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MANUSCRIPT

Inquiry-based mathematics education and attitudes towards mathematics: tracking profiles for teaching

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Abstract
Research on the relationships between the main constructs underlying inquiry-based learning is rarely reported in mathematics education research. Considering this as a complex problem which is worth to be investigated, the present study aims to provide some empirical evidences that might serve as an insight to support further investigations on the relationships between attitudes towards mathematics and inquiry-based learning approaches. Thus, this study adopts a descriptive research design where no variables are manipulated but observed and measured in order to identify changes depicted in data collection. An instructional design focusing on the nature of mathematical inquiry is carried out with the participation of 304 secondary and high school students, and a clustering approach is used to look at how participants are grouped around certain attitudinal profiles before and after such mathematical practice. The results show how the heterogeneity of attitudinal profiles present in the classroom evolves positively in terms of perceived usefulness of mathematics and mathematical self-concept as perception of competence in mathematics. This fact provides some basis that might be used for further research on the idea that certain forms of development in *inquiry-based mathematics education (IBME)* based on greater immersion in the nature and culture of mathematics can help students to improve their attitudes towards mathematics.

Keywords Attitude towards mathematics · Perceived utility of mathematics · Inquiry-based mathematics education

Introduction

Studies on the relationships between inquiry-based approaches and attitudes have a long tradition in the experimental sciences (Toma & Lederman, 2020) but are less developed in mathematics. In recent years, European educational reports have been

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insisting on the need for a renewed pedagogy in the school context that addresses the deficits that are occurring within the STEM disciplines and raises levels of scientific and mathematical literacy. Inquiry-based learning (IBL) is indicated as a priority method at both secondary and university levels (Kogan & Laursen, 2014; Gómez-Chacón et al., 2021), and several studies have been carried out in secondary and high school education analysing the current situation in different countries (Benjumeda & Romero, 2017; Engeln et al., 2013; Gómez-Chacón & De la Fuente, 2019; Maaß & Artigue, 2013; Makar & Fielding-Wells, 2018). Engeln et al. (2013) provide a cross-cultural comparison of the potentials and challenges of implementing inquiry-based mathematics education (onwards IBME) from the perspective of practising teachers in 12 different European countries. The monograph in the journal ZDM (Artigue & Blomhøj, 2013; Maaß & Artigue, 2013) has outlined some of the most important issues of large-scale implementation, noting that IBME is related to a broad set of mathematics education goals, such as improving students' mathematical thinking skills, fostering motivation to learn, equipping them with strategies for further learning in the future, and helping students acquire the skills they need to work as scientists and do research.

Results from meta-analyses of research on inquiry-based learning (Hattie, 2009; Lazonder & Harmsen, 2016) point out the positive influence on domain-specific knowledge and attitudes. Hattie analysed 205 research studies — mostly in science education and sparsely in mathematics education — noting that the effects were greatest at the elementary level, but diminished as students progressed through their school years, and also that the improvement was more noticeable in terms of processes than in terms of contents. Hattie concluded that inquiry-based teaching might be more appropriate when students have a greater cognitive capacity to think critically, producing greater benefits in terms of performance and attitude towards the subject (Hattie, 2009, pp. 209–210).

In this research, inquiry-based teaching is developed by strengthening its cognitive and attitudinal dimensions, fostering habits of mathematical inquiry in the classroom, in the way professional mathematicians work (Gómez-Chacón et al., 2021). The results of different European research and innovation projects such as PRISMAS (Maaß & Reitz-Koncebovski, 2013) or European Project *Partnership for Learning and Teaching in University Mathematics* (PLATINUM) (Gómez-Chacón et al., 2021), whose main purpose is the development of teachers' competences for the design, teaching and evaluation of IBL materials in mathematics, have shown that for a real integration in everyday life, it is necessary to consider among the objectives the development of students' motivation, beliefs and attitudes. Therefore, one of the main current challenges in research is to document and explain the long-term development of learners in terms of the above-mentioned constructs in mathematics educational contexts.

Based on these dimensions and an extensive literature review, it seems pertinent to explore the "transformative potential" of IBME over attitudes towards mathematics when looking at some constructs associated with inquiry-based learning such as vision of mathematics, self-concept and perceived competence, utility value etc. This study raises an overarching research question: Do and, if so, how attitudinal profiles in the mathematics classroom change in inquiry-based teaching contexts?

Thus, we describe how students' attitudinal profiles evolved as an explicit mathematical practice (micro-experiment) focused on the nature of mathematical inquiry, i.e. referring to the practices that mathematicians engage in when creating knowledge (e.g. conjecturing, proving, communicating) and to the human experience of such activity. We consider that the results provided by this study will contribute to the design of teaching routes under IBME.

Theoretical framework

The following is the conceptual framework that justifies our understanding of inquiry-based mathematics education (IBME) and attitudes towards mathematics.

Inquiry-based mathematics education

Numerous models of inquiry-based teaching have been introduced in education (Rönnebeck et al., 2016) and particularly in the area of mathematics (Artigue & Blomhøj, 2013). Artigue and Blomhøj (2013) include as instances of inquiry-based learning (IBME), the teaching proposals made from the following approaches in mathematics education: problem-solving, theory of didactical situations, realistic mathematics education, modelling perspectives, anthropological theory of the didactic and dialogical and critical approaches. They not only compare the different purposes they have through the IBL (the acquisition of knowledge, habits, attitudes, values etc.), but they also focus on the epistemological framework that sustains it.

We have adopted a broad and flexible conceptualisation of IBME along the lines of that put forward by Dorier and Maaß (2014) in the *Encyclopedia of Mathematics Education*, in which it is seen as a form of teaching in which students are invited to work in a similar way to mathematicians.

We think that IBME promotes student engagement and appropriation of a "human view" of mathematics as knowledge to create and discover. To this aim, we consider actively engaging students in the development of conceptual understanding of mathematical through the tasks that are proposed and by getting to know more about mathematicians, their ways of working and their discoveries and creations. We agree with Boaler (2016) when she states that:

> Mathematics is a cultural phenomenon; a set of ideas, connections, and relationships that we can use to make sense of the world. At its core, mathematics is about patterns. We can lay a mathematical lens upon the world, and when we do, we see patterns everywhere; and it is through our understanding of the patterns, developed through mathematical study, that new and powerful knowledge is created. (p. 23)

Integrating a conception of mathematics as a human activity into instructional designs and school mathematics has been found to have positive effects on learning (Gerstenschlager et al., 2021; Watson et al., 2021, Weber et al., 2014). However, operationalising an approach to how mathematicians proceed as an IBME approach in the school context requires concrete tools.

When considering mathematics as a discipline, there are two aspects that may be fruitful to distinguish: the nature of mathematical inquiry and the nature of mathematical knowledge. The first refers to the practices that mathematicians engage in when creating knowledge (e.g. conjecturing, proving, communicating) and to the human experience of such activity (e.g. emotional, affective). The second refers to the nature of the knowledge that mathematicians produce (e.g. mathematical knowledge requires revision and proof). It should be noted that this distinction is not always clear-cut (Pair, 2017). For example, do conjectures fall into one type or the other? It can be argued that conjecturing is an important mathematical practice that plays a role in the creation of mathematical knowledge. On the other hand, established theorems were once conjectures. This would place conjectures in the category of the nature of mathematical knowledge. We note here that although it is valuable to distinguish between the nature of mathematicians' inquiry and the nature of the knowledge that mathematicians produce, the distinctions between knowledge and practice are not clear-cut.

Providing students with information about historical and social facts about mathematicians and their work would enhance students' understanding of the nature of mathematical inquiry and the nature of mathematical knowledge. With the history of mathematics, students can be familiarised with how mathematical ideas develop and the teacher can guide them to appreciate mathematics as a creative activity (Jankvist, 2009 and 2015). We consider that an inquiry learning environment has the potential to affect students' beliefs and attitudes towards mathematics. Part of this empirical research was a micro-experiment on the nature of mathematical inquiry based on mathematical practice using a *Kahoot questionnaire*.

Attitudes towards mathematics

It is easy to see that there is a highly variable range of research with different conceptual approaches to the study of attitudes. Different authors have specified (Di Martino & Zan, 2010; Gómez-Chacón & Marbán, 2019; Gómez-Chacón & De la Fuente, 2019) three ways of defining attitudes towards mathematics:

- A simple definition in which attitude is considered as a degree of positive or negative affect associated with mathematics
- A two-dimensional definition that involves emotions and beliefs, but in which behaviour is not explicitly stated
- A tripartite definition that recognises the three components of attitude: emotional response to mathematics, beliefs about mathematics and mathematics-related behaviour.

We have assumed this third definition and the assessment instruments we have used are consistent with it (Palacios et al., 2014). In this paper, we explore the relationships between the following dimensions: emotional disposition, perceived competence and view of mathematics. Previous research has already shown a potential relationship between attitude and performance (Ma & Kishor, 1997), as well as interconnections between attitudes and beliefs about mathematics teaching (Philipp, 2007), or highlighted relationships between a negative emotional disposition associated with an instrumental view of mathematics and perceived competence (Di Martino & Zan, 2010; Gómez-Chacón, 2010).

We consider attitudes towards mathematics not as a general and unitary trait but as an element formed by differential and specific aspects, a multidimensional consideration of attitude. In our case, the dimensions are *Perception of Mathematical Incompetence, Enjoyment of Mathematics, Perception of Utility* and *Mathematical Self-concept*. The criteria for the selection of these dimensions (factors) were determined by looking at the factors that most frequently appear in literature as part of scales of attitudes towards mathematics measurement and in coherence with the framework of questionnaire 1 (Palacios et al., 2014).

Perception of Mathematical Incompetence focuses on the perception of inability, awkwardness, confusion, difficulty and expectations of failure. This factor was present in classic studies on attitudes (Fennema & Sherman, 1976) and recognised in current research on attitudes (Adelson & Mc-Coach, 2011; Goldin et al., 2016). The second factor, *Enjoyment of Mathematics,* is referred to the positive emotions evoked by the study of mathematics, perception of ease and comfort when solving mathematical problems (Adelson & McCoach, 2011). The third factor, *Perception of Utility*, involves the utility of and need for mathematics (Fennema & Sherman, 1976; Tahara et al., 2010). The fourth and last factor, *Mathematical Self-concept*, is related to the perception of efficacy and/or competence in mathematics (Pietsch et al., 2003).

Mathematics self-concept is one's academic self-perceptions of the mathematic ability in school (Lee & Kung, 2018; Pajares & Miller, 1994; Reyes, 1984). Mathematics self-concept has "to do with how sure a person is of being able to learn new topics in mathematics, perform well in mathematics class, and do well on mathematics tests" (p. 560, Reyes, 1984). It also refers to an individual's perception of her abilities related to mathematics as compared to others (Bong & Skaalvik, 2003). It is related to mathematics achievement (Lee & Kung, 2018), so it is important to understand how students perceived their mathematics learning abilities.

Currently, several studies have further explored the distinction between self-concept and self-efficacy in the academic space (Arens et al., 2020), analysing the similarities and differences between the two. In general, both concepts address students' competence. However, academic self-concept is related to a student's self-perceived competence in an academic domain such as mathematics (Marsh & Craven, 2006). And the academic self-efficacy is understood as the self-perceived confidence to successfully perform a given mathematical task (Bandura, 2001; Zimmerman, 2000). In other words, self-concept is a domain-specific construct, whereas self-efficacy is a domain- and task-specific construct (Arens et al., 2020).

Longitudinal studies examined whether mathematics self-perception (self-concept and self-efficacy) (Skaalvik & Skaalvik, 2011) predicted achievement. Findings on self-concept are related to students' aspirations to pursue STEM (science, technology, engineering and mathematics) degree programmes (Sax et al., 2015) and are positively related to their performance (Lee & Kung, 2018). Moreover, mathematical self-concept and mathematical self-efficacy (i.e. the belief about the student's ability to solve mathematical problems or tasks) predict performance (Kung, 2009). Therefore, it seems relevant to understand mathematical self-concept as a key factor in attitudes towards mathematics as it is a basis for interest in mathematics. In this regard, Bates et al. (2011) and Mato-Vázquez et al. (2014) point out that positive mathematical self-concept is a good predictor of pleasantness towards this subject and that students with negative attitudes present lower confidence in their mathematical abilities, an extremely common phenomenon among secondary school students.

Philipp (2007) also notes that negative attitudes towards mathematics tend to be rooted in their experiences as mathematics learners motivating this study to further explore how prior experiences of the nature of mathematical inquiry impact on changes in emotional response and their view of mathematics. Mathematical literacy includes the ability to use mathematics in a variety of contexts, but also an awareness of the role that mathematics plays in the world (vision of mathematics) OECD (2019). Several studies have shown evidence of causal relationships between perceived self-efficacy and approaches to learning (Zakariya et al., 2020) as well as on perceived usefulness of mathematics as a major component of students' view of mathematics and the evolution in their views on mathematics (Di Martino, 2019). Some authors state that the way in which teachers emotionally and affectively support students determines their school performance in mathematics in terms of perceived efficacy, liking and achievement (Gómez-Chacón & De la Fuente, 2019; Hemmings et al., 2011). Also, the research recognises that the type of tasks poses nuances in the type of impact on the affective dimension. Schukajlow et al. (2012) investigated the effects of teaching modelling problems on students' expectations of enjoyment, interest, valuation and self-efficacy in relation to three types of mathematical problems — intramathematical problems, word problems, and modelling problems — finding the positive impact of the latter type.

In general, it is expected that providing activities related to the real world and to the world of mathematicians will reinforce the learning process of inquiry, improve students' interest and motivation, activate positive attitudes and increase the number of students who enjoy mathematics.

Methodology

Aims and participants

At it has already been stated, the main aim of this study is to gain deeper understanding of the research problem aiming to study potential transformative effects of an IBM approach on attitudes towards mathematics when looking at some constructs associated with inquiry-based teaching and learning in secondary and high school students (ages 12 to 17 years old). The initial willingness to work with an inquiry approach is diagnosed, and then a micro-experiment is developed while measuring different variables that consider attitudes towards mathematics and determining students' attitudinal profiles before and after it. Such micro-experiment was mainly characterised using instructional methods in which students are offered mathematical practice, with more information about the nature of mathematical inquiry and its socio-historical aspects, and by providing a humanistic view of its procedures and contents.

The following research questions are posed: (1) What attitudinal profiles are exhibited by this study group before and after the micro-experiment? (2) How attitudinal profiles evolve under the conditions of such a micro-experiment?

The study was carried out with a group of 304 students (138 girls and 166 boys) from a public secondary school belonging to the Community of Madrid Region. A convenience sampling was then carried out ensuring accessibility to participants. The educational levels of the students were ESO (compulsory secondary education, ages 12–16) and 1st year of high school, age 17 (Fig. 1).

These students are taught in the traditional way, i.e. their teachers explain the subject matter using the blackboard and relying on textbooks to supplement with exercises and examples. Students' familiarity with inquiry-based learning approaches is low.

Design and data analysis

The data in this study are drawn from a larger research project on the creation of mathematical thinking development classrooms and inquiry learning approach (Gómez-Chacón et al., 2021) involving teachers and students.

The study followed the principles of the so-called rapid prototyping methodological approach, in which the unit of analysis was a 2-week micro-experiment. That is, for 2 weeks, a given teacher (or set of teachers) implements practices within their classroom, and we study how students' attitudinal behaviour evolves when this change in practice is implemented: did attitudes and views on mathematics evolve after such instructional experience (micro-experiment) and how? Based on the results of this micro-experiment, adjustments are made to try to reinforce students' inquiry learning, by reinforcing practices, content aspects about the nature of mathematical inquiry (e.g. conjecturing, proving, communicating).

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Level	1º ESO (12-13 years old)	58	19.1	19.1	19.1
	2º ESO (13-14 years old)	67	22.0	22.0	41.1
	3º ESO (14-15 years old)	49	16.1	16.1	57.2
	4º ESO (15-16 years old)	61	20.1	20.1	77.3
	1º High School (16-17 years old)	69	22.7	22.7	100.0
	Total	304	100.0	100.0	

Fig. 1 Study group by educational level

The study involves an initial diagnosis of attitudes towards mathematics of the group through a first questionnaire (questionnaire 1 to be described later) by individual response; the micro-experiment consists of the development of IBME practices through a *Kahoot questionnaire*, working in visibly randomised class sessions and groups (Liljedahl, 2014), and finally a diagnosis of the new situation through a second questionnaire (questionnaire 2 to be described later) by individual response.

It is important at this stage to point out again that the study is conceived in terms of a descriptive design and not a causal one nor even an exploratory one. In this sense, we do not talk about pre-test or post-test measures but instead about starting situation and final situation. Thus, both questionnaires being different avoid problems with testing concerning menacing potential influences from the first test onto the second, and at the same time, they are still useful to look at students' profiles both before the micro-experiment and after it in terms of the evolution or changes of such profiles as a whole and not as individual changes as effect of the experience.

The data provided by the questionnaires were analysed using the statistical analysis package SPSS (version 27). For the calculation of the confirmatory factorial analysis (CFA), the Lisrel 8.8 programme was used. The reliability of the questionnaires is calculated through MacDonald's Omega avoiding an improper use of Cronbach's alpha in this context (McNeish, 2018). The statistical calculations are completed with two *K*-means cluster analyses for each of the scales, based on their normalised factor scores with a mean of 5 and a standard deviation of 1.5, in a range of 1 to 10 points. The aim of this calculation is to assign each subject-student to a single cluster, seeking the greatest possible homogeneity of all its components and whose characteristics are not known beforehand. Subsequently, it is the researcher's task to define these common characteristics and adequately describe their meaning. Cluster analysis was seen as useful in the descriptive context of this study under the assumption that the study group is not homogeneous, being a powerful tool to help to classify data into structures that might be more easily understood and manipulated, providing at the same time a starting point for further research accompanied by deeper analytics that might be performed by means of an exploratory or even causal design. Cross-validated discriminant analysis has been carried out to validate the results obtained from the cluster analysis. Finally, a flow analysis is performed between clusters of the first diagnosis (questionnaire 1) and clusters of the final moment (questionnaire 2).

Instruments

The instruments used are described below.

Questionnaire 1

Questionnaire 1 is a 5-level Likert-type attitudes towards mathematics scale (Palacios et al., 2014) (0 = strongly disagree …. 4 = strongly agree) composed of 32 items (Fig. 3), with contrasted evidence of validity and reliability. The dimensions it assesses are perception of mathematical incompetence, mathematics liking, perceived usefulness of mathematics and mathematical self-concept. This questionnaire is applied before the micro-experiment as an initial diagnosis of the students' attitudes.

Questionnaire 2

In the literature review, we found that there are hardly any validated instruments related to attitudes and beliefs in mathematics that are also conceptually linked to IBL and its potential impacts on students. Moreover, as noted by authors who are developing this linkage with IBME (Pedersen & Haavold, 2022), most existing survey instruments are long, with many items, and therefore laborious to administer. Shortness is important when administering instruments in classroom contexts and with young students, as the longer the inventory, the more costly it is for students to respond (Entwistle & McCune, 2004). This led us to propose a more concise and easier to answer instrument for the final assessment, focusing on aspects of IBME assessment. Such instrument is a final scale for the evaluation of the experience designed as a Likert scale of 4 levels (not at all; a little; quite a lot; a lot) composed of 10 items (Fig. 4).

This questionnaire in relation to IBL is intended to measure students' progress with respect to:

- Vision of mathematics
- Usefulness of mathematics
- Perceived mathematics learning ability

The set of items included in this questionnaire was influenced by the *expectancy-value theory* of Eccles and Wigfield (2002) and by the epistemic characteristics of IBME addressed through the content in Kahoot questionnaire. According to the expectancy-value theory of Eccles and Wigfield (2002), students' motivation to participate in an activity is seen to depend on both their expectations of success and their valuation of the activity. Specifically, the review of the literature on motivation and attitudes and the theoretical underpinning of questionnaire 1 led us to conceptualise motivation and attitudes through three constructs: *perception of mathematics learning ability*, which relates to their perception of their mathematical ability and their expectations of success; perception of the *usefulness of mathematics* which relates to how engaging in mathematics can be useful to the student; and *vision of mathematics*, which is concerned with valuing complexity and a non-routine, non-boring conception of the activity. A teaching approach in which students are provided with information about how mathematicians work and the socio-historical contexts in which mathematical concepts and procedures arise can enhance intrinsic motivation for mathematics.

Kahoot questionnaire: nature of mathematical inquiry

The development of IBME experience was carried out usi*ng Kahoot*, an interactive play platform that allows for the creation of questionnaires with multi-response in which students take an active role. The interactive *Kahoot questionnaire* was designed to target inquiry-based teaching is based on providing students with opportunities to explore and test their own ideas, as well as to verify multiple ways of approaching a solution and an application-oriented view that represents the usefulness of mathematics for real-world problems as a major aspect of the nature of mathematical inquiry.

The questionnaire is made up of 13 items with 4 possible answers (except for the first and the last one, which are 2). The questionnaire items refer to the following dimensions:

> *Nature of mathematical inquiry and mathematical culture*: items 5, 6, 8, 10, 11, 12, related to the origin of numbers, mathematical symbols or questions about very relevant mathematicians and their contribution to mathematics.
> *Mathematical inquiry procedures in areas of mathematics*: items 2, 3, 7, 9, 13, where there are questions about elementary operations, properties of numbers and numbers especially relevant in mathematics (particularly in arithmetic: number e, number pi etc...) and where the contributions of mathematicians reflect the process of inquiry in this discipline.

Mathematical knowledge: items 1, 4 where concepts and definitions must be applied to solve the questions. In our IBME methodological framework, students move repeatedly between making sense of the problem and the challenges they face in making good use of mathematical processes to solve it. For everyone and for the group, this involves exploring, conjecturing and justifying their reasoning, making their decisions explicit during resolution and reflecting on the concepts and resolution processes. The components of the micro-experiment are based on the problem-solving practices and working thinking model of professional mathematicians.

Through the *Kahoot questionnaire*, we encourage students to explicitly consider the work that mathematicians do and how it impacts on their learning situation. In this framework, students interact specifically in the following ways:

- For each question, the teacher presents the whole group with a mathematical question (this can be based on a historical problem/fact). Groups of three to five students are then asked to solve the problem through discussions.
- These teams then put their solutions into practice, review the proposal that emerged from each participant and specify the difficulties encountered.
- Justification of the solution is done internally as part of the problem-solving process (question/task) in each team.
- During small group interactions and whole group checking, the teacher facilitates students' discussion of the significant challenges, ideas and concepts that arise during the question posed.
- Students reflect individually and with their group on what has happened in the class group and expand on new information and facts related to how the mathematicians have previously worked on these problems and concepts.

We note that after the introduction of the problem-question, interactions with the teacher occur according to the needs of the question posed and the students; they do not constitute a linear path, but rather a cyclical process of creativity, information gathering and communication that reflects the real world of mathematics.

In the following, we describe the behaviour of students in response to the question:

> *What is special about the number 2520? Possible answers were: Its square root is an even number / It is a power of a prime number / It is the sum of its positive proper divisors (perfect number) / I can divide it by all numbers from 1 to 10.*

Students have been given the statement "mathematical inquiry involves an exploration of ideas" and asked to explore which of these four answers would be true. It is not a question of answering whether it is true or false, but they are asked to explore the solution and discuss among themselves to justify their answers. When the questionnaire is posed unlike in the other class sessions, they are given limited time.

About 65% of the students seek to answer with pencil and paper by performing calculations, however, some calculations are complex for the time they have available. In this case, only 15% of them manage to answer the question correctly by performing calculations.

About 20% of the total number of students make guesses about what each answer entails and then see how to solve it. To do this, 15% decompose the number into prime factors, which helps them to discard some of the answers and try to arrive at the correct answer. With the decomposition, the first two answers can be discarded, and it can be seen that what characterises this number is that it can be divided by all the numbers from 1 to 10.

There are a number (about 5% of the students) who analyse the answers one by one without taking into account the other possible solutions, so that doing it in isolation does not get to the answer.

Significant is the contributions of some students (15% of students) and the arguments for arriving at the answer, analysing the options and seeing what they entail. For example, for the second option they use the argument that, because it ends in 0, it is divisible by 10 and therefore is not a power of a prime number (and they also contribute that it is divisible by 2, 5 and 10). For the first answer they argue that as it is only divisible by 2 three times, it cannot have an even root (and also give them that it is divisible by 2, 4, 8). This seems to indicate to them that it is possible that it is divisible by all the numbers from 1 to 10, which is not difficult to verify. Indeed, the total number of pupils in this pathway arrives at the correct solution.

In general, the third option, "it is the sum of its positive eigendividers (perfect number)", is complicated both for the students to understand and to check, given how large the number is. This made it possible after the solution to go deeper with the whole class into what a perfect number is and what positive eigendividers are. A positive integer is equal to the sum of its positive eigendividers. Thus, 6 is a perfect number $6=1+2+3$. They were asked to see if 28 was a perfect number, obtaining that $28=1+2+4+4+7+14$. And then the one in the statement was analysed again: the number 2520.

With students at the lower secondary school level, difficulties were encountered in decomposing the number into prime factors, so the decomposition procedure was revisited. Also, a next step of inquiry was to explore the properties of even perfect numbers as triangular numbers and the contributions of Euclid. The mathematician Euclid discovered that the first four perfect numbers are given by the formula $2^{n-1}(2^n-1)$. $n=2$: $2^1\times(2.^2-1)=6$; $n=3$: $2^2\times(2.^3-1)=28$; $n=5$: $2^4\times(2.^5-1)=496$; $n=7$: $2^6\times(2.^7-1)=8128$

Realising that 2^n-1 is a prime number in each case, Euclid proved that the formula $2^{n-1}(2^n-1)$ generates an even perfect number whenever 2^n-1 is prime.

It was stressed that mathematical inquiry is an exploration of ideas, mathematical ideas are part of mathematical identity and knowledge is dynamic. The teacher provides information on how ancient mathematicians made many assumptions about the perfect numbers based on the four they already knew. Many of these assumptions have turned out to be false. One of them was that since 2, 3, 5 and 7 were precisely the first four prime numbers, the fifth perfect number would be obtained with $n=11$, the fifth prime number. However, $2^{11}-1=2047=23\times89$ is not prime and therefore $n=11$ does not produce a perfect number. It is proposed that students work on two of the other wrong assumptions:

- The fifth perfect number would have five digits, since the first four have 1, 2, 3 and 4, respectively.
- The perfect numbers would alternately end in 6 and 8.

Also, for students at high school level, it is extended with the contributions in 1603 when Pietro Cataldi found the sixth and seventh perfect numbers and with Mersenne's prime numbers.

This episode described above attempts to exemplify the nature of mathematical inquiry as creative, emotional and collaborative and the nature of mathematical knowledge as knowledge that is subject to revision (Hersh, 1997); proofs and argumentation are carriers of knowledge (Bleiler-Baxter & Pair, 2017; Weber, 2010), just as formal knowledge in mathematics is informed by conjecture and justification (Burton, 1999). Weber (2010) points out that one of the reasons mathematicians read proofs is to learn new methods and techniques that they can use in their own work, in essence, to fill their mathematical toolbox.

Results and discussion

In this section, several types of results are collected. Firstly, we focus on the results from the application of questionnaire 1 and questionnaire 2, showing the descriptive analyses of the scales that compose them, factor analysis and analysis of variances. Next, the results of the cluster analysis are presented, both of questionnaire 1 initially and of questionnaire 2 after the micro-experiment, for which the normalised factor scores have been taken as variables, and, to describe the evolution and profiles in attitudes, the results of the flow analysis between initial and final clusters are presented. All this has been enriched with the information provided by the *Kahoot questionnaire* applied during the experiment.

Results of the factor analysis of the questionnaires

First, a reliability and validity analysis of the questionnaires was performed. Regarding the first of these, an exploratory factor analysis of the 32 questions of the original Palacios et al. (2014) scale was performed. As a result of this exploratory analysis, a structure was obtained consisting of four factors with eigenvalues greater than 1 and with an adequate fit as summarised in Fig. 2.

Factor 1, named *Perceived incompetence,* encompasses items related to the subject's negative perception of the ability to learn and do mathematics. Factor 2, named *Pleasure* (enjoyment of mathematics), integrates items related to aspects of

	Component			
	1	2	3	4
I tend to have difficulties with mathematics (P29-F1)	.815			
Except in a few cases, no matter how hard I try I can't understand mathematics (P11-F1)	.736			
I feel more awkward in mathematics than most of my classmates (P27-F1)	.731			
No matter what I do, I always get low grades in mathematics (P19-F1)	.713			
I have always had problems with mathematics (P18-F1)	.709			
I tend to feel incapable of solving mathematical problems (P22-F1)	.697			
It will always be difficult for me to learn Mathematics (P9-F1)	.693			
In mathematics, I find it difficult to decide what to do (P23-F1)	.693			
Mathematics confuses me (P28-F1)	.676	.303		
I don't know how to study mathematics (P21-F1)	.609			
In mathematics I often go blank in my mind (P7-F1)	.586			
I am one of those people who was not born to learn mathematics (P26-F1)	.523			
Mathematics is easy (P6-F2)		.462		
I find it fun to study mathematics (P5-F2)		.754		
When I have to study mathematics, I go to the task with some joy (P32-F2)		.726		
I can spend hours studying mathematics and doing mathematics (P31-F2)		.658		
I like mathematics (P1-F2)	.437	.633		
The subject taught in mathematics classes is very interesting (P13-F2)		.622	.360	
Mathematics is one of the most boring subjects (P15-F2)	.367	.577	.441	
I can't stand studying mathematics, even the easy parts (P14-F2)	.346	.575		
If I had the chance, I would sign up for mathematics electives (P16-F2)		.552		
Mathematics is a "drag" (P25-F2)	.364	.540	.397	
I feel comfortable solving mathematics problems (P2-F2)	.470	.478		
I am good at mental arithmetic (P30-F2)	.308	.401		
Mathematics is useless (P3-F3)			.656	
Mathematics should only be present in science degrees (P4-F3)			.602	
Mathematics is useful and necessary in all areas of life (P12-F3)			.561	
It's time for maths class. What a horror! (P8-F2)	.426	.446	.459	
Learning maths is something that only a few people do (P17-F3)			.439	
I can become a good student in mathematics (P24-F4)	.333			.670
If I set my mind to it, I think I would master mathematics well (P10-F4)				.646
I am a good student for my mathematics teachers and professors (P20-F4)	.347	.336		.353
Kaiser-Meyer-Olkin measure of sampling adequacy	.941			
Bartlett's test of sphericity	Chi-squared=4682.189; gl=496; sig.=.000			

Fig. 2 Exploratory factor analysis (EFA) of questionnaire 1

CFA evaluation of questionnaire 1

Modelo	S-B$_{(Chi\text{-}squared)}$; (gl); (p)	RMSEA	NFI	NNFI	CFI	AGFI	AIC
Four factor	872.51; (453); (p =.00)	.059	.95	.97	.98	0.81	1075.94

Fig. 3 CFA evaluation of questionnaire 1

enjoyment and pleasure in mathematics lessons. Factor 3 for *Usefulness* (perceived usefulness of mathematics) associates statements relating to the subject's conception of the applications and usefulness of mathematics. Finally, Factor 4, called *Self-concept*, refers to items related to the individual's positive perception of their skills for mathematics (Fig. 2).

Based on the results of this first exploratory analysis, a confirmatory factor analysis (CFA) was performed with the four factors mentioned above. The results obtained after this analysis show values similar to those originally obtained (Palacios et al., 2014) and always with an adequate fit (Fig. 3).

For reliability, McDonald omega coefficient (McDonald, 1999) was obtained based on the factor loadings of the CFA for each of the four factors present in the scale (Fig. 2). In the four factors, admissible values are obtained for this type of research ($\Omega F1 = 0.92$; $\Omega F2 = 0.92$; $\Omega F3 = 0.80$; $\Omega F4 = 0.80$) (Katz, 2006).

In the case of the questionnaire 2 developed ad hoc for the research, an exploratory factor analysis (EFA) was carried out, without imposing any type of restriction or prior hypothesis. Likewise, the criterion used to determine the number of factors was to maintain those with eigenvalues greater than 1. At first, four factors were extracted with a good fit, but one of the factors had only one item, so it was decided to perform a new calculation with only 3 factors. In this case, the EFA explained a percentage similar to the 4-factor solution (62% of the variance), also with statistical significance and with at least two items each. The results are summarised in Fig. 4.

The first factor, *Vision* (of mathematics), includes items that seek to identify the vision of mathematics and, when applied after the micro-experiment, to see what changes have occurred. The second factor, *Perception of usefulness* (of

	Component		
	1	2	3
It has shown me that mathematics can be easy (C10 – F1)	.793		.301
It has shown me that learning mathematics can be easy (C4 – F1)	.752		.305
It has helped me to see that maths can be fun (C1 – F1)	.731		
Helped me to have a better understanding of mathematics (C6 – F1)	.703	.326	
Shows that mathematics is not as "boring" as it seemed (C5 – F1)	.629		
Shows that there are interesting and motivating contents if they were taught in class (C7 – F1)	.437	.431	.363
Made me see that mathematics is in all aspects of life (C2 – F2)		.844	
It has helped me to see that mathematics is useful (C8 – F2)		.698	
It shows that if the methodology were different, I could master maths (C3 – F3)			.818
It has made me see that I can learn mathematics, even if it's hard (C9 – F3)	.334		.678
Kaiser-Meyer-Olkin measure of sampling adequacy	.879		
Bartlett's test of sphericity	Chi-squared=1005.493; gl=45; sig.=.000		

Fig. 4 Results of the factor analysis of questionnaire 2, rotated component matrix

mathematics), associates items in connection with the usefulness of mathematics. And finally, the third factor, *Perceived learning ability* (of mathematics), refers to the subject's perception of their ability to learn, but with the awareness that it can be a costly learning process that requires effort. Once again, the KMO and Barlett's sphericity tests confirm the sample adequacy for the analysis carried out. The omega coefficient for each factor are admissible for this type of research ($\Omega F1 = 0.83$; $\Omega F2 = 0.74$; $\Omega F3 = 0.66$) (Katz, 2006).

Results of cluster analysis

Questionnaire 1

With the results of questionnaire 1, an initial hierarchical cluster analysis is carried out, taking as variables the normalised factor scores (in a range of 1–10 and with a standard deviation of 0.5) of the four factors obtained and using Euclidean distance. The most commonly used techniques to determine the validity of the clusters, obtained by means of a non-hierarchical procedure, are the analysis of variance and discriminant analysis. With the former, the aim is to determine the differences in the mean values of each factor in each cluster, while the discriminant analysis seeks to determine the percentage of subjects who are correctly assigned (Jain & Dubes, 1988); with respect to the latter, we dedicate a specific section to the analysis.

The results of the analysis of variance show significant differences in all the factors in each of the clusters, as summarised in Fig. 5.

Four clusters are characterised and labelled as follows based on their scores on the four factors of the scale in terms of the mean score on each of them (Fig. 6), thus providing us with four student profiles. These profiles are congruent with those obtained in previous researches (Martinez & Nortes, 2017; Palacios et al., 2014) and close to those obtained by Scofield et al. (2021) and Tulis and Ainley (2011).

Profile 1: *Anti-mathematic*, made up of a total of 66 people (21.71%). These are students who have a negative view of mathematics and therefore do not like it. They also have a very low perception of its usefulness and consider that there are no tangible applications in everyday life. In addition, they consider that they are not able to learn mathematics in a simple way.

Profile 2: *Unable but enthusiastic*, made up of a total of 82 people (26.97%). These are students who do not see themselves as suitable for learning mathematics,

	Factors					
Cluster	F1.- **Incompetence**	F2.- **Pleasure**	F3- **Usefulness**	F4.- **Self-concept**	F	Sig.
P1.- Anti-mathematic	6.07	4.25	3.64	5.06	39.56	.000
P2.- Unable but enthusiastic	4.08	4.72	5.07	3.46	97.24	.000
P3.- Capable but disenchanted	4.35	3.90	5.81	6.25	34.79	.000
P4.- Mathematician	5.49	6.52	5.36	5.47	84.92	.000

Fig. 5 Factors mean differences in each cluster (questionnaire 1)

	Questionnaire 1 factor scores			
Cluster	Incompetence	Pleasure	Usefulness	Self-concept
P1.- Anti-mathematic	Very high	Low	Very low	Low
P2.- Unable but enthusiastic	Very low	High	Low	Very low
P3.- Capable but disenchanted	Low	Very low	Very high	Very high
P4.- Mathematician	High	Very high	High	High

Fig. 6 Delimitation of the meaning of clusters in questionnaire 1

which also means that they do not have a positive view of mathematics. On the contrary, they are students who still have a perception of the usefulness of mathematics that makes them not give up and focus on their learning.

Profile 3: *Capable but disenchanted*, made up of a total of 63 people (20.72%). This is the other side of the previous profile. These are students who see themselves as strong when it comes to learning and capable of it; they are also clear about the usefulness of mathematics and can apply it to real problems. On the other hand, they do not hold the discipline in high regard and do not see much of a future for themselves in studying it.

Finally, Profile 4: *Mathematician*, made up of a total of 93 people (30.6%). It includes students who like mathematics, considering that they learn it relatively easily and that it has a practical utility in nature beyond theoretical procedures. They are therefore students who have a broad conception of mathematics and a positive assessment of it.

Questionnaire 2

Similarly, a hierarchical analysis based on Euclidean distance is carried out with the data from questionnaire 2, which reflects the situation after the experience, taking as variables the standardised scores of the three factors obtained in the factor analysis carried out. As in the case of the first questionnaire, a first validation of the clusters was carried out based on the significance values of the differences in means for the three factors of the second questionnaire in each of the three groupings carried out, with the results presented in Fig. 7.

These three clusters are thus obtained, each of them characterised by standing out clearly from the other two in the score of the corresponding factor, so that they can be labelled with the name given to the factors (Fig. 8).

	Factores				
Cluster	F1. - Vision	F2.- Usefulness	F3.- Learning ability	F	Sig.
Q1.- Confident and mathematical	5.56	4.60	6.51	16.13	.000
Q2.- Discouraged supporter	5.09	3.62	3.72	165.06	.000
Q3.- Pragmatic	4.48	6.23	4.61	187.93	.000

Fig. 7 Factors mean differences in each cluster (questionnaire 2)

Clusters	F1.- Vision	F2.- Usefulness	F3.- Learning ability
Q1.- Confident and mathematical	High	Means	Very high
Q2.- Discouraged supporter	High	Low	Very low
Q3.- Pragmatic	Means	Very high	Means

Fig. 8 Delineating the meaning of clusters in questionnaire 2

The first cluster called "Confident and mathematical" is characterised by the fact that students consider that they are very prepared to learn mathematics, although they do not consider the usefulness of mathematics to be very high. Furthermore, this is supported by the fact that they like mathematics because they have a positive view of mathematics.

The second cluster "Discouraged supporter" are students who hold mathematics in high esteem, but do not see clear usefulness of the content. These students do not consider that they are able to learn mathematics in a fluent way.

Finally, the cluster "Pragmatic" groups together those who are very clear about the usefulness and application of mathematics in different areas of real life. In addition, they have a certain liking for mathematics and see themselves capable of learning it, although they find it difficult, and it is not something that motivates them very much.

Figure 9 illustrates the internal composition of the three clusters and to characterise them based on the different students' scores on the three factors of questionnaire 2.

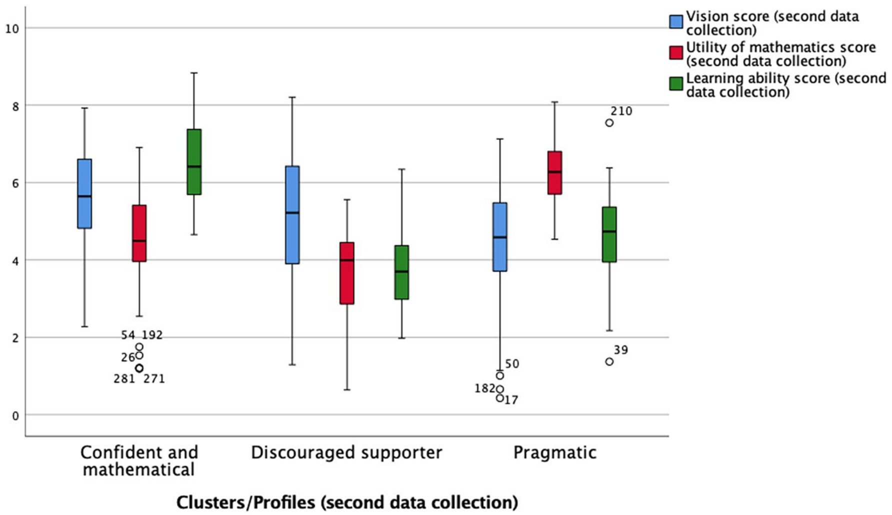

Fig. 9 Results of the analysis of the internal composition of the three factors

As can be seen, although the different box plots obtained are very similar by factors and between clusters in terms of their shape (width, whiskers) and the symmetry shown (except in the case of perceived usefulness), we can observe clear differences in terms of the height at which the different boxes are placed. Thus, the first cluster (*Confident and mathematical*) has most students close to 5 and the learning ability score stands out, with scores of up to 9, the lowest being close to 5. The second cluster (*Discouraged supporter*) reflects greater disparity in the scores of the 3 items, and even more distributed, with the view being the one with the highest score, but at the same time the one with the greatest difference between the highest and lowest scores. The last cluster (*Pragmatic*) also has notable differences in the vision score, but in learning ability, most students are close to the highest score. Of course, the utility scores stand out here. We see that, in all three clusters, the vision score is more unstable and the difference between scores larger, while the utility score tends to be more concentrated.

Results of the cross-validation for the discriminant analysis

As indicated in the "Design and data analysis" section, discriminant analysis was used to validate the results obtained in the cluster analysis. Thus, the original sample was first randomly divided into 5 classes or folds of approximately equal size. Then, iteratively four of such folds were selected to act together as a training set in order to estimate an accurate discriminant function for classification of participants in the groups (clusters), while the remaining fold was considered as a validation set to assess the ability of such function to classify new participants into the groups or clusters. This was done in such a way that each fold played once and only once the role of validation set so that the splitting was performed 5 times, considering for each questionnaire the variables used in the corresponding cluster analysis and the clusters thus obtained for each case. As it can be easily seen, the ratio training-validation was 80–20%, being this a ratio commonly used in literature for splitting the original sample in cross-validation processes. Besides, according to Joseph (2022) and taking into account the size of our original sample, the ratio selected seems to be (quasi)optimal. Then, the recorded predictive performances were averaged. Figures 10 and 11 show the averaged predictive performances for questionnaire 1 together with the results of one of the splits just to illustrate the procedure. For the same purposes, Figs. 12 and 13 show the results for questionnaire 2.

In both cases, the results are highly satisfactory, confirming the adequacy of the results obtained in the previous cluster analysis (see Figs. 10 and 12).

	Split 1	Split 2	Split 3	Split 4	Split 5	Average
Original grouped cases correctly classified	95.1	96.3	94.7	95.1	96.3	95.5
Cross-validates pooled cases correctly classified	96.7	95.1	95.1	96.7	93.3	95.4

Fig. 10 Average of recorded predictive performances (questionnaire 1)

			Assigned group (first data collection)	Predicted group membership				
				Anti-mathematic profile	Unable but enthusiastic	Capable but disenchanted	Mathematician	Total
Selected cases	Original	Count	Anti-mathematic profile	46	2	0	5	53
			Unable but enthusiastic	0	64	0	1	65
			Capable but disenchanted	0	1	50	0	51
			Mathematician	0	0	0	74	74
		%	Anti-mathematic profile	86.8	3.8	.0	9.4	100.0
			Unable but enthusiastic	.0	98.5	.0	1.5	100.0
			Capable but disenchanted	.0	2.0	98.0	0	100.0
			Mathematician	.0	.0	.0	100.0	100.0
Non-selected cases	Cross Validation	Count	Anti-mathematic profile	12	0	0	1	13
			Unable but enthusiastic	0	17	0	0	17
			Capable but disenchanted	0	2	10	0	12
			Mathematician	0	0	0	19	19
		%	Anti-mathematic profile	92.3	0	.0	7.7	100.0
			Unable but enthusiastic	.0	100	0	0	100.0
			Capable but disenchanted	0	16.7	83.3	0	100.0
			Mathematician	.0	.0	.0	100.0	100.0

96.3% of original grouped cases correctly classified.

95.1% of cross-validated pooled cases correctly classified

Fig. 11 Results from split 2 from cross-validation (questionnaire 1)

Results of flow analysis between questionnaire 1 and questionnaire 2 clusters

The analysis of the relationships between the cluster rankings prior and after the micro-experiment of the 304 participating students can be synthesised in the cross-tabulation (Figs. 14 and 15).

Profile 1: “Anti-mathematic”, made up of a total of 66 people (21.71%). These are those who start from a very high negative view of mathematics and a very low conception of usefulness and a low perception of learning ability. The flow shows that 43.9% of the total are in the *Discouraged supporter* profile. It is noteworthy that 31.8% of the group is in the *Pragmatic* profile, when they had started from a very low level, indicating that they have increased their perception of usefulness and their self-perception of learning ability.

Profile 2: “Unable but enthusiastic”, made up of a total of 82 people (26.97%). After the experience, 50% are in the *Pragmatic* profile when at the beginning these

Original grouped cases correctly classified	97.5	98.4	97.9	96.7	98.0	97.7
Cross-validates pooled cases correctly classified	96.7	95.1	100.0	100.0	96.7	97.7

Fig. 12 Average of recorded predictive performances (questionnaire 1)

			Assigned group (second data collection)	Predicted group membership			
				Confident and mathematical	Discouraged supporter	Pragmatic	Total
Selected cases	Original	Count	Confident and mathematical	85	0	0	85
			Discouraged supporter	1	54	4	59
			Pragmatic	0	0	99	99
		%	Confident and mathematical	100.0	.0	.0	100.0
			Discouraged supporter	1.7	91.5	6.8	100.0
			Pragmatic	.0	.0	100.0	100.0
Non-selected cases	Cross Validation	Count	Confident and mathematical	15	0	0	15
			Discouraged supporter	0	22	0	22
			Pragmatic	0	0	24	24
		%	Confident and mathematical	100.0	.0	.0	100.0
			Discouraged supporter	.0	100.0	.0	100.0
			Pragmatic	.0	.0	100.0	100.0

97.9% of original grouped cases correctly classified.
100.0% of cross-validated pooled cases correctly classified

Fig. 13 Results from split 3 from cross-validation (questionnaire 2)

profiles had a low conception of usefulness. This reinforces the idea of the profile that they are students who should be empowered by the usefulness of mathematics so that this fact reinforces them in the other two aspects that are more devalued. The other 50% of this profile is equally divided between the *Discouraged supporter* and the *Confident and mathematical* profiles, when originally, they had these two aspects in very low consideration. Therefore, there is in some way a flow to these groups that can be enhanced.

Profile 3: "Capable but disenchanted", made up of a total of 63 people (20.72%). More than 80% is divided between the *Pragmatic* and the *Confident and mathematical* profiles, which indicates that the initial trend towards the latter is maintained. The flow to the *Discouraged supporter* profile is smaller, although it is significant as it represents 19% of the total and is related to the initial contrast between the three profiles, as they start with a very high usefulness and learning ability. Even so, the *Pragmatic* and *Confident and mathematical* profiles have a medium and high vision respectively, so there is also a change.

Finally, Profile 4: "Mathematician", composed of a total of 93 people (30.6%), is the most numerous. This group is based on the 3 high factors, and we can see that about 80% of the participants are divided between the *Pragmatic* and *Confident and mathematical* profiles. It is worth noting that only 20.4% of this group is situated after the experience in the *Discouraged supporter* profile, which seems to indicate a certain tendency that having a positive view, even in the very high profiles, is rare.

			Assigned group (questionnaire 2)			
			Confident and mathematical	Discouraged supporter	Pragmatic	Total
Assigned group	Anti-mathematic	Count	16	29	21	66
		% within assigned group	24.2%	43.9%	31.8%	100.0%
		% within assigned group (test 2)	16.0%	35.8%	17.1%	21.7%
		% of total	5.3%	9.5%	6.9%	21.7%
	Unable but enthusiastic	Count	20	21	41	82
		% within assigned group	24.4%	25.6%	50.0%	100,0%
		% within assigned group (test 2)	20.0%	25.9%	33.3%	27.0%
		% of total	6.6%	6.9%	13.5%	27.0%
	Capable but disenchanted	Count	23	12	28	63
		% within assigned group	36.5%	19.0%	44.4%	100.0%
		% within assigned group (test 2)	23.0%	14.8%	22.8%	20.7%
		% of total	7.6%	3.9%	9.2%	20.7%
	Mathematician	Count	41	19	33	93
		% within assigned group	44.1%	20.4%	35.5%	100.0%
		% within assigned group (test 2)	41.0%	23.5%	26.8%	30.6%
		% of total	13.5%	6.3%	10.9%	30.6%
Total		Count	100	81	123	304
		% within assigned group	32.9%	26.6%	40.5%	100.0%
		% within assigned group (test 2)	100.0%	100.0%	100.0%	100.0%
		% of total	32.9%	26.6%	40.5%	100.0%

Fig. 14 Cross table assigned group (questionnaire 2)

In summary, the data show that the micro-experiment has had a positive influence on the perceived usefulness of mathematics, and it is in this factor that the flow is maintained or increased considerably in all four groups.

Results of the Kahoot questionnaire analysis

In relation to the way of doing in mathematics and in relation to the nature of mathematical inquiry, we highlight below the most significant evidence observed and positively valued by the students.

Explaining mathematical thinking orally in a group provided them with a tool for reflection with which to analyse and clarify the mathematical reasoning they had used ("explaining it to other people can make you understand it a little better", "it helps me to clarify my ideas when I listen to others").

The correct answer is chosen from a number of alternatives; these challenges students to place the answers in order and to make comparisons between them. In the

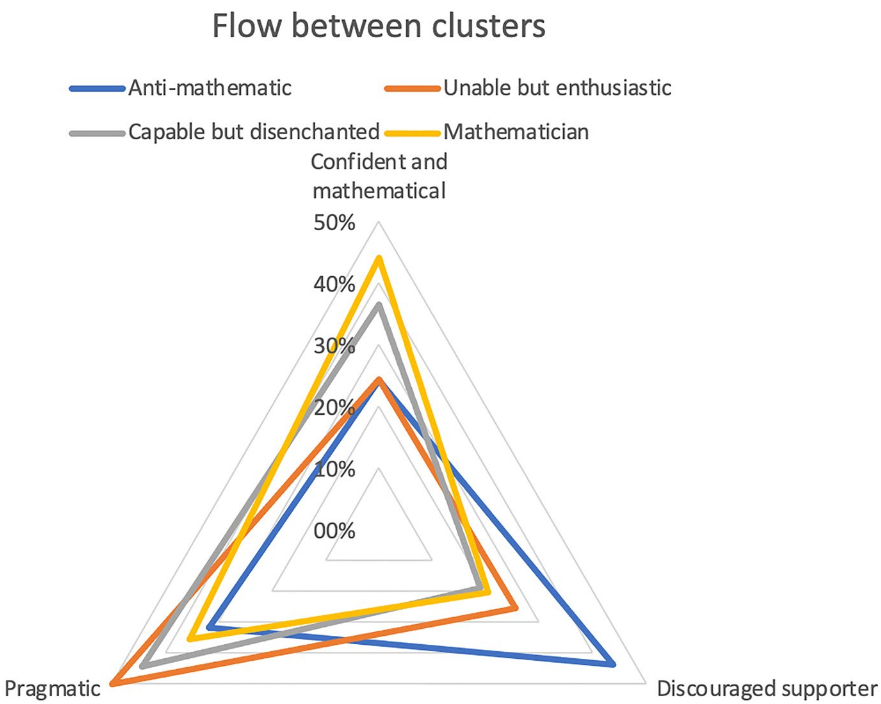

Fig. 15 Radial flow diagram between clusters

problems posed, the students must realise that, for example, when particularising in mathematics, they have to be systematic, integrate properties, rule out cases, etc. In the group, it is difficult to carry out this comparative analysis between the different alternatives; the tendency is to respond in isolation and without systematisation.

The dynamic nature of mathematical knowledge comes as a surprise to more than half of the group. The example shown in the "Instruments" section favours the experience of how mathematical ideas are socially examined through argumentation in an inquiry-oriented classroom, the input of information about the contributions of Euclid and other mathematicians; however, they find it difficult to make a connection with what they do, needing further reinforcement from the teacher that mathematical ideas are part of their identity as learners. The response to the items emphasising ways of proceeding in inquiry and the questions related to the way a mathematician works generated positive reactions in the students towards new discoveries ("How cool is this!", "It's very interesting"). And, above all, they generated new concerns and perspectives on mathematics, influencing their own motivation ("If this was taught in class it would be cool", "When I get home I'm going to look up what you said").

The mechanics of the Kahoot make it easy to integrate elements of mathematical culture. We advocate immersing students in practice by asking broad questions based on these distinctions: "What kind of example is this?"; "What kind of result is this?"; "What kind of concept is being offered here?" The percentage of correct answers to the questions posed concerning mathematical culture is less than 30%; it is not a content previously seen in classroom; however, the proposed methodology achieves a high degree of motivation in the pupils and opens up to a recognition of the usefulness and application of mathematics.

In summary, the *Kahoot questionnaire* seeks brings the student closer to the nature of mathematical inquiry and the nature of mathematical knowledge by offering applications to real models. By answering this questionnaire, students can experience various cognitive processes of evaluation about their learning ability and about their view of mathematics, but not only as an opinion but according to what they are experiencing when solving the questionnaire.

Conclusions

The main objective of this study is to identify potential relationships between attitudes towards mathematics and inquiry-based learning through experiences where students are introduced to nature of mathematical inquiry (e.g. conjecturing, proving, communicating) and the nature of mathematical knowledge and its socio-historical aspects. The attitudes towards mathematics seem to evolve positively when the students can improve by practice with reflection, by an atmosphere of questioning, challenging and reflecting with concrete examples the practices that mathematicians do in when creating knowledge. The main conclusions are the following: (1) positive developments of the perceived usefulness of mathematics and (2) positive development of mathematical self-concept (related to the perception of efficacy and/or competence in mathematics).

As can be seen in the results section, the instructional experience carried out has led to a change in students' attitudes. There is a clear improvement in the perceived usefulness of mathematics factor in all groups. It can also be seen that all profiles have a low positive view of mathematics, except for the profile called "Mathematician". Therefore, it can be said that the profiles tend to have a better perception of usefulness despite having a low positive view of mathematics. It is worth at this point to highlight that some research works already tell us that students need to know the use and life relevance of the mathematics studied in class to pave the way for their effective study, something that goes beyond developing the self-confidence to understand the mathematics itself (Kung, 2009).

Another result derived from this research is related to the mathematical self-concept factor in terms of mathematical ability, mathematical interest and perceived mathematical performance. If we pay attention to the self-perception of learning ability, we also see that it is an attitude of the students that exhibits a somehow more relevant and positive feature after the experience. The knowledge of the nature of mathematical inquiry worked on during the sessions, the extension of information and practice by the teacher after the students' answers to the *Kahoot questionnaire* and the teamwork influence the way they see their own work and process, which improves their perspective of learning (confirms results such as Jankvist, 2009). These didactic variables can inform teachers to improve or diversify their mathematics teaching strategies under IBME approach.

The overall analysis of the results through cluster flow analysis shows different student profiles. These profiles indicate a relationship between students' working styles and mathematical attitudes towards mathematical learning. The typology of attitudes observed has not only a character of appreciation and liking for mathematics but also a markedly cognitive character in relation to the way of valuing and using skills and processes that are important in mathematical work. All this could reinforce the idea that certain forms of intervention based on greater immersion in the nature and culture of mathematics in IBME can help students to improve their attitudes towards mathematics (Engeln et al., 2013; Hattie, 2009). The process of discovery and especially the approach to mathematical culture and the way of working exemplified with mathematicians bring students to new affective dimensions towards mathematics. Moreover, it is confirmed that the mathematical emotional profile is a predictor of school readiness in relation to skills and knowledge (Hidalgo et al., 2005). These results support what Hattie's (2009) meta-analysis shows in relation to attitude improvement. However, these meta-analyses also indicate that it is not only the use of IBL that makes a difference with respect to improving student performance, but the performance of the teacher (Hattie, 2009, p. 243). The interactivity and feedback provided by the *teacher Kahoot questionnaire* may have contributed to the expected benefits. This variable that has not been specifically analysed in this study, which leaves open the question of methods of inquiry and interaction with the teacher (Hemmings et al., 2011).

Regarding the methodology, in this study, the results obtained show that there is consistency between the theoretical factor structure (original scale elaborated by Palacios et al. (2014)) and the empirical factor structure in the questionnaire 1, key aspect for the multidimensional approach to attitudes towards mathematics chosen in this study by means of factors (Tahara et al., 2010). Cluster analysis was also considered useful in the descriptive context of this study, as it allows the data to be classified into structures that can be more easily understood similarities and differences in order to contribute to the design of teaching routes under IBME. Also, the difference in testing (questionnaire 1 and 2), in relation to the observation of student profiles both before and after the micro-experiment, seems relevant in terms of the evolution or changes of these profiles as a whole although it proves limiting for observation of individual changes as an effect of an experimental development.

We note that in addition to the above limitations, the present data do not support interpretation of the results in causal terms, even though such interpretations are based on theoretical analyses of the relations among the observed constructs. Longitudinal studies of the same constructs are called for in future research and as far as IBME is concerned the influence of the teacher's interactions. Even though there is a strong belief that mathematics education can be improved through the implementation of IBME, its effectiveness is not yet firmly established (Bruder & Prescott, 2013).

Finally, to avoid inappropriate generalisations, this study leaves open some questions to be taken into account in a qualitative analyses, such as the influence of the type of activities and the influence of teacher management. It has already been widely reported in the IBME literature that task design can be a determinant of learning success or failure (Gómez-Chacón et al., 2016; Schukajlow et al., 2012). In our case, the choice of activities can determine confidence factors in the use of mathematical knowledge and procedures and in their attitudinal development. Also, the temporal extension of the cycles of experimentation may further substantiate the stability of attitudes. The basic idea of these interventions is that brief reflection on usefulness through mathematical culture and knowledge of the work of mathematicians can initiate a recursive process in which usefulness, motivation and experiences positively influence each other during the following weeks or months.

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Declarations

Ethical approval Not needed.

Conflict of interest The authors declare no competing interests.

Informed consent Not needed.

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