A THEORY OF HARDY SPACES ASSOCIATED TO THE HERZ SPACES

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0. *Introduction*

In [5] Chen and Lau introduced Hardy spaces associated to the Beurling algebras A^p. Their theory was further developed by J. García-Cuerva [9]. The algebras A^p were introduced by A. Buerling [2] in connection with spectral synthesis. They are a nested system of convolution subalgebras of L^1 whose union is L^1. Feichtinger [6] provided the equivalent norm for the A^p which made the extension [9] possible. The associated Hardy spaces HA^p are a nested system of spaces whose union is the ordinary Hardy space H^1. The atomic decomposition for HA^p differs from that of H^1 in that atoms have to be centred at 0 and the size of each atom is given by an L^p-norm controlled as usual by the reciprocal of the measure of its support. This view of HA^p as a kind of H^1 at a point is particularly appealing and casts some light on the general nature of Hardy space theory.

Given this background, it seems to be natural to try to extend the theory to H^q for $q<1$. This extension is the subject of the present paper, which is part of the Ph.D. thesis of the second author. It turns out that the spaces $A_{p,q}$ or $\dot{A}_{p,q}$, which now play the role of the Beurling algebras A^p, had previously been introduced by C. Herz [11] with different notation. Our notation is adapted to the Feichtinger norms which are the most appropriate for our aims. In § 1 we introduce the spaces $A_{p,q}$ and $\dot{A}_{p,q}$ and their duals $B_{p',q}$ and $\dot{B}_{p',q}$ and briefly study those properties which will be important for the development of Hardy space theory. In § 2 we define the Hardy spaces $HA_{p,q}$ and $H\dot{A}_{p,q}$ as spaces of tempered distributions whose non-tangential Poisson maximal function belongs to $A_{p,q}$ or $\dot{A}_{p,q}$ respectively. We obtain several equivalent characterizations including an atomic decomposition, which are collected in Theorem 2.14. This section includes a description of the complex interpolation for these spaces. This problem is more difficult than that for Banach spaces and we solve it along the lines of the method introduced by Calderón and Torchinsky [4]. Finally § 3 is devoted to the Littlewood–Paley characterization of $HA_{p,q}$ for $1<p\leqslant 2$ and $0<q\leqslant 1$. In this way we are able to complete previous results obtained by Lu and Yang [13].

1. *The spaces $\dot{A}_{p,q}$, $A_{p,q}$, $\dot{B}_{p,q}$, $B_{p,q}$: definitions and basic properties*

For $k\in\mathbb{Z}$ define $C_k=\{x\in\mathbb{R}^n\colon 2^{k-1}<|x|\leqslant 2^k\}$ and denote by χ_k the characteristic function of the set C_k.

DEFINITION 1.1. Let $0<q\leqslant 1$ and $q\leqslant p<\infty$.

(a) We shall call $\dot{A}_{p,q}$ the space consisting of those functions $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n-\{0\})$ for which

$$\|f\|_{\dot{A}_{p,q}}=\left\{\sum_{k=-\infty}^{\infty}(2^{kn(1/q-1/p)}\,\|f\chi_k\|_p)^q\right\}^{1/q}<\infty.$$

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(b) We shall call $A_{p,q}$ the space consisting of those functions f on $\mathbb{R}^n$ for which

$$\|f\|_{A_{p,q}}=\left\{\sum_{k=0}^{\infty}(2^{kn(1/q-1/p)}\|f\tilde{\chi}_k\|_p)^q\right\}^{1/q}<\infty,$$

where $\tilde{\chi}_0$ is the characteristic function of $\overline{B(0,1)}$, $B(0,1)$ being the open ball of radius 1 centred at 0, and $\tilde{\chi}_k(x)=\chi_k(x)$ if $k\geqslant 1$.

DEFINITION 1.2. Let $0<q\leqslant 1$ and $1<p<\infty$, and denote by p' the exponent conjugate to p, that is, $1/p+1/p'=1$.

(a) We shall call $\dot{B}_{p,q}$ the space of those functions f on $\mathbb{R}^n$ for which

$$\|f\|_{\dot{B}_{p,q}}=\sup_{k\in\mathbb{Z}}\{2^{-kn(1/q-1/p')}\|f\chi_k\|_p\}<\infty.$$

(b) We shall call $B_{p,q}$ the space of those functions f on $\mathbb{R}^n$ for which

$$\|f\|_{B_{p,q}}=\sup_{k\geqslant 0}\{2^{-kn(1/q-1/p')}\|f\tilde{\chi}_k\|_p\}<\infty.$$

Note that if we define, for $k\in\mathbb{Z}$, $f_k(x)=f(2^kx).\chi_0(x)$ then the spaces we have defined may be viewed as L^p-valued sequence spaces (see [**1**, p. 121] and [**17**] for definitions and basic results). Then

$$\|f\|_{\dot{A}_{p,q}}=\left\{\sum_{k=-\infty}^{\infty}(2^{kn/q}\|f_k\|_{L^p(C_0)})^q\right\}^{1/q}=\|\{f_k\}\|_{l_q^{n/q}(L^p(C_0))}$$

and

$$\|f\|_{\dot{B}_{p,q}}=\sup_{k\in\mathbb{Z}}\{2^{-kn(1/q-1)}\|f_k\|_{L^p(C_0)}\}=\|\{f_k\}\|_{l_\infty^{-n(1/q-1)}(L^p(C_0))}.$$

If we write $\tilde{f}_0(x)=f(x).\tilde{\chi}_0(x)$ and, for $k\in\mathbb{N}$,

$$\tilde{f}_k(x)=\begin{cases}f(2^kx) & \text{if } \frac{1}{2}<|x|\leqslant 1,\\ 0 & \text{if } |x|\leqslant\frac{1}{2},\end{cases}$$

then

$$\|f\|_{A_{p,q}}=\left\{\sum_{k=0}^{\infty}(2^{kn/q}\|\tilde{f}_k\|_{L^p(\overline{B(0,1)})})^q\right\}^{1/q}=\|\{\tilde{f}_k\}\|_{l_q^{n/q}(L^p(\overline{B(0,1)}))},$$

$$\|f\|_{B_{p,q}}=\sup\{2^{-kn(1/q-1)}\|\tilde{f}_k\|_{L^p(\overline{B(0,1)})}\}=\|\{\tilde{f}_k\}\|_{l_\infty^{-n(1/q-1)}(L^p(\overline{B(0,1)}))}.$$

Thus the spaces $\dot{A}_{p,q}$, $A_{p,q}$, $\dot{B}_{p,q}$ and $B_{p,q}$ are complete; moreover $\dot{B}_{p,q}$ and $B_{p,q}$ are Banach.

These spaces are also a particular case of Herz's spaces $\dot{K}_a^{\alpha,\beta}$ and $K_a^{\alpha,\beta}$ (see [**11**]).

The special case where $q=1$, gives $A_{p,1}$, $B_{p',1}$ which are the Beurling Algebras A^p and their duals $B^{p'}$, and the norms used in the definition were introduced by H. Feichtinger [**6**]. Moreover, the Beurling Algebras have been studied by J. García-Cuerva [**9**] and the results obtained there may easily be extended to the homogeneous spaces $\dot{A}^p\equiv\dot{A}_{p,1}$ and $\dot{B}^p\equiv\dot{B}_{p,1}$.

Next we list basic results concerning the spaces $\dot{A}_{p,q}$, $A_{p,q}$, $\dot{B}_{p,q}$, $B_{p,q}$. These are very easy to prove (in brackets we indicate the ideas required to do so).

PROPOSITION 1.3. (a) *Let* $0<q\leqslant 1$ *and* $q<p<\infty$; *then* $A_{p,q}=L^p\cap \dot{A}_{p,q}$.

(b) *If* $p=q$ *then* $\dot{A}_{p,q}=A_{p,q}=L^q$ (definition).

(c) *If* $0<q<p_1\leqslant p_2<\infty$ *then* $\dot{A}_{p_2,q}\subset \dot{A}_{p_1,q}\subset L^q$ *and* $A_{p_2,q}\subset A_{p_1,q}\subset L^q$ (Hölder).

(d) *If* $0<q_1\leqslant q_2\leqslant 1$ *and* $q_2\leqslant p<\infty$ *then* $A_{p,q_1}\subset A_{p,q_2}$ (l^q-properties).

(e) *For* $0<q\leqslant 1<p<\infty$, $\dot{B}_{p,q}\subset B_{p,q}$ (definition).

(f) *If* $1<p_1\leqslant p_2<\infty$ *then* $\dot{B}_{p_2,q}\subset \dot{B}_{p_1,q}$ *and* $B_{p_2,q}\subset B_{p_1,q}$ (Hölder).

(g) *If* $0<q_1\leqslant q_2\leqslant 1$ *then* $B^p\subset B_{p,q_2}\subset B_{p,q_1}$ (definition).

(h) *If* $0<q\leqslant 1<p<\infty$ *then* $L^\infty\subset B_{p,q}$ *and* $L^p\subset B_{p,q}$ (definition).

Note that the results (d) and (g) are false for the corresponding homogeneous spaces.

PROPOSITION 1.4. *If* $0<q\leqslant 1$ *and* $1<p<\infty$ *then* $A_{p,q}\subset A^p\subset L^p\subset B^p\subset B_{p,q}$.

PROPOSITION 1.5. *Let* $0<q\leqslant 1$, $q\leqslant p<\infty$, *and* $\varepsilon>0$. *Then*

(a) $f\in \dot{A}_{p,q}$ *if and only if* $|f|^\varepsilon\in \dot{A}_{p/\varepsilon,q/\varepsilon}$,

(b) $f\in A_{p,q}$ *if and only if* $|f|^\varepsilon\in A_{p/\varepsilon,q/\varepsilon}$.

Note that, since $\sup_{x\in C_k}|f(x)|\leqslant \inf_{x\in C_{k-1}}|f(x)|$, every decreasing function in L^q belongs to $\dot{A}_{p,q}$ provided that $q\geqslant p$.

There is an alternative description of $\dot{B}_{p,q}$ and $B_{p,q}$.

PROPOSITION 1.6. (a) *A function* $f\in \dot{B}_{p,q}$ *if and only if the following quantity is finite:*

$$\sup_{R>0}\left\{|B(0,R)|^{1-1/q}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|f(y)|^p\,dy\right)^{1/p}\right\}.$$

(b) *A function* $f\in B_{p,q}$ *if and only if the following quantity is finite:*

$$\sup_{R\geqslant 1}\left\{|B(0,R)|^{1-1/q}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|f(y)|^p\,dy\right)^{1/p}\right\}.$$

Such quantities are equivalent to the corresponding $\dot{B}_{p,q}$ and $B_{p,q}$ norms.

Proof. We prove part (a); the result for $B_{p,q}$ follows from the same arguments. Let $f\in \dot{B}_{p,q}$. Given $R>0$, choose $k\in\mathbb{Z}$ such that $2^{k-1}<R\leqslant 2^k$. Then

$$\int_{B(0,R)}|f(x)|^p\,dx\leqslant \sum_{j=-\infty}^{k}\int_{C_j}|f(x)|^p\,dx\leqslant \|f\|^p_{\dot{B}_{p,q}}\sum_{j=-\infty}^{k}2^{jnp(1/q-1/p')}$$
$$\leqslant C\,|B(0,R)|^{p(1/q-1/p')}\,\|f\|^p_{\dot{B}_{p,q}}.$$

That is,

$$|B(0,R)|^{1-1/q}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|f(x)|^p\,dx\right)^{1/p}\leqslant C\,\|f\|_{\dot{B}_{p,q}}.$$

Conversely, if we call S the supremum in the proposition then

$$\|f\chi_k\|_p^p \leqslant \int_{B(0,2^k)} |f(x)|^p\,dx \leqslant CS^p 2^{knp(1/q-1/p')}$$

and $\|f\|_{\dot{B}_{p,q}} \leqslant CS$.

Next we shall state a basic duality result. If we denote by X^* the dual space of the space X, we can write:

(1) $(\dot{A}_{p,q})^* = \dot{B}_{p',q}$ and $(A_{p,q})^* = B_{p',q}$ for $0<q\leqslant 1<p<\infty$,

(2) $(\dot{A}_{p,q})^* = (A_{p,q})^* = \{0\}$ for $0<q\leqslant p<1$.

More precisely, we have the following:

THEOREM 1.7. *Let* $0<q\leqslant 1\leqslant p<\infty$.

(a) *For every* $g\in B_{p',q}$ *the functional* Λ_g, *defined by*

$$\Lambda_g(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx,$$

is continuous on $A_{p,q}$ *and its norm in* $(A_{p,q})^*$ *satisfies* $\|\Lambda_g\| \leqslant \|g\|_{B_{p',q}}$. *Conversely, given* $\Lambda\in(A_{p,q})^*$, *there is a unique* $g\in B_{p',q}$ *such that* $\Lambda=\Lambda_g$. *Furthermore,* $\|g\|_{B_{p',q}} \leqslant \|\Lambda\|$. *(Similar results hold for homogeneous spaces.)*

(b) *Let* $0<q\leqslant p<1$. *Then the unique continuous linear functional on* $\dot{A}_{p,q}$ *or* $A_{p,q}$ *is given by* $\Lambda_0(f)\equiv 0$.

Proof. From the descriptions of these spaces as sequence spaces, we get the following statement. Let $0<q\leqslant 1\leqslant p<\infty$, and p' be the exponent conjugate to p; then

$$(\dot{A}_{p,q})^* = (\dot{l}_q^{n/q}(L^p(C_0)))^* = \dot{l}_\infty^{-n(1/q-1)}(L^{p'}(C_0)) = B_{p',q},$$

$$(A_{p,q})^* = (l_q^{n/q}(L^p(\overline{B(0,1)})))^* = l_\infty^{-n(1/q-1)}(L^{p'}(\overline{B(0,1)})) = B_{p',q}.$$

If $0<q\leqslant p<1$, by Proposition 1.3 we have $L^p(B(0,R))\subset A_{p,q}\subset\dot{A}_{p,q}$ and $(L^p(B(0,R)))^*=\{0\}$ whenever $0<p<1$, so that

$$(\dot{A}_{p,q})^* = (A_{p,q})^* = \{0\}.$$

2. *Hardy spaces*

First, we will define all maximal functions which will appear in this section.

Given a fixed ϕ in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$, we can associate with each $f\in\mathscr{S}'(\mathbb{R}^n)$, a function defined on $\mathbb{R}_+^{n+1}$ by $f(x,t)=(f*\phi_t)(x)$ and derive from it the following maximal functions:

(i) the vertical maximal function,

$$\phi^*(f)(x) = \sup_{t>0} |f(x,t)|;$$

(ii) the non-tangential maximal function,

$$\phi_\nabla^*(f)(x) = \sup_{|y-x|<t} |f(y,t)|;$$

(iii) for $N \geq 1$, the non-tangential maximal function of amplitude N,

$$\phi_{\nabla,N}^*(f)(x) = \sup_{|y-x|<Nt} |f(y,t)|;$$

(iv) for $\lambda \geq 1$, the tangential maximal function with exponent λ,

$$\phi_\lambda^{**}f(x) = \sup_{(y,t)\in\mathbb{R}^{n+1}} |f(y,t)| \left(\frac{t}{|x-y|+t}\right)^\lambda.$$

If $f \in L^1(\mathbb{R}^n)$ and $\phi(x) = P(x)$, the Poisson kernel, the above definitions make sense. For this particular case we write

$$P^*(f)(x) = f^+(x), \quad P_\nabla^* f(x) = f^*(x), \quad P_{\nabla,N}^*(f)(x) = f_N^*(x), \quad P_\lambda^{**}(f)(x) = f_\lambda^{**}(x).$$

For $u(x,t)$ a harmonic function on $\mathbb{R}_+^{n+1}$, we shall use the following maximal functions:

(1) the vertical maximal function,

$$m_u^+(x) = \sup_{t>0} |u(x,t)|;$$

(2) the non-tangential maximal function,

$$m_u(x) = \sup_{|y-x|<t} |u(y,t)|;$$

(3) for $N \geq 1$, the non-tangential maximal function of amplitude N,

$$m_u^N(x) = \sup_{|y-x|<Nt} |u(y,t)|;$$

(4) for $\lambda \geq 1$, the tangential maximal function of exponent λ,

$$u_\lambda^{**}(x) = \sup_{(y,t)\in\mathbb{R}^{n+1}} |u(y,t)| \left(\frac{t}{|x-y|+t}\right)^\lambda.$$

Finally, we shall consider the grand maximal function, obtained by taking a large class of functions ϕ. For N, a positive integer, and α, β, multi-indices, let

$$\mathscr{A}_N = \left\{\phi \in \mathscr{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta|\leq N} |x^\alpha D^\beta \phi(x)| \leq 1\right\}.$$

For $f \in \mathscr{S}'(\mathbb{R}^n)$ its grand maximal function will be

$$G_N(f)(x) = \sup_{\phi\in\mathscr{A}_N} |\phi_\nabla^*(f)(x)|.$$

We shall take N sufficiently large and then we shall keep it fixed. There will be no need to retain the subscript N and we shall write simply $G(f)$ and $\mathscr{A}$.

Now we define the Hardy spaces associated to $\dot{A}_{p,q}$ and $A_{p,q}$ starting from harmonic functions. First we state the following result, which is well known for the classical Hardy spaces $H^q(\mathbb{R}^n)$ with $0<q\leq 1$. (See [**8**, pp. 161–172]).

THEOREM 2.1. *Let $u(x,t)$ be a harmonic function in $\mathbb{R}_+^{n+1}$, and $0<q\leq 1$ and*

$q \leqslant p < \infty$. If $m_u(x) \in A_{p,q}$ then $f(x) = \lim_{t\to 0} u(x,t)$ exists in the sense of tempered distributions and determines in a unique way the harmonic function $u(x,t)$. Moreover, $|u(x,t)| \leqslant Ct^{-n/q} \|m_u\|_{A_{p,q}}$. (A similar result holds for $\dot{A}_{p,q}$.)

DEFINITION 2.2. Let $0 < q \leqslant 1$, $q \leqslant p < \infty$ and $f \in \mathscr{S}'(\mathbb{R}^n)$.

(a) We say that f belongs to $H\dot{A}_{p,q}$ if and only if it is the boundary distribution of a harmonic function, $u(x,t)$ in $\mathbb{R}^{n+1}_+$ which satisfies $m_u(x) \in \dot{A}_{p,q}$ and in that case we define $\|f\|_{H\dot{A}_{p,q}} = \|m_u\|_{\dot{A}_{p,q}}$.

(b) We say that f belongs to $HA_{p,q}$ if and only if it is the boundary distribution of a harmonic function, $u(x,t)$ in $\mathbb{R}^{n+1}_+$ which satisfies $m_u(x) \in A_{p,q}$ and in that case we define $\|f\|_{HA_{p,q}} = \|m_u\|_{A_{p,q}}$.

Note that $HA_{p,q} \subset H\dot{A}_{p,q} \subset H^q$, for $0 < q \leqslant 1$, $q \leqslant p < \infty$.

THEOREM 2.3. *Let $0 < q \leqslant 1$ and $q \leqslant p < \infty$. Then $H\dot{A}_{p,q}$ and $HA_{p,q}$ are complete spaces for the quasi-norms given in Definition* 2.2.

We shall give different equivalent characterizations of $H\dot{A}_{p,q}$ and $HA_{p,q}$. First, let us describe some classes of functions which are dense in those spaces.

THEOREM 2.4. *For any $0 < q \leqslant 1$ and $q \leqslant p < \infty$, $L^2 \cap HA_{p,q}$ is dense in $HA_{p,q}$ and $L^2 \cap H\dot{A}_{p,q}$ is dense in $H\dot{A}_{p,q}$.*

Proof. We consider $f \in HA_{p,q}$ and let $u(x,t)$ be the unique harmonic function in $\mathbb{R}^{n+1}_+$ such that $f(x) = \lim_{t\to 0} u(x,t)$ in the sense of distributions.

For $s > 0$ fixed, we set $u_s(x) = u(x,s)$. As $u_s(x) \leqslant m_u(x)$ and since by Theorem 2.1, $|u(x,t)| \leqslant Ct^{-n/q} \|m_u\|_{HA_{p,q}}$, it follows that $u_s(x) \in L^q \cap L^\infty \subset L^2$ and also that $u_s(x) = \lim_{t\to 0} u(x, t+s)$. Since $\|u_s\|_{HA_{p,q}} = \|m_{u_s}\|_{A_{p,q}} \leqslant \|m_u\|_{A_{p,q}}$, we obtain $u_s(x) \in HA_{p,q} \cap L^2$ with $\|u_s\|_{HA_{p,q}} \leqslant \|f\|_{HA_{p,q}}$.

On the other hand, $m_u \in A_{p,q}$ and

$$\sup_{|x-y|<t} |u(y,t) - u(y,t+s)| \leqslant \sup_{|x-y|<t} |u(y,t)| + \sup_{|x-y|<t} |u(y,t+s)| \leqslant Cm_u(x).$$

By the Lebesgue dominated convergence theorem, in order to prove that $\|f - u_s\|_{HA_{p,q}} \to 0$ as $s \to 0$, it suffices to show that

$$m_{u-u_s}(x) = \sup_{|x-y|<t} |u(y,t) - u(y,t+s)| \to 0 \quad \text{a.e. as } s \to 0.$$

But

$$m_{u-u_s}(x) \leqslant \sup_{\substack{|x-y|<t \\ 0<t<\delta}} |u(y,t) - u(y,t+s)| + \sup_{\substack{|x-y|<t \\ \delta<t<T}} |u(y,t) - u(y,t+s)|$$
$$+ \sup_{\substack{|x-y|<t \\ T<t<\infty}} |u(y,t)| + \sup_{\substack{|x-y|<t \\ T<t<\infty}} |u(y,t+s)|.$$

The first term goes to 0 as $\delta, s \to 0$, with $s > \delta$, since we have $|u(y,t)| \leqslant m_u(y)$. As $u(y,t) = \lim_{s\to 0} u(x, t+s)$, it follows that then $|u(y,t) - u(y,t+s)| \to 0$ as $s \to 0$ uniformly in $|x-y| < t$, with $\delta < t < T$. Finally, the third and fourth terms go to zero as T goes to ∞, provided that $s < T < t$, from the estimate $|u(x,t)| \leqslant Ct^{-n/q} \|m_u\|_{HA_{p,q}}$.

THEOREM 2.5. *For $0<q\leq 1$, $q<p<\infty$ and $u(x,t)$ a harmonic function in $\mathbb{R}^{n+1}_+$, the following properties are equivalent:*

(a) $m_u^+\in A_{p,q}$;

(b) $m_u\in A_{p,q}$;

(c) *for any* $N\geq 1$, $m_u^N\in A_{p,q}$.

A similar result holds with $A_{p,q}$ replaced by $\dot{A}_{p,q}$.

Proof. The obvious inequalities $m_u^+(x)\leq m_u(x)\leq m_u^N(x)$ give (c)$\Rightarrow$(b)$\Rightarrow$(a). Let us see that (a)$\Rightarrow$(c).

By a lemma of Hardy and Littlewood, rediscovered by C. Fefferman and E. M. Stein [8] (see alternatively [10, p. 172]), since $u(x,t)$ is harmonic in $\mathbb{R}^{n+1}_+$, if $0<\varepsilon<1$, we have

$$|u(y,t)|^\varepsilon\leq\frac{C_\varepsilon}{|B((y,t),\tfrac12 t)|}\int_{B((y,t),t/2)}|u(z,s)|^\varepsilon\,dz\,ds\leq\frac{C_\varepsilon}{t^n}\int_{|z-y|<t/2}(m_u^+(z))^\varepsilon\,dz.$$

Then

$$(m_u^N(x))^\varepsilon\leq C_\varepsilon\sup_{t>0}\left(\frac{1}{t^n}\int_{|z-x|<(N+1/2)t}(m_u^+(z))^\varepsilon\,dz\right)\leq C_\varepsilon N^n M((m_u^+)^\varepsilon)(x),$$

where M represents the Hardy–Littlewood maximal function; that is,

$$m_u^N(x)\leq C_\varepsilon N^{n/\varepsilon}(M((m_u^+)^\varepsilon)(x))^{1/\varepsilon}.$$

On the other hand, by Proposition 1.5 we have $\|m_u^N\|^q_{A_{p,q}}=\|(m_u^N)^q\|_{A^{p/q}}$. The results stated in [9] for Beurling Algebras give us

$$\|(m_u^N)^q\|_{A^{p/q}}=\sup\left\{\left|\int_{\mathbb{R}^n}(m_u^N)^q(x)g(x)\,dx\right|:\ g\in B^{(p/q)'},\ \|g\|_{B^{(p/q)'}}=1\right\},$$

and taking $\varepsilon<q$, from the theorem of Fefferman and Stein [7], we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(m_u^N)^q(x)g(x)\,dx\right|&\leq C_\varepsilon N^{nq/\varepsilon}\int_{\mathbb{R}^n}(M((m_u^+)^\varepsilon))^{q/\varepsilon}(x)g(x)\,dx\\&\leq C_\varepsilon N^{nq/\varepsilon}\int_{\mathbb{R}^n}(m_u^+)^q(x)Mg(x)\,dx\\&\leq CN^{nq/\varepsilon}\,\|(m_u^+)^q\|_{A^{p/q}}\,\|g\|_{B^{(p/q)'}}.\end{aligned}$$

Therefore we get $\|m_u^N\|_{A_{p,q}}\leq CN^{n+\varepsilon}\,\|m_u^+\|_{A_{p,q}}$ for some $\varepsilon>0$.

THEOREM 2.6. *Let $0<q\leq 1$, $q\leq p<\infty$, and $u(x,t)$ be a harmonic function in $\mathbb{R}^{n+1}_+$. If $m_u(x)\in A_{p,q}$ then there is $M\geq 1$, large enough such that $u_M^{**}(x)\in A_{p,q}$. (A similar result holds for $\dot{A}_{p,q}$.)*

Proof. By the proof of Theorem 2.5, for some $\varepsilon>0$,

$$\|m_u^N\|_{A_{p,q}}\leq CN^{n+\varepsilon}\,\|m_u^+\|_{A_{p,q}}\leq CN^{n+\varepsilon}\,\|m_u\|_{A_{p,q}},$$

where C does not depend on either N or $u(x,t)$. Since

$$u_M^{**}(x)=\sup_{(y,t)\in\mathbb{R}^{n+1}_+}|f(y,t)|\left(\frac{t}{|x-y|+t}\right)^M\leq Cm_u(x)+\sum_{k=0}^\infty 2^{-kM}m_u^{2^{k+1}}(x),$$

we obtain

$$\|u_M^{**}\|_{A_{p,q}} \leq C\, \|m_u\|_{A_{p,q}} \left(1 + C \sum_{k=0}^{\infty} 2^{-kM} 2^{k(n+\varepsilon)}\right).$$

Taking $M > n + \varepsilon$, we see that the sum converges and

$$\|u_M^{**}\|_{A_{p,q}} \leq C\, \|m_u\|_{A_{p,q}}.$$

The same arguments give the proof of the homogeneous result.

LEMMA 2.7. *Let $f \in L^2$. Then $G(f)(x) \leq Cf_M^{**}(x)$ for every $x \in \mathbb{R}^n$.*

Proof. See, for example, [**9**].

THEOREM 2.8. *Let $0 \leq q \leq 1$ and $q \leq p < \infty$. If $f \in HA_{p,q}$ then $G(f) \in A_{p,q}$ and if $f \in H\dot{A}_{p,q}$ then $G(f) \in \dot{A}_{p,q}$.*

Proof. By Lemma 2.7 and Theorem 2.6 we get, for $f \in L^2 \cap HA_{p,q}$,

$$\|G(f)\|_{A_{p,q}} \leq C\, \|f\|_{HA_{p,q}},$$

which, by a density argument, proves the desired result. The result for homogeneous spaces is proved similarly.

We shall give a characterization of $HA_{p,q}$ and $H\dot{A}_{p,q}$ in terms of tangential maximal functions or grand maximal functions. First we give the following definition.

DEFINITION 2.9. Let $a(x)$ be a function defined on $\mathbb{R}^n$, $0 < q \leq 1$, $1 < p < \infty$, and N be a positive integer. Then $a(x)$ is said to be a *central* (p, q, N) *atom* provided that

(i) $a(x)$ lives in a ball centred at 0, say $B(0, R)$,

(ii) $\|a\|_p \leq |B(0, R)|^{1/p - 1/q}$,

(iii) if α is a multi-index with $|\alpha| \leq N$, then $\int_{\mathbb{R}^n} x^\alpha a(x)\, dx = 0$.

Next we shall construct an atomic decomposition for $f \in L^2 \cap HA_{p,q}$ and $f \in L^2 \cap H\dot{A}_{p,q}$.

THEOREM 2.10. *Let $f \in L^2 \cap H\dot{A}_{p,q}$, $0 < q \leq 1$, $1 < p < \infty$, and N be a positive integer. There are a sequence, $\{a_k\}$, of central (p, q, N) atoms, and a sequence $\{\lambda_k\}$ of real numbers satisfying $\sum_{k=0}^{\infty} |\lambda_k|^q < \infty$, such that $f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x)$ in the sense of distributions. If $f \in L^2 \cap HA_{p,q}$ then all of the atoms are supported on balls whose radii are greater than or equal to* 1.

Proof. First we shall construct a partition of unity by smooth, radial, non-negative functions corresponding to the partition $\mathbb{R}^n = \overline{B(0,1)} \cup (\bigcup_{k=1}^{\infty} C_k)$.

Let ψ_0, ψ be a couple of smooth radial functions, with $0 \leqslant \psi_0, \psi \leqslant 1$, such that

(i) the support of ψ_0 is $\tilde{C}_0 = \overline{B(0, 1+\varepsilon)}$, and $\psi_0(x) = 1$ for every $x \in \overline{B(0,1)}$,

(ii) the support of ψ is $\{x \in \mathbb{R}^n\colon -\varepsilon + \frac{1}{2} \leqslant |x| \leqslant 1 + \varepsilon\}$ and $\psi(x) = 1$ for every $x \in C_0$.

For every positive integer k, we define $\psi_k(x) = \psi(2^{-k}x)$ where the support of ψ_k is $\tilde{C}_k = \{x \in \mathbb{R}^n\colon 2^{k-1} - 2^k\varepsilon \leqslant |x| \leqslant 2^k + 2^k\varepsilon\}$ and ψ_k is identically 1 on C_k. If $\varepsilon > 0$ is small enough, then $1 \leqslant \sum_{k=0}^{\infty} \psi_k(x) \leqslant 2$. We get a partition of unity by taking

$$\phi_k(x) = \psi_k(x) \Big/ \sum_{k=0}^{\infty} \psi_k(x).$$

Given $f \in L^2 \cap HA_{p,q}$, write $f(x) = \sum_{k=0}^{\infty} f(x)\phi_k(x)$.

Let $P_k(x)$ be a polynomial of degree N, restricted to $\tilde{C}_k$, satisfying $\int_{\tilde{C}_k} x^\alpha (f(x)\phi_k(x) - P_k(x))\,dx = 0$ for every multi-index α such that $|\alpha| \leqslant N$.

More explicitly $P_k(x) = \sum_{|\alpha| \leqslant N} m_\alpha^k \beta_\alpha^k(x)$ with $m_\alpha^k = |\tilde{C}_k|^{-1} \int_{\tilde{C}_k} f(x)\phi_k(x) x^\alpha\,dx$, and $\beta_\alpha^k(x)$ is the restriction to $\tilde{C}_k$ of a polynomial of degree at most N, uniquely determined by the equations

$$|\tilde{C}_k|^{-1} \int_{\tilde{C}_k} \beta_\alpha^k(x) x^\gamma\,dx = \begin{cases} 0 & \text{when } \alpha \neq \gamma, \\ 1 & \text{when } \alpha = \gamma. \end{cases}$$

Note that by homogeneity, $|\beta_\alpha^k(x)| \leqslant C 2^{-k|\alpha|}$, and we have the following estimate for $P_k(x)$:

$$|P_k(x)| \leqslant C\,|\tilde{C}_k|^{-1} \int_{\tilde{C}_k} |f(x)\phi_k(x)|\,dx.$$

So we write, $f(x) = \sum_{k=0}^{\infty} (f(x)\phi_k(x) - P_k(x)) + \sum_{k=0}^{\infty} P_k(x)$. Note that each term $(f\phi_k - P_k)(x)$ satisfies the cancellation condition (ii) of Definition 2.9 and its support is contained in $B(0, 2^{k+1})$. Moreover, when $p > 1$, the above estimate for $|P_k(x)|$ gives

$$\left(\int_{B(0,2^{k+1})} |f(x)\phi_k(x) - P_k(x)|^p\,dx\right)^{1/p} \leqslant C\left(\int_{\tilde{C}_k} |f(x)\phi_k(x)|^p\,dx\right)^{1/p} \leqslant C \sum_{j=k-1}^{k+1} \|f\tilde{\chi}_j\|_p.$$

Now

$$a_k(x) = (2^{kn(1/q - 1/p)}\,\|f\phi_k - P_k\|_p)^{-1}(f\phi_k - P_k)(x)$$

is a central (p, q, N) atom, with support $B(0, 2^{k+1})$, and

$$\sum_{k=0}^{\infty} (f\phi_k - P_k)(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x) \quad \text{with } \lambda_k = 2^{kn(1/q-1/p)}\,\|f\phi_k - P_k\|_p.$$

Therefore

$$\sum_{k=0}^{\infty} |\lambda_k|^q \leqslant C \sum_{k=0}^{\infty} (2^{kn(1/q-1/p)}\,\|f\tilde{\chi}_k\|_p)^q = C\,\|f\|_{A_{p,q}}^q \leqslant C\,\|G(f)\|_{A_{p,q}}^q \leqslant C\,\|f\|_{HA_{p,q}}^q.$$

Now we have to decompose $\sum_{k=0}^{\infty} P_k(x)$. Note that $G(f) \in A_{p,q}$. Summing by parts we write

$$\sum_{k=0}^{\infty} P_k(x) = \sum_{|\alpha| \leq N} \sum_{k=0}^{\infty} (m_\alpha^k |\tilde{C}_k|)(|\tilde{C}_k|^{-1} \beta_\alpha^k(x))$$

$$= \sum_{|\alpha| \leq N} \left\{ \sum_{k=0}^{\infty} \left(\frac{\beta_\alpha^k(x)}{|\tilde{C}_k|} - \frac{\beta_\alpha^{k+1}(x)}{|\tilde{C}_{k+1}|} \right) \sum_{j=0}^{k} m_\alpha^j |\tilde{C}_j| \right\}$$

$$= \sum_{|\alpha| \leq N} \sum_{k=0}^{\infty} N_\alpha^k \varphi_\alpha^k(x),$$

with $N_\alpha^k = \sum_{j=0}^{k} m_\alpha^j |\tilde{C}_j|$ and $\varphi_\alpha^k(x) = \beta_\alpha^k(x)/|\tilde{C}_k| - \beta_\alpha^{k+1}(x)/|\tilde{C}_{k+1}|$. Writing $h_\alpha^k(x) = N_\alpha^k \varphi_\alpha^k(x)$, we have $\sum_{k=0}^{\infty} P_k(x) = \sum_{|\alpha| \leq N} \sum_{k=0}^{\infty} h_\alpha^k(x)$. Each $h_\alpha^k(x)$ satisfies the cancellation condition and has support contained in $B(0, 2^{k+2})$. To estimate its size note that $\sum_{j=0}^{k} \phi_j(x)$ is essentially a bump function 'adapted' to $B(0, 2^{k+1})$ with $\int_{\mathbb{R}^n} \sum_{j=0}^{k} \phi_j(x)\, dx \sim 2^{kn}$, so that, as in [**8**, p. 184], we obtain

$$\left| \int_{\mathbb{R}^n} f(x) \sum_{j=0}^{k} \phi_j(x)\, dx \right| \leq C 2^{kn} G(f)(x) \chi_{B(0,2^{k+2})}(x).$$

That is,

$$N_\alpha^k \leq C 2^{k(|\alpha|+n)} G(f)(x) \chi_{B(0,2^{k+2})}(x),$$

and, again by homogeneity,

$$|\varphi_\alpha^k(x)| \sim C 2^{-k(|\alpha|+n)} \sum_{j=k-1}^{k+2} \tilde{\chi}_j(x).$$

Therefore, $\|h_\alpha^k\|_p \leq C \sum_{j=k-1}^{k+2} \|G(f)\tilde{\chi}_j\|_p$ and the function

$$b_\alpha^k(x) = (2^{kn(1/q-1/p)} \|h_\alpha^k\|_p)^{-1} h_\alpha^k$$

is a central (p, q, N) atom with support $B(0, 2^{k+2})$. Finally,

$$\sum_{k=0}^{\infty} P_k(x) = \sum_{|\alpha| \leq N} \sum_{k=0}^{\infty} \lambda_\alpha^k b_\alpha^k(x),$$

with

$$\sum_{|\alpha| \leq N} \sum_{k=0}^{\infty} |\lambda_\alpha^k|^q \leq C \sum_{k=0}^{\infty} (2^{kn(1/q-1/p)} \|G(f)\tilde{\chi}_k\|_p)^q \leq C \|G(f)\|_{A_{p,q}}^q \leq C \|f\|_{HA_{p,q}}^q.$$

When $f \in L^2 \cap H\dot{A}_{p,q}$ the atomic decomposition is obtained in a similar way by using the partition of unity ϕ_k $(k \in \mathbb{Z})$ defined as above from all the functions ψ_k $(k \in \mathbb{Z})$.

Note that the atomic radii are greater than or equal to 1 in the $HA_{p,q}$ decomposition.

Next we determine the minimal number of vanishing moments which are needed for an atom to be in $HA_{p,q}$. Let $[x]$ represent the greatest integer smaller than or equal to x.

THEOREM 2.11. *Let $0<q\leqslant 1$, $1<p<\infty$. If $a(x)$ is a central (p,q,N) atom with $N\geqslant[n(1/q-1)]$, then $a^*(x)\in\dot{A}_{p,q}$ and $\|a^*\|_{\dot{A}_{p,q}}\leqslant C$. Moreover, if* $\operatorname{supp} a(x)\subseteq$ *$B(0,R)$ with $R\geqslant 1$, then $a^*(x)\in A_{p,q}$ and $\|a^*\|_{A_{p,q}}\leqslant C$. Here C represents an absolute constant independent of $a(x)$.*

Proof. Let $a(x)$ be a central (p,q,N) atom with support in $B(0,R)$, and j an integer such that $R\sim 2^j$. Then

$$\|a^*\|^q_{\dot{A}_{p,q}}=\sum_{k=-\infty}^{j}(2^{kn(1/q-1/p)}\|a^*\chi_k\|_p)^q+\sum_{k=j+1}^{\infty}(2^{kn(1/q-1/p)}\|a^*\chi_k\|_p)^q.$$

We estimate the first term, using $a^*(x)\leqslant Ma(x)$, the boundedness in L^p $(p>1)$, of M, and the choice of j. We have

$$\sum_{k=-\infty}^{j}(2^{kn(1/q-1/p)}\|a^*\chi_k\|_p)^q\leqslant C\,\|a\|_p^q\,2^{jn(1-q/p)}\leqslant C.$$

If $x\in C_k$ and $k>j$, Taylor's expansion for the Poisson kernel and the cancellation of $a(x)$ yield

$$P_t*a(y)=(-1)^{N+1}\int_{B(0,R)}a(z)\Big(\sum_{|\alpha|=N+1}\frac{1}{\alpha!}D^\alpha P_t(y-\theta z)z^\alpha\Big)\,dz.$$

Since $|\alpha|=N+1$, $|z|<2^j\,|y-x|<t$ and $x\in C_k$, where $k>j$, we have

$$\begin{aligned}\|D^\alpha P_t(y-\theta z)|&\leqslant Ct^{-n-|\alpha|}(1+|y-\theta z|/t)^{-n-|\alpha|}\\&=C(t+|y-\theta z|)^{-n-|\alpha|}\\&\leqslant C(t+|y|-|z|)^{-n-|\alpha|}\\&\leqslant C(|x|-|z|)^{-n-|\alpha|}\\&\leqslant C\,|x|^{-n-|\alpha|},\end{aligned}$$

and $\|a^*\chi_k\|_p\leqslant C\,|B|\,((N+1)/n+1+1/q)2^{-k(n+N+1)}2^{kn/p}$. Therefore, the second term may be estimated by

$$C\,|B|^{((N+1)/n+1-1/q)q}\sum_{k=j+1}^{\infty}2^{-knq((N+1)/n+1-1/q)},$$

which is controlled by an absolute constant provided that $N\geqslant[n(1/q-1)]$.

The same ideas work to prove the result for $A_{p,q}$.

The following result is immediate from a density argument and Theorems 2.10 and 2.11.

THEOREM 2.12. (a) *Given $f\in H\dot{A}_{p,q}$, with $0<q\leqslant 1$ and $1<p<\infty$, there are a sequence $\{a_k\}$ of central (p,q,N) atoms with $N\geqslant[n(1/q-1)]$, and a sequence $\{\lambda_k\}$ of constants satisfying $\sum_k|\lambda_k|^q\leqslant C\,\|f\|^q_{H\dot{A}_{p,q}}$, with C an absolute constant, such* *that* $f(x)=\sum_k \lambda_k a_k(x)$ *in the distribution sense. If* $f\in HA_{p,q}$ *then the central atoms are all supported on balls whose radii are greater than or equal to* 1.

(b) *If* $\{a_k\}$ *is a sequence of central* (p,q,N) *atoms with* $N\geqslant[n(1/q-1)]$, $0<q\leqslant 1$, *and* $1<p<\infty$, *and* $\{\lambda_k\}$ *is a sequence of real numbers satisfying* $\sum_k|\lambda_k|^q<\infty$, *then the sum* $\sum_k\lambda_k a_k(x)$ *converges, in the distribution sense, to* $f\in H\dot{A}_{p,q}$ *and* $\|f\|^q_{H\dot{A}_{p,q}}\leqslant C\sum_k|\lambda_k|^q$. *In particular, if the atomic radii are greater than or equal to* 1 *then the sum converges to* $f\in HA_{p,q}$ *and* $\|f\|^q_{HA_{p,q}}\leqslant C\sum_k|\lambda_k|^q$.

This atomic decomposition allows us to describe $HA_{p,q}$ and $H\dot{A}_{p,q}$ in terms of conjugate harmonic functions in the sense of Stein and Weiss [**15**]. We shall call $F(x,t)=(u_0(x,t),u_1(x,t),\dots,u_n(x,t))$ a system of conjugate harmonic functions and write

$$|F(x,t)|=\left(\sum_{j=0}^{n}|u_j(x,t)|^2\right)^{\frac{1}{2}}.$$

THEOREM 2.13. *Let* $f\in\mathscr{S}'(\mathbb{R}^n)$, $(n-1)/n<q\leqslant 1$, *and* $q<p<\infty$. *If there is a system of conjugate harmonic functions* $F(x,t)$, *in the sense of Stein and Weiss, such that* $\sup_{t>0}\|\,|F(\cdot,t)|\,\|_{A_{p,q}}<\infty$ *and* $\lim_{t\to 0}u_0(x,t)=f(x)$ *in the distribution sense, then* $f\in HA_{p,q}$.

Conversely, given $f\in HA_{p,q}$, *with* $0<q\leqslant 1$ *and* $1<p<\infty$, *there is a system of conjugate harmonic functions, in the sense of Stein and Weiss, satisfying* $\sup_{t>0}\|\,|F(\cdot,t)|\,\|_{A_{p,q}}<\infty$ *and* $\lim_{t\to 0}u_0(x,t)=f(x)$ *in the distribution sense.*

(*Similar results hold for homogeneous spaces.*)

Proof. Note first that the result for $q=1$ was proved in [**9**] and may be extended to the homogeneous spaces $H\dot{A}^p$, where $1<p<\infty$. Since $A_{p,q}\subset L^q$, for $p>q$, we see that $F(x,t)$ belongs to the space of Stein and Weiss. Thus there exists

$$\lim_{t\to 0}F(x,t)=F(x)=(f(x),R_1f(x),\dots,R_nf(x)),$$

where R_i, for $i=1,\dots,n$, are the Riesz transforms.

As $(n-1)/n<q$, taking $\varepsilon>0$ such that $(n-1)/n\leqslant\varepsilon<q$, we see that the function $|F(x,t)|^\varepsilon$ is subharmonic in $\mathbb{R}^{n+1}_+$ and $|F(\cdot,t)|^\varepsilon$ is uniform in $L^{q/\varepsilon}(\mathbb{R}^n)$, so we get $|F(x,t)|^\varepsilon\leqslant P_t*(|F|^\varepsilon)(x)$ (see [**10**, pp. 286, 175]). Consequently,

$$\sup_{|y-x|<t}|F(y,t)|^\varepsilon\leqslant(|F|^\varepsilon)^*(x)\leqslant CM(|F|^\varepsilon)(x)$$

and we obtain $m_{u_0}(x)\leqslant C(M(|F|^\varepsilon)(x))^{1/\varepsilon}$ with $u_0(x,t)\to f(x)$ as $t\to 0$ in the distribution sense.

By Proposition 1.5, using this estimate and the results for Beurling Algebras (see [**9**, proof of Theorem 3.1]) we get

$$\|m_{u_0}\|^q_{A_{p,q}}=\|(m_{u_0})^q\|_{A^{p/q}}\leqslant C\,\|\,|F|^q\|_{A^{p/q}}\leqslant C\,\|\,|F(\cdot,t)|\,\|^q_{A_{p,q}}.$$

Therefore,

$$\|f\|_{HA_{p,q}}\leqslant C\sup_{t>0}\|\,|F(\cdot,t)|\,\|_{A_{p,q}}.$$

To prove the converse, consider $f \in HA_{p,q}$ with $0<q\leqslant 1$ and $1<p<\infty$. Using the atomic decomposition we may reduce the proof to the special case of a central (p, q, N) atom, $a(x)$, with $N\geqslant[n(1/q-1)]$ and $\operatorname{supp}(a(x))\subset B(0,R)$, where $R\geqslant 1$.

Taking $u_0(x,t)=P_t*a(x)$ and $u_j(x,t)=R_j(P_t*a)(x)$, for $j=1,\dots,n$, we have a Stein–Weiss system. We have to prove that $\sup_{t>0}\| |F(\cdot,t)| \|_{A_{p,q}}<C$. As we did in the proof of Theorem 2.11 we get $\|P_t*a\|_{A_{p,q}}<C$, independently of $a(x)$ and t, so we have $m_{u_0}(x)=\sup_{|y-x|<t}|P_t*a(y)|\in A_{p,q}$, and $u_0(x,t)\to a(x)$ as $t\to 0$ in the distribution sense.

To estimate $\|R_j(P_t*a)\|_{A_{p,q}}$, note that $R_j(P_t*a)(x)=Q_{j,t}*a(x)$, for $j=1,\dots,n$, with $Q_{j,t}(x)\sim C_n x_j(t^2+|x|^2)^{-(n+1)/2}$, and this kernel satisfies the same estimates as the Poisson kernel did. So the same ideas as those used in the proof of Theorem 2.11 work in this situation and give us the estimate $\|R_j(P_t*a)\|_{A_{p,q}}<C$ for $j=1,\dots,n$ with C an absolute constant.

If we put together all the results describing $HA_{p,q}$ or $H\dot{A}_{p,q}$, we have the following equivalence theorem.

THEOREM 2.14. *Let $(n-1)/n<q\leqslant 1$, $1<p<\infty$, and $f\in\mathcal{S}'(\mathbb{R}^n)$. Then the following properties are equivalent:*

(a) *there is $F(x,t)=(u_0(x,t),u_1(x,t),\dots,u_n(x,t))$, a Stein–Weiss system such that $\sup_{t>0}\| |F(\cdot,t)| \|_{\dot{A}_{p,q}}<\infty$, where $|F(x,t)|=(\sum_{j=0}^{n}|u_j(x,t)|^2)^{\frac{1}{2}}$ and $u_0(x,t)\to f(x)$ as $t\to 0$ in the distribution sense;*

(b) *$m_u^+(x)\in\dot{A}_{p,q}$, where $u(x,t)$ is a harmonic function such that $u(x,t)\to f(x)$ as $t\to 0$ in the distribution sense;*

(c) *$m_u(x)\in\dot{A}_{p,q}$, where $u(x,t)$ is as in* (b)*;*

(d) *$m_u^N(x)\in\dot{A}_{p,q}$ for every $N\geqslant 1$, where $u(x,t)$ is as above;*

(e) *$G(f)\in\dot{A}_{p,q}$;*

(f) *$f(x)=\sum\lambda_k a_k(x)$, where a_k are central (p,q,N) atoms with $N\geqslant[n(1/q-1)]$ and $\sum|\lambda_k|^q<\infty$.*

Further, the relevant norms in the six properties are equivalent. These are

$$\sup_{t>0}\| |F(\cdot,t)| \|_{\dot{A}_{p,q}},\quad \|m_u^+\|_{\dot{A}_{p,q}},\quad \|m_u\|_{\dot{A}_{p,q}},\quad \|m_u^N\|_{\dot{A}_{p,q}},\quad \|G(f)\|_{\dot{A}_{p,q}},$$

and $\inf(\sum_k|\lambda_k|^q)^{1/q}$, where the infimum is taken over all admissible decompositions.

The same statements with $\dot{A}_{p,q}$ replaced by $A_{p,q}$ are also equivalent to each other. Moreover the atoms which appeared in (f) *have atomic radii greater than or equal to* 1.

The atomic descriptions of $HA_{p,q}$ and $H\dot{A}_{p,q}$ allow us to identify their dual spaces.

DEFINITION 2.15. Let $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, with $1<p<\infty$.

(a) The function f will be said to belong to $\Lambda_{p,q}$, with $0<q\leqslant 1$ and $1<p<\infty$, if and only if for every $R \geqslant 1$ there is a polynomial $P_R^N(f)(x)$ of degree at most $N = [n(1/q - 1)]$ such that

$$\|f\|_{\Lambda_{p,q}} = \sup_{R \geqslant 1} \left\{ |B(0,R)|^{1-1/q} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x) - P_R^N f(x)|^p \, dx \right)^{1/p} \right\} < \infty.$$

(b) The function $f(x)$ will be said to belong to $\dot{\Lambda}_{p,q}$, with $0 < q \leqslant 1$ and $1 < p < \infty$, if and only if for every $R > 0$ there is a polynomial $P_R^N f(x)$ of degree at most N, such that

$$\|f\|_{\dot{\Lambda}_{p,q}} = \sup_{R > 0} \left\{ |B(0,R)|^{1-1/q} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x) - P_R^N f(x)|^p \, dx \right)^{1/p} \right\} < \infty.$$

Identifying functions which differ by a polynomial of degree at most N, almost everywhere, we see that $\Lambda_{p,q}$ and $\dot{\Lambda}_{p,q}$ are Banach spaces.

Note that when $q = 1$, $\Lambda_{p,1}$ is the space CMO^p, introduced in the one-dimensional case by Chen and Lau [**5**] and extended to n-dimensions by García-Cuerva [**9**].

THEOREM 2.16. *Let $0 < q \leqslant 1$ and $1 < p < \infty$, and let p' be the exponent conjugate to p. Then $(HA_{p,q})^* = \Lambda_{p',q}$ and $(H\dot{A}_{p,q})^* = \dot{\Lambda}_{p',q}$. This may be stated more precisely as follows. Given $g \in \Lambda_{p',q}$, the functional L_g, defined over compactly supported functions $f \in HA_{p,q}$ by*

$$L_g(f) = \int_{\mathbb{R}^n} f(x) g(x) \, dx,$$

extends in a unique way to a continuous linear functional $L_g \in (HA_{p,q})^*$ whose norm satisfies $\|L_g\| \leqslant C \, \|g\|_{\Lambda_{p',q}}$. Conversely, given $L \in (HA_{p,q})^*$, there is a unique (up to polynomials of degree at most N) $g \in \Lambda_{p',q}$ such that $L = L_g$. Further, $\|g\|_{\Lambda_{p',q}} \leqslant C \, \|L_g\|$. (Similar results hold for homogeneous spaces.)

Proof. Let $g \in \Lambda_{p',q}$ and $a(x)$ be a central (p, q, N) atom with $\operatorname{supp} a(x) \subset B(0, R)$, where $R \geqslant 1$. By the properties of $a(x)$ and the Hölder inequality, we have

$$\begin{aligned} |L_g(a)| &= \left| \int_{B(0,R)} a(x)(g(x) - P_R^N g(x)) \, dx \right| \\ &\leqslant |B(0,R)|^{1-1/q} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |g(x) - P_R^N g(x)|^{p'} \, dx \right)^{1/p'} \\ &\leqslant \|g\|_{\Lambda_{p',q}}, \end{aligned}$$

where $P_R^N g(x)$ is a polynomial of degree at most N which has, over $B(0, R)$, the same moments as g up to order N.

In general, if $f \in HA_{p,q}$ and has compact support, we can write $f(x) = \sum_k \lambda_k a_k(x)$, where the atoms $\{a_k\}$ are all supported in a fixed ball centred at 0, and $\sum_k |\lambda_k|^q \leqslant C \, \|f\|_{HA_{p,q}}^q$. Since $\|a_k\|_{A_{p,q}} \leqslant C$, the series converges in $A_{p,q}$ and in L^p. Hence as $g \in L_{\mathrm{loc}}^{p'}(\mathbb{R}^n)$, it follows that $L_g(f) = \sum_k \lambda_k L_g(a_k)$ and thence

$$|L_g(f)| \leqslant C \, \|f\|_{HA_{p,q}} \|g\|_{\Lambda_{p',q}}.$$

Now the compactly supported $HA_{p,q}$ functions include the finite linear combinations of atoms and are dense in $HA_{p,q}$. Thus it is possible to extend L_g in a unique way to a continuous linear functional $L_g \in (HA_{p,q})^*$ whose norm satisfies $\|L_g\| \leqslant C \|f\|_{HA_{p,q}}$.

Conversely, given $L \in (HA_{p,q})^*$ and $f \in HA_{p,q}$ we know that $\|L(f)| \leqslant \|L\| \, \|f\|_{HA_{p,q}}$. In particular, when $a(x)$ is a central (p, q, N) atom, $|L(a)| \leqslant C$.

Given $R \geqslant 1$, and $B = B(0, R)$ define the space

$$L^p_N(B) = \left\{ f \in L^p \colon \operatorname{supp} f \subset B, \int x^\alpha f(x)\,dx = 0, |\alpha| \leqslant N \right\}.$$

If $f \in L^p_N(B)$ then $|B|^{1/p-1/q} \|f\|_p^{-1} f(x)$ is a central (p, q, N) atom and $|L(f)| \leqslant C \|L\| \, |B|^{1/q-1/p} \|f\|_p$. Thus, $L \in (HA_{p,q})^*$ gives a continuous linear functional on $L^p_N(B)$ whose norm is bounded by $C \|L\| \, |B|^{1/q-1/p}$. By the Hahn–Banach theorem we extend the functional to $L^p(B)$ with the same norm. The duality between L^p and $L^{p'}$ allows us to represent L over compactly supported functions having vanishing moments to the order N, as

$$L(f) = L_g(f) = \int_{\mathbb{R}^n} f(x) g(x)\,dx$$

for some $g \in L^{p'}_{\text{loc}}(\mathbb{R}^n)$. Let us see that $g \in \Lambda_{p',q}$. For any $R \geqslant 1$,

$$\left(\int_B |g(x) - P^N_R g(x)|^{p'}\,dx \right)^{1/p'}$$
$$= \sup\left\{ \left| \int_B (g - P^N_R g)(x) f(x)\,dx \right| \colon \|f\|_p \leqslant 1, \operatorname{supp} f \subset B \right\}.$$

Since $\|P^N_R f\|_p \leqslant C |B|^{-1/p'} \int_B |f(x)|\,dx$ and $\|f\|_p \leqslant 1$, we get

$$\left| \int_B (g - P^N_R g)(x) f(x)\,dx \right| \leqslant \|g\chi_B\|_{p'} \, \|f - P^N_R f\|_p \leqslant C \|L\| \, |B|^{1/q-1/p}$$

and $\|g\|_{\Lambda_{p',q}} \leqslant C \|L\|$.

Since the space of compactly supported functions having vanishing moments up to order N contains the finite linear combination of atoms, and these are dense in $HA_{p,q}$, we may extend the result to $f \in HA_{p,q}$.

The proof of the homogeneous case is similar.

The atomic characterization of $HA_{p,q}$ and $H\dot{A}_{p,q}$ may be used to get an interpolation result between these spaces by the complex method. Note first that, except for $q = 1$, these are not Banach spaces. The complex method under consideration here is not the classical method by J. L. Lions, A. P. Calderón and S. G. Krejn, but a modification of the usual complex method which can be applied to some concrete quasi-Banach spaces, in particular to $HA_{p,q}$ and $H\dot{A}_{p,q}$. This modification was discovered by A. P. Calderón and A. Torchinsky **[4]** and applied by S. Janson and P. Jones to get interpolation results between classical Hardy spaces H^p **[12]**.

First we describe the method.

DEFINITION 2.17. Consider the closed strip $S = \{z \colon 0 \leqslant \operatorname{Re} z \leqslant 1\}$ in the complex plane. Denote by $\mathring{S}$ the open strip. Let $g(z)$ be an $\mathscr{S}'(\mathbb{R}^n)$-valued function in S and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with compact support such that $\hat{\varphi}(0) \neq 0$. We say that $g(z)$ is an

$\mathscr{S}'(\mathbb{R}^n)$-analytic function in $\mathring{S}$ if $G(x,z)=(g(z)*\varphi)(x)$ satisfies the following properties:

(1) $G(x,z)$ is a uniformly continuous and bounded function in $\mathbb{R}^n\times S$,

(2) $G(x,z)$ is an analytic function in $\mathring{S}$ for every fixed $x\in\mathbb{R}^n$.

DEFINITION 2.18. Let $0<q_0,q_1<1$ and $1<p_0,p_1<\infty$.

(a) We define $\mathscr{F}(HA_{p_0,q_0}, HA_{p_1,q_1})$ to be the space of $\mathscr{S}'(\mathbb{R}^n)$-analytic functions in $\mathring{S}$ satisfying $g(j+it)\in HA_{p_j,q_j}$, with $j=0,1$, for every $t\in\mathbb{R}$ with the norm

$$\|g\|_{\mathscr{F}(HA_{p,q,0},HA_{p,q,1})}=\max_{j=0,1}\left\{\sup_t \|g(j+it)\|_{HA_{p,q,j}}\right\}<\infty,$$

where, for typographic reasons, we sometimes write, for example, $HA_{p,q,j}$ instead of HA_{p_j,q_j}.

(b) We define $\mathscr{F}(H\dot{A}_{p_0,q_0}, H\dot{A}_{p_1,q_1})$ to be the space of $\mathscr{S}'(\mathbb{R}^n)$-analytic functions in $\mathring{S}$ satisfying $g(j+it)\in H\dot{A}_{p_j,q_j}$, with $j=0,1$, for every $t\in\mathbb{R}$ with the norm

$$\|g\|_{\mathscr{F}(H\dot{A}_{p,q,0},H\dot{A}_{p,q,1})}=\max_{j=0,1}\left\{\sup_t \|g(j+it)\|_{H\dot{A}_{p,q,j}}\right\}<\infty.$$

DEFINITION 2.19. Let $0<q_0,q_1<1$, $1<p_0,p_1<\infty$ and $0<\theta<1$.

(a) Define

$$(HA_{p_0,q_0}, HA_{p_1,q_1})_\theta=\{f:\ \exists\, g(z)\in\mathscr{F}(HA_{p_0,q_0}, HA_{p_1,q_1}) \text{ with } f=g(\theta)\}$$

and

$$\|f\|_{(HA_{p,q,0},\, HA_{p,q,1})_\theta}=\inf\{\|g\|_{\mathscr{F}(HA_{p,q,0},\, HA_{p,q,1})}\},$$

where the infimum is taken over all admissible functions $g(z)$ satisfying $g(\theta)=f(x)$.

(b) Define

$$(H\dot{A}_{p_0,q_0}, H\dot{A}_{p_1,q_1})_\theta=\{f:\ \exists\, g(z)\in\mathscr{F}(H\dot{A}_{p_0,q_0}, H\dot{A}_{p_1,q_1}) \text{ with } f=g(\theta)\}$$

and

$$\|f\|_{(H\dot{A}_{p,q,0},H\dot{A}_{p,q,1})_\theta}=\inf\{\|g\|_{\mathscr{F}(H\dot{A}_{p,q,0},H\dot{A}_{p,q,1})}\},$$

where the infimum is taken over all admissible functions $g(z)$ satisfying $g(\theta)=f(x)$.

THEOREM 2.20. *If* $0<q_0,q_1<1$, $1<p_0,p_1<\infty$, $0<\theta<1$ *and*

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$$

then

$$HA_{p,q}=(HA_{p_0,q_0}, HA_{p_1,q_1})_\theta \quad \textit{and} \quad H\dot{A}_{p,q}=(H\dot{A}_{p_0,q_0}, H\dot{A}_{p_1,q_1})_\theta.$$

Before starting the proof of this result we enunciate the following lemma based on the Poisson integral representation formula for the strip S, and whose proof can be found in [**3**, 9.4] and [**16**, pp. 65 and 67].

LEMMA 2.21. *Let* $0<r<\infty$. *Then there exist two positive functions* $\mu_0(\theta,t)$ *and* $\mu_1(\theta,t)$ *in* $(0,1)\times\mathbb{R}$, *such that*

$$|h(z)|^r \leqslant \left(\frac{1}{1-\theta}\int_{\mathbb{R}} |h(it)|^r \mu_0(\theta,t)\,dt\right)^{1-\theta}\left(\frac{1}{\theta}\int_{\mathbb{R}} |h(1+it)|^r \mu_1(\theta,t)\,dt\right)^{\theta},$$

with $\theta=\operatorname{Re} z$, *for any logarithmically subharmonic function* $h(z)$ *in* $\mathring{S}$ *which is uniformly continuous and bounded in S. Furthermore, if* $0<\theta<1$, *then*

$$\frac{1}{1-\theta}\int \mu_0(\theta,t)\,dt=\frac{1}{\theta}\int \mu_1(\theta,t)\,dt=1.$$

Proof of Theorem 2.20. Let $a(x)$ be a central (p,q,M) atom, with $M=\max_{j=0,1}\{[n(1/q_j-1)]\}$ and $\operatorname{supp} a(x)\subset B=B(0,R)$ and suppose that $R\geqslant 1$. For $z\in S$, we write

$$\frac{1}{q(z)}=\frac{1-z}{q_0}+\frac{z}{q_1},\quad \frac{1}{p(z)}=\frac{1-z}{p_0}+\frac{z}{p_1}$$

and define

$$g(z)=|B|^{p/qp(z)-1/q(z)}\left[|a(x)|^{p/p(z)-1}a(x)-P_a(x,z)\right],$$

where $P_a(x,z)$ is a polynomial restricted to B, with the same moments as $|a(x)|^{p/p(z)-1}a(x)$ up to order M, that is,

$$P_a(x,z)=\sum_{|\gamma|\leqslant M} C_\gamma(z)\beta_\gamma(x),$$

with $C_\gamma(z)=|B|^{-1}\int_B x^\gamma |a(x)|^{p/p(z)-1}a(x)\,dx$ and $\beta_\gamma(x)$ a polynomial of degree at most M, restricted to B, which is determined by the equations

$$|B|^{-1}\int_B \beta_\gamma(x)x^\alpha\,dx=\begin{cases}0 & \text{when } \alpha\neq\gamma,\\ 1 & \text{when } \alpha=\gamma,\end{cases}$$

for every multi-index α, such that $|\alpha|\leqslant M$. By homogeneity $|\beta_\gamma(x)|\leqslant C\,|B|^{-|\gamma|/n}$ and

$$|P_a(x,z)|\leqslant C\,|B|^{-1}\int_B |a(x)|^{\operatorname{Re}(p/p(z))}\,dx.$$

To see that $g(x)$ satisfies the properties of an $\mathscr{S}'(\mathbb{R}^n)$-analytic function is not complicated; it follows from the definition of $g(z)$, the above estimate for $|P_a(x,z)|$, and the Mean Value Theorem. Note that, for $z=\theta$, $P_a(x,\theta)$ is identically 0 and $g(\theta)=a(x)$. Moreover, each $g(j+it)$, for $j=0,1$, has M vanishing moments and its support is contained in B. Again, by the estimate for $|P_a(x,z)|$ we get

$$\|\,|a(\cdot)|^{p/p(j+it)}|a(\cdot)|^{-1}a(\cdot)-P_a(\cdot,j+it)\|_{p_j} \leqslant C\left(\int_B |a(x)|^{\operatorname{Re}(p/p(j+it))p_j}\,dx\right)^{1/p_j}=C\,\|a\|_p^{p/p_j},$$

that is, $\|g(j+it)\|_{p_j}\leqslant C\,|B|^{1/p_j-1/q_j}$, for $j=0,1$. Thus $C^{-1}g(j+it)$ is a central

(p_j, q_j, M) atom with support on B and $\|g(j+it)\|_{HA_{p,q,j}} \leqslant C$, for $j = 0, 1$. Therefore, for

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

we have

$$HA_{p,q} \subset (HA_{p_0,q_0}, HA_{p_1,q_1})_\theta.$$

Conversely, let $f(x) \in (HA_{p_0,q_0}, HA_{p_1,q_1})_\theta$. Then there is an $\mathcal{S}'(\mathbb{R}^n)$-analtyic function $g(z) \in \mathcal{F}(HA_{p_0,q_0}, HA_{p_1,q_1})$ such that $g(\theta) = f(x)$ and

$$\|f\|_{(HA_{p,q,0}, HA_{p,q,1})_\theta} = \|g(z)\|_{\mathcal{F}(HA_{p,q,0}, HA_{p,q,1})}.$$

Let us suppose that $\|g(j+it)\|_{HA_{p,q,j}} = 1$ for $j = 0, 1$.

Consider $F(x, s, z) = (g(z) * \varphi_s)(x)$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\varphi}(0) \neq 0$; for $0 < \varepsilon < 1$ define

$$M_\varepsilon(x, z) = \sup_{\substack{|y-x|<s \\ \varepsilon \leqslant s \leqslant 1/\varepsilon}} |F(y, s, z)|.$$

For every fixed $x \in \mathbb{R}^n$, $M_\varepsilon(x, z)$ is a uniform limit of logarithmically subharmonic functions of z in $\mathring{S}$, uniformly continuous and bounded in S, and the same holds for $M_\varepsilon(x, z)$ for every $x \in \mathbb{R}^n$. Moreover, if G represents the grand maximal function, we have $M_\varepsilon(x, t) \leqslant G(g(z))(x)$ and $M_\varepsilon(x, \theta)$ increases to $G(g(\theta))(x)$ as $\varepsilon \to 0$. Thus by Theorem 2.14, in order to show that $f \in HA_{p,q}$, we only need to study the behaviour in $A_{p,q}$ of $M_\varepsilon(x, \theta)$. We shall write

$$I = \left\{ \sum_{k=0}^{\infty} (2^{kn(1/q-1/p)} \|M_\varepsilon(\cdot, \theta)\tilde{\chi}_k\|_p)^q \right\}^{1/q},$$

and $h_k(x, \theta) = M_\varepsilon(x, \theta)\tilde{\chi}_k(x)$ and choose r such that $0 < r < \min\{q_0, q_1\}$. Then

$$I = \left\{ \sum_{k=0}^{\infty} (2^{knr(1/q-1/p)} \|(h_k)^r(\cdot, \theta)\|_{p/r})^{q/r} \right\}^{r/qr}.$$

Using Lemma 2.21, over $h_k(x, \theta)$ we have

$$|h_k(x, \theta)|^r \leqslant (a_k(x))^{1-\theta}(b_k(x))^\theta$$

where

$$a_k(x) = \frac{1}{1-\theta} \int_{\mathbb{R}} |h_k(x, it)|^r \mu_0(\theta, t)\, dt, \quad b_k(x) = \frac{1}{\theta} \int_{\mathbb{R}} |h_k(x, 1+it)|^r \mu_1(\theta, t)\, dt.$$

Using the Hölder inequality for $\|h_k^r(\cdot, \theta)\|_{p/r}$ and the Minkowsky inequality for $\|a_k\|_{p_0/r}$ and $\|b_k\|_{p_1/r}$, and writing

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

we get

$$I \leqslant \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{1-\theta} \int_{\mathbb{R}} 2^{knr(1/q_0 - 1/p_0)} \|h_k(\cdot, it)\|_{p_0}^r \mu_0(\theta, t)\, dt \right)^{q(1-\theta)/r} \right.$$
$$\left. \times \left(\frac{1}{\theta} \int_{\mathbb{R}} 2^{knr(1/q_1 - 1/p_1)} \|h_k(\cdot, 1+it)\|_{p_1}^r \mu_1(\theta, t)\, dt \right)^{q\theta/r} \right\}^{r/qr}.$$

Now, since $1<q/r\leqslant p/r$, we use the Hölder and Minkowsky inequalities (with q instead of p as we had before) and obtain

$$I\leqslant\left\{\frac{1}{1-\theta}\int_{\mathbb{R}}\left[\sum_{k=0}^{\infty}(2^{kn(1/q_0-1/p_0)}\|h_k(\cdot,it)\|_{p_0})^{q_0}\right]^{r/q_0}\mu_0(\theta,t)\,dt\right\}^{(1-\theta)/r}$$
$$\times\left\{\frac{1}{\theta}\int_{\mathbb{R}}\left[\sum_{k=0}^{\infty}(2^{kn(1/q_1-1/p_1)}\|h_k(\cdot,1+it)\|_{p_1})^{q_1}\right]^{r/q_1}\mu_1(\theta,t)\,dt\right\}^{\theta/r}$$
$$=\left(\frac{1}{1-\theta}\int_{\mathbb{R}}\|M_\varepsilon(\cdot,it)\|^r_{A_{p,q,0}}\mu_0(\theta,t)\,dt\right)^{(1-\theta)/r}$$
$$\times\left(\frac{1}{\theta}\int_{\mathbb{R}}\|M_\varepsilon(\cdot,1+it)\|^r_{A_{p,q,1}}\mu_1(\theta,t)\,dt\right)^{\theta/r}.$$

Therefore, the above estimate for $M_\varepsilon(x,z)$ in terms of $G(g(z))(x)$ and Lemma 2.21 gives us

$$I\leqslant\sup_t\{\|G(g(it))\|^{1-\theta}_{A_{p,q,0}}\|G(g(1+it))\|^{\theta}_{A_{p,q,1}}\}$$
$$\leqslant\sup_t\{\|g(it)\|^{1-\theta}_{HA_{p,q,0}}\|g(1+it)\|^{\theta}_{HA_{p,q,1}}\}$$
$$\leqslant\|f\|_{(HA_{p,q,0},HA_{p,q,1})_\theta}=1.$$

Then $\|M_\varepsilon(\cdot,\theta)\|_{A_{p,q}}\leqslant 1$; thus $\|G(g(\theta))\|_{A_{p,q}}=\|G(f)\|_{A_{p,q}}=1$ and finally $\|f\|_{HA_{p,q}}\leqslant 1$.

The proof of the homogeneous result is essentially the same.

3. *The Littlewood–Paley function characterization of* $HA_{p,q}$ *and* $H\dot{A}_{p,q}$ *for* $0<q\leqslant 1$, $1<p\leqslant 2$

Let $f\in L^2\cap L^q$, $0<q\leqslant 1$ and $u(x,t)=f*P_t(x)$ where $P(x)$ is the Poisson kernel, with

$$|\nabla u(x,t)|^2=\left|\frac{\partial u}{\partial t}\right|^2+\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^2.$$

We shall use the following functions:

the Littlewood–Paley g-function

$$g(f)(x)=\left(\int_0^\infty|\nabla u(x,t)|^2\,t\,dt\right)^{\frac12};$$

the Littlewood–Paley g^*_λ-function

$$g^*_\lambda(f)(x)=\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{|y|+t}\right)^{n\lambda}|\nabla u(x-y,t)|^2\,t^{1-n}\,dy\,dt\right)^{\frac12};$$

the Lusin area integral

$$S_\alpha f(x)=\left(\int_{\Gamma_\alpha(x)}|\nabla u(y,t)|^2\,t^{1-n}\,dy\,dt\right)^{\frac12},$$

where $\Gamma_\alpha(x)=\{(y,t)\in\mathbb{R}^{n+1}_+:\ |x-y|<\alpha t,\ t>0\}$.

If $v(x, t) = t\, \partial(f * P_t)(x)/\partial t$, we set

$$S^1_{2\sqrt{n}}\, v(x) = \left(\int_{\Gamma_{2\sqrt{n}}(x)} |v(y, t)|^2\, t^{-n-1}\, dy\, dt\right)^{\frac{1}{2}}.$$

Next we shall construct the atomic decomposition of $HA_{p,q}$ and $H\dot{A}_{p,q}$ by means of the $S^1_{2\sqrt{n}}v(x)$ function.

THEOREM 3.1. *Suppose that* $f \in L^2 \cap L^q$ *and* $0 < q \leqslant 1$. *If* $S^1_{2\sqrt{n}}v(x) \in \dot{A}_{p,q}$ *for* $0 < q \leqslant 1$ *and* $1 < p \leqslant 2$, *then there are a sequence* $\{a_j\}$ *of central* (p, q, N) *atoms with* $N \geqslant [n(1/q - 1)]$ *and a sequence of constants* $\{\lambda_j\}$ *satisfying:*

(1) $f(x) = \sum \lambda_j a_j(x)$;

(2) $\sum |\lambda_j|^q \leqslant C\, \|S^1_{2\sqrt{n}}(v)\|^{g}_{\dot{A}_{p,q}}$, *with C independent of f.*

In particular, if $S^1_{2\sqrt{n}}v(x) \in A_{p,q}$ *for* $0 < q \leqslant 1$ *and* $1 < p \leqslant 2$, *the central atoms are all supported on balls with radii greater than or equal to* 1, *and we have* $\sum |\lambda_j|^q \leqslant C\, \|S^1_{2\sqrt{n}}(v)\|^q_{A_{p,q}}$, *with C independent of f.*

Proof. Consider the closed balls $B_j = \{x \in \mathbb{R}^n\colon\ |x| \leqslant 2^j\}$, for $j \in \mathbb{Z}$, and the cubes $Q_j = \{x \in \mathbb{R}^n\colon\ |x_i| \leqslant 2^j,\ i = 1, \ldots, n\}$, for $j \in \mathbb{Z}$. We shall set $A_j = Q_j - Q_{j-1}$, for $j \in \mathbb{Z}$, and construct a partition of unity in the following way. Let

$$\Omega_k = \{x \in \mathbb{R}^n\colon\ S^1_{2\sqrt{n}}v(x) > 2^k\}, \quad \text{for } k \in \mathbb{Z},$$
$$Q_{i,K} = \{x \in \mathbb{R}^n\colon\ 2^i x - K \in [0, 1)^n\},$$
$$\mathcal{D} = \{Q_{i,K}\colon\ i \in \mathbb{Z},\ K \in \mathbb{Z}^n\}$$

and

$$\mathcal{D}^j_k = \{Q \in \mathcal{D}\colon\ |Q \cap \Omega_k \cap A_j| > |Q|/2^{n+1},$$
$$|Q \cap \Omega_{k+1} \cap A_j| \leqslant |Q|/2^{n+1},\ Q \subset Q_j,\ Q \not\subset Q_{j-1}\}$$

where $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

If $l(Q)$ is the side length of Q, write

$$\tilde{Q} = \{(y, t) \in \mathbb{R}^{n+1}_+\colon\ y \in Q,\ l(Q) < t < 2l(Q)\}.$$

Then $\mathbb{R}^{n+1}_+ = \bigcup_{j=-\infty}^{\infty} \bigcup_{k=-\infty}^{\infty} \bigcup_{Q \in \mathcal{D}^j_k} \tilde{Q}$ is a disjoint partition of $\mathbb{R}^{n+1}_+$. We shall use this partition to obtain the atomic decomposition of f. By Calderón's representation formula, there is a function $\psi \in C^\infty_0(\mathbb{R}^n)$ satisfying

(a) $\operatorname{supp} \psi \subset B(0, 1)$, where ψ is radial and real valued,

(b) $\hat{\psi}(x) = 0$ when $|x| < \varepsilon$ (ε small),

(c) $\int_0^\infty e^{-t} \hat{\psi}(t)\, dt = -1$,

and such that $f(x) = \int_0^\infty \int_{\mathbb{R}^n} v(y, t) \psi_t(x - y) t^{-1}\, dy\, dt$.

Therefore

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{D}^j_k} \int_{\tilde{Q}} v(y, t) \psi_t(x - y) t^{-1}\, dy\, dt = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x),$$

where $a_j(x) = \lambda_j^{-1} \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{D}^j_k} \int_{\tilde{Q}} v(y, t) \psi_t(x - y) t^{-1}\, dy\, dt$, and λ_j is a constant to be determined.

Let $x \in \operatorname{supp} a_j$. Since $\operatorname{supp} \psi \subset B(0, 1)$, we may assume that $|x - y| < t$. For

$(y,t)\in\tilde{Q}$, we have $t<2l(Q)\leqslant 2^{j+1}$, and then $y\in Q_j$ and $|x|\leqslant 2^{j+2}$, so we have $\operatorname{supp} a_j(x)\subset B_{j+2}$. By the property (b) of ψ we have the vanishing moments conditions for $a_j(x)$. To study its L^p-norm we use duality. Now

$$\|a_j\|_p=\sup_{\|\eta\|_{p'}=1}\left\{\left|\int_{\mathbb{R}^n} a_j(x)\eta(x)\,dx\right|\right\}.$$

For convenience we write $\mathcal{U}_k=\bigcup_{Q\in\mathcal{D}_k^j}\tilde{Q}$ in our disjoint partition of $\mathbb{R}^{n+1}_+$. Since $p\leqslant 2$, it follows from Hölder's inequality that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} a_j(x)\eta(x)\,dx\right|\\
&=C\lambda_j^{-1}\left|\sum_{k=-\infty}^{\infty}\int_{\mathcal{U}_k} v(y,t)\psi_t*\eta(y)t^{-1}\,dt\,dy\right|\\
&=C\lambda_j^{-1}\left|\int_{\mathbb{R}^n}\int_0^\infty v(y,t)\chi_{(\bigcup_k\mathcal{U}_k)}(y,t)\psi_t*\eta(y)t^{-1}\,dt\,dy\right|\\
&\leqslant C\lambda_j^{-1}\int_{\mathbb{R}^n}\left(\int_0^\infty |v(y,t)|^2\chi_{(\bigcup_k\mathcal{U}_k)}(y,t)t^{-1}\,dt\right)^{\frac12}\left(\int_0^\infty |\psi_t*\eta(y)|^2t^{-1}\,dt\right)^{\frac12}dy\\
&=C\lambda_j^{-1}\int_{\mathbb{R}^n}\left(\sum_{k=-\infty}^{\infty}\int_0^\infty |v(y,t)|^2\chi_{\mathcal{U}_k}(y,t)t^{-1}\,dt\right)^{\frac12}\left(\int_0^\infty |\psi_t*\eta(y)|^2t^{-1}\,dt\right)^{\frac12}dy\\
&\leqslant C\lambda_j^{-1}\left(\int_{\mathbb{R}^n}\left(\sum_{k=-\infty}^{\infty}\int_0^\infty |v(y,t)|^2\chi_{\mathcal{U}_k}(y,t)t^{-1}\,dt\right)^{p/2}dy\right)^{1/p}\\
&\quad\times\left\{\int_{\mathbb{R}^n}\left(\int_0^\infty |\psi_t*\eta(y)|^2t^{-1}\,dt\right)^{p'/2}dy\right\}^{1/p'}\\
&\leqslant C\lambda_j^{-1}\left\{\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\left[\int_0^\infty |v(y,t)|^2\chi_{\mathcal{U}_k}(y,t)t^{-1}\,dt\right]^{p/2}dy\right\}^{1/p}\\
&\quad\times\left\{\int_{\mathbb{R}^n}\left(\int_0^\infty |\psi_t*\eta(y)|^2t^{-1}\,dt\right)^{p'/2}dy\right\}^{1/p'}.
\end{aligned}$$

Since ψ is a Littlewood–Paley function,

$$\|a_j\|_p\leqslant C\lambda_j^{-1}\left\{\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\left[\int_0^\infty |v(y,t)|^2\chi_{\mathcal{U}_k}(y,t)t^{-1}\,dt\right]^{p/2}dy\right\}^{1/p}.$$

Taking

$$\lambda_j=C\,|B_{j+2}|^{1/q-1/p}\left\{\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\left[\int_0^\infty |v(y,t)|^2\chi_{\mathcal{U}_k}(y,t)t^{-1}\,dt\right]^{p/2}dy\right\}^{1/p}$$

we see that $a_j(x)$ is a central (p,q,N) atom and $f(x)=\sum\lambda_j a_j(x)$.

It remains to estimate $\sum|\lambda_j|^q$. First define $P(\mathcal{U}_k)=\{y\in\mathbb{R}^n\colon (y,t)\in\mathcal{U}_k\}$; then $\chi_{\mathcal{U}_k}(y,t)=\chi_{\mathcal{U}_k}(y,t)\chi_{P(\mathcal{U}_k)}(y)$.

Since $1<p\leqslant 2$, we may use Hölder's inequality and obtain

$$\begin{aligned}
&\left\{\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\left[\int_0^\infty |v(y,t)|^2\chi_{\mathcal{U}_k}(y,t)\chi_{P(\mathcal{U}_k)}(y)t^{-1}\,dt\right]^{p/2}dy\right\}^{1/p}\\
&\qquad\leqslant\left\{\sum_{k=-\infty}^{\infty}\left[\iint_{\mathcal{U}_k}|v(y,t)|^2t^{-1}\,dt\,dy\right]^{p/2}\left(\int_{\mathbb{R}^n}\chi_{P(\mathcal{U}_k)}(y)\,dy\right)^{1-p/2}\right\}^{1/p}.
\end{aligned}$$

Note that for fixed indices k, j, if we have two cubes Q, Q^* in $\mathscr{D}^j_k$, with $Q \cap Q^* \neq \varnothing$, then one of them is completely contained in the other. Picking the larger, we set $P(\mathscr{U}_k)$ as a disjoint union of cubes Q in $\mathscr{D}^j_k$. Therefore,

$$\int_{\mathbb{R}^n} \chi_{P(\mathscr{U}_k)}(y)\,dy = \sum_{\substack{Q \in \mathscr{D}^j_k \\ \text{disj}}} \int_Q dy = \sum_{\substack{Q \in \mathscr{D}^j_k \\ \text{disj}}} |Q| \leqslant C \sum_{\substack{Q \in \mathscr{D}^j_k \\ \text{disj}}} |Q \cap \Omega_k \cap A_j| \leqslant C\,|\Omega_k \cap A_j|.$$

On the other hand, as was proved in [**13**],

$$\left(\int_{\mathscr{U}_k} |v(y,t)|^2 t^{-1}\,dy\,dt\right)^{p/2} \leqslant C 2^{kp}\,|\Omega_k \cap A_j|^{p/2}.$$

Thus $\lambda_j \leqslant C\,|B_{j+2}|^{1/q-1/p} \left(\sum_{k=-\infty}^{\infty} 2^{kp}\,|\Omega_k \cap A_j|\right)^{1/p}$.

Finally, from the definition of Ω_k,

$$\int_{A_j} |S^1_{2\sqrt{n}}\, v(x)|^p\,dx = \sum_{k=-\infty}^{\infty} \int_{(A_j \cap \Omega_k)\setminus(A_j \cap \Omega_{k+1})} |S^1_{2\sqrt{n}} v(x)|^p\,dx$$
$$\geqslant \sum_{k=-\infty}^{\infty} 2^{kp}(|A_j \cap \Omega_k| - |A_j \cap \Omega_{k+1}|) \geqslant C \sum_{k=-\infty}^{\infty} 2^{kp}\,|A_j \cap \Omega_k|,$$

which gives $\lambda_j \leqslant C 2^{jn(1/q-1/p)}\,\|S^1_{2\sqrt{n}} v \chi_{A_j}\|_p$ and also $\sum_{j=-\infty}^{\infty} |\lambda_j|^q \leqslant C\,\|S^1_{2\sqrt{n}} v\|^q_{\dot{A}_{p,q}}$.

If $S^1_{2\sqrt{n}} v(x) \in A_{p,q}$, consider the balls $\tilde{B}_0 = \overline{B(0,1)}$, $\tilde{B}_j = B_j$ for $j \geqslant 1$, the cubes $\hat{Q}_0 = \hat{B}_0$, $\hat{Q}_j = Q_j$ for $j \geqslant 1$, and $\tilde{A}_0 = \hat{Q}_0$, $\tilde{A}_j = A_j$ for $j \geqslant 1$. Construct the decomposition over the disjoint partition of $\mathbb{R}^{n+1}_+$ given by

$$\mathbb{R}^{n+1}_+ = \bigcup_{j=0}^{\infty} \bigcup_{k=-\infty}^{\infty} \bigcup_{\hat{Q} \in \mathscr{D}^j_k} \tilde{Q}.$$

THEOREM 3.2. *Let* $0 < q \leqslant 1$, $1 < p < \infty$ *and* $N = [n(1/q - 1)]$.

(a) *If* $f \in H\dot{A}_{p,q}$ *for* $\lambda > (2N + 3n + 1)/n$ *and* $\alpha > 0$, *we have*

$$\|g^*_\lambda(f)\|_{\dot{A}_{p,q}} \leqslant C_\lambda\,\|f\|_{H\dot{A}_{p,q}}, \quad \|g(f)\|_{\dot{A}_{p,q}} \leqslant C\,\|f\|_{H\dot{A}_{p,q}}, \quad \|S_\alpha(f)\|_{\dot{A}_{p,q}} \leqslant C_\alpha\,\|f\|_{H\dot{A}_{p,q}}.$$

(b) *If* $f \in HA_{p,q}$ *for* $\lambda > (2N + 3n + 1)/n$ *and* $\alpha > 0$, *we have*

$$\|g^*_\lambda(f)\|_{A_{p,q}} \leqslant C_\lambda\,\|f\|_{HA_{p,q}}, \quad \|g(f)\|_{A_{p,q}} \leqslant C\,\|f\|_{HA_{p,q}}, \quad \|S_\alpha(f)\|_{A_{p,q}} \leqslant C_\alpha\,\|f\|_{HA_{p,q}}.$$

Proof. Since $g(f)(x) \leqslant CSf(x) \leqslant C_\lambda g^*_\lambda(f)(x)$ (see [**14**, p. 89]), the result follows from the first inequality.

Let $f(x)$ be a central (p, q, N) atom with $\operatorname{supp} f(x) \subset B(0, R)$ where $R \sim 2^j$ for a fixed $j \in \mathbb{Z}$. Then

$$\|g^*_\lambda(f)\|^q_{\dot{A}_{p,q}} = \sum_{k=-\infty}^{\infty} (2^{kn(1/q-1/p)}\,\|g^*_\lambda(f)\chi_k\|_p)^q.$$

If $k \leqslant j$, then properties of $f(x)$ and the L^p-boundedness $(1 < p)$ of g^*_λ give the estimate

$$\sum_{k=-\infty}^{j} (2^{kn(1/q-1/p)}\,\|g^*_\lambda(f)\chi_k\|_p)^q \leqslant C.$$

If $x \in C_k$ with $k > j$, then we claim that

$$g_\lambda^* f(x) \leqslant C_\lambda\, |B(0,R)|^{(N+1)/n+1-1/q}\, |x|^{-n-N-1},$$

provided that $\lambda > (2N+3n+1)/n$. Therefore, the choice of N and j gives

$$\sum_{k=j+1}^{\infty} (2^{kn(1/q-1/p)}\, \|g_\lambda^*(f)\chi_k\|_p)^q$$

$$\leqslant C_\lambda\, |B(0,R)|^{q((N+1)/n+1-1/q)} \sum_{k=j+1}^{\infty} 2^{-kn((N+1)/n+1-1/q)q} \leqslant C_\lambda,$$

which gives the result.

Let us prove the statement whose truth was claimed above. We shall set

$$K(z) = \left(\frac{\partial P_t}{\partial t}, \frac{\partial P_t}{\partial z_1}, \ldots, \frac{\partial P_t}{\partial z_n}\right)(z), \quad \text{for } z \in \mathbb{R}^n,$$

and call the Poisson kernel P. Consider, for $\lambda \in \mathbb{R}$ fixed, the Hilbert space

$$V_\lambda = \left\{h(y,t)\colon \left[\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{|y|+t}\right)^{n\lambda} |h(y,t)|^2\, t^{1-n}\, dy\, dt\right]^{\frac{1}{2}} < \infty\right\}.$$

Then $g_\lambda^* f(x) = \|K * f(x - \cdot)\|_{v_\lambda}$.

Using the Taylor expansion of $K(z)$ and the atomic properties of $f(x)$, we have

$$g_\lambda^*(f)(x) \leqslant \int_{B(0,R)} |z|^{N+1}\, |f(z)| \left\| \sum_{|\beta|=N+1} D^\beta K(x-y-\theta z)\right\|_{V_\lambda} dz.$$

Standard estimates of the Poisson kernel and the condition $\lambda > (2N+3n+1)/n$ allow us to establish for $x \in C_k$, with $k > j$, that

$$g_\lambda^* f(x) \leqslant C_\lambda\, |B(0,R)|^{(N+1)/n+1-1/q}\, |x|^{-n-N-1},$$

as was claimed.

THEOREM 3.3. *Let $f \in L^2 \cap L^q$ and $0 < q \leqslant 1$. If $\alpha \geqslant 2\sqrt{n}$ and $1 < p < \infty$, then*

$$\|S_\alpha(f)\|_{A_{p,q}} \leqslant C\, \|g(f)\|_{A_{p,q}} \quad \textit{and} \quad \|S_\alpha(f)\|_{\dot{A}_{p,q}} \geqslant C\, \|g(f)\|_{\dot{A}_{p,q}}.$$

Proof. We shall use an argument based on that used by C. Fefferman and E. M. Stein [**8**]. When $f \in L^2 \cap L^p$, $u(x,t) = P_t * f(x)$ is a harmonic function and we define $\mathcal{U}(x,t) = \nabla u(x, t+s)$ and consider the space $\mathcal{H}_2 = L^2(\mathbb{R}^+, s\, ds)$. Then

$$|\mathcal{U}(x,t)|_{\mathcal{H}_2} = \left(\int_0^\infty |\nabla u(x,t+s)|^2\, s\, ds\right)^{\frac{1}{2}} \leqslant g(f)(x)$$

and $\mathcal{U}^+(x) = \sup_{t>0} |\mathcal{U}(x,t)|_{\mathcal{H}_2} = g(f)(x) \in A_{p,q}$. Then there is an $F \in HA_{p,q}$ satisfying $F(x) = \lim_{t \to 0} \mathcal{U}(x,t)$, and from Theorems 2.14 and 3.2 we have

$$\|g(f)\|_{A_{p,q}} = \|\mathcal{U}^+\|_{A_{p,q}} = \|F\|_{HA_{p,q}} \geqslant C_\alpha\, \|S_\alpha(F)\|_{A_{p,q}}.$$

As in [**7**, p. 171], it is possible to show that

$$S_\alpha(F)(x) \geqslant C S_\alpha(f)(x)$$

and the result for $A_{p,q}$ follows.

The homogeneous result is proved in a similar way.

Taking into account the fact that $L^2 \cap L^q$, for $0<q\leqslant 1$, is dense in $HA_{p,q}$ and in $H\dot{A}_{p,q}$, the preceding theorems give the characterization of $HA_{p,q}$ and $H\dot{A}_{p,q}$ for $0<q\leqslant 1$, $1<p\leqslant 2$, in terms of Littlewood–Paley functions.

THEOREM 3.4. *Let* $0<q\leqslant 1$, $1<p\leqslant 2$, *and* $N=[n(1/q-1)]$. *Then*

(a) $f \in HA_{p,q}$ *if and only if* $g(f) \in A_{p,q}$,

(b) $f \in HA_{p,q}$ *if and only if* $S_\alpha(f) \in A_{p,q}$ *where* $\alpha \geqslant 2\sqrt{n}$,

(c) $f \in HA_{p,q}$ *if and only if* $g_\lambda^*(f) \in A_{p,q}$ *where* $\lambda > (2N+3n+1)/n$.

(*The same holds for homogeneous spaces.*)

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