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# On M-Spaces and Banach Spaces

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**Abstract**

We define in this paper the concept of C-space, related with M-spaces and Banach spaces. We obtain various properties on these spaces and propose some open problems.

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**Keywords:** topological spaces, M-spaces, Banach spaces, embedding

## 1 Introduction

There exist three causes that motive this new paper. First, an early theorem of Corson, also the concept of $M$-space (defined by K.Morita), and finally a paper on Banach spaces by the author:

**The Corson Theorem**.[3] *For any covering $\mathcal{U}$ of a infinite dimensional reflexive Banach space $B$, where $\mathcal{U}$ is formed by bounded, convex sets, there is not a point $x$ in $B$ such that each neighborhood of $x$ meets only finitely many members of $\mathcal{U}$, i.e., $\mathcal{U}$ is not locally finite.*

In our paper [6], we study some problems related to the Corson Theorem. In particular, we proved that:"*For every $r \gneqq 0$, there exits an open covering of $c_0$, which is locally finite and is formed by balls of radius $r$*".

We will use in this paper the concept of $M$-space:

**Definition 1.** [7] A paracompact space $X$ is called a $M$-space if there is some perfect map from $X$ onto some metric space.

## 2 Main results.

**Definition 2.** Let $X$ be a topological space. We will say that $X$ is a $C$-space if there is some Banach space $E$ and some perfect map $f$ from $X$ onto $E$ such that exists a locally finite covering of $X$ formed by pre-images of open balls of radius 1 by the map $f$.

**Remarks. 1.** If $X$ is a $C$-space then $X$ is a paracompact $M$-space.
**2.** $c_0$, $E_\infty$, $IR^n$ are $C$-spaces.

**Proposition 1.** *Let $X$ be a topological space, $E$ be a Banach space and $f$ be a perfect map from $X$ onto $E$. Then*
$\mathcal{V} = \{f^{-1}(B_1(x_j))|j \in J\}$ *is a locally finite covering of $X$, if and, only if $\{B_1(x_j)|j \in J\}$is a locally finite covering of $E$.*

**Proof.** ($\Rightarrow$) If $\mathcal{V}$ covers $X$ also $\{B_1(x_j)|j \in J\}$ covers $E$, because $f$ is onto.

For each $z \in E$ and each $x \in f^{-1}(z)$ there exists an open neighborhood $U_z^x$ of $x$, such that meets only finitely members of $\mathcal{V}$. Then $\{U_z^x|x \in f^{-1}(z)\}$ is an open covering of $f^{-1}(z)$, and $f^{-1}(z) \subset \cup_{k=1}^r U_z^{x_k}$ (for some $x_1,\ldots x_r \in f^{-1}(z)$) because $f$ is a perfect map.

Since $f$ is closed, there exists an open neighborhood $W^z$ of $z$ such that $f^{-1}(W^z) \subset \cup_{k=1}^r U_z^{x_k}$.

Then, $f^{-1}(W^z)$ meets only finitely members of $\mathcal{V}$, and also $W^z$ meets only finitely members of$\{B_1(x_j)|j \in J\}$.

($\Leftarrow$) If $\{B_1(x_j)|j \in J\}$ covers $E$, then $\mathcal{V}$ covers $X$.

For each $x \in X$ there exists an open neighborhood $V^{f(x)}$ of $f(x)$ such that meets only finitely members of$\{B_1(x_j)|j \in J\}$.Clearly, $f^{-1}(V^{f(x)})$ is an open neighborhood of $x$ and meets only finitely members of $\mathcal{V}$.

**Corollary 1.** *Let $X$ be a topological space. Then, $X$ is a $C$-space if and only if there exists a Banach space $E$ such that has a locally finite open covering formed by balls of fixed radius, and a perfect map $f$ from $X$ onto $E$.*

**Corollary 2.** *For each compact space $K$, we have that $c_0 \times K$ is a $C$-space.*

**Proof.** Since the projection map $p_1$ is a perfect map from $c_0 \times K$ onto $c_0$.

**Corollary 3.** *For each compact space $K$, we have that $IR^{IN} \times K$ is a $C$-space.*

**Proof.** It follows from the above Corollary, because $c_0$ is homeomorphic to $IR^{IN}$ ( theorems of Kadec [5] and Anderson [1] ).

**Proposition 2.** *Let $X$ be a topological space. If $X$ is separable and $C$-space, then it is homeomorphic to some closed subset of $IR^{IN} \times I^{IN}$.*

**Proof.** Since $X$ is a separable $C$-space, there is some separable Banach space $E$ and some perfect map from $X$ onto $E$. From [8, Theorem 2] it follows that $X$ is homeomorphic to a closed subset in $E \times I^{IN}$. Finally, therems of Kadec [5] and Anderson [1] yield the conclusion.

## 3 Open problems.

1. Let $X$ be a topological space. Have we that $X$ is a $C$-space if and only if $X$ is homeomorphic to a closed subset of $IR^{IN} \times I^{w(X)}$? (where $w(X)$ is the weight of $X$).

2. Have the normed spaces whith locally finite coverings by balls analogous properties to totally bounded spaces?

## 4 References

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