CLASSIFICATION OF SMOOTH CONGRUENCES WITH A FUNDAMENTAL CURVE

by *E. Arrondo**, *M. Bertolini*** and *C. Turrini***

Abstract: We give a classification and a construction of all smooth $(n-1)$-dimensional varieties of lines in $\mathbf{P}^n$ verifying that all their lines meet a curve. This also gives a complete classification of $(n-1)$-scrolls over a curve contained in $G(1,n)$.

Introduction.

The geometry of the line, i.e. the study of the line as the main element, was a very popular subject at the end of last century and beginning of this. A particular attention was given to varieties of lines, also called *congruences*. In other words, a congruence is a subvariety of a Grassmannian $Gr(1,\mathbf{P}^n)$ of lines in $\mathbf{P}^n$. When we will refer to a congruence, we will restrict ourselves to the case of subvarieties of dimension $n-1$ (hence equal to the codimension).

A lot of papers were published at that time by several mathematicians about congruences of lines in $\mathbf{P}^3$. Of important relevance is the work of G. Fano, who made several classifications of these congruences under different assumptions. One fixed assumption in his classification was the non-existence of what was called a *fundamental curve*. A fundamental curve is a curve in $\mathbf{P}^3$ such that all lines of the congruence meet it. The reason for excluding this possibility does not seem to be clear. Recently, after congruences became popular again, M. Gross and the first author classified all those congruences excluded by Fano, i.e they gave a classification of all smooth congruences of lines in $\mathbf{P}^3$ with a fundamental curve.

Not only the congruences of lines in $\mathbf{P}^3$, but also in $\mathbf{P}^4$ (i.e. threefolds in $Gr(1,\mathbf{P}^4)$) came back to the attention of nowadays mathematicians. When the last two authors started the classification of smooth congruences of lines in $\mathbf{P}^4$ with a curve meeting all lines, they found that the same key numerical relation as for $\mathbf{P}^3$ holds (see lemma 4). This was the starting point of this joint work dealing with the general case of smooth congruences of lines in any $\mathbf{P}^n$ with all lines meeting a curve (in the sequel such a curve will be called fundamental, see section 0 for the definition).

In section 1, we deal with the case in which the curve is a line. This is a particular case in the sense that there are infinitely many families of these congruences, while for other curves we find, for each n, a finite number of families. On the other hand, this is

* Departamento de Algebra, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

** Dipartimento di Matematica, Università degli Studi di Milano, via C. Saldini, 50, 20133 Milano, Italy

a generalization of the result in case $n = 3$, which is known under another formulation. Indeed, the Grassmannian $Gr(1, \mathbf{P}^3)$ can be considered, under the Plücker embedding, as a smooth quadric in $\mathbf{P}^5$. Since the variety of the lines meeting a given line is a singular hyperplane section, congruences with a fundamental line can be viewed as surfaces in $\mathbf{P}^4$ contained in a singular quadric. It is then a classical result that goes back to Roth (see [R] §3) that such a smooth surface is either the complete intersection of the quadric and another hypersurface (if it misses the singular point of the quadric) or the rest of a plane under such a complete intersection. We get an analogous result for any n.

In section 2, we give the list of all possible smooth congruences. For the proof we needed to refine some arguments of [A-G]. The base idea is to lift the congruence as a hypersurface in a desingularization of the variety of lines meeting a curve. Then one obtains a numerical relation from the double-point formula (coming from the fact that we have a smooth subvariety of $Gr(1, \mathbf{P}^n)$ with same dimension as codimension), which together with Castelnuovo bound for the genus in two different contexts allows us to conclude.

In section 3 we give an explicit construction for all possible congruences we found in section 2, which completes the classification. We point out that all examples for $n = 3$ (except the congruences of bisecants) extend to any dimension. Moreover an easy argument shows that by the above constructions and [A] we also obtain the complete classification of $(n-1)$-dimensional scrolls over a curve, contained in $G(1, n)$.

The results of this paper are just one way to generalize the classical results on congruences with a fundamental curve. Another natural and interesting generalization, which some of the authors are now studying, are the congruences of lines in $\mathbf{P}^n$ with higher dimensional fundamental locus, e.g. a surface with infinitely many lines through each point of its.

§0. Preliminaries and notations.

As said in the introduction, by a *congruence* we will mean an $(n-1)$-dimensional variety (over the complex numbers) in the Grassmannian variety $G(1, \mathbf{P}^n)$ of lines in $\mathbf{P}^n$. We will denote usually by G to this Grassmannian.

We will write

$\Omega(A, B) = $ Schubert variety of lines in G meeting A and contained in B, where A and B are linear subspaces of $\mathbf{P}^n$ such that $A \subset B$

$\Omega(i, j) = $ class, in the Chow ring of G, of $\Omega(A, B)$, where $\dim A = i$ and $\dim B = j$

We will say that a point P of $\mathbf{P}^n$ is a *k-fundamental point* of a congruence Y if there is a k-dimensional family of lines of Y through P. A curve C of $\mathbf{P}^n$ is a *k-fundamental curve* for Y if all its points are k-fundamental points. In other words an $(n-2)$-fundamental curve is a curve which is met by all the lines of the congruence; in the sequel we will briefly call such a curve a *fundamental curve*.

Given a congruence Y, we will denote

$a = $ number of lines of Y passing through a general point of $\mathbf{P}^n$

$b = $ number of lines of Y contained in a general hyperplane H and meeting a general line of H

$d = $ degree of the curve C

$g = $ geometric genus of C

$e = $ degree of the cone formed by all lines of Y passing through a general point of C

We will not consider the case when Y consists of all lines passing through one point, so that we will assume $e \geq 1$ for any fundamental curve. Note that, although the $(n-1)$-dimensional cycles of G are generated by $\{\Omega(i, n-i) \mid i = 0, \ldots, [\frac{n-1}{2}]\}$, in the case of congruences with fundamental curve the only non-zero intersections with the class of Y are $a = [Y] \cdot \Omega(0, n)$ and $b = [Y] \cdot \Omega(1, n-1)$.

§1. Smooth congruences with a fundamental line.

For this section, we will view G as embedded in a projective space under the Plücker embedding. In this framework, a hypersurface of degree l will mean the intersection of G with such a hypersurface in the Plücker embedding.

Theorem 1. *Let $Y \subseteq G(1, n)$ be a smooth congruence such that all its lines meet a given line $\Lambda \subseteq \mathbf{P}^n$. Denote by Γ the cone $\Omega(\Lambda, \mathbf{P}^n)$. Then either*

  j) $b = a$ and Y is the intersection of Γ with a hypersurface of degree a   *or*

 jj) $b = a - 1$ and Y is linked to an $(n-1)-$fold of degree $n-2$ passing through the vertex of Γ, in the intersection of Γ with a hypersurface of degree $a + 1$ *or*

jjj) The dimension is $n = 3$, $a = b - 1$ and Y is linked under the complete intersection of Γ and a hypersurface of degree b to a plane consisting of all lines passing through a point.

Proof: Since the case $n = 3$ is already known, we will assume $n \geq 4$. A modern proof for the case $n = 3$ can be found in [G1], from where we took the idea for this general proof.

It is easy to see that the Schubert variety $\Gamma = \Omega(\Lambda, \mathbf{P}^n)$ is a cone over the $(n-1)$-fold $B = \mathbf{P}^1 \times \mathbf{P}^{n-2}$ with vertex the point $\lambda \in G(1, n)$ corresponding to Λ. Consider the blow up Γ^* of Γ at λ; then $\Gamma^* \simeq \mathbf{P}\,(O_B \oplus O_B(-1))$ and the exceptional divisor E is isomorphic to B. Put $\mathcal{E} = O_B \oplus O_B(-1)$ and $p : \mathcal{E} \to B$; Then, with standard notations, $Pic(B)$ ($Pic(E)$ resp.) is generated by $\mathbf{P}_1^1 \times \mathbf{P}_2^{n-3}$ and $\mathbf{P}_1^0 \times \mathbf{P}_2^{n-2}$ ($\mathbf{P'}_1^1 \times \mathbf{P'}_2^{n-3}$ and $\mathbf{P'}_1^0 \times \mathbf{P'}_2^{n-2}$

3

resp.), so that $Pic(\Gamma^*)$ is generated by E, $Q = p*(\mathbf{P}_1^1 \times \mathbf{P}_2^{n-3})$ and $Z = \mathbf{P}_1^0 \times \mathbf{P}_2^{n-2}$. Now, for a $(n-1)$-fold $Y^* \subseteq \Gamma^*$ to be mapped to a non singular $(n-1)$-fold $Y \subseteq \Gamma$ there are two possibilities:

i) Y^* does not intersect E, and then the blow up map is an isomorphism between Y^* and Y;

or

ii) Y^* intersects E and then $W = Y^* \cap E$ is contracted to the vertex of Γ which must be a smooth point of Y. So W must be a divisor in Y^* isomorphic to $\mathbf{P}^{n-2}$.

Now suppose that, in $Pic(\Gamma^*)$, it is $Y^* = \alpha E + \beta Q + \gamma Z$, so that the numerical equivalency class of W in E is
$$W = (\beta - \alpha)(\mathbf{P'}_1^1 \times \mathbf{P'}_2^{n-3}) + (\gamma - \alpha)(\mathbf{P'}_1^0 \times \mathbf{P'}_2^{n-2}).$$

In case $j)$, one has $\alpha = \beta = \gamma$, so that Y is the complete intersection of Γ with an hypersurface of degree $\alpha = a$ (case $i)$). In case $jj)$, being $n - 2 > 1$, the only possibility for a divisor W to be a $\mathbf{P}^{n-2}$ embedded into $E = \mathbf{P}^1 \times \mathbf{P}^{n-2}$ is that $W = \mathbf{P}_1^0 \times \mathbf{P}_2^{n-2}$, so that $\beta = \alpha$ and $\gamma = \alpha + 1$. Hence $Y^* = \alpha E + \alpha Q + (\alpha + 1)Z$ and Y is linked, in the complete intersection of Γ with an hypersurface of degree $\alpha + 1$, to the image of $E + Q$ in Γ, i.e. to an $(n-1)$-fold of degree $n - 2$ passing through the vertex of Γ (case $ii)$).

§2. Smooth conguences with a fundamental curve

In this section we will prove the following

Theorem 2. *Let Y be a smooth congruence having an integral curve C as a fundamental curve. Then one of the following holds:*

(i) *$n = 3$ and the congruence consists of the bisecants to either a twisted cubic or an elliptic quartic in $\mathbf{P}^3$.*

(ii) *The curve C is a line (and Y is described in Theorem 1).*

(iii) *The congruence is a scroll (i.e. $e = 1$) and either C is a conic and $a = 1$ or 2, $b = 2$, or C is a plane cubic and $a = b = 3$*

(iv) *The curve C is a plane cubic, $e = 2$, $a = 3$ and $b = 6$*

We will follow several steps in the proof, that we state as lemmas.

Lemma 3. *The fundamental curve C is smooth.*

Proof: Consider the subscheme V of G corresponding to lines in $\mathbf{P}^n$ meeting C. It is singular, and its singular locus is the locus of bisecants to C. This locus has several components: one for each singular point of C (consisting of all lines passing through that point) and another one of the expected dimension two given by the closure of all bisecant lines at smooth points of C. Hence Y is not contained in the singular locus of V unless dim $Y = 2$ and Y is the congruence of bisecants to C. This case has already be studied in [A-G], and corresponds to (i) in the theorem.

Let $f : \tilde{C} \to C$ be the normalization of C, and $L = f^*(\mathcal{O}_C(1))$. The scheme V has a natural desingularization given by $X = \mathbf{P}(f^*(\Omega_{\mathbf{P}^n|C}(2)))$. Call $p : X \to \tilde{C}$ the natural projection. Since by definition of fundamental curve Y is contained in V, it has a lift $\tilde{Y}$ to X. If we are not in case (i), then the map $\pi : \tilde{Y} \to Y$ is birational. If it is not an isomorphism, it must contract a curve E of $\tilde{Y}$ (just applying the Zariski main theorem, because Y is smooth). The curve E meets all fibers of $p_{|\tilde{Y}}$, since none of these fibers is contracted by π. This easily implies that the line represented by the image of E under π is C, which is case (ii) in the theorem.

Hence, assuming that C is not a line we have that π is in fact an isomorphism between $\tilde{Y}$ and Y. This implies that C is smooth. Indeed, if two different points p_1 and p_2 of $\tilde{C}$ go to the same point of C, then the images of the cones at p_1 and p_2 must have some common line. This contradicts the fact that π is an isomorphism. The same argument holds when p_1 and p_2 are infinitely near points. A precise proof can be found in [G2]. The idea is that the tangent space at a point of $\tilde{Y}$ (given by a point p of $\tilde{C}$ and a line l through it) splits as the sum of the tangent space of $\tilde{C}$ at p and the tangent space at l of the cone with vertex p. Since this maps isomorphically to the tangent space of Y, hence the tangent space of $\tilde{C}$ at p also maps isomorphically to the tangent space of C at $f(p)$.

Lemma 4. *Except for cases (i) and (ii), there is a numerical relation*

$$2d^2e^2 - 4de^2 - 2e^2g - 2de - 2eg + 2e^2 + 2e + D(1 + 2e - 2de) + D^2 = 0$$

(This is the same as (*) in [G2] p. 137).

Proof: The map from X to G is given as follows.

First, look at the commutative diagram of exact sequences defining F as a push-out (we will write $\mathcal{O}_X(a,b)$ for $p^*(\mathcal{O}_C(a)) \otimes \mathcal{O}_X(b)$):

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & \Omega_{X/C}(-1,1) & = & \Omega_{X/C}(-1,1) & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & p*\Omega_{\mathbf{P}^n|C}(1,0) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^n}(1)) \otimes \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(1,0) & \longrightarrow & 0 \quad (1) \\
& & \downarrow & & \downarrow \alpha & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X(-1,1) & \longrightarrow & F & \longrightarrow & \mathcal{O}_X(1,0) & \to & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

5

The surjection α gives the map from X to G; in particular α is the pullback on X of the canonical map from $H^0(\mathcal{O}_{\mathbf{P}^n}(1)) \otimes \mathcal{O}_G$ to $\mathcal{Q}$, where $\mathcal{Q}$ is the universal bundle on G. The Chow ring of X is generated by $Pic(C)$ and $t = c_1(\mathcal{O}_X(1))$, which canonically identifies with the pullback of the hyperplane section of G. The relation among them is given by $t^n = (n-1)Lt^{n-1}$, and the class of a point is Lt^{n-1}. Hence, the class of Y in X can be written as $et - D$, where D is, by abusing notation, the pullback of a certain divisor D of C (also, when no confusion arises, we will use D for denoting the degree of the divisor). Let us compute from these data the double-point formula for Y. Writing $N = N_{Y/G}$, we have that $a^2 + b^2 = \deg(c_{n-1}(N))$. Let us compute each of the members in the formula (most relations will come from the exact sequences in diagram (1)).

The pullback to X of the set of lines passing through a point of $\mathbf{P}^3$ is the locus where n sections of the bundle F are dependent. We can apply then Porteous formula (see, for example [F] Thm. 14.4, from where we also keep the notation for the Schur polynomial) and get that

$$a = \Delta_1^{(n-1)}(F_{|Y}) \cap [Y] = (t^{n-1} - (n-2)Lt^{n-2})(et - D) = ed - D$$

(where we used the notations in [F], and the fact that in our case $\Delta_1^i(F) = t^i - (i-1)Lt^{i-1}$, as can be easely checked by induction on i)

The second degree b is easily computed by geometrical means. It is the number of lines of Y contained in a general hyperplane H of $\mathbf{P}^{n-1}$ and meeting a general line r of H. The intersection of H with C consists of d points, and through each of them, lines of Y form a cone of degree e, thus meeting L in e points. Hence $b = ed$.

Let us now compute the total Chern polynomial of N, $c(N) = c(T_{G|Y})c(T_Y)^{-1}$. We use the fact that $T_G = F \otimes T_{X/C}(1, -1)$ and the fact that F appears as an extension in (1), as well as the fact that $N_{Y/X} = \mathcal{O}_X(et - D)$ to conclude that

$$c(N) = c(p^*(T_{\mathbf{P}^n|C}))c(\mathcal{O}_X(2, -1))^{-1}c(\mathcal{O}_X(et - D))c(p^*T_C)^{-1}.$$

Now, it is a straightforward calculation to show that

$$c_{n-1}(N) = (e+1)t^{n-1} + (5e - ne - n + 3)Lt^{n-2} + (e+1)Kt^{n-2} - Dt^{n-2}$$

(where K represents the canonical divisor of C). Now, its degree comes from multiplying in the Chow ring of X the above class with the class $et - D$ of Y, to obtain

$$\deg(c_{n-1}(N)) = (4e^2 + 2e)d + e(e+1)(2g-2) - (2e+1)D$$

Identifying this with $a^2 + b^2$ and substituting the values of a and b we finally get

$$2d^2e^2 - 4de^2 - 2e^2g - 2de - 2eg + 2e^2 + 2e + D(1 + 2e - 2de) + D^2 = 0$$

Lemma 5. *The curve C is plane and hence the relation in lemma 4 becomes*

$$de(d-1)(e-1) = D(2de - D - 2e - 1) \tag{2}$$

Proof: It is the same as in [G2] by getting a contradiction between the relation in lemma 4 and Castelnuovo bound for the genus of non-plane curves.

Lemma 6. *In the same hypothesis of lemmas 4 and 5, $b \leq 2a$ and hence $e \leq 3$.*

Proof: This is just as in [A-G], by applying Hurwitz theorem to the map from the curve of lines of Y in P (the plane containing C) to C. The second inequality is a consequence of the first together with (2).

We now complete the proof of the theorem by analyizing separately each of the possible values for e.

1) If $e = 1$, then the congruence Y is a scroll of $\mathbf{P}^{n-2}$'s over the curve C. More precisely, through each point Q of C, the lines of the congruence Y are those contained in a given hyperplane H_Q of $\mathbf{P}^n$ containing Q. This distribution of hyperplanes is given by the epimorphism in the exact sequence (defining the vector bundle E):

$$0 \to \mathcal{O}_C(D) \to \Omega_{\mathbf{P}^n}(2) \otimes \mathcal{O}_C \to E \to 0 \qquad (3)$$

(recall that Y is the zero locus of $p^*(\mathcal{O}_C(D)) \to p^*(\Omega_{\mathbf{P}^n}(2) \otimes \mathcal{O}_C) \to \mathcal{O}_X(0,1)$). Since the curve C is plane, we can build the following commutative diagram (defining the sheaf $\mathcal{F}$):

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & \mathcal{O}_C(L)^{\oplus n-2} & = & \mathcal{O}_C(L)^{\oplus n-2} & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{O}_C(D) & \to & \Omega_{\mathbf{P}^n}(2) \otimes \mathcal{O}_C & \to & E & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_C(D) & \to & \Omega_{\mathbf{P}^2}(2) \otimes \mathcal{O}_C & \to & \mathcal{F} & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
\qquad (4)
$$

The injectivity in the right column in (4) comes because otherwise the sheaf $\mathcal{F}$ would have rank two, just implying that all lines in the plane P are in the congruence.

Putting $e = 1$ in expression (2), we get that either $\deg D = 0$ (and hence $a = b = d$) or $a + b = 3$. In this second case, from lemma 6 the only possiblity is $a = 1$, $b = d = 2$. So we assume that $\deg D = 0$.

Assume the sheaf $\mathcal{F}$ appearing in (4) has torsion at some point p of C. This would imply that for each other q in C, the line joining p and q is in the congruence. For $d > 2$ this contradicts the fact proved in lemma 3 that π is an isomorphism. Hence, $\mathcal{F}$ is an invertible sheaf in this case, so it must be $\mathcal{F} = \mathcal{O}_C(L - D)$, by looking at the first Chern

class. This gives the immersion of C into $\check{\mathbf{P}}^2$ defined by assigning to each point in C the only line through it contained in P. For $d > 3$ there is only one g_d^2 on C (see [A-C-G-H] page 56), so that $L = L - D$ and thus $D = 0$. This would imply that all lines in P pass through a fix point of C, which is absurd. Hence, $d \leq 3$.

2) If $e = 2$, it happens as in [A-G] that the curve (of degree a) W in the projective space of hyperquadrics in $\mathbf{P}^n$ must be plane (just using Castelnuovo bound together with lemma 5). This plane cannot be contained in the locus K of singular quadrics whose vertex is at a given plane P (same proof as in [A-G], using that the embedded tangent space of K at a cone contains the space of quadrics passing through the vertex of the cone). This locus K is defined by the maximal minors of a $3 \times (n+1)$ matrix of linear forms. Hence, $a \leq 3$, and the only numerical solutions for (a, b) are $(2, 6)$ and $(3, 6)$. The first one is ruled out by lemma 6.

3) If $e = 3$, the curve W in the space of hypercubics is contained in a linear space A of dimension three. This has to meet K (space of cones with vertex at the plane P) along a curve. Indeed, the embedded tangent space to K at a cubic cone is contained in the space of cubic hypersurfaces that are singular at the vertex of the cone. Assume that the intersection of K and A contains a surface S. Take a point p of C and let p' the point of W representing the cone with vertex p. By assuption, there is a plane in A through p' (the tangent plane to S at p') consisting of singular cubics at p. If $a > 3$, this plane must meet W in at least another point q' (for general p). Hence the cone at the corresponding point q of C is singular at p. When we vary the point p, if the point q also varies, this means that all cones are singular, which is absurd (the general fiber of the map $\tilde{Y} \to C$ is smooth). This means that there is a point q in C whose cone through it is singular along all points of C. In particular the pencil of lines through q and contained in P should be in the congruence, which is also absurd.

Now K is defined by the maximal minors of a $3 \times \binom{n+2}{2}$ matrix of linear forms (this matrix corresponds to the derivatives of forms of degree 3 with respect to the directions given by the plane P). Hence, W is contained in the locus of a $\mathbf{P}^3$ given by the maximal minors of a 3×4 matrix of linear forms, hence $a \leq 6$. The only numerical solutions of (2) are then $(a, b) = (3, 12), (4, 12)$, which are again impossible by lemma 6.

§3. Explicit constructions.

Lemma 7. *Let $C \subseteq \mathbf{P}^n$ be a non singular plane curve of degree d. Then $\Omega_{\mathbf{P}^n}(2)_{|C} \cong (\oplus^{n-2}\mathcal{O}_C(L)) \oplus \mathcal{S}(P)$ where L is an hyperplane section of C, E is a divisor on C of degree 1 and $\mathcal{S}$ is a normalized rank 2 vector bundle of degree $d-2$ on C with $\wedge^2\mathcal{S} \cong \mathcal{O}_C(L-2P)$.*

Proof: First of all, notice that

8

$$\Omega_{\mathbf{P}^n | \mathbf{P}^2} \cong (\oplus^{n-2} \mathcal{O}_{\mathbf{P}^2}(-1)) \oplus \Omega_{\mathbf{P}^2}.$$

Twisting by $\mathcal{O}(2)$ and restricting to C, we get

$$\Omega_{\mathbf{P}^n}(2)_{|C} \cong (\oplus^{n-2} \mathcal{O}_C(L)) \oplus \Omega_{\mathbf{P}^2}(2)_{|C}.$$

If Λ denotes the plane containing C and

$$f : X = \Omega_{\mathbf{P}^n}(2)_{|C} \to G(1, n)$$

is the projection, denote by $Y_0 = f^{-1}(\Omega(1, \Lambda))$ the ruled surface of lines of X contained in Λ. Let $\mathcal{S}$ be the normalized bundle defined by $Y_0 = \mathbf{P}(\mathcal{S})$. By the above, Y_0 can be identified with $\mathbf{P}(\Omega_\Lambda(2)_{|C}) \subseteq \mathbf{P}(\Omega_{\mathbf{P}^n}(2)_{|C})$, which implies (3). Now, with the same arguments as in [A-G], one shows that $deg\mathcal{S} = d - 2$, and that the numerical equivalence class of the divisor inducing the map $Y_0 \to \Omega(1, \Lambda)$ is $C_0 + f$. By the way notice also that the class of Y_0 in X is the $(n-2) - th$ Chern class of $(\oplus^{n-2} \mathcal{O}_X(t - p^*L))$.

Construction of case $(3, 3)$

Here $\Omega_{\mathbf{P}^n}(2)_{|C} \cong (\oplus^{n-2} \mathcal{O}_C(L)) \oplus \mathcal{F}(P)$ as in Lemma 7, with $degL = 3$, $degP = 1$, $deg\mathcal{F} = 1$ and $\wedge^2 \mathcal{F} \cong \mathcal{O}_C(L - 2P)$, so that Y_0 is the elliptic ruled surface of invariant -1

Notice that, C being a plane curve, the image of X in the grassmannian $G(1, n) \in \mathbf{P}^{\frac{n(n+1)}{2} - 1}$ is in fact contained in a $\mathbf{P}^{3n-4}$, so that, since $dim|t| = 3n - 4$, the embedding of X into $G(1, n)$ is given by the whole linear system $|t|$.

As shown in Lemma 4, the class of Y in $X = \mathbf{P}(\Omega_{\mathbf{P}^n}(2)_{|C})$, can be written as $t - p^*D$, D being a degree 0 divisor on C.

As D is a divisor of degree 0, then $\Omega_{\mathbf{P}^n}(2)_{|C}(-D)$ is generated by global sections, so $|t - p^*D|$ is base point free and a general member is smooth.

Recall from §2 that we have an exact sequence given by the right column of (4) and that we showed that $\mathcal{F} = \mathcal{O}_C(L - D)$ and $D \neq 0$. This implies that $Y = \mathbf{P}(E) = \mathbf{P}(\mathcal{O}_C(L - D) \oplus \mathcal{O}_C(L)^{n-2})$. Since it is mapped to the Plücker embedding of G by the complete very ample linear series $|\,t\,|$, then it is an embedding in G.

For the geometric construction, we observe first that the hyperplanes formed at each point must all contain a fixed linear space Λ of codimension three, which is disjoint with the plane P containing C. Indeed, this comes from the splitting of E. Hence these hyperplanes are defined as the span of Λ and some a line in P. The way of constructing these lines is given in [Go] in the following way. Take a group structure on C such that the origin is an inflection point. Now take a point σ such that $3\sigma = L - D$ and fix a point R outside C. For each point p in C, let A and B the points of C in the line defined by R and $-p + 2\sigma$.

9

Then, the line associated to p is the one defined by $-A - \sigma$ and $-B - \sigma$. We do not know how to prove this directly, but just as in [G1] checking that this construction corresponds to the above description and that, for fixed D, the dimension is the same (dimension two for the choice R and also for a section of $\Omega(2)_{|C}(-D)$).

Construction of case $(2,2)$

Let C be a conic, contained in a plane L in $\mathbf{P}^n$. By Lemma 7, the sheaf $\Omega_{\mathbf{P}^n}(2)_{|C}$ is isomorphic to $\oplus^{n-2}\mathcal{O}_{\mathbf{P}^1}(2) \oplus (\oplus^2\mathcal{O}_{\mathbf{P}^1}(1))$.

As $X = \mathbf{P}(\Omega_{\mathbf{P}^n}(2)_{|C})$ we can write $X = \mathbf{P}(\oplus^{n-2}\mathcal{O}_{\mathbf{P}^1}(2)\oplus(\oplus^2\mathcal{O}_{\mathbf{P}^1}(1)))$. Let us consider now the linear system $|t|$. As $h^0(\mathcal{O}_X(t)) = 3n - 2$ and $G(1,n) \subset \mathbf{P}^{\frac{n(n+1)}{2}-2}$, the map $f : X \to G(1,n)$ is induced by a $(3n-3)$-dimensional subspace V of $H^0(\mathcal{O}_X(t))$. More precisely, $V = H^0(\mathcal{O}_{\mathbf{P}^1}(2)^{n-2}) \oplus V'$, where V' is a three-dimensional subspace of $H^0(\mathcal{O}_{\mathbf{P}^1}(1)^2)$.

Claim. *A smooth scroll with $a = b = 2$ comes from an element $Y \in |t|$ of one of the following types:*
 (i) $Y = \mathbf{P}(E)$ with $E = \oplus^{n-1}\mathcal{O}_{\mathbf{P}^1}(2)$ i.e. $\mathbf{P}^1 \times \mathbf{P}^{n-2}$ embedded in $\mathbf{P}^{3n-4}$ via $\mathcal{O}(2,1)$
 (ii) $Y = \mathbf{P}(E)$ with $E = \oplus^{n-3}\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(3)$.

Proof of claim: The exact sequence (3) in §2 becomes

$$0 \to \mathcal{O}_{\mathbf{P}^1} \to \oplus^{n-2}\mathcal{O}_{\mathbf{P}^1}(2) \oplus (\oplus^2\mathcal{O}_{\mathbf{P}^1}(1)) \to E \to 0.$$

We look at the different possibilities for E depending on the composed map

$$\varphi : \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1}(2)^{n-2} \oplus \mathcal{O}_{\mathbf{P}^1}(1)^2 \to \mathcal{O}_{\mathbf{P}^1}(1)^2$$

 (i) If φ is an injective morphism of bundles, then E is an extension of $\mathrm{coker}\varphi = \mathcal{O}_{\mathbf{P}^1}(2)$ by $\mathcal{O}_{\mathbf{P}^1}(2)^{n-2}$, hence $E = \oplus^{n-1}\mathcal{O}_{\mathbf{P}^1}(2)$
 (ii) If φ is (after changing basis) zero on one factor $\mathcal{O}_{\mathbf{P}^1}(1)$ and different from zero on the other, then E contains the first $\mathcal{O}_{\mathbf{P}^1}(1)$ as a direct summand. The complement of this will be an extension of $\mathcal{O}_{\mathbf{P}^1}(3)$ (this is the cokernel for a general monomorphism of bundles $\mathcal{O}_{\mathbf{P}^1}\text{to}\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2)$ by $\mathcal{O}_{\mathbf{P}^1}(2)^{n-3}$, so that $E = \oplus^{n-3}\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)) \oplus \mathcal{O}_{\mathbf{P}^1}(3))$.
 (iii) If $\varphi = 0$, then $Y = \mathbf{P}(E)$ must contain $\mathbf{P}(\Omega_{\mathbf{P}^2}(2)_{|C})$, which would imply that all the lines in L through any point of C are in the congruence. This is a contradiction.

The case (ii) occurs on an open subset of the locus defined by a quadric equation in V'. Hence, we can find a variety Y in case (ii) not defined by an element of V. We want to prove now that if Y is not in V, it is mapped isomorphically in $G(1,n)$.

Let us consider the sequence

$$0 \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X(t)) \to H^0(\mathcal{O}_Y(t)) \to 0.$$

Now Y not being in V, V is mapped onto $H^0(\mathcal{O}_Y(t))$ so that the immersion of Y into the grassmannian is determined by a complete linear system, which is in fact very ample.

This concludes the proof.

We now furnish an example of geometric construction of the two possible scrolls of bedegree (2,2) quoted in cases (i) and (ii) of the claim above, generalizing some examples of [Go] to the case of higher dimension.

Case (i). Let L be a 2-plane into $\mathbf{P}^n$ and C a smooth conic on it, and let Λ be a $\mathbf{P}^{n-3}$ skew with L. For each point $p \in C$ take the tangent line t_p to C at p, consider the $\mathbf{P}^{n-1} = [\Lambda, t_p]$ spanned by Λ and t_p and denote by Σ_p the $\mathbf{P}^{n-2}$ of the lines through p contained in $[\Lambda, t_p]$. A model for Y is generated by the Σ_p's when p varies in$C$.

Case (ii). Let L and C be as above and take a point p_0 in C. Each $\mathbf{P}^{n-2} = \Sigma_p$ of the lines of the congruence Y through a point p of C generates a $\mathbf{P}^{n-1}$ passing through p_0, the $\mathbf{P}^{n-1}$ generated by Σ_{p_0} is tangent to C at p_0 but does not contain L.

Construction of case $(1, 2)$

In this case, we obtain that the exact sequence (3) in §2 is

$$0 \to \mathcal{O}_{\mathbf{P}^1}(1) \to \mathcal{O}_{\mathbf{P}^1}(1)^2 \oplus \mathcal{O}_{\mathbf{P}^1}(2)^{n-2} \to E \to 0$$

From this we immediately get (since the map $\mathcal{O}_{\mathbf{P}^1}(1) \to \mathcal{O}_{\mathbf{P}^1}(1)^2$ is not zero) that $Y = \mathbf{P}(E) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2)^{n-2})$. The map to G is given by the complete linear series of the canonical $\mathcal{O}(1)$ for E, which is very ample, so that we get a smooth congruence.

The geometric interpretation is as follows. The curve in $\check{\mathbf{P}}^n$ of the hyperplanes defined at points of C has degree $a = 1$, so that all these planes must be contained in a linear subspace Λ of codimension two. Hence all lines in the congruence meet C and Λ. The set of lines meeting both C and Λ defines a congruence with $a = b = 2$, so that it contains another congruence with $a = 1$ and $b = 0$. This implies that Λ meets C in one point and Y consists of the closure of all lines meeting Λ and C in points different from $C \cap \Lambda$.

Remark: For $n \neq 4$ the only $(n-1)$-dimensional scrolls over a curve contained in $G(1, n)$ are the three ones constructed above (cases $(3, 3)$, $(2, 2)$ and $(1, 2)$). Indeed it is easy to see that, for $n \neq 4$, the only linear subspaces of dimension $n - 2$ contained in $G(1, n)$ are the Schubert cycles $\Omega(0, n)$.

For $n = 4$ the linear subspaces of dimension 2 contained in $G(1, 4)$ are the Schubert cycles $\Omega(0, 4)$ and $\Omega(1, 3)$. Hence in this case one has also to consider the scrolls whose

fibers are $\Omega(1,3)$, described in [A], which are without a fundamental curve and therefore not included in our paper.

This completes the classification.

Construction of case $(3,6)$

Let $C \subseteq \mathbf{P}^n$ be the non singular plane cubic quoted in theorem 2, case (iv). Then $e = 2, d = 3, D = 3$. Now, with notation of Lemma 7 we have $deg\mathcal{F} = +1$, i.e. Y_0 is the elliptic ruled surface of invariant -1. As shown in Lemma 4, the class of Y in $X = \mathbf{P}(\Omega_{\mathbf{P}^n}(2)_{|C})$, can be written as $2t - p^*D$.
So we study the linear system $L = |2t - p^*D|$, on X. We have

$$h^0(\mathcal{O}_X(2t - p^*D)) = \tfrac{3(n+1)(n-2)}{2} + h^0(S^2(\mathcal{F})(2P - D)).$$

Since $e = -1$, there exists a curve numerically equivalent to $2C_0 - f$ (see [E] lemma 1.4), where C_0 and f denote respectively a fundamental section and a fibre of Y_0, so that we can fix a D such that $h^0(\mathcal{O}_X(2t - p^*D)) \geq \tfrac{3(n+1)(n-2)}{2} + 1$.

Now we show that the surface Y_0 is not contained in the base locus of L. To do it, consider the ideal $\mathcal{I}$ of Y_0 and the exact sequence

$$0 \to \mathcal{I}(2t - p^*D) \to \mathcal{O}_X(2t - p^*D) \to \mathcal{O}_{Y_0}(2t - p^*D) \to 0.$$

Notice that for $i > 0$, $R^i p_* \mathcal{I}(2t - p^*D) = 0$, since the fibre of this sheaf at a point p_0 of C is $H^i(\mathcal{J}(2))$, $\mathcal{J}$ denoting the ideal of the $\mathbf{P}^1$ defined by Y_0 in the fibre $\mathbf{P}^{n-1}$ of X at p_0. Therefore one can apply p_* to the exact sequence above to get the sequence

$$0 \to p_* \mathcal{I}(2t - p^*D) \to S^2(\Omega_{\mathbf{P}^n}(2)_{|C})(-D) \to S^2(\Omega_{\mathbf{P}^2}(2)_{|C})(-D) \to 0$$

which splits; this easily implies that $h^0(p_* \mathcal{I}(2t - p^*D)) = h^0(\mathcal{I}(2t - p^*D)) = h^0(\mathcal{O}_X(2t - p^*D)) - 1 = \tfrac{3(n+1)(n-2)}{2}$, so that Y_0 is not contained in the base locus of $L = |2t - p^*D|$.

As $Y \in |2t - p^*D|$ and $[Y_0] = t^{n-2} - 3(n-2)t^{n-1}p^*P$, as remarked in Lemma 7 then $Y \cap Y_0$ is a curve C' numerically equivalent to $2C_0 - f$, since $C'^2 = Y^2 \cdot Y_0 = 4t^n - 12(n-1)t^{n-1}p^*P = 0$.

As C' is the unique effective divisor in its linear equivalence class, it is contained in the base locus of the system $|2t - p^*D|$. Now restricting this system to a fiber $p^{-1}(P)$, at a point $P \in C$, we get the following exact sequence:

$$0 \to \mathcal{O}_X(2t - p^*(D + P)) \to \mathcal{O}_X(2t - p^*D) \to \mathcal{O}_{p^{-1}(P)}(2) \to 0$$

by which we get that the image of $H^0(\mathcal{O}_X(2t - p^*D))$ in $H^0(\mathcal{O}_{p^{-1}(P)}(2))$ has dimension $\tfrac{(n^2+n-4)}{2}$. Hence $|2t - p^*D|$ cuts out on the fiber a system of dimension $\tfrac{(n-2)(n+3)}{2}$ of

quadrics which has at least two base-points, hence just the two points of $Y \cap Y_0 \cap p^{-1}(P)$. Thus $Y \cap Y_0$ is the base locus of $|2t - p^* D|$, and Y is smooth away from $Y \cap Y_0$.

With the same argument as in [A-G] we can conclude that Y is embedded in $G(1, n)$ by the map $f : X \rightarrow G(1, n)$ as a smooth congruence of bidegree $(3, 6)$.

References:

[A] Alzati, A., *3-Scroll immersi in $G(1, 4)$*, Ann. Univ. Ferrara, **32** (1986), 45-54.

[A-C-G-H] Arbarello, E.– Cornalba, M.– Griffiths, P.– Harris, J., *Geometry of Algebraic Curves*, Vol. I Grund. der Math. Wissen. **267** Springer-Verlag (1985).

[A-G] Arrondo, E.– Gross, M., *On smooth surfaces in $Gr(1, \mathbf{P}^3)$ with a fundamental curve*, to appear in Man. Math.

[E] Ein, L., *Non-degenerate surfaces of degree $n + 3$ in $\mathbf{P}^n$*, Crelle Journal Reine Angew. Math. **351** (1984), 1-11.

[F1] Fano, G., *Sulle congruenze di rette del terzo ordine prive di linea singolare*, Att. Acc. di Scienze Torino **29** (1984), 474-493.

[F2] Fano, G., *Nuove ricerche sulle congruenze di rette del 3^0 ordine prive di linea singolare*, Memoria della Reale Acad. Sc. Torino (2) **51** (1902), 1-79.

[F] Fulton, W., *Intersection Theory*, Ergebnisse (3) **2**, Springer (1984).

[Go] Goldstein, N. *Scroll surfaces in $Gr(1, \mathbf{P}^3)$*, Conference on Algebraic Varieties of small dimension (Turin 1985), Rend. Sem. Mat. Univ. Politecnica, Special Issue (1987) 69-75.

[G1] Gross, M., *Surfaces in the four-dimensional Grassmannian*, Ph. D. thesis, Berkeley (1990).

[G2] Gross, M., *The distribution of bidegrees of smooth surfaces in $Gr(1, \mathbf{P}^3)$*, Math. Ann. **292** (1992), 127-147.

[R] Roth, L., *On the projective classification of surfaces*, Proc. London Math. Soc., **42** (1937), 142-170.

E-mail of the authors:

arrondo@mat.ucm.es

bertolin@vmimat.mat.unimi.it

turrini@vmimat.mat.unimi.it