Bounded duality in topological Abelian groups

Dedicated to Juan Carlos Ferrando, celebrating his successful academic and research career

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Abstract

We define the β-duality for topological Abelian groups by means of the notion of Hejcman of boundedness in uniform spaces.
A real locally convex space considered as an Abelian group is β-reflexive iff it is reflexive in the ordinary sense for locally convex spaces. Thus, β-reflexivity is the natural extension to Abelian topological groups of the well-known notion of reflexivity. We prove:
1) A locally compact Abelian group is β-reflexive.
2) A β-reflexive metrizable group is reflexive in Pontryagin sense.
3) The β-bidual of a metrizable group is also a metrizable group.

1 Introduction

Duality in locally convex spaces has been a flourishing topic in the second half of the 20th century. The monograph written by Grothendieck after a course explained in Sao Paulo in 1954 is a starting point for an abundant literature on this field.

The class of Abelian topological groups encloses the topological vector spaces as a distinguished subclass. In particular, the locally quasi convex Abelian groups generalize the locally convex spaces. Thus, notions of reflexivity and duality are natural also in the category of locally quasi-convex groups.

The seminal work of Pontryagin in Locally Compact Abelian groups has been an inspiring point in considering duality for general Abelian topological groups. Kaplan, Varopoulos, Smith can be honoured as the initiators of this project. The cornerstone of their developments is to consider the compact-open topology in the character group of a topological group. In previous work we have either generalized or detected obstructions to extend important theorems on locally convex spaces, like Mackey Theorem, Grothendieck's Completeness Theorem, Banach-Dieudonné Theorem, Eberlein Smulyan or Dunford Pettis Theorem to the class of locally quasi-convex groups (See [8], [7], [5], [6], [10]) . Also the definition of Schwartz spaces has given rise to that of Schwartz groups [3].

In the present paper, after considering the notion of boundedness modelled in the definition given by Hejcman for uniform spaces in [9], we start dealing with a new duality for Abelian topological groups. We have called it β-duality and in some sense is the most natural one when one tries to extend some results of Functional Analysis. In fact, we prove that a locally convex space E is reflexive iff it is β-reflexive considered as a topological group. We also prove that the β-bidual of a metrizable group is again metrizable. This duality opens a new field of research, and the present paper is only an introduction to it.

2 Notation, definitions and remarks

Let us collect some facts and notation concerning group dualities. As usual, $\mathbb{T}$ denotes the multiplicative group of all complex numbers with modulus 1, with the topology induced by the Euclidean on $\mathbb{C}$. We use the notation $\mathbb{T}_+ = \{t \in \mathbb{T} : \mathrm{Re}\, t \geq 0\}$, where "Re" denotes the real part of a complex number. For an Abelian group G, $\mathrm{Hom}(G, \mathbb{T})$ denotes the group of all homomorphisms $\chi : G \to \mathbb{T}$ (also called *characters*), with multiplication defined pointwise. If G is a topological Abelian group, $G^\wedge$ will stand for the character group of G which is the subgroup of $\mathrm{Hom}(G, \mathbb{T})$ formed by all continuous characters.

Let (G, H) be a group duality (i. e. a pair formed by an Abelian group G and a subgroup H of $\mathrm{Hom}(G, \mathbb{T})$). If H separates the points of G, we say that the duality is *separating.*

The *polar* of a set $A \subset G$ with respect to the duality (G, H) is the set

$$A^\triangleright = \{\chi \in H : \chi(A) \subset \mathbb{T}_+\}$$

The *inverse polar* of a set $B \subset H$ with respect to the duality (G, H) is the set

$$B^\triangleleft = \{x \in G : \chi(x) \in \mathbb{T}_+ \quad \forall \chi \in B\}.$$

It is known ((1.4) in [4]) that any character contained in the polar of a zero neighborhood is continuous.

A subset A of H is equicontinuous if there exists a zero neighborhood in G such that $A \subset U^\triangleright$, or equivalently if $A^\triangleleft$ is a zero neighborhood in G (see [8]).

A set $A \subset G$ is said to be *quasi-convex with respect to the duality* (G, H) if $A^{\triangleright\triangleleft} = A$, that is, if $A = \bigcap_{\chi \in A^\triangleright} \chi^{-1}(\mathbb{T}_+)$. If G is a topological Abelian group, quasi-convex subsets of G with respect to the duality $(G, G^\wedge)$ will be said simply to be *quasi-convex.*

A topological Abelian group is called *locally quasi-convex* if it has a basis of neighborhoods of zero formed by quasi-convex sets. This notion is originally due to Vilenkin, and later recovered in [4]. Locally quasi-convex groups play a similar role in the category of topological Abelian groups as locally convex spaces in that of topological vector spaces. In fact, it is well known (see [4]) that a topological vector space is locally convex if and only is it is locally quasi-convex as topological group.

Given an Abelian group G and a symmetric subset $U \subset G$ such that $0 \in U$, we consider the following sequence of subsets of G:

$$U_{(n)} = \{x \in G : x \in U,\ 2x \in U,\ \ldots,\ nx \in U\}, \quad n \in \mathbb{N}.$$

It is natural to put $U_{(\infty)} := \{x \in G : nx \in U\ \forall n \in \mathbb{N}\}$. Clearly $0 \in U_{(n)}$ for every $n \in \mathbb{N} \cup \{\infty\}$, and $U_{(\infty)} \subset U_{(n+1)} \subset U_{(n)}$ for every $n \in \mathbb{N}$.

For a nonempty subset $B \subset G$ and for a natural number n, nB will denote the set $\{nx : x \in B\}$. Obviously $nB \subset B + B \overset{n}{+ \ldots +} B$.

We now recall the notion of topology of uniform convergence on a family $\mathfrak{S}$ or $\tau_{\mathfrak{S}}$-topology, as it appears in [8].

A nonempty family $\mathfrak{S}$ of subsets of G is called *well-directed* if the following conditions hold:

a) For $B_1, B_2 \in \mathfrak{S}$, there exists $B_3 \in \mathfrak{S}$ such that $B_1 \cup B_2 \subset B_3$.

b) For $B \in \mathfrak{S}$ and $n \in \mathbb{N}$, there exists $A \in \mathfrak{S}$, such that $nB \subset A$.

If $\mathfrak{S}$ is the family of all nonempty finite subsets, or of all compact subsets of G, $\mathfrak{S}$ is well-directed.

Let (G, H) be a group duality and $\mathfrak{S}$ a well-directed family of nonempty subsets of G. Since $\mathbb{T}$ is a metric space, we can consider in $H \subset \mathbb{T}^G$ the topology $\tau_{\mathfrak{S}}(H, G)$ of uniform convergence on the sets $A \in \mathfrak{S}$. It will be called an $\mathfrak{S}$-*topology*, and it is a group topology. If $\mathfrak{S}$ covers G, then $\tau_{\mathfrak{S}}(H, G)$ is Hausdorff.

In the same fashion, if $\mathfrak{S}'$ is a well-directed family of nonempty subsets of H, and $\alpha : G \to Hom(H, \mathbb{T})$ the natural homomorphism, the preimage topology $\alpha^{-1}(\tau_{\mathfrak{S}'}(\alpha(G), H))$ will be denoted by $\tau_{\mathfrak{S}'}(G, H)$ and called the $\mathfrak{S}'$-*topology* of G. Clearly a $\mathfrak{S}'$-topology in G is a group topology and it is Hausdorff if H separates the points of G and $\mathfrak{S}'$ covers H.

Let G be a group, H a subgroup of $Hom(G, \mathbb{T})$, $\mathfrak{S}$ and $\mathfrak{S}'$ well-directed families of nonempty subsets of G and H respectively. The following facts hold:

a) The collection
$$\mathcal{B} = \{B^{\triangleleft} : B \in \mathfrak{S}'\}$$
is a fundamental system of neighborhoods of the neutral element e_G in the topology $\tau_{\mathfrak{S}'}(G, H)$. In particular, $\tau_{\mathfrak{S}'}(G, H)$ is a locally quasi-convex topology.

b) The collection

$$\mathcal{A} = \{A^{\triangleright} : A \in \mathfrak{S}\}$$

is a fundamental system of neighborhoods of the neutral element e_H in the topology $\tau_{\mathfrak{S}}(H, G)$. In particular, $\tau_{\mathfrak{S}}(H, G)$ is locally quasi-convex.

In order to introduce the β–duality, we first recall the notion of H–bounded subset (adapted from the definition given by Hejcman for uniform spaces).

Let (G, τ) be a topological group. A subset B of G is *H–bounded* if for every zero neighborhood U there exists a finite set $F \subset G$ and some $n \in \mathbb{N}$ such that $B \subset F + U + \overset{n}{\ldots} + U$.

The family of H–bounded sets of G is well-directed and the group $G^{\wedge}$ can be endowed with the topology τ_β of uniform convergence on the H–bounded sets of G. Let us write $G^{\wedge}_{\beta} := (G, \tau_\beta)$. We will say that the group (G, τ) is *β-reflexive* if the canonical evaluation mapping $j_G : G \to (G^{\wedge}_{\beta})^{\wedge}_{\beta}$ defined by $j_G(x)(\chi) = \chi(x)$ is a topological isomorphism.

The notion of H–bounded set is the natural extension to the class of topological groups of the well-known notion of bounded subsets in the context of topological vector spaces as indicated in the next statement, whose proof is easy.

Lemma 2.1. *If E is a locally convex vector space, $A \subset E$ is bounded in E if and only if A is H-bounded in E as a topological Abelian group.*

If E is a real locally convex vector space we denote by E^* the dual vector space of continuous linear forms and by E^*_β the dual vector space E^* endowed with the topology of uniform convergence on the bounded subsets of E.

Observe that the collection

$$\{B^\circ : B \quad \text{bounded set of} \quad E\}$$

is a fundamental system of neighborhoods of zero for the topology τ_β, where $B^\circ = \{f \in E^* : |f(x)| \leq 1 \text{ for all } \quad x \in B\}$.

As usual we say that E is reflexive if $J_E : E \to (E^*_\beta)^*_\beta$ defined by $J_E(x)(f) = f(x)$ is a topological isomorphism.

Let ρ be the universal covering map $\rho : \mathbb{R} \to \mathbb{T}$ given by $\rho(s) = exp(2\pi i s)$.

Proposition 2.2. *Let E be a locally convex vector space, then $\rho_E : E^*_\beta \to E^{\wedge}_{\beta}$ given by $\rho_E(f) = \rho \circ f$ is a topological isomorphism. Consequently, E is reflexive as a vector space iff E is β-reflexive as a topological group,*

Proof. It was proved by Smith in [12] that $\rho_E : E^* \to E^{\wedge}$ is an algebraic isomorphism. The continuity of ρ_E is clear because $\rho_E((4B)^\circ) \subset B^{\triangleright}$ for every subset B of E. In order to see that ρ_E is open, take a neighborhood of zero U of E^*_β. There exists a bounded subset B of E such that $B^\circ \subset 4U$.

The set $H = \{tu \in E : t \in [-1,1], u \in B\}$ is the balanced hull of B and therefore H is bounded (see [11, 5.1]); so $H^{\triangleright}$ is a neighborhood of zero in $E^{\wedge}_{\beta}$. Moreover $\rho_E((4H)^{\circ}) = H^{\triangleright}$ because H is balanced (see [8, 1.11]). Thus $H^{\triangleright} = \rho_E(\frac{1}{4}H^{\circ}) = \rho_E(\frac{1}{4}B^{\circ}) \subset \rho_E(U)$ and $\rho_E(U)$ is a neighborhood of zero in $E^{\wedge}_{\beta}$.

□

Definition 2.3. *[3] Let G be a Hausdorff topological Abelian group. We say that G is a Schwartz group if for every symmetric neighborhood of zero U in G there exists another neighborhood of zero V in G and a sequence (F_n) of finite subsets of G such that*

$$V \subset F_n + U_{(n)} \quad \textit{for every} \quad n \in \mathbb{N}.$$

Remark 2.4. *From the definition it follows directly that locally precompact Abelian groups (in particular, locally compact Abelian groups) are Schwartz groups.*

Observe that precompact subsets of an Abelian topological group are H-bounded. The converse holds for Schwartz groups, as proved in [3, 3.8]. We include below the proof for the reader's convenience.

Proposition 2.5. *Let G be a locally quasi-convex Schwartz group and B an H-bounded subset of G. Then B is precompact.*

Proof. Given a zero neighborhood $U \subset G$ (which can be chosen quasi-convex), we must find a finite $F \subset G$ such that $B \subset F + U$. Since G is a Schwartz group, there exist another neighborhood $V \subset G$ and a sequence (F_n) of finite subsets of G such that $V \subset F_n + U_{(n)}$ for every $n \in \mathbb{N}$. On the other hand, by the H–boundedness of B there exist a finite set $F_0 \subset G$ and an $m \in \mathbb{N}$ such that $B \subset F_0 + V + \overset{m}{\cdots} + V$. Hence

$$\begin{aligned} B &\subset F_0 + V + \overset{m}{\cdots} + V \\ &\subset F_0 + (F_m + U_{(m)}) + \overset{m}{\cdots} + (F_m + U_{(m)}) \\ &\subset F_0 + (F_m + \overset{m}{\cdots} + F_m) + (U_{(m)} + \overset{m}{\cdots} + U_{(m)}) \\ &\subset F_0 + (F_m + \overset{m}{\cdots} + F_m) + U \end{aligned}$$

The fact that $U_{(m)} + \cdots + U_{(m)} \subset U$ derives from the quasi-convexity of U. Taking $F = F_0 + (F_m + \overset{m}{\cdots} + F_m)$, we have $B \subset F + U$. □

Proposition 2.6. *[3, 4.2] Let E be a locally convex space. The following statements are equivalent:*

- *(a) E is a Schwartz space.*
- *(b) The additive topological Abelian group underlying E is a locally quasi-convex Schwartz group.*

Proof. Among the known definitions of Schwartz spaces, we will use the following: a (real) locally convex space E is a Schwartz space if for every zero neighborhood $U \subset E$ there exists another zero neighborhood $V \subset E$ such that for every $\alpha > 0$ there exists a finite subset $F_\alpha \subset E$ with $V \subset F_\alpha + \alpha U$.

(a)⇒(b): This implication is quite trivial: without lost of generality, assume that U is a circled neighborhood of zero, specialize $\alpha = 1/n$ and observe that $U_{(n)} = 1/nU$ holds.

(b)⇒(a): Fix an absolutely convex zero neighborhood $U \subset E$. There exists another neighborhood V and a sequence F_n of finite subsets of E such that $V \subset F_n + U_{(n)}$ for every $n \in \mathbb{N}$. Let $\alpha > 0$ and $n \in \mathbb{N}$ such that $1/n < \alpha$. Then

$$V \subset F_n + U_{(n)} \subset F_n + \alpha U.$$

□

Proposition 2.7. *[3, 5.5] Every Abelian topological group G which is a hemicompact k–space (in particular, every character group of a metrizable group), endowed with the topology of uniform convergence on compact sets is a Schwartz group.*

Theorem 2.8. *Every locally compact Abelian group G is β - reflexive.*

Proof. Every locally compact Abelian group G is locally quasi-convex and Schwartz, so by Proposition 2.5 the H-bounded sets of G are precompact, and by the completeness of G they are relatively compact. Therefore in the group of characters $G^\wedge$, the compact-open topology coincides with the topology of uniform convergence on H–bounded sets. Thus, $G^\wedge_\beta$ is again locally compact and consequently, the compact-open topology in $(G^\wedge_\beta)^\wedge$ coincides again with the topology of uniform convergence on H-bounded sets. In other words, $(G^\wedge_\beta)^\wedge_\beta$ is the Pontryagin bidual of G. As G is Pontryagin reflexive, it is also β-reflexive. □

Proposition 2.9. *If G is a topological Abelian group, then $j_\beta : G \longrightarrow (G^\wedge_\beta)^\wedge_\beta$ is continuous iff every H–bounded set of $G^\wedge_\beta$ is equicontinuous.*

Proof. Suppose j_β is continuous and let B be an H–bounded subset of $G^\wedge_\beta$. Then, $B^\triangleright$ is a zero neighborhood in $(G^\wedge_\beta)^\wedge_\beta$. Since j_β is continuous, $j_\beta^{-1}(B^\triangleright) = B^\triangleleft$ is a zero neighborhood in G and therefore B is equicontinuous. Conversely, take a zero neighborhood U in $(G^\wedge_\beta)^\wedge_\beta$. There exists an H-bounded subset in $G^\wedge_\beta$ such that $B^\triangleright \subset U$. From the equality $j_\beta^{-1}(B^\triangleright) = B^\triangleleft$, we have that $j_\beta(B^\triangleleft) \subset U$. Thus, j_β is continuous.

□

Proposition 2.10. *Let G be a locally quasi-convex topological Abelian group such that every equicontinuous subset of $G^\wedge$ is H-bounded in $G^\wedge_\beta$. Then, $j_G : G \longrightarrow (G^\wedge_\beta)^\wedge_\beta$ is open onto its image.*

Proof. Let U be a quasi-convex zero neighborhood in G. Clearly $U^{\triangleright}$ is equicontinuous, and H-bounded by the assumption. Thus, $(U^{\triangleright})^{\triangleright}$ is a zero neighborhood in $(G^{\wedge}_{\beta})^{\wedge}_{\beta}$. Since $j_G(U) = (U^{\triangleright})^{\triangleright} \cap j_G(G)$, j_β is relatively open. □

Remark 2.11. The equicontinuous subsets of the dual of a topological group may not be τ_β-bounded, as we explain right below. However, they are relatively compact with respect to the compact- open topology and therefore τ_{co}- bounded.

In [9, 3.8] it is provided an example of a metrizable locally bounded group G, whose dual is not locally bounded with respect to the bounded convergence topology τ_β. A *locally bounded group* is a topological group which has a bounded neighborhood of zero. Denote by V a bounded neighborhood of zero in the above mentioned example G. Clearly $V^{\triangleright}$ is an equicontinuous subset of $G^{\wedge}$ which is not bounded, for otherwise $G^{\wedge}$ should be locally bounded.

Theorem 2.12. *Every metrizable, β-reflexive topological Abelian group is Pontryagin reflexive.*

Proof. Let G be a β-reflexive group, that is, $j_G : G \longrightarrow (G^{\wedge}_{\beta})^{\wedge}_{\beta}$ is a topological isomorphism. We must see that $\alpha_G : G \longrightarrow (G^{\wedge}_{co})^{\wedge}_{co}$ is also a topological isomorphism. Observe that $(G^{\wedge}_{\beta})^{\wedge}_{\beta}$ is locally quasi-convex, so G is locally quasi-convex which implies that $\alpha_G : G \longrightarrow (G^{\wedge}_{co})^{\wedge}_{co}$ is relatively open and one to one. The continuity of α_G follows from the assumption that G is metrizable and [1, 5.11] applies.

The surjectivity of α_G derives from the fact that $\tau_{co} \leq \tau_\beta$, and therefore $(G^{\wedge}_{co})^{\wedge} \subset (G^{\wedge}_{\beta})^{\wedge}$, which by the hypothesis can be identified with G. □

Theorem 2.13. *If G is a metrizable topological Abelian group, then $j_\beta : G \longrightarrow (G^{\wedge}_{\beta})^{\wedge}_{\beta}$ is continuous and $(G^{\wedge}_{\beta})^{\wedge}_{\beta}$ is also metrizable.*

Proof. Let $B \subset G^{\wedge}$ be an H-bounded set in $G^{\wedge}_{\beta}$. In particular, it is H-bounded in $G^{\wedge}_{co}$ and by Propositions 2.5 and 2.7 it is precompact in $G^{\wedge}_{co}$. By the completeness of the latter, it is relatively compact and therefore equicontinuous. This proves that j_G is continuous.
Let $\{V_n \, , \, n \in \mathbb{N}\}$ be a decreasing basis of neighborhoods of zero in G. In order to see that $(G^{\wedge}_{\beta})^{\wedge}_{\beta}$ is metrizable, we only need to prove that $\{V_n^{\triangleright\triangleright} \, , \, n \in \mathbb{N}\}$ is a basis of neighborhoods of zero in $(G^{\wedge}_{\beta})^{\wedge}_{\beta}$. So let W be a zero neighborhood in $(G^{\wedge}_{\beta})^{\wedge}_{\beta}$. There exists a bounded set B of $G^{\wedge}_{\beta}$ such that $B^{\triangleright} \subset W$. Since B is equicontinuous $B^{\triangleleft}$ is a zero neighborhood in G, and consequently $V_m \subset B^{\triangleleft}$, for some some $m \in \mathbb{N}$. Thus, $V_m^{\triangleright\triangleright} \subset (B^{\triangleleft})^{\triangleright\triangleright} = B^{\triangleright} \subset W$. □

The previous theorem uses a strong tool, namely the fact that the dual of a metrizable group is a Schwartz group. For topological vector spaces, a direct route to obtain a similar result can be described as follows.

Theorem 2.14. *If X is a metrizable topological vector space, then $(X^*_\beta)^*_\beta$ is also metrizable. If $J : X \longrightarrow (X^*_\beta)^*_\beta$ is onto, and X^* separates points of X, $(X^*_\beta)^*_\beta$ can be identified with the locally convex modification of X (via the algebraic isomorphism J).*

First we prove some auxiliary results.

Proposition 2.15. *If X is a metrizable topological vector space and X^*_{co} its dual endowed with the compact-open topology, then every bounded subset $B \subset X^*_{co}$ is equicontinuous.*

Proof. Let $\{V_n \ , \ n \in \mathbb{N}\}$ be a decreasing basis of neighborhood of zero in X. Clearly $\{\frac{1}{n}V_n \ , \ n \in \mathbb{N}\}$ is also a decreasing basis of neighborhood of zero in X.

Assume by contradiction that B is X^*_c-bounded but not equicontinuous. Then for all $n \in \mathbb{N}$ we can find $f_n \in B$ such that $f_n(\frac{1}{n}V_n) \not\subset [-1, 1]$. Pick $x_n \in V_n$ such that $f_n(\frac{1}{n}x_n) \notin [-1, 1]$. Clearly $x_n \longrightarrow 0$ by the election of $\{V_n \ , \ n \in \mathbb{N}\}$. Thus $S := \{x_n \ , \ n \in \mathbb{N}\} \cup \{0\}$ is a compact subset of X and S° a neighborhood of zero in τ_{co}. On one hand we have $\frac{1}{n}f_n \notin S^\circ$ for all $n \in \mathbb{N}$ and on the other hand B is bounded in X^*_c, which implies the existence of λ_0 such that $B \subset \lambda S^\circ$ for all $\lambda \geq \lambda_0$. If $n \geq \lambda_0$, $B \subset nS^\circ$ and thus $\dfrac{f_n}{n} \in S^\circ$, a contradiction.

□

Corollary 2.16. *Let X be a metrizable topological vector space and $B \subset X^*$ bounded in τ_β. Then B is equicontinuous.*

Proof. This derives from the fact that $\tau_{co} \leq \tau_\beta$ and the previous proposition.

□

Lemma 2.17. *Let X be a metrizable topological vector space. The canonical mapping $J : X \longrightarrow (X^*_\beta)^*_\beta$ is continuous.*

Proof. Let L be a neighborhood of zero in $(X^*_\beta)^*_\beta$. We may assume that $L = B^\circ$ where B is a bounded subset of X^*_β. Clearly $J^{-1}(B^\circ) =^\circ B$, being ${}^\circ B$ the inverse polar of B.

From Corollary 2.16, B is equicontinuous, and therefore $J^{-1}(B^\circ) =^\circ B$ is a neighborhood of zero in X.

□

Proposition 2.18. *Let X be a topological vector space and $V \subset X$ a neighborhood of zero. Then V° is bounded in X^*_β.*

Proof. We need to prove that for every bounded set $B \subset X$ there is a real number $r > 0$ such that $V^\circ \subset rB^\circ$. Fix such a B. Boundedness of B implies that there exist $s > 0$ such that $B \subset sV$. Taking polars we get $(sV)^\circ \subset B^\circ$, thus $V^\circ \subset sB^\circ$. ☐

Proposition 2.19. *Let X be a metrizable topological vector space. Then X^*_β has a fundamental sequence of bounded sets.*

Proof. Let $\{V_n \, : \, n \in \mathbb{N}\}$ be a decreasing basis of zero neighborhood in X. Fix a bounded set B in X^*_β. By Corollary 2.16, it is equicontinuous. Thus, there exists $n \in \mathbb{N}$ such that $B \subset V_n^\circ$.

On the other hand, by Proposition 2.18, V_n° is bounded in X^*_β. Thus

$$V_1^\circ \subset V_2^\circ \subset \cdots \subset V_n^\circ \subset \cdots$$

form a countable family of bounded subsets of X^*_β which is fundamental. The term "fundamental" is referred in the Literature also by "swallows bounded sets". ☐

Proof of Theorem 2.14 . Since $(X^*_\beta)^*_\beta$ has a countable basis of neighborhoods, namely $\{(V_n^\circ)^\circ \; , \; n \in \mathbb{N}\}$, it is metrizable. If J is onto, we can conclude that $(X^*_\beta)^*_\beta$ is the locally convex modification of (X, τ).

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