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TESIS DOCTORAL

Generalized Hermit polynomials in the description of Chevishev-like polynomials

(Generalización de polinomios de Hermite en la descripción de polinomios de tipo Chevishev

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Directores

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Memoria para optar al grado de Doctor en Matemáticas.

Presentada por:
Clemente Cesarano

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Clemente Cesarano

Contents

Prefacio

La teoría de funciones especiales multidimensionles o con índices múltiples se ve reflejada en la literatura especializada únicamente para casos muy particulares (véanse, por ejemplo, [1] y las referencias que cita, [2], [3], [4], [5], [6], [7], [8]). La única excepción en este campo es la clase de polinomios de Hermite, que fue introducida desde un principio en su caso general y puede hallarse en el libro clásico de P. Appell y J. Kampé de Fériet [9].

Se ha demostrado ([10]) que los polinomios de Hermite juegan un papel fundamental en la extensión de las funciones especiales clásicas al caso multidimensional o con índices múltiples.

Algunos trabajos en esta dirección apenas dan una idea parcial de las amplias posibilidades que esta teoría permite (véanse, por ejemplo, [10], [11], [12]).

Motivación

Sobre la base de las consideraciones anteriores, el presente trabajo se presenta en este marco y está dedicado a derivar las principales propiedades de los polinomios de Hermite con índices múltiples o multidimensionales que se establecen usando los polinomios generalizados de Hermite como herramienta.

A partir de los polinomios de Hermite, ha sido posible obtener algunas extensiones de algunos conjuntos especiales de funciones incluyendo: las funciones de Bessel [12], los polinomios de Bernoulli [13], de Dickson [14], de Appell [15] , de Laguerre [16], y algunas nuevas familias de polinomios, denominadas *híbridas*, por verificar propiedades que son típicas tanto de los polinomios de Hermite como de los de Laguerre [17].

Objetivos

En este trabajo establecemos cómo es posible, partiendo de los polinomios con índices múltiples de Hermite, introducir los polinomios de Chebyshev de tipo multidimensional de primera y segunda especie y algunas de sus generalizaciones.
En particular, introducimos los polinomios de Chebyshev con índices múltiples o multidimensionales mediante una transformación integral adecuada, a través de un enfoque simbólico de la transformada de Laplace.
Primero, presentamos en la introducción algunas técnicas operativas importantes que permiten comprender mejor los conceptos y el formalismo correspondiente que hemos utilizado para obtener muchas de las relaciones que aparecen en los capítulos siguientes.

Resultados

El primer capítulo está dedicado a una presentación general de las familias de los polinomios de Hermite en una y dos variables. Si bien el tema no es novedoso y puede encontrarse en el libro antes citado de P. Appell y J. Kampé de Fériet [9], nuestro enfoque es bastante distinto y lo basamos fundamentalmente en funciones generadoras y reglas operativas. En nuestra opinión este método es muy flexible y permite obtener de manera sencilla las principales relaciones que a menudo se obtienen con más dificultad usando los medios tradicionales.
Hemos enunciado explícitamente los resultados generales para el caso de polinomios en dos variables, pero los procedimientos relevantes pueden extenderse fácilmente a varias variables. Más aún: presentamos algunas aplicaciones interesantes de polinomios de Hermite a las funciones de Bessel, como una descripción de varias propiedades de funciones de Bessel de dos variables mediante polinomios de Hermite en dos variables. En el último apartado tratamos concisamente la potente herramienta conocida como Principio de Monomialidad, aplicada con los polinomios de Hermite ([18], [19], [20], [21],

[22], [23], [24], [25]).

En el segundo capítulo extendemos la teoría de polinomios generalizados de Hermite al caso con índices múltiples. En particular presentamos los polinomios *vectoriales* de Hermite en dos variables y con dos índices y deducimos muchas propiedades interesantes que muestran el paralelismo con el caso ordinario ([22], [23], [26], [27], [28], [29], [30], [31], [32], [33], [34]).

En el capítulo tres discutimos una aplicación importante de los polinomios de Hermite con índices múltiples. De hecho este capítulo está dedicado a la teoría de funciones de Hermite bi-ortogonales, que supone una herramienta fundamental para describir funciones del oscilador armónico ([10], [11], [22], [35]).

En el cuarto capítulo presentamos la teoría clásica de los polinomios de Chebyshev. Empezamos definiendo una familia de polinomios complejos que incluyen ambos polinomios de Chebyshev clásicos de primera y de segunda especie, relacionados con las partes real e imaginaria. Este enfoque es original y permite derivar muchas de las funciones generatrices. Las relaciones entre las dos especies de familias de Chebyshev son esencialemente recientes ([36], [37], [38]).

En el quinto capítulo presentamos algunos resultados novedosos en relación con los polinomios de Chebyshev en varias variables y con índices múltiples. El capítulo tiene dos partes: en la primera introducimos los polinomios de Chebyshev de segunda especie en dos variables y muchas de sus propiedades, como la representación integral y la función generatriz. Mostramos unas relaciones de recurrencia y obtenemos unas ecuaciones en derivadas parciales y una conexión con los polinomios de Hermite mediante la acción de una función Gama de Euler sobre un operador diferencial. La segunda parte extiende los resultados mencionados a una clase más general de polinomios de tipo Chebyshev relacionados con los polinomios de Gould-Hopper Hermite ([2], [39], [40], [41], [42], [43]).

En el sexto y último capítulo extendemos los resultados anteriores para incluir los polinomios de la familia de Hermite considerados por A. Wünsche y, usando exponenciales truncadas de polinomios, obtener generalizaciones adicionales para los polinomios de tipo Chebyshev ([40], [42], [44], [45], [46]). Es importante señalar que la introducción de la mayor parte de las funciones antes descritas viene motivada por su uso en la resolución explícita de problemas físicos. Los polinomios con índices múltiples de Hermite se usan para estudiar: la distribución de campos de radiación coherente (y no coherente) en Óptica Cuántica, los sistemas acoplados multidimensionales en problemas de radiación electromagnética, los fenómenos relevantes en la propagación de ondas. Los polinomios de Laguerre de órdenes superiores se usan en el calculo de momentos de radiaciones caóticas y las funciones multidimensionales de Bessel en el estudio de la teoría de láseres. Por último, aunque no menos importante, los polinomios de Chebyshev se aplican usualmente en teoría de la aproximación.

Publicaciones

Este trabajo fue comenzado en octubre de 1999 en el Departamento de Matemáticas de la Universidad de Ulm (Alemania) en colaboración con la Unidad de Física Teórica del centro de investigación ENEA Frascati (Italia), bajo la dirección del profesor Werner Balser (Universidad de Ulm) y el doctor Giuseppe Dattoli (ENEA). Posterirormente, la tesis ha sido desarrollada en la Universidad Complutense de Madrid y conjuntamente en la Universidad Politécnica de Madrid bajo la dirección del profesor Luis Vázquez (Complutense) y del profesor Salvador Jiménez (Politécnica). Esta tesis trata principalmente de las aplicacionese de los polinomios de Hermite para el estudio de los polinomios de Chebyshev y, en particular, de las familias generalizadas de polinomios de Hermite que se usan para obtener significativas representaciones integrales para los polinomios de Chebyshev ordinarios así como los multidimensionales.

A lo largo de esta década se han obtenido numerosas publicaciones sobre polinomios de Hermite en relación tanto con los aspectos de sus técnicas operacionales como con sus aplicaciones a la descripción de las representaciones integrales de los polinomios de Chebyshev. A continuación sigue una relación de mis contribuciones en diferentes trabajos publicados. Está ordenada según los temas tratados en cada parte de esta tesis.

Prefacio e introducción

En el prefacio y en la introducción que le sigue, presentamos muchas de las propiedades que satisfacen los polinomios de Hermite y la posibilidad de deducir algunas extensiones para muchas familias de funciones especiales y de polinomios ortogonales. Específicamente, los resultados obtenidos se refieren a los polinomios de Bernoulli, Dickson, Appell, Laguerre y Bernestein, y a varias familias nuevas generadas por polinomios de Hermite y de Laguerre, llamadas de polinomios híbridos.

I G. Dattoli, S. Lorenzutta and C. Cesarano, *Finite Sums and Generalized Forms of Bernoulli Polynomials*, Rend. Mat., Serie VII, **19** (1999), 385–391.

II G. Dattoli, P.E. Ricci and C. Cesarano, *A Note on Multi-index Polynomials of Dickson Type and their Applications in Quantum Optics*, J. Comput. Appl. Math., **145** (2002), 417–424.

III G. Dattoli, S. Lorenzutta, C. Cesarano and P.E. Ricci, *Second level exponentials and families of Appell polynomials*, Int. Transf. Spec. Funct., **13** (2002), 521–527.

IV G. Dattoli, H.M. Srivastava and C. Cesarano, *On a New Family of Laguerre Polynomials*, Accad. Sc. di Torino, Atti Sc. Fis., **132** (2000), 223–230.

V G. Dattoli, S. Lorenzutta, C. Cesarano, *Bernestein polynomials and operational methods*, J. Comp. Anal. Appl., **8** (2006), 369–377.

VI G. Dattoli, S. Lorenzutta, P.E. Ricci and C. Cesarano, *On a Family of Hybrid Polynomials*, Integral Transforms and Special Functions, **15** (2004), 485–490.

Capítulo I

En este primer capítulo estudiamos los polinomios generalizados de Hermite y varias propiedades relacionadas de utilidad. También de deducimos muchas identidades relevantes, utilizando un enfoque de tipo operatorio. Aparecen en las siguientes publicaciones.

I G. Dattoli, S. Lorenzutta and C. Cesarano, *Generalized polynomials and new families of generating functions*, Annali dellUniversit di Ferrara, Sez. VII Sc. Mat., **XLVII** (2001), 57–61.

II C. Cesarano, *Hermite polynomials and some generalizations on the heat equations*, Int. J. of Systems Applications, Engineering & Development, **8** (2014), 193–197.

También presenta algunos resultados que muestran la relación entre las funciones de Bessel y los polinomios de Hermite y que han sido publicados aquí:

III C. Cesarano and D. Assante, *A note on generalized Bessel functions*, Int. J. of Mathematical Models and Methods in Applkied Sciences, **7** (2013), 625–629.

Finalmente, queremos señalar cómo el intrumento de monomialidad supone una ayuda eficaz para el estudio de los polinomios de Hermite, como se muestra en las siguientes publicaiones:

IV C. Cesarano, *Monomiality Principle and related operational techniques for Orthogonal Polynomials and Special Functions*, Int. J. of Pure Mathematics, **1** (2014), 1–7.

V C. Cesarano, *Operational techniques for the solution of interpolation problems in applied mathematics and economics*, on *Recent Researches in Applied Economics and Management*, WSEAS Press, **1** (2013), 475–479.

Capítulo II

En el segundo capítulo extendemos las clases de polinomios de Hermite generalizados y discutimos los casos con índices múltiples. En particular, describimos con gran detalle los llamados polinomios de Hermite vectoriales (con dos índices y dos variables). Muchos de los resultados presentados han aparecido en las siguientes publicaciones.

I G. Dattoli, A. Torre, S. Lorenzutta and C. Cesarano, *Generalized polynomials and operatorial identities*, Accad. Sc. di Torino Atti Sc. Fis., **132** (2000), 231–249.

II G. Dattoli, P.E. Ricci and C. Cesarano, *The Bessel functions and the Hermite polynomials from a unified point of view*, Applicable Analysis, **80** (2001), 379–384.

III C. Cesarano, *A note on generalized Hermite polynomials*, Int. J. of applied Math. and Informatics, **8** (2014), 1–6.

IV C. Cesarano, G.M. Cennamo and L. Placidi, *Humbert Polynomials and Functions in Terms of Hermite Polynomials Towards Applications to Wave Propagation*, WSEAS Transactions on Mathematics, **13** (2014), 595–602.

V G. Dattoli, C. Cesarano, P.E. Ricci and L. Vazquez, *Fractional derivatives: integral representations and generalized polynomials*, J. Concrete and Applicable Mathematics, **2** (2004), 59–66.

VI G. Dattoli, C. Cesarano, P.E. Ricci and L. Vazquez, *Special Polynomials and Fractional Calculus*, Math. & Comput. Modelling, **37** (2003), 729–733.

Capítulo III

El tercer capítulo presenta las aplicaciones de los polinomios de Hermite para describir con detalle el concepto de bi-ortogonalidad relativa a las funciones de Hermite. Los resultados de este capítulo están en las siguientes publicaciones.

I C. Cesarano, *Humbert polynomials and functions in terms of Hermite polynomials*, on *Recent Advances in Mathematics, Statistics and Economics*, Venice, Italy, March 15-17, 2014, 28–33.

II C. Cesarano, *Operational methods for Hermite polynomials*, on *Recent Advances in Mathematics, Statistics and Economics*, Venice, Italy, March 15-17, 2014, 57–61.

III C. Cesarano, C. Fornaro and L. Vazquez, *Operational results in bi-orthogonal Hermite functions*, Acta Mathematica Uni. Comenianae, presentado para su publicación (2014).

Capítulo IV

En el capítulo IV se presentan los polinomios de Chebyshev. Aparte de la descripción de la teoría clásica, demostramos importantes identidades relativas a las representaciones integrales y a la definición de algunos casos especiales de los polinomios de Chebyshev generalizados. Los resultados más interesantes se reflejan en las siguientes publicaciones.

I G. Dattoli, D. Sacchetti and C. Cesarano, *A note on Chebyshev polynomials*, Annali dell'Università di Ferrara, Sez. VII Sc. Mat., **XLVII** (2001), 107–115.

II C. Cesarano, *Indentities and generating functions on Chebyshev polynomials*, Georgian Math. J., **19** (2012), 427–440.

III C. Cesarano and C. Fornaro, *Operational Identities on Generalized Two-Variable Chebyshev Polynomials*, International Journal of Pure and Applied Mathematics, **100** (2015), 59–74.

Capítulo V

El quinto capítulo presnta generalizaciones interesantes sobre los polinomios de Chebyshev en dos variables y analiza las representaciones integrales relacionadas. Los siguientes artículos han presentado los resultados descritos.

I G. Dattoli, C. Cesarano and S. Lorenzutta, *From Hermite to Humbert Polynomials*, Rend. Ist. Mat. Univ. Trieste, **XXXV** (2003), 37–48.

II C. Cesarano, *Generalized Chebyshev polynomials*, Hacettepe Journal of Mathematics and Statistics, **43** (2014), 731–740.

III C. Cesarano, *Generalizations of two-variable Chebyshev and Gegenbauer polynomials*, Int. J. of Applied Mathematics & Statistics (IJAMAS), **53** (2015), 1–7.

Capítulo VI

Unas representaciones adicionales para los polinomios generalizados de Chebyshev (en una y en dos variables) se describen en el sexto y último capítulo, por medio de nuevas clases de polinomios y de polinomios de Hermite truncados, publicados en los siguientes artículos.

I G. Dattoli and C. Cesarano, *On a new family of Hermite polynomials associated to parabolic cylinder functions*, Applied Mathematics and Computation, **141** (2003), 143–149.

II G. Dattoli, C. Cesarano and D. Sacchetti, *A note on truncated polynomials*, Appl. Math. and Comput., **134** (2003), 595–605.

Preface

The theory of multidimensional or multi-index special functions can be found in literature only in very particular cases (see e.g. [1] and the references therein, [2], [3], [4], [5], [6], [7], [8]). The only exception in this field, is the class of Hermite polynomials, which was introduced from the beginning in the general case and can be found in a classical book of P. Appell and J. Kampé de Fériet [9].

It has also been showed [10] that the Hermite polynomials play a fundamental role in the extension of the classical special functions to the multidimensional or multi-index case.

Some works in this direction give only a partial idea of the wide scenario opened by this theory (see e.g. [10], [11], [12]).

Motivation

On the basis of previous considerations the present dissertation is to be considered in the above mentioned framework, and is devoted to the derivation of the main properties of the multi-index or multi-dimensional Chebyshev polynomials, by using the generalized Hermite polynomials as tool.

Starting from the Hermite polynomials it has already been possible to obtain some extensions of some classical special sets of functions, including: the Bessel functions [12], the Bernoulli [13], Dickson [14], Appell [15] , Laguerre [16] polynomials, and some new families of polynomials, called *hybrid*, since they verify properties which are typical both of the Hermite and the Laguerre polynomials [17] .

Objectives

In this dissertation we show that, starting from the multi-index Hermite polynomials, it is possible to introduce the Chebyshev polynomials of multi-dimensional type of first and second kind, and some of their generalizations. In particular, the multi-index or multi-dimensional Chebyshev polynomials are introduced by using a suitable integral transform, via a symbolic approach to the Laplace transform.
We firstly present in the introduction some important operational techniques, to better understand the concepts and the related formalism that we use to derive many of the relations involved in the following chapters.

Results

The first chapter is devoted to a general presentation of the families of Hermite polynomials of the one and two variables. The subject is not new, since it can be found in the above mentioned book of P. Appell and J. Kampé de Fériet [9], but our approach is quite different, being substantially based on generating functions and operatorial rules. In our opinion this method is very flexible and permits in a simple way the derivation of the principal relations which sometimes can be hardly achieved by using traditional means.
General results are explicitly stated mainly in the case of the two variable polynomials, but the relevant procedures could be easily extended to several variables. Moreover, we present some interesting applications of Hermite polynomials to the Bessel functions, that is a description of some properties of two-variable Bessel functions in terms of two-variable Hermite polynomials. In the last section, there is a brief discussion on that powerful tool that is recognized as Monomiality Principle in application with the Hermite polynomials ([18], [19], [20], [21], [22], [23], [24], [25]).
In the second chapter we extend the theory of generalized Hermite polynomials to the multi-index case. In particular we present the *vectorial* Hermite polynomials of two variables and two indexes and we deduce many interest-

ing properties to show the parallelism with the ordinary case ([22], [23], [26], [27], [28], [29], [30], [31], [32], [33], [34]).

In Chapter III, we discuss an important application of the multi-index Hermite polynomials. In fact, this chapter is devoted to the theory of bi-orthogonal Hermite functions which represent a fundamental tool in the description of harmonic oscillator functions ([10], [11], [22], [35]).

The fourth chapter presents the classical theory of Chebyshev polynomials starting from the definition of a family of complex polynomials including both the first and second kind classical Chebyshev ones, which are related to its real and imaginary part. This point of view is original and permits to derive a lot of generating functions and relations between the two kinds Chebyshev families which are essentially recent ([36], [37], [38]).

In the fifth chapter some new results related to the multivariables and multi-index Chebyshev polynomials are presented. This chapter contains two sections. In the first one, the two-variable second kind Chebyshev polynomials are introduced and many properties, such as the integral representation, generating function, recurrence relations and partial differential equation are derived and a connection with Hermite polynomials through the action of Gamma function on a differential operator is shown. The second section extends the above results to a more general class of Chebyshev-like polynomials related to the Gould-Hopper Hermite polynomials ([2], [39], [40], [41], [42], [43])

In the sixth and last chapter the results of the preceding ones are extended in order to include the polynomials of the Hermite family considered by A. Wünsche and by using the truncated exponential polynomials to obtain further generalizations for the Chebyshev-like polynomials ([40], [42], [44], [45], [46]).

We want to point out that the introduction of the most part of the above functions was motivated by their use in the explicit solution of physical problems, as the multi-index Hermite polynomials are used in order to study: the distribution of coherent (or not coherent) radiation fields in quantum optics, the multidimensional coupled systems for electromagnetic radiation problems, the relevant wave propagation phenomena, the higher order Laguerre polynomials are used for the computation of moments of chaotic radiations and the multidimensional Bessel functions was used in the study of lasers theory. Last but not least, the Chebyshev polynomials are traditionally applied to the approximation theory.

Publications

This work began in October 1999 at the Department of Mathematics at the University of Ulm (Germany) in collaboration with the Unit of Theoretical Physics of the research center ENEA Frascati (Italy) under the supervision of professor Werner Balser (University of Ulm) and doctor Giuseppe Dattoli (ENEA). Consequently, the thesis has been developed at Universidad Complutense de Madrid joint with Universidad Politécnica de Madrid under the supervision of professor Luis Vázquez (Complutense) and professor Salvador Jiménez (Politécnica). This thesis discusses mainly the applications of the Hermite polynomials for the study of Chebyshev polynomials and in particular the generalized families of Hermite polynomials are used to derive interesting integral representations for ordinary and multidimensional Chebyshev polynomials. In the course of this decade numerous publications were obtained on Hermite polynomials, related both to the aspects of their operational techniques and to their applications to the description of the integral representations of the Chebyshev polynomials.

Introduction

In order to better understand the properties and the related operational relations that we will discuss in later chapters, it is appropriate to highlight some important operational techniques that involve exponential operators. This introduction consists of two sections: the first will present the properties of translation related to the exponential operators, while the second will show some relevant identities in common use for the study of special functions and orthogonal polynomials.

0.1 Translation operators

In this section we will introduce the formalism and the techniques of the exponential operators; we will consider real functions, which are analytic in a neighborhood of the origin, but it is easy to generalize the properties that we will discuss to the complex case. With such hypothesis the generic function $f(x)$ can be expanded in Taylor series, in particular we can write:

$$f(x+\lambda) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} f^{(n)}(x) \tag{0.1.1}$$

where λ is a continuous parameter.
We start to discuss the so called *shift* or *translation* operator $e^{\lambda \frac{d}{dx}}$, where again λ is a continuous parameter; its action on a function $f(x)$, analytic in a neighborhood of the origin, produces a shift of the variable x by the parameter λ:

$$e^{\lambda \frac{d}{dx}} f(x) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \frac{d^n}{dx^n} f(x) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} f^{(n)}(x) \tag{0.1.2}$$

and then from (0.1.1), gives:

$$e^{\lambda \frac{d}{dx}} f(x) = f(x + \lambda). \tag{0.1.3}$$

Proposition 0.1

Let x be a real variable and $f(x)$ a real function, analytic in a neighborhood of the origin. Then the following identities hold:

$$e^{\lambda x \frac{d}{dx}} f(x) = f\left(e^{\lambda} x\right) \tag{0.1.4}$$

where λ is a continuous parameter,

$$e^{\lambda x^2 \frac{d}{dx}} f(x) = f\left(\frac{x}{1 - \lambda x}\right) \tag{0.1.5}$$

where λ is a continuous parameter and $|x| < \frac{1}{|\lambda|}$,

$$e^{\lambda x^n \frac{d}{dx}} f(x) = f\left(\frac{x}{\sqrt[n-1]{1 - (n-1)\lambda x^{n-1}}}\right) \tag{0.1.6}$$

where $|x| < \left(\sqrt[n-1]{\frac{1}{(n-1)|\lambda|}}\right)$ and λ continuous parameter.

Proof

By setting $x = e^{\theta}$, where θ is a real variable, we note that:

$$\frac{d}{d\theta} = \frac{d}{dx}\frac{dx}{d\theta} = e^{\theta}\frac{d}{dx} = x\frac{d}{dx}.$$

By applying the operator $e^{\lambda x \frac{d}{dx}}$ on the function $f(x)$ and by using the above relation, we obtain:

$$e^{\lambda x \frac{d}{dx}} f(x) = e^{\lambda \frac{d}{d\theta}} f\left(e^{\theta}\right)$$

and from the relation stated in equation (0.1.3), we get:

$$e^{\lambda \frac{d}{d\theta}} f\left(e^{\theta}\right) = f\left(e^{\lambda+\theta}\right) = f\left(e^{\lambda} x\right)$$

which gives equation (0.1.4).

To derive the second identity of the statement, we set:

$$x = -\frac{1}{\xi}$$

where ξ is a real variable non equal to zero; since:

$$\frac{d}{d\xi} = \frac{d}{dx}\frac{dx}{d\xi} = \frac{1}{\xi^2}\frac{d}{dx} = x^2\frac{d}{dx}$$

we find:

$$e^{\lambda x^2 \frac{d}{dx}} f(x) = e^{\lambda \frac{d}{d\xi}} f\left(-\frac{1}{\xi}\right) = f\left(-\frac{1}{\xi+\lambda}\right)$$

after using the relation (0.1.3). By exploiting the r.h.s. of the above relation, we have:

$$e^{\lambda x^2 \frac{d}{dx}} f(x) = f\left(\frac{1}{\frac{1}{x}-\lambda}\right)$$

and then equation (0.1.5) immediately follows when:

$$|x| < \frac{1}{|\lambda|}$$

to guarantee the analyticity of the function.

The last relation can be obtained to follow the same procedure outlined above. By setting:

$$x = \left(\sqrt[n-1]{\frac{1}{\xi}}\right)$$

where ξ is a real variable, $\xi \neq 0$, and by using again the identity (0.1.3), we easily state equation (0.1.6) with its restriction to guarantee the analyticity of $f(x)$.

To generalize the action of the shift operator we look at the operator of the form:

$$e^{\lambda q(x) \frac{d}{dx}} \tag{0.1.7}$$

where λ is a continuous parameter and the function $q(x)$ must satisfy some properties.

Proposition 0.2

Let x be a real variable, λ also a real parameter and let a function $f(x)$ analytic in a neighborhood of the origin. Then the follow relation holds:

$$e^{\lambda q(x) \frac{d}{dx}} f(x) = f\left(\varphi\left(\varphi^{-1}(x)+\lambda\right)\right) \tag{0.1.8}$$

where $\varphi(\theta)$ is a real function, which is invertible in a neighborhood of the origin and satisfies the identity:

$$\varphi^{'}(\theta) = q\left(\varphi(\theta)\right). \tag{0.1.9}$$

Proof

By choosing the follow change of variables:

$$x = \varphi(\theta)$$

and by noting that $\varphi(\theta)$ satisfies the (0.1.9), we get:

$$\frac{d\theta}{dx} = \frac{1}{q(x)}$$

that is:

$$q(x) = \frac{dx}{d\theta}$$

and finally:

$$q(x)\frac{d}{dx} = \frac{d}{d\theta}.$$

We can now calculate the action of the shift operator. We have:

$$e^{\lambda q(x)\frac{d}{dx}} f(x) = e^{\lambda \frac{d}{d\theta}} f\left(\varphi(\theta)\right)$$

and by using the (0.1.3), we write:

$$e^{\lambda q(x)\frac{d}{dx}} f(x) = f\left(\varphi\left(\theta + \lambda\right)\right). \tag{0.1.10}$$

By noting that the function $\varphi(\theta)$ is invertible in a neighborhood of the origin and by indicating with $\varphi^{-1}(x) = \theta$ its inverse, we find:

$$f\left(\varphi\left(\theta + \lambda\right)\right) = f\left(\varphi\left(\varphi^{-1}(x) + \lambda\right)\right)$$

that is the thesis.

This result can be used to define a more complicated shift operator. We have in fact:

Definition 0.1

Let x be a real variable, λ also a real parameter and let the functions $v(x)$ and $q(x)$. We define the operator:

$$E(x;\lambda) := e^{\lambda\left(v(x)+q(x)\frac{d}{dx}\right)} \tag{0.1.11}$$

such that:

$$e^{\lambda\left(v(x)+q(x)\frac{d}{dx}\right)}x = x(\lambda)g(\lambda). \tag{0.1.12}$$

The function $x(\lambda)$ and $g(\lambda)$ must satisfy the follow system of first order differential equations:

$$\begin{cases} \frac{d}{d\lambda}x(\lambda) = q(x(\lambda)), & x(0) = x_0 \\ \frac{d}{d\lambda}g(\lambda) = v(x(\lambda))g(\lambda), & g(0) = 1 \end{cases} \tag{0.1.13}$$

The identity (0.1.11) can be generalized to obtain the formula:

$$e^{\lambda\left(v(x)+q(x)\frac{d}{dx}\right)}x^n = (x(\lambda))^n g(\lambda) \tag{0.1.14}$$

and then, we can state the general statement:

$$e^{\lambda\left(v(x)+q(x)\frac{d}{dx}\right)}f(x) = f(x(\lambda))g(\lambda) \tag{0.1.15}$$

where $f(x)$ is a real function analytic in the origin.

0.2 Disentangling rules

In this section we will present some useful result regarding the rules and the properties satisfy by the exponential operators. We firstly note that, in general, the exponential of two operators $\widehat{A}$ and $\widehat{B}$ does not satisfy the identity:

$$e^{\widehat{A}+\widehat{B}} = e^{\widehat{A}}e^{\widehat{B}}$$

as to the scalar case. There are many results which allow to calculate the *compensation* between the first and second member of the above relation, by using the value of the commutator of the operators:

$$\left[\widehat{A},\widehat{B}\right] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}.$$

Theorem 0.1

Let x be a real variable, λ also a real parameter and let a function $f(x)$ analytic in a neighborhood of the origin. Then, the following relation holds:

$$e^{\lambda\left(x+\frac{d}{dx}\right)} = e^{\frac{\lambda^2}{2}} e^{\lambda x} e^{\lambda \frac{d}{dx}}. \tag{0.2.1}$$

Proof

By using the action of the operator $E(x;\lambda)$ stated in the previous section and in particular by using the identity (0.1.11), we have:

$$e^{\lambda\left(x+\frac{d}{dx}\right)} f(x) = f(x(\lambda))g(\lambda) \tag{0.2.2}$$

and then the system (0.1.13) reads:

$$\begin{cases} \frac{d}{d\lambda}x(\lambda) = 1 \\ x(0) = x \end{cases} \tag{0.2.3}$$

where $q(x) = 1$ and $v(x) = x$. By solving the system (0.2.3) we find:

$$x(\lambda) = \lambda + x$$

and then:

$$\frac{d}{d\lambda}g(\lambda) = (\lambda + x)g(\lambda),$$

$$\log g(\lambda) = \frac{\lambda^2}{2} + \lambda x,$$

which gives:

$$g(\lambda) = e^{\frac{\lambda^2}{2}+\lambda x} = e^{\frac{\lambda^2}{2}} e^{\lambda x}. \tag{0.2.4}$$

By substituting the above result in the (0.2.2), we finally get:

$$e^{\lambda\left(x+\frac{d}{dx}\right)} f(x) = e^{\frac{\lambda^2}{2}} e^{\lambda x} f(x+\lambda) = e^{\frac{\lambda^2}{2}} e^{\lambda x} e^{\lambda \frac{d}{dx}}$$

which is the thesis.

The above result can be generalized given the fundamental statement:

Theorem 0.2

Let $\widehat{A}$ and $\widehat{B}$ two generic operators such that:

$$\left[\widehat{A}, \widehat{B}\right] = k,$$

$$\left[k,\widehat{A}\right]=\left[k,\widehat{B}\right]=0,$$

where k is the commutator, usually a real number.

Then they satisfy the identity:

$$e^{\widehat{A}+\widehat{B}}=e^{-\frac{k}{2}}e^{\widehat{A}}e^{\widehat{B}}. \tag{0.2.5}$$

From the results stated in the previous section it is also possible to derive another important relation for the exponential operators.

Theorem 0.3

Let x a real variable, λ also a real parameter and let a function $f(x)$ analytic in a neighborhood of the origin. Then the follow relation holds:

$$e^{\lambda\left(x+x\frac{d}{dx}\right)}f(x)=e^{x\left(e^{\lambda}-1\right)}e^{\lambda x\frac{d}{dx}}f(x). \tag{0.2.6}$$

Proof

Let the following operators:

$$\begin{aligned}\widehat{A}&=\lambda x\\ \widehat{B}&=\lambda x\frac{d}{dx}\end{aligned} \tag{0.2.7}$$

which give:

$$\left[\widehat{A},\widehat{B}\right]=-\lambda^2x=-\lambda\widehat{A}.$$

We consider the following exponential operator:

$$e^{\lambda\left(x+x\frac{d}{dx}\right)} \tag{0.2.8}$$

and after setting $q(x)=x$ and $v(x)=x$, we obtain in the system (0.1.13):

$$\begin{cases}\frac{d}{d\lambda}x(\lambda)=x(\lambda), \qquad x(0)=x_0\\ \frac{d}{d\lambda}g(\lambda)=x(\lambda)g(\lambda), \qquad g(0)=1\end{cases} \tag{0.2.9}$$

From the first equation of the above system, we get:

$$x(\lambda)=xe^{\lambda}$$

and then the second equation gives:

$$\begin{aligned} \frac{d}{d\lambda} g(\lambda) &= x e^{\lambda} g(\lambda) \\ g(0) &= 1 \end{aligned}$$

finally:

$$g(\lambda) = e^{x e^{\lambda} - x} = e^{x\left(e^{\lambda}-1\right)}. \tag{0.2.10}$$

By using the previous results, we can write the operator in equation (0.2.8) in the form:

$$e^{\lambda\left(x + x\frac{d}{dx}\right)} f(x) = e^{x\left(e^{\lambda}-1\right)} f\left(x e^{\lambda}\right)$$

and from the (0.1.4), we obtain:

$$e^{\lambda\left(x + x\frac{d}{dx}\right)} f(x) = e^{x\left(e^{\lambda}-1\right)} e^{\lambda x \frac{d}{dx}} f(x).$$

This result can be also generalized. In fact the operational identity of the above theorem can be written as:

$$e^{\lambda\left(x + x\frac{d}{dx}\right)} = e^{x\left(e^{\lambda}-1\right)} e^{\lambda x \frac{d}{dx}}. \tag{0.2.11}$$

Theorem 0.4

Let $\widehat{A}$ and $\widehat{B}$ be two generic operators such that:

$$\left[\widehat{A}, \widehat{B}\right] = -\lambda \widehat{A}.$$

where λ is a continuous parameter, such that $\lambda \neq 0$.
Then the *Sack Identity* holds [10]:

$$e^{\widehat{A}+\widehat{B}} = e^{\frac{e^{\lambda}-1}{\lambda}\widehat{A}} e^{\widehat{B}}. \tag{0.2.12}$$

The proof of relation (0.2.12) is an immediate consequence of the previous theorem; it is in fact enough to note that:

$$x = \frac{1}{\lambda}\widehat{A}$$

in relation (0.2.11).

Theorem 0.5

Let $\widehat{A}$ and $\widehat{B}$ be two generic operators and let λ a continuous parameter. Suppose that the operators $\widehat{A}$ and $\widehat{B}$ are independent to λ; then the following *Hausdorff Identity* holds:

$$e^{\lambda\widehat{A}}\widehat{B}e^{-\lambda\widehat{A}} = \widehat{B} + \lambda\left[\widehat{A},\widehat{B}\right] + \frac{\lambda^2}{2!}\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right] + \frac{\lambda^3}{3!}\left[\widehat{A},\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right]\right] + ... \tag{0.2.13}$$

Proof

We first note that the operators:

$$\widehat{A} \qquad \text{and} \qquad e^{\lambda\widehat{A}}$$

commute, since the operator $e^{\lambda\widehat{A}}$ can be written in terms of powers of $\widehat{A}$. By writing the Taylor series of the l.h.s. of equation (0.2.13) with initial point $\lambda = 0$, we get:

$$e^{\lambda\widehat{A}}\widehat{B}e^{-\lambda\widehat{A}} = \sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\frac{d^k}{d\lambda^k}\left(e^{\lambda\widehat{A}}\widehat{B}e^{-\lambda\widehat{A}}\right)_{\lambda=0}. \tag{0.2.14}$$

It is easy to note that:

$$\left(e^{\lambda\widehat{A}}\widehat{B}e^{-\lambda\widehat{A}}\right)_{\lambda=0} = \widehat{B}$$

and:

$$\frac{d}{d\lambda}\left(e^{\lambda\widehat{A}}\widehat{B}e^{-\lambda\widehat{A}}\right)_{\lambda=0} = \left(e^{\lambda\widehat{A}}\widehat{A}\widehat{B}e^{-\lambda\widehat{A}} - e^{\lambda\widehat{A}}\widehat{B}\widehat{A}e^{-\lambda\widehat{A}}\right)_{\lambda=0} =$$

$$= \left(e^{\lambda\widehat{A}}\left[\widehat{A},\widehat{B}\right]e^{-\lambda\widehat{A}}\right)_{\lambda=0} = \left[\widehat{A},\widehat{B}\right],$$

$$\frac{d^2}{d\lambda^2}\left(e^{\lambda\widehat{A}}\widehat{B}e^{-\lambda\widehat{A}}\right)_{\lambda=0} = \left(e^{\lambda\widehat{A}}\widehat{A}\left[\widehat{A},\widehat{B}\right]e^{-\lambda\widehat{A}} - e^{\lambda\widehat{A}}\left[\widehat{A},\widehat{B}\right]\widehat{A}e^{-\lambda\widehat{A}}\right)_{\lambda=0} = \left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right].$$

By using the induction it is possible to state the coefficients of the series in the r.h.s of the relation (0.2.13) and then the thesis immediately follows.
It is important to note that the above result is interesting when the operators $\widehat{A}$ and $\widehat{B}$ don't commute; otherwise equation (0.2.13) reduces to the case $\widehat{B} = \widehat{B}$.
The Hausdorff identity can be used in many applications. We can note for example, that $\forall m \in \mathbb{N}$ the follow identity holds:

$$e^{\lambda\frac{d^m}{dx^m}}(1) = \sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\frac{d^{km}}{dx^{km}}(1) = 1, \tag{0.2.15}$$

where x is a real number and λ a continuous parameter. In the case $m = 2$. we have:

$$e^{\lambda \frac{d^2}{dx^2}} x = \left(e^{\lambda \frac{d^2}{dx^2}} x e^{-\lambda \frac{d^2}{dx^2}} \right) e^{\lambda \frac{d^2}{dx^2}} (1) = e^{\lambda \frac{d^2}{dx^2}} x e^{-\lambda \frac{d^2}{dx^2}} (1). \tag{0.2.16}$$

We can use the Hausdorff identity on the operator:

$$e^{\lambda \frac{d^2}{dx^2}} x e^{-\lambda \frac{d^2}{dx^2}}$$

by assuming the following relations:

$$\widehat{A} = \frac{d^2}{dx^2}, \qquad \widehat{B} = x.$$

We obtain, in fact:

$$e^{\lambda \frac{d^2}{dx^2}} x e^{-\lambda \frac{d^2}{dx^2}} = x + 2\lambda \frac{d}{dx} \tag{0.2.17}$$

since:

$$\left[\widehat{A}, \widehat{B}\right] = 2\frac{d}{dx}$$

and:

$$\begin{aligned} \left[\widehat{A}, \left[\widehat{A}, \widehat{B}\right]\right] &= 0 \\ \left[\widehat{A}, \left[\widehat{A}, \left[\widehat{A}, \widehat{B}\right]\right]\right] &= 0 \\ \left[\widehat{A}, \ldots \left[\widehat{A}, \left[\widehat{A}, \widehat{B}\right]\right]\right] &= 0. \end{aligned}$$

The relation (0.2.17) with equation (0.2.16) allow us to state the important relation:

$$e^{\lambda \frac{d^2}{dx^2}} x = \left(x + 2\lambda \frac{d}{dx} \right) (1) = x. \tag{0.2.18}$$

It is easy to generalize the above identity to have:

$$e^{\lambda \frac{d^2}{dx^2}} x^k = \left(x + 2\lambda \frac{d}{dx} \right)^k (1) = x^k. \tag{0.2.19}$$

Moreover, the Hausdorff identity can be used also for a generic function $f(x)$ which is analytic in the origin; in fact, by applying the operator in equation (0.2.15), for $m = 2$, to the function $f(x)$, we have:

$$e^{\lambda \frac{d^2}{dx^2}} f(x) = f\left(x + 2\lambda \frac{d}{dx} \right) (1) \tag{0.2.20}$$

and by choosing $f(x) = e^x$, we obtain, by using the Weyl Identity:

$$e^{\lambda \frac{d^2}{dx^2}} e^x = e^{x+2\lambda \frac{d}{dx}}(1) = e^{\lambda} e^x e^{2\lambda \frac{d}{dx}}(1) = e^{\lambda + x} \tag{0.2.21}$$

since we have:

$$\left[x, 2\lambda \frac{d}{dx}\right] = -2\lambda.$$

We can also note that the relation (0.2.21) can be deduced from the general definition:

$$e^{\lambda \frac{d^m}{dx^m}} e^x = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{d^{km}}{dx^{km}} e^x = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} e^x = e^{\lambda + x} \tag{0.2.22}$$

which holds $\forall m \in \mathbb{N}$.

The relation (0.2.20) can be also seen as a generalization of the following result.

Theorem 0.6

Let $\widehat{A}$ and $\widehat{B}$ two generic operators, such that $\left[\widehat{A}, \widehat{B}\right] = 1$. The following identity holds:

$$e^{\widehat{A}^m} f\left(\widehat{B}\right) = f\left(\widehat{B} + m\widehat{A}^{m-1}\right) e^{\widehat{A}^m} \tag{0.2.23}$$

where the function $f(x)$ is analytic in the origin and $m \in \mathbb{N}$.

By setting $\widehat{A} = \frac{d}{dx}$, $\widehat{B} = x$, in the above relation immediately follows that:

$$e^{\lambda \frac{d^m}{dx^m}} f(x) = f\left(x + m\lambda \frac{d^{m-1}}{dx^{m-1}}\right)(1) \tag{0.2.24}$$

since:

$$\left[x, \frac{d}{dx}\right] = 1. \tag{0.2.25}$$

The equation (0.2.24) is easily recognized as a generalization of the identity (0.2.20). We can also note that equation (0.2.24) can be also written in the form:

$$e^{\lambda \frac{d^m}{dx^m}} f(x) = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} f^{(km)}(x) = f\left(x + m\lambda \frac{d^{m-1}}{dx^{m-1}}\right)(1) \tag{0.2.26}$$

and then, by setting $m = 1$, we immediately obtain the relation in equation (0.1.3).

Chapter I

Generalized two-variable Hermite polynomials

In this first chapter, we will introduce the generalized two-variable Hermite polynomials. We will present different types of these families of Hermite polynomials and we will discuss some interesting operational properties. We introduce the two-variable Hermite polynomials by using the techniques of the translation operator, but we also present them through the method of the generating function. The ordinary Hermite polynomials will be derived as particular case of the generalized two-variable Hermite polynomials. In Section I.2 we will see how the Hermite polynomials represent an useful tool to describe the structure and the relevant properties of the generalized Bessel functions. Finally, in Section I.4, we will introduce the concept of Monomiality Principle and we will show how is possible to derive many relations involving the generalized Hermite polynomials by using its formalism and the related techniques.

I.1 Introduction to the Hermite polynomials

To introduce the ordinary one-variable Hermite polynomials and the related generalized two-variable of the Gould-Hopper type, we can use the formalism and the techniques of the exponential operators. We have seen in (0.1.1) how to represent by Taylor's series an analytic function $f(x)$ adding a parameter λ to the variable: $f(x+\lambda)$. We have also seen in (0.1.2) how to define the shift or translation operator that acts on a value $f(x)$ and gives the shifted value $f(x+\lambda)$. We will limit ourselves to real domain, assuming that λ is a real number and $f(x)$ is also analytic in $x+\lambda$ without any other restriction. The action of the exponential operator on an analytic function $f(x)$ produces a shift of the variable x by λ.

The two-variable Hermite polynomials can be defined by using the relation stated in (0.1.2), after noting that:

$$e^{yD} f(x) = f(x+y) = \sum_{n=0}^{+\infty} \frac{y^n}{n!} f^{(n)}(x) \tag{I.1.1}$$

and then: $f(x) = x^m$ implies $e^{yD} x^m = (x+y)^m$

$f(x) = \sum_{m=0}^{+\infty} a_m x^m$ implies $e^{yD} f(x) = \sum_{m=0}^{+\infty} a_m (x+y)^m$.

The previous procedure can be easily generalized to exponential operators containing higher derivatives. In fact by considering the second derivative, we can generalize the (0.1.2) as follows:

$$e^{yD^2} f(x) = \sum_{n=0}^{+\infty} \frac{y^n}{n!} f^{(2n)}(x) \tag{I.1.2}$$

and by noting that:

$$D^{2n} x^m = \frac{m!}{(m-2n)!} x^{m-2n} \tag{I.1.3}$$

we have:

$$e^{yD^2} x^m = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{y^n}{n!} \frac{m!}{(m-2n)!} x^{m-2n}. \tag{I.1.4}$$

The above identity shows the general action of the exponential operator; we can use it to formally introduce the generalized two-variable Hermite polynomials.

Definition I.1

The two-variable Hermite Polynomials $H_m^{(2)}(x,y)$ of Kampé de Fériet form are defined by the following formula:

$$H_m^{(2)}(x,y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!} y^n x^{m-2n} \tag{I.1.5}$$

It is important to note that, assuming $f(x) = \sum_{m=0}^{+\infty} a_m x^m$, we can obtain from (I.1.2), the identity:

$$e^{yD^2} f(x) = \sum_{m=0}^{+\infty} a_m H_m^{(2)}(x,y). \tag{I.1.6}$$

From the above definition we can state an elementary form of this kind of Hermite polynomials; in fact by the identity (I.1.1), we immediately obtain:

$$H_m^{(1)}(x,y) = (x+y)^m \tag{I.1.7}$$

which can also be recast in the form:

$$e^{yD} f(x) = \sum_{m=0}^{+\infty} a_m H_m^{(1)}(x,y). \tag{I.1.8}$$

In the following we will indicate the two-variable Hermite polynomials of Kampé de Fériet form by using the symbol $He_m(x,y)$ instead than $H_m^{(2)}(x,y)$. The two-variable Hermite polynomials $He_m(x,y)$ are linked to the ordinary Hermite polynomials by the following relations:

$$He_m\left(x, -\frac{1}{2}\right) = He_m(x) \tag{I.1.9}$$

where:

$$He_m(x) = m! \sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^r x^{n-2r}}{r!(n-2r)!2^r} \tag{I.1.10}$$

and

$$He_m(2x,-1) = H_m(x) \tag{I.1.11}$$

where:

$$H_m(x) = m! \sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^r (2x)^{n-2r}}{r!(n-2r)!} \tag{I.1.12}$$

It is also important to note that the Hermite polynomials $He_m(x,y)$ satisfy the relation:

$$He_m(x,0) = x^m. \tag{I.1.13}$$

Proposition I.1

The polynomials $He_m(x,y)$ solve the following partial differential equation:

$$\frac{\partial^2}{\partial x^2}He_m(x,y) = \frac{\partial}{\partial y}He_m(x,y) \tag{I.1.14}$$

Proof

By deriving, separately with respect to x and to y, in (I.1.5), we obtain:

$$\begin{aligned} \frac{\partial}{\partial x}He_m(x,y) &= mHe_{m-1}(x,y) \\ \frac{\partial}{\partial y}He_m(x,y) &= He_{m-2}(x,y). \end{aligned} \tag{I.1.15}$$

From the first of the above relation, by deriving again with respect to x and by noting the second relation, we end up with eq. (I.1.14).

Proposition I.1 helps us to derive an important operational rule for the Hermite polynomials $He_m(x,y)$. In fact, by considering the differential equation (I.1.14) as linear ordinary in the variable y and by reminding the (I.1.13), we can immediately state the following relation:

$$He_m(x,y) = e^{y\frac{\partial^2}{\partial x^2}}x^m. \tag{I.1.16}$$

The generating function of the above Hermite polynomials can be stated in many ways, we have in fact:

Proposition I.2

The polynomials $He_m(x,y)$ satisfy the following differential difference equation:

$$\begin{aligned} \frac{d}{dz}Y_n(z) &= anY_{n-1}(z) + bn(n-1)Y_{n-2}(z) \\ Y_n(0) &= \delta_{n,0} \end{aligned} \tag{I.1.17}$$

where a and b are real numbers.

Proof

By using the generating function method, by putting:

$$G(z;t) = \sum_{n=0}^{+\infty}\frac{t^n}{n!}Y_n(z) \tag{I.1.18}$$

with t continuous variable, we can rewrite the (I.1.17) in the form:

$$\begin{aligned} \frac{d}{dz}G(z;t) &= \left(at+bt^2\right)G(z;t) \\ G(0;t) &= 1 \end{aligned} \tag{I.1.19}$$

that is a linear ordinary differential equation and then its solution reads:

$$G(z;t)=\exp\left(xt+yt^2\right) \tag{I.1.20}$$

where we have put $az=x$ and $bz=y$. Finally, by exploiting the r.h.s of the previous relation we find the thesis and also the relation linking the Hermite polynomials and their generating function:

$$\exp\left(xt+yt^2\right)=\sum_{m=0}^{+\infty}\frac{t^m}{m!}He_m(x,y). \tag{I.1.21}$$

It also could be interesting to explore a different class of generalized two-variable Hermite polynomials. From the differential difference equation exposed in Proposition I.2, we can consider an its slight modification:

$$\begin{aligned} \frac{d}{dz}Y_n(z) &= 2anY_{n-1}(z)-bn(n-1)Y_{n-2}(z) \\ Y_n(0) &= \delta_{n,0} \end{aligned} \tag{I.1.22}$$

(where, again, a and b are real numbers) and by following the same procedure of the generating function method, showed before, we can immediately write:

$$\begin{aligned} \frac{d}{dz}G(z;t) &= \left(2at-bt^2\right)G(z;t) \\ G(0;t) &= 1 \end{aligned} \tag{I.1.23}$$

that is, as in Proposition I.2, a linear differential equation, whose solution reads:

$$G(z;t)=\exp\left(2xt-yt^2\right) \tag{I.1.24}$$

where, again, $az=x$ and $bz=y$. By exploited the r.h.s of the previous relation we can introduce the following generalized two-variable Hermite polynomials:

$$\exp\left(2xt-yt^2\right)=\sum_{m=0}^{+\infty}\frac{t^m}{m!}H_m(x,y). \tag{I.1.25}$$

From the Cauchy problem (I.1.23), by exploiting the terms in the previous relation, we end up to state the explicit form of the polynomial $H_m(x,y)$:

$$H_m(x,y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!}(-y)^n(2x)^{m-2n} \tag{I.1.26}$$

It could be interesting to explore a different class of Hermite polynomials. We can start to observe that the generating function of the polynomials $H_m(x,y)$ can be modified in the form:

$$\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) = \sum_{m=0}^{+\infty} \frac{t^m}{m!} He'_m(x,y) \tag{I.1.27}$$

and then we can define the two-variable Hermite polynomials of the type $He'_m(x,y)$.

The previous relation allows us to write their generating function as follows:

$$\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) = \exp\left(xt + yt^2 - \frac{t^2}{2} - \frac{t^4}{2}\right) \tag{I.1.28}$$

and by exploiting the exponential functions on the r.h.s., we get:

$$\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n\left(\frac{x}{2}, -y\right) \sum_{r=0}^{+\infty} \frac{t^{2r}}{r!} H_r\left(-\frac{1}{4}, \frac{1}{2}\right). \tag{I.1.29}$$

After rearranging the indexes, we can finally state the relation linking the present two classes of Hermite polynomials:

$$He'_m(x,y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!} H_{m-2n}\left(\frac{x}{2}, -y\right) H_n\left(-\frac{1}{4}, \frac{1}{2}\right). \tag{I.1.30}$$

We can now prove useful relations related to the polynomials $H_m(x,y)$.

Proposition I.3

The polynomials $H_m(x,y)$ satisfy the following recurrence relations:

$$\begin{aligned} \frac{\partial}{\partial x} H_m(x,y) &= 2mH_{m-1}(x,y) \\ \frac{\partial}{\partial y} H_m(x,y) &= -m(m-1)H_{m-2}(x,y). \end{aligned} \tag{I.1.31}$$

Proof

Deriving equation (I.1.25) with respect to x, we have:

$$2t\exp\left(2xt - yt^2\right) = \sum_{m=0}^{+\infty} \frac{t^m}{m!}\frac{\partial}{\partial x}H_m(x,y) \tag{I.1.32}$$

and then, applying again the equation (I.1.25), we obtain:

$$2\sum_{m=0}^{+\infty} \frac{t^{m+1}}{n!}H_m(x,y) = \sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{\partial}{\partial x}H_m(x,y) \tag{I.1.33}$$

and this proves the first of the relations in the statement. Using again the relation linking the Hermite polynomials of the type $H_m(x,y)$ and their generating function and deriving with respect to t, we get:

$$(2x - 2yt)\exp\left(2xt - yt^2\right) = \sum_{m=0}^{+\infty} \frac{mt^{m-1}}{m!}H_m(x,y) \tag{I.1.34}$$

then:

$$2x\sum_{m=0}^{+\infty} \frac{t^m}{m!}H_m(x,y) - 2y\sum_{m=0}^{+\infty} \frac{t^{m+1}}{m!}H_m(x,y) = \sum_{m=0}^{+\infty} \frac{mt^{m-1}}{m!}H_m(x,y) \tag{I.1.35}$$

and this proves the second recurrence relation of the proposition.
The proposition shown before could be used to state the analogous result contained in Proposition I.1; we can prove in fact:

Proposition I.4

The polynomials $H_m(x,y)$ solve the following partial differential equation:

$$-\frac{1}{4}\frac{\partial^2}{\partial x^2}H_m(x,y) = \frac{\partial}{\partial y}H_m(x,y) \tag{I.1.36}$$

Proof
From the first recurrence relation contained in Proposition I.3, by deriving again with respect to x, we get:

$$\frac{\partial^2}{\partial x^2}H_m(x,y) = 2m\frac{\partial}{\partial x}H_{m-1}(x,y) \tag{I.1.37}$$

and by applying the second recurrence relation into previous proposition, we write:

$$\frac{\partial^2}{\partial x^2}H_m(x,y) = 4m(m-1)H_{m-2}(x,y) \tag{I.1.38}$$

that is:

$$-\frac{1}{4}\frac{\partial^2}{\partial x^2}H_m(x,y) = -m(m-1)H_{m-2}(x,y) \tag{I.1.39}$$

and after subsituting the second relation stated in Proposition I.3, we end up with the thesis.

It is immediate to note that the differential equation contained in the above proposition could be read as an ordinary linear differential equation in the variable y; then by noting that:

$$H_m(x,0) = (2x)^m \tag{I.1.40}$$

we can conclude that the Hermite polynomials of the form $H_m(x,y)$ satisfy the following operational relation:

$$H_m(x,y) = e^{-\frac{1}{4}\frac{\partial^2}{\partial x^2}}(2x)^m. \tag{I.1.41}$$

This last relation could be used to obtain a different representation of the Hermite polynomials of the form $H_m(x,y)$. In fact by exploiting the exponential on the r.h.s. of (I.1.41), we have:

$$\exp\left(-\frac{y}{4}\frac{\partial^2}{\partial x^2}\right) = \sum_{n=0}^{+\infty}(-1)^n\left(\frac{y}{4}\right)^n\frac{\partial^{2n}}{\partial x^{2n}} \tag{I.1.42}$$

and then, the expression into the (I.1.41), reads:

$$H_m(x,y) = \left[\sum_{n=0}^{+\infty}(-1)^n\left(\frac{y}{4}\right)^n\frac{\partial^{2n}}{\partial x^{2n}}\right](2x)^m. \tag{I.1.43}$$

After observing that the effect of the derivative on $(2x)^m$ is trivial when $2n > m$, we can conclude that:

$$H_m(x,y) = \left[\sum_{n=0}^{[m/2]}(-1)^n\left(\frac{y}{4}\right)^n\frac{\partial^{2n}}{\partial x^{2n}}\right](2x)^m. \tag{I.1.44}$$

In this first section we have introduced the two-variable Hermite polynomials of type $He_m(x,y)$ and $H_m(x,y)$, by discussing their basic properties; we have also stated the link between them. Before approaching the study of the operational identities regarding these families of Hermite polynomials, we want to show an useful relation with the generalized Bessel functions.

I.2 Generalized Hermite polynomials and Bessel functions

In this section we will explore some interesting relations linking the generalized two-variable Hermite polynomials of the type $He_m(x,y)$ and the generalized cylindrical Bessel functions of two variables $J_m(x,y)$. We remind that the generating function of the Bessel function $J_m(x,y)$ is of the form:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right]=\sum_{m=-\infty}^{+\infty} t^m J_m(x,y) \tag{I.2.1}$$

(where t is a continuous parameter) and its explicit form reads:

$$J_m(x,y)=\sum_{m=-\infty}^{+\infty} J_{m-2n}(x)J_n(y) \tag{I.2.2}$$

(with n a natural number) where the ordinary cylindrical Bessel function has the following expression:

$$J_m(x)=\sum_{r=0}^{+\infty}\frac{(-1)^r}{(m+r)!\,r!}\left(\frac{x}{2}\right)^{m+2r}. \tag{I.2.3}$$

We start to observe that the generalized two-variable Bessel function can be expressed in terms of generalized Hermite polynomials of the type $He_m(x,y)$. In fact, by noting that the argument of the exponential in the equation (I.2.1) can be recast in the form:

$$\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)=\frac{1}{2}\left(xt+yt^2\right)-\frac{1}{2}\left(\frac{x}{t}-\frac{y}{t^2}\right) \tag{I.2.4}$$

and, by noting that the generating function of the Hermite polynomials has the following expression (see eq.(I.1.21)):

$$\exp\left(xt+yt^2\right)=\sum_{n=0}^{+\infty}\frac{t^m}{m!}He_m(x,y) \tag{I.2.5}$$

we can write, by using equation (I.2.1):

$$\sum_{m=-\infty}^{+\infty} t^m J_m\left(x,y\right)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}He_n\left(\frac{x}{2},\frac{y}{2}\right)\sum_{r=0}^{+\infty}\frac{t^{-r}}{2!}He_r\left(-\frac{x}{2},-\frac{y}{2}\right). \tag{I.2.6}$$

After setting $m = n - r$, rearranging the indices and equating the terms of same power, we obtain:

$$J_m(x,y) = \sum_{r=0}^{+\infty} \frac{1}{(m+r)!\, r!} He_{m+r}\left(\frac{x}{2}, \frac{y}{2}\right) He_r\left(-\frac{x}{2}, -\frac{y}{2}\right) \tag{I.2.7}$$

for $m \geq 0$.

It is important to note, that the above relation gives a representation of the generalized Bessel function only on the semiaxis $(0, +\infty)$, due to the nature of the Hermite polynomials. Nevertheless, after noting that the generalized Bessel function verify the property:

$$J_{-m}(x,y) = J_m(-x,-y) \tag{I.2.8}$$

we can give an expression of the relation (I.2.7), for the negative integer, in the following way:

$$J_{-m}(x,y) = \sum_{r=0}^{+\infty} \frac{1}{(m+r)!\, r!} He_{m+r}\left(-\frac{x}{2}, -\frac{y}{2}\right) He_r\left(\frac{x}{2}, \frac{y}{2}\right). \tag{I.2.9}$$

The above equation gives a representation of the Bessel Function $J_m(x,y)$ in terms of the Hermite polynomials $He_m(x,y)$ on the real axis $(0, +\infty)$ and then, it also gives (by using relation (I.2.7)) a representation of the Bessel function on the negative axis $(-\infty, 0)$.

We can conclude that the relation contained in the (I.2.9) is the complete expression of the representation of the generalized two-variable Bessel function in terms of the Hermite polynomials of the type $He_m(x,y)$.

It is worth noting that the multiplication and addition theorems related to generalized Bessel functions $J_m(x,y)$ are an important tool to derive many operational indentities involving the family of Bessel functions and, in general, the related differential equations. We can now derive the cited theorems in terms of the Hemite polynomials of the type $He_m(x,y)$. We remind that, the multiplication theorem for the generalized two-variable Bessel function, is given by the formula:

$$J_m(\lambda x, \mu y) = \lambda^n \sum_{p=0}^{+\infty} J_{m+p}\left(x, y; \frac{\mu}{\lambda^2}\right) F_p\left(x, y; \frac{\lambda^2}{\mu}\right) \tag{I.2.10}$$

where $F_p\left(x,y;\frac{\lambda^2}{\mu}\right)=\sum_{v=0}^{[p/2]}\frac{(1-\lambda^2)^{p-2v}(1-\mu^2)}{(p-2v)!\,v!}\left(\frac{x}{2}\right)^{p-2v}\left(\frac{\lambda^2}{\mu}\right)^{v}\left(\frac{y}{2}\right)^{v}$
$\lambda,\mu\in\mathbb{R}-\{-\infty,+\infty\}$.
By manipulating the term $F_p\left(x,y;\frac{\lambda^2}{\mu}\right)$, we get:

$$\sum_{v=0}^{[p/2]}\frac{1}{(p-2v)!\,v!}\left[\frac{x}{2}(1-\lambda^2)\right]^{p-2v}\left[\frac{y}{2}(1-\mu^2)\frac{\lambda^2}{\mu}\right]^{v}=\frac{1}{p!}H_p\left[\frac{x}{2}(1-\lambda^2),\frac{y}{2}(1-\mu^2)\frac{\lambda^2}{\mu}\right] \tag{I.2.11}$$

and finally, we obtain the expression of the multiplication theorem in the form:

$$J_m(\lambda x,\mu y)=\lambda^m\sum_{p=0}^{+\infty}\frac{1}{p!}J_{m+p}\left(x,y;\frac{\mu}{\lambda^2}\right)\,H_m\left[\frac{x}{2}(1-\lambda^2),\frac{y}{2}(1-\mu^2)\frac{\lambda^2}{\mu}\right]. \tag{I.2.12}$$

The addition theorems relevant to Bessel function of the form $J_m(x,y)$ are a generalization of the Neumann and Graf formulae related to the ordinary cylindrical one-variable Bessel function of the first type. In particular, for the Neumann addition theorem, we have the following statement:

$$J_m(x\pm u,y\pm v)=\sum_{r=-\infty}^{+\infty}J_{m-r}(x,y)\,J_{\pm r}(u,v) \tag{I.2.13}$$

by reminding the property:

$$J_r(\pm u,\pm v)=J_{\pm r}(u,v) \tag{I.2.14}$$

which becames from relation (I.2.8). Since the summation in the (I.2.13) run on the index r, we can redefine that index itself on the functions $J_{m-r}(x,y)$ and $J_{\pm r}(u,v)$, to better present the Neumann theorem.
We have, in fact:

$$J_m(x\pm u,y\pm v)=\sum_{r=-\infty}^{+\infty}J_{m\mp r}(x,y)\,J_r(u,v). \tag{I.2.15}$$

By following the same procedure, it is possible to generalize the Graf addition formula as extension of the Neumann addition theorem. By using the Graf formula related to the one-variable Bessel functions, we immediately get:

$$\sum_{r=-\infty}^{+\infty}\xi^r J_{m+r}(x,y)\,J_r(u,v)= \tag{I.2.16}$$

$$=\left(\frac{x-\frac{u}{\xi}}{x-\xi u}\right)^{\frac{m}{2}}J_m\left[w(x,y;\xi),\bar{w}(y,v;\xi^2);\left(\frac{x-\xi u}{x-\frac{u}{\xi}}\right)\left(\frac{y-\frac{v}{\xi^2}}{y-\xi^2 v}\right)^{\frac{1}{2}}\right]$$

where:

$\xi \in \mathbb{R}$, $|\xi| < +\infty$

$w(x,u;\xi) = \left[\left(x - \frac{u}{\xi}\right)(x - \xi u)\right]^{\frac{1}{2}}$

$\bar{w}(y,v;\xi^2) = \left[\left(y - \frac{v}{\xi^2}\right)(y - \xi^2 v)\right]^{\frac{1}{2}}$.

The above relation could be cast in terms of the generalized Hermite polynomials of the type $He_m(x,y)$. We start to introduce the following function, by setting:

$$G_m(x,y,u,v;\xi) = \sum_{r=-\infty}^{+\infty} \xi^r J_{m+r}(x,y)\, J_r(u,v) \tag{I.2.17}$$

where x, y, u, v are real numbers and $\xi \in \mathbb{R}$, $|\xi| < +\infty$ is a parameter. Without prejudicing the generality, it is possible to determine its generating function, by putting:

$$\sum_{m=-\infty}^{+\infty} t^m G_m(x,y,u,v;\xi) = \sum_{m=-\infty}^{+\infty} t^m \sum_{r=-\infty}^{+\infty} \xi^r J_{m+r}(x,y)\, J_r(u,v) \tag{I.2.18}$$

which, once setting $n = m + r$, can be recast in the following form:

$$\sum_{m=-\infty}^{+\infty} t^m G_m(x,y,u,v;\xi) = \sum_{n=-\infty}^{+\infty} t^n J_n(x,y) \sum_{r=-\infty}^{+\infty} \left(\frac{\xi}{t}\right)^r J_r(u,v). \tag{I.2.19}$$

The above relation gives us the expression of the generating functions of the Bessel functions $J_n(x,y)$ and $J_r(u,v)$, by paying attention to consider as parameter the ratio in the second summation. We can now explicit the expressions of the generating functions in the r.h.s of the previous relation, to get:

$$\sum_{m=-\infty}^{+\infty} t^m G_m(x,y,u,v;\xi) = \tag{I.2.20}$$

$$= \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^2 - \frac{1}{t^2}\right) + \frac{u}{2}\left(\frac{\xi}{t} - \frac{t}{\xi}\right) + \frac{v}{2}\left(\frac{\xi^2}{t^2} - \frac{t^2}{\xi^2}\right)\right].$$

The argument of the exponetial on the r.h.s. of the above expression, could be recast in the form:

$$A = \frac{x}{2}t - \frac{x}{2}\frac{1}{t} + \frac{y}{2}t^2 - \frac{y}{2}\frac{1}{t^2} + \frac{u}{2}\frac{\xi}{t} - \frac{u}{2\xi}t + \frac{v\xi^2}{2t^2} - \frac{vt^2}{2\xi^2} \tag{I.2.21}$$

and then:

$$A = \left(\frac{x}{2} - \frac{u}{2\xi}\right)t + \left(\frac{y}{2} - \frac{v}{2\xi^2}\right)t^2 - \left(\frac{x}{2} - \frac{u\xi}{2}\right)\frac{1}{t} - \left(\frac{y}{2} - \frac{v\xi^2}{2}\right)\frac{1}{t^2} \tag{I.2.22}$$

It is evident that the previous relation can be recognized as the argument of the exponential of the generating function for particular generalized Hermite polynomials of the form $He_m(x,y)$ and then, by using (I.2.20), we can write:

$$\sum_{m=-\infty}^{+\infty} t^m G_m(x,y,u,v;\xi) = \tag{I.2.23}$$

$$= \sum_{m=-\infty}^{+\infty} t^m \sum_{r=0}^{+\infty} \frac{1}{(m+r)!\,r!} He_{m+r}\left[\left(\frac{x}{2}-\frac{u}{2\xi}\right),\left(\frac{y}{2}-\frac{v}{2\xi^2}\right)\right] \cdot$$
$$\cdot He_r\left[-\left(\frac{x}{2}-\frac{u\xi}{2}\right),-\left(\frac{y}{2}-\frac{v\xi^2}{2}\right)\right]. \tag{I.2.24}$$

By equating the terms of the same power of m, after substituting the expression of the function $G_m(x,y,u,v;\xi)$ in terms of the Bessel functions (see equation (I.2.18)), we can state the Graf addition formula related to generalized Bessel functions of the form $J_m(x,y)$ in terms of the Hermite polynomials of the type $He_m(x,y)$:

$$\sum_{s=-\infty}^{+\infty} \xi^s J_{m+s}(x,y)\, J_s(u,v) = \tag{I.2.25}$$

$$= \sum_{r=0}^{+\infty} \frac{1}{(m+r)!\,r!} He_{m+r}\left[\left(\frac{x}{2}-\frac{u}{2\xi}\right),\left(\frac{y}{2}-\frac{v}{2\xi^2}\right)\right] \cdot$$
$$\cdot He\left[-\left(\frac{x}{2}-\frac{u\xi}{2}\right),-\left(\frac{y}{2}-\frac{v\xi^2}{2}\right)\right]. \tag{I.2.26}$$

After explored the properties related to the Hermite polynomials of type $He_m(x,y)$ in the description of the addition formulae related to the generalized two-variable Bessel functions, we can deal with the operational rules satisfied by the Hermite polynomials of different types, we have introduced in the first section.

I.3 Operatorial identities for Hermite polynomials

The use of the operational identities may significantly simplify the study of Hermite generating functions and the discovery of new relations, hardly achievable by using conventional means. Before entering in the two-variable case of the Hermite polynomials, we will introduce some identities related to the Hermite polynomials of type $H_n(x)$ (see (I.1.12)) that will be largely exploited in this section.

By remembering the following identity:

$$e^{-\frac{1}{4}\frac{d^2}{dx^2}}(2x)^m = \left(2x - \frac{d}{dx}\right)^m (1) \tag{I.3.1}$$

we can immediately state the following relation.

Proposition I.5

The operational definition of the polynomials $H_n(x)$ reads:

$$e^{-\frac{1}{4}\frac{d^2}{dx^2}}(2x)^m = H_m(x) \tag{I.3.2}$$

Proof

By exploiting the r.h.s of the (I.3.1), we immediately obtain the Burchnall identity:

$$\left(2x - \frac{d}{dx}\right)^n = n!\sum_{s=0}^{n}(-1)^s \frac{1}{(n-s)!s!}H_{n-s}(x)\frac{d^s}{dx^s} \tag{I.3.3}$$

after using the decoupling Weyl identity, since the commutator of the operators of l.h.s. is not zero. The derivative operator of the (I.3.3) gives a non trivial contribution only in the case $s = 0$ and then we can conclude with:

$$\left(2x - \frac{d}{dx}\right)^m (1) = H_m(x) \tag{I.3.4}$$

which proves the statement.

The relation (I.3.2) can be also derived from the explicit form of Hermite polynomials as in the case of the Hermite polynomials of the type $H_m(x, y)$, that has been proved in the first section (see eq. (I.1.41)).

The Burchnall identity can be also inverted to give another important relation for the Hermite polynomials $H_m(x)$; we find in fact:

Proposition I.6

The polynomials $H_m(x)$ satisfy the following operational identity:

$$H_m\left(x + \frac{1}{2}\frac{d}{dx}\right) = \sum_{s=0}^{m}\binom{m}{s}(2x)^{m-s}\frac{d^s}{dx^s}. \tag{I.3.5}$$

Proof

By multiplying the l.h.s. of the above relation by $\frac{t^n}{n!}$ and then summing up, we obtain:

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} H_m \left(x + \frac{1}{2} \frac{d}{dx} \right) = e^{2\left(x+\frac{1}{2}\right)\left(\frac{d}{dx}\right)t - t^2}. \tag{I.3.6}$$

By using the Weyl identity, the r.h.s. of the equation (I.3.6) reads:

$$e^{2\left(x+\frac{1}{2}\right)\left(\frac{d}{dx}\right)t - t^2} = e^{2xt} e^{t\frac{d}{dx}} \tag{I.3.7}$$

and from (I.3.5) the result immediately follows, after expanding the r.h.s and by equating the like $t-$powers.

The previous results can be used to derive some addition and multiplication relations for the Hermite polynomials.

Proposition I.7

The polynomials $H_m(x)$ satisfy the following identity $\forall\ n, m \in \mathbb{N}$:

$$H_{n+m}(x) = \sum_{s=0}^{\min(n,m)} (-2)^s \binom{n}{s} \binom{m}{s} s! H_{n-s}(x) H_{m-s}(x). \tag{I.3.8}$$

Proof

By using the identity (I.3.4), contained in Proposition I.5, we can write:

$$H_{n+m}(x) = \left(2x - \frac{d}{dx}\right)^n \left(2x - \frac{d}{dx}\right)^m = \left(2x - \frac{d}{dx}\right)^n H_m(x) \tag{I.3.9}$$

and by exploiting the r.h.s. of the above relation, we find:

$$H_{n+m}(x) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} H_{n-s}(x) \frac{d^s}{dx^s} H_m(x). \tag{I.3.10}$$

After noting that the following operational identity holds:

$$\frac{d^s}{dx^s} H_m(x) = \frac{2^s m!}{(m-s)!} H_{m-s}(x) \tag{I.3.11}$$

we immediately obtain the statement.

From the above proposition we can immediately derive as a particular case, the following identity:

$$H_{2m}(x) = (-1)^m 2^m (m!)^2 \sum_{s=0}^{m} \frac{(-1)^s \left[H_s(x)\right]^2}{2^s (s!)^2 (n-s)!}. \tag{I.3.12}$$

The use of the identity (I.3.5), stated in Proposition I.6, can be exploited to obtain the inverse of relation contained in eq. (I.3.12). We have indeed:

Proposition I.8

Given the Hermite polynomial $H_m(x)$, the square $[H_m(x)]^2$ can be written as:

$$H_m(x)H_m(x) = [H_n(x)]^2 = 2^m(m!)^2 \sum_{s=0}^{m} \frac{H_{2m}(x)}{2^s(s!)^2(n-s)!}. \qquad \text{(I.3.13)}$$

Proof

We can write:

$$[H_m(x)]^2 = e^{-\frac{1}{4}\frac{d^2}{dx^2}} \left[H_m\left(x + \frac{1}{2}\frac{d}{dx}\right) H_m\left(x + \frac{1}{2}\frac{d}{dx}\right) \right]. \qquad \text{(I.3.14)}$$

By using the relation (I.3.5), we find, after manipulating the r.h.s.:

$$[H_m(x)]^2 = e^{-\frac{1}{4}\frac{d^2}{dx^2}} \left[2^m(m!)^2 \sum_{s=0}^{m} \frac{(2x)^{2m}}{2^s(s!)^2(m-s)!} \right] \qquad \text{(I.3.15)}$$

and then, from the Burchnall identity (I.3.1), the thesis.

In the first section we have introduced the two-variable Hermite polynomials of the Kampé de Fériet type, for which the generating function writes:

$$e^{xt+yt^2} = \sum_{m=0}^{+\infty} \frac{t^m}{m!} He_m(x,y) \qquad \text{(I.3.16)}$$

and their explicit form reads:

$$He_m(x,y) = m! \sum_{s=0}^{\left[\frac{m}{2}\right]} \frac{x^{m-2s}y^s}{(m-2s)!s!}. \qquad \text{(I.3.17)}$$

A generalization of the identities (I.3.1), (I.3.2) and (I.3.5) can be immediately obtained for the above Hermite polynomials. We have:

Proposition I.9

The Hermite polynomials satisfy the following relation:

$$\left(x + 2y\frac{\partial}{\partial x}\right)^m (1) = \sum_{s=0}^{m} (2y)^s \binom{m}{s} He_m(x,y)\frac{\partial^s}{\partial x^s}(1). \qquad \text{(I.3.18)}$$

Proof

By multiplying the l.h.s. of the above equation by $\frac{t^m}{m!}$ and then summing up, we find:

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} \left(x + 2y \frac{\partial}{\partial x} \right)^m = e^{t\left(x+2y\frac{\partial}{\partial x}\right)}(1). \tag{I.3.19}$$

To develope the exponential in the r.h.s. of the (I.3.19) we need to apply the Weyl identity and then we have to calculate the commutator of the two operators:

$$\left[tx, t2y \frac{\partial}{\partial x} \right] = -2t^2 y \tag{I.3.20}$$

which help us to write:

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} \left(x + 2y \frac{\partial}{\partial x} \right)^m = e^{xt+yt^2} e^{2ty\frac{\partial}{\partial x}}(1). \tag{I.3.21}$$

After expanding and manipulating the r.h.s. of the previous relation and by equating the like t powers we find immediately the (I.3.18).

In the first section we have stated the operational definition of the polynomials $He_n(x, y)$ through the equation (I.1.16): it is easy to note this is a trivial consequence of the generalization of the Burchnall type identity:

$$e^{y\frac{\partial^2}{\partial x^2}} x^m = \left(x + 2y \frac{\partial}{\partial x} \right)^m. \tag{I.3.22}$$

In fact from the above relation and from the statement of Proposition I.9, we can write:

$$e^{y\frac{\partial^2}{\partial x^2}} x^m = \sum_{s=0}^{m} (2y)^s \binom{m}{s} He_m(x, y) \frac{\partial^s}{\partial x^s}(1) \tag{I.3.23}$$

and by noting that the r.h.s. of the above relation is not zero only for $s = 0$, we can immediately obtain the (I.3.18).

By following the same procedure used to state the relation (I.3.5), we can derive the inverse of the generalized Burchnall type identity, that is:

$$He_m \left(x - 2y \frac{\partial}{\partial x}, y \right) = \sum_{s=0}^{m} (-2y)^s \binom{m}{s} x^{m-s} \frac{\partial^s}{\partial x^s}. \tag{I.3.24}$$

We can also generalize the multiplication rules obtained for the Hermite polynomials $H_m(x)$, stated in Proposition I.7.

Proposition I.10

Given the Kampé de Fériet Hermite polynomials $He_m(x,y)$. We have:

$$He_{n+m}(x,y) = m!n! \sum_{s=0}^{\min(n,m)} (2y)^s \frac{He_{n-s}(x,y)He_{m-s}(x,y)}{(n-s)!(m-s)!s!}. \tag{I.3.25}$$

Proof

By using the relations stated in (I.3.18), (I.3.22) and (I.3.23), we can write:

$$He_{n+m}(x,y) = \left(x + 2y\frac{\partial}{\partial x}\right)^n He_m(x,y) \tag{I.3.26}$$

and then:

$$He_{n+m}(x,y) = \sum_{s=0}^{n} (2y)^s \binom{n}{s} He_n(x,y)\frac{\partial^s}{\partial x^s}He_m(x,y). \tag{I.3.27}$$

By noting that:

$$\frac{\partial^s}{\partial x^s}x^m = \frac{m!}{(m-2s)!}x^{m-2s} \tag{I.3.28}$$

we obtain:

$$\frac{\partial^s}{\partial x^s}He_m(x,y) = \frac{m!}{(m-s)!}He_{m-s}(x,y). \tag{I.3.29}$$

After substituting the above relation in the (I.3.27) and rearranging the terms we immediately obtain the thesis.

It is also possible to prove the inverse of the identity stated in the above proposition. In fact, by noting that:

$$He_n(x,y)He_m(x,y) = e^{y\frac{\partial^2}{\partial x^2}}\left[He_n\left(x - 2y\frac{\partial}{\partial x}, y\right) He_m\left(x - 2y\frac{\partial}{\partial x}, y\right)\right] \tag{I.3.30}$$

we can exploite the r.h.s. of above relation by using the equation written in (I.3.24), to obtain:

$$He_n(x,y)He_m(x,y) = e^{y\frac{\partial^2}{\partial x^2}}\left[\sum_{s=0}^{n}(-2y)^s\binom{n}{s}x^{n-s}\frac{\partial^s}{\partial x^s}x^m\right] \tag{I.3.31}$$

and then we can finally write:

$$He_n(x,y)He_m(x,y) = n!m! \sum_{s=0}^{\min(n,m)} (-2y)^s \frac{H_{n+m-2s}(x,y)}{(n-s)!(m-s)!s!}. \tag{I.3.32}$$

The previous identity and the equation (I.3.25) can be easily used to derive the particular case for $n = m$. We have in fact, from the (I.3.25):

$$He_{2m}(x,y) = 2^m(m!)^2\sum_{s=0}^{m}\frac{[He_s(x,y)]^2}{(s)!^2(m-s)!2^s} \tag{I.3.33}$$

and, for $n = m$, in the (I.3.32) we have:

$$[He_m(x,y)]^2 = (-2y)^m (m!)^2 \sum_{s=0}^{m} \frac{(-1)^s He_{2s}(x,y)}{(m-s)!(s!)^2 2^s}. \tag{I.3.34}$$

Before concluding this section we want prove two other important relations satisfied by the Hermite polynomials $He_n(x,y)$.

Proposition I.11

The Hermite polynomials $He_m(x,y)$ solve the following differential equation:

$$2y\frac{\partial^2}{\partial x^2}He_m(x,y) + x\frac{\partial}{\partial x}He_m(x,y) = mHe_m(x,y). \tag{I.3.35}$$

Proof

By using the results derived from Proposition I.9, we can easily write that:

$$\left(x + 2y\frac{\partial}{\partial x}\right)He_m(x,y) = He_{m+1}(x,y) \tag{I.3.36}$$

and from the first of the recurrence relations stated in (I.1.15):

$$\frac{\partial}{\partial x}He_m(x,y) = mHe_{m-1}(x,y) \tag{I.3.37}$$

we have:

$$\left(x + 2y\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)He_m(x,y) = mHe_m(x,y) \tag{I.3.38}$$

which is the thesis.

From this statement an important recurrence relation can also be derived. By exploiting, in fact, the relation (I.3.36), we obtain:

$$He_{m+1}(x,y) = xHe_m(x,y) + 2y\frac{\partial}{\partial x}He_m(x,y) \tag{I.3.39}$$

and then from the (I.3.37) we can conclude with:

$$He_{m+1}(x,y) = xHe_m(x,y) + 2myHe_{m-1}(x,y). \tag{I.3.40}$$

I.4 Monomiality Principle and Hermite polynomials

In this section we will present the concepts and the related aspects of the monomiality principle to explore different approaches for Hermite polynomials.

The associated operational calculus introduced by the monomiality principle allows us to reformulate the theory of the generalized Hermite polynomials from a unified point of view. In fact, these are indeed shown to be particular cases of more general polynomials and can be also used to derive classes of isospectral problems. Many properties of conventional and generalized orthogonal polynomials have been shown to be derivable, in a straightforward way, within an operational framework, which is a consequence of the monomiality principle. Before investigating the case of the generalized Hermite polynomials, let us briefly discuss about the Monomiality Principle. By *quasi-monomial* we mean any expression characterized by an integer n, satisfying the relations: $\hat{M}f_n = f_{n+1}$ $\hat{P}f_n = nf_{n-1}$ where $\hat{M}$ and $\hat{P}$ play the role of multiplicative and derivative operators. An example of quasi-monomial is provided by:

$$_{\delta}x_n = \prod_{m=0}^{n}(x - m\delta)$$

whose associated multiplication and derivative operators read:
$\hat{M} = xe^{\delta\frac{d}{dx}}$
$\hat{P} = \frac{e^{\delta\frac{d}{dx}-1}}{\delta}$, for $\delta \neq 0$
It is worth noting that, when $\delta = 0$, then:

$$_{\delta}x_n = x^n$$

and:

$$\hat{M} = x$$
$$\hat{P} = \frac{d}{dx}.$$

More generally, a given polynomial $p_n(x)$, $n \in \mathbb{N}$, $x \in \mathbb{C}$ can be considered a quasi-monomial if two operators $\hat{M}$ and $\hat{P}$ called multiplicative and derivative operators respectively, can be defined in such a way that:

$$\hat{M}p_n(x) = p_{n+1}(x) \tag{I.4.1}$$

$$\hat{P}p_n(x) = np_{n-1}(x) \tag{I.4.2}$$

with:

$$\left[\hat{M}, \hat{P}\right] = \hat{M}\hat{P} - \hat{P}\hat{M} = \hat{1}$$

that is $\hat{M}$, $\hat{P}$ and $\hat{1}$ satisfy a Weyl group structure with respect to commutation operation. The *rules* we have just established can be exploited to completely characterize the family of polynomials; we note indeed that, if $\hat{M}$ and $\hat{P}$ have a differential realization, the polynomial $p_n(x)$ satisfy the differential equation:

$$\hat{M}\hat{P}p_n(x) = p_n(x).$$

If $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as:

$$\hat{M}^n(1) = p_n(x).$$

If $p_0(x) = 1$, then the generating function of $p_n(x)$ can always be cast in the form:

$$e^{t\hat{M}}(1) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} p_n(x)$$

where $t \in \mathbb{R}$.

The Hermite polynomials are an examples of quasi-monomial. It is therefore possible to show that their properties can be derived by using the monomiality principle. We have introduced, in Section I.1, through the Definition I.1, the generalized Hermite polynomials of two-variable:

$$He_m(x,y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!} y^n x^{m-2n}$$

and it is immediate to prove that they are quasi-monomial under the action of the operators:

$$\hat{M} = x + 2y\frac{\partial}{\partial x} \tag{I.4.3}$$

$$\hat{P} = \frac{\partial}{\partial x}. \tag{I.4.4}$$

According to the previous statements, we easily obtain:

Differential equation

$$\left(2y\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x}\right) He_m(x,y) = mHe_{m-1}(x,y) \tag{I.4.5}$$

Generating function

$$e^{t\left(x+2y\frac{\partial}{\partial x}\right)}(1)=\sum_{m=0}^{+\infty}\frac{t^m}{m!}He_m(x,y) \tag{I.4.6}$$

where $t\in\mathbb{R}$, $|t|<+\infty$.

We have proved in the first section that the generalized Hermite polynomials of two variables satisfies the heat equation (see Proposition I.1). It is possible to derive a different proof, by using the formalism of the monomiality principle. In fact, the proof is just a consequence of the structure of the generating function itself. By keeping, indeed the derivatives of both sides of the (I.4.6) with respect to t and then equating the t-like powers, we find:

$$\frac{\partial}{\partial y}He_m(x,y)=m(m-1)He_{m-2}(x,y),$$

$$\frac{\partial}{\partial x}He_m(x,y)=mHe_{m-1}(x,y),$$

from which the heat equation follows. This statement allows a further important result, indeed by regarding it as an ordinary first order equation in the variable y and by treating the differential operators as an ordinary number, we can write the polynomials $He_m(x,y)$ in terms of the following operational definition:

$$He_m(x,y)=e^{y\frac{\partial^2}{\partial x^2}}He_m(x,0)=e^{y\frac{\partial^2}{\partial x^2}}x^m$$

that is exactly the same conclusion obtained in Proposition I.1, where we have used the techniques of the exponential operators. The considerations presented above for the generalized Hermite polynomials of the type $He_m(x,y)$, confirm that the majority of the properties of families of polynomials, recognized as quasi-monomial, can be deduced, quite straightforwardly, by using operational rules associated with the relevant multiplication and derivative operators. Furthermore, they suggest that we can introduce or *define* families of isospectral problems by exploiting the correspondence:

$$\hat{M}\to x,\hat{P}\to\frac{\partial}{\partial x},p_n(x)\to x^n.$$

We can therefore use the polynomials

$$p_n(x)$$

as a basis to introduce *new* functions with eigenvalues corresponding to the ordinary case. In Section I.2 we have presented the ordinary and generalized Bessel functions and we have discussed some of their properties. An useful example of the applications of the monomiality principle powerful tool to investigate some isospectral problems, is provided by a p-based Bessel function, defined as:

$$ {}_pJ_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r p_{n+2r}}{2^{n+2r} r!(n+r)!} $$

which is easily shown to satisfy the equation:

$$ \left[\hat{M}\hat{P}\hat{M}\hat{P} - \left(\hat{M}^2 - n^2\right)\right] = {}_pJ_n(x) = 0. $$

Since the generating function of the ordinary cylindrical Bessel function is:

$$ \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(x), $$

we can cast the relevant p-based Bessel function as:

$$ \exp\left[\frac{\hat{M}}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n {}_pJ_n(x). $$

We can then introduce particular p-based Bessel functions, by using the generalized Hermite polynomials, since we have proved that they satisfied the rules of the monomiality principle. In fact, in the case of Hermite-based Bessel function, we can immediately obtain:

$$ \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{4}\left(t - \frac{1}{t}\right)^2\right] = \sum_{n=-\infty}^{+\infty} t^n {}_HJ_n(x, y) $$

which is a trivial consequence of the structure of the multiplicative operator related to Hermite polynomials and to Weyl's decoupling rule. The last identity can be exploited to derive the series expansion definition:

$$ {}_HJ_n(x, y) = \sum_{r=0}^{+\infty} \frac{(-1)^r H_{n+2r}(x, y)}{2^{n+2r} r!(n+r)!} $$

and the link with the two-variable Bessel function is given by:

$$ {}_HJ_n(x, y) = e^y \sum_{r=0}^{+\infty} J_{n+2r}(x, 2y) \frac{(-y)^2}{r!}. $$

As we have seen, the monomiality principle is an important and powerful tool to investigate the structures and the related properties of many classes of special functions and orthogonal polynomials. We had limited ourselves to the cases of Bessel functions and Hermite polynomials, since we will exploite these results in the next chapters.

Chapter II

Multi-index Hermite polynomials

This chapter is devoted to the description of a special class of Hermite polynomials, which, in some way, we can identify as *vectorial* Hermite polynomials. The concept behind this kind of Hermite polynomials is essentially based on increasing simultaneously both the index and the number of variables. These polynomials will be used in the third chapter to define some special functions recognized as Hermite bi-orthogonal functions. We will begin the chapter by presenting the generalized Hermite polynomials of type $H_n^{(m)}(x, y)$ that will be used to describe and simplify some relevant properties of Chebyshev polynomials in the next chapters.

II.1 Hermite polynomials of type $H_n^{(m)}(x, y)$

In Section I.1 we have introduced the two-variable Hermite polynomials $He_m(x, y)$ by using the concepts and the formalism of the translation operator (see Definition I.1). More in general the above Hermite polynomials can be derived, as a particular case, from a more general class of polynomials recognized as belonging to the Hermite family that we will discuss in this section. To introduce this generalized class of Hermite polynomials we adopt the same procedure used in the first section of Chapter I.

Definition II.1

We will call Hermite polynomials of the type $H_n^{(m)}(x,y)$, the polynomials defined by the formula:

$$H_n^{(m)}(x,y) = \sum_{s=0}^{\left[\frac{n}{m}\right]} \frac{n!}{s!(n-ms)!} y^s x^{n-ms}. \tag{II.1.1}$$

It is easy to recognize that the above definition comes from relations (I.1.1) and (I.1.4). In fact, by noting that:

$$e^{yD^m} f(x) = \sum_{n=0}^{+\infty} \frac{y^n}{n!} f^{(mn)}(x) \tag{II.1.2}$$

and

$$D^{ms} x^n = n(n-1)\dots(n-ms+1)x^{n-ms} = \frac{n!}{(n-ms)!} x^{n-ms} \tag{II.1.3}$$

for $s = 0, 1, \dots \left[\frac{n}{m}\right]$ we obtain:

$$e^{yD^m} x^n = \sum_{s=0}^{\left[\frac{n}{m}\right]} \frac{y^s}{s!} \frac{n!}{(n-ms)!} x^{n-ms}. \tag{II.1.4}$$

It also interesting to note that the Hermite polynomials $H_n^{(m)}(x,y)$ can also be introduced using directly their generating function; in fact by exploiting the exponential $\exp(xt + yt^m)$ we can immediately recognize the identity:

$$e^{xt+yt^m} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n^{(m)}(x,y) \tag{II.1.5}$$

and then by setting $m = 2$ we obtain the generating function of the Hermite polynomials $He_n(x,y)$ (see (I.1.21)).

Proposition II.1

The polynomials $H_n^{(m)}(x,y)$ satisfy the following partial differential equation:

$$\frac{\partial}{\partial y} H_n^{(m)}(x,y) = \frac{\partial^m}{\partial x^m} H_n^{(m)}(x,y) \tag{II.1.6}$$

Proof

From (II.1.5), by differentiating with respect to y, we find:

$$\sum_{n=0}^{+\infty} \frac{t^{n+m}}{n!} H_n^{(m)}(x,y) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \frac{\partial}{\partial y} H_n^{(m)}(x,y) \qquad \text{(II.1.7)}$$

after manipulating the l.h.s. of the above equation and by equating the like t powers, we can immediately write:

$$\frac{n!}{(n-m)!} H_{n-m}^{(m)}(x,y) = \frac{\partial}{\partial y} H_n^{(m)}(x,y). \qquad \text{(II.1.8)}$$

Otherwise, by deriving m-times with respect to x in the (II.1.5), we have:

$$\begin{aligned} \frac{\partial}{\partial x} H_n^{(m)}(x,y) &= n H_{n-1}^{(m)}(x,y) \\ \frac{\partial^m}{\partial x^m} H_n^{(m)}(x,y) &= \frac{n!}{(n-m)!} H_{n-m}^{(m)}(x,y) \end{aligned} \qquad \text{(II.1.9)}$$

and then by comparing the second equation of (II.1.9) with equation (II.1.8), we immediately obtain the partial differential equation (II.1.6).

It is worth emphasizing from the previous proof two important recurrence relations related to the polynomials $H_n^{(m)}(x,y)$; we have proved, in fact that:

$$\begin{aligned} \frac{\partial}{\partial y} H_n^{(m)}(x,y) &= \frac{n!}{(n-m)!} H_{n-m}^{(m)}(x,y) \\ \frac{\partial}{\partial x} H_n^{(m)}(x,y) &= n H_{n-1}^{(m)}(x,y). \end{aligned} \qquad \text{(II.1.10)}$$

Proposition II.1 allows us to derive a similar operational definition for the Hermite polynomials $H_n^{(m)}(x,y)$ as in the case of the two-variable Kampé de Fériet polynomials. We note in fact that for $y = 0$ in equation (II.1.1), we have:

$$H_n^{(m)}(x,0) = x^n. \qquad \text{(II.1.11)}$$

By considering the equation in (II.1.6) an ordinary differential equation in the variable y, we can immediately conclude that, since it is linear and of the first order, the solution can be expressed as:

$$H_n^{(m)}(x,y) = e^{y \frac{\partial^m}{\partial x^m}} x^n \qquad \text{(II.1.12)}$$

or, in more explicit terms:

$$H_n^{(m)}(x,y) = \left[\sum_{s=0}^{\left[\frac{n}{m}\right]} \frac{y^r}{s!} \left(\frac{\partial}{\partial x} \right)^{ms} \right] x^n. \qquad \text{(II.1.13)}$$

In Section I.4, we have seen that the generalized Hermite polynomials in two variables of the form $He_m(x,y)$ can be considered quasi-monomial under the action of two specified operators. The two-variable, m-th order Hermite polynomials of type $H_n^{(m)}(x,y)$ are also quasi-monomial under the action of the following operators:

$$\hat{M} = x + my\frac{\partial^{m-1}}{\partial x^{m-1}}, \hat{P} = \frac{\partial}{\partial x}.$$

It is also possible to generalize this class of m-order Hermite polynomials, introducing the m-variable Hermite polynomials of order m, by setting:

$$H_n^{(m)}(x_1,\ldots,x_m) = n!\sum_{r=0}^{[s/2]}\frac{H_{n-mr}^{(m-1)}(x_1,\ldots,x_m)x_m^r}{r!(n-mr)!}. \tag{II.1.14}$$

This family of Hermite polynomials is also quasi-monomial with the related operators:

$$\hat{M} = x_1 + \sum_{s=2}^{m} sx_s\frac{\partial^{s-1}}{\partial x_1^{s-1}}, \hat{P} = \frac{\partial}{\partial x}.$$

By noting that a generic polynomial $p_n(x)$ recognized as quasi monomial satisfy the identity:

$$\hat{M}\hat{P}p_n(x) = p_n(x) \tag{II.1.15}$$

we immediately find that the above families of Hermite polynomials of order m solve the following differential equations:

$$\left(my\frac{\partial^m}{\partial x^m} + x\frac{\partial}{\partial x}\right)H_n^{(m)}(x,y) = nH_n^{(m)}(x,y) \tag{II.1.16}$$

$$\left(\sum_{s=2}^{m} sx_s\frac{\partial^s}{\partial x_1^s} + x_1\frac{\partial^s}{\partial x_1}\right)H_n^{(m)}(x_1,\ldots,x_m) = nH_n^{(m)}(x_1,\ldots,x_m). \tag{II.1.17}$$

In Section I.3 we have shown the Burchnall identity and we have derived other interesting formulae relevant to the ordinary Hermite polynomials of the type $He_m(x,y)$. To obtain the cited results for the Hermite polynomials of order m, it is necessary to make some considerations regarding the Weyl identities.

Proposition II.2

Let ξ be a real parameter, then the following identity holds:

$$\exp\left(\xi\left(x+\frac{\partial^n}{\partial x^n}\right)\right)=\exp\left(x\xi+\frac{\xi^{n+1}}{n+1}\right)\cdot\exp\left(\sum_{r=0}^{(n-1)}\frac{n!\xi^{r+1}}{(n-r)!(r+1)!}\left(\frac{\partial}{\partial x}\right)^{n-r}\right) \tag{II.1.18}$$

Proof

We start to consider this exponential operator:

$$S\left(\hat{A},\hat{B};\xi\right)=e^{\xi\left(\hat{A}+\hat{B}^n\right)} \tag{II.1.19}$$

where ξ is a real number and $\hat{A}$ and $\hat{B}$ denote operators such that:

$$[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}=\hat{k}$$

with k commuting with both of them. The decoupling theorem for the exponential operator introduced above, can be proved as follows. By taking the derivative of both sides with respect to ξ, we get:

$$\frac{\partial}{\partial\xi}S\left(\hat{A},\hat{B};\xi\right)=\left(\hat{A}+\hat{B}^n\right)S\left(\hat{A},\hat{B};\xi\right). \tag{II.1.20}$$

After setting:

$$S\left(\hat{A},\hat{B};\xi\right)=e^{\xi\hat{A}}\Sigma \tag{II.1.21}$$

and by using the relation:

$$e^{-\xi\hat{A}}\hat{B}^n e^{\xi\hat{A}}=\left(\hat{B}-\xi k\right)^n \tag{II.1.22}$$

we finally find:

$$\frac{\partial}{\partial\xi}\Sigma=\left(\hat{B}-\xi k\right)^n\Sigma \tag{II.1.23}$$

which can be easily integrated. Thus getting in conclusion:

$$S\left(\hat{A},\hat{B};\xi\right)=\exp(\xi\hat{A})\cdot\exp\left[\sum_{r=0}^{n}\binom{n}{r}\frac{\hat{B}^{n-r}k^r\xi^{r+1}}{r+1}(-1)^r\right]. \tag{II.1.24}$$

It is immediate to note that the thesis follows as a particular case with:

$$\begin{aligned}\hat{A}&=x\\ \hat{B}&=\tfrac{\partial}{\partial x}\end{aligned} \tag{II.1.25}$$

The generalization of the Weyl identity, which we have proved above, allows us to derive the following generalized Burchnall identity:

$$\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)^n=\sum_{r=0}^{n}\binom{n}{r}H_{n-r}^{(m)}(x,y)H_r^{(m-1)}(x,y)\left(\{G\}_{s=0}^{m-2}\right) \quad \text{(II.1.26)}$$

where we have indicated with G the expression:

$$G=\frac{m(m-1)!y}{(m-1-s)!\,(s+1)!}\frac{\partial^{m-1-s}}{\partial x^{m-1-s}}. \quad \text{(II.1.27)}$$

We note that the Burchnall-type identity in (II.1.26), for $m=3$, specializes as:

$$\left(x+3y\frac{\partial^2}{\partial x^2}\right)^2=\sum_{r=0}^{n}\binom{n}{r}H_{n-r}^{(3)}(x,y)H_r\left(3y\frac{\partial^2}{\partial x^2},3y\frac{\partial}{\partial x}\right). \quad \text{(II.1.28)}$$

An immediate application of these last identities is the derivation of the following Nielsen formula:

$$H_{2n}^{(m)}(x,y)=\sum_{r=0}^{n}\binom{n}{r}H_{n-r}^{(m)}(x,y)F_{n,r}^{(m-1)}(x,y) \quad \text{(II.1.29)}$$

where

$$F_{n,r}^{(m-1)}(x,y)=H_r^{(m-1)}\left[\left\{\frac{m(m-1)!\,y}{(m-1-r)!\,(r+1)!}\frac{\partial^{m-1-s}}{\partial x^{m-1-s}}\right\}_{r=0}^{m-2}\right]H_n^{(m)}(x,y), \quad \text{(II.1.30)}$$

and in case $m=3$, we get indeed:

$$F_{n,r}^{(2)}(x,y)=s!\sum_{r=0}^{[s/2]}\frac{(3y)^{s-r}(2s-3r)!\,H_{n-(2s-3r)}^{(3)}(x,y)}{(s-2r)!\,r!\,[n-(2s-3r)]!}. \quad \text{(II.1.31)}$$

A further application of the so far developed method is associated with the derivation of generating functions of the type:

$$G_l^{(m)}(x,y;t)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}H_{n+l}^{(m)}(x,y). \quad \text{(II.1.32)}$$

In fact, by noting that:

$$e^{\alpha\frac{\partial^s}{\partial x_1^s}}H_n^{(m)}(x_1,\dots,x_m)=\begin{cases}H_n^{(m)}(x_1,\dots,\alpha+x_s,\dots,x_m),\\ H_n^{(m)}(x_1,\dots,x_m,\dots,\alpha),\end{cases} \quad \text{(II.1.33)}$$

and by using the generalized Burchnall identity (II.1.26), we obtain:

$$G_l^{(m)}(x,y;t)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)^n H_l^{(m)}(x,y) \quad \text{(II.1.34)}$$

that is:

$$G_l^{(m)}(x,y;t)=e^{\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)t}H_l^{(m)}(x,y). \quad \text{(II.1.35)}$$

II.2 Two-index, two-variable Hermite polynomials

In Section I.1 we have introduced the one-variable, one-index Hermite polynomials $He_n(x)$ as a particular case of the polynomials $He_n(x, y)$. It is possible to use these polynomials to introduce a new class of Hermite polynomials with two indexes and two variables, which are a vectorial extension of the polynomials $He_n(x)$. This means that these polynomials have a couple of indexes that act on a couple of variables or, the same could be seen as a bi-dimensional index that acts on a bi-dimensional variable.
Let be the positive quadratic form:

$$\begin{aligned} q(x,y) &= ax^2 + 2bxy + cy^2 \\ a, c &> 0 \\ \Delta = ac - b^2 &> 0 \end{aligned} \tag{II.2.1}$$

where a, b, c are real numbers. The associated matrix reads:

$$\hat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{II.2.2}$$

and, since (II.2.1) holds and $\Delta = |\hat{M}| > 0$, is an invertible matrix. Let be now a vector $\underline{z} = \binom{x}{y}$in space $\mathbb{R}^2$, it immediately follows that:

$$\begin{aligned} q(\underline{z}) &= \underline{z}^t \hat{M} \underline{z} \\ q(\underline{z}) &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2 \end{aligned} \tag{II.2.3}$$

with these assumptions, we can now introduce the generalized two-index, two-variable Hermite polynomials.

Definition II.2

Let $\underline{z} = \binom{z}{y}$ and $\underline{w} = \binom{t}{u}$ be two vectors of space $\mathbb{R}^2$. We will name as two-index, two-variable Hermite polynomials $H_{m,n}(x, y)$ the polynomials defined by the following generating function:

$$e^{\underline{z}^t \hat{M} \underline{w} - \frac{1}{2}\underline{w}^t \hat{M} \underline{w}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n}(x, y). \tag{II.2.4}$$

By using the properties of matrix $\hat{M}$, in particular its invertibility, we can define the associated polynomials of $H_{m,n}(x, y)$. By noting in fact that the adjunct quadratic form of $q(\underline{z})$ writes:

$$\overline{q}\,(\underline{z}) = \underline{z}^t \hat{M}^{-1} \underline{z} \tag{II.2.5}$$

we have that the two-index, two-variable associated Hermite polynomials $G_{m,n}(x, y)$, are defined by the following generating function:

$$e^{\underline{v}^t \hat{M}^{-1} \underline{k} - \frac{1}{2} \underline{k} \hat{M}^{-1} \underline{k}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{r^m}{m!} \frac{s^n}{n!} G_{m,n}(x, y) \tag{II.2.6}$$

where $\underline{k} = \binom{r}{s}$ and $\underline{v} = \binom{\xi}{\eta}$ such that $\underline{v} = \hat{M} \underline{z}$.
After manipulating the exponent of the l.h.s, we can write the above relation in a more convenient form:

$$e^{\underline{z}^t \underline{k} - \frac{1}{2} \underline{k} \hat{M}^{-1} \underline{k}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{r^m}{m!} \frac{s^n}{n!} G_{m,n}(x, y). \tag{II.2.7}$$

The Hermite polynomials $H_{m,n}(x, y)$ and their associated $G_{m,n}(x, y)$ satisfy some important recurrence relations that we will expose in the following.

Proposition II.3

Given the polynomials $H_{m,n}(x, y)$, we have:

$$H_{m+1,n}(x, y) = (ax + by) H_{m,n}(x, y) - a\, m\, H_{m-1,n}(x, y) - b\, n\, H_{m,n-1}(x, y) \tag{II.2.8}$$

and:

$$H_{m,n+1}(x, y) = (bx + cy) H_{m,n}(x, y) - b\, m\, H_{m-1,n}(x, y) - c\, n\, H_{m,n-1}(x, y) \tag{II.2.9}$$

where a, b, c are the real numbers defined in (II.2.1).

Proof

By deriving with respect to t in (II.2.4), we note that:

$$\frac{\partial}{\partial t} \left[\underline{z}^t \hat{M} \underline{w} - \frac{1}{2} \underline{w}^t \hat{M} \underline{w} \right] \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n}(x, y) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} m \frac{t^{m-1}}{m!} \frac{u^n}{n!} H_{m,n}(x, y) \tag{II.2.10}$$

and by exploiting the l.h.s., we have:

$$l.h.s. = \begin{pmatrix} x & y \end{pmatrix} \hat{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tag{II.2.11}$$
$$- \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \end{pmatrix} \hat{M} \begin{pmatrix} t \\ u \end{pmatrix} + \begin{pmatrix} t & u \end{pmatrix} \hat{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!} H_{m,n}(x,y)$$

that is:

$$(ax+by-at-bu)\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!} H_{m,n}(x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} m\frac{t^{m-1}}{m!}\frac{u^n}{n!} H_{m,n}(x,y). \tag{II.2.12}$$

Expanding the l.h.s of the above equation and by equating the like t powers, we immediately obtain the relation (II.2.8).

Following the same procedure, but by deriving with respect to u in the (II.2.4), we have:

$$(bx+cy)-\frac{1}{2}(2bt-2cu)\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!} H_{m,n}(x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} n\frac{t^m}{m!}\frac{u^{n-1}}{n!} H_{m,n}(x,y) \tag{II.2.13}$$

and then the (II.2.9) follows.

In the previous proof we have used the formalism of the vectorial derivation; this technique can also be used to derive the important operational rules satisfied by the Hermite polynomials $H_{m,n}(x,y)$. We can prove in fact that there exist the following shift operators acting on the polynomials $H_{m,n}(x,y)$ in the following way:

$$\begin{aligned} \hat{E}_{\pm,0}\left[H_{m,n}(x,y)\right] &= H_{m\pm1,n}(x,y) \\ \hat{E}_{0,\pm}\left[H_{m,n}(x,y)\right] &= H_{m,n\pm1}(x,y) \end{aligned} \tag{II.2.14}$$

It is important to note that the above operators depend on a discrete parameter. In fact, operators $\hat{E}_{\pm,0}$ depend on index m, while operators $\hat{E}_{0,\pm}$ depend on index n. To explicit the structure of the operators presented above, we prove some important relations involving the two-index, two-variable Hermite polynomials of the form $H_{m,n}(x,y)$.

Proposition II.4

The polynomials $H_{m,n}(x,y)$ satisfy the following recurrence relations:

$$\frac{\partial}{\partial x}H_{m,n}(x,y) = amH_{m-1,n}(x,y) + bnH_{m,n-1}(x,y) \qquad \text{(II.2.15)}$$

and:

$$\frac{\partial}{\partial y}H_{m,n}(x,y) = bmH_{m-1,n}(x,y) + cnH_{m,n-1}(x,y) \qquad \text{(II.2.16)}$$

Proof

By deriving with respect to x in the (II.2.4), we obtain:

$$\frac{\partial}{\partial x}\left[\underline{z}^t\hat{M}\underline{w} - \frac{1}{2}\underline{w}^t\hat{M}\underline{w}\right]\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}H_{m,n}(x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}\frac{\partial}{\partial x}H_{m,n}(x,y). \qquad \text{(II.2.17)}$$

We can note that the derivative in l.h.s. can be exploited in the form:

$$\frac{\partial}{\partial x}\left[\underline{z}^t\hat{M}\underline{w} - \frac{1}{2}\underline{w}^t\hat{M}\underline{w}\right] = \frac{\partial}{\partial x}\left[\underline{z}^t\hat{M}\underline{w}\right] = \begin{pmatrix} 1 & 0 \end{pmatrix}\hat{M}\begin{pmatrix} t \\ u \end{pmatrix} = at + bu \qquad \text{(II.2.18)}$$

and then expression (II.2.17) reads:

$$a\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^{m+1}}{m!}\frac{u^n}{n!}H_{m,n}(x,y) + b\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^{n+1}}{n!}H_{m,n}(x,y) = \qquad \text{(II.2.19)}$$

$$= \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}\frac{\partial}{\partial x}H_{m,n}(x,y)$$

which proves (II.2.15).

In an analogous way the relation (II.2.16) can be stated. In fact by deriving, again (II.2.4) with respect to y and by noting that:

$$\frac{\partial}{\partial y}\left[\underline{z}^t\hat{M}\underline{w}\right] = \begin{pmatrix} 0 & 1 \end{pmatrix}\hat{M}\begin{pmatrix} t \\ u \end{pmatrix} = bt + cu \qquad \text{(II.2.20)}$$

we immediately obtain:

$$b\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^{m+1}}{m!}\frac{u^n}{n!}H_{m,n}(x,y) + c\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^{n+1}}{n!}H_{m,n}(x,y) = \qquad \text{(II.2.21)}$$

$$= \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}\frac{\partial}{\partial y}H_{m,n}(x,y)$$

and then (II.2.16).

The four recurrence relations, stated through Propositions II.3 and II.4 help us to explicit the shift operators introduced in the (II.2.14). In fact by noting that the (II.2.8) can be written in the form:

$$amH_{m-1,n}(x,y) = (ax+by)H_{m,n}(x,y) - H_{m+1,n}(x,y) - bnH_{m,n-1}(x,y) \tag{II.2.22}$$

and by using the (II.2.15), we obtain:

$$\left[(ax+by) - \frac{\partial}{\partial x}\right] H_{m,n}(x,y = H_{m+1,n}(x,y). \tag{II.2.23}$$

By combining the relations (II.2.9) and (II.2.16) in the same way as above, we have:

$$\left[(bx+cy) - \frac{\partial}{\partial y}\right] H_{m,n}(x,y = H_{m,n+1}(x,y). \tag{II.2.24}$$

Expression (II.2.15) can be also recast in the form:

$$mH_{m-1,n}(x,y) = -\frac{b}{a}nH_{m,n-1}(x,y) + \frac{1}{a}\frac{\partial}{\partial x}H_{m,n}(x,y) \tag{II.2.25}$$

which once combined with (II.2.16), gives:

$$\left(\frac{\partial}{\partial y} - \frac{b}{a}\frac{\partial}{\partial x}\right) H_{m,n}(x,y) = n\left(-\frac{b^2}{a} + c\right) H_{m,n-1}(x,y) \tag{II.2.26}$$

and finally:

$$-\frac{1}{n\Delta}\left(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}\right) H_{m,n}(x,y) = H_{m,n-1}(x,y). \tag{II.2.27}$$

Following the same procedure, combining again relation (II.2.15) and (II.2.16), we can state the last important identity:

$$\frac{1}{m\Delta}\left(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right) H_{m,n}(x,y) = H_{m-1,n}(x,y). \tag{II.2.28}$$

Definition II.3

Given the Hermite polynomials $H_{m,n}(x,y)$, we define the related shift operators, by setting:

$$\begin{aligned} \hat{E}_{+,0} &= (ax+by) - \frac{\partial}{\partial x} \\ \hat{E}_{0,+} &= (bx+cy) - \frac{\partial}{\partial y} \end{aligned} \tag{II.2.29}$$

and

$$\hat{E}_{-,0} = \frac{1}{m\Delta}\left(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right) \qquad \text{(II.2.30)}$$

$$\hat{E}_{0,-} = -\frac{1}{n\Delta}\left(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}\right).$$

It immediately follows that such operators satisfy, as defined, the shifting indicated in the (II.2.14). It is also important to note that the above operators are parameter-dependent; in particular operators $\hat{E}_{\pm,0}$ are depending on index m and operators $\hat{E}_{0,\pm}$ on index n. For example, the relation:

$$\hat{E}_{0,-}\left[H_{m,n+1}(x,y)\right] = H_{m,n}(x,y)$$

must be read as:

$$\left[-\frac{1}{(n+1)\Delta}\left(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}\right)\right] H_{m,n+1}(x,y) = H_{m,n}(x,y).$$

The shift operators help us to prove an important result of the Hermite polynomials $H_{m,n}(x,y)$.

Proposition II.5

The following partial differential equation:

$$\left[-\underline{\partial}_z^t \hat{M}^{-1}\underline{\partial}_z + \underline{z}^t\underline{\partial}_z\right] f_{m,n}(x,y) = (m+n) f_{m,n}(x,y) \qquad \text{(II.2.31)}$$

where $\underline{\partial}_z = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$,

is solved by polynomials $H_{m,n}(x,y)$.

Proof

By using shift operators, we have:

$$\hat{E}_{-,0}\left[\hat{E}_{+,0} H_{m,n}(x,y)\right] = H_{m,n}(x,y) \qquad \text{(II.2.32)}$$

and:

$$\hat{E}_{0,-}\left[\hat{E}_{0,+} H_{m,n}(x,y)\right] = H_{m,n}(x,y). \qquad \text{(II.2.33)}$$

By expliciting (II.2.32), we obtain:

$$\left[acx\frac{\partial}{\partial x} - b^2 y\frac{\partial}{\partial y} + bcy\frac{\partial}{\partial x} - abx\frac{\partial}{\partial y} - c\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x \partial y}\right] H_{m,n}(x,y) = \Delta m H_{m,n}(x,y) \qquad \text{(II.2.34)}$$

and from (II.2.33):

$$-\left[b^2x\frac{\partial}{\partial x}+bcy\frac{\partial}{\partial x}-b\frac{\partial^2}{\partial x\partial y}-abx\frac{\partial}{\partial y}-acy\frac{\partial}{\partial y}+a\frac{\partial^2}{\partial y^2}\right]H_{m,n}(x,y)=\Delta nH_{m,n}(x,y). \tag{II.2.35}$$

By summing up the last expressions, we have:

$$\left[acx\frac{\partial}{\partial x}-b^2y\frac{\partial}{\partial y}-c\frac{\partial^2}{\partial x^2}+2b\frac{\partial^2}{\partial x\partial y}-b^2x\frac{\partial}{\partial x}+acy\frac{\partial}{\partial y}-a\frac{\partial^2}{\partial y^2}\right]H_{m,n}(x,y)= \tag{II.2.36}$$

$$=\Delta(m+n)H_{m,n}(x,y)$$

and after rearranging the terms in the l.h.s., we can write:

$$\left\{\frac{1}{\Delta}\left(-c\frac{\partial^2}{\partial x^2}+2b\frac{\partial^2}{\partial x\partial y}-a\frac{\partial^2}{\partial y^2}\right)+ \right. \tag{II.2.37}$$

$$\left.+\frac{1}{ac-b^2}\left[\left(ac-b^2\right)x\frac{\partial}{\partial x}+\left(ac-b^2\right)y\frac{\partial}{\partial y}\right]\right\}H_{m,n}(x,y)=(m+n)H_{m,n}(x,y).$$

By considering that:

$$\begin{aligned}\hat{M}^{-1}&=\frac{1}{\Delta}\begin{pmatrix}c & -b\\ -b & a\end{pmatrix}\\ \underline{\partial}_z^t&=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}\end{aligned}$$

we immediately obtain:

$$\begin{aligned}\frac{1}{\Delta}\left(-c\frac{\partial^2}{\partial x^2}+2b\frac{\partial^2}{\partial x\partial y}-a\frac{\partial^2}{\partial y^2}\right)&=-\underline{\partial}_z^t\hat{M}^{-1}\partial_z\\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}&=\underline{z}^t\underline{\partial}_z\end{aligned}$$

and then, we can write the expression (II.2.37) in the compact form:

$$\left[-\underline{\partial}_z^t\hat{M}^{-1}\underline{\partial}_z+\underline{z}^t\underline{\partial}_z\right]H_{m,n}(x,y)=(m+n)H_{m,n}(x,y) \tag{II.2.38}$$

which is exactly relation (II.2.31).

II.3 Operatorial relations for Hermite polynomials of type $H_{m,n}(x,y)$

In this section we will derive a number of identities regarding the polynomials $H_{m,n}(x,y)$, which are strictly derived from the analogous rules stated in

Section I.3 for the two-variable Hermite polynomials of the type $He_n(x,y)$. In particular by using the above mentioned identities as the Weyl decoupling rules and the generalized Crofton identity:

$$\begin{aligned} e^{\hat{A}+\hat{B}} &= e^{-\frac{k}{2}} e^{\hat{A}} e^{\hat{B}} \\ e^{\hat{A}^m} f\left(\hat{B}\right) &= f\left(\hat{B} + mk\hat{A}^{m-1}\right) e^{\hat{A}^m} \end{aligned} \tag{II.3.1}$$

where $\hat{A}$ and $\hat{B}$ are two operators such that their commutator is a real number k (or any operator commuting either with $\hat{A}$ and $\hat{B}$). The first identity we derive is an extension of the two-dimensional case of the Burchnall identity.

Proposition II.6

The two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ satisfy the following identity:

$$\hat{I}_{m,n} = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} (-1)^{r+s} H_{m-r,n-s}(x,y) \frac{\partial^{r+s}}{\partial x^r \partial y^s} \tag{II.3.2}$$

where:

$$\hat{I}_{m,n} = \left(ax + by - \frac{\partial}{\partial x}\right)^m \left(bx + cy - \frac{\partial}{\partial y}\right)^n, \tag{II.3.3}$$

and where a, b and c are real numbers.

Proof

By multiplying both sides of the operator relation (II.3.3) by $\frac{t^m}{m!}$ and $\frac{u^n}{n!}$ and then, choosing the same values in (II.3.2), we have:

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} \hat{I}_{m,n} = e^{t\left(ax+by-\frac{\partial}{\partial x}\right)} e^{u\left(bx+cy-\frac{\partial}{\partial y}\right)}. \tag{II.3.4}$$

After applying to the r.h.s. of the above relation the identities written in (II.3.1), we obtain:

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} \hat{I}_{m,n} = e^{\underline{z}^t \hat{M} \underline{w} - \frac{1}{2} \underline{w}^t \hat{M} \underline{w}} e^{-t\frac{\partial}{\partial x}} e^{-u\frac{\partial}{\partial y}}. \tag{II.3.5}$$

By expanding the r.h.s. of the (II.3.5), by rearranging the sums and by equating the like $(t,u)-$power coefficients, we end up with (II.3.2).

It is interesting to note that the above identity can be used to derive the standard Burchnall identity. In fact to obtain the polynomials $H_n(x)$ it is enough to setting $a = 1$ and $y = 0$ in the definition of the vectorial polynomials $H_{m,n}(x, y)$; and then by setting the same positions in the (II.3.2), we have the Burchnall identity for the one-index, one-variable Hermite polynomials:

$$\hat{I}_m = \sum_{r=0}^{m} \binom{m}{r} (-1)^r H_{m-r}(x) \frac{\partial^r}{\partial x^r}. \tag{II.3.6}$$

It is also possible to extend the Burchnall identity to the associated Hermite polynomials $G_{m,n}(x, y)$. By using the link stated in the equation (II.2.6), we can introduce the operator:

$$\hat{L}_{m.,n} = \left[x + \frac{1}{\Delta}\left(b\frac{\partial}{\partial y} - c\frac{\partial}{\partial x}\right)\right]^m \left[y + \frac{1}{\Delta}\left(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}\right)\right]^n \tag{II.3.7}$$

and by following the same procedure, leading to equation (II.3.5), we obtain:

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!}\hat{L}_{m,n} = \tag{II.3.8}$$

$$= \exp\left[(tx + uy) - \frac{1}{2}\underline{w}^t \hat{M} \underline{w}\right] \cdot \exp\left[\frac{t}{\Delta}\left(b\frac{\partial}{\partial y} - c\frac{\partial}{\partial x}\right) + \frac{u}{\Delta}\left(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}\right)\right].$$

Furthermore, by noting that $\underline{z}^t \underline{w} = xt + yu$, we have:

$$e^{(tx+uy)-\frac{1}{2}\underline{w}^t \hat{M} \underline{w}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!} G_{m,n}(x, y) \tag{II.3.9}$$

and by exploiting the consequences of the (II.2.6), we can note that:

$$\begin{aligned} \xi &= ax + by \\ \eta &= bx + cy \end{aligned} \tag{II.3.10}$$

which implies:

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{c}{\Delta}\frac{\partial}{\partial x} - \frac{b}{\Delta}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \eta} &= \frac{a}{\Delta}\frac{\partial}{\partial y} - \frac{b}{\Delta}\frac{\partial}{\partial x} \end{aligned} \tag{II.3.11}$$

and then, we end up with the identity:

$$\hat{L}_{m,n} = \sum_{r=0}^{m}\sum_{s=0}^{n} \binom{m}{r}\binom{n}{s} (-1)^{r+s} G_{m-r,n-s}(x, y) \frac{\partial^{r+s}}{\partial \xi^r \partial \eta^s} \tag{II.3.12}$$

which is the extension of the Burchnall type identity to the Hermite polynomials $G_{m,n}(x, y)$.

As for the case of the polynomials $H_n(x)$ and $He_n(x,y)$, the Burchnall identity can be used to derive important operational relations for the related polynomials. In particular it is very interesting to state analogous operational definitions for the polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$. By noting that the following relation holds:

$$\begin{aligned}&\exp\left[-\tfrac{1}{2}\left(\partial_x \partial_y\right)\hat{M}^{-1}\tbinom{\partial_x}{\partial_y}\right](ax+by)^m(bx+cy)^n = \\ &= \hat{I}_{m,n}\exp\left[-\tfrac{1}{2}\left(\partial_x \partial_y\right)\hat{M}^{-1}\tbinom{\partial_x}{\partial_y}\right]\end{aligned} \tag{II.3.13}$$

which is a consequence of the second equation (II.3.1) and using now the identity (II.3.2), along with the assumption that the exponential operator is acting on its r.h.s. only, we obtain the identity:

$$\exp\left[-\frac{1}{2}\left(\partial_x \partial_y\right)\hat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right](ax+by)^m(bx+cy)^n = H_{m,n}(x,y). \tag{II.3.14}$$

For the associated Hermite polynomials $G_{m,n}(x,y)$, we first note the following relation:

$$\begin{aligned}&e^{-\frac{1}{2}(\partial_x \partial_y)\hat{M}^{-1}\binom{\partial_x}{\partial_y}}\cdot x^m y^n = \left[x+\frac{1}{\Delta_M}\left(b\partial_y - c\partial_x\right)\right]^m \cdot \\ &\cdot\left[y+\frac{1}{\Delta_M}\left(b\partial_x - a\partial_y\right)\right]^n\cdot\exp\left[-\frac{1}{2}\left(\partial_x \partial_y\right)\hat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right]\end{aligned} \tag{II.3.15}$$

where the terms in the r.h.s of the above equation, other than the exponential, correspond to the dual of $\hat{I}_{m,n}$ defined in (II.2.3), that is the operator $\hat{L}_{m,n}$ previously presented. It is evident that equation (II.3.15) can be exploited to conclude that:

$$\exp\left[-\frac{1}{2}\left(\partial_x \partial_y\right)\hat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right]x^m y^n = G_{m,n}(x,y) \tag{II.3.16}$$

which is the analogous of equation (II.3.14) and holds under the same conditions. We now want to emphasize that both equations (II.3.14) and (II.3.16) indicate that both the polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ are solutions of particular partial differential equations.

Theorem II.1

The two-index, two-variable Hermite polynomials of form $H_{m,n}(x,y)$ and their associated $G_{m,n}(x,y)$ solve the following partial differential equation:

$$\frac{\partial}{\partial \tau} S_{m,n}(x,y;\tau) = -\frac{1}{2}\left(\partial_x \partial_y\right) \hat{M}^{-1} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} S_{m,n}(x,y;\tau) \tag{II.3.17}$$

satisfying the conditions at $\tau = 0$

$$S_{m,n}(x,y;0) = \begin{cases} \xi^m \eta^n, \text{ when } S_{m,n} = H_{m,n} \\ x^m y^n, \text{ when } S_{m,n} = G_{m,n} \end{cases} \tag{II.3.18}$$

The proof is an immediate consequence of the equations (II.3.14) and (II.3.16). Polynomials $H_{m,n}(x,y)$ and their associated $G_{m,n}(x,y)$ can be treated in a more flexible way using a particular class of the Hermite polynomials.

Definition II.4

Let be the real variables x, y, ξ, η and χ, we will call five-variable, two-index Hermite polynomials $H_{m,n}(x,y;\xi,\eta|\chi)$ the ones defined by the following generating function:

$$e^{xt+yt^2+\xi\tau+\eta\tau^2+\chi t\tau} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{\tau^n}{n!} H_{m,n}(x,y;\xi,\eta|\chi) \tag{II.3.19}$$

where t and τ are continuous variables such that $|t|, |\tau| < +\infty$.

From the above definition, by expanding the l.h.s., we can immediately obtain the explicit form of the polynomials $H_{m,n}(x,y;\xi,\eta|\chi)$. In fact, by noting that:

$$e^{xt+yt^2}, e^{\xi\tau+\eta\tau^2}$$

are the generating functions of the two-variable Hermite polynomials of the type $He_n(x,y)$, we have:

$$H_{m,n}(x,y;\xi,\eta|\chi) = m!n! \sum_{q=0}^{\min(m,n)} \frac{\chi^q He_{m-q}(x,y) He_{n-q}(\xi,\eta)}{q!(m-q)!(n-q)!}. \tag{II.3.20}$$

Theorem II.2

Hermite polynomials $H_{m,n}(x,y)$ can be written as polynomials $H_{m,n}(x,y;\xi,\eta|\chi)$, according to:

$$H_{m,n}(x,y) = H_{m,n}\left(ax+by, -\frac{1}{2}a; bx+cy, -\frac{1}{2}c| - b\right). \tag{II.3.21}$$

Proof

By manipulating the generating function of polynomials $H_{m,n}(x,y)$, given in (II.2.4), we can write:

$$\exp\left[\underline{z}^t\hat{M}\underline{w}-\frac{1}{2}\underline{w}^t\hat{M}\underline{w}\right]= \tag{II.3.22}$$

$$=\exp\left\{(ax+by)t+(bx+cy)u-\frac{1}{2}\left[(at+bu)t+(bt+cu)u\right]\right\}$$

and by remembering that $\underline{z}=\binom{x}{y}$ and $\underline{w}=\binom{t}{u}$ are two vectors of space $\mathbb{R}^2$ and $\hat{M}$ is given by (II.2.2) with a,b,c real numbers, such that:

$$\begin{aligned} a,c &> 0 \\ ac-b^2 &> 0 \end{aligned} \tag{II.3.23}$$

then we obtain:

$$\exp\left[\underline{z}^t\hat{M}\underline{w}-\frac{1}{2}\underline{w}^t\hat{M}\underline{w}\right]= \tag{II.3.24}$$

$$=\exp\left(axt+byt-\frac{1}{2}at^2+bxu+cyu-\frac{1}{2}cu^2-but\right)=$$

$$=\exp\left[(ax+by)\,t-\frac{1}{2}at^2\right]\cdot\exp\left[(bx+cy)\,u-\frac{1}{2}cu^2\right]\cdot\exp\left(-but\right)$$

and by appropriately treating the variables in the exponential of the r.h.s., we write:

$$\exp\left[\underline{z}^t\hat{M}\underline{w}-\frac{1}{2}\underline{w}^t\hat{M}\underline{w}\right]= \tag{II.3.25}$$

$$=\sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty}\sum_{q=0}^{+\infty}(-1)^q\frac{t^{r+q}}{r!}\frac{u^{s+q}}{s!}\frac{b^q}{q!}He_r\left(ax+by,-\frac{1}{2}a\right)He_s\left(bx+cy,-\frac{1}{2}c\right).$$

By setting:

$$\begin{aligned} r+q &= m \\ s+q &= n \end{aligned}$$

the relation (II.3.25) can be written in a more convenient form:

$$\exp\left[\underline{z}^t\hat{M}\underline{w}-\frac{1}{2}\underline{w}^t\hat{M}\underline{w}\right]= \tag{II.3.26}$$

$$= \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} t^m u^n \sum_{q=0}^{+\infty} \frac{(-1)^q b^q}{q!(m-q)!(n-q)!} He_{m-q}\left(ax+by, -\frac{1}{2}a\right) He_{n-q}\left(bx+cy, -\frac{1}{2}c\right)$$

and from the (II.3.19), after equating the like t and u powers, we obtain the statement.

The opposite is not true, in general, unless the conditions:

$$\begin{aligned} ac - b^2 &> 0 \\ a, c &> 0 \end{aligned}$$

can be ensured.

It is also interesting to note that polynomials $H_{m,n}(x, y; \xi, \eta|\chi)$ help us to write the summing relations stated in Section I.3 for the Hermite polynomials of type $He_n(x, y)$. In fact, by setting:

$$\begin{aligned} x &= \xi \\ y &= \eta \\ \chi &= 2y \end{aligned}$$

the five-variable, two-index Hermite polynomials become:

$$H_{m,n}(x, y; \xi, \eta|\chi) = H_{m,n}(x, y|2y) = m!n! \sum_{q=0}^{\min(m,n)} \frac{(2y)^q He_{m-q}(x,y) He_{n-q}(x,y)}{q!(m-q)!(n-q)!}. \tag{II.3.27}$$

It immediately follows from Proposition I.10 that:

$$H_{m,n}(x, y|2y) = He_{m+n}(x, y). \tag{II.3.28}$$

In Section II.2 we have introduced the two-index, two-variable associated Hermite polynomials $G_{m,n}(x, y)$ and we have presented their generating function through the following relation:

$$e^{\underline{z}^t \underline{k} - \frac{1}{2}\underline{k}\hat{M}^{-1}\underline{k}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{r^m}{m!} \frac{s^n}{n!} G_{m,n}(x, y) \tag{II.3.29}$$

where $\binom{x}{y}$ and $\underline{k} = \binom{r}{s}$ are the two vectors from the adjoint quadratic form

$$\overline{q}(\underline{z}) = \underline{z}^t \hat{M}^{-1} \underline{z} \tag{II.3.30}$$

as we have shown in Definition II.2. We can now explore how it is possible to represent the associated Hermite polynomials $G_{m,n}(x,y)$ in terms of the five-variable Hermite polynomials of the form $H_{m,n}\left(x,y;\xi,\eta|\chi\right)$.

Corollary II.1

Associated Hermite polynomials $G_{m,n}(x,y)$ can be written in terms of the five-variable, two-index Hermite polynomials of the form $H_{m,n}\left(x,y;\xi,\eta|\chi\right)$.

Proof

Expliciting the generating function of polynomials $G_{m,n}(x,y)$, we have:

$$\exp\left[\underline{z}^t\underline{k}-\frac{1}{2}\underline{k}^t\hat{M}^{-1}\underline{k}\right]=\exp\left[\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}-\frac{1}{2}\begin{pmatrix} r & s \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1}\begin{pmatrix} r \\ s \end{pmatrix}\right] \tag{II.3.31}$$

By manipulating the r.h.s of above equation, we obtain:

$$\begin{aligned}
&\exp\left[\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}-\tfrac{1}{2}\begin{pmatrix} r & s \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1}\begin{pmatrix} r \\ s \end{pmatrix}\right]= \\
&=\exp\left[xr+ys-\tfrac{1}{2}\begin{pmatrix} r & s \end{pmatrix}\tfrac{1}{\Delta}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}\right]= \\
&=\exp\left[xr+ys-\tfrac{1}{2\Delta}\begin{pmatrix} cr-bs & -br+as \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}\right]
\end{aligned} \tag{II.3.32}$$

and then:

$$\exp\left[\underline{z}^t\underline{k}-\frac{1}{2}\underline{k}^t\hat{M}^{-1}\underline{k}\right]=\exp\left[xr+ys-\frac{1}{2\Delta}cr^2+\frac{1}{\Delta}brs-\frac{1}{2\Delta}as^2\right]. \tag{II.3.33}$$

The expression on the r.h.s. of the above relation can be recast in a convenient form, by setting:

$$\exp\left[xr+ys-\frac{1}{2\Delta}cr^2+\frac{1}{\Delta}brs-\frac{1}{2\Delta}as^2\right]= \tag{II.3.34}$$

$$=\exp\left[xr+\left(-\frac{1}{2\Delta}c\right)r^2\right]\exp\left[ys+\left(-\frac{1}{2\Delta}a\right)s^2\right]\exp\left[\frac{1}{\Delta}brs\right].$$

The first two exponential on the r.h.s of the previous equation are the generating function of the generalized two-variable Hermite polynomials discussed in Section I.1 (see equation (I.1.21)) and, since the third exponential could be expanded in Taylor series, we get:

$$\exp\left[\underline{z}^t\underline{k} - \frac{1}{2}\underline{k}^t\hat{M}^{-1}\underline{k}\right] = \sum_{m=0}^{+\infty}\frac{r^m}{m!}He_m\left(x, -\frac{1}{2\Delta}c\right)\sum_{n=0}^{+\infty}\frac{s^n}{n!}He_n\left(y, -\frac{1}{2\Delta}a\right)\sum_{q=0}^{+\infty}\frac{1}{q!}\frac{b^q}{\Delta}r^q s^q. \tag{II.3.35}$$

The r.h.s. of the above equation can be recast in the form:

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{q=0}^{+\infty}\frac{r^{m+q}}{m!}\frac{s^{n+q}}{n!}\frac{1}{q!}\frac{b^q}{\Delta}He_m\left(x, -\frac{1}{2\Delta}c\right)He_n\left(y, -\frac{1}{2\Delta}a\right). \tag{II.3.36}$$

By setting $m + q = k$ and $n + q = j$, we can write:

$$\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{q=0}^{+\infty}\frac{r^k}{(k-q)!}\frac{s^j}{(j-q)!}\frac{1}{q!}\frac{b^q}{\Delta}He_{k-q}\left(x, -\frac{1}{2\Delta}c\right)He_{j-q}\left(y, -\frac{1}{2\Delta}a\right) \tag{II.3.37}$$

and, without loss of generality, we can set $k = m$ and $j = n$ to obtain:

$$\exp\left[\underline{z}^t\underline{k} - \frac{1}{2}\underline{k}^t\hat{M}^{-1}\underline{k}\right] = \tag{II.3.38}$$

$$= \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{q=0}^{+\infty}\frac{r^m}{(m-q)!}\frac{s^n}{(n-q)!}\frac{1}{q!}\frac{b^q}{\Delta}He_{m-q}\left(x, -\frac{1}{2\Delta}c\right)He_{n-q}\left(y, -\frac{1}{2\Delta}a\right).$$

In this section, Definition II.4, we have introduced the five-variable, two-index Hermite polynomials of the form $H_{m,n}(x, y; \xi, \eta|\chi)$ and their explicit form reads:

$$H_{m,n}(x, y; \xi, \eta|\chi) = m!n!\sum_{q=0}^{\min(m,n)}\frac{\chi^q He_{m-q}(x, y)He_{n-q}(\xi, \eta)}{q!(m-q)!(n-q)!}. \tag{II.3.39}$$

We can observe that the expression on r.h.s. of equation (II.3.36) can be recognized as an Hermite polynomials $H_{m,n}(x, y; \xi, \eta|\chi)$, by setting:

$x \to x$

$y \to -\frac{c}{2\Delta}$

$\xi \to y$

$\eta \to -\frac{a}{2\Delta}$

$\chi \to \frac{b}{\Delta}$

and then we can conclude with:

$$\exp\left[\underline{z}^t\underline{k} - \frac{1}{2}\underline{k}^t\hat{M}^{-1}\underline{k}\right] = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{r^m}{m!}\frac{s^n}{n!}H_{m,n}\left(x, -\frac{c}{2\Delta}; y, -\frac{a}{2\Delta}|\frac{b}{\Delta}\right). \tag{II.3.40}$$

Since the generating function of the associated Hermite polynomials $G_{m,n}(x,y)$ has the form:

$$e^{\underline{z}^t\underline{k}-\frac{1}{2}\underline{k}\hat{M}^{-1}\underline{k}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{r^m}{m!}\frac{s^n}{n!}G_{m,n}(x,y) \tag{II.3.41}$$

we can easily obtain the thesis of the statement:

$$G_{m,n}(x,y) = H_{m,n}\left(x, -\frac{c}{2\Delta}; y, -\frac{a}{2\Delta}|\frac{b}{\Delta}\right). \tag{II.3.42}$$

The operational results obtained for the two-index, two-variable Hermite polynomials of the type $H_{m,n}(x,y)$ and for their associated $G_{m,n}(x,y)$ in terms of the five-variable, two-index Hermite polynomials $H_{m,n}\left(x,y;\xi,\eta|\chi\right)$, suggest us to generalize this special class of Hermite polynomials, to explore other relevant identities involving the Hermite polynomials of different type. Let us remember that the two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ have been introduced in Definition II.2, through their generating function, that is:

$$e^{\underline{z}^t\hat{M}\underline{w}-\frac{1}{2}\underline{w}^t\hat{M}\underline{w}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}H_{m,n}(x,y). \tag{II.3.43}$$

We can generalize the polynomials $H_{m,n}(x,y)$, by acting directly on the above expression.

Definition II.5

Let ρ be a real number such that $|\rho| < +\infty$ and let $\underline{z} = \binom{x}{y}$ and $\underline{w} = \binom{t}{u}$ be two vectors of space $\mathbb{R}^2$. We will call the generalized two-index, two-variable Hermite polynomials $H_{m,n}(x,y;\rho)$ as the polynomials defined by the following generating function:

$$e^{\underline{z}^t\hat{M}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{M}\underline{w}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}H_{m,n}(x,y;\rho). \tag{II.3.44}$$

By expanding the generating function on the above definition, we have:

$$\exp\left[\underline{z}^t\hat{M}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{M}\underline{w}\right] = \tag{II.3.45}$$

$$= \exp\left[\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} t \\ u \end{pmatrix} + \frac{1}{2}\rho\begin{pmatrix} t & u \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} t \\ u \end{pmatrix}\right]$$

and the argument of the exponential on the r.h.s. of the previous identity, can be written as:

$$(ax+by)t+(bx+cy)u+\frac{1}{2}\rho a t^2+\frac{1}{2}\rho u t b+\frac{1}{2}\rho t b u+\frac{1}{2}\rho c u^2. \tag{II.3.46}$$

It is immediate to note that the terms in the above relation could be recast in a convenient form, after remembering the structure of the generating function of the two-variable Hermite polynomials $He_m(x,y)$ presented in the first chapter:

$$\exp\left(xt+yt^2\right)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}H_n(x,y). \tag{II.3.47}$$

We have indeed, that:

$$\exp\left[\underline{z}^t\hat{H}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{H}\underline{w}\right]= \tag{II.3.48}$$
$$=\exp\left[(ax+by)\,t+\left(\frac{1}{2}\rho a\right)t^2\right]\exp\left[(bx+cy)\,u+\left(\frac{1}{2}\rho c\right)u^2\right]\exp\left(\rho b t u\right)$$

and then, by expliciting the exponentials:

$$\exp\left[\underline{z}^t\hat{H}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{H}\underline{w}\right]= \tag{II.3.49}$$
$$=\sum_{m=0}^{+\infty}\frac{t^m}{m!}H_m\left(ax+by,\frac{1}{2}\rho a\right)\sum_{n=0}^{+\infty}\frac{u^n}{n!}H_n\left(bx+cy,\frac{1}{2}\rho c\right)\sum_{q=0}^{+\infty}\frac{\rho^q b^q}{q!}t^q u^q$$

that is:

$$\exp\left[\underline{z}^t\hat{H}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{H}\underline{w}\right]= \tag{II.3.50}$$
$$=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{q=0}^{+\infty}\frac{t^{m+q}}{m!}\frac{u^{n+q}}{n!}\frac{\rho^q b^q}{q!}H_m\left(ax+by,\frac{1}{2}\rho a\right)H_n\left(bx+cy,\frac{1}{2}\rho c\right).$$

By setting $m+q=k$ and $n+q=j$, we can rearrange the above expression in the form:

$$\exp\left[\underline{z}^t\hat{H}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{H}\underline{w}\right]= \tag{II.3.51}$$
$$=\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{q=0}^{+\infty}\frac{t^k}{(k-q)!}\frac{u^j}{j!(j-q)!}\frac{\rho^q b^q}{q!}H_{k-q}\left(ax+by,\frac{1}{2}\rho a\right)H_{j-q}\left(bx+cy,\frac{1}{2}\rho c\right)$$

and by manipulating the terms involving the factorial, we end up with:

$$\exp\left[\underline{z}^t\hat{H}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{H}\underline{w}\right]= \tag{II.3.52}$$
$$=\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{t^k}{k!(k-q)!}\frac{u^j}{j!(j-q)!}\cdot$$
$$\cdot\sum_{q=0}^{+\infty}\frac{k!j!}{q!}H_{k-q}\left(ax+by,\frac{1}{2}\rho a\right)H_{j-q}\left(bx+cy,\frac{1}{2}\rho c\right)\rho^q b^q$$

that is:

$$\exp\left[\underline{z}^t\hat{H}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{H}\underline{w}\right]= \tag{II.3.53}$$
$$=\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{t^k}{k!}\frac{u^j}{j!}\cdot$$
$$\cdot\sum_{q=0}^{\min(k,j)}q!\begin{pmatrix}k\\q\end{pmatrix}\begin{pmatrix}j\\q\end{pmatrix}H_{k-q}\left(ax+by,\frac{1}{2}\rho a\right)H_{j-q}\left(bx+cy,\frac{1}{2}\rho c\right)\rho^q b^q.$$

Without loss of generality, we can set $k=m$ and $j=n$.

$$\exp\left[\underline{z}^t\hat{H}\underline{w}+\frac{1}{2}\rho\underline{w}^t\hat{H}\underline{w}\right]= \tag{II.3.54}$$
$$=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{t^m}{m!}\frac{u^n}{n!}\cdot$$
$$\cdot\sum_{q=0}^{\min(m,n)}q!\begin{pmatrix}m\\q\end{pmatrix}\begin{pmatrix}n\\q\end{pmatrix}H_{m-q}\left(ax+by,\frac{1}{2}\rho a\right)H_{n-q}\left(bx+cy,\frac{1}{2}\rho c\right)\rho^q b^q.$$

From Definition II.5, we can conclude that the explicit form of the generalized two-index, two-variable Hermite polynomials of the type $H_{m,n}(x,y;\rho)$ is given by the following relation:

$$H_{m,n}(x,y;\rho)= \tag{II.3.55}$$
$$=\sum_{q=0}^{\min(m,n)}q!\begin{pmatrix}m\\q\end{pmatrix}\begin{pmatrix}n\\q\end{pmatrix}H_{m-q}\left(ax+by,\frac{1}{2}\rho a\right)H_{n-q}\left(bx+cy,\frac{1}{2}\rho c\right)\rho^q b^q.$$

We note that the polynomials $H_{m,n}(x,y;\rho)$ are also a generalization of the five-variable, two-index Hermite polynomials $H_{m,n}(x,y;\xi,\eta|\chi)$ whose the explict form has been stated in equation (II.3.20). In fact, by setting $\eta = \rho$, we immediately obtain:

$$H_{m,n}\left(x, y; \rho\right) = H_{m,n}\left(ax + by, \frac{1}{2}\rho a; bx + cy, \frac{1}{2}\rho c | \rho b\right). \qquad \text{(II.3.56)}$$

The same considerations could be done relatively to the associated Hermite polynomials of the form $G_{m,n}(x, y)$. We can in fact introduce the generalized two-index, two-variable Hermite polynomials by setting:

$$e^{\underline{z}^t \underline{k} - \rho \underline{k} \hat{M}^{-1} \underline{k}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{r^m}{m!} \frac{s^n}{n!} G_{m,n}(x, y; \rho) \qquad \text{(II.3.57)}$$

where, again, ρ is a real number and $\underline{z} = \binom{x}{y}$ and $\underline{k} = \binom{r}{s}$ two vectors of the space $\mathbb{R}^2$.

By following the same procedure shown above, for the polynomials $H_{m,n}(x, y; \rho)$, we can easily state the explicit form of the polynomials $G_{m,n}(x, y; \rho)$:

$$G_{m,n}\left(x, y; \rho\right) = \sum_{q=0}^{\min(m,n)} q! \begin{pmatrix} m \\ q \end{pmatrix} \begin{pmatrix} n \\ q \end{pmatrix} H_{m-q}\left(x, -\frac{1}{\Delta}\rho c\right) H_{n-q}\left(y, -\frac{1}{\Delta}\rho a\right) \rho^q b^q. \qquad \text{(II.3.58)}$$

In Corollary II.1, we have shown that the associated Hermite polynomials $G_{m,n}(x, y)$ can be written in terms of the polynomials $H_{m,n}\left(x, y; \xi, \eta | \chi\right)$ and we have just seen the same, relative to the generalized Hermite polynomials $H_{m,n}(x, y; \rho)$; it is evident that we can represent the generalized associated Hermite polynomials of the form $G_{m,n}(x, y; \rho)$ in terms of the five-variable, two-index Hermite polynomials $H_{m,n}\left(x, y; \xi, \eta | \chi\right)$. In fact, by setting:

$x \to x$

$y \to -\frac{c\rho}{\Delta}$

$\xi \to y$

$\eta \to -\frac{a\rho}{\Delta}$

$\chi \to b\rho$

we can easily conclude with:

$$G_{m,n}\left(x, y; \rho\right) = H_{m,n}\left(x, -\frac{c\rho}{\Delta}; y, \frac{a\rho}{\Delta} | \rho b\right). \qquad \text{(II.3.59)}$$

The properties of the Hermite vectorial polynomials $H_{m,n}(x, y)$ and $G_{m,n}(x, y)$ are very useful and we will base on them to derive some important generalizations in the next chapter, regarding the concepts of bi-orthogonality for some particular Hermite functions.

Chapter III

Orthogonal Hermite Functions

The concepts and techniques presented in the previous two chapters will be used here to introduce the Hermite functions which satisfy the properties of orthogonality. In particular, we first discuss the Hermite orthogonal functions obtained from the ordinary Hermite polynomials, and then we generalize to the two variables case. Subsequently, on the basis of what is shown in Chapter II, relatively to the vectorial Hermite polynomials, we will introduce the bi-orthogonal Hermite functions and derive some relevant operational properties.

III.1 Orthogonal Hermite functions of one and two variables

In Chapter I we have introduced the two-variable Hermite polynomials $He_m(x, y)$ and the ordinary one-variable Hermite polynomials $He_m(x)$, whose their explicit forms has been given in (I.1.5) and (I.1.10).
It is immediately to note that the generating function of the Hermite polynomials $He_m(x)$ has the form:

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x), \qquad \text{(III.1.1)}$$

since, as we have shown in Proposition I.2, relatively to the two-variable Hermite polynomials of the type $He_m(x, y)$, they solve the following differential difference equation:

$$\frac{d}{dx}D_n(x) = nD_{n-1}(x) \qquad \text{(III.1.2)}$$
$$D_n(0) = \frac{n!(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!2^{\frac{n}{2}}}$$

where n is even.

It is well known that the most important property staisfied by the Hermite polynomials is the orthogonality. By using this important aspect we will introduce the related Hermite functions to derive many other relations involving the Hermite polynomials of the type $He_m(x)$. We start to prove an important identity for the ordinary Hermite polynomials.

Proposition III.1

The ordinary Hermite polynomials $He_m(x)$ satisfy the following Rodrigues formula:

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \left(\frac{d}{dx}\right)^n \left(e^{-\frac{x^2}{2}}\right) \qquad \text{(III.1.3)}$$

Proof

By starting from the generating function relation, presented above (see eq.(III.1.1)), we can manipulate the argument of the exponential to obtain:

$$e^{-\frac{x^2}{2}+xt-\frac{t^2}{2}} e^{\frac{x^2}{2}} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x) \qquad \text{(III.1.4)}$$

and then:

$$e^{-\frac{1}{2}(x-t)^2} = e^{-\frac{x^2}{2}} \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x). \qquad \text{(III.1.5)}$$

In Introduction, we have presented the shift operator for a function $f(x)$, which is analytic in a neighborhood of the origin, and we have seen its action as given by (0.1.2).

After the above considerations, we can recast the l.h.s. of the relation (III.1.5) in the following form:

$$e^{-\frac{1}{2}(x-t)^2} = e^{-t\frac{d}{dx}} \left(e^{-\frac{x^2}{2}}\right) \qquad \text{(III.1.6)}$$

and then, we have:

$$e^{\frac{x^2}{2}} e^{-t\frac{d}{dx}} \left(e^{-\frac{x^2}{2}} \right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x). \tag{III.1.7}$$

The exponential operator in the previous equation can be expanded to obtain:

$$e^{\frac{x^2}{2}} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n t^n}{n!} \left(\frac{d}{dx} \right)^n \right] \left(e^{-\frac{x^2}{2}} \right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x). \tag{III.1.8}$$

At this point, we note that the terms acting on the exponential function $e^{-\frac{x^2}{2}}$ give only an operational contribute except for the term $\left(\frac{d}{dx}\right)^n$; we can rewrite the previous relation in the following more convenient form:

$$e^{\frac{x^2}{2}} \sum_{n=0}^{+\infty} \frac{(-1)^n t^n}{n!} \left(\frac{d}{dx} \right)^n \left(e^{-\frac{x^2}{2}} \right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x). \tag{III.1.9}$$

Finally, by equating the terms of the same power of n we immediately obtain the thesis of the proposition, that is the Rodrigues formula.
It is well known that the orthogonal polynomials are defined through a weight function and they are determinated to less than a constant; since the Hermite polynomials belong to the family of classical orthogonal polynomials, they are defined, as we have seen in the previous chapters, as solution of a ordinary differential equation of hypergeometric type. We can now investigate the properties and the related relations of the Hermite polynomials under the point of view of their orthogonality.

Proposition III.2

The ordinary Hermite polynomials are orthogonal on the interval $(-\infty, +\infty)$ respect to the weight function

$$e^{-\frac{x^2}{2}} \tag{III.1.10}$$

that is:

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} He_n(x) He_m(x) dx = n! \sqrt{2\pi} \delta_{n,m}. \tag{III.1.11}$$

Proof

By using the Rodrigues Formula, we can recast the integral of the statement in the following form:

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} He_n(x) He_m(x) dx = (-1)^n \int_{-\infty}^{+\infty} \left(\frac{d}{dx}\right)^n \left(e^{-\frac{x^2}{2}}\right) He_m(x) dx. \tag{III.1.12}$$

By solving the integral on the r.h.s. of the above equation, by using the method by parts, we get:

$$(-1)^n \int_{-\infty}^{+\infty} \left(\frac{d}{dx}\right) \left(e^{-\frac{x^2}{2}}\right) He_m(x) dx = \tag{III.1.13}$$

$$= (-1)^n \left[\left(\frac{d}{dx}\right) \left(e^{-\frac{x^2}{2}}\right) He_m(x) - \int_{-\infty}^{+\infty} \left(\frac{d}{dx}\right) \left(e^{-\frac{x^2}{2}}\right) \frac{d}{dx} He_m(x) dx \right]_{-\infty}^{+\infty}$$

and then:

$$(-1)^n \int_{-\infty}^{+\infty} \left(\frac{d}{dx}\right)^n \left(e^{-\frac{x^2}{2}}\right) He_m(x) dx = \tag{III.1.14}$$

$$= (-1)^n \left\{ \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left[\left(\frac{d}{dx}\right)^{n-1} \left(e^{-\frac{x^2}{2}}\right) He_m(x) \right]_a^b + \right.$$

$$\left. -(-1) \left[- \int_{-\infty}^{+\infty} \left(\frac{d}{dx}\right)^{n-1} \left(e^{-\frac{x^2}{2}}\right) \frac{d}{dx} He_m(x) dx \right] \right\}.$$

By noting that the limit in the r.h.s. of the previous relation gives zero and by using the recurrence relation satisfied by the ordinary Hermite polynomials:

$$\frac{d}{dx} He_n(x) = n He_{n-1}(x) \tag{III.1.15}$$

we can obtain the following expression:

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} He_n(x) He_m(x) dx = (-1)^{n+1} m! \int_{-\infty}^{+\infty} \left(\frac{d}{dx}\right)^{n-m} \left(e^{-\frac{x^2}{2}}\right) dx. \tag{III.1.16}$$

Regarding the integral on the r.h.s. of the above relation, we note that:

$$\int_{-\infty}^{+\infty} \left(\frac{d}{dx}\right)^s \left(e^{-\frac{x^2}{2}}\right) dx = 0 \tag{III.1.17}$$

after setting $n-m=s$, assuming $n \neq m$, while, if $n=m$, we have:

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \tag{III.1.18}$$

and then the proposition is completely proved.

The orthogonality property satisfied by the Hermite polynomials $He_m(x)$ suggests us to introduce a family of functions, based on the Hermite polynomials themselves in such a way as to derive similar properties.

Definition III.1

Let be the ordinary Hermite polynomials of the type $He_m(x)$, we will call one-variable Hermite function, the function defined by the following relation:

$$he_m(x) = \left(\frac{1}{\sqrt{2\pi} m!}\right)^{\frac{1}{2}} He_m(x) e^{-\frac{x^2}{4}}. \tag{III.1.19}$$

Proposition III.3

The one-variable Hermite functions of the type $he_m(x)$ are orthonormal on the interval $(-\infty, +\infty)$, that is:

$$\int_{-\infty}^{+\infty} he_n(x) he_m(x) dx = \delta_{n,m}. \tag{III.1.20}$$

Proof

By substituting in the integral the explicit form of the Hermite functions $he_m(x)$, we get:

$$\int_{-\infty}^{+\infty} he_n(x) he_m(x) dx = \tag{III.1.21}$$

$$\int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi} n!}\right)\left(\frac{1}{\sqrt{2\pi} m!}\right) e^{-\frac{x^2}{4}} e^{-\frac{x^2}{4}} He_n(x) He_m(x) dx =$$

$$= \left(\frac{1}{2\pi n! m!}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e He_n(x) He_m(x) dx.$$

Since the Hermite polynomials $He_m(x)$ are orthogonal on the interval $(-\infty, +\infty)$ with the weight function $e^{-\frac{x^2}{2}}$ (see Proposition III.2), we obtain:

$$\int_{-\infty}^{+\infty} he_n(x) he_m(x) dx = \left(\frac{1}{n!m!}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2\pi}} n! \sqrt{2\pi}\delta_{n,m} = \sqrt{\frac{n!}{m!}}\delta_{n,m} \qquad \text{(III.1.22)}$$

and then the thesis follows immediately.

Proposition III.4

The one-variable orthogonal Hermite functions $he_m(x)$ satisfied the following recurrence relations:

$$2\frac{d}{dx} he_m(x) = \sqrt{m} he_{m-1}(x) - \sqrt{m+1} he_{m+1}(x), \qquad \text{(III.1.23)}$$

$$x he_m(x) = \sqrt{m} he_{m-1}(x) + \sqrt{m+1} he_{m+1}(x). \qquad \text{(III.1.24)}$$

Proof

By deriving with respect to x both sides of equation (III.1.19), we have:

$$\frac{d}{dx} hm_n(x) = \left(\frac{1}{\sqrt{2\pi} m!}\right)^{\frac{1}{2}} \frac{d}{dx}\left(He_m(x) e^{-\frac{x^2}{4}}\right) \qquad \text{(III.1.25)}$$

and by using the recurrence relation (III.1.15) showed in Proposition III.2, we can write:

$$\frac{d}{dx} he_m(x) = \left(\frac{1}{\sqrt{2\pi} m!}\right)^{\frac{1}{2}} \left[m He_{m-1}(x) e^{-\frac{x^2}{4}} - \frac{x}{2} He_m(x) e^{-\frac{x^2}{4}} \right]. \qquad \text{(III.1.26)}$$

Futhermore, it is easy to prove that the ordinary Hermite polynomial $He_m(x)$ satisfies the following relation:

$$He_m(x) = \frac{1}{x}\left[m He_{m-1}(x) + He_{m+1}(x) \right] \qquad \text{(III.1.27)}$$

which helps us to write (III.1.26) in the form:

$$\frac{d}{dx} he_m(x) = \left(\frac{1}{\sqrt{2\pi} m!}\right)^{\frac{1}{2}} \left[m He_{m-1}(x) e^{-\frac{x^2}{4}} - \left(\frac{1}{2} e^{-\frac{x^2}{4}} \left(m He_{m-1}(x) + He_{m+1}(x) \right) \right) \right]. \qquad \text{(III.1.28)}$$

By substituting the expression of the Hermite polynomials $He_{m-1}(x)$ and $He_{m+1}(x)$ in terms of the orthogonal Hermite functions $he_m(x)$ (see Definition III.1), we get:

$$\frac{d}{dx}he_m(x) = \tag{III.1.29}$$

$$\left(\frac{1}{\sqrt{2\pi}m!}\right)^{\frac{1}{2}}\left[me^{-\frac{x^2}{4}}\left(\sqrt{2\pi}(m-1)!\right)^{\frac{1}{2}}e^{\frac{x^2}{4}}he_{m-1}(x)+\right.$$
$$\left.-\frac{1}{2}e^{-\frac{x^2}{4}}n\left(\sqrt{2\pi}(m-1)!\right)^{\frac{1}{2}}e^{\frac{x^2}{4}}he_{m-1}(x)-\frac{1}{2}e^{-\frac{x^2}{4}}\left(\sqrt{2\pi}(m+1)!\right)^{\frac{1}{2}}e^{\frac{x^2}{4}}he_{m+1}(x)\right]$$

and then:

$$\frac{d}{dx}he_m(x) = \frac{m[(m-1)!]^{\frac{1}{2}}}{(m!)^{\frac{1}{2}}}he_{m-1}(x) - \frac{1}{2}\frac{m[(m-1)!]^{\frac{1}{2}}}{(m!)^{\frac{1}{2}}}he_{m-1}(x) - \frac{1}{2}\frac{m[(m+1)!]^{\frac{1}{2}}}{(m!)^{\frac{1}{2}}}he_{m+1}(x) \tag{III.1.30}$$

which proves the first relation of the statement. To completely prove the proposition, we start to note that the recurrence relation (III.1.27) verified by the Hermite polynomials $He_m(x)$, can be recast in the following form:

$$xHe_n(x) = He_{n+1}(x) + nHe_{n-1}(x) \tag{III.1.31}$$

and by substituting the expressions of the ordinary Hermite polynomials in terms of the related Hermite functions, we immediately obtain:

$$x\left[\left(\sqrt{2\pi}m!\right)^{\frac{1}{2}}e^{\frac{x^2}{4}}he_m(x)\right] = \tag{III.1.32}$$
$$= \left(\sqrt{2\pi}(m+1)!\right)^{\frac{1}{2}}e^{\frac{x^2}{4}}he_{m+1}(x) + m\left(\sqrt{2\pi}(m-1)!\right)^{\frac{1}{2}}e^{\frac{x^2}{4}}he_{m-1}(x)$$

which prove the second recurrence relations and then the thesis.

In the second chapter, we have presented the shift operators related to the two-index, two-variable Hermite polynomials of the type $H_{m,n}(x,y)$ which helped us to prove important realtions, in particular they solved partial differential equations contained in Proposition II.5. We can follow the same procedure to explore the differential characteristics involving the orthogonal Hermite functions of the type $he_m(x)$. By manipulating the recurrence relations stated in Proposition III.4, we easily obtain:

$$\left(\frac{d}{dx}+\frac{x}{2}\right)he_m(x) = \sqrt{m}he_{m-1}(x) \tag{III.1.33}$$

$$\left(-\frac{d}{dx}+\frac{x}{2}\right)he_m(x) = \sqrt{m+1}he_{m+1}(x) \tag{III.1.34}$$

which show the action on the Hermite function. Then, we can explicit set:

$$\hat{a}_- = \left(\frac{d}{dx} + \frac{x}{2}\right) \qquad \text{(III.1.35)}$$
$$\hat{a}_+ = \left(-\frac{d}{dx} + \frac{x}{2}\right)$$

and rewrite in a formal way the previous relations:

$$\hat{a}_- he_m(x) = \sqrt{m} he_{m-1}(x) \qquad \text{(III.1.36)}$$
$$\hat{a}_+ he_m(x) = \sqrt{m+1} he_{m+1}(x).$$

The shift operators introduced in Chapter II (see Definition II.3) were dependent on discrete parameters related to the Hermite polynomials $H_{m,n}(x,y)$, while the above operators do not change with the index function. It is immediate to note that the following relation holds:

$$\hat{a}_+ \hat{a}_- he_m(x) = m he_m(x) \qquad \text{(III.1.37)}$$

which can be used to state the following result:

Theorem III.1

The one-variable orthogonal Hermite functions $he_m(x)$ solved the following ordinary differential equations:

$$\left[\frac{d^2}{dx^2} - \frac{x^2}{4} + \left(m + \frac{1}{2}\right)\right] he_m(x) = 0. \qquad \text{(III.1.38)}$$

Proof

Expliciting the operatorial relation (III.1.37), we have:

$$\left(-\frac{d}{dx} + \frac{x}{2}\right)\left(\frac{d}{dx} + \frac{x}{2}\right) he_m(x) = m he_m(x) \qquad \text{(III.1.39)}$$

then:

$$\left(-\frac{d^2}{dx^2} - \frac{x}{2}\frac{d}{dx} - \frac{1}{2} + \frac{x}{2}\frac{d}{dx} + \frac{x^2}{4}\right) he_m(x) = m he_m(x) \qquad \text{(III.1.40)}$$

and finally:

$$\left(-\frac{d^2}{dx^2} + \frac{x^2}{4} - \frac{1}{2} - m\right) he_m(x) = 0 \qquad \text{(III.1.41)}$$

which completely proves the statement of the theorem.

At the beginning of this chapter, we have presented the generating function of the Hermite polynomial $He_m(x)$ and in Definition III.1 we have introduced the orthogonal Hermite function $he_m(x)$, based on the ordinary Hermite polynomials. It is now possible to derive the generating function for these type of Hermite functions, by manipulating the relations (III.1.1) and (III.1.19). We have in fact:

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{m=0}^{+\infty} \frac{t^m}{m!}\left(\sqrt{2\pi}m!\right)^{\frac{1}{2}} e^{\frac{x^2}{4}} he_m(x) \tag{III.1.42}$$

which immediately gives the link between the Hermite function $he_m(x)$ and its generating function, that is:

$$\frac{1}{\left(\sqrt{2\pi}\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-t)^2 - \frac{x^2}{4}\right) = \sum_{m=0}^{+\infty} \frac{t^m}{(m!)^{\frac{1}{2}}} he_m(x) e^{-\frac{x^2}{2}}. \tag{III.1.43}$$

In the same way, we can derive the analogous Rodrigues formula for the orthogonal Hermite functions. In fact, by substituting in the Rodrigues formula, related to the ordinary Hermite polynomials $He_m(x)$ stated in Proposition III.1, the expression of the polynomials $He_m(x)$ in terms of the functions $he_m(x)$ (see equation (III.1.19)), we get:

$$\left(\sqrt{2\pi}m!\right)^{\frac{1}{2}} e^{\frac{x^2}{4}} he_m(x) = (-1)^m e^{\frac{x^2}{2}} \left(\frac{d}{dx}\right)^m \left(e^{-\frac{x^2}{2}}\right) \tag{III.1.44}$$

and rearranging the terms we end up with the following expression:

$$he_m(x) = \frac{1}{\left(\sqrt{2\pi}\right)^{\frac{1}{2}}} (-1)^m \frac{1}{(m!)^{\frac{1}{2}}} e^{\frac{x^2}{4}} \left(\frac{d}{dx}\right)^m \left(e^{-\frac{x^2}{2}}\right) \tag{III.1.45}$$

which represent the Rodrigues formula for the orthogonal Hermite functions $he_m(x)$.

In Chapter I, we have presented the generalized two-variable Hermite polynomials $He_m(x, y)$ and we have derived as simpliest case the ordinary Hermite polynomials $He_m(x)$; we have also introduced the two-variable Hermite polynomials of the type $H_m(x, y)$ and we have shown their explicit form in (I.1.26).

Since we have introduced the orthogonal Hermite functions of one variable, by using the structure and the properties of the ordinary Hermite polynomials $He_m(x)$, we expect that it is also possible to define analogous Hermite functions of two variables, which are orthogonal, using the generalized two-variable Hermite polynomials. This is obviously possible, but we will face the question starting directly by the definition of the one-variable Hermite functions $he_m(x)$.

Definition III.2

Let be x and y two real variables and let $he_m(x)$ be the one-variable Hermite function. We define the two-variable Hermite function $he_m(x, y)$, as the function given by the following expression:

$$he_m(x,y) = \sum_{r=0}^{[m/2]} \sqrt{\frac{m!}{(m-2r!r!}} he_{m-2r}(x) he_r(y). \tag{III.1.46}$$

Theorem III.2

The two-variable Hermite functions $he_m(x, y)$ are orthogonal functions on the interval $(-\infty,+\infty)$.

Proof

We have to prove that the following integral:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} he_n(x,y) he_m(x,y) dx \tag{III.1.47}$$

is a finite number, thus the functions are orthogonal. By substituting the explicit expression of the two-variable Hermite functions $he_m(x, y)$ given in Definition III.2, we get:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} he_n(x,y) he_m(x,y) dx = \tag{III.1.48}$$

$$\sum_{r=0}^{[n/2]} \sum_{s=0}^{[m/2]} \sqrt{\frac{n!m!}{(n-2r)!(m-2s)!r!s!}} \int_{-\infty}^{+\infty} he_{n-2r}(x) he_{m-2s}(x) dx \int_{-\infty}^{+\infty} he_r(y) he_s(y) dy$$

and since the one-variable Hermite functions are orthonormal on the interval $(-\infty,+\infty)$, (see equation (III.1.20)), we have:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} he_n(x,y) he_m(x,y) dx = \tag{III.1.49}$$

$$= \sum_{r=0}^{[n/2]} \sum_{s=0}^{[m/2]} \sqrt{\frac{n!m!}{(n-2r)!(m-2s)!r!s!}} \int_{-\infty}^{+\infty} he_{n-2r}(x) he_{m-2s}(x) dx \delta_{r,s}.$$

We note that, in the above summations, all the terms are zero except when $r = s$ and then we can rewrite the previous relation in the form:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} he_n(x,y) he_m(x,y) dx = \tag{III.1.50}$$

$$= \sum_{r=0}^{[n/2]} \sum_{r=0}^{[m/2]} \sqrt{\frac{n!m!}{((n-2r)!)^2 (r!)^2}} \int_{-\infty}^{+\infty} he_{n-2r}(x) he_{m-2r}(x) dx$$

and by applying again the orthonormal property of the one-variable Hermite functions $he_m(x)$, we similarly obtain:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} he_n(x,y) he_m(x,y) dx = \sum_{r=0}^{[n/2]} \sum_{r=0}^{[m/2]} \sqrt{\frac{n!m!}{((n-2r)!)^2 (r!)^2}} \delta_{n,m}. \tag{III.1.51}$$

Also in this case, the only non trivial value it is obtained for $n = m$, and so we can conclude with:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} he_n(x,y) he_n(x,y) dx = \sum_{r=0}^{[n/2]} \frac{n!}{(n-2r)!r!} \tag{III.1.52}$$

which proves the orthogonality of the two-variable Hermite functions $he_m(x, y)$. It could be useful observe that the term obtained in the proof of Theorem III.2, (see equation (III.1.52)) can be read as a special case of the two-variable Hermite polynomials of the type $H_m(x, y)$:

$$H_n\left(\frac{1}{2}, -1\right) = \sum_{r=0}^{[n/2]} \frac{n!}{(n-2r)!r!}. \tag{III.1.53}$$

We can derive the generating function for the two-variable orthogonal Hermite functions $he_m(x, y)$, by using the structure and the identities of the Hermite polynomials. For this purpose, we use a different class of two-variable Hermite polynomials, introduced in the first chapter, which we have indicated with $He'_m(x, y)$, with the generating function given by (I.1.27).

By manipulating the argument of the exponential, we obtain:

$$\begin{aligned}
&\exp\left(xt-\frac{t^2}{2}+yt^2-\frac{t^4}{2}\right)= \\
&=\exp\left(xt-\frac{t^2}{2}\right)\exp\left(yt^2-\frac{t^4}{2}\right)= \\
&=\sum_{m=0}^{+\infty}\frac{t^m}{m!}He_m(x)\sum_{r=0}^{+\infty}\frac{t^{2r}}{r!}He_r(y)
\end{aligned} \tag{III.1.54}$$

and by setting $m+2r=n$, after rearranging the indexes in the above summations, we end up with:

$$He'_m(x,y)=m!\sum_{r=0}^{[m/2]}\frac{1}{(m-2r)!r!}He_{m-2r}(x)He_r(y) \tag{III.1.55}$$

which gives an expression of the two-variable Hermite polynomials $He'_m(x,y)$ in terms of the ordinary one-variable Hermite polynomials. We will use the relation showed above to state the link between the two-variable orthogonal Hermite functions $he_m(x,y)$ and their generating function.

We start substituting in the definition of the functions $he_m(x,y)$ (see equation (III.1.46)) the expression of the one-variable orthogonal Hermite functions $he_m(x)$ given in Definition III.1:

$$\begin{aligned}
&he_m(x,y)= \\
&=\sum_{r=0}^{[m/2]}\sqrt{\frac{m!}{(m-2r)!r!}}\left(\frac{1}{\sqrt{2\pi}(m-2r)!}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{2\pi}r!}\right)^{\frac{1}{2}}e^{-\frac{x^2}{4}}e^{-\frac{y^2}{4}}He_{m-2r}(x)He_r(y)
\end{aligned} \tag{III.1.56}$$

which gives:

$$he_m(x,y)=e^{-\frac{x^2}{4}}e^{-\frac{y^2}{4}}\frac{1}{\sqrt{2\pi}}\sqrt{m!}\sum_{r=0}^{[m/2]}\frac{1}{(m-2r)!r!}He_{m-2r}(x)He_r(y) \tag{III.1.57}$$

and by substituting expression (III.1.55) stated above, we have:

$$he_m(x,y)=\frac{\sqrt{m!}}{\sqrt{2\pi}}e^{-\frac{x^2}{4}}e^{-\frac{y^2}{4}}\frac{He'_m(x,y)}{m!}. \tag{III.1.58}$$

By expliciting the two-variable Hermite polynomials $He'_m(x,y)$ in terms of the Hermite functions $he_m(x,y)$, the previous equation reads:

$$He'_m(x,y)=\frac{m!}{\sqrt{m!}}\sqrt{2\pi}e^{\frac{x^2+y^2}{4}}he_m(x,y) \tag{III.1.59}$$

that once replaced in the expression (III.1.54), gives:

$$\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) = \sum_{m=0}^{+\infty} \frac{t^m}{m!}\left[\frac{m!}{\sqrt{m!}}\sqrt{2\pi}e^{\frac{x^2+y^2}{4}} he_m(x,y)\right] \quad \text{(III.1.60)}$$

and then, we can finally state the expression of the generating function of the two-variable Hermite functions $he_m(x,y)$:

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-t)-\frac{1}{2}(y-t^2)}e^{\frac{x^2+y}{4}} = \sum_{m=0}^{+\infty} \frac{t^m}{\sqrt{m!}} he_m(x,y). \quad \text{(III.1.61)}$$

In the first chapter, we have proved many recurrence relations involving the generalized two-variable Hermite polynomials of type $He_m(x,y)$ and $H_m(x,y)$, and we have stated interesting identities from the operational point of view. It is possible to generalized those results to derive similar relations for the two-variable Hermite functions $he_m(x,y)$. The starting point is the link between the polynomials $He^{'}_m(x,y)$ and the functions $he_m(x,y)$ showed above, in equations (III.1.58) and (III.1.59). By deriving with respect to x in relation (III.1.58), we have:

$$\frac{\partial}{\partial x} he_m(x,y) = \frac{1}{\sqrt{2\pi m!}}e^{-\frac{y^2}{4}}\left[-\frac{x}{2}e^{-\frac{x^2}{4}}He^{'}_m(x,y) + e^{-\frac{x^2}{4}}\frac{\partial}{\partial x}He^{'}_m(x,y)\right] \quad \text{(III.1.62)}$$

and then:

$$\frac{\partial}{\partial x} he_m(x,y) = -\frac{1}{\sqrt{2\pi m!}}\frac{x}{2}e^{-\frac{x^2+y^2}{4}}He^{'}_m(x,y) + \frac{1}{\sqrt{2\pi m!}}e^{-\frac{x^2+y^2}{4}}mHe^{'}_{m-1}(x,y)(x,y) \quad \text{(III.1.63)}$$

to conclude with the following generalization:

$$\frac{\partial}{\partial x} he_m(x,y) = -\frac{x}{2}he_m(x,y) + \sqrt{m}he_{m-1}(x,y). \quad \text{(III.1.64)}$$

In the same way it is possible to state an analogous recurrence relation satisfied by the Hermite functions $he_m(x,y)$. In fact, by deriving with respect to y in equation (III.1.58), we obtain:

$$\frac{\partial}{\partial y} he_m(x,y) = \frac{1}{\sqrt{2\pi m!}}e^{-\frac{x^2}{4}}\left[-\frac{y}{2}e^{-\frac{x^2}{4}}He^{'}_m(x,y) + e^{-\frac{y^2}{4}}\frac{\partial}{\partial y}He^{'}_m(x,y)\right] \quad \text{(III.1.65)}$$

which gives:

$$\frac{\partial}{\partial y} he_m(x,y) = -\frac{1}{\sqrt{2\pi m!}}\frac{y}{2}e^{-\frac{x^2+y^2}{4}}He^{'}_m(x,y) + \frac{1}{\sqrt{2\pi m!}}e^{-\frac{x^2+y^2}{4}}m(m-1)He^{'}_{m-2}(x,y). \quad \text{(III.1.66)}$$

By using the identity in equation (III.1.59), we can finally state the second generalized recurrence relation for the two-variable Hermite functions $he_m(x,y)$:

$$\frac{\partial}{\partial y}he_m(x,y) = -\frac{y}{2}he_m(x,y) + \sqrt{m(m-1)}he_{m-2}(x,y). \tag{III.1.67}$$

A further recurrence relation involving the Hermite functions $he_m(x,y)$ can be deduced by operating directly in the equation linking the generalized Hermite polynomials of the type $He^{'}_m(x,y)$ and its generating function. We remind, once again, that the generating function of the polynomials $He^{'}_m(x,y)$ has the following expression:

$$\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) = \sum_{m=0}^{+\infty} \frac{t^m}{m!}He^{'}_m(x,y) \tag{III.1.68}$$

and by deriving both sides respect to t, we obtain:

$$(x - t + 2yt - 2t^3)\sum_{m=0}^{+\infty} \frac{t^m}{m!}He^{'}_m(x,y) = \sum_{m=0}^{+\infty} m\frac{t^{m-1}}{m!}He^{'}_m(x,y). \tag{III.1.69}$$

By exploiting the terms in the above relation, we can write:

$$\begin{aligned} & x\sum_{m=0}^{+\infty} \frac{t^m}{m!}He^{'}_m(x,y) - \sum_{m=0}^{+\infty} \frac{t^{m+1}}{m!}He^{'}_m(x,y) + \\ & +2y\sum_{m=0}^{+\infty} \frac{t^{m+1}}{m!}He^{'}_m(x,y) - 2\sum_{m=0}^{+\infty} \frac{t^{m+3}}{m!}He^{'}_m(x,y) = \\ & = \sum_{m=0}^{+\infty} m\frac{t^{m-1}}{m!}He^{'}_m(x,y) \end{aligned} \tag{III.1.70}$$

and by equating the terms of the same power of m, we have:

$$x\frac{He^{'}_m(x,y)}{m!} + (2y-1)\frac{He^{'}_{m-1}(x,y)}{(m-1)!} - 2\frac{He^{'}_{m-3}(x,y)}{(m-3)!} = \frac{m+1}{(m+1)!}He^{'}_{m+1}(x,y) \tag{III.1.71}$$

which gives the important recurrence relation for the generalized Hermite polynomials $He^{'}_m(x,y)$:

$$xHe^{'}_m(x,y) + (2y-1)mHe^{'}_{m-1}(x,y) - 2\left[m(m-1)(m-2)\right]He^{'}_{m-3}(x,y) = He^{'}_{m+1}(x,y). \tag{III.1.72}$$

We can use the relation stated above to derive the analogous identity for the two-variable Hermite functions $he_m(x,y)$. In fact, by substituting the expression of the Hermite polynomials $He'_m(x,y)$ in terms of the Hermite functions $he_m(x,y)$, given by equation (III.1.58), we have:

$$\begin{aligned} &x\sqrt{m!}\sqrt{2\pi}e^{\frac{x^2+y^2}{4}}he_m(x,y)+(2y-1)m\sqrt{(m-1)!}\sqrt{2\pi}e^{\frac{x^2+y^2}{4}}he_{m-1}(x,y)+ \\ &\quad -2\left[m(m-1)(m-2)\right]\sqrt{(m-3)!}\sqrt{2\pi}e^{\frac{x^2+y^2}{4}}he_{m-3}(x,y)= \\ &\quad =\sqrt{(m+1)!}\sqrt{2\pi}e^{\frac{x^2+y^2}{4}}he_{m+1}(x,y) \end{aligned} \tag{III.1.73}$$

and we can finally conclude with:

$$\begin{aligned} &xhe_m(x,y)+(2y-1)\sqrt{m}he_{m-1}(x,y)-2\sqrt{m(m-1)(m-2)}he_{m-3}(x,y)= \\ &\quad =\sqrt{m+1}he_{m+1}(x,y). \end{aligned} \tag{III.1.74}$$

In this first section we have presented the properties of the Hermite functions of one and two variable. The structure of the presentation has been based on the concepts and the related operational properties, both of the ordinary one-variable Hermite polynomials and of the generalized two-variable Hermite polinomials of different types. By following the same idea used in the second chapter to introduce the two-index, two-variable Hermite polynomials of type $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$, we try to generalize the operational techniques related to the property of orthogonality satisfied by the Hermite polynomials of type $He_m(x)$ and $He'_m(x,y)$ for the case of vectorial Hermite polynomials. In the next section we will see how this type of generalization will determine a different type of feature linked to the concept of orthogonality.

III.2 Bi-orthogonal Hermite functions

In Chapter II, we have presented the two-index, two-variable Hermite polynomials of the type $H_{m,n}(x,y)$ and we have defined their associated $G_{m,n}(x,y)$, by deriving many properties and interesting identities for both type of generalized vectorial polynomials. It is now interesting to explore the possibility to find similar Hermite functions as those defined in the previous section of the present chapter, to obtain an extension of the concepts and the related identities satisfied from the Hermite polynomials $H_{m,n}(x,y)$ and their associated

$G_{m,n}(x,y)$. The structure of the vectorial extension of Hermite polynomials is based on the fact that a vector index acts on a vector variable or, what is the same, a couple of indexes act on a couple of variables. We have seen that many of the properties satisfied by this family of Hermite polynomials could be referred to the analogous ones satisfied by the ordinary Hermite polynomials of type $He_m(x)$ and their generalizations, but the cited properties, relevant to the polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$, have been deduced without making use of $He_m(x)$ properties, this means that they could not be obtained as natural extensions of those relevant to one-index Hermite polynomials. This suggests that we can not expect the same relation linking the two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ and the related Hermite functions we are going to define; we also see that the concept of orthogonality is not the same of that existing for the one-index Hermite polynomials of type $He_m(x)$ and $He'_m(x,y)$. We start indeed from this last point: we will prove that the vectorial Hermite polynomials of the type $H_{m,n}(x,y)$ and their associated $G_{m,n}(x,y)$ satisfied a bi-orthogonality condition instead the orthogonality condition, in the sense that the polynomials $H_{m,n}(x,y)$ are orthogonal with respect to the associated polynomials $G_{m,n}(x,y)$.

Theorem III.3

The two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ and their related associated $G_{m,n}(x,y)$ satisfy the following bi-orthogonality condition:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx H_{m,n}(x,y) G_{r,s}(x,y) e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} = \frac{2\pi}{\sqrt{\Delta}} m! n! \delta_{m,r} \delta_{n,s} \qquad \text{(III.2.1)}$$

where: $\underline{z} = \binom{x}{y}$ is a vector of space $\mathbb{R}^2$, and $\hat{M}$ is the matrix associated to the quadratic form

$$\begin{aligned} q(x,y) &= ax^2 + 2bxy + cy^2 \qquad \text{(III.2.2)} \\ a, c &> 0 \\ ac - b^2 &> 0 \end{aligned}$$

with a, b, c real numbers.

Proof

We have defined the Hermite polynomials $H_{m,n}(x,y)$ through their generating function in Definition II.2:

$$e^{\underline{z}^t \hat{M}\underline{w} - \frac{1}{2}\underline{w}^t \hat{M}\underline{w}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!} H_{m,n}(x,y) \tag{III.2.3}$$

where $\underline{z} = \binom{x}{y}$ and $\underline{w} = \binom{t}{u}$ are two vectors of space $\mathbb{R}^2$.

It is possible to recast the above equation in a more convenient form, by acting on the argument of the exponential, we have indeed:

$$e^{-\frac{1}{2}\left[(\underline{z}-\underline{w})^t \hat{M}(\underline{z}-\underline{w})\right]} = e^{-\frac{1}{2}\left[\underline{z}^t \hat{M}\underline{z}\right]} \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!} H_{m,n}(x,y) \tag{III.2.4}$$

which better outline the analogy between the structure of the generating functions related to the ordinary Hermite polynomials $He_m(x)$ and the two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$. This aspect allows us to obtain a generalization of Rodrigues formula showed for the ordinary Hermite polynomials. In fact by acting directly on the statement contained in Proposition III.1, we immediately have:

$$H_{m,n}(x,y) = (-1)^{m+n} e^{\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)} \frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[e^{-\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\right] \tag{III.2.5}$$

which represent the Rodrigues formula related to the Hermite polynomials $H_{m,n}(x,y)$.

The above identity could be recast in the following form:

$$e^{-\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)} H_{m,n}(x,y) = (-1)^{m+n} \frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[e^{-\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\right] \tag{III.2.6}$$

which allows us to rewrite the integral in the statement in an operational form:

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left[(-1)^{m+n} \frac{\partial^{m+n}}{\partial x^m \partial y^n}\left(e^{-\frac{1}{2}\underline{z}^t \hat{M}\underline{z}}\right) G_{r,s}(x,y)\right]. \tag{III.2.7}$$

We first start to evaluate the integral with respect to variable y:

$$(-1)^{m+n} \int_{-\infty}^{+\infty} \left[\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left(e^{-\frac{1}{2}\underline{z}^t \hat{M}\underline{z}}\right) G_{r,s}(x,y)\right] dy \tag{III.2.8}$$

which, integrating by parts, gives:

$$(-1)^{m+n}\int_{-\infty}^{+\infty}\left[\frac{\partial^{m+n}}{\partial x^m\partial y^n}\left(e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)G_{r,s}(x,y)\right]dy = \tag{III.2.9}$$

$$= (-1)^{m+(n+1)}\int_{-\infty}^{+\infty}\left[\frac{\partial^{m+(n-1)}}{\partial x^m\partial y^{n-1}}\left(e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\frac{\partial}{\partial y}G_{r,s}(x,y)\right]dy.$$

In Chapter II, we have proved many operational relations involving the Hermite polynomials $H_{m,n}(x,y)$ and their associated $G_{m,n}(x,y)$, by exploiting the formalism of the vectorial derivation. In this way, by deriving with respect to y in equation (II.2.7) of Definition II.2, we immediately get:

$$\frac{\partial}{\partial y}\sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty}\frac{k^m}{r!}\frac{h^n}{s!}G_{r,s}(x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{k^m}{r!}\frac{h^n}{s!}\frac{\partial}{\partial y}G_{r,s}(x,y) \tag{III.2.10}$$

that is:

$$sG_{r,s-1}(x,y) = \frac{\partial}{\partial y}G_{r,s}(x,y). \tag{III.2.11}$$

By substituting this last expression into the integral, we have the following relation:

$$(-1)^{m+n}\int_{-\infty}^{+\infty}\left[\frac{\partial^{m+n}}{\partial x^m\partial y^n}\left(e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)G_{r,s}(x,y)\right]dy = \tag{III.2.12}$$

$$= (-1)^{m+(n+1)}s\int_{-\infty}^{+\infty}\left[\frac{\partial^{m+(n-1)}}{\partial x^m\partial y^{n-1}}\left(e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)G_{r,s-1}(x,y)\right]dy.$$

Without loss of generality, we can suppose that $n \geq s$ and then, iterating the process on the index s, we finally obtain:

$$(-1)^{m+n}\int_{-\infty}^{+\infty}\left[\frac{\partial^{m+n}}{\partial x^m\partial y^n}\left(e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)G_{r,s}(x,y)\right]dy = \tag{III.2.13}$$

$$= (-1)^{m+(n+s)}s!\int_{-\infty}^{+\infty}\left[\frac{\partial^{m+(n-s)}}{\partial x^m\partial y^{n-s}}\left(e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)G_{r,0}(x,y)\right]dy$$

which is not trivial if and only if $n = s$.

Let $n = s$, the double integral in the statement, becomes:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx H_{m,n}(x,y) G_{r,s}(x,y) e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} = \tag{III.2.14}$$

$$= (-1)^{m+(n-s)} n! \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left[\frac{\partial^m}{\partial x^m} \left(e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} \right) G_{r,0}(x,y) \right]$$

which, once integrated by parts with respect to the variable x, gives:

$$(-1)^{m+(n+s)} n! \left\{ \int_{-\infty}^{+\infty} \left[\lim_{\substack{a\to-\infty \\ b\to+\infty}} \left(\frac{\partial^{m-1}}{\partial x^{m-1}} \left(e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} \right) G_{r,0}(x,y) \right) \right]_a^b + \right. \tag{III.2.15}$$

$$\left. -(-1) \left[- \int_{-\infty}^{+\infty} \frac{\partial^{m-1}}{\partial x^{m-1}} \left(e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} \right) \frac{\partial}{\partial x} G_{r,0}(x,y) \right] dy \right\} .$$

By operating in the same way seen in equation (III.2.11), regarding the partial derivative acts on the polynomial $G_{r,0}(x,y)$, we have:

$$\frac{\partial}{\partial x} G_{r,0}(x,y) = r! G_{r-1,0}(x,y) \tag{III.2.16}$$

which, once substituting in the integral, gives:

$$(-1)^{(m+1)+(n+s)} n! r! \left[\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \frac{\partial^{m-1}}{\partial x^{m-1}} \left(e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} \right) G_{r-1,0}(x,y) dx \right) dy \right] . \tag{III.2.17}$$

We can suppose $m \geq r$ and by iterating the process we end up with:

$$(-1)^{(m+r)+(n+s)} n! r! \left[\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \frac{\partial^{m-r}}{\partial x^{m-r}} \left(e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} \right) G_{0,0}(x,y) dx \right) dy \right] \tag{III.2.18}$$

where it is immediate to observe that the integral provides a zero result when m is not equal to r. By assuming that $m = r$, we can conclude with:

$$(-1)^{2m+2n} n! m! \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} dx \right) dy. \tag{III.2.19}$$

By noting that the term:

$$(-1)^{2m+2n} = (-1)^{2(m+n)} \tag{III.2.20}$$

is positive whatever the values of n and m, and by the fact that:

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}dx\right)dy=\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(ax^2+2abxy+cy^2\right)}dx\right)dy=2\pi\frac{1}{\sqrt{\Delta}} \tag{III.2.21}$$

we finally obtain:

$$(-1)^{2m+2n}n!m!\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}dx\right)dy=n!m!2\pi\frac{1}{\sqrt{\Delta}} \tag{III.2.22}$$

that is:

$$\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dxH_{m,n}(x,y)G_{r,s}(x,y)e^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}=n!m!2\pi\frac{1}{\sqrt{\Delta}} \tag{III.2.23}$$

which proves the theorem.

In the previous section, we have used the orthogonality property, satisfied by the one-index Hermite polynomials of the type $He_m(x)$ and $He^{'}_m(x,y)$, to introduce the Hermite functions in one and two variables $he_m(x)$ and $he_m(x,y)$. In the same way, we can use the result proved in the above theorem to define functions based on the two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ and their associated $G_{m,n}(x,y)$, which can verify the bi-orthogonality property.

Definition III.3

Let be the Hermite polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$, we will call two-index, two-variable Hermite functions, the functions defined in the following way:

$$\overline{H}_{m,n}(x,y)=\frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}H_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}} \tag{III.2.24}$$

$$\overline{G}_{m,n}(x,y)=\frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}G_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}} \tag{III.2.25}$$

It is evident that the two-index, two-variable Hermite functions are bi-orthogonal and in particular bi-orthonormal. We have, in fact, by applying the result of

Theorem III.3:

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \overline{H}_{m,n}(x,y)\, \overline{G}_{r,s}(x,y) = \tag{III.2.26}$$

$$= \frac{\sqrt{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\frac{1}{\sqrt{r!s!}}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy H_{m,n}(x,y) G_{r,s}(x,y) e^{-\frac{1}{2}\underline{z}^t \hat{M}\underline{z}}$$

and, then:

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \overline{H}_{m,n}(x,y)\, \overline{G}_{r,s}(x,y) = \tag{III.2.27}$$

$$= \frac{\sqrt{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\frac{1}{\sqrt{r!s!}} m!n! \frac{2\pi}{\sqrt{\Delta}}\delta_{m,r}\delta_{n,s} = \delta_{m,r}\delta_{n,s}.$$

In Section II.3, we have discussed the theory and some applications of the two-index, two-variable Hermite polynomials of the type $H_{m,n}(x,y)$ and the related associated $G_{m,n}(x,y)$. These family of Hermite polynomials has been introduced by operating a dimensional increase on the standard Hermite polynomials $He_m(x)$, by using a two-dimensional vector index acting on a two-dimensional vector variable; the structure used to define the Hermite polynomials of the form $H_{m,n}(x,y)$ is based on a quadratic form and then on a two-dimensional matrix, which is invertible. This last fact has suggested to explore the possibility to introduce a slightly different polynomials recognized as Hermite-type, so that we have defined the associated two-index, two-variable Hermite polynomials of type $G_{m,n}(x,y)$. It is evident that, many of the properties deduced for these polynomials belonging to the class of the generalized Hermite polynomials, are a generalization of the same relations presented and discussed for the ordinary Hermite polynomials $He_m(x)$ and they have been described for both the $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ Hermite polynomials. Through Theorem II.1 it has been shown that the polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ solved the same partial differential equation, only for different initial conditions, which proved the structural link between the two-index, two-variable Hermite polynomials and their associated. Since we have defined the two-index, two-variable Hermite functions $\overline{H}_{m,n}(x,y)$ and $\overline{G}_{m,n}(x,y)$ by using the related Hermite polynomials $H_{m,n}(x,y)$ and

$G_{m,n}(x,y)$, we expect to deduce similar relations which have involved the above bi-orthogonal Hermite functions and finally to obtain a partial differential equation solved by the Hermite functions of type $\overline{H}_{m,n}(x,y)$ and $\overline{G}_{m,n}(x,y)$.

Proposition III.5

The Hermite functions $\overline{H}_{m,n}(x,y)$ satisfied the following recurrence relations:

$$\left[\frac{\partial}{\partial x}+\frac{1}{2}(ax+by)\right]\overline{H}_{m,n}(x,y)=a\sqrt{m}\overline{H}_{m-1,n}(x,y)+b\sqrt{n}\overline{H}_{m,n-1}(x,y)\,, \tag{III.2.28}$$

and

$$\left[\frac{\partial}{\partial y}+\frac{1}{2}(bx+cy)\right]\overline{H}_{m,n}(x,y)=b\sqrt{m}\overline{H}_{m-1,n}(x,y)+c\sqrt{n}\overline{H}_{m,n-1}(x,y)\,. \tag{III.2.29}$$

Proof

By deriving with respect to x in the definition of the Hermite function $\overline{H}_{m,n}(x,y)$, we have:

$$\frac{\partial}{\partial x}\overline{H}_{m,n}(x,y)=\frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\frac{\partial}{\partial x}\left(H_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right). \tag{III.2.30}$$

Let us study the derivative of the r.h.s. of above equation, obtaining:

$$\frac{\partial}{\partial x}\left(H_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right)=\left(\frac{\partial}{\partial x}H_{m,n}(x,y)\right)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}+H_{m,n}(x,y)\frac{\partial}{\partial x}e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}} \tag{III.2.31}$$

and by applying the recurrence relation (II.2.15), we finally have:

$$\begin{aligned}\frac{\partial}{\partial x}&\left(H_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right)= \\ &=\left(amH_{m-1,n}(x,y)+bnH_{m,n-1}(x,y)\right)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}+ \\ &-\frac{1}{4}H_{m,n}(x,y)\left[\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\right]e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}.\end{aligned} \tag{III.2.32}$$

By noting that the two-index, two-variable Hermite polynomials of type $H_{m,n}(x,y)$ can be expressed in terms of the Hermite function $\overline{H}_{m,n}(x,y)$ and by making the appropriate manipulations, we end up with:

$$\begin{aligned}\frac{\partial}{\partial x}&\left(H_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right)= \\ &=am\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{(m-1)!n!}\,\overline{H}_{m-1,n}(x,y)+bn\sqrt{m!(n-1)!}\,\overline{H}_{m,n-1}(x,y)+ \\ &-\frac{1}{2}\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{m!n!}\,\overline{H}_{m,n}(x,y)(ax+by)\end{aligned} \tag{III.2.33}$$

and then:

$$\frac{\partial}{\partial x}\overline{H}_{m,n}(x,y) = \tag{III.2.34}$$
$$= \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{m!n!}\,\cdot$$
$$\cdot\left[a\sqrt{m}\,\overline{H}_{m-1,n}(x,y) + b\sqrt{n}\,\overline{H}_{m,n-1}(x,y) - \frac{1}{2}\overline{H}_{m,n}(x,y)(ax+by)\right]$$

which proves the first recurrence relation in the statement. To show the second relation, we derive with respect to y again in the definition of the Hermite functions $\overline{H}_{m,n}(x,y)$ (see eq.(III.2.24)):

$$\frac{\partial}{\partial y}\overline{H}_{m,n}(x,y) = \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}\frac{\partial}{\partial y}\left(H_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right) \tag{III.2.35}$$

and by following the same procedure used above, we can easily prove the second recurrence relations.

We have introduced the Hermite functions and their adjoint by using he structure of the Hermite polynomials of type $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$. As we have seen in the above statement, it is possible to derive similar relations for these Hermite functions of type $\overline{H}_{m,n}(x,y)$, by using the techniques and the operational properties of two-index, two-variable Hermite polynomials. By following this approach, from Proposition II.3, stated in Chapter II, we have:

Proposition III.6

The bi-orthogonal Hermite functions of type $\overline{H}_{m,n}(x,y)$ verify the following relations:

$$\sqrt{m+1}\,\overline{H}_{m+1,n}(x,y) = \tag{III.2.36}$$
$$= (ax+by)\overline{H}_{m,n}(x,y) - a\sqrt{m}\,\overline{H}_{m-1,n}(x,y) - b\sqrt{n}\,\overline{H}_{m,n-1}(x,y)\,,$$

$$\sqrt{n+1}\,\overline{H}_{m,n+1}(x,y) = \tag{III.2.37}$$
$$= (bx+cy)\overline{H}_{m,n}(x,y) - b\sqrt{m}\,\overline{H}_{m-1,n}(x,y) - c\sqrt{n}\,\overline{H}_{m,n-1}(x,y)\,.$$

Proof

We start to note that the Hermite function of indexes $m+1$, n reads:

$$\overline{H}_{m+1,n}\left(x,y\right)=\frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m+1}}\frac{1}{\sqrt{m!n!}}H_{m+1,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}} \tag{III.2.38}$$

where we can substitute the first recurrence relation stated in Proposition II.3 related to the Hermite polynomials of type $H_{m,n}(x,y)$:

$$\begin{aligned}&\sqrt{m+1}\,\overline{H}_{m+1,n}\left(x,y\right)=\\&=\frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}e^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\left[(ax+by)H_{m,n}(x,y)-amH_{m-1,n}(x,y)-bnH_{m,n-1}(x,y)\right].\end{aligned} \tag{III.2.39}$$

By using the definition of Hermite function $\overline{H}_{m,n}\left(x,y\right)$, the above equation can be written in the form:

$$\begin{aligned}&\sqrt{m+1}\,\overline{H}_{m+1,n}\left(x,y\right)=\\&=\frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}e^{-\frac{1}{4}\underline{z}^t\,\hat{M}\underline{z}}\left[(ax+by)e^{\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\sqrt{m!n!}\,\overline{H}_{m,n}\left(x,y\right)+\right.\\&-am\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{(m-1)!n!}e^{\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\,\overline{H}_{m-1,n}\left(x,y\right)+\\&\left.-bn\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{m!(n-1)!}e^{\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\,\overline{H}_{m,n-1}\left(x,y\right)\right]\end{aligned} \tag{III.2.40}$$

that is, once recast, the first expression of the present proposition.
In an analogous way it is possible to prove the second recurrence relation by using again the equations stated in Proposition II.3.
The relations derived in the above propositions can be used to define useful operators acting on the Hermite functions of type $\overline{H}_{m,n}\left(x,y\right)$. By following the same procedure outlined in Chapter II for the two-index, two-variable Hermite polynomials of the form $H_{m,n}(x,y)$, we will see that it is possible to state similar differential relations involving the bi-orthogonal Hermite functions. Manipulating (III.2.36), we have:

$$a\sqrt{m}\,\overline{H}_{m-1,n}\left(x,y\right)=(ax+by)\overline{H}_{m,n}(x,y)-b\sqrt{n}\,\overline{H}_{m,n-1}(x,y)-\sqrt{m+1}\,\overline{H}_{m+1,n}(x,y) \tag{III.2.41}$$

which, once substituted in (III.2.28), gives:

$$\left[-\frac{\partial}{\partial x}+\frac{1}{2}(ax+by)\right]\overline{H}_{m,n}(x,y)=\sqrt{m+1}\,\overline{H}_{m+1,n}(x,y). \tag{III.2.42}$$

In the same way, by using (III.2.37) and (III.2.29), we obtain:

$$\left[\frac{1}{2}(bx+cy)-\frac{\partial}{\partial y}\right]\overline{H}_{m,n}(x,y)=\sqrt{n+1}\,\overline{H}_{m,n+1}(x,y). \tag{III.2.43}$$

The recurrence relations stated in the previous Propositions III.5 and III.6, can be also used to derive further differential expressions regarding the bi-orthogonal Hermite functions. In fact, by following the same procedure used above, in particular alternately combining the first and the second expression of Proposition III.6 with the second and the first of Proposition III.5, it is possible to complete the characterization with regard to the differential properties satisfied by the Hermite functions of type $\overline{H}_{m,n}(x,y)$. We have, indeed, the following relations:

$$\left[-\frac{1}{\Delta}\left(b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right)+\frac{1}{2}y\right]\overline{H}_{m,n}(x,y)=\sqrt{n}\,\overline{H}_{m,n-1}(x,y), \tag{III.2.44}$$

$$\left[-\frac{1}{\Delta}\left(c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right)+\frac{1}{2}x\right]\overline{H}_{m,n}(x,y)=\sqrt{m}\,\overline{H}_{m-1,n}(x,y). \tag{III.2.45}$$

It is evident the analogy between the four relations above presented and the expressions proved in Chapter II regarding the two-index, two-variable Hermite polynomials of type $H_{m,n}(x,y)$, that is, the recurrence relations contained in Proposition II.3 and Proposition II.4. They have, in fact, the same structure and then the differential expressions in them suggest to introduce similar operators acting on the bi-orthogonal Hermite functions.

Definition III.4

Given the Hermite functions $\overline{H}_{m,n}(x,y)$, we define the related shift operators, by setting:

$$\hat{a}_{+,0} = \frac{1}{2}(ax+by)-\frac{\partial}{\partial x} \tag{III.2.46}$$

$$\hat{a}_{0,+} = \frac{1}{2}(bx+cy)-\frac{\partial}{\partial y} \tag{III.2.47}$$

and

$$\hat{a}_{-,0} = \frac{1}{\Delta}\left(c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right)+\frac{1}{2}x \tag{III.2.48}$$

$$\hat{a}_{0,-} = -\frac{1}{\Delta}\left(b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right)+\frac{1}{2}y \tag{III.2.49}$$

where $\Delta = ac - b^2$.

The above operators are free from any parameter, not presenting any index variable in their structure; are therefore different from the shift operators defined in the second chapter (Definition II.3) acting on Hermite polynomials of type $H_{m,n}(x, y)$.

It could be useful to summarize the action of these operators:

$$\begin{cases} \hat{a}_{+,0}\overline{H}_{m,n}(x,y) = \sqrt{m+1}\ \overline{H}_{m+1,n}(x,y) \\ \hat{a}_{0,+}\overline{H}_{m,n}(x,y) = \sqrt{n+1}\ \overline{H}_{m,n+1}(x,y) \\ \hat{a}_{-,0}\overline{H}_{m,n}(x,y) = \sqrt{m}\ \overline{H}_{m-1,n}(x,y) \\ \hat{a}_{0,-}\overline{H}_{m,n}(x,y) = \sqrt{n}\ \overline{H}_{m,n-1}(x,y) \end{cases} \tag{III.2.50}$$

As mentioned above and by virtue of the relations established above, we can proceed to state the important result concerning the partial differential equation solved by the bi-orthogonal Hermite functions $\overline{H}_{m,n}(x, y)$ and $\overline{G}_{m,n}(x, y)$. We will proceed by presenting the results for the Hermite functions of type $\overline{H}_{m,n}(x, y)$ and later we will discuss the case for the related associated Hermite functions.

Theorem III.4

The bi-orthogonal Hermite functions solve the following partial differential equation:

$$\left[-\underline{\partial}_z^t \hat{M}^{-1}\underline{\partial}_z - \left(m+n+1-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}\right)\right]\overline{H}_{m,n}(x,y) = 0 \tag{III.2.51}$$

where:

$$\underline{\partial}_z = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \tag{III.2.52}$$

Proof

We proceed in a similar way to what was done for the demonstration of Proposition II.5, as shown in Chapter II. We start to consider the following operational relations, deriving from the above considerations:

$$\hat{a}_{+,0}\left[\hat{a}_{-,0}\overline{H}_{m,n}(x,y)\right] = m\overline{H}_{m,n}(x,y) \tag{III.2.53}$$

$$\hat{a}_{0,+}\left[\hat{a}_{0,-}\overline{H}_{m,n}(x,y)\right]=n\overline{H}_{m,n}(x,y) \tag{III.2.54}$$

which can be explicited to obtain:

$$\left[\frac{1}{2}(ax+by)-\frac{\partial}{\partial x}\right]\left[\frac{1}{\Delta}\left(c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right)+\frac{1}{2}x\right]\overline{H}_{m,n}(x,y)=m\overline{H}_{m,n}(x,y) \tag{III.2.55}$$

$$\left[\frac{1}{2}(bx+cy)-\frac{\partial}{\partial y}\right]\left[-\frac{1}{\Delta}\left(b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right)+\frac{1}{2}y\right]\overline{H}_{m,n}(x,y)=n\overline{H}_{m,n}(x,y). \tag{III.2.56}$$

The operator in the first of the above relations can be recast in the form:

$$\begin{aligned}&\frac{1}{2\Delta}\left[c(ax+by)\frac{\partial}{\partial x}\right]-\frac{1}{2\Delta}\left[b(ax+by)\frac{\partial}{\partial y}\right]+\frac{1}{4}\left(ax^2+bxy\right)+\\&\quad+\frac{1}{\Delta}\left(b\frac{\partial^2}{\partial x\partial y}-c\frac{\partial^2}{\partial x^2}\right)-\frac{1}{2}-\frac{1}{2}x\frac{\partial}{\partial x}\end{aligned} \tag{III.2.57}$$

and regarding the the second equations, we can rewrite the operator as follows:

$$\begin{aligned}&-\frac{1}{2\Delta}\left[b(bx+cy)\frac{\partial}{\partial x}-a(bx+cy)\frac{\partial}{\partial y}\right]+\frac{1}{4}y\left(bx+cy\right)+\\&\quad+\frac{1}{\Delta}\left(b\frac{\partial^2}{\partial x\partial y}-a\frac{\partial^2}{\partial y^2}\right)-\frac{1}{2}-\frac{1}{2}y\frac{\partial}{\partial y}.\end{aligned} \tag{III.2.58}$$

After substituting the above expressions in the operational relations (III.2.56) and summing these relations member to member, we obtain:

$$\begin{aligned}&\left\{\frac{1}{2\Delta}\left[c(ax+by)\frac{\partial}{\partial x}\right]-\frac{1}{2\Delta}\left[b(ax+by)\frac{\partial}{\partial y}\right]+\frac{1}{4}\left(ax^2+bxy\right)+\right.\\&\quad+\frac{1}{\Delta}\left(b\frac{\partial^2}{\partial x\partial y}-c\frac{\partial^2}{\partial x^2}\right)-\frac{1}{2}-\frac{1}{2}x\frac{\partial}{\partial x}-\frac{1}{2\Delta}\left[b(bx+cy)\frac{\partial}{\partial x}-a(bx+cy)\frac{\partial}{\partial y}\right]+\\&\quad\left.+\frac{1}{4}y\left(bx+cy\right)+\frac{1}{\Delta}\left(b\frac{\partial^2}{\partial x\partial y}-a\frac{\partial^2}{\partial y^2}\right)-\frac{1}{2}-\frac{1}{2}y\frac{\partial}{\partial y}\right\}\overline{H}_{m,n}(x,y)=\\&\quad=(m+n)\overline{H}_{m,n}(x,y).\end{aligned} \tag{III.2.59}$$

We note that, by using the definition of two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ (see Chapter II), the following relations hold:

$$\frac{1}{4}\left(ax^2+2bxy+cy^2\right)=\frac{1}{4}\underline{z}^t\hat{M}\underline{z} \tag{III.2.60}$$

$$-\underline{\partial}_z^t\hat{M}^{-1}\underline{\partial}_z=-\frac{1}{\Delta}\left(c\frac{\partial^2}{\partial x^2}-2b\frac{\partial^2}{\partial x\partial y}+a\frac{\partial^2}{\partial y^2}\right) \tag{III.2.61}$$

and then we can recast the operator in the l.h.s of equation (III.2.61) in the following form:

$$\left[-\underline{\partial}_z^t \hat{M}^{-1}\underline{\partial}_z + \frac{1}{4}\underline{z}^t \hat{M}\underline{z} - 1\right]\overline{H}_{m,n}(x,y) = (m+n)\overline{H}_{m,n}(x,y) \qquad \text{(III.2.62)}$$

which easily gives the statement of the theorem.

We can now establish analogous results for the adjoint bi-orthogonal Hermite functions of type $\overline{G}_{m,n}(x,y)$. By considering the link that exists between the two-index, two-variable Hermite polynomials and their adjoint and moreover between the present Hermite functions and the related associated functions, we proceed in a non-repetitive way, but by acting directly on the operators presented in Definition III.4. We start to consider the following vectorial operator:

$$\hat{a}^+ = \begin{pmatrix} \hat{a}_{+,0} \\ \hat{a}_{0,+} \end{pmatrix} \qquad \text{(III.2.63)}$$

and we can easily prove that:

$$\hat{a}^+ = \frac{1}{2}\hat{M}\underline{z} - \underline{\partial}_z. \qquad \text{(III.2.64)}$$

In fact, the r.h.s. in the above equation can be explicited to have:

$$\frac{1}{2}\hat{M}\underline{z} - \underline{\partial}_z = \frac{1}{2}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} ax+by \\ bx+cy \end{pmatrix} - \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \qquad \text{(III.2.65)}$$

and then, by the first relation in Definition III.4, we find:

$$\begin{aligned} \tfrac{1}{2}(ax+by) - \tfrac{\partial}{\partial x} &= \hat{a}_{+,0} \\ \tfrac{1}{2}(bx+cy) - \tfrac{\partial}{\partial y} &= \hat{a}_{0,+} \end{aligned} \qquad \text{(III.2.66)}$$

which proves the statement. In the same way, by setting:

$$\hat{a}^- = \begin{pmatrix} \hat{a}_{-,0} \\ \hat{a}_{0,-} \end{pmatrix} \qquad \text{(III.2.67)}$$

we further obtain the relation:

$$\hat{a}^- = \hat{M}^{-1}\underline{\partial}_z + \frac{1}{2}\underline{z}. \qquad \text{(III.2.68)}$$

We use now the two vector operators defined above, for the Hermite functions of type $\overline{H}_{m,n}(x,y)$, to determine the corresponding creation and annihilation operators for the associated Hermite functions $\overline{G}_{m,n}(x,y)$. In the second chapter, we have seen that the structural difference between the two-index, two-variable Hermite polynomials and their associated, is essentially different in the matrix of its quadratic form that defines them. Otherwise, the Hermite functions have been defined by using the Hermite polynomials of type $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$, this aspect suggests us to define the creation and annihilation operators for the bi-orthogonal Hermite functions of type $\overline{G}_{m,n}(x,y)$, by modifying directly the corresponding operators obtained for the Hermite functions $\overline{H}_{m,n}(x,y)$.

We remind that the adjoint quadratic form of the two-index, two-variable Hermite polynomials of type $H_{m,n}(x,y)$, has been expressed by (eq. (II.2.5)):

$$\overline{q}\left(\underline{z}\right) = \underline{z}^t \hat{M}^{-1} \underline{z} \tag{III.2.69}$$

which introduced the vectorial variable $\underline{v} = \hat{M}\underline{z}$, where $\underline{v} = \binom{\xi}{\eta}$ to define the associated Hermite polynomials of the form $G_{m,n}(x,y)$. By using the above relations, we introduce the operators regarding the associated Hermite functions $\overline{G}_{m,n}(x,y)$, by setting:

$$\hat{B}^{+} = \frac{1}{2}\hat{M}^{-1}\underline{v} - \underline{\partial}_v \tag{III.2.70}$$

and

$$\hat{B}^{-} = \hat{M}\underline{\partial}_v + \frac{1}{2}\underline{v}. \tag{III.2.71}$$

It is evident that the above expressions are referred to the vectorial variable $\underline{v}$ and then we need to explict the creation and annihilation operators related to the associated Hermite functions $\overline{G}_{m,n}(x,y)$ in terms of the vectorial variable $\underline{z}$. By using the link between the variables $\underline{z}$ and $\underline{v}$, we immediately get:

$$\frac{1}{2}\hat{M}^{-1}\underline{v} - \underline{\partial}_v = \frac{1}{2}\underline{z} - \hat{M}^{-1}\underline{\partial}_z \tag{III.2.72}$$

$$\hat{M}\underline{\partial}_v + \frac{1}{2}\underline{v} = \underline{\partial}_z + \frac{1}{2}\hat{M}\,\underline{z} \tag{III.2.73}$$

and then, we can rewrite the creation and annihilation operators in the following form:

$$\hat{B}^{+} = \frac{1}{2}\,\underline{z} - \hat{M}^{-1}\underline{\partial}_z \tag{III.2.74}$$

$$\hat{B}^{-} = \underline{\partial}_z + \frac{1}{2}\hat{M}\,\underline{z} \tag{III.2.75}$$

It is now possible to obtain an explicit form of the creation and annihilation operators related to the associated Hermite functions $\overline{G}_{m,n}(x,y)$. From the first expression, we have:

$$\hat{B}^{+} = \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{\Delta}\begin{pmatrix} -c & b \\ b & -a \end{pmatrix}\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{cases} \frac{1}{2}x - \frac{1}{\Delta}\left(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right) \\ \frac{1}{2}y - \frac{1}{\Delta}\left(-b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right) \end{cases} \tag{III.2.76}$$

and, in analogous way for the second operator:

$$\hat{B}^{-} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{\partial}{\partial x} + \frac{1}{2}(ax + by) \\ \frac{\partial}{\partial y} + \frac{1}{2}(bx + cy) \end{cases} \tag{III.2.77}$$

We can finally state the explicit form for the creation and annihilation operators related to the Hermite functions $\overline{G}_{m,n}(x,y)$. For the creation operators, we obtain:

$$\begin{aligned} \hat{B}_{+,0} &= \tfrac{1}{2}x - \tfrac{1}{\Delta}\left(c\tfrac{\partial}{\partial x} - b\tfrac{\partial}{\partial y}\right) \\ \hat{B}_{0,+} &= \tfrac{1}{2}y - \tfrac{1}{\Delta}\left(-b\tfrac{\partial}{\partial x} + a\tfrac{\partial}{\partial y}\right) \end{aligned} \tag{III.2.78}$$

and, similarly, for the annihilation operators, we get:

$$\begin{aligned} \hat{B}_{-,0} &= \tfrac{\partial}{\partial x} + \tfrac{1}{2}(ax + by) \\ \hat{B}_{0,-} &= \tfrac{\partial}{\partial y} + \tfrac{1}{2}(bx + cy) \end{aligned} \tag{III.2.79}$$

We have defined the above operators by using the concepts and the related formalism of the creation and annihilation operators introduced for the Hermite bi-orthogonal functions of type $\overline{H}_{m,n}(x,y)$. We expect that these operators produce the same effect on the associated Hermite functions of the form $\overline{G}_{m,n}(x,y)$. In fact, we immediately obtain the fundamental relations:

$$\begin{cases} \hat{B}_{+,0}\,\overline{G}_{m,n}(x,y) = \sqrt{m+1}\,\overline{G}_{m+1,n}(x,y) \\ \hat{B}_{0,+}\,\overline{G}_{m,n}(x,y) = \sqrt{n+1}\,\overline{G}_{m,n+1}(x,y) \\ \hat{B}_{-,0}\,\overline{G}_{m,n}(x,y) = \sqrt{m}\,\overline{G}_{m-1,n}(x,y) \\ \hat{B}_{0,-}\,\overline{G}_{m,n}(x,y) = \sqrt{n}\,\overline{G}_{m,n-1}(x,y) \end{cases} \tag{III.2.80}$$

which confirms that the operators defined in the relations (III.2.78) and (III.2.79) are exactly the creation and annihilation operators related to the associatede Hermite functions of the functions $\overline{H}_{m,n}(x, y)$. The results exposed in this chapter have been used in many physics applications, but we wish to observe, however, the considerable importance that assume from the purely mathematical point of view. Several developments can still be derived by using the concepts presented in the above lines.

Chapter IV

Chebyshev polynomials and integral representations

After a lengthy treatise on Hermite polynomials in the previous chapters, we begin the discussion of the Chebyshev polynomials. In this chapter we will introduce the Chebyshev polynomials of the first and second kind, and discuss their basic properties. From the third section we will deal with their integral representations and what will be done with the help of the concepts and operational techniques of the Hermite polynomials. We will see that many of the properties verified by the Chebyshev polynomials can be deduced in an immediate way, thanks to the relations satisfied by the Hermite polynomials. Not only that, the use of Hermite polynomials makes it possible to establish new relationships and to introduce families of generalized Chebyshev polynomials. In fact we will present the Chebyshev polynomials in two variables and one parameter that allow us to obtain some special families of Gegenbauer polynomials. Finally, on the basis of what we saw in the second chapter, and with the support of the generalized Hermite polynomials of type $H_n^{(m)}(x, y)$, we will also introduce generalizations of Chebyshev polynomials in several indexes.

IV.1 Chebyshev polynomials

There are a number of distinct families of polynomials that go by the name of Chebyshev polynomials. The Chebyshev polynomials *par excellence* can be defined by:

Definition IV.1

Let x be a real variable, we call Chebyshev polynomials of first kind, the polynomials defined by the following relation:

$$T_n(x) = \cos(n \arccos(x)). \tag{IV.1.1}$$

In the same way we can also introduce the second kind Chebyshev polynomials, by using again the link with the circular functions.

Definition IV.2

Let x be a real variable, we call Chebyshev polynomials of second kind, the polynomials:

$$U_n(x) = \frac{\sin\left[(n+1)\arccos(x)\right]}{\sqrt{1-x^2}}. \tag{IV.1.2}$$

The study of the properties of the Chebyshev polynomials can be simplified by introducing the following complex quantity:

$$\mathbf{T}_n(x) = \exp\left[in(\arccos(x)\right] \tag{IV.1.3}$$

so that:

$$\begin{aligned} \mathrm{Re}\left[\mathbf{T}_n(x)\right] &= \cos(n \arccos(x)) \\ \mathrm{Im}\left[\mathbf{T}_n(x)\right] &= \sin(n \arccos(x)). \end{aligned} \tag{IV.1.4}$$

The above relations can be recast directly in terms of the Chebyshev polynomials of the first and second kind. In fact, by noting that the second kind Chebyshev polynomials of degree $n-1$ reads:

$$U_{n-1}(x) = \frac{\sin(n \arccos(x))}{\sqrt{1-x^2}} \tag{IV.1.5}$$

we can immediately conclude that:

$$\begin{aligned} T_n(x) &= \operatorname{Re}[\mathbf{T}_n(x)] \\ U_{n-1}(x) &= \frac{\operatorname{Im}[\mathbf{T}_n(x)]}{\sqrt{1-x^2}} \end{aligned} \tag{IV.1.6}$$

To derive the related generating functions of the Chebyshev polynomials of the first and second kind, we can consider the generating functions of the complex quantity, introduced in (IV.1.3); let, in fact, be the real number ξ, such that $|\xi| < 1$, we can immediately write:

$$\sum_{n=0}^{+\infty} \xi^n \mathbf{T}_n(x) = \sum_{n=0}^{+\infty} \left(\xi e^{i \arccos(x)}\right)^n = \frac{1}{1 - \xi e^{i \arccos(x)}}. \tag{IV.1.7}$$

Proposition IV.1

Let be $\xi \in \mathbb{R}$, such that $|\xi| < 1$; the generating function of Chebyshev polynomials of the first kind reads:

$$\sum_{n=0}^{+\infty} \xi^n T_n(x) = \frac{1 - \xi x}{1 - 2\xi x + \xi^2}. \tag{IV.1.8}$$

Proof

By using the link stated in equation (IV.1.6) and by (IV.1.7), for a real number ξ, such that $|\xi| < 1$, we can write:

$$\sum_{n=0}^{+\infty} \xi^n T_n(x) = \sum_{n=0}^{+\infty} \xi^n \operatorname{Re}[\mathbf{T}_n(x)] = \operatorname{Re}\left[\frac{1}{1 - \xi e^{i \arccos(x)}}\right]. \tag{IV.1.9}$$

By manipulating the r.h.s. of the previous relation, we find:

$$\operatorname{Re}\left[\frac{1}{1 - \xi e^{i \arccos(x)}}\right] = \operatorname{Re}\left\{\frac{[1 - \xi \cos(\arccos(x))] + i\xi \sin(\arccos(x))}{[1 - \xi \cos(\arccos(x))]^2 + \xi^2 \sin^2(\arccos(x))}\right\} \tag{IV.1.10}$$

that is:

$$\operatorname{Re}\left[\frac{1}{1 - \xi e^{i \arccos(x)}}\right] = \frac{1 - \xi \cos(\arccos(x))}{1 - 2\xi \cos(\arccos(x)) + \xi^2} \tag{IV.1.11}$$

and then, we immediately obtain (IV.1.8).

By following the same procedure, we can also derive the related generating function for the Chebyshev polynomials $U_n(x)$.

It is easy, in fact, to note, from the second equation in (IV.1.6) and from (IV.1.7), that:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1}(x) = \sum_{n=0}^{+\infty} \xi^n \frac{\mathrm{Im}\left[\mathbf{T}_n(x)\right]}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \mathrm{Im}\left[\frac{1}{1-\xi e^{i \arccos(x)}}\right]. \tag{IV.1.12}$$

By using the same manipulation exploited in the previous proposition, we end up with:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1}(x) = \frac{\xi}{1-2\xi x+\xi^2} \tag{IV.1.13}$$

which is the generating function of the Chebyshev polynomials of the second kind of degree $n-1$, with again $|\xi| < 1$.

It is also possible to derive different generating functions for these families of Chebyshev polynomials, by using the property of the complex quantity in (IV.1.7). In fact by noting that:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} \mathbf{T}_n(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\xi e^{i \arccos(x)}\right)^n = \exp\left[\xi e^{i \arccos(x)}\right]$$

we have:

Proposition IV.2

For the Chebyshev polynomials of the first and second kind, the following results hold:

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x) &= e^{\xi x} \cos\left(\xi\sqrt{1-x^2}\right) \\ \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(x) &= e^{\xi x} \frac{\sin\left(\xi\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} \end{aligned} \tag{IV.1.14}$$

where $|\xi| < 1$.

Proof

From the identity:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x) = \mathrm{Re}\left\{\exp\left[\xi e^{i \arccos(x)}\right]\right\} \tag{IV.1.15}$$

after setting:

$$\psi = \arccos(x)$$

we can rearrange the r.h.s. in the following from:

$$\mathrm{Re}\left\{\exp\left[\xi\left(\cos(\psi)+i\sin(\psi)\right)\right]\right\}=\exp\left(\xi\cos(\psi)\right)\mathrm{Re}\left[\exp\left(i\xi\sin(\psi)\right)\right]. \tag{IV.1.16}$$

By noting that:

$$\mathrm{Re}\left[\exp\left(i\xi\sin(\psi)\right)\right]=\mathrm{Re}\left[\cos\left(\xi\sin(\psi)\right)+i\sin\left(\xi\sin(\psi)\right)\right]=\cos\left(\xi\sin(\psi)\right) \tag{IV.1.17}$$

we immediately obtain the first equation in (IV.1.14).

For the Chebyshev polynomials of th esecond kind, by using the complex quantity $\mathbf{T}_n(x)$, we write:

$$\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1}(x)=\frac{1}{\sqrt{1-x^2}}\mathrm{Im}\left\{\exp\left[\xi e^{i\arccos(x)}\right]\right\}. \tag{IV.1.18}$$

By using the same setting $\psi=\arccos(x)$, we can write the r.h.s. of the above identity in the form:

$$\frac{1}{\sqrt{1-x^2}}\mathrm{Im}\left\{\exp\left[\xi\left(\cos(\psi)+i\sin(\psi)\right)\right]\right\}=\frac{1}{\sqrt{1-x^2}}\exp\left(\xi\cos(\psi)\right)\sin\left(\xi\sin(\psi)\right) \tag{IV.1.19}$$

and then the second of (IV.1.14) immediately follows.

The use of the complex representation of Chebyshev polynomials can be also exploited to derive less trivial relations involving first and second kind Chebyshev polynomials. From definition of the first kind Chebyshev polynomials, given in (IV.1.1), we can generalize it, by putting:

$$T_{n+l}(x)=\left[\cos(n+l)\arccos(x)\right] \tag{IV.1.20}$$

and, from (IV.1.3), we can immediately write:

$$\mathbf{T}_{n+l}(x)=\exp\left[i(n+l)\arccos(x)\right] \tag{IV.1.21}$$

then:

$$\begin{aligned}\mathrm{Re}\left[\mathbf{T}_{n+l}(x)\right]&=T_{n+l}(x)\\ \mathrm{Im}\left[\mathbf{T}_{n+l}(x)\right]&=\frac{U_{n-1+l}(x)}{\sqrt{1-x^2}}.\end{aligned} \tag{IV.1.22}$$

By using the same procedure exploited in Propositions IV.1 and IV.2 to derive the generating functions of the polynomials $T_n(x)$ and $U_n(x)$, we can state the following results.

Proposition IV.3

Let be $\xi \in \mathbb{R}$, such that $|\xi| < 1$; the following identities hold:

$$\sum_{n=0}^{+\infty} \xi^n T_{n+l}(x) = \frac{(1-\xi x)\, T_l(x) - \xi\,(1-x^2)\, U_{l-1}}{1-2\xi x+\xi^2} \tag{IV.1.23}$$

and:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1+l}(x) = \frac{\xi T_l(x) + (1-\xi x)\, U_{l-1}}{1-2\xi x+\xi^2}. \tag{IV.1.24}$$

Proof

From the previous results, it is easy to note that:

$$\sum_{n=0}^{+\infty} \xi^n \mathbf{T}_{n+l}(x) = \sum_{n=0}^{+\infty} \xi^n e^{in\arccos(x)} e^{il\arccos(x)} = e^{il\arccos(x)} \frac{1}{1-\xi e^{i\arccos(x)}}. \tag{IV.1.25}$$

Otherwise:

$$e^{il\arccos(x)} = \cos(l\arccos(x)) + i\sin(\arccos(x)) \tag{IV.1.26}$$

and so:

$$\sum_{n=0}^{+\infty} \xi^n T_{n+l}(x) = \sum_{n=0}^{+\infty} \xi^n \mathrm{Re}\left[\mathbf{T}_{n+l}(x)\right] = \mathrm{Re}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right]. \tag{IV.1.27}$$

The r.h.s. can be rearranged in the more convenient form:

$$\mathrm{Re}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right] = \tag{IV.1.28}$$

$$= \mathrm{Re}\left\{\frac{[\cos(l\arccos(x)) + i\sin(\arccos(x))]\,[1-\xi x + i\xi\sin(\arccos(x)]}{1-2\xi x+\xi^2}\right\}$$

to give:

$$\mathrm{Re}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right] = \frac{(1-\xi x)\, T_l(x) - \xi\,(1-x^2)\, U_{l-1}}{1-2\xi x+\xi^2} \tag{IV.1.29}$$

which proves the first statement.

In an analogous way, we note that:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1+l}(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{+\infty} \xi^n \mathrm{Im}\left[\mathbf{T}_{n+l}(x)\right] = \tag{IV.1.30}$$

$$= \operatorname{Im}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right]$$

and by following the same procedure, we immediately obtain (IV.1.24).

The corresponding generating functions stated in Proposition IV.2, for the Chebyshev polynomials are also easily obtained.

Proposition IV.4

For a real ξ, $|\xi| < 1$, the polynomials $T_n(x)$ and $U_n(x)$ satisfy the following relations:

$$\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}T_{n+l}(x) = e^{\xi x}\left[\cos\left(\xi\sqrt{1-x^2}\right)T_l(x) - \sqrt{1-x^2}\sin\left(\xi\sqrt{1-x^2}\right)U_{l-1}(x)\right] \tag{IV.1.31}$$

and:

$$\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1+l}(x) = e^{\xi x}\left[\sqrt{1-x^2}\cos\left(\xi\sqrt{1-x^2}\right)U_{l-1}(x) + \sin\left(\xi\sqrt{1-x^2}\right)T_l(x)\right]. \tag{IV.1.32}$$

Proof

From (IV.1.15) it follows that:

$$\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}T_{n+l}(x) = \operatorname{Re}\left[e^{il\arccos(x)}e^{\xi e^{i\arccos(x)}}\right] \tag{IV.1.33}$$

or in a more convenient form, by setting $\psi = \arccos(x)$:

$$\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}T_{n+l}(x) = \operatorname{Re}\left\{[\cos(l\psi) + i\sin(l\psi)]\left[e^{\xi\cos\psi}e^{\xi i\sin\psi}\right]\right\}. \tag{IV.1.34}$$

By exploiting the r.h.s., we obtain:

$$\operatorname{Re}\left\{[\cos(l\psi) + i\sin(l\psi)]\left[e^{\xi\cos\psi}e^{\xi i\sin\psi}\right]\right\} = \tag{IV.1.35}$$

$$= \operatorname{Re}\left\{\cos(l\psi)\,e^{\xi\cos(\psi)}\left[\cos(\xi\sin(\psi)) + i\sin(\xi\sin(\psi))\right] + \right.$$

$$\left. + \sin(l\psi)\,e^{\xi\cos(\psi)}\left[\cos(\xi\sin(\psi)) + i\sin(\xi\sin(\psi))\right]\right\}$$

and then, after substituting the previous setting of ψ:

$$\operatorname{Re}\left\{[\cos(l\arccos(x)) + i\sin(l\arccos(x))]\left[e^{\xi x}e^{\xi i\sin(\arccos(x))}\right]\right\} = \tag{IV.1.36}$$

$$= e^{\xi x}\left[\cos\left(\xi\sqrt{1-x^2}\right)T_l(x) - \sqrt{1-x^2}\sin\left(\xi\sqrt{1-x^2}\right)U_{l-1}(x)\right]$$

that is equation (IV.1.31).

Regarding the second statement, we have:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1+l}(x) = \frac{1}{\sqrt{1-x^2}} \mathrm{Im}\left[e^{il \arccos(x)} e^{\xi e^{i \arccos(x)}} \right] \qquad \text{(IV.1.37)}$$

and it is easy, by following the same procedure previously outlined, to state the second identity (IV.1.32).

In the next sections it will be shown that the simple method we have proposed in these introductory remarks offers a fairly important tool of analysis for wide classes of properties of the Chebyshev polynomials.

IV.2 Products of Chebyshev polynomials

In this section we will show some important identities related to the generating functions of products of Chebyshev polynomials. We introduce the following results.

Proposition IV.5

For the polynomials $T_n(x)$ and $U_n(x)$ and for their complex representation $\mathbf{T}_n(x)$, the following identities are true:

$$\begin{aligned} |\mathbf{T}_n(x)|^2 &= [T_n(x)]^2 + \left(1-x^2\right)[U_{n-1}(x)]^2 = 1, \\ \mathrm{Re}\,[\mathbf{T}_n(x)]^2 &= [T_n(x)]^2 - \left(1-x^2\right)[U_{n-1}(x)]^2, \\ \mathrm{Im}\,[\mathbf{T}_n(x)]^2 &= 2\sqrt{1-x^2}\,T_n(x)U_{n-1}(x). \end{aligned} \qquad \text{(IV.2.1)}$$

Proof

By noting that:

$$|\mathbf{T}_n(x)|^2 = \mathrm{Re}\mathbf{T}_n(x)^2 + \mathrm{Im}\mathbf{T}_n(x)^2 \qquad \text{(IV.2.2)}$$

that is:

$$|\mathbf{T}_n(x)|^2 = [T_n(x)]^2 + \left(1-x^2\right)[U_{n-1}(x)]^2 . \qquad \text{(IV.2.3)}$$

After substituting the explicit forms of the polynomials $T_n(x)$ and $U_n(x)$, we obtain the first of (IV.2.1).

We can also note that:

$$[\mathbf{T}_n(x)]^2 = \left[T_n(x) + i\sqrt{1-x^2}U_{n-1}(x)\right]^2 \tag{IV.2.4}$$

and by expanding the r.h.s.:

$$[\mathbf{T}_n(x)]^2 = [T_n(x)]^2 + i2\sqrt{1-x^2}T_n(x)U_{n-1}(x) - \left(1-x^2\right)[U_{n-1}(x)]^2 \tag{IV.2.5}$$

which once separated into its real and imaginary part allows us to recognize the second and the third identities of the statement.

From (IV.2.1) it also follows that:

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\xi^n}{n!}|\mathbf{T}_n(x)|^2 &= \exp(\xi) \\ \sum_{n=0}^{+\infty} \frac{\xi^n}{n!}|\mathbf{T}_n(x)|^2 &= \exp\left[\xi \exp\left(2i\arccos(x)\right)\right]. \end{aligned} \tag{IV.2.6}$$

(IV.2.1) and (IV.2.6) can be used to state further relations linking the Chebyshev polynomials of the first and second kind. We have in fact:

Proposition IV.6

The polynomials $T_n(x)$ and $U_n(x)$ satisfy the following identities:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!}|T_n(x)|^2 = \frac{1}{2}\left[e^{\xi} + e^{\xi\left(2x^2-1\right)}\cos\left(2\xi x\sqrt{1-x^2}\right)\right] \tag{IV.2.7}$$

and:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!}|U_{n-1}(x)|^2 = \frac{1}{2\left(1-x^2\right)}\left[e^{\xi} - e^{\xi\left(2x^2-1\right)}\cos\left(2\xi x\sqrt{1-x^2}\right)\right]. \tag{IV.2.8}$$

Proof

By summing term to term the first two identities of (IV.2.1), we have:

$$2T_n^2(x) = |\mathbf{T}_n(x)|^2 + \mathrm{Re}\mathbf{T}_n^2(x). \tag{IV.2.9}$$

By multiplying both sides of the previous relation by $\frac{\xi^n}{n!}$ and then summing up, we find:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!}T_n^2(x) = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!}|\mathbf{T}_n(x)|^2 + \sum_{n=0}^{+\infty} \frac{\xi^n}{n!}\mathrm{Re}\mathbf{T}_n^2(x) \tag{IV.2.10}$$

and by using (IV.2.6), we can write:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!}T_n^2(x) = \exp(\xi) + \mathrm{Re}\exp\left[\xi\exp(2i\arccos(x))\right]. \tag{IV.2.11}$$

By expanding the r.h.s. of the above identity, we obtain:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!}T_n^2(x) = \exp(\xi) + \exp\left[\xi\left(2x^2-1\right)\right]\mathrm{Re}\left[\exp\left(i2\xi x\sqrt{1-x^2}\right)\right] \tag{IV.2.12}$$

that is:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!}T_n^2(x) = \tag{IV.2.13}$$

$$= \exp(\xi) + \exp\left[\xi\left(2x^2-1\right)\right]\mathrm{Re}\left[\cos\left(2\xi x\sqrt{1-x^2}\right) + i\sin\left(2\xi x\sqrt{1-x^2}\right)\right],$$

which proves equation (IV.2.7).

The second identity of the statement can be derived in the same way; in fact it is enough to note that by subtracting term to term the first two relations of (IV.2.1), we find:

$$2\left(1-x^2\right)U_{n-1}^2(x) = |\mathbf{T}_n(x)|^2 - \mathrm{Re}\mathbf{T}_n^2(x) \tag{IV.2.14}$$

and by following the same procedure, used above, we obtain (IV.2.8).

The last equation of (IV.2.1) allows us to state the additional identity:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!}T_n(x)U_{n-1}(x) = \frac{\exp\left[\xi\left(2x^2-1\right)\right]}{2\sqrt{1-x^2}}\sin\left(2\xi x\sqrt{1-x^2}\right). \tag{IV.2.15}$$

In the previous section we have derived different generating functions for the Chebyshev polynomials $T_n(x)$ and $U_{n-1}(x)$; we can generalize those results for their products. We firstly note that, from (IV.2.1) and from the choice of ξ, $|\xi|<1$, that:

$$\xi|\mathbf{T}_n(x)|^2 < 1$$

we have:

$$\sum_{n=0}^{+\infty} \xi^n|\mathbf{T}_n(x)|^2 = \frac{1}{1-\xi}. \tag{IV.2.16}$$

Otherwise, can also be noted that:

$$\mathbf{T}_n^2(x) = \left[\exp\left(i\arccos(x)\right)^n\right]^2 \leq 1$$

and since $|\xi| < 1$, it follows that:

$$\sum_{n=0}^{+\infty} \xi^n \mathbf{T}_n^2(x) = \frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}. \tag{IV.2.17}$$

Proposition IV.7

Let be $\xi \in \mathbb{R}$, $|\xi| < 1$; the following identities hold:

$$\sum_{n=0}^{+\infty} \xi^n T_n^2(x) = \frac{1}{2} \frac{1}{1-\xi} \left[1 + \frac{(1-\xi)\left(1 - \xi\left(2x^2 - 1\right)\right)}{1 - 2\xi\left(2x^2-1\right) + \xi^2} \right] \tag{IV.2.18}$$

and:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1}^2(x) = \frac{1}{2\left(1-x^2\right)} \frac{1}{1-\xi} \left[1 - \frac{(1-\xi)\left(1 - \xi\left(2x^2 - 1\right)\right)}{1 - 2\xi\left(2x^2-1\right) + \xi^2} \right]. \tag{IV.2.19}$$

Proof

By multiplying both sides of (IV.2.9) by ξ^n and then summing up, we find:

$$2\sum_{n=0}^{+\infty} \xi^n T_n^2(x) = \sum_{n=0}^{+\infty} \xi^n |\mathbf{T}_n(x)|^2 + \sum_{n=0}^{+\infty} \xi^n \mathrm{Re}\mathbf{T}_n^2(x) \tag{IV.2.20}$$

and from the (IV.2.16) and (IV.2.17), we can write:

$$2\sum_{n=0}^{+\infty} \xi^n T_n^2(x) = \frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right]. \tag{IV.2.21}$$

Setting $\psi = \arccos(x)$, the r.h.s. of the above relation can be recast in the form:

$$\frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right] = \frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1 - \xi e^{-i\psi}}{\left(1 - \xi e^{i\psi}\right)\left(1 - \xi e^{-i\psi}\right)}\right]. \tag{IV.2.22}$$

After exploiting the r.h.s., rewriting in terms of x, we obtain:

$$\frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right] = \frac{1}{1-\xi} + \frac{1 - \xi\left(2x^2-1\right)}{1 - 2\xi\left[\cos\left(2\arccos(x)\right)\right] + \xi^2} \tag{IV.2.23}$$

which gives (IV.2.18).

From (IV.2.14) and using again (IV.2.16) and (IV.2.17), we have:

$$2\left(1-x^2\right)\sum_{n=0}^{+\infty} \xi^n U_{n-1}^2(x) = \frac{1}{1-\xi} - \mathrm{Re}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right] \tag{IV.2.24}$$

which once exploited gives us (IV.2.19).

It is also easy to note that, as for (IV.2.15), we can state, additionally, the identity:

$$\sum_{n=0}^{+\infty} \xi^n T_n(x) U_{n-1}(x) = \frac{x\xi}{\left[1 - 2\xi\left(2x^2 - 1\right) + \xi^2\right]}. \tag{IV.2.25}$$

In fact, by multiplying both sides of the third equation of (IV.2.1) by ξ^n and then summing up, we obtain:

$$2\sqrt{1-x^2} \sum_{n=0}^{+\infty} \xi^n T_n(x) U_{n-1}(x) = \operatorname{Im}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right] \tag{IV.2.26}$$

which, by using the same procedure exploited in the above proposition, gives (IV.2.25).

In the first section (see (IV.1.3)) we have introduced the complex quantity $\mathbf{T}_n(x)$ to better derive the properties of the Chebyshev polynomials $T_n(x)$ and $U_{n-1}(x)$. To deduce further properties involving generating functions of Chebyshev polynomials, we will indicate with $\overline{\mathbf{T}}_n(x)$ the complex conjugation of the Chebyshev representation $\mathbf{T}_n(x)$.

By using the identities stated in (IV.2.1), we can immediately obtain:

$$\begin{aligned}
\operatorname{Re}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right] &= T_n(x)T_n(y) + \sqrt{\left(1-x^2\right)\left(1-y^2\right)}U_{n-1}(x)U_{n-1}(y) \\
\operatorname{Im}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right] &= \sqrt{1-x^2}U_{n-1}(x)T_n(y) - \sqrt{1-y^2}U_{n-1}(y)T_n(x)
\end{aligned}$$

and:

$$\begin{aligned}
\operatorname{Re}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right] &= T_n(x)T_n(y) - \sqrt{\left(1-x^2\right)\left(1-y^2\right)}U_{n-1}(x)U_{n-1}(y) \\
\operatorname{Im}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right] &= \sqrt{1-x^2}U_{n-1}(x)T_n(y) - \sqrt{1-y^2}U_{n-1}(y)T_n(x).
\end{aligned}$$

Theorem IV.1

Let be $\xi \in \mathbb{R}$, $|\xi| < 1$, the polynomials $T_n(x)$ and $U_n(x)$ satisfy the following identities:

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x)T_n(y) &= \frac{1}{2}\left[e^{\xi F_+}\cos\left(\xi G_-\right) + e^{\xi F_-}\cos\left(\xi G_+\right)\right] && \text{(IV.2.27)} \\
\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(x)U_{n-1}(y) &= -\frac{1}{2}\frac{\left[e^{\xi F_-}\cos\left(\xi G_+\right) + e^{\xi F_+}\cos\left(\xi G_-\right)\right]}{\sqrt{1-x^2}\left(1-y^2\right)}
\end{aligned}$$

where:

$$F_{\pm} = xy \pm \sqrt{(1-x^2)(1-y^2)}, \qquad \text{(IV.2.28)}$$
$$G_{\pm} = y\sqrt{1-x^2} \pm \sqrt{1-y^2}.$$

Proof

From the relations involving the complex quantity and its conjugate, we find:

$$2T_n(x)T_n(y) = \mathrm{Re}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right] + \mathrm{Re}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]. \qquad \text{(IV.2.29)}$$

By multiplying both sides by $\frac{\xi^n}{n!}$ and summing up, after setting $\psi = \arccos(x)$, $\phi = \arccos(y)$, it follows that:

$$2\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}T_n(x)T_n(y) = \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right] + \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right]. \qquad \text{(IV.2.30)}$$

By exploiting the r.h.s of the above equation we obtain:

$$\begin{aligned}
&\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right] + \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right] = \qquad \text{(IV.2.31)}\\
&= \mathrm{Re}\left\{\exp\left[\xi\left(\cos\psi + i\sin\psi\right)\left(\cos\phi - i\sin\phi\right)\right]\right\} + \\
&+\mathrm{Re}\left\{\exp\left[\xi\left(\cos\psi + i\sin\psi\right)\left(\cos\phi + i\sin\phi\right)\right]\right\}
\end{aligned}$$

which gives, after substituting the values of x and y:

$$\begin{aligned}
&\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right] + \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right] = \qquad \text{(IV.2.32)}\\
&= \mathrm{Re}\left\{\exp\left[\xi\left(xy - ix\sqrt{1-y^2} + iy\sqrt{1-x^2} + \sqrt{1-x^2}\sqrt{1-y^2}\right)\right]\right\} + \\
&+\mathrm{Re}\left\{\exp\left[\xi\left(xy + ix\sqrt{1-y^2} + iy\sqrt{1-x^2} - \sqrt{1-x^2}\sqrt{1-y^2}\right)\right]\right\}.
\end{aligned}$$

By using the identities in (IV.2.28), the above relation can be recast in the more convenient form:

$$\begin{aligned}
&\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right] + \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right] = \\
&= e^{\xi F_+}\mathrm{Re}\left[\cos\left(\xi y\sqrt{1-x^2}\right)\cos\left(\xi x\sqrt{1-y^2}\right) - i\cos\left(\xi y\sqrt{1-x^2}\right)\sin\left(\xi x\sqrt{1-y^2}\right) + \right.\\
&\left. + i\cos\left(\xi x\sqrt{1-y^2}\right)\sin\left(\xi y\sqrt{1-x^2}\right) + \sin\left(\xi y\sqrt{1-x^2}\right)\sin\left(\xi x\sqrt{1-y^2}\right)\right] + \\
&+e^{\xi F_-}\mathrm{Re}\left[\cos\left(\xi x\sqrt{1-y^2}\right)\cos\left(\xi y\sqrt{1-x^2}\right) + i\cos\left(\xi x\sqrt{1-y^2}\right)\sin\left(\xi y\sqrt{1-x^2}\right) + \right.\\
&\left. + i\cos\left(\xi y\sqrt{1-x^2}\right)\sin\left(\xi x\sqrt{1-y^2}\right) - \sin\left(\xi x\sqrt{1-y^2}\right)\sin\left(\xi y\sqrt{1-x^2}\right)\right].
\end{aligned}$$

Remembering that:

$$\begin{aligned}\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) &= \cos(\alpha+\beta)\\ \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) &= \cos(\alpha-\beta)\end{aligned}$$

we can rearrange the r.h.s. of the previous equation in the form:

$$\begin{aligned}&\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right]+\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right]= \\ &= e^{\xi F_+}\cos\left[\xi\left(y\sqrt{1-x^2}-x\sqrt{1-y^2}\right)\right]+e^{\xi F_-}\cos\left[\xi\left(y\sqrt{1-x^2}+x\sqrt{1-y^2}\right)\right]\end{aligned} \tag{IV.2.33}$$

and immediately follows the first one of (IV.2.27).

Otherwise, it is easy to note that:

$$\begin{aligned}&2\sqrt{(1-x^2)(1-y^2)}U_{n-1}(x)U_{n-1}(y)= \\ &= \mathrm{Re}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right]-\mathrm{Re}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]\end{aligned} \tag{IV.2.34}$$

which, once following the same procedure previous exploited, gives:

$$2\sqrt{(1-x^2)(1-y^2)}\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1}(x)U_{n-1}(y)=\exp\left[\xi\left(e^{i\psi}e^{-i\phi}\right)\right]-\exp\left[\xi\left(e^{i\psi}e^{i\phi}\right)\right] \tag{IV.2.35}$$

and then, the second of the (IV.2.27) can easily be derived.

These results can be used to find similar identities linking products of the polynomials $T_n(x)$ and $U_n(x)$. Regarding the imaginary part, we note that:

$$\begin{aligned}2\sqrt{1-x^2}U_{n-1}(x)T_n(y) &= \mathrm{Im}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]+\mathrm{Im}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right]\\ 2\sqrt{1-y^2}U_{n-1}(y)T_n(x) &= \mathrm{Im}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]-\mathrm{Im}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right].\end{aligned} \tag{IV.2.36}$$

By using again the setting in (IV.2.28) and the above identities we can state the following result.

Theorem IV.2

Let be $\xi\in\mathbb{R}$, $|\xi|<1$, the polynomials $T_n(x)$ and $U_n(x)$ satisfy the following identities, involving products in $T-U$:

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1}(x)T_n(y) &= \frac{1}{2}\frac{e^{\xi F_+}\sin\left(\xi G_-\right)+e^{\xi F_-}\sin\left(\xi G_+\right)}{\sqrt{1-x^2}}\\ \sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1}(y)T_n(x) &= \frac{1}{2}\frac{e^{\xi F_-}\sin\left(\xi G_+\right)+e^{\xi F_+}\sin\left(\xi G_-\right)}{\sqrt{1-y^2}}\end{aligned} \tag{IV.2.37}$$

Proof

From the (IV.2.36), we get:

$$\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1}(x)T_n(y)=\frac{1}{2\sqrt{1-x^2}}\mathrm{Im}\left[\exp\left(\xi e^{i\psi}e^{-i\phi}\right)\right]+\mathrm{Im}\left[\exp\left(\xi e^{i\psi}e^{i\phi}\right)\right] \tag{IV.2.38}$$

$$\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1}(y)T_n(x)=\frac{1}{2\sqrt{1-y^2}}\mathrm{Im}\left[\exp\left(\xi e^{i\psi}e^{i\phi}\right)\right]-\mathrm{Im}\left[\exp\left(\xi e^{i\psi}e^{-i\phi}\right)\right] \tag{IV.2.39}$$

where is $\psi=\arccos(x)$ and $\phi=\arccos(y)$. By following the same procedure used in the previous theorem we easily obtain the thesis.

The relations stated in Proposition IV.7 can be extended to the two-variable case. By noting in fact that:

$$|\mathbf{T}(x)|=|\exp\left(i\arccos(x)\right)|=1$$

and by choosing $|\xi|<1$, we have:

$$\xi|\mathbf{T}(x)||\mathbf{T}(y)|<1$$

and finally:

$$\sum_{n=0}^{+\infty}\xi^n\mathbf{T}_n(x)\mathbf{T}_n(y)=\frac{1}{1-\xi\left(e^{i\arccos(x)}\right)\left(e^{i\arccos(y)}\right)} \tag{IV.2.40}$$

IV.3 Integral representations

In this section we will introduce new representations of Chebyshev polynomials, by using the Hermite polynomials and the method of the generating function.

In Section IV.1 we have introduced the second kind Chebyshev polynomials $U_n(x)$ (see Definition IV.2); by exploiting the relation (IV.1.2) we can immediately get the follow explicit form:

$$U_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(n-k)!(2x)^{n-2k}}{k!(n-2k)!}. \tag{IV.3.1}$$

Proposition IV.8

The second kind Chebyshev polynomials satisfy the following integral representation:

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^n H_n\left(2x, -\frac{1}{t}\right) dt. \tag{IV.3.2}$$

Proof

By noting that:

$$n! = \int_0^{+\infty} e^{-t}t^n dt$$

for $k \leq n$ we can write:

$$(n-k)! = \int_0^{+\infty} e^{-t}t^{n-k} dt. \tag{IV.3.3}$$

From the explicit form of the Chebyshev polynomials $U_n(x)$, given in (IV.3.1), and by recalling the standard form of the two-variable Hermite polynomials:

$$He_n(x,y) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{y^k x^{n-2k}}{k!(n-2k)!}$$

after a substitution and a manipulation, we can immediately write:

$$U_n(x) = \int_0^{+\infty} e^{-t}t^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k t^{-k}(2x)^{n-2k}}{k!(n-2k)!} dt$$

and then the thesis.

By following the same procedure we can also obtain an analogous integral representation for the Chebyshev polynomials of the first kind $T_n(x)$, introduced in Definition IV.1. In fact, it is easy to derive their explicit form:

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k-1)!(2x)^{n-2k}}{k!(n-2k)!} \tag{IV.3.4}$$

and then, by using the same relations written in the previous proposition, we have:

$$T_n(x) = \frac{1}{2(n-1)!}\int_0^{+\infty} e^{-t}t^{n-1} H_n\left(2x, -\frac{1}{t}\right) dt. \tag{IV.3.5}$$

In the first chapter we have stated some useful operational results regarding the two-variable Hermite polynomials; in particular their recurrence relations can be used to state important results linking the Chebyshev polynomials of the first and second kind.

Theorem IV.3

The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ satisfy the following recurrence relations:

$$\begin{aligned} \frac{d}{dx}U_n(x) &= nW_{n-1}(x) \\ U_{n+1}(x) &= xW_n(x) - \frac{n}{n+1}W_{n-1}(x) \end{aligned} \tag{IV.3.6}$$

and:

$$T_{n+1}(x) = xU_n(x) - U_{n-1}(x) \tag{IV.3.7}$$

where:

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}He_n\left(2x, -\frac{1}{t}\right)dt.$$

Proof

The recurrence relations for the standard Hermite polynomials $He_n(x,y)$ stated in the first chapter (see Proposition I.1), can be costumized in the form:

$$\begin{aligned} \left[(2x) + \left(-\frac{1}{t}\right)\frac{\partial}{\partial x}\right]He_n\left(2x, -\frac{1}{t}\right) &= He_{n+1}\left(2x, -\frac{1}{t}\right) \\ \frac{1}{2}\frac{\partial}{\partial x}He_n\left(2x, -\frac{1}{t}\right) &= nHe_{n-1}\left(2x, -\frac{1}{t}\right). \end{aligned} \tag{IV.3.8}$$

From the integral representations stated in the relations (IV.3.2) and (IV.3.5), relevant to the Chebyshev polynomials of the first and second kind, and by using the second of the identities written above, we obtain:

$$\frac{d}{dx}U_n(x) = \frac{2n}{n!}\int_0^{+\infty} e^{-t}t^nHe_{n-1}\left(2x, -\frac{1}{t}\right)dt \tag{IV.3.9}$$

and:

$$\frac{d}{dx}T_n(x) = \frac{n}{(n-1)!}\int_0^{+\infty} e^{-t}t^{n-1}He_{n-1}\left(2x, -\frac{1}{t}\right)dt. \tag{IV.3.10}$$

It is easy to note that the above relations give a link between the polynomials $T_n(x)$ and $U_n(x)$; in fact, since:

$$U_{n-1}(x) = \frac{1}{(n-1)!}\int_0^{+\infty} e^{-t}t^{n-1}He_{n-1}\left(2x, -\frac{1}{t}\right)dt$$

we immediately obtain:

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x). \qquad \text{(IV.3.11)}$$

By applying the Hermite multiplication operator to the second kind Chebyshev polynomials, shown in the first of the identities (IV.3.8), we can write:

$$U_{n+1}(x) = \frac{1}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}\left[(2x) + \left(-\frac{1}{t}\right)\frac{\partial}{\partial x}\right] He_n\left(2x, -\frac{1}{t}\right) dt$$

that is:

$$U_{n+1}(x) = \qquad \text{(IV.3.12)}$$

$$= x\frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1} He_n\left(2x, -\frac{1}{t}\right) dt - \frac{n}{n+1}\frac{2}{n!}\int_0^{+\infty} e^{-t}t^{n} He_{n-1}\left(2x, -\frac{1}{t}\right) dt.$$

The second member of the r.h.s. of the above relation suggest us to introduce the following polynomials:

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1} He_n\left(2x, -\frac{1}{t}\right) dt \qquad \text{(IV.3.13)}$$

recognized as belonging to the families of the Chebyshev polynomials. Thus, from relation (IV.3.9), we have:

$$\frac{d}{dx}U_n(x) = nW_{n-1}(x) \qquad \text{(IV.3.14)}$$

and, from identity (IV.3.12), we get:

$$U_{n+1}(x) = xW_n(x) - \frac{n}{n+1}W_{n-1}(x). \qquad \text{(IV.3.15)}$$

Finally, by using the Hermite multiplication operator for the first kind Chebyshev polynomials, we can write:

$$T_{n+1}(x) = \frac{1}{2n!}\int_0^{+\infty} e^{-t}t^{n}\left[(2x) + \left(-\frac{1}{t}\right)\frac{\partial}{\partial x}\right] He_n\left(2x, -\frac{1}{t}\right) dt \qquad \text{(IV.3.16)}$$

and then, after expanding the r.h.s. of the above relation, we can find:

$$T_{n+1}(x) = xU_n(x) - U_{n-1}(x) \qquad \text{(IV.3.17)}$$

which completely proves the theorem.

We have stated the generating functions for the Chebyshev polynomials of the first and second kind (see eqs. (IV.1.8) and (IV.1.13)), by introducing

the complex quantity $\mathbf{T}_n(x)$. It can be also possible to derive a slight different relations linking the Chebyshev polynomials and and their generating functions, by using the integral representations and the related recurrence relations.
We note indeed, for the Chebyshev polynomials $U_n(x)$, that by multiplying both sides of equation (IV.3.2) by ξ^n, $|\xi| < 1$ and by summing up over n, it follows that:

$$\sum_{n=0}^{+\infty} \xi^n U_n(x) = \int_0^{+\infty} e^{-t} \sum_{n=0}^{+\infty} \frac{(t\xi)^n}{n!} He_n\left(2x, -\frac{1}{t}\right) dt. \tag{IV.3.18}$$

By recalling the generating function of the polynomials $He_n(x, y)$ stated in the relation (I.1.21) and by integrating over t, we end up with:

$$\sum_{n=0}^{+\infty} \xi^n U_n(x) = \frac{1}{1 - 2\xi x + \xi^2}. \tag{IV.3.19}$$

We can now state the related generating function for the first kind Chebyshev polynomials $T_n(x)$ and for the polynomials $W_n(x)$, by using the results proved in the previous theorem.

Corollary IV.1

Let be $x, \xi \in \mathbb{R}$, such that $|x| < 1, |\xi| < 1$; the generating functions of the polynomials $T_n(x)$ and $W_n(x)$ are:

$$\sum_{n=0}^{+\infty} \xi^n T_{n+1}(x) = \frac{x - \xi}{1 - 2\xi x + \xi^2} \tag{IV.3.20}$$

and:

$$\sum_{n=0}^{+\infty} (n+1)(n+2)\xi^n W_{n+1}(x) = \frac{8(x - \xi)}{(1 - 2\xi x + \xi^2)^3}. \tag{IV.3.21}$$

Proof

By multiplying both sides of relation (IV.3.7) by ξ^n and by summing up over n, we obtain:

$$\sum_{n=0}^{+\infty} \xi^n T_{n+1}(x) = x \sum_{n=0}^{+\infty} \xi^n U_n(x) - \sum_{n=0}^{+\infty} \xi^n U_{n-1}(x)$$

that is:

$$\sum_{n=0}^{+\infty} \xi^n T_{n+1}(x) = \frac{x}{1 - 2\xi x + \xi^2} - \frac{\xi}{1 - 2\xi x + \xi^2}$$

which gives (IV.3.20).

In the same way, by multiplying both sides of the second relation stated in (IV.3.6) by ξ^n and by summing up over n, we get:

$$\sum_{n=0}^{+\infty} \xi^n U_{n+1}(x) = x \sum_{n=0}^{+\infty} \xi^n W_n(x) - \sum_{n=0}^{+\infty} \frac{n}{n+1} \xi^n W_{n-1}(x)$$

and then the thesis.

These results allow us to note that the use of integral representations relating Chebyshev and Hermite polynomials is a fairly important tool of analysis allowing the derivation of a wealth of relations between first and second kind Chebyshev polynomials and the Chebyshev-like polynomials $W_n(x)$.

In (IV.3.2), we have introduced an integral representation for the second kind Chebyshev polynomials $U_n(x)$; it is also possible to state a different representation by using quite the same procedure. In fact, by using their explicit form stated in (IV.3.1), we can immediately write:

$$U_n(x) = \frac{1}{n!} \int_0^{+\infty} e^{-t} He_n(2xt, -t) dt. \qquad \text{(IV.3.22)}$$

The above relation can be used to introduce a generalization of the polynomials $U_n(x)$.

Definition IV.3

Let be x, y real variables and let α a real parameter, we call generalized Chebyshev polynomials of second kind, the polynomials defined by the following relation:

$$U_n(x, y; \alpha) = \frac{1}{n!} \int_0^{+\infty} e^{-\alpha t} He_n(2xt, -yt) dt. \qquad \text{(IV.3.23)}$$

By using the recurrence relations relevant to the two-variable Hermite polynomials, shown in the first chapter, we can state the following result.

Proposition IV.9

The generalized Chebyshev polynomials $U_n(x, y; \alpha)$ satisfy the following recurrence relations:

$$\begin{aligned}\frac{\partial}{\partial y}U_n(x,y;\alpha) &= \frac{\partial}{\partial \alpha}U_{n-2}(x,y;\alpha) \\ \frac{\partial}{\partial x}U_n(x,y;\alpha) &= -2\frac{\partial}{\partial \alpha}U_{n-1}(x,y;\alpha).\end{aligned} \tag{IV.3.24}$$

Proof

By deriving respect to y in relation (IV.3.23), we get:

$$\frac{\partial}{\partial y}U_n(x,y;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t}\frac{\partial}{\partial y}He_n(2xt,-yt)dt$$

and since (see Proposition I.3):

$$\frac{\partial}{\partial y}He_n(2xt,-yt) = (-t)n(n-1)He_{n-2}(2xt,-yt)$$

we obtain:

$$\frac{\partial}{\partial y}U_n(x,y;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t}(-t)n(n-1)He_{n-2}(2xt,-yt)dt$$

which gives the first of (IV.3.24).

The second relation can be obtained in the same way, by noting that (see Proposition I.3):

$$\frac{\partial}{\partial x}He_n(2xt,-yt) = (-2t)nHe_{n-1}(2xt,-yt).$$

Proposition IV.10

The generalized Chebyshev polynomials $U_n(x,y;\alpha)$ satisfy the following Cauchy problem:

$$\begin{cases} \frac{\partial^2}{\partial x^2}U_n(x,y;\alpha) = -4\frac{\partial^2}{\partial\alpha\partial y}U_n(x,y;\alpha) \\ U_n(x,0;\alpha) = \frac{(2x)^n}{\alpha^{n+1}} \end{cases}. \tag{IV.3.25}$$

Proof

By deriving with respect to x in the second identity of (IV.3.24), we find:

$$\frac{\partial^2}{\partial x^2}U_n(x,y;\alpha) = -4\frac{\partial}{\partial \alpha}\left(\frac{\partial}{\partial \alpha}U_{n-2}(x,y;\alpha)\right)$$

and then, since:

$$\frac{\partial}{\partial \alpha}U_{n-2}(x,y;\alpha) = \frac{\partial}{\partial y}U_n(x,y;\alpha)$$

we obtain:

$$\frac{\partial^2}{\partial x^2} U_n(x,y;\alpha) = -4\frac{\partial^2}{\partial\alpha\partial y} U_n(x,y;\alpha). \tag{IV.3.26}$$

By setting $y = 0$ in relation (IV.3.23), we have:

$$U_n(x,0;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} H_n(2xt,0)dt$$

and since (see eq. (I.1.13)):

$$He_n(2xt,0) = (2xt)^n$$

we find:

$$U_n(x,0;\alpha) = \frac{(2x)^n}{n!}\int_0^{+\infty} e^{-\alpha t} t^n dt$$

that is:

$$U_n(x,0;\alpha) = \frac{(2x)^n}{\alpha^{n+1}}. \tag{IV.3.27}$$

The partial differential equation, stated in (IV.3.26), can be viewed as a first order ordinary differential equation for the variable y; and then by using the initial condition given by (IV.3.27), we can state the solution:

$$U_n(x,y;\alpha) = e^{\frac{y}{4}\widehat{D}_\alpha^{-1}\frac{\partial^2}{\partial x^2}}\frac{(2x)^n}{\alpha^{n+1}} \tag{IV.3.28}$$

which completely proves the proposition, where the symbol $\widehat{D}_\alpha^{-1}$ denotes the inverse of the derivative.

We have introduced the generalized Chebyshev polynomials $U_n(x,y;\alpha)$ by using a different integral form of the standard second kind Chebyshev polynomials, defined in eq.(IV.3.23). By using the integral representation stated in Proposition IV.8 for the polynomials $U_n(x)$, and the related representation for the Chebyshev polynomials of the first kind $T_n(x)$ and the polynomials $W_n(x)$, we can introduce the following generalizations:

$$U_n(x,y;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} t^n He_n\left(2x,-\frac{y}{t}\right)dt, \tag{IV.3.29}$$

$$T_n(x,y;\alpha) = \frac{1}{2(n-1)!}\int_0^{+\infty} e^{-\alpha t} t^{n-1} He_n\left(2x,-\frac{y}{t}\right)dt \tag{IV.3.30}$$

and:

$$W_n(x,y;\alpha) = \frac{1}{(n+1)!}\int_0^{+\infty} e^{-\alpha t} t^{n+1} He_n\left(2x, -\frac{y}{t}\right) dt. \qquad \text{(IV.3.31)}$$

Proposition IV.11

The generalized Chebyshev polynomials satisfy the following recurrence relations:

$$\begin{aligned} \frac{\partial}{\partial\alpha}U_n(x,y;\alpha) &= -\frac{1}{2}(n+1)W_n(x,y;\alpha) \\ \frac{\partial}{\partial\alpha}T_n(x,y;\alpha) &= -\frac{n}{2}U_n(x,y;\alpha). \end{aligned} \qquad \text{(IV.3.32)}$$

Proof

By deriving with respect to α in relation (IV.3.29), we find:

$$\frac{\partial}{\partial\alpha}U_n(x,y;\alpha) = -\frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} t^{n+1} He_n\left(2x, -\frac{y}{t}\right) dt$$

and using the first equation in (IV.3.32), the result immediately follows.
In the same way by following the same procedure with identity (IV.3.30), we have:

$$\frac{\partial}{\partial\alpha}T_n(x,y;\alpha) = -\frac{1}{2(n-1)!}\int_0^{+\infty} e^{-\alpha t} t^{n} H_n\left(2x, -\frac{y}{t}\right) dt$$

and then the thesis.
It is worth noting that the Chebyshev polynomials can be viewed as a particular case of the Gegenbauer polynomials.

Definition IV.4

Let be x and μ real variables, we call $n-th$ order Gegenbauer polynomials, the polynomials defined by the following relation:

$$C_n^{(\mu)}(x) = \frac{1}{\Gamma(\mu)}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2x)^{n-2k}\Gamma(n-k+\mu)}{k!(n-2k)!} \qquad \text{(IV.3.33)}$$

where $\Gamma(\mu)$ is the Euler function.
By recalling the integral representation of the above Euler function:

$$\Gamma(\mu) = \int_0^{+\infty} e^{-t} t^{\mu-1} dt \qquad \text{(IV.3.34)}$$

and by using the same arguments exploited for the Chebyshev case (see eqs. (IV.3.2) and (IV.3.3)), we can state the integral representation for the Gegenbauer polynomials:

$$C_n^{(\mu)}(x) = \frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-t} t^{n+\mu-1} He_n\left(2x, -\frac{1}{t}\right) dt. \tag{IV.3.35}$$

We can also generalized the Gegenbauer polynomials by using their integral representation.

Definition IV.5

Let be x, y real variables and let α be a real parameter, we say generalized Gegenbauer polynomials, the polynomials defined by the following relation:

$$C_n^{(\mu)}(x, y; \alpha) = \frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu-1} He_n\left(2x, -\frac{y}{t}\right) dt. \tag{IV.3.36}$$

The above integral representation is a very flexible tool; in fact it can be exploited to derive interesting relations regarding the Gegenbauer polynomials and also the Chebyshev polynomials.

Proposition IV.12

Let be $\xi \in \mathbb{R}$, such that $|\xi| < 1$ and $\mu \neq 0$. The generating function of the polynomials $C_n^{(\mu)}(x, y; \alpha)$ is given by:

$$\sum_{n=0}^{+\infty} \xi^n C_n^{(\mu)}(x, y; \alpha) = \frac{1}{[\alpha - 2x\xi + y\xi^2]^{\mu}}. \tag{IV.3.37}$$

Proof

By multiplying both sides of identity (IV.3.36), by ξ^n and by summing up over n, we get:

$$\sum_{n=0}^{+\infty} \xi^n C_n^{(\mu)}(x, y; \alpha) = \int_0^{+\infty} \sum_{n=0}^{+\infty} \frac{\xi^n t^n}{n!\Gamma(\mu)} e^{-\alpha t} t^{\mu-1} He_n\left(2x, -\frac{y}{t}\right) dt$$

and by noting that:

$$\sum_{n=0}^{+\infty} \frac{(\xi t)^n}{n!} He_n\left(2x, -\frac{y}{t}\right) = \exp\left[\xi\,(2xt) + \xi^2(-yt)\right]$$

we can write:

$$\sum_{n=0}^{+\infty} \xi^n C_n^{(\mu)}(x,y;\alpha) = \int_0^{+\infty} \frac{1}{\Gamma(\mu)} e^{-\alpha t} e^{\xi(2xt)+\xi^2(-yt)} t^{\mu-1} dt. \qquad \text{(IV.3.38)}$$

Finally, by integrating over t and by using the integral representation of the Euler function, we obtain the thesis.

Proposition IV.13

The generalized second kind Chebyshev polynomials and the generalized Gegenbauer polynomials satisfy the following recurrence relation:

$$(-1)^m \frac{\partial^m}{\partial \alpha^m} U_n(x,y;\alpha) = m! C_n^{(m+1)}(x,y;\alpha). \qquad \text{(IV.3.39)}$$

Proof

By deriving with respect to α in relation (IV.3.29), m-times, we get:

$$\frac{\partial^m}{\partial \alpha^m} U_n(x,y;\alpha) = \frac{(-1)^m}{n!} \int_0^{+\infty} e^{-\alpha t} t^{n+m} He_n\left(2x, -\frac{y}{t}\right) dt.$$

The r.h.s. of the above identity can be written in the form:

$$\frac{(-1)^m}{n!} \int_0^{+\infty} e^{-\alpha t} t^{n+m} He_n\left(2x, -\frac{y}{t}\right) dt = \frac{(-1)^m m!}{n! m!} \int_0^{+\infty} e^{-\alpha t} t^{n+m} He_n\left(2x, -\frac{y}{t}\right) dt$$

and then the thesis.

In this section we have proved some important relations regarding the Chebyshev polynomials of first and second kind. In particular, in Theorem IV.3 we have proved the related recurrence relations by using modified relations of the two-variable Hermite polynomials. From (IV.3.8) it is easy to note that:

$$\left[(2x) + \left(-\frac{y}{t}\right)\frac{\partial}{\partial x}\right] He_n\left(2x, -\frac{y}{t}\right) = He_{n+1}\left(2x, -\frac{y}{t}\right) \qquad \text{(IV.3.40)}$$

which can be used to derive the following results.

Theorem IV.4

The generalized Gegenbauer polynomials $C_n^{(\mu)}(x,y;\alpha)$ satisfy the recurrence relations:

$$\frac{n+1}{2\mu} C_{n+1}^{(\mu)}(x,y;\alpha) = x C_n^{(\mu+1)}(x,y;\alpha) - y C_{n-1}^{(\mu+1)}(x,y;\alpha) \qquad \text{(IV.3.41)}$$

and:

$$\frac{\partial}{\partial y} C_n^{(\mu)}(x, y; \alpha) = -\mu C_{n-2}^{(\mu+1)}(x, y; \alpha). \qquad \text{(IV.3.42)}$$

Proof

By using relation (IV.3.40), we can write the generalized Gegenbauer polynomial of order $n+1$, in the form:

$$C_{n+1}^{(\mu)}(x, y; \alpha) = \qquad \text{(IV.3.43)}$$

$$= \frac{1}{(n+1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} \left[(2x) + \left(-\frac{y}{t}\right) \frac{\partial}{\partial x}\right] He_n \left(2x, -\frac{y}{t}\right) dt.$$

After expanding the r.h.s of the above identity, we get:

$$C_{n+1}^{(\mu)}(x, y; \alpha) = \qquad \text{(IV.3.44)}$$

$$= \frac{1}{(n+1)!\Gamma(\mu)} \left[\int_0^{+\infty} e^{-\alpha t} t^{n+\mu} (2x) He_n \left(2x, -\frac{y}{t}\right) dt - \right.$$

$$\left. + \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} y(2n) He_{n-1} \left(2x, -\frac{y}{t}\right) dt\right]$$

and then:

$$C_{n+1}^{(\mu)}(x, y; \alpha) = \qquad \text{(IV.3.45)}$$

$$= \frac{2x}{(n+1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} He_n \left(2x, -\frac{y}{t}\right) dt -$$

$$+ \frac{2yn}{(n+1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} He_{n-1} \left(2x, -\frac{y}{t}\right) dt.$$

We can rearrange the above relation in the form:

$$\frac{n+1}{2} C_{n+1}^{(\mu)}(x, y; \alpha) =$$

$$= x \frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} He_n \left(2x, -\frac{y}{t}\right) dt -$$

$$+ y \frac{1}{(n-1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} He_{n-1} \left(2x, -\frac{y}{t}\right) dt$$

and finally:

$$\frac{n+1}{2\mu} C_{n+1}^{(\mu)}(x, y; \alpha) =$$

$$= x \frac{1}{n!\Gamma(\mu+1)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} He_n \left(2x, -\frac{y}{t}\right) dt -$$

$$+ y \frac{1}{(n-1)!\Gamma(\mu+1)} \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} He_{n-1} \left(2x, -\frac{y}{t}\right) dt$$

which proves (IV.3.41).

To show the recurrence relation in (IV.3.42), it is important to note that:

$$\frac{\partial}{\partial y} He_n\left(2x, -\frac{y}{t}\right) = -\frac{n(n-1)}{t} He_{n-2}\left(2x, -\frac{y}{t}\right). \tag{IV.3.46}$$

In fact, by deriving with respect to y in (IV.3.36), we get:

$$\frac{\partial}{\partial y} C_n^{(\mu)}(x, y; \alpha) = \frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu-1} \frac{\partial}{\partial y} He_n\left(2x, -\frac{y}{t}\right) dt$$

and by using (IV.3.46), we can write:

$$\frac{\partial}{\partial y} C_n^{(\mu)}(x, y; \alpha) = -\frac{n(n-1)}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n-2+\mu} He_{n-2}\left(2x, -\frac{y}{t}\right) dt$$

which immediately gives the thesis.

IV.4 Further generalizations

By using the operational rules and the related formalism of the two-variable Hermite polynomials, introduced in the first chapter, we have derived the definitions and some interesting properties for the first and second kind Chebyshev polynomials and for ordinary and generalized Gegenbauer polynomials. Now, we can use the Hermite polynomials of the type $H_n^{(m)}(x, y)$, which have been defined in the first section of Chapter II, to introduce a further generalization of the Chebyshev and Gegenbauer polynomials.

Definition IV.6

Let be x, y real variables and let α a real parameter, we say generalized, m-order, two-variable, second kind Chebyshev polynomials, the polynomials defined by the relation:

$${}_mU_n(x, y; \alpha) = \frac{1}{n!} \int_0^{+\infty} e^{-\alpha t} t^n H_n^{(m)}\left(mx, -\frac{y}{t^{m-1}}\right) dt. \tag{IV.4.1}$$

Proposition IV.14

The generating function of the generalized Chebyshev polynomials of the type ${}_mU_n(x, y; \alpha)$, is given by:

$$\sum_{n=0}^{+\infty} \xi^n \left[{}_mU_n(x, y; \alpha)\right] = \frac{1}{\alpha - mx\xi + y\xi^m} \tag{IV.4.2}$$

where $\xi \in \mathbb{R}$, $|\xi| < 1$.

Proof

Let be $\xi \in \mathbb{R}$, $|\xi| < 1$, by multiplying both sides of the relation (IV.4.1) by ξ^n and by summing up over n, we have:

$$\sum_{n=0}^{+\infty} \xi^n \left[{}_mU_n(x,y;\alpha)\right] = \int_0^{+\infty} e^{-\alpha t} \sum_{n=0}^{+\infty} \frac{(\xi t)^n}{n!} H_n^{(m)}\left(mx, -\frac{y}{t^{m-1}}\right) dt. \quad \text{(IV.4.3)}$$

By noting that the generating function of the Hermite polynomials of the type $H_n^{(m)}(x,y)$, (see eq. (II.1.5)), is:

$$\sum_{n=0}^{+\infty} \frac{(\xi t)^n}{n!} H_n^{(m)}\left(mx, -\frac{y}{t^{m-1}}\right) = \exp\left[mx(\xi t) - y\left(t\xi^m\right)\right]$$

we obtain, in (IV.4.3):

$$\sum_{n=0}^{+\infty} \xi^n \left[{}_mU_n(x,y;\alpha)\right] = \int_0^{+\infty} e^{-\alpha t} e^{(mx\xi - y\xi^m)t} dt$$

which, once integrating over t, gives the the statement (IV.4.2).

Definition IV.7

Let x, y be real variables, α a real parameter and $\mu \in \mathbb{R}$, $\mu > 0$, we say generalized, m-order Gegenbauer polynomials, the polynomials defined by the relation:

$${}_mC_n^{(\mu)}(x,y;\alpha) = \frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu-1} H_n^{(m)}\left(mx, -\frac{y}{t^{m-1}}\right) dt. \quad \text{(IV.4.4)}$$

It is easy to state, by following the same procedure used for the polynomials ${}_mU_n(x,y;\alpha)$, the generating function of the above generalized Gegenbauer polynomials. We have in fact:

$$\sum_{n=0}^{+\infty} \xi^n \left[{}_mC_n^{(\mu)}(x,y;\alpha)\right] = \frac{1}{(\alpha - mx\xi + y\xi^m)^\mu} \quad \text{(IV.4.5)}$$

where, again $\xi \in \mathbb{R}$, $|\xi| < 1$.

By using the recurrence relations stated for the Hermite polynomials of the type $H_n^{(m)}(x,y)$, see (II.1.9), we can derive, as for the generalized Gegenbauer polynomials of the type $C_n^{(\mu)}(x,y;\alpha)$, see Theorem IV.4 the following important identities for the polynomials ${}_mC_n^{(\mu)}(x,y;\alpha)$.

Theorem IV.5

The generalized, m-order Gegenbauer polynomials $C_n^{(\mu)}(x,y;\alpha)$ satisfy the recurrence relations:

$$\begin{aligned}
\frac{n+1}{m\mu}\left[{}_mC_{n+1}^{(\mu)}(x,y;\alpha)\right] &= x\left[{}_mC_n^{(\mu+1)}(x,y;\alpha)\right] - y\left[{}_mC_{n-m+1}^{(\mu+1)}(x,y;\alpha)\right] \\
\frac{\partial}{\partial y}\left[{}_mC_n^{(\mu)}(x,y;\alpha)\right] &= -\mu\left[{}_mC_{n-m}^{(\mu+1)}(x,y;\alpha)\right].
\end{aligned} \tag{IV.4.6}$$

Proof

The proof of this statement is equivalent to that of Theorem IV.4 ; it is enough to note that the polynomials $H_n^{(m)}(x,y)$ satisfy the following identities:

$$\begin{aligned}
\frac{\partial}{\partial x}H_n^{(m)}\left(mx,-\frac{y}{t^{m-1}}\right) &= nmH_{n-1}^{(m)}\left(mx,-\frac{y}{t^{m-1}}\right) \\
\frac{\partial}{\partial y}H_n^{(m)}\left(mx,-\frac{y}{t^{m-1}}\right) &= -\frac{n(n-1)}{t^{m-1}}H_{n-2}^{(m)}\left(mx,-\frac{y}{t^{m-1}}\right).
\end{aligned}$$

The generalization of the Gegenbauer polynomials was also given by Gould [2], but the procedure here described can be considered complementary to that of Gould; in particular the using of the Hermite polynomials and their related properties provides benefits to derive known and unknown relations. For instance we can also use the Gegenbauer polynomials, introduced in equation (IV.3.33) or (IV.3.35) to find further links between the ordinary Chebyshev polynomials.

In fact, by setting $\mu = 2$, in relation (IV.3.35), we have:

$$C_n^{(2)}(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^{n+1}He_n\left(2x,-\frac{1}{t}\right)dt$$

which can be easily recognized as:

$$C_n^{(2)}(x) = \frac{n+1}{2}W_n(x) \tag{IV.4.7}$$

where the polynomials $W_n(x)$ have been specified in (IV.3.13).

Proposition IV.15

The Chebyshev polynomials of the first and second kind satisfy the following recurrence relation:

$$(n+1)W_n(x) = \frac{x}{1-x^2}U_{n+1}(x) - \frac{n+2}{1-x^2}T_{n+2}(x). \qquad \text{(IV.4.8)}$$

Proof

By recalling the definitions of the first and second kind Chebyshev polynomials in terms of the circular functions (eqs. (IV.1.1) and (IV.1.2)):

$$\begin{aligned} T_n(x) &= \cos n(\arccos(x)) \\ U_n(x) &= \frac{\sin\left[(n+1)\arccos(x)\right]}{\sqrt{1-x^2}} \end{aligned}$$

and from identity (IV.3.14), we get:

$$(n+1)W_n(x) = \frac{d}{dx}\left\{\frac{\sin\left[(n+2)\arccos(x)\right]}{\sqrt{1-x^2}}\right\}.$$

By exploiting the r.h.s. of the above equation, we obtain:

$$(n+1)W_n(x) = \frac{x}{1-x^2}\frac{\sin\left[(n+2)\arccos(x)\right]}{\sqrt{1-x^2}} - \frac{n+2}{1-x^2}\cos\left[(n+2)\arccos(x)\right]. \qquad \text{(IV.4.9)}$$

which immediately proves the statement.

In this first chapter dedicated to the Chebyshev polynomials, we have seen interesting integral representations related to ordinary and generalized Chebyshev polynomials. The common denominator with respect to which were derived properties are the plethora of identities and operational relations satisfied by the Hermite polynomials in their different forms. In the next chapters, we will investigate families of polynomials that can be traced to the Chebyshev polynomials, always operating with the aid of the Hermite polynomials.

Chapter V

Generalized two-variable Chebyshev polynomials

In this chapter we will discuss the two-variable Chebyshev polynomials. The approach will be based on generalized two-variable Hermite polynomials, introduced in the first chapter and the integral representations stated in the previous chapter concerning the Chebyshev polynomials. In addition, expanding what we have seen previously, we will discuss both the Chebyshev polynomials in several indices and the Gegenbauer polynomials. A fundamental role, as we will see, is played by the powerful operational techniques verified by the families of generalized Hermite polynomials.

V.1 Two-variable Chebyshev polynomials

In the previous chapter, we have introduced the second kind Chebyshev polynomials by Definition IV.2 and we have also derived their explicit form:

$$U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k)!(2x)^{n-2k}}{k!(n-2k)!} \tag{V.1.1}$$

By exploiting the method of the integral representation, we have also defined the two-variable, one-parameter, second kind Chebyshev polynomials $U_n(x, y; \alpha)$:

$$U_n(x, y; \alpha) = \frac{1}{n!} \int_0^{+\infty} e^{-\alpha t} He_n\,(2xt, -yt)\, dt \tag{V.1.2}$$

by using the related formula of the one-variable case:

$$U_n(x) = \frac{1}{n!} \int_0^{+\infty} e^{-t} He_n\,(2xt, -t)\,dt \tag{V.1.3}$$

In this chapter, we will introduce further generalizations of the Chebyshev polynomials by using again the method of the integral representation. Let us introduce the two-variable Chebyshev polynomials of the second kind:

Definition V.1

Let be x and y two real variables, we say generalized two-variable Chebyshev polynomials of the second kind, the polynomials defined by the following relation:

$$U_n(x, y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(n-k)!x^{n-2k}y^k}{k!(n-2k)!}. \tag{V.1.4}$$

It is easy to note that the above polynomials can be derived directly from the explicit form of the standard second kind Chebyshev polynomials, or by using the integral representation of the polynomials $U_n(x, y; \alpha)$. We have, in fact:

Proposition V.1

The generalized Chebyshev polynomials $U_n(x, y)$ satisfy the following integral representation:

$$U_n(x, y) = \frac{1}{n!} \int_0^{+\infty} e^{-t} t^n He_n\left(x, \frac{y}{t}\right) dt. \tag{V.1.5}$$

Proof

By noting that:

$$(n-k)! = \int_0^{+\infty} e^{-t} t^{n-k} dt$$

we can write in relation (V.1.5):

$$U_n(x, y) = \int_0^{+\infty} e^{-t} t^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2k} y^k t^{-k}}{k!(n-2k)!} dt.$$

By recalling that the Hermite polynomials $He_n(x, y)$, reads:

$$He_n(x, y) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2k} y^k}{k!(n-2k)!}$$

we immediately get the thesis.

The above result help us to state the link between the Chebyshev polynomials $U_n(x,y)$ and the slight different one-parameter Chebyshev polynomials $U_n(x,y;\alpha)$ presented in the previous chapter.

Proposition V.2

The polynomials $U_n(x,y)$ and $U_n(x,y;\alpha)$, satisfy the following equation:

$$U_n\left(\frac{x}{2},-y;1\right)=U_n(x,y). \tag{V.1.6}$$

Proof

The statement is immediately derived. It is enough to substitute y with $-y$ and to set $\alpha=1$ in the identity (V.1.2), that is:

$$U_n\left(\frac{x}{2},-y;1\right)=\frac{1}{n!}\int_0^{+\infty}e^{-t}He_n\left(xt,yt\right)dt. \tag{V.1.7}$$

By expanding the Hermite polynomial in the r.h.s. of the above relation, we get:

$$He_n\left(xt,yt\right)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2k}t^{n-2k}y^k t^k}{k!(n-2k)!}=t^n n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2k}\left(\frac{y}{t}\right)^k}{k!(n-2k)!}=t^n He_n\left(x,\frac{y}{t}\right) \tag{V.1.8}$$

and then the thesis.

In Chapter IV, it has been shown that most of the properties of the Chebyshev polynomials $U_n(x)$ and $U_n(x,y;\alpha)$, can be directly inferred from those of the ordinary and the generalized two-variable Hermite polynomials and from their integral representations. We can also use the integral representation of the generalized two-variable second kind Chebyshev polynomials to introduce the analogous generalization of the first kind Chebyshev polynomials.

Definition V.2

Let be x and y two real variables, we call generalized two-variable Chebyshev polynomials of the first kind, the polynomials defined by the following integral representation:

$$T_n(x,y)=\frac{1}{(n-1)!}\int_0^{+\infty}e^{-t}t^{n-1}He_n\left(x,\frac{y}{t}\right)dt. \tag{V.1.9}$$

Theorem V.1

The generalized Chebyshev polynomials $U_n(x,y)$ and $T_n(x,y)$ satisfy the following recurrence relations:

$$2y\frac{\partial}{\partial x}U_{n-1}(x,y) = \left(n - x\frac{\partial}{\partial x}\right)U_n(x,y) \tag{V.1.10}$$

and:

$$T_{n+1}(x,y) = xU_n(x,y) + 2yU_{n-1}(x,y). \tag{V.1.11}$$

Proof

By using the recurrence relations related to the two-variable Hermite polynomials, stated in Propositions I.9 and I.11, we can derive the following identities relevant to the polynomials $He_n\left(x, \frac{y}{t}\right)$; we have, in fact:

$$\left[x + 2\left(\frac{y}{t}\right)\frac{\partial}{\partial x}\right]He_n\left(x,\frac{y}{t}\right) = He_{n+1}\left(x,\frac{y}{t}\right) \tag{V.1.12}$$

$$\frac{\partial}{\partial x}He_n\left(x,\frac{y}{t}\right) = nHe_{n-1}\left(x,\frac{y}{t}\right).$$

It is also important to note that, from the partial differential equation stated in the relation (I.3.35) (see Proposition I.11), we can write:

$$\left[2\left(\frac{y}{t}\right)\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - n\right]He_n\left(x,\frac{y}{t}\right) = 0. \tag{V.1.13}$$

By exploiting the above identity, we obtain:

$$\frac{2y}{t}\frac{\partial^2}{\partial x^2}He_n\left(x,\frac{y}{t}\right) = \left(n - x\frac{\partial}{\partial x}\right)He_n\left(x,\frac{y}{t}\right) \tag{V.1.14}$$

and from the second identity in (V.1.12), we have:

$$2n\frac{y}{t}\frac{\partial}{\partial x}He_{n-1}\left(x,\frac{y}{t}\right) = \left(n - x\frac{\partial}{\partial x}\right)He_n\left(x,\frac{y}{t}\right). \tag{V.1.15}$$

From the integral representation of the generalized Chebyshev polynomials $U_n(x,y)$ (see eq. (V.1.5)), we can write:

$$\left(n - x\frac{\partial}{\partial x}\right)U_n\left(x,y\right) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^n\left(n - x\frac{\partial}{\partial x}\right)He_n\left(x,\frac{y}{t}\right)dt \tag{V.1.16}$$

which gives:

$$\left(n - x\frac{\partial}{\partial x}\right)U_n\left(x,y\right) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^n 2n\frac{y}{t}\frac{\partial}{\partial x}He_{n-1}\left(x,\frac{y}{t}\right)dt \tag{V.1.17}$$

and then:

$$\left(n - x\frac{\partial}{\partial x}\right) U_n\left(x, y\right) = \frac{2y}{(n-1)!}\frac{\partial}{\partial x}\int_0^{+\infty} e^{-t}t^{n-1}He_{n-1}\left(x, \frac{y}{t}\right) dt \quad \text{(V.1.18)}$$

that corresponds to (V.1.10).

By using the integral representation of the polynomials $T_n(x, y)$ and from the first of the identities in (V.1.12), we can write:

$$T_{n+1}(x, y) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^n \left[x + 2\left(\frac{y}{t}\right)\frac{\partial}{\partial x}\right] He_n\left(x, \frac{y}{t}\right) dt \qquad \text{(V.1.19)}$$

which, once expanded the r.h.s, by using the second of the recurrence in (V.1.12), gives:

$$T_{n+1}(x, y) = \frac{x}{n!}\int_0^{+\infty} e^{-t}t^n He_n\left(x, \frac{y}{t}\right) dt + \frac{2yn}{n!}\int_0^{+\infty} e^{-t}t^{n-1}He_{n-1}\left(x, \frac{y}{t}\right) dt$$

and then statement (V.1.11) immediately follows.

It is easy also to note that from the relations (V.1.10) and (V.1.11) the other recurrence can be proved:

$$U_{n+1}(x, y) = xU_n(x, y) + yU_{n-1}(x, y). \qquad \text{(V.1.20)}$$

The recurrences (V.1.10) and (V.1.19) can be exploited to define rising and lowering operators for generalized Chebyshev polynomials; indeed, by using the operator $\widehat{D}_x^{-1}$, denoting a kind of inverse derivative, we can immediately write, from (V.1.10):

$$U_{n-1}(x, y) = \frac{1}{2y}\widehat{D}_x^{-1}\left[n - x\frac{\partial}{\partial x}\right] U_n(x, y) \qquad \text{(V.1.21)}$$

and, from (V.1.19):

$$U_{n+1}(x, y) = \left[x + \frac{1}{2}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)\right] U_n(x, y). \qquad \text{(V.1.22)}$$

We can now use these last relations to introduce the following operators:

$$\widehat{E}_+ = x + \frac{1}{2}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right) \qquad \text{(V.1.23)}$$

$$\widehat{E}_- = \frac{1}{2y}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)$$

whose action can be written as:

$$\hat{E}_{+} U_n(x,y) = U_{n+1}(x,y) \tag{V.1.24}$$

$$\hat{E}_{-} U_n(x,y) = U_{n-1}(x,y)$$

which can be exploited to derive the differential equation satisfied by the generalized two-variable Chebyshev polynomials $U_n(x,y)$.

Theorem V.2

The polynomials $U_n(x,y)$ satisfy the following partial differential equation:

$$\left[\left(4y + x^2\right)\frac{\partial^2}{\partial x^2} + 3x\frac{\partial}{\partial x} - n(n+2)\right] U_n(x,y) = 0. \tag{V.1.25}$$

Proof

By using the *rising* and the *lowering* operators defined in (V.1.23), we can immediately write:

$$\hat{E}_{-}\left[\hat{E}_{+} U_n(x,y)\right] = U_n(x,y) \tag{V.1.26}$$

which can be expanded to give:

$$\frac{1}{2y}\hat{D}_x^{-1}\left[\left((n+1) - x\frac{\partial}{\partial x}\right)\right]\left[x + \frac{1}{2}\hat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)\right] U_n(x,y) = U_n(x,y). \tag{V.1.27}$$

By noting that:

$$\frac{\partial}{\partial x}\hat{D}_x^{-1} = \hat{1} \tag{V.1.28}$$

we can derive with respect to x in relation (V.1.16), to get:

$$\left[(n+1) - x\frac{\partial}{\partial x}\right]\left[x + \frac{1}{2}\hat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)\right] U_n(x,y) = 2y\frac{\partial}{\partial x}U_n(x,y)$$

that is:

$$\left[(n+1)x + \frac{n+1}{2}\hat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right) - x\frac{\partial}{\partial x}x - \frac{1}{2}x\left(n - x\frac{\partial}{\partial x}\right)\right] U_n(x,y) = \tag{V.1.29}$$

$$= 2y\frac{\partial}{\partial x}U_n(x,y)$$

and again:

$$\left[(n+1)\frac{\partial}{\partial x}x + \frac{n+1}{2}\left(n - x\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}x\frac{\partial}{\partial x}x - \frac{1}{2}\frac{\partial}{\partial x}x\left(n - x\frac{\partial}{\partial x}\right)\right] U_n(x,y) = \tag{V.1.30}$$

$$= 2y\frac{\partial^2}{\partial x^2}U_n(x,y).$$

By noting that:

$$x\frac{\partial}{\partial x} = \frac{\partial}{\partial x}x - 1 \tag{V.1.31}$$

we can rewrite (V.1.27) in the form:

$$\left[(n+1)\left(1 + x\frac{\partial}{\partial x}\right) + \frac{n+1}{2}\left(n - x\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}x\left(1 + x\frac{\partial}{\partial x}\right) + \right.$$

$$\left. -\frac{1}{2}\frac{\partial}{\partial x}x\left(n - x\frac{\partial}{\partial x}\right)\right]U_n(x,y) = 2y\frac{\partial^2}{\partial x^2}U_n(x,y) \tag{V.1.32}$$

which can be further expanded, giving:

$$\left[(n+1) + (n+1)x\frac{\partial}{\partial x} + \frac{n(n+1)}{2} - \frac{n+1}{2}x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x^2\frac{\partial}{\partial x} + \right. \tag{V.1.33}$$

$$\left. -\frac{n}{2}\frac{\partial}{\partial x}x + \frac{1}{2}\frac{\partial}{\partial x}x^2\frac{\partial}{\partial x}\right]U_n(x,y) = 2y\frac{\partial^2}{\partial x^2}U_n(x,y). \tag{V.1.34}$$

The above identity can be also recast in a more convenient form; indeed, by noting that:

$$\frac{\partial}{\partial x}x^2\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial x^2}$$

we write:

$$\left\{-\frac{1}{2}x^2\frac{\partial^2}{\partial x^2} + \left[\frac{2(n+1) - (n+1) - 4 - n}{2}\right]x\frac{\partial}{\partial x} + \right.$$

$$\left. + \left[\frac{2(n+1) + n(n+1) - 2 - n}{2}\right]\right\}U_n(x,y) = 2y\frac{\partial^2}{\partial x^2}U_n(x,y)$$

and finally:

$$\left[-\frac{1}{2}x^2\frac{\partial^2}{\partial x^2} - \frac{3}{2}x\frac{\partial}{\partial x} + \frac{n(n+2)}{2}\right]U_n(x,y) = 2y\frac{\partial^2}{\partial x^2}U_n(x,y) \tag{V.1.35}$$

which immediately gives the statement.

In the previous chapter we have defined the one-variable Chebyshev polynomials of second kind in many different ways; in particular in identity (IV.3.1), we have obtained their explicit form:

$$U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k)!(2x)^{n-2k}}{k!(n-2k)!}.$$

It is easy to note that the generalized Chebyshev polynomials $U_n(x,y)$ are linked to the ordinary polynomials $U_n(x)$, by the following relation:

$$U_n(2x,-1) = U_n(x) \tag{V.1.36}$$

and also by the formula:

$$U_n(x,y) = (-1)^n y^{\frac{n}{2}} U_n\left(\frac{ix}{2\sqrt{y}}\right). \tag{V.1.37}$$

These relations can be used to better clarify the role of the integral transform of the Chebyshev polynomials; we can in fact derive a different integral representation for the polynomials $U_n(x)$ and an important operational identity.

Proposition V.3

The polynomials $U_n(x)$ satisfy the following relations:

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^{\frac{n}{2}} He_n\left(\sqrt{t}x\right) dt \tag{V.1.38}$$

and:

$$U_n(x) = \frac{\Gamma\left(1+\frac{1}{2}\left(n + x\frac{\partial}{\partial x}\right)\right)}{n!} He_n(x). \tag{V.1.39}$$

Proof

The first identity can be obtained from (V.1.5), by using relation (V.1.31). In fact, by setting:

$$\begin{cases} x \to 2x \\ y \to -1 \end{cases}$$

we have:

$$U_n(x) = U_n(2x,-1) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^n H_n\left(2x, -\frac{1}{t}\right) dt$$

and since:

$$He_n\left(2x, -\frac{1}{t}\right) = t^{-\frac{1}{2}} He_n\left(\sqrt{t}x\right)$$

we immediately get (V.1.38).

To derive the second statement we note that the dilatation operator acts on a generic function $f(x)$ as:

$$e^{\lambda x \frac{d}{dx}} f(x) = f(e^{\lambda} x) \tag{V.1.40}$$

where λ is also a real variable. We can rewrite identity (V.1.38) in the form:

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^{\frac{n}{2}} e^{\ln(\sqrt{t})x\frac{d}{dx}} \left[He_n\left(x\right)\right] dt$$

that is:

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^{\frac{n}{2}} t^{\frac{1}{2}} e^{x\frac{d}{dx}} \left[He_n\left(x\right)\right] dt. \qquad \text{(V.1.41)}$$

By noting that the Euler-function reads:

$$\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt$$

we immediately recast (V.1.41) to obtain the thesis.

The integral representation of the generalized Chebyshev polynomials can be also used to state their generating function.

Proposition V.4

Let be $\xi \in \mathbb{R}$, such that $|\xi| < 1$; the generating function of the generalized two-variable Chebyshev polynomials of second kind, reads:

$$\sum_{n=0}^{+\infty} \xi^n U_n(x,y) = \frac{1}{1 - x\xi - y\xi^2}. \qquad \text{(V.1.42)}$$

Proof

From relation (V.1.5), multiplying by ξ^n, $|\xi| < 1$, and by summing up over n, we have:

$$\sum_{n=0}^{+\infty} \xi^n U_n(x,y) = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} \int_0^{+\infty} e^{-t} t^n He_n\left(x, \frac{y}{t}\right) dt$$

and then:

$$\sum_{n=0}^{+\infty} \xi^n U_n(x,y) = \int_0^{+\infty} e^{-t} \sum_{n=0}^{+\infty} \frac{(t\xi)^n}{n!} He_n\left(x, \frac{y}{t}\right) dt. \qquad \text{(V.1.43)}$$

The above relation can be immediately simplify by using the link between the Hermite polynomials and their generating function, shown in Section I.1 (see eq. (I.1.21)), that is:

$$\sum_{n=0}^{+\infty} \xi^n U_n(x,y) = \int_0^{+\infty} e^{-t} e^{t(x\xi)+t(y\xi^2)} dt$$

which gives the thesis.

V.2 Generalized two-variable Chebyshev polynomials

In the second chapter, we have introduced the generalized Hermite polynomials of the type $H_n^{(m)}(x, y)$, by using the formalism of the translation operator (see Definition II.1). We want introduce an extension of the generalized two-variable Chebyshev polynomials discussed in previous section by using the structure and the related properties of the $m-th$ order Hermite polynomials mentioned before. It is possible to derive this generalization in many cases; in particular we can directly define the new generalized Chebyshev polynomials and then to state the link with the Hermite polynomials $H_n^{(m)}(x, y)$.

Definition V.3

We will call generalized two-variable $m^{th} - order$ Chebyshev polynomials the polynomials defined by the formula:

$$U_n^{(m)}(x, y) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(n-k)! x^{n-mk} y^k}{k!(n-mk)!} \tag{V.2.1}$$

where $x, y \in \mathbb{R}$ and $n, m \in \mathbb{N}$.
By using the Hermite polynomials $H_n^{(m)}(x, y)$ we can immediately derive the integral representation for the Chebyshev polynomials of the type $U_n^{(m)}(x, y)$.

Proposition V.5

The generalized Chebyshev polynomials $U_n^{(m)}(x, y)$ satisfy the following integral representation:

$$U_n^{(m)}(x, y) = \frac{1}{n!} \int_0^{+\infty} e^{-t} t^n H_n^{(m)}\left(x, \frac{y}{t}\right) dt. \tag{V.2.2}$$

Proof
By following the same procedure used to state the analogous result related to the Chebyshev polynomials $U_n(x, y)$, we can write, from identity (V.2.1):

$$U_n^{(m)}(x, y) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \int_0^{+\infty} e^{-t} t^{n-k} \frac{x^{n-mk} y^k}{k!(n-mk)!} dt \tag{V.2.3}$$

since:

$$(n-k)! = \int_0^{+\infty} e^{-t} t^{n-k} dt.$$

By manipulating relation (V.2.3), we find:

$$U_n^{(m)}(x,y) = \int_0^{+\infty} e^{-t} t^n \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{x^{n-mk}}{k!(n-mk)!} \left(\frac{y}{t}\right)^k dt$$

and, from the explicit form of the polynomials $H_n^{(m)}(x,y)$, (see eq. (II.1.1)), the thesis immediately follows.

The concepts and the formalism used to derive the properties of the polynomials $U_n(x,y)$ can be easily extended to explore the nature and the characteristic of the present Chebyshev polynomials $U_n^{(m)}(x,y)$. In this sense, since we have obtained the important link stated in the above result, we will use the structure and the properties of the Hermite polynomials of the type $H_n^{(m)}(x,y)$.

Proposition V.6

Let be ξ a real number such that $|\xi| < 1$; the generalized Chebyshev polynomials $U_n^{(m)}(x,y)$ admit the follow generating function:

$$\sum_{n=0}^{+\infty} \xi^n U_n^{(m)}(x,y) = \int_0^{+\infty} e^{-t(1-x\xi)} e^{y\xi^m t^{m-1}} dt. \qquad \text{(V.2.4)}$$

Proof

In (II.1.5) we heve stated the expression of the generating function of the polynomials $H_n^{(m)}(x,y)$:

$$e^{xt+yt^m} = \sum_{n=0}^{+\infty} \frac{t}{n!} H_n^{(m)}(x,y) \qquad \text{(V.2.5)}$$

From the integral representation of the Chebyshev polynomials $U_n^{(m)}(x,y)$, (see eq.(V.2.2)), after multiplying by ξ^n, with $|\xi| < 1$ and summing up over n, we obtain:

$$\sum_{n=0}^{+\infty} \xi^n U_n^{(m)}(x,y) = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} \int_0^{+\infty} e^{-t} t^n H_n^{(m)}\left(x, \frac{y}{t}\right) dt.$$

The above relation can be rearranged in the form:

$$\sum_{n=0}^{+\infty} \xi^n U_n^{(m)}(x,y) = \int_0^{+\infty} e^{-t} \sum_{n=0}^{+\infty} \frac{(\xi t)^n}{n!} H_n^{(m)}\left(x, \frac{y}{t}\right) dt$$

and then from (V.2.5), we have:

$$\sum_{n=0}^{+\infty} \xi^n U_n^{(m)}(x,y) = \int_0^{+\infty} e^{-t} e^{(xt)\xi + (yt^{m-1})\xi^m} dt. \tag{V.2.6}$$

The above identity is formally the statement (V.2.4).

It is important to note that the integral:

$$\int_0^{+\infty} e^{-t(1-x\xi)} e^{y\xi^m t^{m-1}} dt$$

diverges when $y > 0$. Otherwise for the values of m greather than 2 and $y < 0$, the integal representation of the polynomials $U_n^{(m)}(x,y)$ can be written in the form:

$$\sum_{n=0}^{+\infty} \xi^n U_n^{(m)}(x,y) = \frac{1}{1 + x\xi + y\xi^m}. \tag{V.2.7}$$

In Section II.1 we have stated the important recurrence relations for the Hermite polynomials $H_n^{(m)}(x,y)$, (see eq. (II.1.10)), which can be customed in the form:

$$\frac{\partial}{\partial x} H_n^{(m)}(x, \frac{y}{t}) = n H_{n-1}^{(m)}(x, \frac{y}{t}) \tag{V.2.8}$$

$$\left(x + m\frac{y}{t}\frac{\partial^{m-1}}{\partial x^{m-1}}\right) H_n^{(m)}(x, \frac{y}{t}) = H_{n+1}^{(m)}(x, \frac{y}{t}).$$

These relations can be used to generalize the recurrence relations stated in the previous section (see eqs.(V.1.10) and (V.1.19)) for the Chebyshev polynomials $U_n(x,y)$.

Proposition V.7

The generalized Chebyshev polynomials of the type $U_n^{(m)}(x,y)$ satisfy the following identities:

$$my\frac{\partial^{m-1}}{\partial x^{m-1}} U_{n-1}^{(m)}(x,y) = \left(n - x\frac{\partial}{\partial x}\right) U_n^{(m)}(x,y) \tag{V.2.9}$$

$$U_{n+1}^{(m)}(x,y) = xU_n^{(m)}(x,y) + (m-1)y\frac{\partial^{m-2}}{\partial x^{m-2}} U_{n-1}^{(m)}(x,y).$$

Proof

From the integral representation of the Chebyshev polynomials $U_n^{(m)}(x,y)$, stated in (V.2.2), by substituting the multiplicative recurrence relation related to the Hermite polynomials $H_n^{(m)}(x,\frac{y}{t})$, (eq. (V.2.8)), we have:

$$U_n^{(m)}(x,y) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^n \left(x + m\frac{y}{t}\frac{\partial^{m-1}}{\partial x^{m-1}}\right) H_{n-1}^{(m)}\left(x,\frac{y}{t}\right)dt. \qquad \text{(V.2.10)}$$

The r.h.s. of the above identy can be exploited to give:

$$\begin{aligned} U_n^{(m)}(x,y) &= \frac{x}{n!}\int_0^{+\infty} e^{-t}t^n H_{n-1}^{(m)}\left(x,\frac{y}{t}\right)dt + \\ &+\frac{my}{n!}\int_0^{+\infty} e^{-t}t^{n-1}\frac{\partial^{m-1}}{\partial x^{m-1}}H_{n-1}^{(m)}\left(x,\frac{y}{t}\right)dt \end{aligned}$$

and then:

$$\begin{aligned} &U_n^{(m)}(x,y) - \frac{x}{n(n!)}\int_0^{+\infty} e^{-t}t^n\frac{\partial}{\partial x}H_n^{(m)}\left(x,\frac{y}{t}\right)dt = \\ &= \frac{my}{n(n-1)!}\int_0^{+\infty} e^{-t}t^{n-1}\frac{\partial^{m-1}}{\partial x^{m-1}}H_{n-1}^{(m)}\left(x,\frac{y}{t}\right)dt. \end{aligned}$$

In the relation above, the first of the identities of the statement can be recognized.

To prove the second identity of this proposition we can use the induction over m, by noting that it is a formally extension of recurrence relation (V.1.19) related to the polynomials $U_n(x,y)$.

The relations obtained in the above result can be used to better clarify the recurrence of the polynomials $U_n^{(m)}(x,y)$. We note in fact that the second identity in (V.2.9) can be written as:

$$U_{n+1}^{(m)}(x,y) = xU_n^{(m)}(x,y) + (m-1)y\widehat{D}_x^{-1}\frac{\partial^{m-1}}{\partial x^{m-1}}U_{n-1}^{(m)}(x,y)$$

and by using the first one in (V.2.9), we get:

$$U_{n+1}^{(m)}(x,y) = xU_n^{(m)}(x,y) + \frac{m-1}{m}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)U_n^{(m)}(x,y).$$

The above identity and the first of (V.2.9), allow us to define the rising and lowering operators related to the generalized Chebyshev polynomials $U_n^{(m)}(x,y)$, by setting:

$$\widehat{E}_+ = x + \frac{m-1}{m}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right) \qquad \text{(V.2.11)}$$

$$\widehat{E}_- = \frac{1}{my}\widehat{D}_x^{-(m-1)}\left(n - x\frac{\partial}{\partial x}\right)$$

which, as we have noted before, act on the polynomials $U_n^{(m)}(x,y)$ as follows:

$$\widehat{E}_+ U_n^{(m)}(x,y) = U_{n+1}^{(m)}(x,y) \qquad \text{(V.2.12)}$$

$$\widehat{E}_- U_n^{(m)}(x,y) = U_{n-1}^{(m)}(x,y).$$

Proceeding as in the case of the generalized Chebyshev polynomials of the type $U_n(x,y)$, we can prove the following important result.

Theorem V.3

The polynomials $U_n^{(m)}(x,y)$ satisfy the following partial differential equation:

$$\left[my\frac{\partial^m}{\partial x^m} + \frac{x^2}{m}\frac{\partial^2}{\partial x^2} + \left(1 + n - \frac{2n-1}{m}\right)x\frac{\partial}{\partial x} - n\left(1 + \frac{n(m-1)}{m}\right)\right]U_n^{(m)}(x,y) = 0 \qquad \text{(V.2.13)}$$

Proof

By using the structure of the rising and lowering operators defined above, we can immediately write the relation:

$$\widehat{E}_-\widehat{E}_+ U_n^{(m)}(x,y) = U_n^{(m)}(x,y)$$

which in explicit forms, reads:

$$\left[\frac{1}{my}\widehat{D}_x^{-(m-1)}\left((n+1) - x\frac{\partial}{\partial x}\right)\right]\left[x + \frac{m-1}{m}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)\right]U_n^{(m)}(x,y) = U_n^{(m)}(x,y) \qquad \text{(V.2.14)}$$

and then:

$$\widehat{D}_x^{-(m-1)}\left((n+1) - x\frac{\partial}{\partial x}\right)\left[x + \frac{m-1}{m}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)\right]U_n^{(m)}(x,y) = myU_n^{(m)}(x,y). \qquad \text{(V.2.15)}$$

It is easy note that the following operational identities hold:

$$\begin{aligned} \frac{\partial^{m-1}}{\partial x^{m-1}}\widehat{D}_x^{-(m-1)} &= \widehat{1} \\ \widehat{D}_x^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} &= \widehat{1} \end{aligned}$$

and then, by deriving $m - times$ with respect to x in equation (V.2.15), we obtain:

$$\left((n+1) - x\frac{\partial}{\partial x}\right)\left[x + \frac{m-1}{m}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)\right]U_n^{(m)}(x,y) = my\frac{\partial^{m-1}}{\partial x^{m-1}}U_n^{(m)}(x,y). \tag{V.2.16}$$

By considering only the operators acting on the above relation, we can write:

$$\begin{aligned} my\frac{\partial^{m-1}}{\partial x^{m-1}} &= \left((n+1) - x\frac{\partial}{\partial x}\right)\left[x + \frac{m-1}{m}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right)\right] \\ &= (n+1)x + (n+1)\frac{m-1}{m}\widehat{D}_x^{-1}\left(n - x\frac{\partial}{\partial x}\right) - x\frac{\partial}{\partial x}x + \\ &\quad -x\frac{m-1}{m}\left(n - x\frac{\partial}{\partial x}\right) \end{aligned}$$

and by deriving again with respect to x, we get:

$$my\frac{\partial^m}{\partial x^m} = (n+1)\frac{\partial}{\partial x}x + (n+1)\frac{m-1}{m}\left(n - x\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}x\frac{\partial}{\partial x}x + \tag{V.2.17}$$
$$-\frac{\partial}{\partial x}x\frac{m-1}{m}\left(n - x\frac{\partial}{\partial x}\right).$$

By noting that the following identities hold:

$$\begin{aligned} \frac{\partial}{\partial x}x &= 1 + x\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x}x^2\frac{\partial}{\partial x} &= 2x\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial x^2} \end{aligned}$$

we can rearranged identity (V.2.17) in the form:

$$my\frac{\partial^m}{\partial x^m} = (n+1)\left(1 + x\frac{\partial}{\partial x}\right) + \frac{(n+1)(m-1)}{m}\left(n - x\frac{\partial}{\partial x}\right) +$$
$$-\frac{\partial}{\partial x}x\left(1 + x\frac{\partial}{\partial x}\right) + \frac{n(m-1)}{m}\frac{\partial}{\partial x}x + \frac{m-1}{m}\frac{\partial}{\partial x}x^2\frac{\partial}{\partial x}$$

then:

$$my\frac{\partial^m}{\partial x^m} = (n+1) + (n+1)x\frac{\partial}{\partial x} + \frac{n(n+1)(m-1)}{m} - \frac{(n+1)(m-1)}{m}x\frac{\partial}{\partial x} +$$
$$-\frac{\partial}{\partial x}x - \frac{\partial}{\partial x}x^2\frac{\partial}{\partial x} - \frac{n(m-1)}{m} - \frac{n(m-1)}{m}x\frac{\partial}{\partial x} + \frac{m-1}{m}\frac{\partial}{\partial x}x^2\frac{\partial}{\partial x}$$

and finally:

$$my\frac{\partial^m}{\partial x^m} = -\frac{x^2}{m}\frac{\partial^2}{\partial x^2} + \left(-n - 1 + \frac{2n-1}{m}\right)x\frac{\partial}{\partial x} + n\left(1 + \frac{n(m-1)}{m}\right). \tag{V.2.18}$$

By substituting the above relation in (V.2.16) we immediately obtain the thesis. In this chapter we have presented the generalized two-variable Chebyshev polynomials of first and second kind of type $U_n(x, y)$ and $T_n(x, y)$, and we have discussed, in particular, some interesting integral representations. In Section V.2 we have also studied a further generalization of two-variable Chebyshev polynomials by introducing the polynomials $U_n^{(m)}(x, y)$ and we have also deduced some interesting properties by using the structure and the operational relations satisfied by the Hermite polynomials of the form $H_n^{(m)}(x, y)$, introduced in Chapter II. It is evident that these families of Chebyshev polynomials represent a relevant generalization of the ordinary first and second kind Chebyshev polynomials $T_n(x)$ and $U_n(x)$ (see Chapter IV) and then we can recognize the generalizations presented in this chapter as Chebyshev-like polynomials. We will discuss some other relevant link between the Chebyshev-like polynomials with other families of special functions in the next chapter, with particular attention to their integral representations.

Chapter VI

Chebyshev-like polynomials

In the previous two chapters, dedicated to the theory of Chebyshev polynomials, we have presented some generalizations of this family of polynomials and, in particular, we have described their integral representations. The integral representations that we have deduced, first for the ordinary Chebyshev polynomials of first and second kind (see Section IV.3), are based on the operational relations satisfied by the generalized Hermite polynomials of different types. Moreover, the integral representation technique has been also used to introduce a generalization of Chebyshev polynomials as in the case that appears in Definition IV.3:

$$U_n(x, y; \alpha) = \frac{1}{n!} \int_0^{+\infty} e^{-\alpha t} He_n(2xt, -yt) dt,$$

and, by following the same procedure, we have defined the generalized Chebyshev polynomials of type $T_n(x, y; \alpha)$ and $W_n(x, y; \alpha)$ through the equations (IV.3.30) and (IV.3.31). Furthermore, by using the generalized Hermite polynomials of type $H_n^{(m)}(x, y)$ (see Section II.1), we have also introduced another generalization involving the Chebyshev polynomials:

$${}_mU_n(x, y; \alpha) = \frac{1}{n!} \int_0^{+\infty} e^{-\alpha t} t^n H_n^{(m)} \left(mx, -\frac{y}{t^{m-1}} \right) dt.$$

It is evident that the previous generalizations, obtained by using the generalized Hermite polynomials and the integral representation technique, have led to families of Chebyshev polynomials directly related the ordinary case, i.e.

the polynomials of type $U_n(x)$ and $T_n(x)$. In Chapter V, instead, we have introduced the generalized two-variable Chebyshev polynomials of type:

$$U_n(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(n-k)!x^{n-2k}y^k}{k!(n-2k)!}$$

and, of course, we have derived the related integral representation. From this, we have also defined the generalized two-variable Chebyshev polynomials of type $T_n(x,y)$ (see Definition V.2). It has been outlined in the previous chapter that these families of polynomials could be called as Chebyshev-like polynomials, since they present a substantial generalization of the ordinary Chebyshev polynomials introduced in the fourth chapter; a fortiori, the polynomials defined through the relation:

$$U_n^{(m)}(x,y) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(n-k)!x^{n-mk}y^k}{k!(n-mk)!}$$

and the related properties, further show a different nature with respect to the ordinary Chebyshev polynomials, while on the contrary prove their similarity to the generalized Hermite polynomials $H_n^{(m)}(x,y)$. In this last chapter, we will show further integral representations for the Chebyshev-like polynomials by using, again, the properties of some special Hermite polynomials, but also with the help of a generalized class of exponential truncated polynomials which will be briefly described.

VI.1 Hermite polynomials and parabolic cylinder functions

In Chapter I we have presented the generalized two-variable Hermite polynomials of the type $He_m(x,y)$:

$$He_m(x,y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!} y^n x^{m-2n} \qquad \text{(VI.1.1)}$$

and we have derived the explicit form of the ordinary Hermite polynomials $He_m(x)$:

$$He_m(x) = m! \sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^r x^{n-2r}}{r!(n-2r)!2^r}. \tag{VI.1.2}$$

We have also seen that the link between these two classes of Hemite polynomials is expressed by the following relation:

$$He_m\left(x, -\frac{1}{2}\right) = He_m(x). \tag{VI.1.3}$$

We want now to introduce a further generalization of the ordinary Hermite polynomials which is associated to the parabolic cylinder functions. We remind that the parabolic cylinder functions are often denoted by the symbol $D_\nu(x)$ and is defined for all real values of ν and x, but the related properties are different in the case that the order is positive or negative. There are the following integral expression to define the parabolic cylinder functions for the different values of the index ν. For all value of x, we have:

$$D_\nu(x) = \sqrt{\frac{2}{\pi}} \exp\left(\frac{x^2}{4}\right) \int_0^{+\infty} t^\nu \exp\left(\frac{-t^2}{2}\right) \cos\left(xt - \frac{\nu\pi}{2}\right) dt, \quad \nu > 1 \tag{VI.1.4}$$

$$D_\nu(x) = \frac{1}{\Gamma(-\nu)} \exp\left(\frac{-x^2}{4}\right) \int_0^{+\infty} t^{-\nu-1} \exp\left(\frac{-t^2}{2} - xt\right) dt, \quad \nu < 0 \tag{VI.1.5}$$

while for positive values of x:

$$D_\nu(x) = \frac{1}{\Gamma(-\nu)} \exp\left(\frac{-x^2}{4}\right) \int_0^{+\infty} \frac{\exp(-t)}{\sqrt{x^2+2t}\left(\sqrt{x^2+2t}-x\right)^{\nu+1}} dt, \quad \nu < 0. \tag{VI.1.6}$$

The parabolic cylinder functions are also solutions of the differential equation:

$$\frac{d^2y}{dx^2} + \left(ax^2 + bx + c\right) y = 0 \tag{VI.1.7}$$

that can be represented in the two real standard forms:

$$\frac{d^2y}{dx^2} - \left(\frac{1}{4}x^2 + a\right) y = 0 \tag{VI.1.8}$$

$$\frac{d^2y}{dx^2} + \left(\frac{1}{4}x^2 - a\right) y = 0. \tag{VI.1.9}$$

It could be useful to note that, if the index is an integer, we can define the related generating function, that is:

$$\exp\left(xt - \frac{x^2}{4} - \frac{t^2}{2}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} D_n(x), \tag{VI.1.10}$$

which is similar to the generating functions shown for the various types of Hermite polynomials, as we have seen in Chapter I. We remind, in fact, that the generalized two-variable Hermite polynomials have the following generating functions:

$$\exp\left(xt + yt^2\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x,y), \tag{VI.1.11}$$

$$\exp\left(2xt - yt^2\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x,y), \tag{VI.1.12}$$

$$\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) = \sum_{m=0}^{+\infty} \frac{t^m}{m!} He_m^{'}(x,y), \tag{VI.1.13}$$

and the ordinary Hermite polynomials of one-variable, that we have presented in the third chapter, have the form:

$$He_m(x) = m! \sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^r x^{n-2r}}{r!(n-2r)!2^r}, \tag{VI.1.14}$$

with the generating function defined by:

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x). \tag{VI.1.15}$$

Given the similarity between the parabolic cylinder functions and the Hermite polynomials, it is interesting to explore the possibility to derive some useful relations linking the parabolic cylinder functions and a special class of Hermite polynomials. We start to present a *new* class of Hermite polynomials, introduced by A. Wunsche, by putting directly their explicit form:

$$He_n^{\nu}(x) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r}{(n-2r)!r!} a_{r,n}^{\nu} x^{n-2r} \tag{VI.1.16}$$

where:

$$a_{r,n}^{\nu} = \sum_{j=0}^{xr} \binom{r}{j} \frac{(n-j)!}{2^{r-j} n!} \frac{(\nu-1+j)!}{(\nu-1)!}. \tag{VI.1.17}$$

These Hermite polynomials could be reduced to the ordinary Hermite polynomials $He_n(x)$ for $\nu = 0$ and it can be specified through the operational rule:

$$He_n^\nu(x) = \left({}_1F_1\left(\nu; -n; -\frac{\partial^2}{\partial x^2}\right)\right) He_n(x) \qquad \text{(VI.1.18)}$$

with ${}_1F_1(\alpha; \beta; \gamma)$ being the confluent hypergeometric function [1].
We have shown that the generalized two-variable Hermite polynomials of type $He_n(x,y)$ solve the differential equation (I.3.35) (see Proposition I.11), and then can also be represented by the following operational relation:

$$He_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^r x^{n-2r}}{(n-2r)!r!} = (-i)^n y^{\frac{n}{2}} He_n\left(\frac{ix}{2\sqrt{y}}\right). \qquad \text{(VI.1.19)}$$

We will see that also the generalized Hermite polynomials of the type $He_n^\nu(x)$ solve a slightly different differential equation. We will use the method of integral transform, by applying the operational techniques shown in the previous chapters related to the Hermite polynomials and to the Chebyshev polynomials.
By noting that, for $k \leq n$:

$$(n-k)! = \int_0^{+\infty} e^{-t} t^{n-k} dt \qquad \text{(VI.1.20)}$$

we can recast equation (VI.1.16) in the form:

$$He_n^\nu(x) = \frac{1}{(\nu-1)!} \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r x^{n-2r}}{(n-2r)!r!} \int_0^{+\infty} dt \int_0^{+\infty} t^n u^{\nu-1} e^{-(t+u)} \left(\frac{1}{2} + \frac{u}{t}\right)^r du \qquad \text{(VI.1.21)}$$

and by using the equation (VI.1.19), we obtain:

$$He_n^\nu(x) = \frac{1}{(\nu-1)!} \frac{1}{n!} \int_0^{+\infty} dt \int_0^{+\infty} t^n u^{\nu-1} e^{-(t+u)} He_n\left(x, -\left(\frac{1}{2} + \frac{u}{t}\right)\right) du \qquad \text{(VI.1.22)}$$

Theorem VI.1

The generalized Hermite polynomials of the form $He_n^\nu(x)$ solve the following differential equation:

$$\left(\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x} + n\right) He_n^\nu(x) = -2\nu He_{n+1}^{\nu+1}(x). \qquad \text{(VI.1.23)}$$

Proof

We note that relation (I.3.35), related to the Hermite polynomials $He_n(x, \frac{1}{2}+\frac{u}{t})$, once introduced in equation (VI.1.22) gives:

$$H\frac{1}{(\nu-1)!}\frac{1}{n!}\int_0^{+\infty} dt \int_0^{+\infty} t^n u^{\nu-1} e^{-(t+u)} \cdot \tag{VI.1.24}$$
$$\cdot \left[-2\left(\frac{1}{2}+\frac{u}{t}\right)\left(\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x} + n\right)\right] \cdot$$
$$\cdot He_n\left(x, -\left(\frac{1}{2}+\frac{u}{t}\right)\right) du = 0$$

and then, from the relation shown in equation (VI.1.18) follows the thesis. Relation (VI.1.22) it is also useful to state the generating function for the generalized Hermite polynomials of type $He_n^\nu(x)$. In fact, by remembering that the generalized two-variable Hermite polynomials have the following generating function:

$$\exp\left(xt + yt^2\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} He_n(x, y) \tag{VI.1.25}$$

and by multiplying both sides of (VI.1.22) by $\xi^{\nu-1}$ and u^n and then summing up on the (ν, n) indexes, we find:

$$\sum_{\nu=1}^{+\infty}\sum_{n=0}^{+\infty} \xi^{\nu-1} u^n He_n^\nu(x) = \int_0^{+\infty} \frac{e^{-(1-xu)t-\frac{1}{2}t^2u^2}}{1-\xi+tu^2} dt, \quad |\xi| < 1. \tag{VI.1.26}$$

It also interesting to explore the properties related to the coefficients of the generalized Hermite polynomials of type $He_n^\nu(x)$. From equation (VI.1.17), it is easy to write:

$$a_{r,n}^\nu = \frac{(-1)^r}{n!(\nu-1)!} \int_0^{+\infty} dt \int_0^{+\infty} t^n u^{\nu-1} e^{-(t+u)} \left(\frac{1}{2}+\frac{u}{t}\right)^r du \tag{VI.1.27}$$

and from the above expression it is possible to consider the following generalization of the coefficients $a_{r,n}^\nu$:

$$a_{r,n}^\nu(\alpha,\beta) = \frac{(-1)^r}{n!(\nu-1)!} \int_0^{+\infty} dt \int_0^{+\infty} t^n u^{\nu-1} e^{-(t+u)} e^{-\frac{\beta}{t}} e^{-\alpha\left(\frac{1}{2}+\frac{u}{t}\right)} \left(\frac{1}{2}+\frac{u}{t}\right)^r du. \tag{VI.1.28}$$

It is immediate to derive the following recurrence relations related to the above coefficients:

$$n\frac{\partial}{\partial\beta}a^{\nu}_{r,n}(\alpha,\beta) = -a^{\nu}_{r,n-1}(\alpha,\beta) \qquad \text{(VI.1.29)}$$

$$\frac{\partial}{\partial\alpha}a^{\nu}_{r,n}(\alpha,\beta) = -a^{\nu}_{r+1,n}(\alpha,\beta)$$

VI.2 Truncated polynomials

In the evaluation of integrals involving products of special function, the truncated polynomials play a role of crucial importance. In this section we will show some relevant properties of these polynomials to apply in the treatment of the integral representations of multi-variable Chebyshev polynomials.

Definition VI.1

Let x be a real variable, we will say *truncated exponential polynomials* the first $(n+1)$ terms of the Mac Laurin series for e^x:

$$e_n(x) = \sum_{r=0}^{n}\frac{x^r}{r!}. \qquad \text{(VI.2.1)}$$

From the above definition, it is immediate to obtain the following integral representation for the truncated polynomials:

$$e_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-\xi}(x+\xi)^n d\xi \qquad \text{(VI.2.2)}$$

which is a consequence of the relation:

$$n! = \int_0^{+\infty} e^{-\xi}\xi^n d\xi \qquad \text{(VI.2.3)}$$

shown in Chapter IV (see equation (IV.3.3)). By proceeding in the same way as what we have presented for the Hermite polynomials and Chebyshev polynomials in the previous chapters, we can immediately derive the following generating function:

$$\frac{e^{tx}}{1-t} = \sum_{n=0}^{+\infty} t^n e_n(x). \qquad \text{(VI.2.4)}$$

Proposition VI.1

The truncated exponential polynomials solve the following differential equation:

$$\left[x\frac{d^2}{dx^2}-(n+x)\frac{d}{dx}+n\right]e_n(x)=0. \tag{VI.2.5}$$

Proof

By taking the derivative with respect to t and x of both sides of equation (VI.2.4), we obtain the recurrence relations:

$$\begin{aligned} e_{n+1}(x) &= \left[1+\frac{x}{n+1}\left(1-\frac{d}{dx}\right)\right]e_n(x) && \text{(VI.2.6)} \\ e_{n-1}(x) &= \frac{d}{dx}e_n(x) && \text{(VI.2.7)} \end{aligned}$$

which help us to define the shifting operators:

$$\begin{aligned} \hat{E}_+ &= 1+\frac{x}{n+1}\left(1-\frac{d}{dx}\right) && \text{(VI.2.8)} \\ \hat{E}_- &= \frac{d}{dx}. \end{aligned}$$

We note that, as for the operators related to the vectorial Hermite polynomials $H_{m,n}(x,y)$ (see Section II.2), they depend on a discrete parameter. By using the relation:

$$\hat{E}_+\left[\hat{E}_-e_n(x)\right]=e_n(x) \tag{VI.2.9}$$

we immediately obtain the thesis.

By noting that the Appell polynomials [9] are generated by the following generating function:

$$A(t)e^{xt}=\sum_{n=0}^{+\infty}\frac{t^n}{n!}p_n(x) \tag{VI.2.10}$$

it is evident that the truncated exponential polynomials can be framed within the context of Appell polynomials.

From the integral representation of polynomials $e_n(x)$ stated in equation (VI.2.2), we can introduce the following generalization:

$$e_n^{(\alpha)}(x)=\frac{1}{n!}\int_0^{+\infty}e^{-\xi}\xi^{\alpha}(x+\xi)^n d\xi \tag{VI.2.11}$$

where α is a real number. From the above equation we can formally define these generalized truncated polynomials.

Definition VI.2

Let α be a real number, we will call associated truncated exponential polynomials $e_n^{(\alpha)}(x)$ the polynomials defined by the following generating function:

$$\frac{e^{tx}\Gamma(\alpha+1)}{(1-t)^{\alpha+1}} = \sum_{n=0}^{+\infty} t^n e_n^{(\alpha)}(x). \tag{VI.2.12}$$

From the above definition we immediately get their explicit forms:

$$e_n^{(\alpha)}(x) = \sum_{s=0}^{n} \frac{x^s \Gamma(n-s+\alpha+1)}{s!(n-s)!}. \tag{VI.2.13}$$

It is interesting to note that the above polynomials allow us to obtain a particular formula of addition, that is:

$$e_n^{(\alpha+\beta+2)}(x+y) = \sum_{s=0}^{n} e_{n-s}^{(\alpha)}(x) e_s^{(\beta)}(y). \tag{VI.2.14}$$

Definition VI.3

Let be the generalized two-variable Hermite polynomials of type $He_n(x,y)$, we will call generalized truncated exponential polynomials, the polynomials defined by the following integral representation:

$$_{[2]}e_n(x) = \frac{1}{n!} \int_0^{+\infty} e^{-\xi} He_n(x,\xi) d\xi. \tag{VI.2.15}$$

By remembering the explicit form of the Hermite polynomials $He_n(x,y)$ (see eq. (VI.1.1)), we easily obtain:

$$_{[2]}e_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r}}{(n-2r)!} \tag{VI.2.16}$$

and, from the generating function of the generalized two-variable Hermite polynomials shown in equation (VI.1.25), we get:

$$\frac{e^{tx}}{1-t^2} = \sum_{n=0}^{+\infty} t^n \, _{[2]}e_n(x). \tag{VI.2.17}$$

The properties verified by the Hermite polynomials of type $He_n(x,y)$ can be used to derive analogous relations for these type of truncated exponential polynomials.

Proposition VI.2

The generalized truncated exponential polynomials ${}_{[2]}e_n(x)$ satisfied the following differential equation:

$$\left[x\frac{d^3}{dx^3} - n\frac{d^2}{dx^2} - x\frac{d}{dx} + n\right] {}_{[2]}e_n(x) = 0. \qquad \text{(VI.2.18)}$$

Proof

By using the recurrence relations related to the Hermite polynomials $He_n(x, y)$ (see equations (I.3.36) and (I.3.37)), we have:

$${}_{[2]}e_{n+1}(x) = \left[\frac{d}{dx} + \frac{x}{n+1}\left(1 - \frac{d^2}{dx^2}\right)\right] {}_{[2]}e_n(x) \qquad \text{(VI.2.19)}$$

$${}_{[2]}e_{n-1}(x) = \frac{d}{dx}{}_{[2]}e_n(x) \qquad \text{(VI.2.20)}$$

which once combined, immediately give the thesis.

In the second chapter we have introduced the generalized Hermite polynomials of type $H_n^{(m)}(x, y)$ and we have derived some interesting properties. We can now use this class of Hermite polynomials to define a further class of truncated exponential polynomials.

Definition VI.4

We will call $m-$ *order* truncated exponential polynomials ${}_{[m]}e_n(x)$, the polynomials defined by the following relation:

$${}_{[m]}e_n(x) = \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{x^{n-mr}}{(n-mr)!}. \qquad \text{(VI.2.21)}$$

Without presenting the proof, being a direct consequence of the properties of the Hermite polynomials $H_n^{(m)}(x, y)$ and the technique previously used for the generalized truncated polynomials of type ${}_{[2]}e_n(x)$ (see Definition VI.2), we list the relevant relations they satisfied:

$${}_{[m]}e_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-\xi} H_n^{(m)}(x, \xi)d\xi, \qquad \text{(VI.2.22)}$$

$$\frac{e^{tx}}{1-t^m} = \sum_{n=0}^{+\infty} t^n {}_{[m]}e_n(x), \qquad \text{(VI.2.23)}$$

$$\left[x\frac{d^{m+1}}{dx^{m+1}} - n\frac{d^m}{dx^m} - x\frac{d}{dx} + n\right]{}_{[m]}e_n(x) = 0. \tag{VI.2.24}$$

It is possible, in analogy to the case $m = 1$, to introduce the associated truncated polynomials of order m, by setting:

$$_{[m]}e_n^{(\alpha)}(x) = \sum_{s=0}^{\left[\frac{n}{m}\right]} \frac{x^{(n-ms)}\Gamma(s+\alpha+1)}{s!(n-ms)!} \tag{VI.2.25}$$

whose properties can be easily derived by following the same procedure used above.

Definition VI.5

Let be the generalized two-variable Hermite polynomials of type $H_n^{(m)}(x,y)$, we will call generalized two-variable truncated exponential polynomials, the polynomials expressed by the following integral representation:

$$_{[m]}e_n(x,y) = \frac{1}{n!}\int_0^{+\infty} e^{-\xi} H_n^{(m)}(x+2y\xi, -\xi)d\xi. \tag{VI.2.26}$$

By using the relations verified by the Hermite polynomials, and in particular their generating function and the identity shown in Proposition II.1, we obtain:

$$\frac{e^{tx}}{1-2yt+t^2} = \sum_{n=0}^{+\infty} t^n {}_{[2]}e_n(x,y). \tag{VI.2.27}$$

In the next section we will use the truncated exponential polynomials to derive some interesting integral relations for the generalized Chebyshev polynomials.

VI.3 Further integral representations

In Chapter IV we have presented some integral representations for the Chebyshev polynomials of first and second kind (see equations (IV.3.2) and (IV.3.5)) and also for their some generalized forms as in the case of the Chebyshev polynomials $U_n(x,y,\alpha)$ and ${}_mU_n(x,y,\alpha)$ (see equations (IV.3.29) and (IV.4.1)). Moreover, in the fifth chapter we have introduced the two-variable, second kind Chebyshev polynomials and we have also derived their respectively integral representations; we have also defined the two-variable first kind

Chebyshev polynomials directly through the integral representation. In this section we describe how the generalized Hermite polynomials of type $He_n^{\nu}(x)$ could be useful to obtain different forms of integral representations for the various Chebyshev-like polynomials discussed in the previous chapters. We start to note that the generalized two-variable Hermite polynomials satisfy the following operational relation:

$$\exp\left(z\frac{\partial^2}{\partial x^2}\right)He_n(x,y) = He_n(x,y+z) \tag{VI.3.1}$$

which is a consequence of the action of the translation operator presented in Introduction. By substituting the above equation in (VI.1.22), we get immediately:

$$He_n^{\nu}(x) = \frac{e^{\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)}}{(\nu-1)!}\int_0^{+\infty} e^{-u}u^{\nu-1}U_n\left(\frac{x}{2},u\right)du \tag{VI.3.2}$$

where the Chebyshev polynomials $U_n(x,y)$ have been introduced in Definition V.1. The equation (VI.3.2) can be written in the form:

$$\exp\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)He_n^{\nu}(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r(n-r)!(\nu-1+r)!}{n!r!(\nu-1)!}\frac{n!}{(n-2r)!}x^{n-2r} \tag{VI.3.3}$$

which gives this different interesting form:

$$\exp\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)He_n^{\nu}(x) = {}_1F_1\left(\nu;-n;-\frac{\partial^2}{\partial x^2}\right)x^n. \tag{VI.3.4}$$

By assuming $|t|<1$ in equation (VI.3.2), we can also write this also relevant integral representation for the two-variable second kind Chebyshev polynomials:

$$\sum_{\nu=1}^{+\infty} t^{\nu-1}He_n^{\nu}(x) = \exp\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\frac{1}{1-t}\int_0^{+\infty} e^{-u}U_n\left(\frac{x}{2},\frac{u}{1-t}\right)du. \tag{VI.3.5}$$

By noting that the r.h.s. of the above equation reads:

$$\int_0^{+\infty} e^{-u}U_n\left(\frac{x}{2},\frac{u}{1-t}\right)du = \sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r(n-r)!}{(n-2r)!(1-t)^r}x^{n-2r} \tag{VI.3.6}$$

and since the two-variable Hermite polynomials $He_n(x,y)$ verified the relation stated in equation (I.1.16), we have:

$$\sum_{\nu=1}^{+\infty} t^{\nu-1}He_n^{\nu}(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r(n-r)!He_{n-2r}\left(x,-\frac{1}{2}\right)}{(n-2r)!(1-t)^{r+1}}. \tag{VI.3.7}$$

In the previous section we have presented the truncated exponential polynomials and in particular we have defined the generalized two-variable truncated polynomials of order 2, that is ${}_{[2]}e_n(x,y)$ (see Definition VI.4). It is easy to note that the above truncated polynomials reduce to the second kind Chebyshev polynomials, by setting $x=0$:

$$U_n(y) = {}_{[2]}e_n(0,y). \tag{VI.3.8}$$

The r.h.s. of equation (VI.3.8) can be recognized as belonging to the family of associated truncated exponential polynomials and it can also be seen as a Chebyshev-like polynomials by the following integral relation:

$${}_{[2]}U_n(x,y) = \int_0^{+\infty} e^{-t} t^n {}_{[2]}e_n\left(x, \frac{y}{t}\right) dt. \tag{VI.3.9}$$

Finally we can derive the generating function of this class of Chebyshev-like polynomials:

$$\sum_{n=0}^{+\infty} \xi^n {}_{[2]}U_n(x,y) = \int_0^{+\infty} e^{-t} \frac{e^{x\xi t}}{1 - y\xi^2 t} dt. \tag{VI.3.10}$$

It is evident that the concepts and the related properties of the exponential truncated polynomials are a powerful tool to derive some interesting properties for the generalizations of Chebyshev polynomials, i.e. Chebyshev-like polynomials and, even before, for the generalized Hermite polynomials. This suggest that further progress can be made with respect to the integral representations which involve Chebyshev and Hermite polynomials of generalized type.

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