. By the arguments in the previous paragraph, for each $j \in I_1$, $\mu(j) = \nu(j)$. Hence, $\nu^{-1}(s_2) \subseteq \mu^{-1}(s_2)$. By Lemma 5, $\#\mu^{-1}(s_2) = \#\nu^{-1}(s_2)$, and therefore $\mu^{-1}(s_2) = \nu^{-1}(s_2)$.

We can apply the same argument iteratively to conclude that all students in any path in $G(\mu, \lambda)$ have the same assignment under μ and ν. The students who are not in a path in $G(\mu, \lambda)$, are contained in I_1 and have the same assignment in both μ and ν. We conclude that $\mu = \nu$ and ν does not Pareto dominate μ. □

To conclude the proof of of Theorem 1, by Corollary 5, $\{(\mu_0, \lambda_0), (\mu_1, \lambda_1), \ldots, (\mu_T, \lambda_T)\}$ is a sequence of extended matching such that (μ_t, λ_t) is a fair Pareto improvements of $(\mu_{t-1}, \lambda_{t-1})$. Therefore, (μ_T, λ_T) is a justifiable Pareto improvement of (μ, ω). By Lemma 7, (μ_T, λ_T) does not admit any further fair Pareto improvement.

6.2 Proof of Theorem 2.

Theorem 2 is an immediate consequence of the following intermediate result. For every justifiable Pareto improvement of an initial (ex-post) stable extended matching, there is an application of the SETC that obtains an extended matching with the same matching after a finite number of steps.

Proposition 4. *Let (μ, λ) be an (ex-post) stable extended matching and (μ', λ') a justifiable Pareto improvement of (μ, λ). There exist $t \in \mathbb{N}$ and an extended matching (μ_t, λ_t) such that (μ_t, λ_t) is the outcome obtained at step t of an application of an SETC algorithm, $\mu_t = \mu'$, and (μ_t, λ_t) is characteristic-wise equivalent to (μ', λ').*

The key step in the proof of Proposition 4 is checking that the matching of any fair Pareto improvement from an arbitrary (ex-post) stable extended matching (μ, λ) can be obtained as well as a series of fair Pareto improvements generated after a series of steps of the application of an algorithm in the SETC class. Lemma 8 analyzes the structure of fair Pareto improvements and is a crucial first step for the construction of improvement cycles of $G(\mu, \lambda)$.

Lemma 8. *Let (μ, λ) be an (ex-post) stable extended matching and $(\nu, \bar{\lambda})$ a fair Pareto improvement of (μ, λ). There exists a finite set of disjoint cycles of students $\Phi = \{\phi_1, \ldots, \phi_m\}$ such that for each $i \notin \cup_{\phi \in \Phi} N(\phi)$, $\nu(i) = \mu(i)$, and for each $j \in \cup_{\phi \in \Phi} N(\phi)$, there are j' and $m' \leq m$ with $jj' \in \phi_{m'}$ and $\nu(j) = \mu(j')$.*

Proof. Let $N \subseteq I$ be the set of students who either strictly prefer their assignment under ν to the assignment under μ or such that $\lambda(i) \neq \bar{\lambda}(i)$. Let us partition the set N in three disjointed sets N_1, N_2, N_3 defined by:

$$\begin{aligned}
N_1 &= \{i \in N : \ \mu(i) = \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},\\
N_2 &= \{i \in N : \ \mu(i) \neq \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},\\
N_3 &= \{i \in N : \ \mu(i) \neq \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) = \lambda^{\nu(i)}(i)\},
\end{aligned}$$

Let $n = \#N$ and index the students in N in such that for each $x, x', x'' \in \{1, \ldots, n\}$, if $i_x \in N_1$, $i_{x'} \in N_2$, $i_{x''} \in N_3$ then $x < x' < x''$. Moreover, for each $x, y \in \mathbb{N}$ such that $i_x, i_y \in N_3$ and $\nu(i_x) = \nu(i_y)$, if $x < y$ then $(i_x, \bar{\lambda}^{\nu(i_x)}(i_x)) \succ_{\nu(i_x)} (i_y, \bar{\lambda}^{\nu(i_x)}(i_y))$.

Let $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda}']$ be a directed graph with vertices $i \in I$ and such that its edges are constructed sequentially in the following way. For each $x \in \{1, \ldots, n\}$:

i) If $i_x \in N_1$, i_x points to student j if and only if $i \neq j$ and $\bar{\lambda}^{\nu(i_x)}(i_x) = \lambda^{\nu(i_x)}(j)$.

ii) If $i_x \in N_2$, i_x points to student j if and only if $i \neq j$ and $\bar{\lambda}^{\nu(i_x)}(i_x) = \lambda^{\nu(i_x)}(j)$.

iii) If $i_x \in N_3$, i_x points to the student $j \in N$ such that $\mu(j) = \nu(i_x)$, and j has not been pointed by any i_y with $y < x$. [18]

Students that do not belong to N do not point to any other student. Note that for each $i \in N$, i always points to a student in N.

In the graph $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda})]$, each student is pointed to by a unique student and points to a unique student in N. Since N is finite, there is at least a cycle in the graph $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda})]$. Moreover, each student in N is in a cycle and no two cycles intersect. By construction, the matching ν is obtained by assigning each student to the school to which the student she points to is initially assigned. $\square$

Note that the cycles defined in Lemma 8 need not to define improvement cycles of $G(\mu, \lambda)$. In Lemma 9, we show that if $(\nu, \bar{\lambda})$ is a fair improvement cycle of (μ, λ), then the graph $G(\mu, \lambda)$ indeed has at least one improvement cycle involving students that participate in the fair Pareto improvement $(\nu, \bar{\lambda})$. Without loss of generality, we can assume that none of those cycles exclusively involves students assigned to the same school according to the matching μ.[19]

Lemma 9. *Let (μ, λ) be an (ex-post) stable extended matching. If $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ), then there exist a finite sequence of improvement cycles $\{\gamma_1, \ldots, \gamma_{t^*}\}$ and an allocation of transferable characteristics $\tilde{\lambda}$ such that:*

- *γ_1 is an improvement cycle of $G(\mu, \lambda)$.*
- *For each $t \in \{2, \ldots, t^*\}$, γ_t is an improvement cycle of $G(\gamma_{t-1} \circ \ldots \circ \gamma_1 \circ (\mu, \lambda))$.*
- *$(\nu, \tilde{\lambda}) = \gamma_{t^*} \circ \ldots \circ \ldots \circ \gamma_1 \circ (\mu, \lambda)$.*
- *$(\nu, \tilde{\lambda})$ is characteristic-wise equivalent to $(\nu, \bar{\lambda})$.*

Proof. Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be the set of cycles of students defined in Lemma 8. Since $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ), we can construct a set of pairs consisting of disjoint cycles and allocations of transferable characteristics restricted to the students

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[18] Note that since $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ) such a student j exists for each $i_x \in N_3$.

[19] It may be the case that some of the cycles defined in the proof of Lemma 8 involve students assigned to the same school according to the initial extended matching (μ, λ). Such a cycle would never be solved at any stage of an algorithm in the SETC class. However, we could construct a extended matching (ν, λ') that is characteristic-wise equivalent to $(\nu, \bar{\lambda})$ by setting $\lambda'(i) = \lambda(i)$ for each student involved in the non-improving cycle.

involved in the cycle, $\Pi = \big\{(\phi_1, \bar{\lambda} \mid_{N(\phi_1)}), \ldots, (\phi_M, \bar{\lambda} \mid_{N(\phi_M)})\big\}$. The result is trivial in the case when all the pairs in Π are improvement cycles of the graph $G(\mu, \lambda)$, but it is possible that no pair in the set Π is an improvement cycle of $G(\mu, \lambda)$. We proceed through a series of steps. Let $N = \cup_{\phi \in \Phi} N(\phi)$ be the set of students involved in cycles in Φ. First, we prove that there is at least an improvement cycle γ_1 in $G(\mu, \lambda)$ involving only students in N. Next, we prove that solving any of such improvement cycles leads to an extended matching, $(\mu_1, \lambda_1) = \gamma_1 \circ (\mu, \lambda)$, such that $(\nu, \bar{\lambda})$ (weakly) Pareto dominates (μ_1, λ_1). Finally, we check that for each such extended matching (μ_1, λ_1), there is an extended matching $(\nu, \hat{\lambda})$ such that $(\nu, \hat{\lambda})$ is characteristic-wise equivalent to $(\nu, \bar{\lambda})$ and a fair Pareto improvement of (μ_1, λ_1), which allows to repeat the argument iteratively as many times as necessary.

Step 1. There is an improvement cycle in $G(\mu, \lambda)$ that only involves students in N.

The result is trivial in the case where some element of Π is an improvement cycle of $G(\mu, \lambda)$. To prove the alternative case, we assume that none of the elements of Π appears in $G(\mu, \lambda)$.

To show the existence of an improvement cycle in $G(\mu, \lambda)$ first we prove that for any $\phi \in \Phi$ and any $ij \in \phi$, there exists some $k \in I$ such that $kj \in G(\mu, \lambda)$ and $k'k \in \phi'$ for some $k' \in I$ and $\phi' \in \Phi$. Consider an arbitrary $\phi \in \Phi$ and $ij \in \phi$. We consider two cases:

Case 1. If $i \in X_{(\mu,\lambda)}(j)$, then $ij \in G(\mu, \lambda)$ by construction. Moreover, i is involved in cycle ϕ, which implies there exists $k' \in I$ with $k'i \in \phi' \in \Phi$.

Case 2. If $i \notin X_{(\mu,\lambda)}(j)$, there exists a student i' such that $i' \in \tilde{D}_{(\mu,\lambda)}(j)$ and

$$(i', \lambda^{\mu(j)}(i')) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i, \lambda^{\mu(j)}(i)).$$

Let $k \in \tilde{D}_{(\mu,\lambda)}(j)$ be such that for each $i' \in \tilde{D}_{\mu,\lambda)}(j)$,

$$(k, \max\{\lambda^{\mu(j)}(k), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i', \max\{\lambda^{\mu(j)}(i'), \lambda^{\mu(j)}(j)\}).$$

Note that this student k exists because school priorities are complete and transitive. Note also that $k \in X_{(\mu,\lambda)}(j)$, and therefore $kj \in G(\mu, \lambda)$. Finally, we check that k is in a cycle in Φ. That is, there is $\phi' \in \Phi$ such that $k'k \in \phi'$ for some $k' \in I$. Assume to the contrary that $\mu(k) = \nu(k)$, $\lambda^{\mu(k)}(k) = \bar{\lambda}^{\mu(k)}(k)$, and $\mu(j)\ P_k\ \mu(k) = \nu(k)$. Note that since $ij \in \phi$, by Lemma 8, $\nu(i) = \mu(j)$ and $\bar{\lambda}^{\mu(j)}(i) \in \{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}$. Since $k \in X_{(\mu,\lambda)}(j)$, $i \notin X_{(\mu,\lambda)}(j)$, then $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, which is a contradiction, since $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ), and $(\nu, \bar{\lambda})$ is (ex-post) stable. Thus, $\nu(k)\ P_k\ \mu(k)$, which implies that k is in a cycle in Φ.

Thus, for each student j who is in an cycle $\phi \in \Phi$, there exists another student k such that $kj \in G(\mu, \lambda)$ and k is in an cycle $\phi' \in \Phi$. Since the set of students in a cycle is finite and each student is pointed to at least by another student in N, there exists at least a cycle in $G(\mu, \lambda)$. Note that if for each $ij \in \phi$, $i \in X_{(\mu,\lambda)}(j)$, then ϕ is a cycle of $G(\mu, \lambda)$, and since $\phi \in \Phi$, there is $i'j' \in \phi$ such that $\mu(i') \neq \mu(j')$. If there is $ij \in \phi$, $i \notin X_{(\mu,\lambda)}(j)$, then there is $k \in X_{(\mu,\lambda)}(j)$ with $\mu(k) \neq \mu(j)$. Therefore, each cycle of $G(\mu, \lambda)$ involving student j is an improvement cycle of $G(\mu, \lambda)$, which suffices to prove Step 1.

In Step 1 we have shown the existence of an improvement cycle in $G(\mu, \lambda)$ involving only students in N. In the remaining steps we focus on cycles with specific characteristics. Construct the graph $\bar{G}(\mu, \lambda)$ in the following way. Let $j \in N$. There is only one edge $ij \in \bar{G}(\mu, \lambda)$ and it is selected in the following way:

i) If $ij \in G(\mu, \lambda)$ and for each $i' \in N \cap \tilde{D}_{(\mu,\lambda)}(j)$ we have $(i, \bar{\lambda}^{\nu(i)}(i)) \succsim_{\nu(i)} (i', \lambda^{\mu(j)}(i'))$, then $ij \in \bar{G}(\mu, \lambda)$.

ii) If either $ij \notin G(\mu, \lambda)$ or $ij \in G(\mu, \lambda)$ and there is $i' \in N \cap \tilde{D}_{(\mu,\lambda)}(j)$ such that $(i', \lambda^{\mu(j)}(i')) \succ_{\mu(j)} (i, \bar{\lambda}^{\nu(i)}(i))$, then let $i^* \in N \cap \tilde{D}_{(\mu,\lambda)}(j)$ be the student such that for each $i' \in N \cap \tilde{D}_{(\mu,\lambda)}(j)$, $(i^*, \lambda^{\mu(j)}(i^*)) \succsim_{\nu(i)} (i, \bar{\lambda}^{\nu(i)}(i'))$, and $i^*j \in \bar{G}(\mu, \lambda)$.

That is, the graph $\bar{G}(\mu, \lambda)$ is constructed in such a way that, for each student j if the link $ij \in \phi$ also belongs to $G(\mu, \lambda)$, then $ij \in \bar{G}(\mu, \lambda)$; otherwise, j is pointed to by the student in $X_{(\mu,\lambda)}(j)$ with the highest transferable characteristic at $\mu(j)$. Note that $ij \in \bar{G}(\mu, \lambda)$ implies $ij \in G(\mu, \lambda)$. By the argument in the proof of Step 1, $\bar{G}(\mu, \lambda)$ has at least one cycle that by construction defines an improvement cycle of $G(\mu, \lambda)$. Let $\gamma_1 = (\phi^*, \lambda_1 |_{N(\phi^*)})$ be a specific improvement cycle in $\bar{G}(\mu, \lambda)$ where $\lambda_1 |_{N(\phi^*)}$ is defined in such a way that for each $i \in N(\phi^*)$:

i) if $ij \in \phi$ for some $(\phi, \bar{\lambda} |_{N(\phi)})$, then $\lambda_1^{\mu(j)}(i) = \bar{\lambda}^{\mu(j)}(i)$.

ii) otherwise:

  - if for each $j' \in \tilde{D}_{(\mu,\lambda)}(j)$, $(i, \lambda^{\mu(j)}(j)) \succsim_{\mu(j)} (j', \lambda^{\mu(j)}(j'))$, then $\lambda_1^{\mu(j)}(i) = \lambda^{\mu(j)}(j)$;
  - if for some $j' \in \tilde{D}_{(\mu,\lambda)}(j)$, $(j', \lambda^{\mu(j)}(j')) \succ_{\mu(j)} (i, \lambda^{\mu(j)}(j))$, then $\lambda_1^{\mu(j)}(i) = \lambda^{\mu(j)}(i)$.

By focusing on the improvement cycle γ_1, we consider cycles that coincide as much as possible to the cycles obtained in Lemma 8, and we avoid the exchange of seats among students without a simultaneous exchange of transferable characteristics whenever possible.

Let $(\mu_1, \lambda_1) = \gamma_1 \circ (\mu, \lambda)$. Since (μ_1, λ_1) is the outcome of solving an improvement cycle of $G(\mu, \lambda)$, by Corollary 5, (μ_1, λ_1) Pareto dominates (μ, λ). Hence, we focus on proving that $(\nu, \bar{\lambda})$ (weakly) Pareto dominates (μ_1, λ_1).

Step 2. For each $i \in I$, $\nu(i)\ R_i\ \mu_1(i)\ R_i\ \mu(i)$.

Let $\gamma_1 = (\phi^*, \lambda_1 \mid_{N(\phi^*)})$. Note that if $i \notin N(\phi^*)$, $\mu_1(i) = \mu(i)$ and since $(\nu, \bar{\lambda})$ Pareto dominates (μ, λ), we have that $\nu(i)\ R_i\ \mu_1(i)\ R_i\ \mu(i)$. Hence, assume $i \in N(\phi^*)$ and let $j \in N(\phi^*)$ be such that $ij \in \phi^*$. Note that $\mu_1(i) = \mu(j)$. We consider two cases

Case 1. If $ij \in \phi'$ for some $\phi' \in \Phi$, then $\nu(i) = \mu_1(i) = \mu(j)$.

Case 2. If $ij \notin \phi'$ for any $\phi' \in \Phi$, we claim that $\nu(i)\ R_i\ \mu(j)$. Suppose that $\mu(j)\ P_i\ \nu(i)$. That is, $i \in \tilde{D}_{(\nu,\bar{\lambda})}(j)$. Note that since (μ, λ) is (ex-post) stable, and $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ), we have that $(\nu, \bar{\lambda})$ is (ex-post) stable. Consider the student $k \in I$ such that $kj \in \phi'$ for some $\phi' \in \Phi$, so $\nu(k) = \mu(j)$. By the definition of γ_1, since $ij \in \phi^*$, $ij \in \bar{G}(\mu, \lambda)$ and $kj \notin \bar{G}(\mu, \lambda)$, we have that $(i, \lambda^{\mu(j)}(i)) \succ_{\mu(j)} (k, \bar{\lambda}^{\mu(j)}(k))$, which is a contradiction because $(\nu, \bar{\lambda})$ is (ex-post) stable.

Thus, each student j involved in γ_1 weakly prefers $\nu(j)$ to $\mu_1(j)$ to $\mu(j)$. Each remaining student is assigned to the same school to which she is assigned under μ which implies that the matching (μ_1, λ_1) Pareto dominates (μ, λ) and it is (weakly) Pareto dominated by $(\nu, \bar{\lambda})$.

Step 3. There is an extended matching $(\nu, \tilde{\lambda})$ such that $(\nu, \tilde{\lambda})$ is a fair Pareto improvement of (μ_1, λ_1) and $(\nu, \tilde{\lambda})$ is a characteristic-wise equivalent to $(\nu, \bar{\lambda})$.

The result is immediate if $\gamma_1 \in \Pi$. Hence assume that $\gamma_1 \notin \Pi$. Let $\gamma_1 = (\phi^*, \lambda_1 \mid_{N(\phi)})$ Construct an allocation of transferable characteristics $\tilde{\lambda}$ such that for each $i \notin N(\phi^*)$, $\tilde{\lambda}(i) = \bar{\lambda}(i)$, and for each $j \in N(\phi^*)$, for each $s \in S$ with $\mu_1(j)\ R_j\ s$, $\tilde{\lambda}^s(i) = \lambda_1^s(i)$, and for each $s' \in S$ with $s'\ P_j\ \mu_1(s)$, $\tilde{\lambda}^s(i) = \bar{\lambda}^s(i)$. Note that for each $j \notin N(\phi^*)$, $\mu_1(j) = \mu(j)$ and $\lambda_1(j) = \lambda(j)$, while for each $i \in N(\phi^*)$ and each s with $s\ P_i\ \mu_1(i)$, $\lambda_1^s(i) = \lambda^s(i)$. Hence, $\tilde{\lambda}$ is properly defined as an allocation of transferable characteristics. By the definition of the improvement cycle γ_1, if there is j and s such that for some i, $s = \nu(i) = \mu(j)$ and $\bar{\lambda}^s(i) = \lambda^s(j)$, but there is no $j' \in N$ with $\lambda_1^s(j') = \lambda^s(j)$, there is $k \in N$ with $kj \in \phi$ and $\lambda^s(k) > \lambda^s(j)$. Thus, $(\nu, \tilde{\lambda})$ is characteristic-wise equivalent to $(\nu, \bar{\lambda})$.

We next check that $(\nu, \tilde{\lambda})$ is a fair Pareto improvement of (μ_1, λ_1). By Step 2, $(\nu, \tilde{\lambda})$ Pareto dominates (μ_1, λ_1). Since $(\nu, \bar{\lambda})$ is (ex-post) stable and $(\nu, \tilde{\lambda})$ is characteristic-wise equivalent to $(\nu, \bar{\lambda})$, $(\nu, \tilde{\lambda})$ is (ex-post) stable. Hence, it only remains to check that for each $i \in I$ and $s \notin \{\mu_1(i), \nu(i)\}$, $\lambda_1^s(i) = \tilde{\lambda}^s(i)$. By Lemma 8, there is a set of disjoint cycles Φ such that for each cycle $\phi \in \Phi$ and each student $i \in I$ if $ij \in \phi$, $\nu(i) = \mu(j)$ and $\bar{\lambda}^{\nu(i)}(i) = \lambda^{\mu(j)}(j)$, and for each $s' \notin \{\nu(i), \mu(j)\}$, $\bar{\lambda}^s(i) = \lambda^s(i)$. Note that $\phi^* \notin \Phi$, but $N(\phi^*) \subseteq N = \cup_{\phi \in \Phi} N(\phi)$. Let $i \in I$. Assume first that $i \notin N(\phi^*)$, $\mu_1(I) = \mu(i)$, $\lambda_1(i) = \lambda(i)$, and $\bar{\lambda}(i) = \tilde{\lambda}(i)$. Next, assume that $i \in N(\phi^*)$. Note first that by the definition of $\tilde{\lambda}$ for each school s such that $\mu_1(i)\ P_i\ s$, $\lambda_1^s(i) = \tilde{\lambda}_1^S(i)$. Hence, we only have to check the possible changes in schools that are at least as good as $\mu_1(i)$. Let $j \in N$ be such that $ij \in \phi^*$. If $ij \in \phi$ for some $\phi \in \Phi$ by the definition of γ_1, $\mu_1(i) = \nu(i)$ and $\lambda_1(i) = \tilde{\lambda}(i)$. Next, assume that $ij \notin \phi$ for each $\phi \in \Phi$. Note that according to the matching μ_1, i occupies j position at $\mu(j)$. Let $j' \in N$ be such that $j'j \in \phi$ for some $\phi \in \Phi$. Then, $\nu(j') = \mu(j) = \mu_1(i)$. By the definitions of $\lambda_1 \mid_{N(\phi^*)}$, and $\tilde{\lambda}$, since $\lambda_1^{\mu(j)}(i) \neq \lambda^{\mu(j)}(j)$ implies that $\lambda^{\mu(j)}(i) > \lambda^{\mu(j)}(j)$, we have that $\tilde{\lambda}^{\mu(j)}(j') \in \{\lambda_1^{\mu(j)}(i), \lambda^{\mu(j)}(j')\}$. Finally, consider $j'' \in N$ such that $ij'' \in \phi'$ for some $\phi' \in \Phi$, and therefore $\nu(i) = \mu(j'')$. Note that if $j'' \notin N(\phi^*)$, then $\tilde{\lambda}(j'') = \lambda(j'')$. By the definition of $\tilde{\lambda}$, for each $s \notin \{\mu_1(i), \mu_1(j'')\}$, $\lambda_1^s(i) = \tilde{\lambda}^s(i)$. To conclude, assume that $j'' \in N(\phi^*)$. Let j^* be such that $j^*j'' \in \phi^*$. That is, j^* takes the position of j'' at school $\mu(j'')$ and the corresponding transferable characteristics (unless $\lambda^{\mu(j'')}(j^*) > \lambda^{\mu(j'')}(j'')$). Therefore, by the definition of $\tilde{\lambda}$, for each $s \notin \{\mu_1(i), \mu_1(j^*)\}$, $\lambda_1^s(i) = \tilde{\lambda}^s(i)$.

We are now in condition to conclude the proof of Lemma 9. The result is trivial in the case where all the elements of Π appear in $G(\mu, \lambda)$. In that case, the elements of Π are independent improvement cycles of $G(\mu, \lambda)$ involving disjoint sets of students. Solving the improvement cycle in an arbitrary order obtains $(\nu, \bar{\lambda})$. Hence, assume to the contrary that no pair in Π is an improvement cycle of $G(\mu, \lambda)$. This assumption is without loss of generality because of the following observation. If a pair $(\phi, \bar{\lambda} \mid_{N(\phi)}) \in \Pi$ is an improvement cycle of $G(\mu, \lambda)$, then this improvement cycle is solved first. Since all the pairs in Π involve disjoint sets of students and whenever there are two students forming a link in $G(\mu, \lambda)$, and those students are not involved in the cycle $\phi \in \Phi$, then the link also appears in $G((\phi, \bar{\lambda} \mid_{N(\phi)}) \circ (\mu, \lambda))$. Following this logic, whenever a subset of cycles Φ appear in $G(\mu, \lambda)$, these cycles are solved first, until the point when no improvement cycles of $G(\mu, \lambda)$ remains. In that case, by Step 1, we can find an improvement cycle in $G(\mu, \lambda)$ involving only students in N. By Step 2, the extended matching obtained solving any such improvement cycle is not Pareto dominated by $(\nu, \bar{\lambda})$, and by Step 3, it admits a fair Pareto improvement by an extended matching that is characteristic-wise equivalent to $(\nu, \bar{\lambda})$. We can repeat the argument as many times as necessary solving improvement cycles until we obtain an extended matching that is characteristic-wise equivalent to $(\nu, \bar{\lambda})$. □

Lemma 10. *Let* (μ, λ) *be an (ex-post) stable extended matching, and let* (μ_1, λ_1) *and* $(\mu_1, \hat{\lambda}_1)$ *be two possible outcomes at the first stage of the SETC. If for each* $i \in I$, $\lambda_1^{\mu_1(i)}(i) \geq \hat{\lambda}_1^{\mu_1(i)}(i)$ *and there is a cycle* ϕ *in* $G(\mu_1, \hat{\lambda}_1)$*; then the cycle* ϕ *also appears in* $G(\mu_1, \lambda_1)$.

Proof. Note that for each $i \in I$ with $\hat{\lambda}_1^{\mu_1(i)}(i) \in \{\lambda^{\mu_1(i)}(i), \lambda^{\mu_1(i)}(j)\}$ for some j with $\mu_1(i) = \mu(j)$, For each i and each j' such that $\mu_1(j')\ P_i\ \mu_1(i)$, $\hat{\lambda}_1^{\mu_1(j')}(i) = \lambda_1^{\mu_1(j')}(i)$ If a student $j'' \in X_{(\mu_1, \hat{\lambda})}(i)$, then also $j'' \in X_{(\mu_1, \lambda_1)}(i)$, because $\lambda_1^{\mu_1(i)}(i) \geq \hat{\lambda}_1^{\mu_1(i)}(i)$. Hence, if there is an improvement cycle involving a cycle ϕ in $G(\mu_1, \hat{\lambda}_1)$, then $G(\mu_1, \lambda_1)$ admits an improvement cycle involving the cycle ϕ. □

Proof of Proposition 4. Let $(\nu, \bar{\lambda})$ be a justifiable Pareto improvement of (μ, ω). There is a sequence of extended matchings, $\{(\mu'_0, \lambda'_0), (\mu'_1, \lambda'_1), \ldots, (\mu'_{t*}, \lambda'_{t^*})\}$ such that improvements that $(\mu'_0, \lambda'_0) = (\mu, \lambda)$, $(\mu'_{t^*}, \lambda'_{t^*}) = (\mu', \lambda')$ and for each $t \in \{1, \ldots, t^*\}$, (μ'_t, λ'_t) is a fair Pareto Improvement of $(\mu'_{t-1}, \lambda'_{t-1})$. By Lemma 9, the application of an SETC algorithm starting at (μ, λ) yields after a finite number of steps an extended matching (μ'_1, λ^*_1) that is characteristic-wise equivalent to (μ'_1, λ'_1). Applying the argument of Step 3 in Lemma 9, we can construct an extended matching characteristic-wise equivalent to (μ'_2, λ'_2) such that is a fair Pareto improvement of (μ'_1, λ^*_1). By Lemma 10, if there is an improvement cycle in $G(\mu'_1, \lambda_1)$, then there is an improvement cycle in $G(\mu'_1, \lambda^*_1)$ involving the same cycle of students. We can repeat the argument as many times as necessary to obtain the result. □

Proof of Theorem 2. Let (μ_t, λ_t) be the extended matching obtained after a series of t steps of the application of a SETC algorithm such that (μ_t, λ_t) is characteristic-wise equivalent to (μ', λ'). Either the algorithm stops at step t and the (μ_t, λ_t) is contrained efficient, or (μ_t, λ_t) admits a fair Pareto improvement and the final outcome of the SETC Pareto dominates (μ', λ'). □

6.3 Proof of the Remaining Results

Proof of Proposition 1. Let A be an algorithm in the SETC, define the SOTC mechanism Ψ that for each profile of students' preferences selects the matching obtained through the application of A at that preference profile. By Theorem 1, for each preference profile the extended matching selected by Ψ is (ex-post) stable and constrained efficient. For each $P \in \mathcal{P}$, $\Psi(P) = (\mu, \lambda)$ (weakly) Pareto dominates the SOSEM, and for some $P' \in \mathcal{P}$, $\Psi(P')$ Pareto dominates the SOSEM. By Abdulkadiroğlu et al. (2009), the SOSEM is in the Pareto frontier of the set of mechanisms that satisfy *strategy-proofness*. Hence, Ψ violates *strategy-proofness*. □

Proof of Lemma 1. Let $s = \mu(j)$. Since $i \in \tilde{D}_{(\mu,\lambda)}(j)$, we have that $s \; P_i \; \mu(i)$. Since (μ, λ) is (ex-post) stable, for each $j' \neq i$ such that $s \; P_{j'} \; \mu(j')$, $(j, \lambda^s(j)) \succ_s (j', \lambda^s(j'))$. Therefore, since priorities are fully transferable, we have $(i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (j', \lambda^s(j'))$ and $i \in X_{(\mu,\lambda)}(j)$. □

Proof of Proposition 3. We start the proof with a description of an algorithm that obtains the EADA matching in problems without transferable characteristic and the algorithm in the SETC class that translate it to the extended matching framework.

(Simplified) Efficiency Adjusted Deferred Acceptance Algorithm (EADA). Tang and Yu (2014):

Given a matching μ, a school s is ***underdemanded in relation to*** μ if no student prefers s to the school to which they are assigned by μ. The simplified EADA algorithm works by executing the student-proposing DA algorithm iteratively after sequentially altering the preferences of students assigned to underdemanded schools. Starting with the SOSEM, as a first step, the student-proposing DA algorithm is executed a second time with the students previously assigned to underdemanded schools listing those schools as their top choices. Therefore, in this second stage, students at underdemanded schools retain their seats, and their potential priorities at schools where they cannot obtain seats become ineffective. This process is repeated until there are no underdemanded schools.

Next, we propose the EADA-SETC algorithm, a specific SETC algorithm that under transferable priorities selects the matching obtained by the (simplified) EADA algorithm. The successive selection of cycles utilized by this algorithm requires identifying the students assigned to underdemanded schools, dropping the potential cycles involving those students, and of the remaining cycles, solving those that would satisfy the school priorities under the initial allocation of transferable characteristic for the students who are not assigned to underdemanded schools first. This process is equivalent to running the student-proposing DA algorithm when students who are assigned to underdemanded schools report that those underdemanded schools are their most preferred alternative. Running this process as many times as necessary leads to a constrained efficient extended matching with the matching process of the EADA algorithm.

EADA-SETC Algorithm:

Step *0*. Let $(\mu_0, \lambda_0) = (\mu^{SO}_\omega, \omega)$, $I_0 = I$, and let $U_0 = \{s \in S_0 : \text{ for each } j \in I,\ \mu_0(j)\ R_j\ s\}$, be the set of underdemanded schools at μ_0.

Step *t*≥*1*. Given $(\mu_{t-1}, \lambda_{t-1})$:

Stage *t.0*. If $\cup_{\tau=0}^{t-1} U_\tau = S$, the algorithm stops and $(\mu_{t-1}, \lambda_{t-1})$ is the final outcome. If $\cup_{\tau=0}^{t-1} U_\tau \neq S$, let $(\mu_t^0, \lambda_t^0) = (\mu_{t-1}, \lambda_{t-1})$, $I_t = I \backslash \big\{i \in I : \ \mu_{t-1}(i) \in \big(\cup_{\tau=0}^{t-1} U_\tau\big) \cup \{\varnothing\}\big\}$, and move to stage $t.1$.

Stage *t.t' (t'*≥*1)*. For each extended matching (μ, λ), let the graph $G_t(\mu, \lambda)$ be such that for each $i, j \in I$, $ij \in G_t(\mu, \lambda)$ if and only if $i, j \in I_t$ and for each $i' \in \tilde{D}(\mu, \lambda)(j) \cap I_t$, $(i, \lambda^{\mu(j)}(i)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$.

- If there is one or more cycles at $G_t(\mu_t^{t'-1}, \lambda_t^{t'-1})$, solve one of the cycles at $G_t(\mu_t^{t'-1}, \lambda_t^{t'-1})$, for example, γ; let $(\mu_t^{t'}, \lambda_t^{t'}) = \gamma \circ (\mu_t^{t'-1}, \lambda_t^{t'-1})$ and move to Stage $t.(t'+1)$.

- If there is no cycle at $G_t(\mu_t^{t'-1}, \lambda_t^{t'-1})$, let $(\mu_t, \lambda_t) = (\mu_t^{t'-1}, \lambda_t^{t'-1})$, and let

  $$U_t = \left\{ s \in S \setminus \left(\cup_{\tau=0}^{t-1} U_\tau \right) : \text{ for each } i \in I_{t-1},\ \mu_t(j)\ R_j\ s \right\},$$

  and move to step $t+1$.

Note that for each step $t \geq 1$, $\cup_{\tau=0}^{t-1} U_\tau$ is the set of underdemanded schools at μ_{t-1}, and I_t are the set of students who are not assigned to underdemanded schools at μ_{t-1}.

We now check that for each school choice problem with transferable characteristics and fully transferable priorities the extended matching obtained by the EADA-SETC algorithm selects the EADA matching.

Note first that by Lemma 1, for each $i, j \in I$, k and t, $ij \in G_t(\mu, \lambda)$ implies $ij \in G(\mu, \lambda)$. Thus, since (μ_0, λ_0) is (ex-post) stable, by Corollary 1, $(\mu_t^{t'}, \lambda_t^{t'})$ is also (ex-post) stable. Since for each t, t', and $j \in I_t$, there is at most another student i such that $ij \in G_t(\mu_t^{t'}, \lambda_t^{t'})$. This fact implies that for each t, t', all the cycles in $G_t(\mu_t^{t'}, \lambda_t^{t'})$ are disjoint, that is, iff ϕ and ϕ' are cycles in $G_t(\mu_t^{t'}, \lambda_t^{t'})$, then $\phi \cap \phi' = \varnothing$. Note also that since (μ_0, λ_0) is the SOSEM $(\mu_\omega^{SO}, \omega)$, and $(\mu_\omega^{SO}, \omega)$ Pareto dominates every other (ex-post) stable extended matching (μ, ω), $U_0 \neq \varnothing$, because the existence of a cycle in $G_0(\mu_\omega^{SO}, \omega)$ implies that there is an extended matching that Pareto dominates the SOSEM that is (ex-post) stable according to the school priorities defined under the initial allocation of transferable characteristics ω. Moreover, if at some t, $\cup_{\tau=0}^{t-1} U_\tau = S$, by an argument similar to those in the proof of Lemma 7 and since priorities are fully transferable, then $(\mu_{t-1}, \lambda_{t-1})$ does not admit any improvement cycle, and no extended matching (μ', λ') Pareto dominates $(\mu_{t-1}, \lambda_{t-1})$. Hence, the algorithm selects a constrained efficient extended matching.

If $U_0 = S$, then every student is assigned to her best preferred school, and the algorithm stops immediately, $(\mu_\omega^{SO}, \omega)$ is constrained efficient, and μ_ω^{SO} coincides with the outcome of the EADA algorithm. If $U_0 \neq \varnothing$, note that (μ_0, λ_0) is (ex-post) stable but it is not constrained efficient. We prove the result by comparing the graph $G_1(\mu_0.\lambda_0)$ defined at step 1 of the EADA-SETC algorithm with the directed application graph associated to (μ_0, λ_0) obtained for an alternative school choice problem for specific student preferences and school priorities.

Consider the school choice problem with transferable characteristics

$$(I, S, (q_s)_{s \in S}, (R_i^*)_{i \in I}, \omega, (\succ_s)_{s \in S}),$$

such that for each $i \in I_1$, $R_i^* = R_i$ and for each $j \notin I_1$ R_j^* is such that for each $s \in S \setminus \{\mu_0(j)\}$, $\mu_0(j)\ P_j^*\ s$. and for each $i, j \in I$ and each allocation of transferable characteristics λ, $(i, \lambda^s(i)) \succsim_s^* (j, \lambda^s(j))$ if and only if $(i, \omega^s(i)) \succsim_s^* (j, \omega^s(j))$. That is, students assigned to underdemanded schools under μ_0 consider that school as the best possible alternative, and transferable characteristics are irrelevant for school priorities. For each extended matching (μ, λ), let us denote by $G^*(\mu, \lambda)$ the directed application graph associated with (μ, λ) for the problem $(I, S, (q_s)_{s \in S}, (R_i^*)_{i \in I}, \omega, (\succ_s)_{s \in S})$ Note that $G^*(\mu_0, \lambda_0)$ coincides with $G_1(\mu_0, \lambda_0)$. By Theorem 1 and Corollary 1, starting with an (ex-post) stable extended matching, the SETC algorithm obtains a constrained efficient extended matching. Note that under the new student preferences and school priorities, since the transferable characteristics are irrelevant, the student-proposing DA algorithm obtains the unique constrained efficient matching (see Gale and Shapley, 1962). This fact also implies that the order in which the cycles are solved at any stage $1.t$ is irrelevant and a unique extended matching (μ_1, λ_1) is obtained, and μ_1 coincides with the matching of the SOSEM for the school choice problem with student preferences $(R_i^*)_{i \in I}$.

We can iteratively repeat the argument as many times as necessary for each $t \geq 1$, and $G^*(\mu_t, \lambda_t)$ coincides with $G_t(\mu_t, \lambda_t)$, until for some $t \geq 0$, $U_t = \cup_{\tau=0}^{t-1} U_\tau = S$, which completes the proof. □

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Appendix A. Obtaining Individually Rational and Nonwasteful Extended Matchings. (Not for Publication).

In this appendix we consider the possibility of obtaining fair Pareto improvements over extended matchings that are neither individually rational nor nonwasteful.

It is immediate to obtain a fair Pareto improvement of an extended matching that is not individually rational. Simply, let (μ, ω) be such that for some $i \in I$, $\{\varnothing\} \succ_i \mu(i)$. The extended matching (μ', ω) such that for each $i \in I$ $\mu'(i) = \{\varnothing\}$ if $\{\varnothing\}\ P_i\ \mu(i)$ and $\mu'(i) = \mu(i)$ otherwise is an individually rational extended matching and a fair Pareto improvement over (μ, ω).

Next we present an algorithm that for every individually rational extended matching, obtains a nonwasteful extended matching that is a fair Pareto improvement of the initial extended matching.

For each extended matching (μ, λ) let

$$F(\mu, \lambda) = \left\{ s \in S :\ \#\mu^{-1}(s) < q_s \text{ and there is } i \in I \text{ with } s\ P_i\ \mu(i) \right\}.$$

That is, $F(\mu, \lambda)$ is the set of demanded schools with available positions at the extended matching (μ, λ).

Student Fair Refilling Algorithm

Step 0: Let (μ_0, λ) be an individually rational extended matching.

Step $t \geq 1$: Given the extended matching (μ_{t-1}, λ),

- If $F(\mu_{t-1}, \lambda) = \varnothing$, then the algorithm stops and (μ_{t-1}, λ) is the obtained extended matching.
- If $F(\mu_{t-1}, \lambda) \neq \varnothing$, then pick an arbitrary school $\hat{s} \in F(\mu_{t-1}, \lambda)$, let $\hat{\imath}_t$ be the student such that $\hat{s}\ P_{\hat{\imath}_t}\ \mu_{t-1}(\hat{\imath}_t)$ and for each $j \in I$ with $\hat{s}\ P_j\ \mu_{t-1}(j)$, $(\hat{\imath}_t, \lambda^{\hat{s}}(\hat{\imath}_t)) \succsim_{\hat{s}} (j, \lambda^{\hat{s}}(j))$, and let let μ_t be defined by
  i) $\mu_t(\hat{\imath}_t) = \hat{s}$, and
  ii) for each $i \in I \setminus \{\hat{\imath}_t\}$, $\mu_k(i) = \mu_{k-1}(i)$,

  and move to step $t + 1$.

This algorithm fills an empty position at a time but a new vacant may open. Since the students that obtain a new position strictly improve, the algorithm eventually stops after a finite number of steps and obtains an individually rational and nonwasteful extended matching. Note that, since the algorithm does not perform any change in the allocation of transferable characteristics, at each step the algorithm obtains a fair Pareto improvement of the initial extended matching.

Remark 4. *Let (μ, ω) be an individually rational extended matching. Then any outcome of the student fair refilling algorithm with $(\mu_0, \omega) = (\mu, \omega)$ is an individually rational and nonwasteful extended matching and a fair Pareto improvement over (μ, λ).*

Appendix B. On Constrained Efficient Extended Matchings and SETC Outcomes. (Not for Publication)

In this appendix, we provide the complete description of the example of a constrained efficient extended matching that is Pareto dominated by the outcome of the application of a SETC algorithm (see footnote 15).

Example 7. *Let $I = \{i_1, i_2, i_3, i_4, i_5, i'_1, i'_2, i'_3, i_4, i'_5, j\}$, $S = \{s_1, s_2, s_3, s_4, s_5, s'_1, s'_2, s'_3, s'_4, s'_5, \sigma\}$, and $q_s = 1$ for $s \in S$. The relevant student preferences are as follows:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_j
s_2	s_3	s'_1	s_2	s_4	s_1
s_1	s_5	s_1	s_3	s_5	s'_1
$\{\varnothing\}$	s_1	s_3	s_5	$\{\varnothing\}$	σ
	s_2	$\{\varnothing\}$	s_4		$\{\varnothing\}$
	$\{\varnothing\}$		$\{\varnothing\}$		

$P_{i'_1}$	$P_{i'_2}$	$P_{i'_3}$	$P_{i'_4}$	$P_{i'_5}$
s'_2	s'_3	s_1	s'_2	s'_4
s'_1	s'_5	s'_1	s'_3	s'_5
$\{\varnothing\}$	s'_1	s'_3	s'_5	$\{\varnothing\}$
	s'_2	$\{\varnothing\}$	s'_4	
	$\{\varnothing\}$		$\{\varnothing\}$	

The relevant initial allocation of transferable characteristics and relevant school priorities are:

$\omega^s(i)$	i_1	i_2	i_3	i_4	i_5	j
s_1	$\mathbf{0}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{5}^{s_1}$
s_2	$\mathbf{2}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$	$\cdots$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{2}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{0}^{s_3}$	$\cdots$
s_4	$\mathbf{1}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{0}^{s_4}$	$\cdots$
s_5	$\mathbf{1}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{2}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{0}^{s_5}$	$\cdots$
s'_1	$\cdots$	$\cdots$	$-\mathbf{1}^{s'_1}$	$\cdots$	$\cdots$	$\mathbf{5}^{s'_1}$
σ	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbf{5}^{\sigma}$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
$(i_1, \mathbf{0}^{s_1})$	$(i_2, \mathbf{4}^{s_2})$	$(i_3, \mathbf{4}^{s_3})$	$(i_4, \mathbf{0}^{s_4})$	$(i_5, \mathbf{0}^{s_5})$
$(i_2, \mathbf{4}^{s_1})$	$(i_1, \mathbf{4}^{s_2})$	$(i_2, \mathbf{4}^{s_3})$	$(i_5, \mathbf{4}^{s_4})$	$(i_2, \mathbf{4}^{s_5})$
$(i_3, \mathbf{3}^{s_1})$	$(i_4, \mathbf{3}^{s_2})$	$(i_4, \mathbf{4}^{s_3})$		$(i_4, \mathbf{3}^{s_5})$
$(i'_3, \mathbf{3}^{s_1})$	$(i_1, \mathbf{2}^{s_2})$	$(i_4, \mathbf{3}^{s_3})$		
$(j, \mathbf{5}^{s_1})$		$(i_2, \mathbf{2}^{s_3})$		
$(i'_3, \mathbf{0}^{s_1})$				
$(i'_3, -\mathbf{1}^{s_1})$				

$\omega^s(i)$	i'_1	i'_2	i'_3	i'_4	i'_5	j
s'_1	$\mathbf{0}^{s'_1}$	$\mathbf{4}^{s'_1}$	$\mathbf{3}^{s'_1}$	$\mathbf{2}^{s'_1}$	$\mathbf{1}^{s'_1}$	$\mathbf{5}^{s'_1}$
s'_2	$\mathbf{2}^{s'_2}$	$\mathbf{4}^{s'_2}$	$\mathbf{1}^{s'_2}$	$\mathbf{3}^{s'_2}$	$\mathbf{0}^{s'_2}$	$\cdots$
s'_3	$\mathbf{1}^{s'_3}$	$\mathbf{2}^{s'_3}$	$\mathbf{4}^{s'_3}$	$\mathbf{3}^{s'_3}$	$\mathbf{0}^{s'_3}$	$\cdots$
s'_4	$\mathbf{1}^{s'_4}$	$\mathbf{2}^{s'_4}$	$\mathbf{3}^{s'_4}$	$\mathbf{4}^{s'_4}$	$\mathbf{0}^{s'_4}$	$\cdots$
s'_5	$\mathbf{1}^{s'_5}$	$\mathbf{4}^{s'_5}$	$\mathbf{2}^{s'_5}$	$\mathbf{3}^{s'_5}$	$\mathbf{0}^{s'_5}$	$\cdots$
s_1	$\cdots$	$\cdots$	$-\mathbf{1}^{s_1}$	$\cdots$	$\cdots$	$\mathbf{5}^{s_1}$

$\succ_{s'_1}$	$\succ_{s'_2}$	$\succ_{s'_3}$	$\succ_{s'_4}$	$\succ_{s'_5}$
$(i'_1, \mathbf{0}^{s'_1})$	$(i'_2, \mathbf{4}^{s'_2})$	$(i'_3, \mathbf{4}^{s'_3})$	$(i'_4, \mathbf{0}^{s'_4})$	$(i'_5, \mathbf{0}^{s'_5})$
$(i'_2, \mathbf{4}^{s'_1})$	$(i'_1, \mathbf{4}^{s'_2})$	$(i'_2, \mathbf{4}^{s'_3})$	$(i'_5, \mathbf{4}^{s'_4})$	$(i'_2, \mathbf{4}^{s'_5})$
$(i'_3, \mathbf{3}^{s'_1})$	$(i'_4, \mathbf{3}^{s'_2})$	$(i'_4, \mathbf{4}^{s'_3})$		$(i'_4, \mathbf{3}^{s'_5})$
$(i_3, \mathbf{3}^{s'_1})$	$(i'_1, \mathbf{2}^{s_2})$	$(i'_4, \mathbf{3}^{s'_3})$		
$(j, \mathbf{5}^{s'_1})$		$(i'_2, \mathbf{2}^{s'_3})$		
$(i_3, \mathbf{0}^{s'_1})$				
$(i_3, -\mathbf{1}^{s'_1})$				

The SOSEM $(\mu^{SO}_{\omega}, \omega)$ *is defined by*

$$\mu^{SO}_{\omega} = [(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_5), (i'_1, s'_1), (i'_2, s'_2), (i'_3, s'_3), (i'_4, s'_4), (i'_5, s'_5), (j, \sigma)] .$$

Consider now the extended matching (μ', λ) *with*

$$\mu' = [(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_5), (i_5, s_4), (i'_1, s'_2), (i'_2, s'_3), (i'_3, s'_1), (i'_4, s'_5), (i'_5, s'_4), (j, \sigma)] .$$

and λ *be such that* $\lambda(j) = \omega(j)$ *and:*

$\lambda^s(i)$	i_1	i_2	i_3	i_4	i_5
s_1	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{0}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{4}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{2}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{1}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{0}^{s_4}$	$\mathbf{4}^{s_4}$
s_5	$\mathbf{1}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{2}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{0}^{s_5}$
s'_1			$-\mathbf{1}^{s'_1}$		
$\lambda^s(i)$	i'_1	i'_2	i'_3	i'_4	i'_5
s'_1	$\mathbf{3}^{s'_1}$	$\mathbf{4}^{s'_1}$	$\mathbf{0}^{s'_1}$	$\mathbf{2}^{s'_1}$	$\mathbf{1}^{s'_1}$
s'_2	$\mathbf{4}^{s'_2}$	$\mathbf{2}^{s'_2}$	$\mathbf{1}^{s'_2}$	$\mathbf{3}^{s'_2}$	$\mathbf{0}^{s'_2}$
s'_3	$\mathbf{1}^{s'_3}$	$\mathbf{4}^{s'_3}$	$\mathbf{2}^{s'_3}$	$\mathbf{3}^{s'_3}$	$\mathbf{0}^{s'_3}$
s'_4	$\mathbf{1}^{s'_4}$	$\mathbf{2}^{s'_4}$	$\mathbf{3}^{s'_4}$	$\mathbf{0}^{s'_4}$	$\mathbf{4}^{s'_4}$
s'_5	$\mathbf{1}^{s'_5}$	$\mathbf{4}^{s'_5}$	$\mathbf{2}^{s'_5}$	$\mathbf{3}^{s'_5}$	$\mathbf{0}^{s'_5}$
s_1			$-\mathbf{1}^{s_1}$		

The extended matching (μ', λ) is a fair Pareto improvement of $(\mu_\omega^{SO}, \omega)$, and it is a constrained efficient extended matching. However, (μ', λ) cannot be obtained by the application of the SETC algorithm. With the arguments of Example 6, after a series of steps, an application of the SETC algorithm may obtain the extended matching $(\mu', \bar\lambda)$ that is characteristic-wise equivalent to (μ', λ), with $\bar\lambda$ being defined by $\bar\lambda(j) = \omega(j)$ and

$\bar\lambda^s(i)$	i_1	i_2	i_3	i_4	i_5
s_1	$\mathbf{0}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{4}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{2}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{1}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{0}^{s_4}$	$\mathbf{4}^{s_4}$
s_5	$\mathbf{1}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{2}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{0}^{s_5}$
s'_1			$-\mathbf{1}^{s'_1}$		

$\bar\lambda^s(i)$	i'_1	i'_2	i'_3	i'_4	i'_5
s'_1	$\mathbf{0}^{s'_1}$	$\mathbf{3}^{s'_1}$	$\mathbf{4}^{s'_1}$	$\mathbf{2}^{s'_1}$	$\mathbf{1}^{s'_1}$
s'_2	$\mathbf{4}^{s'_2}$	$\mathbf{2}^{s'_2}$	$\mathbf{1}^{s'_2}$	$\mathbf{3}^{s'_2}$	$\mathbf{0}^{s'_2}$
s'_3	$\mathbf{1}^{s'_3}$	$\mathbf{4}^{s'_3}$	$\mathbf{2}^{s'_3}$	$\mathbf{3}^{s'_3}$	$\mathbf{0}^{s'_3}$
s'_4	$\mathbf{1}^{s'_4}$	$\mathbf{2}^{s'_4}$	$\mathbf{3}^{s'_4}$	$\mathbf{0}^{s'_4}$	$\mathbf{4}^{s'_4}$
s'_5	$\mathbf{1}^{s'_5}$	$\mathbf{4}^{s'_5}$	$\mathbf{2}^{s'_5}$	$\mathbf{3}^{s'_5}$	$\mathbf{0}^{s'_5}$
s_1			$-\mathbf{1}^{s_1}$		

The extended matching (μ', λ) is not constrained efficient because an additional exchange between students i_3 and i'_3 is possible. In fact ,the extended matching $(\mu'', \hat\lambda)$ defined by

$$\mu'' = [(i_1, s_2), (i_2, s_3), (i_3, s'_1), (i_4, s_5), (i_5, s_4), (i'_1, s'_2), (i'_2, s'_3), (i'_3, s_1), (i'_4, s'_5), (i'_5, s'_4), (j, \sigma)],$$

and $\hat\lambda(i) = \bar\lambda(i)$ for each $i \notin \{i_3, i'_3\}$ and $\hat\lambda^{s_1}(i'_3) = \bar\lambda^{s_1}(i'_3) = \mathbf{4}^{s_1}$ and $\hat\lambda^{s'_1}(i_3) = \bar\lambda^{s'_1}(i'_3) = \mathbf{4}^{s'_1}$, is the outcome of an application of the SETC algorithm starting at the SOSEM. The extended matching $(\mu'', \hat\lambda)$ Pareto dominates the constrained efficient extended matching (μ', λ).

Appendix C. On Transferable Characteristics for Multiple Schools. (Not for Publication).

Example 8 illustrates the difficulties that may arise when characteristics are not school specific. It can be the case that some students could swap seats at their initially assigned and transferable characteristics, all the remaining students priorities at those schools are respected. However, if transferable characteristics are not school specific, the new allocation of transferable characteristics may trigger a sequence of violations of fairness in other schools.

Example 8. *Let* $I = \{i_1, i_2, i_3, i_4, i_5\}$*,* $S = \{s_1, s_2, s_3, s_4, s_5\}$*,* $q_{s_x} = 1$ *for* $x = 1, \ldots, 5$*. The students' relevant preferences are:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
s_4	s_1	s_1	s_4	s_5
s_2	s_2	s_2	s_5	s_4
s_1	s_3	s_3	$\{\varnothing\}$	$\{\varnothing\}$
$\{\varnothing\}$	$\{\varnothing\}$	$\{\varnothing\}$		

The transferable characteristics for school s_2 *determine the priorities of school l* s_4*. Hence, the relevant initial allocation of transferable characteristics and the relevant school priorities are:*

$\omega^s(i)$	i_1	i_2	i_3	i_4	i_5
s_1	$\mathbf{4}^{s_1}$	$\mathbf{0}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{0}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$
s_3	$\cdots$	$\mathbf{0}^{s_3}$	$\mathbf{4}^{s_3}$	$\cdots$	$\cdots$
s_4	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
s_5	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
$(i_1, \mathbf{4}^{s_1})$	$(i_1, \mathbf{4}^{s_2})$	$(i_2, \mathbf{0}^{s_3})$	$(i_5, \mathbf{1}^{s_2})$	$(i_4, \mathbf{0}^{s_5})$
$(i_4, \mathbf{1}^{s_1})$	$(i_2, \mathbf{4}^{s_2})$	$(i_3, \mathbf{4}^{s_3})$	$(i_1, \mathbf{4}^{s_2})$	$(i_5, \mathbf{4}^{s_5})$
$(i_2, \mathbf{4}^{s_1})$	$(i_3, \mathbf{3}^{s_2})$		$(i_4, \mathbf{2}^{s_2})$	
$(i_3, \mathbf{3}^{s_1})$	$(i_1, \mathbf{0}^{s_2})$		$(i_1, \mathbf{0}^{s_2})$	
$(i_2, \mathbf{0}^{s_1})$				

and $\mu^{SO}_{\omega} = [(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_5)]$.

When students i_1 *and* i_2 *exchange their transferable characteristics, the allocation of exchangeable characteristics is* λ'*.*

$\lambda^s(i)$	i_1	i_2	i_3	i_4	i_5
s_1	$\mathbf{0}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{4}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

With the exchange of transferable characteristics, students i_1 *and* i_2 *could improve by exchanging their respective seats at* s_1 *and* s_2*. The extended matching* (μ', λ) *with*

$$\mu' = [(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_5)],$$

is a fair Pareto improvement of the SOSEM. However, the extended matching (μ', λ') *is not (ex-post) stable. After obtaining* $\mathbf{4}^{s_2}$*, student* i_1 *would have justified envy of* i_4 *at* s_4*. In fact,* $\mu^{SO}_{\lambda'} = [(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_5), (i_5, s_4)]$*. That is, the initial Pareto improvement generated by the exchange of transferable characteristics initiates a chain reaction that leads to an extended matching where the students initiating the exchange are not better off.*

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