

## On Covering Properties\*)

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Using AULL's concept of an  $\alpha$ -paracompact subset, the various axioms of separation and embeddings, we define in this paper some classes of topological spaces: the classes  $\Gamma^*(T_i)$  for each  $i = 2, 3, 3a, 4, 5, 5a$  and  $\Gamma^*(T_j^*)$  for each  $j = 4, 5, 5a$ . (Throughout this paper a  $T_3$  space is a regular  $T_0$  space, a  $T_{3a}$  space is a completely regular  $T_0$  space, a  $T_4$  space is a normal  $T_1$  space, a  $T_5$  space is a hereditarily normal  $T_1$  space, a  $T_{5a}$  space is a perfectly normal  $T_0$  space, a  $T_4^*$  space is a collectionwise normal  $T_1$  space, a  $T_5^*$  space is a hereditarily collectionwise normal  $T_1$  space and a  $T_{5a}^*$  space is a  $T_5^*$  space in which all closed subsets are  $G_\delta$  (Cf. [4] II)).

These classes allow us to give characterizations of compact, LINDELÖF and paracompact spaces in the realm of  $T_2$  spaces. In fact: "a  $T_{3a}$  space is compact if and only if it is  $\alpha$ -paracompact in each  $T_{3a}$  space in which it is embedded as a closed subset" (Proposition 1.2), "a  $T_4$  space is LINDELÖF if and only if it is  $\alpha$ -paracompact in each  $T_4$  space in which is embedded as a closed subset" (Proposition 1.11) and "a collectionwise normal  $T_1$  space is paracompact if and only if it is  $\alpha$ -paracompact in each collectionwise normal  $T_1$  space in which is embedded as a closed subset" (Proposition 1.16).

Analogously, we define in Section 2 of this paper the classes  $\Pi^*(T_i)$  (for each  $i = 2, 3, 3a, 4, 5, 5a$ ) and  $\Pi^*(T_j^*)$  (for each  $j = 4, 5, 5a$ ) related to TELGÁRSKY's class  $\Pi^*$ .

### 1. The classes $\Gamma^*(T_i)$ and $\Gamma^*(T_j^*)$

C. E. AULL defined in [1] the notion of an  $\alpha$ -paracompact subset. A subset  $E$  of a topological space  $X$  is said to be  $\alpha$ -paracompact in  $X$  if every covering of  $E$  by open subsets of  $X$  has a refinement by open subsets of  $X$  which is locally finite in  $X$  and which covers  $E$ .  $\alpha$ -paracompact subsets have been studied in [1] and [3]. Using this

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concept and the various axioms of separation we shall define some classes of topological spaces. These classes give characterizations of compact, LINDELÖF, and paracompact spaces in the realm of  $T_2$  spaces.

**Definition 1.** For every  $i = 2, 3, 3a, 4, 5, 5a$ , we shall denote by  $\Gamma^*(T_i)$  the class of all  $T_i$  spaces which are  $\alpha$ -paracompact in each  $T_i$  space in which are embedded as closed subsets.

**Definition 2.** For every  $j = 4, 5, 5a$ , we shall denote by  $\Gamma^*(T_j^*)$  the class of all  $T_j^*$  spaces which are  $\alpha$ -paracompact in each  $T_j^*$  space in which are embedded as closed subsets.

**Remark.** If  $C$  is the class of compact  $T_2$  spaces then  $C \subset \Gamma^*(T_2) \subset \Gamma^*(T_3) \subset \Gamma^*(T_{3a}) \subset \Gamma^*(T_4) \subset \Gamma^*(T_4^*)$ .

**1.1 Proposition.** The classes  $C$ ,  $\Gamma^*(T_2)$ ,  $\Gamma^*(T_3)$  and  $\Gamma^*(T_{3a})$  coincide.

**Proof.** It suffices to show that  $\Gamma^*(T_{3a}) \subset C$ . Let  $X$  be a paracompact  $T_2$  space. If  $X \notin C$  then  $X$  is not countably compact and the discrete space of natural numbers  $N$  is embedded in  $X$  as a closed subset. Let  $j$  be the embedding of  $N$  in  $X$ .

Let  $X \in \Gamma^*(T_{3a})$ , let  $Z$  be a  $T_{3a}$  space such that  $N$  is embedded in  $Z$  as closed subset, let  $Z + X$  be the topological sum, let  $j_1: Z \hookrightarrow Z + X$ ,  $j_2: X \hookrightarrow Z + X$  be the embeddings and let  $Z \cup_j X$  be the adjunction space determined by  $Z$ ,  $X$  and  $j$ . Since  $Z$  and  $X$  are  $T_2$ , and the natural quotient mapping  $q: Z + X \rightarrow Z \cup_j X$  is perfect, then  $Z \cup_j X$  is completely regular. Let  $C$  be a closed subset of  $Z \cup_j X$  and  $p \in Z \cup_j X$  such that  $p \notin C$ . Then  $q^{-1}(C) \cap q^{-1}(p) = \emptyset$ . If  $p \in q(j_1(Z \setminus N))$  there exists a continuous mapping  $f_1: j_1(Z) \rightarrow [0, 1]$  such that  $f_1((q^{-1}(C) \cap j_1(Z)) \cup j_1(N)) = \{0\}$  and  $f_1(t) = 1$  (where  $q(t) = p$ ). The continuous mapping  $f: Z + X \rightarrow [0, 1]$  such that  $f|_{j_1(Z)} = f_1$  and  $f|_{j_2(X)} = 0$  defines a continuous mapping  $\bar{f}: Z \cup_j X \rightarrow [0, 1]$  such that  $\bar{f}(C) = \{0\}$  and  $\bar{f}(p) = 1$ . If  $p \in q(j_2(X \setminus N))$  the proof is similar to the above because  $X$  is  $T_{3a}$ . If  $p = q(j_1(n)) = q(j_2(n))$  where  $n \in N$  there exists a continuous mapping  $f_1: j_1(Z) \rightarrow [0, 1]$  such that  $f_1(j_1(n)) = 1$  and  $f_1((q^{-1}(C) \cap j_1(Z)) \cup j_1(N \setminus \{n\})) = \{0\}$  and there exists a continuous mapping  $f_2: j_2(X) \rightarrow [0, 1]$  such that  $f_2(j_2(n)) = 1$  and  $f_2((q^{-1}(C) \cap j_2(X)) \cup j_2(X \setminus \{n\})) = \{0\}$  then the continuous mapping  $f: Z + X \rightarrow [0, 1]$  such that  $f|_{j_1(Z)} = f_1$  and  $f|_{j_2(X)} = f_2$  defines a continuous mapping  $\bar{f}: Z \cup_j X \rightarrow [0, 1]$  such that  $\bar{f}(C) = \{0\}$  and  $\bar{f}(p) = 1$ . Thus  $Z \cup_j X$  is  $T_{3a}$ .

$$\begin{array}{ccc} Z & \xrightarrow{q \circ j_1} & Z \cup_j X \\ \cup & & \uparrow q \circ j_2 \\ N & \hookrightarrow & X \end{array}$$

Since  $q \circ j_2: X \rightarrow Z \cup_j X$  is a homeomorphic embedding,  $(q \circ j_2)(X)$  is  $\alpha$ -paracompact in  $Z \cup_j X$ . Let  $\mathcal{U}$  be a covering of  $N$  by open subsets of  $Z$ , then  $\{(q \circ j_1)(U) \mid U \in \mathcal{U}\}$  is a covering of  $(q \circ j_1)(N)$  by open subsets of  $(q \circ j_1)(Z)$ . For every  $U \in \mathcal{U}$  there exists an open subset  $U^*$  of  $Z \cup_j X$  such that  $U^* \cap (q \circ j_1)(Z) = (q \circ j_1)(U)$ . Let  $\mathcal{U}^*$  be the family of these subsets  $U^*$ , there is a refinement  $\mathcal{W}$  of  $\mathcal{A} = \mathcal{U}^* \cup \{(Z \cup_j X) \setminus (q \circ j_1)(N)\}$  locally finite in  $Z \cup_j X$  which covers  $(q \circ j_2)(X)$  by open subsets of  $Z \cup_j X$ . The family  $\{W \cap (q \circ j_1)(Z) \mid W \in \mathcal{W}, W \cap (q \circ j_1)(N) \neq \emptyset\}$  is locally finite in  $(q \circ j_1)(Z)$  and covers

$(q \circ j_1)(N)$  by open subsets of  $(q \circ j_1)(Z)$ . Finally

$$\mathcal{V} = \{(q \circ j_1)^{-1}(W \cap (q \circ j_1)(Z)) \mid W \in \mathcal{W}, W \cap (q \circ j_1)(N) \neq \emptyset\}$$

is a refinement of  $\mathcal{U}$  by open subsets of  $Z$ , locally finite in  $Z$ , which covers  $N$ .

Thus,  $N$  is a member of the class  $I^*(T_{3a})$ . Let  $\omega$  be the initial number of cardinality  $\aleph_0$ , let  $\Omega$  be the smallest uncountably ordinal number and let  $\omega' = [0, \omega]$  and  $\Omega' = [0, \Omega]$ . The space  $T = \Omega' \times \omega' \setminus \{(\Omega, \omega)\}$  is called the Tychonoff plank.  $A = \{\Omega\} \times [0, \omega)$  and  $B = [0, \Omega) \times \{\omega\}$  are closed subsets of  $T$  which are not strongly separated, then  $A$  is not an  $\alpha$ -paracompact subset in  $T$  (Theorem 1.3 in [3]). Since  $A$  is homeomorphic to  $N$  and  $T$  is  $T_{3a}$ , is  $N \notin I^*(T_{3a})$ . The contradiction follows from the hypothesis " $X$  is a member of  $I^*(T_{3a})$ ". Hence, if  $X$  is paracompact,  $T_2$  space and  $X \notin C$  then  $X \notin I^*(T_{3a})$ , i.e.:  $I^*(T_{3a}) \subset C$ .

**1.2. Proposition.** *A  $T_i$  space is compact if and only if it is  $\alpha$ -paracompact in each  $T_i$  space in which it is embedded as a closed subset. (For every  $i = 2, 3, 3a$ ).*

**1.3. Remark.**  $I^*(T_4) \neq C$ . In fact  $N \notin C$  and  $N$  is  $\alpha$ -paracompact in each  $T_4$  space  $Z$  such that  $N$  is embedded in  $Z$  as a closed subset (by b) of 1.3 in [3]).

**1.4. Proposition.** *If  $X$  is a LINDELÖF  $T_3$  space then  $X \in I^*(T_4)$ .*

**Proof.** The proposition follows from b) of 1.3. in [3].

**1.5. Proposition.** *If  $X \in I^*(T_4)$  and  $Y$  is closed subset of  $X$  then  $Y \in I^*(T_4)$ .*

**Proof.** Remark that  $Y$  is  $T_4$  and let  $Z$  be a  $T_4$  space such that  $Y$  is embedded in  $Z$  as a closed subset. Let  $j: Y \rightarrow X$  be the embedding of  $Y$  in  $X$  and let  $q: Z + X \rightarrow Z \cup_j X$  be the natural quotient mapping onto the adjunction space determined by  $Z$ ,  $X$  and  $j$ . Since  $q$  is a perfect mapping,  $Z \cup_j X$  is  $T_4$ . Then  $(q \circ j_2)(X)$  is  $\alpha$ -paracompact in  $Z \cup_j X$  and (analogously to the proof of 1.1)  $Y$  is  $\alpha$ -paracompact in  $Z$ .

**1.6. Proposition.** *Let  $X$  be a paracompact  $T_1$  space. Then  $X$  is a LINDELÖF space if and only if for every cardinal number  $\aleph > \aleph_0$  the discrete space of cardinality,  $\aleph, D(\aleph)$  is not embedded in  $X$  as a closed subset.*

**Proof.** If  $X$  is a LINDELÖF space and  $\aleph > \aleph_0$ ,  $D(\aleph)$  is not embedded in  $X$  as a closed set since a closed subset of a LINDELÖF space is LINDELÖF.

Conversely, let  $X$  be a paracompact space. Then  $X$  is a LINDELÖF space iff for every open covering of  $X$  there exists an open, locally finite, and countable refinement. Thus, if  $X$  is a paracompact  $T_1$  space and is not a LINDELÖF space, there exists an open covering  $\mathcal{U}_0$  of  $X$  such that any open refinement of  $\mathcal{U}_0$  which is locally finite is not countable. Let  $\mathcal{V} = \{V_j\}_{j \in J}$  be an open refinement of  $\mathcal{U}_0$  which is locally finite. Then  $\text{card}(J) > \aleph_0$ . For each  $V_j$  which is non-empty pick by the axiom of choice,  $x_j \in V_j$ . Then  $\{x_j\}_{j \in J}$  is a closed subset of  $X$  (because  $X$  is a  $T_1$  space and  $\mathcal{V}$  is locally finite). Then  $\{x_j\}_{j \in J}$  is a discrete space of cardinality  $\text{card}(J)$ .

**1.7. Proposition.**  *$I^*(T_4)$  is the class of LINDELÖF  $T_3$  spaces.*

**Proof.** For any  $\aleph > \aleph_0$ ,  $D(\aleph) \notin I^*(T_4)$ . In fact for every  $\aleph > \aleph_0$  exists a  $T_3$  space  $X$  such that  $D(\aleph)$  is closed in  $X$  and there is not a family  $\{G_j\}_{j \in J}$  of pairwise disjoint open subsets of  $X$  such that  $\{j\} \subset G_j$  for every  $j \in D(\aleph)$  ( $X$  is the Bing space, see [2] pp. 381–382 or [4] II pp. 100–102). Then it follows from Corollary 11B of [1] that  $D(\aleph)$  is not  $\alpha$ -paracompact in  $X$ .

Let  $Z \in \Gamma^*(T_4)$ . Then  $Z$  is a paracompact  $T_3$  space. From 1.6 and 1.5 and the above paragraph it follows that  $Z$  is a LINDELÖF space.

The fact that every LINDELÖF  $T_3$  space is in  $\Gamma^*(T_4)$  was proved in 1.4.

**1.8. Proposition.** *Let  $X$  be a topological space. Then  $X$  is a LINDELÖF  $T_{5a}$  space if and only if  $X$  is a hereditarily LINDELÖF  $T_3$  space.*

**Proof.** If  $X$  is a LINDELÖF  $T_{5a}$  space then  $X$  is  $T_3$  and if  $G$  is an open subset of  $X$ ,  $G = \bigcup_{n \in \mathbb{N}} F_n$  where  $F_n$  is closed for each  $n \in \mathbb{N}$ . Then  $G$  is a LINDELÖF space and every subspace of  $X$  is a LINDELÖF space.

Conversely, let  $X$  be a hereditarily LINDELÖF  $T_3$  space. Then  $X$  is  $T_4$  and for every closed subset  $F$  of  $X$  and each  $x \in X \setminus F$  there exists an open neighbourhood  $V^x$  of  $x$  such that  $\overline{V^x} \cap F = \emptyset$ . Thus  $X \setminus F = \bigcup_{n \in \mathbb{N}} \overline{V^{x_n}}$ , and  $F = \bigcap_{n \in \mathbb{N}} (X \setminus \overline{V^{x_n}})$  for some sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $X \setminus F$ . Hence,  $X$  is  $T_{5a}$ .

**1.9. Proposition.** 1)  $\Gamma^*(T_5)$  is the class of LINDELÖF  $T_5$  spaces.

2)  $\Gamma^*(T_{5a})$  is the class of hereditarily LINDELÖF  $T_3$  spaces.

**Proof.** If  $X$  is a LINDELÖF  $T_5$  space (hereditarily LINDELÖF  $T_3$  space) then  $X \in \Gamma^*(T_4)$  and  $X$  is hereditarily normal (perfectly normal), thus  $X \in \Gamma^*(T_5)$  ( $X \in \Gamma^*(T_{5a})$ ).

Since  $T_5$  and  $T_{5a}$  spaces are invariants of perfect mappings and the generalized BING spaces (see [4] II p. 128) are  $T_5$  and  $T_{5a}$  but are not collectionwise normal, analogously to 1.5 and 1.7, the assertion follows.

**1.10. Remark.**  $\Gamma^*(T_4) \supset \Gamma^*(T_5) \supset \Gamma^*(T_{5a})$  and these three classes are distinct. In fact: the cube  $[0, 1]^{\mathbb{R}}$  is a member of  $\Gamma^*(T_4)$  which is not in  $\Gamma^*(T_5)$  and the space  $[0, \Omega]$  is a member of  $\Gamma^*(T_5)$  which is not in  $\Gamma^*(T_{5a})$ .

**1.11. Proposition.** *A  $T_j$  space is a LINDELÖF space if and only if it is  $\alpha$ -paracompact in each  $T_j$  space in which it is embedded as a closed subset (For every  $j = 4, 5, 5a$ ).*

**1.12. Remark.** There are spaces in the class  $\Gamma^*(T_4^*)$  which are not LINDELÖF spaces. In fact, for each  $\aleph > \aleph_0$ ,  $D(\aleph) \in \Gamma^*(T_4^*)$  because  $D(\aleph)$  is  $T_4^*$  and if  $X$  is a  $T_4^*$  space in which  $D(\aleph)$  is a closed subset, there exists a discrete family  $\{A_j\}_{j \in D(\aleph)}$  of open subsets of  $X$  such that  $\{j\} \subset A_j$  for each  $j \in D(\aleph)$ , then  $\bigcup_{j \in D(\aleph)} \{j\} = D(\aleph)$  is  $\alpha$ -paracompact in  $X$  (1.1 in [3]).

**1.13. Proposition.** *Let  $X$  be a regular space and  $E$  a subset of  $X$ . Suppose*

- a)  $E$  is weakly paracompact,
- b)  $E$  is an  $\alpha$ -collectionwise normal subset in  $X$  ([1]) and
- c) for every open subset  $U$  of  $X$  such that  $E \subset U$  there is an open subset  $V$  of  $X$  such that  $E \subset V \subset \overline{V} \subset U$ .

Then,  $E$  is  $\alpha$ -paracompact in  $X$ .

**Proof.** Let  $\mathcal{U}$  be a covering of  $E$  by open subsets of  $X$ . There exists a point-finite refinement  $\mathcal{O}^*$  of  $\mathcal{U}^* = \{U \cap E \mid U \in \mathcal{U}, U \cap E \neq \emptyset\}$  by open subsets of  $E$ , which covers  $E$ .

For each  $V \cap E \in \mathcal{O}^*$  there is  $U_V \cap E \in \mathcal{U}^*$  such that  $V \cap E \subset U_V \cap E$ . Let  $\mathcal{O} = \{V \cap U_V \mid V \cap E \in \mathcal{O}^*, U_V \cap E \in \mathcal{U}^*, V \cap E \subset U_V \cap E\}$ . Clearly,  $\mathcal{O}$  is a refinement of

$\mathcal{U}$  by open subsets of  $X$  and is point-finite in  $E$ . We denote  $\mathcal{O} = \{0_s\}_{s \in S}$ . Analogously to the proof of the Theorem of MICHAEL-NAGAMI ([2] pp. 400–401) for each  $k \in \mathbb{N} \cup \{0\}$  there is a discrete family  $\{A_T\}_{T \in \mathcal{F}_{k+1}}$  of closed subsets of  $E$  such that  $A_T \subset 0_{s_T}$  for some  $0_{s_T} \in \mathcal{O}$ . As  $E$  is  $\alpha$ -collectionwise normal in  $X$ , for each  $k \in \mathbb{N} \cup \{0\}$ , there exists a discrete family  $\{G_T\}_{T \in \mathcal{F}_{k+1}}$  of open subsets of  $X$  such that  $G_T \supset A_T$  for every  $T \in \mathcal{F}_{k+1}$ . For each  $k \in \mathbb{N} \cup \{0\}$  and for each  $T \in \mathcal{F}_{k+1}$ , let  $V_T = G_T \cap 0_{s_T}$ . Then  $\mathcal{V}_{k+1} = \{V_T\}_{T \in \mathcal{F}_{k+1}}$  is a discrete family of open subsets of  $X$  and  $\bigcup_{i \in \mathbb{N}} \mathcal{V}_i = \mathcal{V}$  is a refinement of  $\mathcal{U}$ ,  $\sigma$ -discrete in  $X$ , by open subsets of  $X$ , which covers  $E$ . Moreover, we have the hypothesis c). From Theorem 1.3 in [3] it follows that  $E$  is  $\alpha$ -paracompact in  $X$ .

**1.14. Proposition.** *If  $X$  is a paracompact  $T_2$  space then  $X \in \Gamma^*(T_4^*)$ .*

*Proof.* The Proposition follows from 1.13.

**1.15. Proposition.**  *$\Gamma^*(T_4^*)$  is the class of paracompact  $T_2$  spaces.*

**1.16. Proposition.** *A collectionwise normal  $T_1$  space is a paracompact space if and only if it is  $\alpha$ -paracompact in each collectionwise normal  $T_1$  space in which it is embedded as a closed subset.*

**1.17. Proposition.** 1)  $\Gamma^*(T_5^*)$  is the class of paracompact  $T_5^*$  spaces. 2)  $\Gamma^*(T_{5a}^*)$  is the class of hereditarily paracompact  $T_2$  spaces in which all closed subsets are  $G_\delta$  (which is the class of paracompact  $T_{5a}$  spaces).

*Proof.* 1) Follows from 1.14.

2) Clearly  $\Gamma^*(T_{5a}^*)$  is the class of paracompact  $T_{5a}$  spaces. If  $X$  is a paracompact  $T_{5a}$  space, for every open subset  $G$  of  $X$ ,  $G$  is  $F_\sigma$ . Hence  $G$  is paracompact and  $X$  is hereditarily paracompact. Hence,  $X$  is hereditarily collectionwise normal. The converse is obvious.

**1.18. Remark.**  $\Gamma^*(T_4^*) \supset \Gamma^*(T_5^*) \supset \Gamma^*(T_{5a}^*)$  and these three classes are distinct. In fact: the ALEXANDROFF compactification  $X^*$  of  $D(\mathfrak{M})$  (where  $\mathfrak{M} > \aleph_0$ ) is a member of  $\Gamma^*(T_5^*)$  which is not in  $\Gamma^*(T_{5a}^*)$  (because  $X^*$  is  $T_5$  and hereditarily paracompact and is not  $T_{5a}$ ), and the cube  $[0, 1]^{\mathbb{R}}$  is a member of  $\Gamma^*(T_4^*)$  which is not in  $\Gamma^*(T_5^*)$ .

**1.19. Proposition.** *A  $T_j^*$  space is paracompact if and only if it is an  $\alpha$ -paracompact subset in each  $T_j^*$  space in which it is embedded as a closed subset. (For every  $j = 5, 5a$ )*

## 2. The classes $\Pi^*(T_i)$ and $\Pi^*(T_j^*)$

R. TELGÁRSKY in [5] defined the concept of a well-situated subset. A subset  $E$  of a topological space  $X$  is said to be well-situated in  $X$  if for every paracompact  $T_2$  space  $Y$ ,  $E \times Y$  is  $\alpha$ -paracompact in  $X \times Y$ . Moreover, he denoted (in [5]) by  $\Pi^*$  the class of all paracompact  $T_2$  spaces which are well-situated in each paracompact  $T_2$  space in which they are embedded as closed subsets. We shall define some classes of topological spaces related to the class  $\Pi^*$ .

**Definition 3.** For every  $i = 2, 3a, 3, 4, 5, 5a$  we shall denote by  $\Pi^*(T_i)$  the class of all  $T_i$  spaces which are well-situated in each  $T_i$  space in which they are embedded as closed subsets.

**Definition 4.** For every  $j = 4, 5, 5a$ , we shall denote  $\Pi^*(T_j^*)$  the class of all  $T_j^*$  spaces which are well-situated in each  $T_j^*$  space in which they are embedded as closed subsets.

**2.1. Remarks.** 1) a) For every  $i = 2, 3, 3a, 4, 5, 5a$  we have  $\Pi^*(T_i) \subset \Gamma^*(T_i)$ . b) For every  $j = 4, 5, 5a$  we have  $\Pi^*(T_j^*) \subset \Gamma^*(T_j^*)$ .

2) For every  $i = 2, 3, 3a, 4$  we have  $\Pi^*(T_i) \subset \Pi^*$  and  $\Pi^*(T_4^*) \subset \Pi^*$ .

**2.2. Proposition.** The classes  $C$ ,  $\Pi^*(T_2)$ ,  $\Pi^*(T_3)$  and  $\Pi^*(T_{3a})$  coincide.

**Proof.** Clearly  $\Pi^*(T_2) \subset \Pi^*(T_3) \subset \Pi^*(T_{3a}) \subset \Gamma^*(T_{3a}) = C$ .

We shall show that  $C \subset \Pi^*(T_2)$ . Let  $X$  be a compact  $T_2$  space. Let  $Z$  be a  $T_2$  space such that  $X$  is embedded as a closed subset in  $Z$  and let  $Y$  be any paracompact  $T_2$  space. Let  $\mathcal{A}$  be any covering of  $X \times Y$  by open subsets of  $Z \times Y$ . For each  $y \in Y$  and for each  $x \in X$  there is an open neighbourhood  $V_y^x$  of  $x$  in  $Z$  and there is an open neighbourhood  $V_x^y$  of  $y$  in  $Y$  such that  $V_y^x \times V_x^y$  is contained in some member of  $\mathcal{A}$ , whence there are  $x_1(y), \dots, x_{n(y)}(y)$  such that  $X \subset V_{y_1}^{x_1(y)} \cup \dots \cup V_{y_{n(y)}}^{x_{n(y)}(y)}$  let  $V^y = V_{x_1(y)}^{y_1} \cap \dots \cap V_{x_{n(y)}(y)}^{y_{n(y)}}$ . Thus, there is an open refinement  $\mathcal{V} = \{V_j\}_{j \in J}$  of  $\{V^y\}_{y \in Y}$ , locally finite in  $Y$ . For each  $j \in J$  we pick a  $y_j \in Y$  such that  $V_j \subset V^{y_j}$ . Clearly  $\bigcup_{j \in J} \{V_{y_j}^{x_{n(y_j)}(y_j)} \times V_j, \dots, V_{y_j}^{x_1(y_j)} \times V_j\}$  is a refinement of  $\mathcal{A}$ , by open subsets of  $Z \times Y$ , locally finite in  $Z \times Y$ , which covers  $X \times Y$ .

**2.3. Remarks.** 1)  $\Pi^*(T_4) \neq C$ , 2)  $\Pi^*(T_4) \neq \Gamma^*(T_4)$ , 3)  $\Pi^*(T_4) \subsetneq \Pi^*(T_4^*)$ , 4)  $\Pi^*(T_4^*) \neq \Gamma^*(T_4^*)$ .

**Proofs.** 1)  $N \in \Pi^*(T_4)$ . For let  $Z$  be a  $T_4$  space such that  $N$  is closed subset in  $Z$ . Then  $N$  is  $\alpha$ -paracompact in  $Z$  by 1.4. Thus, there exists a family  $\{G_n\}_{n \in N}$  of open subsets of  $Z$ , discrete in  $Z$  such that  $n \in G_n$  for each  $n \in N$  (Corollary 11B in [1]). Then, for any paracompact  $T_2$  space  $Y$ ,  $\{G_n \times Y\}_{n \in N}$  is a discrete family which covers  $N \times Y$  by open subsets of  $Z \times Y$ . Since  $\{n\} \in \Pi^*(T_2)$  for each  $n \in N$ ,  $\{n\}$  is well-situated in  $Z$  for each  $n \in N$ . Hence  $N$  is well-situated in  $Z$ .

2) The space of rational numbers  $\mathbb{Q}$  is a member of the class  $\Gamma^*(T_4)$  (Proposition 1.4) but  $\mathbb{Q} \notin \Pi^*(T_4)$  because  $\mathbb{Q} \notin \Pi^*$  ([5]).

3) Obviously  $\Pi^*(T_4) \subset \Pi^*(T_4^*)$ . Moreover  $D(\mathfrak{M})$ , where  $\mathfrak{M} > \aleph_0$  is a member of the class  $\Pi^*(T_4^*)$  and is not a member of the class  $\Pi^*(T_4)$ , because  $D(\mathfrak{M})$  where  $\mathfrak{M} > \aleph_0$  is not a LINDELÖF space.

4)  $\mathbb{Q} \in \Gamma^*(T_4^*)$  and  $\mathbb{Q} \notin \Pi^*(T_4^*)$ .

**Notes.** R. TELGÁRSKY in [5] defined and studied the notion of a  $C$ -scattered space.  $X$  is said to be  $C$ -scattered if for each non-void closed subset  $E$  of  $X$  there is a point  $x \in E$  and an open neighbourhood  $U^x$  of  $x$  for which  $\overline{U^x} \cap E$  is compact. Also he denoted (in [6]) by  $SC$  the class of all  $C$ -scattered spaces.

**2.4. Proposition.** For every  $i = 4, 5, 5a$ , we have  $\Gamma^*(T_i) \cap SC \subset \Pi^*(T_i)$  and for every  $i = 4, 5, 5a$ , we have  $\Gamma^*(T_i^*) \cap SC \subset \Pi^*(T_i^*)$ .

**Proof.** The proof is analogous to that of Theorem 2.3 of TELGÁRSKY ([5]). Let  $X$  be a  $C$ -scattered space which belongs to  $\Gamma^*(T_i)$  ( $\Gamma^*(T_i^*)$ ). The proof proceeds by transfinite induction.

If  $X^{(0)} = \emptyset$ , then  $X = \emptyset$  and hence the assertion is true.

If  $X^{(\alpha+1)} = \emptyset$ , then  $X^{(\alpha)}$  is locally compact and by Theorem 1.6 of TELGÁRSKY ([5]) there is a locally finite closed covering  $\{R_i\}_{i \in J}$  of  $X$  such that every  $R_i^{(\alpha)}$  is compact.

Let  $Z$  be a  $T_i$  space ( $T_i^*$  space) in which  $X$  is a closed subset. Then  $\{R_j\}_{j \in J}$  is locally finite in  $Z$  and  $X$  is  $\alpha$ -paracompact in  $Z$ . Hence (by Proposition 1.5 of [3]) there exists a family  $\{V_j\}_{j \in J}$  of open subsets of  $Z$ , locally finite in  $Z$ , such that  $R_j \subset V_j$  for each  $j \in J$ . Thus, for any paracompact  $T_2$  space  $Y$ ,  $\{V_j \times Y\}_{j \in J}$  is a locally finite family of open subsets of  $Z \times Y$ , which covers  $X \times Y$ . It suffices to prove that each  $R_i$  is well-situated in  $Z$ . To simplify the notation let us put  $F = R_i$ . Let  $\mathcal{A}$  be any open covering of  $F \times Y$  in  $Z \times Y$ . For each  $y \in Y$  the set  $F^{(\alpha)} \times \{y\}$  is compact; hence there is an open subset  $U_y$  of  $Z$  and an open subset  $V_y$  of  $Y$  such that

$$F^{(\alpha)} \times \{y\} \subset U_y \times V_y \subset \overline{U_y} \times \overline{V_y} \subset \bigcup_{A \in \mathcal{A}_y} A$$

where  $\mathcal{A}_y$  is some finite subfamily of  $\mathcal{A}$ . Clearly,  $\{V_y\}_{y \in Y}$  is an open covering of  $Y$ . Let  $\mathcal{B}$  be an open, locally finite refinement of  $\{V_y\}_{y \in Y}$ . We may assume that  $\mathcal{B}$  is irreducible. For each  $B \in \mathcal{B}$  we pick a  $y_B \in Y$  such that  $B \subset V_{y_B}$  and  $\{V_{y_B}\}_{B \in \mathcal{B}}$  is also a covering of  $Y$ , then it has a shrinking  $\mathcal{B}^*$  such that for each  $B \in \mathcal{B}$  there is  $B' \in \mathcal{B}^*$  such that  $\overline{B'} \subset B$ . Since  $F^{(\alpha)} \subset U_{y_B}$ , we have  $(F \setminus U_{y_B})^{(\alpha)} = \emptyset$ . Hence, by the inductive assumption,  $F \setminus U_{y_B}$  is well-situated in  $Z$ . Then there exists a refinement  $\mathcal{C}_B$  of  $\mathcal{A}$ , by open subsets of  $Z \times Y$ , locally finite in  $Z \times Y$ , which covers  $(F \setminus U_{y_B}) \times B$  and such that  $\bigcup_{C \in \mathcal{C}_B} C \subset Z \times B$ . Thus,

$$\{(U_{y_B} \times B) \cap A / A \in \mathcal{A}_{y_B}, B \in \mathcal{B}\} \cup \left( \bigcap_{B \in \mathcal{B}} \mathcal{C}_B \right)$$

is a refinement of  $\mathcal{A}$  by open subsets of  $Z \times Y$ , locally finite in  $Z \times Y$ , which covers  $F \times Y$ .

If  $X^{(\alpha)} = \emptyset$  for the ordinal limit number  $\alpha$ , then by Theorem 1.6 of [5] there is a locally finite closed covering  $\{R_j\}_{j \in J}$  of  $X$  such that for each  $j \in J$  there is a  $\beta < \alpha$  such that  $R_j^{(\beta)} = \emptyset$ .

Hence, every  $R_j$  is well-situated in  $Z$  by the inductive assumption. Thus from Proposition 1.5 in [3] it follows that  $X$  is well-situated in  $Z$ .

**Problem 1.** For every  $i = 4, 5, 5a$ , does the class  $\Gamma^*(T_i) \cap SC$  coincide with the class  $\Pi^*(T_i)$ ? (Analogous problem for  $\Gamma^*(T_i^*)$ ).

**Problem 2.** For every  $i = 4, 5, 5a$ , does the class  $\Pi^*(T_i)$  coincide with the class  $\Gamma^*(T_i) \cap \Pi^*$ ? (Analogous problem for  $\Gamma^*(T_i^*)$ ).

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