CALDERÓN-ZYGMUND OPERATORS AND COMMUTATORS ON WEIGHTED LORENTZ SPACES

MARÍA J. CARRO*, HONGLIANG LI†, JAVIER SORIA*, AND QINXIU SUN‡

Abstract. We find necessary conditions (which are also sufficient, for some particular cases) for a pair of weights u and w such that a Calderón-Zygmund operator T, or its commutator $[b, T]$, with $b \in BMO$, is bounded on the weighted Lorentz spaces $\Lambda_u^p(w)$, for $1 < p < \infty$. This result completes the study already known for the Hardy-Littlewood maximal operator and the Hilbert transform, and hence unifies the weighted theories for the A_p and B_p classes.

Dedicated to our mentor, tutor and, above all, dear friend Guido Weiss, thanks to whom we are a little "weisser".

1. Introduction

The main goal of this work is to complete the unification of classical results about the boundedness of Calderón-Zygmund operators T on different weighted settings, like the Lebesgue spaces $L^p(u)$ or the Lorentz spaces $\Lambda^p(w)$, into a general framework involving both theories. This is done by considering the so-called weighted Lorentz spaces $\Lambda_u^p(w)$ defined as the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which (see Section 2 for the rigorous definition and the meaning of the different notations):

$$\|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty f_u^*(t)^p w(t)dt\right)^{1/p} < \infty.$$

Recall that, if $w = 1$, then $\Lambda_u^p(w) = L^p(u)$ and if $u = 1$, then $\Lambda_u^p(w) = \Lambda^p(w)$.

It is well-known [7] that, for this kind of singular operators, the boundedness of $T : L^p(u) \to L^p(u)$ holds if u is in the A_p class of weights, $1 < p < \infty$:

$$u \in A_p \iff \sup_{\{Q:Q \text{ is a cube}\}} \left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q u^{-1/(p-1)}(x)\,dx\right)^{p-1} < \infty.$$

For the case $p = \infty$, we define $A_\infty = \cup_{p>1} A_p$. The A_p class was introduced by Muckenhoupt [14] and characterized as follows: for every $p > 1$,

$$M : L^p(u) \longrightarrow L^p(u) \quad \iff \quad u \in A_p,$$

where M is the Hardy-Littlewood maximal operator

$$Mf(x) = \sup\left\{\frac{1}{|Q|}\int_Q |f(y)|\,dy : Q \text{ is a cube in } \mathbb{R}^n \text{ and } x \in Q\right\}.$$

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Moreover, in the case $T = H$, the Hilbert transform,

$$Hf(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy,$$

it was proved in [11] that

$$H : L^p(u) \longrightarrow L^p(u) \iff u \in A_p. \tag{1}$$

If we now consider the case of the weighted Lorentz spaces $\Lambda^p(w)$, Neugebauer proved in [15] that

$$H : \Lambda^p(w) \to \Lambda^p(w) \text{ if and only if } w \in B_\infty^* \cap B_p, \tag{2}$$

for all $1 < p < \infty$, where the Ariño-Muckenhoupt B_p class is characterized [3] by the condition

$$r^p \int_r^\infty \frac{w(t)}{t^p} dt \le C \int_0^t w(s) ds, \quad \text{for all } r > 0, \tag{3}$$

and the B_∞^* class is defined as

$$\int_0^t \frac{W(s)}{s} ds \le C W(t), \quad \text{with } W(t) = \int_0^t w(s) ds.$$

It is well-known that there are many different characterizations of the B_p class (see, e.g., [3, 5, 17]). In particular, (3) is equivalent to the boundedness of

$$A : L^p_{\text{dec}}(w) \to L^p(w),$$

where A is the Hardy operator defined by

$$Af(x) = \frac{1}{x} \int_0^x f(t) dt, \ x > 0,$$

and $L^p_{\text{dec}}(w)$ is the cone of all non-increasing functions in $L^p(w)$ [3]. Hence, since the decreasing rearrangement (see Section 2 for the definition) of M satisfies that [4, Theorem III.3.8]

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) ds,$$

it immediately holds that

$$w \in B_p \iff M : \Lambda^p(w) \longrightarrow \Lambda^p(w).$$

A simple remark that we will need later on is that the primitive of a B_p-weight satisfies the Δ_2-condition; namely, if $w \in B_p$, then

$$W(2t) \le C W(t), \quad \text{for every } t > 0.$$

On the other hand, the condition $w \in B_\infty^*$ is equivalent [15] to the boundedness of

$$A^* : L^p_{\text{dec}}(w) \to L^p(w),$$

with A^* the adjoint Hardy operator

$$A^* f(x) = \int_x^\infty f(t) \frac{dt}{t}, \ x > 0.$$

Hence, condition (2) follows from the inequality [4, Theorem III.4.8]

$$(Hf)^*(t) \lesssim \frac{1}{t} \int_0^t f^*(s) ds + \int_t^\infty f^*(s) \frac{ds}{s},$$

as well as the converse inequality [4, Proposition III.4.10]. Taking into considerations the equivalences (1) and (2), it is a natural question to characterize the weights u and w so that

$$M, H, T : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w) \tag{4}$$

are bounded operators. The complete answer for the operator M (see Section 2 for the definitions involved) was given in [5, Theorem 3.3.5]: for every $p > 1$,

$$M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w) \quad \iff \quad w \in B_p(u), \tag{5}$$

where $w \in B_p(u)$ if there exists $q \in (0, p)$ such that, for every finite family of cubes $(Q_j)_{j=1}^J$ and every family of measurable sets $(E_j)_{j=1}^J$, with $E_j \subset Q_j$, for every j, we have that

$$\frac{W\big(u\big(\bigcup_j Q_j\big)\big)}{W\big(u\big(\bigcup_j E_j\big)\big)} \le C \max_j \left(\frac{|Q_j|}{|E_j|}\right)^q.$$

Finally, the solution for the Hilbert transform H can be found in [1]: for every $p > 1$,

$$H : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w) \quad \iff \quad u \in A_\infty, w \in B^*_\infty, \text{ and } w \in B_p(u).$$

It is worth mentioning that, contrary to the case of the Hilbert transform, the boundedness of M on $\Lambda^p_u(w)$ does not imply that $u \in A_\infty$ (there are examples for which u may not even satisfy the doubling condition [5, Theorem 3.3.10]).

The main result of this paper is to prove that these conditions also extend to Calderón-Zygmund operators T. Since they are also necessary for the case $T = H$, we see that they cannot be improved, for a general T, and hence we fully characterize (4) in this setting.

Theorem 1.1 (Main result)**.** *Let $1 < p < \infty$ and let T be a Calderón-Zygmund operator. Then, if $u \in A_\infty$, $w \in B^*_\infty$, and $w \in B_p(u)$, we have that*

$$T : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w).$$

We will also obtain the analogous result for the corresponding BMO commutator:

Theorem 1.2. *Let $b \in BMO(\mathbb{R}^n)$ and let T be a Calderón-Zygmund operator. Then, if $u \in A_\infty$, $w \in B^*_\infty$, and $w \in B_p(u)$, we have that*

$$\|[b, T](f)\|_{\Lambda^p_u(w)} \le C\|b\|_{BMO}\|f\|_{\Lambda^p_u(w)}.$$

Remark 1.3. If $w = 1$ and $p > 1$, it holds that $w \in B_p(u)$ is equivalent to $u \in A_p$, and if $u = 1$, then $B_p(1) = B_p$. Thus, Theorem 1.1 recovers the previous results on $L^p(u)$ and $\Lambda^p(w)$ spaces [11, 1] and unifies the classical theories on A_p and B_p weights.

The paper is organized as follows: In Section 2 we will present all the definitions we need and some technical lemmas, and Section 3 is devoted to the proofs of the main results.

As usual the symbol $A \lesssim B$ will be used to indicate that there exists a positive constants C independent of all parameters involved so that $A \le CB$; similarly, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Previous definitions and technical lemmas

Functions $u : \mathbb{R}^n \to \mathbb{R}_+$ and $w : \mathbb{R}_+ \to \mathbb{R}_+$ are called weights whenever they are Lebesgue measurable functions, positive, not identically zero, and locally integrable. If w is a weight on $\mathbb{R}_+$, we denote by $W(t) = \int_0^t w(s)\, ds$ its primitive, and observe that $W(t) < \infty$, for all $t > 0$.

Given a weight u in $\mathbb{R}^n$ and a measurable function f, its decreasing rearrangement f_u^* is defined as [4]

$$f_u^*(t) = \inf\{s : \lambda_f^u(s) \le t\},\ \ t \ge 0,$$

where

$$\lambda_f^u(s) = u(\{x \in X : |f(x)| > s\}),\ \ s \ge 0$$

is the distribution function of f with respect to the measure $u(x)dx$. In particular, we write $f_u^* = f^*$ and $\lambda_f^u = \lambda_f$, if $u = 1$.

Definition 2.1 ([6, 5])**.** Let u be a weight in $\mathbb{R}^n$ and w a weight in $\mathbb{R}_+$. For every $p > 1$, the weighted Lorentz space $\Lambda_u^p(w)$ is the class of all measurable functions f such that

$$\|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty f_u^*(t)^p w(t) dt\right)^{1/p} < \infty.$$

As we have already observed, with $u = 1$, $\Lambda_1^p(w) = \Lambda^p(w)$ is the weighted Lorentz space [13]; if $u = 1$ and $w(t) = t^{p/q-1}$, then we recover the classical Lorentz spaces $L^{q,p}$ [18, Chapter V§3], and with $w = 1$,

$$\|f\|_{\Lambda_u^p(1)} = \left(\int_0^\infty f_u^*(t)^p dt\right)^{1/p} = \left(p\int_0^\infty t^{p-1}\lambda_f^u(t) dt\right)^{1/p}$$
$$= \left(\int_{\mathbb{R}^n} |f(x)|^p u(x)\, dx\right)^{1/p} = \|f\|_{L^p(u)}.$$

Definition 2.2. [10, Definition 4.1.2] A function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x, x) : x \in \mathbb{R}^n\}$ is called a standard kernel if there exists $\delta, A > 0$ satisfying the size condition

$$|K(x, y)| \le \frac{A}{|x - y|^n}, \tag{6}$$

and the regularity condition

$$|K(x, y) - K(x', y)| \le \frac{A|x - x'|^\delta}{(|x - y| + |x' - y|)^{n+\delta}},$$

when $|x - x'| \le \frac{1}{2}\max(|x - y|, |x' - y|)$ and

$$|K(x, y) - K(x, y')| \le \frac{A|y - y'|^\delta}{(|x - y| + |x - y'|)^{n+\delta}},$$

when $|y - y'| \le \frac{1}{2}\max(|x - y|, |x - y'|)$. The class of all standard kernels with constants δ and A is denoted by $SK(\delta, A)$.

Definition 2.3. [10, Definition 4.1.8] Let $0 < \delta, A < \infty$ and $K \in SK(\delta, A)$. T is called a Calderón-Zygmund operator associated with the standard kernel K if T is defined on the class of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$, admits a bounded extension on $L^2(\mathbb{R}^n)$

$$\|Tf\|_{L^2} \le B\|f\|_{L^2}, \qquad \text{for all } f \in L^2(\mathbb{R}^n),$$

and

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy,$$

for all $f\in C_0^\infty(\mathbb{R}^n)$ and $x\notin \operatorname{supp}(f)$.

We recall that the first example of a Calderón-Zygmund operator is the Hilbert transform. Also, for a locally integrable function b, the commutator of T and b is defined as

$$[b,T](f)=bT(f)-T(bf).$$

For a general Calderón-Zygmund operator T, it is well-known that T is bounded on $L^p(\mathbb{R}^n)$, for all $p\in(1,\infty)$ [10, Theorem 4.2.2]. Coifman and Fefferman [7] proved that T is bounded on the weighted Lebesgue space $L^p(u)$, when $u\in A_p$ and $1<p<\infty$. For the commutator, Coifman, Rochberg and Weiss [8] proved that $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$, if b is a $BMO(\mathbb{R}^n)$ function. In the sequel, we always assume $b\in BMO(\mathbb{R}^n)$ in the commutator $[b,T]$.

For $\beta>0$, let M_β be the modified Hardy-Littlewood maximal function

$$M_\beta(f)(x)=M(|f|^\beta)^{1/\beta}(x)=\left(\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)|^\beta dy\right)^{1/\beta},$$

and let $M_\beta^\sharp$ be the sharp maximal function

$$M_\beta^\sharp(f)(x)=\sup_{x\in Q}\inf_c\left(\frac{1}{|Q|}\int_Q\Big||f(y)|^\beta-|c|^\beta\Big|dy\right)^{1/\beta},$$

where Q is a cube in $\mathbb{R}^n$.

The following two lemmas describe some relations among Calderón-Zygmund operators, commutators, the Hardy-Littlewood maximal function and the sharp maximal functions.

Lemma 2.4. [2]. *Let T be a Calderón-Zygmund operator and $0<\beta<1$. Then*

$$M_\beta^\sharp(T(f))(x)\le CM(f)(x)$$

for all bounded functions f with compact support.

Lemma 2.5. [16]. *Let T be a Calderón-Zygmund operator, $b\in BMO(\mathbb{R}^n)$, and $0<\beta<\varepsilon$. Then*

$$M_\beta^\sharp([b,T](f))(x)\le C\|b\|_{BMO}(M_\varepsilon(T(f))(x)+M^2(f)(x)),$$

for all bounded functions f with compact support, where $M^2(f)=M(M(f))$.

Lemma 2.6. Let $1<p<\infty$, $u\in A_\infty$, $w\in B_\infty^*$, and $W\in\Delta_2$. Then

$$\|M(f)\|_{\Lambda_u^p(w)}\lesssim\|M^\sharp(f)\|_{\Lambda_u^p(w)},\tag{7}$$

if $\|M(f)\|_{\Lambda_u^p(w)}<\infty$.

Proof. Let us recall the following good-λ inequality [9, Lemma 4.2]: if $u\in A_\infty$, there exist $C>0$ and $\rho>0$, such that for all $t,\ \gamma>0$:

$$u(\{x:M(f)(x)>2t,M^\sharp(f)(x)\le\gamma t\})\le C\gamma^\rho u(\{x:M(f)>t\})\tag{8}$$

for all locally integrable functions f on $\mathbb{R}^n$. By [5, Proposition 2.2.5], we know that

$$\begin{aligned}\|M(f)\|^p_{\Lambda^p_u(w)} &= \int_0^\infty p\,W(u(\{x : M(f)(x) > t\}))t^{p-1}dt \\ &= p\,2^p \int_0^\infty W(u(\{x : M(f)(x) > 2t\}))t^{p-1}dt.\end{aligned}$$

Since $W \in \Delta_2$, then

$$W(s+t) \le c(W(s) + W(t)), \text{ for all } s, t > 0,$$

and hence, using (8), we obtain

$$\begin{aligned}\|M(f)\|^p_{\Lambda^p_u(w)} &\le p\,2^p c \int_0^\infty W(u(\{x : M(f)(x) > 2t, M^\sharp(f)(x) \le \gamma t\}))t^{p-1}dt \\ &\quad + p\,2^p c \int_0^\infty W(u(\{x : M(f)(x) > 2t, M^\sharp(f)(x) > \gamma t\}))t^{p-1}dt \\ &\le p\,2^p c \int_0^\infty W(C\gamma^\rho u(\{x : M(f)(x) > t\}))t^{p-1}dt \\ &\quad + p\,2^p c \int_0^\infty W(u(\{x : M^\sharp(f)(x) > \gamma t\}))t^{p-1}dt \\ &= p\,2^p c \int_0^\infty W(C\gamma^\rho u(\{x : M(f)(x) > t\}))t^{p-1}dt \\ &\quad + (2/\gamma)^p c\|M^\sharp(f)\|^p_{\Lambda^p_u(w)}.\end{aligned}$$

To estimate the last integral, we use the following result, which actually gives a characterization of B^*_∞ [1, Proposition 2.3 (vi)]: if $w \in B^*_\infty$, then for every $\varepsilon > 0$, there exists an $\alpha > 0$ such that $W(t) \le \varepsilon W(s)$, provided $t \le \alpha s$. Thus, with $\varepsilon = (2^{p+1}c)^{-1}$, there exists an $\alpha > 0$ such that, if $C\gamma^\rho \le \alpha$, then

$$W(C\gamma^\rho u(\{x : M(f)(x) > t\})) \le (2^{p+1}c)^{-1}\, W(u(\{x : M(f)(x) > t\}))).$$

Therefore, taking $\gamma \le (\alpha/C)^{1/\rho}$ in (8), we finally obtain

$$\begin{aligned}\|M(f)\|^p_{\Lambda^p_u(w)} &\le p\,2^p c(2^{p+1}c)^{-1} \int_0^\infty W(u(\{x : M(f)(x) > t\}))t^{p-1}dt \\ &\quad + (2/\gamma)^p c\|M^\sharp(f)\|^p_{\Lambda^p_u(w)} \\ &= 2^{-1}\|M(f)\|^p_{\Lambda^p_u(w)} + (2/\gamma)^p c\|M^\sharp(f)\|^p_{\Lambda^p_u(w)}.\end{aligned}$$

Since $\|M(f)\|_{\Lambda^p_u(w)} < \infty$, we conclude (7). □

By a homogeneity argument, it is easy to see that (7) also holds for the modified operators M_β and $M^\sharp_\beta$.

The following result gives a useful estimate we will need to prove our main theorem:

Lemma 2.7. *If $w \in B^*_\infty$, there exists $\varepsilon > 0$ so that*

$$\int_0^1 \frac{W(t)}{t^{1+\varepsilon}}dt < \infty.$$

Proof. Using the hypothesis and integration by parts, we have that, for $0 < a < 1$,

$$\begin{aligned}\int_a^1 \frac{W(t)}{t^{1+\varepsilon}}dt &= \int_0^1 \frac{W(s)}{s}ds - \frac{1}{a^\varepsilon}\int_0^a \frac{W(s)}{s}ds + \varepsilon\int_a^1 \left(\int_0^t \frac{W(s)}{s}ds\right)t^{-\varepsilon-1}dt\\ &\lesssim \int_0^1 \frac{W(s)}{s}ds + C\varepsilon\int_a^1 \frac{W(t)}{t^{1+\varepsilon}}dt.\end{aligned}$$

Taking $\varepsilon > 0$ so that $C\varepsilon < 1$, we obtain that

$$\int_a^1 \frac{W(t)}{t^{1+\varepsilon}}dt \leq \frac{1}{1-C\varepsilon}\int_0^1 \frac{W(s)}{s}ds < \infty$$

and letting $a \to 0$, we get the result. □

3. Proof of the main results

Before we start, let us observe that if $w \in B_\infty^*$, then $w \notin L^1(\mathbb{R}_+)$. This follows easily form the fact that [15, p. 143]

$$\int_0^r \log\left(\frac{r}{x}\right)w(x)\,dx \lesssim \int_0^r w(x)\,dx.$$

Hence,

$$\infty = \lim_{r\to\infty}\int_1^2 \log\left(\frac{r}{2}\right)w(x)\,dx \lesssim \int_0^\infty w(x)\,dx.$$

Thus, we conclude that the class of all bounded functions with compact support is dense in the weighted Lorentz spaces $\Lambda_u^p(w)$ [5, 12], result that we will use in the proofs of main results.

Proof of Theorem 1.1. Let $0 < \beta < 1$. Since $w \in B_p(u)$, its primitive satisfies the doubling condition $W \in \Delta_2$ [5, Lemma 3.3.1]. Now, using that $w \in B_\infty^*$, by Lemma 2.6, Lemma 2.4 and again by condition $w \in B_p(u)$, which is equivalent to (5), we get that, if $\|M_\beta T(f)\|_{\Lambda_u^p(w)} < \infty$, then

$$\|T(f)\|_{\Lambda_u^p(w)} \leq \|M_\beta(T(f))\|_{\Lambda_u^p(w)} \lesssim \|M_\beta^\sharp(T(f))\|_{\Lambda_u^p(w)} \lesssim \|M(f)\|_{\Lambda_u^p(w)} \lesssim \|f\|_{\Lambda_u^p(w)}.$$

Since $w \in B_p(u)$ implies $w \in B_{p/\beta}(u)$ [5, Theorem 3.3.5], we have that

$$\|M_\beta(T(f))\|_{\Lambda_u^p(w)} = \|M(|T(f)|^\beta)\|_{\Lambda_u^{p/\beta}(w)}^{1/\beta} \leq C\||T(f)|^\beta\|_{\Lambda_u^{p/\beta}(w)}^{1/\beta} = C\|T(f)\|_{\Lambda_u^p(w)}.$$

So, to obtain

$$\|T(f)\|_{\Lambda_u^p(w)} \leq C\|f\|_{\Lambda_u^p(w)}$$

we only need to show that $\|T(f)\|_{\Lambda_u^p(w)} < \infty$.

Let $R > 0$ so that $\operatorname{supp} f \subset B(0,R)$ and set $B = B(0,R)$. Then

$$\begin{aligned}\|T(f)\|_{\Lambda_u^p(w)} &\lesssim \left(\int_0^\infty W(u(\{x \in 3B : |T(f)(x)| > t\}))t^{p-1}dt\right)^{1/p}\\ &\quad + \left(\int_0^\infty W(u(\{x \in \mathbb{R}^n\backslash 3B : |T(f)(x)| > t\}))t^{p-1}dt\right)^{1/p}\\ &= I_1 + I_2.\end{aligned}$$

For I_2, it is easy to see, by the condition (6) of T that

$$|T(f)(x)| \le CM(f)(x),$$

since if $x \in (3B)^c$ and $y \in B$, we have that

$$|x| \ge 3R, \quad |x-y| \ge |x| - R \ge |x|/2, \text{ and } B \subset B(0,|x|).$$

Therefore,

$$|T(f)(x)| \lesssim \int_B |x-y|^{-n}|f(y)|dy \lesssim \frac{1}{|x|^n}\int_{B(0,|x|)}|f(y)|dy \lesssim M(f)(x),$$

and hence, since $w \in B_p(u)$,

$$I_2 \lesssim \|M(f)\|_{\Lambda^p_u(w)} \lesssim \|f\|_{\Lambda^p_u(w)} < \infty.$$

To estimate I_1, using Lemma 2.7, let us take $r > 1$ big enough to have that

$$\int_0^1 \frac{W(t)}{t^{p/r+1}}dt < \infty.$$

Then,

$$\begin{aligned} I_1^p &= \int_0^\infty W\left(\int_{\{x\in 3B:|T(f)(x)|>t\}} u(x)dx\right) t^{p-1}dt \\ &\lesssim W(u(3B)) + \int_1^\infty W\left(\int_{\{x\in 3B:|T(f)(x)|>t\}} u(x)dx\right) t^{p-1}dt. \end{aligned}$$

Since the condition $u \in A_\infty$ implies that $u \in A_r$, we get the classical estimate $T : L^r(u) \to L^{r,\infty}(u)$. Hence, for every $t > 0$,

$$\int_{\{x\in 3B:|T(f)(x)|>t\}} u(x)dx \lesssim \int_{\{x\in\mathbb{R}^n:|T(f)(x)|>t\}} u(x)dx \lesssim \frac{\|f\|^r_{L^r(u)}}{t^r},$$

and using that $W \in \Delta_2$,

$$\begin{aligned} &\int_1^\infty W\left(\int_{\{x\in 3B:|T(f)(x)|>t\}} u(x)dx\right) t^{p-1}dt \lesssim \int_1^\infty W\left(\frac{\|f\|^r_{L^r(u)}}{t^r}\right) t^{p-1}dt \\ &\lesssim \int_1^\infty W\left(\frac{1}{t^r}\right) t^{p-1}dt \approx \int_0^1 \frac{W(t)}{t^{p/r+1}}dt < \infty. \end{aligned}$$

□

Proof of Theorem 1.2. First, let b be a bounded function and f be a bounded function with compact support. Let $0 < \beta < \varepsilon < 1$. Since T is bounded on $\Lambda^p_u(w)$ by Theorem 1.1, Tf lies in $\Lambda^p_u(w)$ and thus $bT(f)$ lies in $\Lambda^p_u(w)$ as well. Likewise $T(bf)$ lies in $\Lambda^p_u(w)$. Hence $\|[b,T](f)\|_{\Lambda^p_u(w)} < \infty$ and thus $\|M_\beta[b,T](f)\|_{\Lambda^p_u(w)} < \infty$, by the assumption $w \in B_p(u)$. Then, Lemma 2.6, Lemma 2.5, the condition $w \in B_p(u)$, and Theorem 1.1 lead us to

$$\begin{aligned} \|[b,T](f)\|_{\Lambda^p_u(w)} &\le \|M_\beta[b,T](f)\|_{\Lambda^p_u(w)} \lesssim \|M^\sharp_\beta[b,T](f)\|_{\Lambda^p_u(w)} \\ &\lesssim \|b\|_{BMO}\Big(\|M_\varepsilon(T(f))\|_{\Lambda^p_u(w)} + \|M^2(f)\|_{\Lambda^p_u(w)}\Big) \\ &\lesssim \|b\|_{BMO}\Big(\|T(f)\|_{\Lambda^p_u(w)} + \|f\|_{\Lambda^p_u(w)}\Big) \lesssim \|b\|_{BMO}\|f\|_{\Lambda^p_u(w)}. \end{aligned}$$

In general, if b is any function in BMO, define

$$b_N(x) = \begin{cases} N, & \text{if } b(x) > N, \\ b(x), & \text{if } |b(x)| \le N, \\ -N, & \text{if } b(x) < -N. \end{cases}$$

It is obvious that $b_N \to b$ pointwisely and $\|b_N\|_{BMO} \le \|b\|_{BMO}$ ([10, Exercise 3.1.4]). The Lebesgue dominated convergence theorem gives that $b_N \to b$ in $L^2(\text{supp}(f))$. Hence $b_k f \to bf$ in $L^2(\mathbb{R}^n)$ and $T(b_k f) \to T(bf)$ in $L^2(\mathbb{R}^n)$, by the boundedness of T on $L^2(\mathbb{R}^n)$. Thus, we deduce that there exists a subsequence of integers k_j, for which $T(b_{k_j} f) \to T(bf)$, a.e. For this subsequence, we obtain $[b_{k_j}, T](f) \to [b, T](f)$ a.e. Now, by Fatou's lemma in the weighted Lorentz spaces $\Lambda_u^p(w)$,

$$\begin{aligned} \|[b,T](f)\|_{\Lambda_u^p(w)} &\le \liminf_{j\to\infty} \|[b_{k_j}, T](f)\|_{\Lambda_u^p(w)} \\ &\lesssim \liminf_{j\to\infty} \|b_{k_j}\|_{BMO} \|f\|_{\Lambda_u^p(w)} \lesssim \|b\|_{BMO} \|f\|_{\Lambda_u^p(w)}. \end{aligned}$$

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Department of Analysis and Applied Mathematics, Complutense University of Madrid, 28040 Madrid, Spain.
Email address: mjcarro@ucm.es

Department of Mathematics, Zhejiang International Studies University, Hangzhou, 310012, China.
Email address: honglli@126.com

Interdisciplinary Mathematics Institute (IMI), Department of Analysis and Applied Mathematics, Complutense University of Madrid, 28040 Madrid, Spain.
Email address: javier.soria@ucm.es

Department of Mathematics, Zhejiang University of Science and Technology, Hangzhou, 310023, China.
Email address: qxsun@126.com