SCHWARTZMAN CYCLES AND ERGODIC SOLENOIDS

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ABSTRACT. We extend Schwartzman theory beyond dimension 1 and provide a unified treatment of Ruelle-Sullivan and Schwartzman theories via Birkhoff's ergodic theorem for the class of immersions of solenoids with a trapping region.

1. INTRODUCTION

This is the second paper of a series of articles [1, 2, 3, 4] in which we aim to give a geometric realization of *real* homology classes in smooth manifolds. This paper is devoted to the definition of Schwartzman homology classes and its relationship with the generalized currents associated to solenoids defined in [1].

Let M be a smooth manifold. A closed oriented submanifold $N \subset M$ of dimension $k \geq 0$ determines a homology class in $H_k(M, \mathbb{Z})$. This homology class in $H_k(M, \mathbb{R})$, as dual of De Rham cohomology, is explicitly given by integration of the restriction to N of differential k-forms on M. Unfortunately, because of topological reasons dating back to Thom [7], not all integer homology classes in $H_k(M, \mathbb{Z})$ can be realized in such a way. Geometrically, we can realize any class in $H_k(M, \mathbb{Z})$ by topological k-chains. The real homology $H_k(M, \mathbb{R})$ classes are only realized by formal combinations with real coefficients of k-cells. This is not satisfactory for various reasons. In particular, for diverse purposes it is important to have an explicit realization, as geometric as possible, of real homology classes.

The first contribution in this direction came in 1957 from the work of S. Schwartzman [6]. Schwartzman showed how, by a limiting procedure, one-dimensional curves embedded in M can define a real homology class in $H_1(M, \mathbb{R})$. More precisely, he proved that this happens for almost all curves solutions to a differential equation admitting an invariant ergodic probability measure. Schwartzman's idea consists on integrating 1-forms over large pieces of the parametrized curve and normalizing this integral by the length of the parametrization. Under suitable conditions, the limit exists and defines an element of the dual of $H^1(M, \mathbb{R})$, i.e. an element of $H_1(M, \mathbb{R})$. This procedure is equivalent to the more geometric one of closing large pieces of the curve by relatively short closing paths. The closed curve obtained defines an integer homology class. The normalization by the length of the parameter range provides a class in $H_1(M, \mathbb{R})$. Under suitable hypothesis, there exists a unique limit in real homology when the pieces exhaust the parametrized curve, and this limit is independent of the closing procedure. In sections 4 and 5, we shall study this circle of ideas in great generality. In section 4 we shall define Schwartzman cycles for parametrized and unparametrized curves in M, and study their properties. In section 5, we explore an alternative route to define real homology classes associated to curves in M by using the universal covering $\pi : \tilde{M} \to M$.

It is natural to ask whether it is possible to realize every real homology class using Schwartz-man limits. By the result of [3], we can realize any real homology class by the generalized current associated to an immersed oriented uniquely ergodic solenoid. A *solenoid* (see [1]) is an abstract laminated space endowed with a transversal structure. For these oriented solenoids we can consider k-forms that we can integrate provided that we are given a transversal measure invariant by the holonomy group. An immersion of a solenoid S into M is a regular map $f : S \to M$ that is an immersion in each leaf. If the solenoid S is endowed with a transversal measure $\mu = (\mu_T)$, then any smooth k-form in M can be pulled back to S by f and integrated. The resulting numerical value only depends on the cohomology class of the k-form. Therefore we have defined a closed current that we denote by (f, S_μ) and that call a generalized current [1]. It defines a homology class $[f, S_\mu] \in H_k(M, \mathbb{R})$. This is reviewed in section 2.

In section 6, we study the relation between the generalized current defined by an immersed oriented measured 1-solenoid S_μ and the Schwartzman measure defined by any one of its leaves. The relationship is best expressed for ergodic and uniquely ergodic solenoids. In the first case, almost all μ_T-leaves define Schwartzman classes which represent $[f, S_\mu]$. In the second case, the property holds for all leaves.

Section 7 is devoted to the generalization of the Schwartzman theory to higher dimensions. For a complete k-dimensional immersed submanifold $N \subset M$ of a Riemannian manifold, we define a Schwartzman class by taking large balls, closing them with small caps, normalizing the homology class thus obtained and finally taking the limit. This process is only possible when such capping exist. If S is a k-solenoid immersed in M, one would naturally expect that there is some relation between the generalized currents and the Schwartzman current (if defined) of the leaves. The main result is that there is such relation for the class of minimal, ergodic solenoids with a trapping region (see definition 7.9). For such solenoids, the holonomy group is generated by a single map. Then the bridge between generalized currents and Schwartzman currents of the leaves is provided by Birkhoff's ergodic theorem. We prove the following:

Theorem 1.1. *Let S_μ be an oriented and minimal solenoid endowed with an ergodic transversal measure μ, and possessing a trapping region W. Let $f : S_\mu \to M$ be an immersion of S_μ into M such that $f(W)$ is contained in a ball. Then for μ_T-almost all leaves $l \subset S_\mu$, the Schwartzman homology class of $f(l) \subset M$ is well defined and coincides with the homology class $[f, S_\mu]$.*

We are particularly interested in uniquely ergodic solenoids, with only one ergodic transversal measure. As is well known, in this situation we have uniform convergence of Birkhoff's sums, which implies the stronger result:

Theorem 1.2. *Let S_μ be a minimal, oriented and uniquely ergodic solenoid which has a trapping region W. Let $f : S_\mu \to M$ be an immersion of S_μ into M such that $f(W)$ is contained in a ball. Then for all leaves $l \subset S_\mu$, the Schwartzman homology class of $f(l) \subset M$ is well defined and coincides with the homology class $[f, S_\mu]$.*

2. Solenoids and generalized currents

Let us review the main concepts introduced in [1], and that we shall use later in this paper.

Definition 2.1. *A k-solenoid, where $k \geq 0$, of class $C^{r,s}$, is a compact Hausdorff space endowed with an atlas of flow-boxes $\mathcal{A} = \{(U_i, \varphi_i)\}$,*

$$\varphi_i : U_i \to D^k \times K(U_i),$$

where D^k is the k-dimensional open ball, and $K(U_i) \subset \mathbb{R}^l$ is the transversal set of the flow-box. The changes of charts $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are of the form

(1)
$$\varphi_{ij}(x, y) = (X(x, y), Y(y)),$$

where $X(x, y)$ is of class $C^{r,s}$ and $Y(y)$ is of class C^s.

Let S be a k-solenoid, and $U \cong D^k \times K(U)$ be a flow-box for S. The sets $L_y = D^k \times \{y\}$ are called the (local) leaves of the flow-box. A leaf $l \subset S$ of the solenoid is a connected k-dimensional manifold whose intersection with any flow-box is a collection of local leaves. The solenoid is oriented if the leaves are oriented (in a transversally continuous way).

A transversal for S is a subset T which is a finite union of transversals of flow-boxes. Given two local transversals T_1 and T_2 and a path contained in a leaf from a point of T_1 to a point of T_2, there is a well-defined holonomy map $h : T_1 \to T_2$. The holonomy maps form a pseudo-group.

A k-solenoid S is minimal if it does not contain a proper sub-solenoid. By [1, section 2], minimal solenoids exist. If S is minimal, then any transversal is a global transversal, i.e., it intersects all leaves. In the special case of an oriented minimal 1-solenoid, the holonomy return map associated to a local transversal,

$$R_T : T \to T$$

is known as the Poincaré return map (see [1, Section 4]).

Definition 2.2. *Let S be a k-solenoid. A transversal measure $\mu = (\mu_T)$ for S associates to any local transversal T a locally finite measure μ_T supported on T, which are invariant by the holonomy pseudogroup, i.e. if $h : T_1 \to T_2$ is a holonomy map, then $h_* \mu_{T_1} = \mu_{T_2}$.*

We denote by S_μ a k-solenoid S endowed with a transversal measure $\mu = (\mu_T)$. We refer to S_μ as a measured solenoid. Observe that for any transversal measure $\mu = (\mu_T)$ the scalar multiple $c\,\mu = (c\,\mu_T)$, where $c > 0$, is also a transversal measure. Notice that there is no natural scalar normalization of transversal measures.

Definition 2.3. (Transverse ergodicity) *A transversal measure $\mu = (\mu_T)$ on a solenoid S is ergodic if for any Borel set $A \subset T$ invariant by the pseudo-group of holonomy maps on T, we have*

$$\mu_T(A) = 0 \quad \text{or} \quad \mu_T(A) = \mu_T(T).$$

We say that S_μ is an ergodic solenoid.

Definition 2.4. *Let S be a k-solenoid. The solenoid S is uniquely ergodic if it has a unique (up to scalars) transversal measure μ and its support is the whole of S.*

Now let M be a smooth manifold of dimension n. An immersion of a k-solenoid S into M, with $k < n$, is a smooth map $f : S \to M$ such that the differential restricted to the tangent spaces of leaves has rank k at every point of S. The solenoid $f : S \to M$ is transversally immersed if for any flow-box $U \subset S$ and chart $V \subset M$, the map $f : U = D^k \times K(U) \to V \subset \mathbb{R}^n$ is an embedding, and the images of the leaves intersect transversally in M. If moreover f is injective, then we say that the solenoid is embedded.

Note that under a transversal immersion, resp. an embedding, $f : S \to M$, the images of the leaves are immersed, resp. injectively immersed, submanifolds.

Let $\mathcal{C}_k(M)$ denote the space of k-dimensional currents on M.

Definition 2.5. *Let S_μ be an oriented measured k-solenoid. An immersion $f : S \to M$ defines a generalized Ruelle-Sullivan current $(f, S_\mu) \in \mathcal{C}_k(M)$ as follows. Let $S = \bigcup_i S_i$ be a measurable partition such that each S_i is contained in a flow-box U_i. For $\omega \in \Omega^k(M)$, we define*

$$\langle (f, S_\mu), \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{L_y \cap S_i} f^* \omega \right) \, d\mu_{K(U_i)}(y) \,,$$

where L_y denotes the horizontal disk of the flow-box.

In [1] it is proved that (f, S_μ) is a closed current. Therefore, it defines a real homology class

$$[f, S_\mu] \in H_k(M, \mathbb{R}) \,.$$

In their original article [5], Ruelle and Sullivan defined this notion for the restricted class of solenoids embedded in M.

3. SCHWARTZMAN MEASURES

Let S be a Riemannian k-solenoid, that is, a solenoid endowed with a Riemmanian metric on each leaf. In some situations, we may define transversal measures associated to S by considering large chunks of a single leaf $l \subset S$. These will be called Schwartzman measures. We start by recalling some notions from [1, Section 6].

Definition 3.1. (daval measures) *Let μ be a measure supported on S. The measure μ is a daval measure if it desintegrates as volume along leaves of S, i.e. for any flow-box (U, φ) with local transversal $T = \varphi^{-1}(\{0\} \times K(U))$, we have a measure $\mu_{U,T}$ supported on T such that for any Borel set $A \subset U$*

$$(2) \qquad\qquad \mu(A) = \int_T \mathrm{Vol}_k(A_y) \, d\mu_{U,T}(y) \,,$$

where $A_y = A \cap \varphi^{-1}(D^k \times \{y\}) \subset U$.

We denote by $\mathcal{M}_{\mathcal{L}}(S)$ the space of probability daval measures, by $\mathcal{M}_{\mathcal{T}}(S)$ the space of (non-zero) transversal measures on S, and by $\overline{\mathcal{M}}_{\mathcal{T}}(S)$ the quotient of $\mathcal{M}_{\mathcal{T}}(S)$ by positive scalars. The following result is Theorem 6.8 in [1].

Theorem 3.2. (Tranverse measures of the Riemannian solenoid) *There is a one-to-one correspondence between transversal measures (μ_T) and finite daval measures μ. Furthermore, there is an isomorphism*

$$\overline{\mathcal{M}}_{\mathcal{T}}(S) \cong \mathcal{M}_{\mathcal{L}}(S) \,.$$

The correspondence follows from equation (2). If S is a uniquely ergodic Riemannian solenoid, then the above result allows to normalize the transversal measure in a unique way, by imposing that the corresponding daval measure has total mass 1.

Now we introduce a subclass of solenoids for which daval measures do exist.

Definition 3.3. (Controlled growth solenoids) *Let S be a Riemannian solenoid. Fix a leaf $l \subset S$ and an exhaustion (C_n) by subsets of l. For a flow-box (U, φ) write*

$$C_n \cap U = A_n \cup B_n \,,$$

where A_n is composed by all full disks $L_y = \varphi^{-1}(D^k \times \{y\})$ contained in C_n, and B_n contains those connected components B of $C_n \cap U$ such that $B \neq L_y \cap U$ for any y. The solenoid S has controlled growth with respect to l and (C_n) if for any flow-box U in a finite covering of S

$$\lim_{n \to +\infty} \frac{\mathrm{Vol}_k(B_n)}{\mathrm{Vol}_k(A_n)} = 0 \,.$$

The solenoid S has controlled growth if S contains a leaf l and an exhaustion (C_n) such that S has controlled growth with respect to l and (C_n).

For a Riemannian solenoid S, it is natural to consider the exhaustion by Riemannian balls $B(x_0, R_n)$ in a leaf l centered at a point $x_0 \in l$ and with $R_n \to +\infty$, and test the controlled growth condition with respect to such exhaustions.

The controlled growth condition depends a priori on the Riemannian metric. As we see next, it guarantees the existence of daval measures, hence the existence of transversal measures on S. Indeed the measures we construct are Schwartzman measures defined as:

Definition 3.4. (Schwartzman limits and measures) *We say that a measure μ is a Schwartzman measure if it is obtained as the limit*

$$\mu = \lim_{n \to +\infty} \mu_n \,,$$

where the measures (μ_n) are the normalized k-volume of the exhaustion (C_n) (that is, μ_n are normalized to have total mass 1). We denote by $\mathcal{M}_\mathcal{S}(S)$ the space of (probability) Schwartzman measures.

Compactness of probability measures show:

Proposition 3.5. *There are always Schwartzman measures on S,*

$$\mathcal{M}_\mathcal{S}(S) \neq \emptyset \,.$$

Theorem 3.6. *If S is a solenoid with controlled growth, then any Schwartzman measure is a daval measure,*

$$\mathcal{M}_\mathcal{S}(S) \subset \mathcal{M}_\mathcal{L}(S) \,.$$

In particular, $\mathcal{M}_\mathcal{L}(S) \neq \emptyset$ and S admits transversal measures.

Proof. Let $\mu_n \to \mu$ be a Schwartzman limit as in definition 3.4. For any flow-box U we prove that μ desintegrates as volume on leaves of U. Since S has controlled growth, pick a leaf and an exhaustion which satisfy the controlled growth condition. Let

$$C_n \cap U = A_n \cup B_n \,,$$

be the decomposition for $C_n \cap U$ described before. The set A_n is composed of a finite number of horizontal disks. We define a new measure ν_n with support in U which is the restriction of μ_n to A_n, i.e. it is proportional to the k-volume on horizontal disks. The measure ν_n desintegrates as volume on leaves in U. The transversal measure is a finite sum of Dirac measures. Moreover the controlled growth condition implies that (ν_n) and $(\mu_{n|U})$ must converge to the same limit. But we know that $\mathcal{M}_\mathcal{L}(S)$ is closed, thus the limit measure $\mu_{|U}$ desintegrates on leaves in U. So μ is a daval measure. $\qquad\square$

For uniquely ergodic solenoids we have:

Corollary 3.7. *The volume μ of a uniquely ergodic solenoid with controlled growth is the unique Schwartzman measure. Therefore there is only one Schwartzman limit*

$$\mu = \lim_{n \to +\infty} \mu_n \,,$$

which is independent of the leaf and the exhaustion.

Proof. There are always Schwartzman limits. Theorem 3.6 shows that any such limit μ desintegrates as volume on leaves. Thus the measure μ defines the unique (up to scalars) transversal measure (μ_T). But, conversely, the transversal measure determines the measure μ uniquely. Therefore there is only possible limit μ, which is the volume of the uniquely ergodic solenoid. □

4. Schwartzman clusters and asymptotic cycles

Let M be a compact C^∞ Riemannian manifold. Observe that since $H_1(M, \mathbb{R})$ is a finite dimensional real vector space, it comes equipped with a unique topological vector space structure.

The map $\gamma \mapsto [\gamma]$ that associates to each loop its homology class in $H_1(M, \mathbb{Z}) \subset H_1(M, \mathbb{R})$ is continuous when the space of loops is endowed with the Hausdorff topology. Therefore, by compactness, oriented rectifiable loops in M of uniformly bounded length define a bounded set in $H_1(M, \mathbb{R})$.

We have a more precise quantitative version of this result.

Lemma 4.1. *Let (γ_n) be a sequence of oriented rectifiable loops in M, and (t_n) be a sequence with $t_n > 0$ and $t_n \to +\infty$. If*

$$\lim_{n \to +\infty} \frac{l(\gamma_n)}{t_n} = 0 \,,$$

then in $H_1(M, \mathbb{R})$ we have

$$\lim_{n \to +\infty} \frac{[\gamma_n]}{t_n} = 0 \,.$$

Proof. Via the map

$$\omega \mapsto \int_\gamma \omega \,,$$

each loop γ defines a linear map L_γ on $H^1(M, \mathbb{R})$ that only depends on the homology class of γ. We can extend this map to $\mathbb{R} \otimes H_1(M, \mathbb{Z})$ by

$$c \otimes \gamma \mapsto c \cdot L_\gamma \,.$$

We have the isomorphism

$$H_1(M, \mathbb{R}) = \mathbb{R} \otimes H_1(M, \mathbb{Z}) \cong \left(H^1(M, \mathbb{R}) \right)^* \,.$$

The Riemannian metric gives a C^0-norm on forms. We consider the norm in $H^1(M, \mathbb{R})$ given as

$$||[\omega]||_{C^0} = \min_{\omega \in [\omega]} ||\omega|| \,,$$

and the associated operator norm in $H_1(M, \mathbb{R}) \cong \left(H^1(M, \mathbb{R}) \right)^*$.

We have

$$|L_\gamma([\omega])| = \left| \int_\gamma \omega \right| \le l(\gamma) ||\omega||_{C^0} \le l(\gamma) ||[\omega]||_{C^0} \,,$$

so

$$||L_\gamma|| \le l(\gamma) \,.$$

Hence $l(\gamma_n)/t_n \to 0$ implies $L_{\gamma_n}/t_n \to 0$ which is equivalent to $[\gamma_n]/t_n \to 0$. $\qquad \square$

Definition 4.2. (Schwartzman asymptotic 1-cycles) *Let c be a parametrized continuous curve $c : \mathbb{R} \to M$ defining an immersion of $\mathbb{R}$. For $s, t \in \mathbb{R}$, $s < t$, we choose a rectifiable oriented curve $\gamma_{s,t}$ joining $c(s)$ to $c(t)$ such that*

$$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{l(\gamma_{s,t})}{t - s} = 0 \,.$$

The parametrized curve c is a Schwartzman asymptotic 1-cycle if the juxtaposition of $c|_{[s,t]}$ and $\gamma_{s,t}$, denoted $c_{s,t}$ (which is a 1-cycle), defines a homology class $[c_{s,t}] \in H_1(M, \mathbb{Z})$ such that the limit

(3) $$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{[c_{s,t}]}{t - s} \in H_1(M, \mathbb{R})$$

exists.

We define the Schwartzman asymptotic homology class as

$$[c] := \lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{[c_{s,t}]}{t - s} \,.$$

Thanks to lemma 4.1 this definition does not depend on the choice of the closing curves $(\gamma_{s,t})$. If we take another choice $(\gamma'_{s,t})$, then as homology classes,

$$[c_{s,t}] = [c'_{s,t}] + [\gamma'_{s,t} - \gamma_{s,t}] \,,$$

and

$$\frac{l(\gamma'_{s,t} - \gamma_{s,t})}{t - s} = \frac{l(\gamma'_{s,t})}{t - s} + \frac{l(\gamma_{s,t})}{t - s} \to 0 \,,$$

as $t \to \infty$, $s \to -\infty$. By lemma 4.1,

$$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{[\gamma_{s,t} - \gamma'_{s,t}]}{t - s} = 0 \,,$$

thus

$$[c] = \lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{[c_{s,t}]}{t - s} = \lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{[c'_{s,t}]}{t - s} \,.$$

Note that we do not assume that $c(\mathbb{R})$ is an embedding of $\mathbb{R}$, i.e. $c(\mathbb{R})$ could be a loop. In that case, the Schwartzman asymptotic homology class coincides with a scalar multiple (the scalar depending on the parametrization) of the integer homology class $[c(\mathbb{R})]$. This shows that the Schwartzman homology class is a generalization to the case of immersions $c : \mathbb{R} \to M$. More precisely we have:

Proposition 4.3. *If $c : \mathbb{R} \to M$ is a loop then it is a Schwartzman asymptotic 1-cycle and the Schwartzman asymptotic homology class is a scalar multiple of the homology class of the loop $[c(\mathbb{R})] \in H_1(M, \mathbb{Z})$.*

If $c : \mathbb{R} \to M$ is a rectifiable loop with its arc-length parametrization, and $l(c)$ is the length of the loop c, then

$$[c] = \frac{1}{l(c)} [c(\mathbb{R})] .$$

Proof. Let $t_0 > 0$ be the minimal period of the map $c : \mathbb{R} \to M$. Then

$$[c_{s,t}] = \left[\frac{t - s}{t_0} \right] [c(\mathbb{R})] + O(1) .$$

Then

$$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{[c_{s,t}]}{t - s} = \frac{1}{t_0} [c(\mathbb{R})] .$$

When $c : \mathbb{R} \to M$ is the arc-length parametrization of a rectifiable loop, the period t_0 coincides with the length of the loop. $\qquad \square$

We will assume also in the definition of Schwartzman asymptotic 1-cycle that we choose $(\gamma_{s,t})$ such that $l(\gamma_{s,t})/(t - s) \to 0$ uniformly and separately on s and t when $t \to +\infty$ and $s \to -\infty$. For simplicity we can decide to choose always $\gamma_{s,t}$ with uniformly bounded length, and even with $\{\gamma_{s,t} ; s < t\}$ contained in a compact subset of the space of continua of M. Then the uniform boundedness will hold for any Riemannian metric and the notions defined will not depend on the Riemannian structure.

Definition 4.4. (Positive and negative asymptotic cycles) *Under the assumptions of definition 4.2, if the limit*

(4)
$$\lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s} \in H_1(M, \mathbb{R})$$

exists then it does not depend on s, and we say that the parametrized curve c defines a positive asympotic cycle. The positive Schwartzman homology class is defined as

$$[c_+] = \lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s} .$$

The definition of negative asymptotic cycle and negative Schwartzman homology class is the same but taking $s \to -\infty$,

$$[c_-] = \lim_{s \to -\infty} \frac{[c_{s,t}]}{t - s} .$$

The independence of the limit (4) on s follows from

$$\lim_{t \to +\infty} \frac{[c_{s',t}]}{t - s'} = \lim_{t \to +\infty} \frac{[c_{s,t}] + [c_{s',s}] + O(1)}{t - s} \cdot \frac{t - s}{t - s'} = \lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s} .$$

Proposition 4.5. *A parametrized curve c is a Schwartzman asymptotic 1-cycle if and only if it is both a positive and a negative asymptotic cycle and*

$$[c_+] = [c_-] .$$

In that case we have

$$[c] = [c_+] = [c_-] .$$

Proof. If c is a Schwartzman asymptotic 1-cycle, then for $t \to +\infty$ take $s \to -\infty$ very slowly, say satisfying the relation $t = s^2 \, l(c_{|[s,0]})$, which defines $s = s(t) < 0$ uniquely as a function of $t > 0$. Then

$$[c] = \lim_{\substack{t \to \infty \\ s=s(t) \to -\infty}} \frac{[c_{s,t}]}{t-s} = \lim_{t \to +\infty} \frac{[c_{s,0}] + [c_{0,t}] + O(1)}{t-s}$$

$$= \lim_{t \to +\infty} \left(\frac{[c_{s,0}] + O(1)}{t} + \frac{[c_{0,t}]}{t} \right) \frac{t}{t-s} = \lim_{t \to +\infty} \frac{[c_{0,t}]}{t},$$

since $\frac{t}{t-s} \to 1$ because $\frac{s}{t} \to 0$, and $\frac{[c_{s,0}]}{t} \to 0$ by lemma 4.1. So c is a positive asymptotic cycle and $[c] = [c_+]$. Analogously, c is a negative asymptotic cycle and $[c] = [c_-]$.

Conversely, assume that c is a positive and negative asymptotic cycle with $[c_+] = [c_-]$. For t large we have

$$\frac{[c_{0,t}]}{t} = [c_+] + o(1).$$

For $-s$ large we have

$$\frac{[c_{s,0}]}{-s} = [c_-] + o(1).$$

Now

$$\frac{[c_{s,t}]}{t-s} = \frac{-s}{t-s} \cdot \frac{[c_{s,0}]}{-s} + \frac{t}{t-s} \cdot \frac{[c_{0,t}]}{t} + \frac{O(1)}{t-s} = \frac{-s}{t-s}[c_+] + \frac{t}{t-s}[c_-] + o(1).$$

As $[c_+] = [c_-]$, we get that this limit exists and equals $[c] = [c_+] = [c_-]$. $\qquad\square$

Definition 4.6. (Schwartzman clusters) *Under the assumptions of definition 4.2, we can consider, regardless of whether (3) exists or not, all possible limits*

$$(5) \qquad\qquad \lim_{n \to +\infty} \frac{[c_{s_n,t_n}]}{t_n - s_n} \in H_1(M, \mathbb{R}),$$

with $t_n \to +\infty$ and $s_n \to -\infty$, that is, the derived set of $([c_{s,t}]/(t-s))_{t \to \infty, s \to -\infty}$. The limits (5) are called Schwartzman asymptotic homology classes of c, and they form the Schwartzman cluster of c,

$$\mathcal{C}(c) \subset H_1(M, \mathbb{R}).$$

A Schwartzman asymptotic homology class (5) is balanced when the two limits

$$\lim_{n \to +\infty} \frac{[c_{0,t_n}]}{t_n} \in H_1(M, \mathbb{R}),$$

and

$$\lim_{n \to +\infty} \frac{[c_{s_n,0}]}{-s_n} \in H_1(M, \mathbb{R}),$$

do exist in $H_1(M, \mathbb{R})$. We denote by $\mathcal{C}_b(c) \subset \mathcal{C}(c) \subset H_1(M, \mathbb{R})$ the set of those balanced Schwartzman asymptotic homology classes. The set $\mathcal{C}_b(c)$ is named the balanced Schwartzman cluster.

We define also the positive and negative Schwartzman clusters, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$, by taking only limits $t_n \to +\infty$ and $s_n \to -\infty$ respectively.

Proposition 4.7. *The Schwartzman clusters $\mathcal{C}(c)$, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are closed subsets of $H_1(M, \mathbb{R})$.*

If $\{[c_{s,t}]/(t-s); s < t\}$ is bounded in $H_1(M, \mathbb{R})$, then the Schwartzman clusters $\mathcal{C}(c)$, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are non-empty, compact and connected subsets of $H_1(M, \mathbb{R})$.

Proof. The Schwartzman cluster $\mathcal{C}(c)$ is the derived set of

$$([c_{s,t}]/(t-s))_{t\to\infty, s\to-\infty} \,,$$

in $H_1(M, \mathbb{R})$, hence closed.

Under the boundedness assumption, non-emptiness and compactness follow. Also the oscillation of $([c_{s,t}])_{s,t}$ is bounded by the size of $[\gamma_{s,t}]$. Therefore the magnitude of the oscillation of $([c_{s,t}]/(t-s))_{s,t}$ tends to 0 as $t\to\infty$, $s\to-\infty$. This forces the derived set to be connected under the boundedness assumption, since it is ϵ-connected for each $\epsilon > 0$. (A compact metric space is ϵ-connected for all $\epsilon > 0$ if and only if it is connected.)

Also $\mathcal{C}_+(c)$, resp. $\mathcal{C}_-(c)$, is closed because it is the derived set of

$$([c_{0,t}]/t)_{t\to\infty} \,,$$

resp.

$$([c_{s,0}]/(-s))_{s\to-\infty} \,,$$

in $H_1(M, \mathbb{R})$. Non-emptiness, compactness and connectedness under the boundedness assumption follow for the cluster sets $\mathcal{C}_\pm(c)$ in the same way as for $\mathcal{C}(c)$. □

Note that all these cluster sets may be empty if the parametrization is too fast.

The balanced Schwartzman cluster $\mathcal{C}_b(c)$ does not need to be closed, as shown in the following counter-example.

Counter-example 4.8. *We consider the torus $M = \mathbb{T}^2$. We identify $H_1(M, \mathbb{R}) \cong \mathbb{R}^2$, with $H_1(M, \mathbb{Z})$ corresponding to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Consider a line l in $H_1(M, \mathbb{R}^2)$ of irrational slope passing through the origin, $y = \sqrt{2}\, x$ for example. We can find a sequence of pairs of points $(a_n, b_n) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ in the open lower half plane H_l determined by the line l, such that the sequence of segments $[a_n, b_n]$ do converge to the line l, and the middle point $(a_n + b_n)/2 \to 0$ (this is an easy exercise in diophantine approximation). We assume that the first coordinate of b_n tends to $+\infty$, and the first coordinate of a_n tends to $-\infty$. Now we can construct a parametrized curve c on $\mathbb{T}^2$ such that for all $n \geq 1$ there are an infinite number of times $t_{n,i} \to +\infty$ with $[c_{0,t_{n,i}}]/t_{n,i} = b_n$, and for an infinite number of times $s_{n,i} \to -\infty$, $[c_{s_{n,i},0}]/(-s_{n,i}) = a_n$. Thus in homology the curve c oscillates wildly. We can adjust the velocity of the parametrization so that $-s_{n,i} = t_{n,i}$. Hence for these times*

$$\frac{[c_{s_{n,i}, t_{n,i}}]}{t_{n,i} - s_{n,i}} = \frac{a_n(-s_{n,i}) + b_n(t_{n,i}) + O(1)}{t_{n,i} - s_{n,i}} \to \frac{a_n + b_n}{2} \,,$$

when $i \to +\infty$, and the two ends balance each other. We have great freedom in constructing c, so that we may arrange to have always $[c_{s,t}] \subset H_l$. Then we get that $0 \in \mathcal{C}(c)$ and all $(a_n + b_n)/2 \in \mathcal{C}_b(c)$ but $0 \notin \mathcal{C}_b(c)$.

We have that c is a Schwartzman asymptotic 1-cycle (resp. positive, negative) if and only if $\mathcal{C}(c)$ (resp. $\mathcal{C}_+(c)$, $\mathcal{C}_-(c)$) is reduced to one point. In that case the Schwartzman asymptotic 1-cycle is balanced. The next result generalizes proposition 4.5. We need first a definition.

Definition 4.9. *Let $A, B \subset V$ be subsets of a real vector space V. For $a, b \in V$ the segment $[a, b] \subset V$ is the convex hull of $\{a, b\}$ in V. The additive hull of A and B is*

$$A \widehat{+} B = \bigcup_{\substack{a \in A \\ b \in B}} [a, b] \,.$$

Proposition 4.10. *The Schwartzman balanced cluster $\mathcal{C}_b(c)$ is contained in the additive hull of $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$*

$$\mathcal{C}_b(c) \subset \mathcal{C}_+(c) \widehat{+} \mathcal{C}_-(c) \,.$$

Moreover, for each $a \in \mathcal{C}_+(c)$ and $b \in \mathcal{C}_-(c)$, we have

$$\mathcal{C}_b(c) \cap [a, b] \neq \emptyset \,.$$

Proof. Let $x \in \mathcal{C}_b(c)$,

$$x = \lim_{n \to +\infty} \frac{[c_{s_n, t_n}]}{t_n - s_n} \,.$$

We write

$$\frac{[c_{s_n, t_n}]}{t_n - s_n} = \frac{[c_{s_n, 0}]}{-s_n} \cdot \frac{-s_n}{t_n - s_n} + \frac{[c_{0, t_n}]}{t_n} \cdot \frac{t_n}{t_n - s_n} + o(1) \,,$$

and the first statement follows.

For the second, consider

$$a = \lim_{n \to +\infty} \frac{[c_{0, t_n}]}{t_n} \in \mathcal{C}_+(c) \,,$$

and

$$b = \lim_{n \to +\infty} \frac{[c_{s_n, 0}]}{-s_n} \in \mathcal{C}_-(c) \,.$$

Then taking any accumulation point $\tau \in [0, 1]$ of the sequence $(t_n/(t_n - s_n))_n \subset [0, 1]$ and taking subsequences in the above formulas, we get a balanced Schwartzman homology class

$$c = \tau a + (1 - \tau) b \in \mathcal{C}_b(c) \,.$$

$\square$

Corollary 4.11. *If $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are non-empty, then $\mathcal{C}_b(c)$ is non-empty, and therefore $\mathcal{C}(c)$ is also non-empty.*

Note that we can have $\mathcal{C}_+(c) = \mathcal{C}_-(c) = \emptyset$ (then $\mathcal{C}_b(c) = \emptyset$) but $\mathcal{C}(c) \neq \emptyset$ (modify appropriately counter-example 4.8).

There is one situation where we can assert that the balanced Schwartzman cluster set is closed.

Proposition 4.12. *If $B = \{[c_{s,t}]/(t - s); s < t\} \subset H_1(M, \mathbb{R})$ is a bounded set, then $\mathcal{C}(c)$, $\mathcal{C}_+(c)$, $\mathcal{C}_-(c)$ and $\mathcal{C}_b(c)$ are all compact sets. More precisely, they are all contained in the convex hull of $\overline{B}$.*

Proof. Obviously $\mathcal{C}(c)$, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are bounded as cluster sets of bounded sets, hence compact by proposition 4.7.

In order to prove that $\mathcal{C}_b(c)$ is bounded, we observe that the additive hull of bounded sets is bounded, therefore boundedness follows from proposition 4.10. We show that $\mathcal{C}_b(c)$ is closed. Since $\mathcal{C}_b(c) \subset \mathcal{C}(c)$ and $\mathcal{C}(c)$ is closed, any accumulation point x of $\mathcal{C}_b(c)$ is in $\mathcal{C}(c)$. Let

$$x = \lim_{n \to +\infty} \frac{[c_{s_n, t_n}]}{t_n - s_n} \,,$$

and write as before

$$\frac{[c_{s_n, t_n}]}{t_n - s_n} = \frac{[c_{s_n, 0}]}{-s_n} \cdot \frac{-s_n}{t_n - s_n} + \frac{[c_{0, t_n}]}{t_n} \cdot \frac{t_n}{t_n - s_n} + o(1) \,.$$

Note that $([c_{s_n,0}]/(-s_n))_n$ and $([c_{0,t_n}]/t_n)_n$ stay bounded. Therefore we can extract converging subsequences and also for the sequence $(t_n/(t_n - s_n))_n \subset [0, 1]$. The limit along these subsequences $t_{n_k} \to +\infty$ and $s_{n_k} \to -\infty$ give the same Schwartzman homology class x which turns out to be balanced.

The final statement follows from the above proofs. $\qquad\square$

The situation described in proposition 4.12 is indeed quite natural. It arises each time that M is a Riemannian manifold and c is an arc-length parametrization of a rectifiable curve. In the following proposition we make use of the natural norm $||\cdot||$ in the homology of a Riemannian manifold defined in the Appendix.

Proposition 4.13. *Let M be a Riemannian manifold and denote by $||\cdot||$ the norm in homology. If c is a rectifiable curve parametrized by arc-length then the cluster sets $\mathcal{C}(c)$, $\mathcal{C}_+(c)$, $\mathcal{C}_-(c)$ and $\mathcal{C}_b(c)$ are compact subsets of $\bar{B}(0, 1)$, the closed ball of radius 1 for the norm in homology.*

So $\mathcal{C}(c)$ and $\mathcal{C}_\pm(c)$ are non-empty, compact and connected, and $\mathcal{C}_b(c)$ is non-empty and compact.

Proof. Observe that we have
$$l(c_{s,t}) = l(c|_{[s,t]}) + l(\gamma_{s,t}) = t - s + l(\gamma_{s,t})\,.$$
Thus
$$l([c_{s,t}]) \leq t - s + l(\gamma_{s,t})\,.$$
By theorem A.4,
$$||[c_{s,t}]|| \leq t - s + l(\gamma_{s,t})\,,$$
and
$$\left\|\frac{[c_{s,t}]}{t-s}\right\| \leq 1 + \frac{l(\gamma_{s,t})}{t-s}\,.$$
Since $\frac{l(\gamma_{s,t})}{t-s} \to 0$ uniformly, we get that $B = \{[c_{s,t}]/(t-s); s < t\} \subset H_1(M, \mathbb{R})$ is a bounded set.

By proposition 4.7, $\mathcal{C}(c)$ and $\mathcal{C}_\pm(c)$ are non-empty, compact and connected. By corollary 4.11, $\mathcal{C}_b(c)$ is non-empty and by proposition 4.12, it is compact. $\qquad\square$

Obviously the previous notions depend heavily on the parametrization. For a non-parametrized curve we can also define Schwartzman cluster sets.

Definition 4.14. *For a non-parametrized oriented curve $c \subset M$, we define the Schwartzman cluster $\mathcal{C}(c)$ as the union of the Schwartzman clusters for all orientation preserving parametrizations of c. We define the positive $\mathcal{C}_+(c)$, resp. negative $\mathcal{C}_-(c)$, Schwartzman cluster set as the union of all positive, resp. negative, Schwartzman cluster sets for all orientation preserving parametrizations.*

Proposition 4.15. *For an oriented curve $c \subset M$ the Schwartzman clusters $\mathcal{C}(c)$, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are non-empty closed cones of $H_1(M, \mathbb{R})$. These cones are degenerate (i.e. reduced to $\{0\}$) if and only if $\{[c_{s,t}]; s < t\}$ is a bounded subset of $H_1(M, \mathbb{Z})$.*

Proof. We can choose the closing curves $\gamma_{s,t}$ only depending on $c(s)$ and $c(t)$ and not on the parameter values s and t, nor on the parametrization. Then the integer homology class $[c_{s,t}]$ only depends on the points $c(s)$ and $c(t)$ and not on the parametrization. Therefore, we can adjust the speed of the parametrization so that $[c_{s,t}]/(t - s)$ remains in a ball centered at 0. This shows that $\mathcal{C}(c)$ is not empty. Adjusting the speed of the parametrization we equally get

that it contains elements that are not 0, provided that the set $\{[c_{s,t}]; s < t\}$ is not bounded in $H_1(M, \mathbb{Z})$. Certainly, if $\{[c_{s,t}]; s < t\}$ is bounded, all the cluster sets are reduced to $\{0\}$. Observe also that if $a \in \mathcal{C}(c)$ then any multiple λa, $\lambda > 0$, belongs to $\mathcal{C}(c)$, by considering the new parametrization with velocity multiplied by λ. So $\mathcal{C}(c)$ is a cone in $H_1(M, \mathbb{R})$.

Now we prove that $\mathcal{C}(c)$ is closed. Let $a_n \in \mathcal{C}(c)$ with $a_n \to a \in H_1(M, \mathbb{R})$. For each n we can choose a parametrization of c, say $c^{(n)} = \tilde{c} \circ \psi_n$ (here $\tilde{c}$ is a fixed parametrization and ψ_n is an orientation preserving homeomorphism of $\mathbb{R}$), and parameters s_n and t_n such that $||[c^{(n)}_{s_n, t_n}] - a|| \leq 1/n$ (considering any fixed norm in $H_1(M, \mathbb{R})$). For each n we can choose t_n as large as we like, and s_n negative as we like. Choose them inductively such that (t_n) and $(\psi_n(t_n))$ are both increasing sequences converging to $+\infty$, and (s_n) and $(\psi_n(s_n))$ are both decreasing sequences converging to $-\infty$. Construct a homeomorphism ψ of $\mathbb{R}$ with $\psi(t_n) = \psi_n(t_n)$ and $\psi(s_n) = \psi_n(s_n)$. It is clear that a is obtained as Schwartzman limit for the parametrization $\tilde{c} \circ \psi$ at parameters s_n, t_n.

The proofs for $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are similar. $\qquad\square$

Remark 4.16. The image of these cluster sets in the projective space $\mathbb{P}H_1(M, \mathbb{R})$ is not necessarily connected: On the torus $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, choose a curve in $\mathbb{R}^2$ that oscillates between the half y-axis $\{y > 0\}$ and the half x-axis $\{x > 0\}$, remaining in a small neighborhood of these axes and being unbounded for $t \to +\infty$, and being bounded when $s \to -\infty$. Then its Schwartzman cluster consists of two lines through 0 in $H_1(\mathbb{T}^2, \mathbb{R}) \cong \mathbb{R}^2$, and its projection in the projective space consists of two distinct points.

Remark 4.17. Let c be a parametrized Schwartzman asymptotic 1-cycle, and consider the unparametrized oriented curve defined by c, denoted by $\bar{c}$. Assume that the asymptotic Schwartzman homology class is $a = [c] \neq 0$. Then

$$\mathcal{C}_\pm(\bar{c}) = \mathcal{C}(\bar{c}) = \mathbb{R}_{\geq 0} \cdot a \,,$$

as a subset of $H_1(M, \mathbb{R})$. This follows since any parametrization of $\bar{c}$ is of the form $c' = c \circ \psi$, where $\psi : \mathbb{R} \to \mathbb{R}$ is a positively oriented homeomorphism of $\mathbb{R}$. Then

$$(6) \qquad \frac{c'_{s,t}}{t - s} = \frac{c_{\psi(s), \psi(t)}}{\psi(t) - \psi(s)} \cdot \frac{\psi(t) - \psi(s)}{t - s} \,.$$

The first term in the right hand side tends to a when $t \to +\infty$, $s \to -\infty$. If the left hand side is to converge, then the second term in the right hand side stays bounded. After extracting a subsequence, it converges to some $\lambda \geq 0$. Hence (6) converges to λa.

We define now the notion of asymptotically homotopic curves.

Definition 4.18. (Asymptotic homotopy) *Let $c_0, c_1 : \mathbb{R} \to M$ be two parametrized curves. They are asymptotically homotopic if there exists a continuous family c_u, $u \in [0, 1]$, interpolating between c_0 and c_1, such that*

$$c : \mathbb{R} \times [0, 1] \to M \,, \ \ c(t, u) = c_u(t) \,,$$

satisfies that $\delta_t(u) = c(t, u)$, $u \in [0, 1]$ is rectifiable with

$$(7) \qquad\qquad\qquad l(\delta_t) = o(|t|) \,.$$

Two oriented curves are asymptotically homotopic if they have orientation preserving parametrizations that are asymptotically homotopic.

Proposition 4.19. *If c_0 and c_1 are asymptotically homotopic parametrized curves then their cluster sets coincide:*

$$\mathcal{C}_\pm(c_0) = \mathcal{C}_\pm(c_1),$$
$$\mathcal{C}_b(c_0) = \mathcal{C}_b(c_1),$$
$$\mathcal{C}(c_0) = \mathcal{C}(c_1).$$

If c_0 and c_1 are asymptotically homotopic oriented curves then their cluters sets coincide:

$$\mathcal{C}_\pm(c_0) = \mathcal{C}_\pm(c_1),$$
$$\mathcal{C}(c_0) = \mathcal{C}(c_1).$$

Proof. For parametrized curves we have

$$[c_{0,s,t}] = [c_{1,s,t}] + [\delta_s - \gamma_{1,s,t} - \delta_t + \gamma_{0,s,t}].$$

The length of the displacement by the homotopy is bounded by (7), so

$$l(\delta_s - \gamma_{1,s,t} - \delta_t + \gamma_{0,s,t}) = l(\gamma_{1,s,t}) + l(\gamma_{0,s,t}) + o(|t| + |s|),$$

thus

$$\frac{[c_{0,s,t}]}{t-s} = \frac{[c_{1,s,t}]}{t-s} + o(1).$$

For non-parametrized curves, the homotopy between two particular parametrizations yields a one-to-one correspondence between points in the curves

$$c_0(t) \mapsto c_1(t).$$

Using this correspondence, we have a correspondence of pairs of points $(a,b) = (c_0(s), c_0(t))$ with pairs of points $(a', b') = (c_1(s), c_1(t))$. Thus if the sequence of pairs of points (a_n, b_n) gives a cluster value for c_0, then the corresponding sequence (a'_n, b'_n) gives a proportional cluster value, since (with obvious notation)

$$[c_{0,a_n,b_n}] = [c_{1,a'_n,b'_n}] + O(1).$$

So we can always normalize the speed of the parametrization of c_1 in order to assure that the limit value is the same. This proves that the clusters sets coincide. $\qquad\square$

5. CALIBRATING FUNCTIONS

Let M be a C^∞ smooth compact manifold. We define now the notion of calibrating function.

Let $\pi : \tilde{M} \to M$ be the universal cover of M and let Γ be the group of deck transformations of the cover.

Fix a point $\tilde{x}_0 \in \tilde{M}$ and $x_0 = \pi(\tilde{x}_0)$. There is a faithful and transitive action of Γ in the fiber $\pi^{-1}(x_0)$ induced by the action of Γ in $\tilde{M}$, and we have a group isomorphism $\Gamma \cong \pi_1(M, x_0)$. Thus from the group homomorphism

$$\pi_1(M, x_0) \to H_1(M, \mathbb{Z}),$$

we get a group homomorphism

$$\rho : \Gamma \to H_1(M, \mathbb{Z}).$$

Definition 5.1. (Calibrating function) *A map* $\Phi : \tilde{M} \to H_1(M, \mathbb{R})$ *is a calibrating function if the diagram*

$$\begin{array}{ccc} \Gamma \cong \pi_1(M, x_0) & \hookrightarrow & \tilde{M} \\ \rho \downarrow & & \downarrow \Phi \\ H_1(M, \mathbb{Z}) & \to & H_1(M, \mathbb{R}) \end{array}$$

is commutative and Φ *is equivariant for the action of* Γ *on* $\tilde{M}$, *i.e. for any* $g \in \Gamma$ *and* $\tilde{x} \in \tilde{M}$,

$$\Phi(g \cdot \tilde{x}) = \Phi(\tilde{x}) + \rho(g) \,.$$

If $\tilde{x}_0 \in \tilde{M}$ *we say that the calibrating function* Φ *is associated to* $\tilde{x}_0$ *if* $\Phi(\tilde{x}_0) = 0$.

Proposition 5.2. *There are smooth calibrating functions associated to any point* $\tilde{x}_0 \in \tilde{M}$.

Proof. Fix a smooth non-negative function $\varphi : \tilde{M} \to \mathbb{R}$ with compact support $K = \overline{U}$ with $U = \{\varphi > 0\}$ such that $\pi(U) = M$. Moreover, we can request that $U \cap \pi^{-1}(x_0) = \{\tilde{x}_0\}$.

For any $g_0 \in \Gamma$, define $\varphi_{g_0}(\tilde{x}) = \varphi(g_0^{-1} \cdot \tilde{x})$. The support of φ_{g_0} is $g_0 K$, and $(g_0 K)_{g_0 \in \Gamma}$ is a locally finite covering of $\tilde{M}$, as follows from the compactness of K. Set

$$\psi_{g_0}(\tilde{x}) := \frac{\varphi_{g_0}(\tilde{x})}{\sum_{g \in \Gamma} \varphi_g(\tilde{x})} \,.$$

Then $\psi_{g_0}(\tilde{x}) = \psi_e(g_0^{-1} \cdot \tilde{x})$ and

$$\sum_{g \in \Gamma} \psi_g \equiv 1 \,.$$

Also ψ_{g_0} has compact support $g_0 K$, and it is a smooth function since the denominator is strictly positive (because $\pi(U) = M$) and it is at each point a finite sum of smooth functions.

We define the map

$$\Phi : \tilde{M} \to H_1(M, \mathbb{R}) \,,$$

by

$$\Phi(\tilde{x}) = \sum_{g \in \Gamma} \psi_g(\tilde{x}) \, \rho(g) \,.$$

We check that Φ is a calibrating function:

$$\begin{aligned} \Phi(g \cdot \tilde{x}) &= \sum_{h \in \Gamma} \psi_h(g \cdot \tilde{x}) \, \rho(h) \\ &= \sum_{h \in \Gamma} \psi_{g^{-1}h}(\tilde{x}) \, (\rho(g) + \rho(g^{-1}h)) \\ &= \sum_{h' \in \Gamma} \psi_{h'}(\tilde{x}) \, \rho(g) \; + \; \sum_{h' \in \Gamma} \psi_{h'}(\tilde{x}) \, \rho(h') \\ &= \rho(g) + \Phi(\tilde{x}) \,. \end{aligned}$$

Notice that by construction $\Phi(\tilde{x}_0) = 0$. $\qquad \square$

We note also that choosing a function ϕ of rapid decay, we may do a similar construction, as long as $\sum_{g \in \Gamma} \phi_g$ is summable (we may need to add a translation to Φ in order to ensure $\Phi(\tilde{x}_0) = 0$).

Observe that the calibrating property implies that for a curve $\gamma : [a, b] \to M$, the quantity $\Phi(\tilde{\gamma}(b)) - \Phi(\tilde{\gamma}(a))$ does not depend on the lift $\tilde{\gamma}$ of γ, because for another choice $\tilde{\gamma}'$, we would have for some $g \in \Gamma$,

$$\tilde{\gamma}'(a) = g \cdot \tilde{\gamma}(a) \,,$$

and
$$\tilde{\gamma}'(b) = g \cdot \tilde{\gamma}(b) \,.$$

Therefore
$$\Phi(\tilde{\gamma}'(b)) - \Phi(\tilde{\gamma}'(a)) = \Phi(g \cdot \tilde{\gamma}(b)) - \Phi(g \cdot \tilde{\gamma}(a)) = \Phi(\tilde{\gamma}(b)) - \Phi(\tilde{\gamma}(a)) \,.$$

This justifies the next definition.

Definition 5.3. *Given a calibrating function* Φ, *for any curve* $\gamma : [a,b] \to M$, *we define* $\Phi(\gamma) := \Phi(\tilde{\gamma}(b)) - \Phi(\tilde{\gamma}(a))$ *for any lift* $\tilde{\gamma}$ *of* γ.

Proposition 5.4. *For any loop* $\gamma \subset M$ *we have*
$$\Phi(\gamma) = [\gamma] \in H_1(M,\mathbb{Z}) \,.$$

Proof. Modifying γ, but without changing its endpoints nor $\Phi(\gamma)$ nor $[\gamma]$, we can assume that $x_0 \in \gamma$. Since $\Gamma \cong \pi_1(M, x_0)$, let $h_0 \in \Gamma$ be the element corresponding to γ. Then γ lifts to a curve joining $\tilde{x}_0$ to $h_0 \cdot \tilde{x}_0$, and
$$\Phi(\gamma) = \Phi(h_0 \cdot \tilde{x}_0) - \Phi(\tilde{x}_0) = \rho(h_0) = [\gamma] \in H_1(M,\mathbb{Z}) \,.$$
$\square$

Proposition 5.5. *We assume that* M *is endowed with a Riemannian metric and that the calibrating function* Φ *is smooth. Then for any rectifiable curve* γ *we have*
$$|\Phi(\gamma)| \leq C \cdot l(\gamma) \,,$$
where $l(\gamma)$ *is the length of* γ, *and* $C > 0$ *is a positive constant depending only on the metric.*

Proof. The calibrating function Φ is a smooth function on $\tilde{M}$ and Γ-equivariant, hence it is bounded as well as its derivatives. The result follows. $\square$

Example 5.6. *For* $M = \mathbb{T}$, $\tilde{M} = \mathbb{R}$, $H_1(M,\mathbb{Z}) = \mathbb{Z} \subset \mathbb{R} = H_1(M,\mathbb{R})$, $\Gamma = \mathbb{Z}$ *and* $\rho : \Gamma \to H_1(M,\mathbb{Z})$ *is given (with these identifications) by* $\rho(n) = n$. *We can take* $\varphi(x) = |1 - x|$, *for* $x \in [-1, 1]$, *and* $\varphi(x) = 0$ *elsewhere. Then*
$$\sum_{n=-\infty}^{\infty} \varphi(x - n) = 1 \,,$$
and
$$\psi_n(x) = \varphi_n(x) = \varphi(x - n) \,.$$
Therefore we get the calibrating function
$$\Phi(x) = \sum_{n=-\infty}^{\infty} \varphi(x - n) \, n = x \,.$$

It is a smooth calibrating function (despite that φ *is not).*

A similar construction works for higher dimensional tori.

Proposition 5.7. *Let* $c : \mathbb{R} \to M$ *be a* C^1 *curve. Consider two sequences* (s_n) *and* (t_n) *such that* $s_n < t_n$, $s_n \to -\infty$, *and* $t_n \to +\infty$.

Then the following conditions are equivalent:

(1) *The limit*
$$[c] = \lim_{n \to +\infty} \frac{[c_{s_n, t_n}]}{t_n - s_n} \in H_1(M, \mathbb{R})$$
exists.

(2) *The limit*
$$[c]_\Phi = \lim_{n \to \infty} \frac{\Phi(c_{|[s_n, t_n]})}{t_n - s_n} \in H_1(M, \mathbb{R})$$
exists.

(3) *For any closed 1-form $\alpha \in \Omega^1(M)$, the limit*
$$[c](\alpha) = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha$$
exists.

(4) *For any cohomology class $[\alpha] \in H^1(M, \mathbb{R})$, the limit*
$$[c][\alpha] = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha$$
exists, and does not depend on the closed 1-form $\alpha \in \Omega^1(M)$ representing the cohomology class.

(5) *For any continuous map $f : M \to \mathbb{T}$, let $\widetilde{f \circ c} : \mathbb{R} \to \mathbb{R}$ be a lift of $f \circ c$, the limit*
$$\rho(f) = \lim_{n \to +\infty} \frac{\widetilde{f \circ c}(t_n) - \widetilde{f \circ c}(s_n)}{t_n - s_n}$$
exists.

(6) *For any (two-sided, embedded, transversally oriented) hypersurface $H \subset M$ such that all intersections $c(\mathbb{R}) \cap H$ are transverse, the limit*
$$[c] \cdot [H] = \lim_{n \to \infty} \frac{\#\{u \in [s_n, t_n] \ ; \ c(u) \in H\}}{t_n - s_n}$$
exists. The notation $\#$ means a signed count of intersection points.

When these conditions hold, we have $[c] = [c]_\Phi$ for any calibrating function Φ. If $\alpha \in \Omega^1(M)$ is a closed form, then $[c](\alpha) = [c][\alpha] = \langle [c], [\alpha] \rangle$. If $f : M \to \mathbb{T}$ is a continuous map and $a = f^*[dx] \in H^1(M, \mathbb{Z})$ is the pull-back of the generator $[dx] \in H^1(\mathbb{T}, \mathbb{Z})$, and H is a hypersurface such that $[H]$ is the Poincaré dual of a, then $\langle [c], a \rangle = \rho(f) = [c] \cdot [H]$.

Proof. The equivalence of (1) and (2) follows from the properties of Φ. Let $c : \mathbb{R} \to M$ be a curve. Then
$$\Phi(c_{|[s_n, t_n]}) = \Phi([c_{s_n, t_n}]) - \Phi(\gamma_{s_n, t_n}) = [c_{s_n, t_n}] + O(l(\gamma_{s_n, t_n})).$$
Dividing by $t_n - s_n$ and passing to the limit the equivalence of (1) and (2) follows.

We prove that (1) is equivalent to (3). First note that
$$\left| \int_{\gamma_{s_n, t_n}} \alpha \right| \le C \, l(\gamma_{s_n, t_n}) \, \|\alpha\|_{C^0} .$$
We have when $t_n - s_n \to +\infty$,
$$\frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha = \frac{1}{t_n - s_n} \int_{c_{s_n, t_n}} \alpha + O\left(\frac{l(\gamma_{s_n, t_n})}{t_n - s_n} \right) = \frac{[c_{s_n, t_n}](\alpha)}{t_n - s_n} + o(1) .$$

and the equivalence of (1) and (3) results.

The equivalence of (3) and (4) results from the fact that the limit

$$[c](\alpha) = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha$$

does not depend on the representative of the cohomology class $a = [\alpha]$. If $\beta = \alpha + d\phi$, with $\phi : M \to \mathbb{R}$ smooth, then $[c](\alpha) = [c](\beta)$ since

$$[c](d\phi) = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} d\phi = \lim_{n \to \infty} \frac{\phi(c(t_n)) - \phi(c(s_n))}{t_n - s_n} \to 0,$$

since ϕ is bounded. Also $[c][\alpha] = [c](\alpha)$.

We turn now to (4) implies (5). First note that there is an identification $H^1(M, \mathbb{Z}) \cong [M, K(\mathbb{Z}, 1)] = [M, \mathbb{T}]$, where any cohomology class $[\alpha] \in H^1(M, \mathbb{Z})$ is associated to a (homotopy class of a) map $f : M \to \mathbb{T}$ such that $[\alpha] = f^*[\mathbb{T}]$, where $[\mathbb{T}] \in H^1(\mathbb{T}, \mathbb{Z})$ is the fundamental class. To prove (5), assume first that f is smooth. With the identification $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the class $f^*(dx) = df \in \Omega^1(M)$ represents $[\alpha]$. Therefore

(8)
$$\frac{\widetilde{f \circ c}(t_n) - \widetilde{f \circ c}(s_n)}{t_n - s_n} = \frac{1}{t_n - s_n} \int_{[s_n, t_n]} d(f \circ c) =$$
$$= \frac{1}{t_n - s_n} \int_{[s_n, t_n]} (df)(c') = \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} df ,$$

and from the existence of the limit in (4) we get the limit in (5) that we identify as

$$\rho(f) = [c][df] .$$

If f is only continuous, we approximate it by a smooth function, which does not change the limit in (5).

Conversely, if (5) holds, then any integer cohomology class admits a representative of the form $\alpha = df$, where $f : M \to \mathbb{T}$ is a smooth map. Then using (8) we have

$$\frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha \to \rho(f) .$$

So the limit in (4) exists for $\alpha = df$. This implies that the limit in (4) exists for any closed $\alpha \in \Omega^1(M)$, since $H^1(M, \mathbb{Z})$ spans $H^1(M, \mathbb{R})$.

We check the equivalence of (5) and (6). First, let us see that (6) implies (5). As before, it is enough to prove (5) for a smooth map $f : M \to \mathbb{T}$. Let $x_0 \in \mathbb{T}$ be a regular value of f, so that $H = f^{-1}(x_0) \subset M$ is a smooth (two-sided) hypersurface. Then $[H]$ represents the Poincaré dual of $[df] \in H^1(M, \mathbb{Z})$. Choose x_0 such that it is also a regular value of $f \circ c$, so all the intersections of $c(\mathbb{R})$ with H are transverse. Now for any $s < t$,

$$[c_{s,t}] \cdot [H] = \#c([s, t]) \cap H + \#\gamma_{s,t} \cap H ,$$

where $\#$ denotes signed count of intersection points (we may assume that all intersections of $\gamma_{s,t}$ and H are transverse, by a small perturbation of $\gamma_{s,t}$; also we do not count the extremes of $\gamma_{s,t}$ in $\#\gamma_{s,t} \cap H$ in case that either $c(s) \in H$ or $c(t) \in H$).

Now

$$\#c([s, t]) \cap H = [\widetilde{f \circ c}(t)] + [-\widetilde{f \circ c}(s)] = \widetilde{f \circ c}(t) - \widetilde{f \circ c}(s) + O(1),$$

where $[\cdot]$ denotes the integer part, and $|\#\gamma_{s,t} \cap H|$ is bounded by the total variation of $\widetilde{f \circ \gamma_{s,t}}$, which is bounded by the maximum of df times the total length of $\gamma_{s,t}$, which is $o(t-s)$ by assumption. Hence

$$\lim_{n \to +\infty} \frac{\widetilde{f \circ c}(t_n) - \widetilde{f \circ c}(s_n)}{t_n - s_n} = \lim_{n \to +\infty} \frac{\#c([s_n, t_n]) \cap H}{t_n - s_n}$$

exists.

Conversely, if (5) holds, consider a two-sided embedded topological hypersurface $H \subset M$. Then there is a collar $[0,1] \times H$ embedded in M such that H is identified with $\{\frac{1}{2}\} \times H$. There exists a continuous map $f : M \to \mathbb{T}$ such that $H = f^{-1}(x_0)$ for $x_0 = \frac{1}{2} \in \mathbb{T}$, constructed by sending $[0,1] \times H \to [0,1] \to \mathbb{T}$ and collapsing the complement of $[0,1] \times H$ to 0.

Now if all intersections of $c(\mathbb{R})$ and H are transverse, that means that for any $t \in \mathbb{R}$ such that $c(t) \in H$, we have that $c(t-\epsilon)$ and $c(t+\epsilon)$ are at opposite sides of the collar, for $\epsilon > 0$ small (the sign of the intersection point is given by the direction of the crossing). So $f(c(s))$ crosses x_0 increasingly or decreasingly (according to the sign of the intersection). Hence

$$\frac{\#\{u \in [s_n, t_n] \ ; \ c(u) \in H\}}{t_n - s_n} = \frac{\widetilde{f \circ c}(t_n) - \widetilde{f \circ c}(s_n)}{t_n - s_n} + o(1).$$

The required limit exists. $\qquad\square$

Remark 5.8. Proposition 5.7 holds if we only assume the curve c to be rectifiable.

Corollary 5.9. *Let $c : \mathbb{R} \to M$ be a C^1 curve. The following conditions are equivalent:*

(1) *The curve c is a Schwartzman asymptotic cycle.*

(2) *The limit*

$$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{\Phi(c_{|[s,t]})}{t-s} \in H_1(M, \mathbb{R})$$

exists.

(3) *For any closed 1-form $\alpha \in \Omega^1(M)$, the limit*

$$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{1}{t-s} \int_{c([s,t])} \alpha$$

exists.

(4) *For any cohomology class $[\alpha] \in H^1(M, \mathbb{R})$, the limit*

$$[c][\alpha] = \lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{1}{t-s} \int_{c([s,t])} \alpha$$

exists, and does not depend on the closed 1-form $\alpha \in \Omega^1(M)$ representing the cohomology class.

(5) *For any continuous map $f : M \to \mathbb{T}$, let $\widetilde{f \circ c} : \mathbb{R} \to \mathbb{R}$ be a lift of $f \circ c$, we have that the limit*

$$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{\widetilde{f \circ c}(t) - \widetilde{f \circ c}(s)}{t-s}$$

exists.

(6) *For a (two-sided, embedded, transversally oriented) hypersurface $H \subset M$ such that all intersections $c(\mathbb{R}) \cap H$ are transverse, the limit*

$$\lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{\#\{u \in [s,t]\,;\, c(u) \in H\}}{t-s}$$

exists.

When c is a Schwartzman asymptotic cycle, we have $[c] = [c]_\Phi$ for any calibrating function Φ. If $\alpha \in \Omega^1(M)$ is a closed form then

$$[c](\alpha) = [c][\alpha] = \langle [\alpha], [c] \rangle\,.$$

If $f : M \to \mathbb{T}$ and $a = f^*[dx] \in H^1(M, \mathbb{Z})$, where $[dx] \in H^1(\mathbb{T}, \mathbb{Z})$ is the generator, and $H \subset M$ is a hypersurface such that $[H]$ is the Poincaré dual of a, then we have

$$\langle [c], [\alpha] \rangle = \rho(f) = [c] \cdot [H]\,.$$

6. Schwartzman 1-dimensional cycles

We assume that M is a compact C^∞ Riemannian manifold, with Riemannian metric g.

Definition 6.1. (Schwartzman representation of homology classes) *Let $f : S \to M$ be an immersion in M of an oriented 1-solenoid S. Then S is a Riemannian solenoid with the pull-back metric f^*g.*

(1) *If S is endowed with a transversal measure $\mu = (\mu_T) \in \mathcal{M}_{\mathcal{T}}(S)$, the immersed measured solenoid $f : S_\mu \to M$ represents a homology class $a \in H_1(M, \mathbb{R})$ if for (μ_T)-almost all leaves $c : \mathbb{R} \to S$, parametrized positively and by arc-length, we have that $f \circ c$ is a Schwartzman asymptotic 1-cycle with $[f \circ c] = a$.*

(2) *The immersed solenoid $f : S \to M$ fully represents a homology class $a \in H_1(M, \mathbb{R})$ if for all leaves $c : \mathbb{R} \to S$, parametrized positively and by arc-length, we have that $f \circ c$ is a Schwartzman asymptotic 1-cycle with $[f \circ c] = a$.*

Note that if $f : S \to M$ fully represents an homology class $a \in H_1(M, \mathbb{R})$, then for all oriented leaves $c \subset S$, we have that $f \circ c$ is a Schwartzman asymptotic cycle and

$$\mathcal{C}_+(f \circ c) = \mathcal{C}_-(f \circ c) = \mathcal{C}(f \circ c) = \mathbb{R}_{\geq 0} \cdot a \subset H_1(M, \mathbb{R})\,,$$

by remark 4.17.

Observe that contrary to what happens with Ruelle-Sullivan cycles, we can have an immersed solenoid fully representing an homology class without the need of a transversal measure on S.

Definition 6.2. (Cluster of an immersed solenoid) *Let $f : S \to M$ be an immersion in M of an oriented 1-solenoid S. The homology cluster of (f, S), denoted by $\mathcal{C}(f, S) \subset H_1(M, \mathbb{R})$, is defined as the derived set of $([(f \circ c)_{s,t}]/(t-s))_{c, t \to \infty, s \to -\infty}$, taken over all images of orientation preserving parametrizations c of all leaves of S, and $t \to +\infty$ and $s \to -\infty$. Analogously, we define the corresponding positive and negative clusters.*

The Riemannian cluster of (f, S), denoted by $\mathcal{C}^g(f, S)$, is defined in a similar way, using arc-length orientation preserving parametrizations. Analogously, we define the positive, negative and balanced Riemannian clusters.

As in section 4, we can prove with arguments analogous to those of propositions 4.13 and 4.15 :

Proposition 6.3. *The homology clusters $\mathcal{C}(f, S)$, $\mathcal{C}_{\pm}(f, S)$ are non-empty, closed cones of $H_1(M, \mathbb{R})$. If these cones are non-degenerate, their images in $\mathbb{P}H_1(M, \mathbb{R})$ are non-empty and compact sets.*

The Riemannian homology clusters $\mathcal{C}^g(f, S)$, $\mathcal{C}^g_{\pm}(f, S)$ are non-empty, compact and connected subsets of $H_1(M, \mathbb{R})$.

The following proposition is clear, and gives the relationship with the clusters of the images by f of the leaves of S.

Proposition 6.4. *Let $f : S \to M$ be an immersion in M of an oriented 1-solenoid S. We have*

$$\bigcup_{c \subset S} \mathcal{C}(f \circ c) \subset \mathcal{C}(f, S),$$

where the union runs over all parametrizations of leaves of S. We also have

$$\bigcup_{c \subset S} \mathcal{C}_{\pm}(f \circ c) \subset \mathcal{C}_{\pm}(f, S),$$

and

$$\bigcup_{c \subset S} \mathcal{C}_b(f \circ c) \subset \mathcal{C}_b(f, S).$$

And similarly for all Riemanniann clusters with $\mathcal{C}_*(f \circ c)$ denoting the Schwartzman clusters for the arc-length parametrization.

We recall that given an immersion $f : S \to M$ of an oriented 1-solenoid, S becomes a Riemannian solenoid and theorem 3.2 gives a one-to-one correspondence between the space of transversal measures (up to scalar normalization) and the space of daval measures,

$$\overline{\mathcal{M}_{\mathcal{T}}}(S) \cong \mathcal{M}_{\mathcal{L}}(S).$$

Moreover, in the case of 1-solenoids that we consider here, they do satisfy the controlled growth condition of definition 3.3. Therefore all Schwartzman measures desintegrate as length on leaves by theorem 3.6.

Giving any transversal measure μ we can consider the associated generalized current $(f, S_\mu) \in \mathcal{C}_k(M)$.

Definition 6.5. *We define the Ruelle-Sullivan map*

$$\Psi : \mathcal{M}_{\mathcal{T}}(S) \to H_1(M, \mathbb{R})$$

by

$$\mu \mapsto \Psi(\mu) = [f, S_\mu].$$

The Ruelle-Sullivan cluster cone of (f, S) is the image of Ψ

$$\mathcal{C}_{RS}(f, S) = \Psi(\mathcal{M}_{\mathcal{T}}(S)) = \{[f, S_\mu]; \mu \in \mathcal{M}_{\mathcal{T}}(S)\} \subset H_1(M, \mathbb{R}).$$

The Ruelle-Sullivan cluster set is

$$\mathbb{P}\mathcal{C}_{RS}(f, S) \cong \{[f, S_\mu]; \mu \in \mathcal{M}_{\mathcal{L}}(S)\} \subset H_1(M, \mathbb{R}),$$

i.e. using transversal measures which are normalized (using the Riemannian metric of M).

Proposition 6.6. *Let $\mathcal{V}_T(S)$ be the set of all signed measures, with finite absolute measure and invariant by holonomy, on the solenoid S. The Ruelle-Sullivan map Ψ extended by linearity to $\mathcal{V}_T(S)$ is a linear continuous operator,*

$$\Psi : \mathcal{V}_T(S) \to H_1(M, \mathbb{R}).$$

Proof. Coming back to the definition of generalized current, it is clear that $\mu \mapsto [f, S_\mu]$ is linear in flow-boxes, therefore globally. It is also continuous because if $\mu_n \to \mu$, then $[f, S_{\mu_n}] \to [f, S_\mu]$ as can be seen in a fixed flow-box covering of S. □

Corollary 6.7. *The Ruelle-Sullivan cluster $\mathcal{C}_{RS}(f, S)$ is a non-empty, convex, compact cone of $H_1(M, \mathbb{R})$. Extremal points of the convex set $\mathcal{C}_{RS}(f, S)$ come from the generalized currents of ergodic measures in $\mathcal{M}_\mathcal{L}(S)$.*

Proof. Since $\mathcal{M}_\mathcal{L}(S)$ is non-empty, convex and compact set, its image by the continuous linear map Ψ is also a non-empty, convex and compact set. Any extremal point of $\mathcal{C}_{RS}(f, S)$ must have an extremal point of $\mathcal{M}_\mathcal{L}(S)$ in its pre-image, and these are the ergodic measures in $\mathcal{M}_\mathcal{L}(S)$ (according to the identification of $\mathcal{M}_\mathcal{L}(S)$ to $\overline{\mathcal{M}_T}(S)$ and by proposition 5.11 in [1]). □

It is natural to investigate the relation between the Schwartzman cluster and the Ruelle-Sullivan cluster.

Theorem 6.8. *Let S be a 1-solenoid. For any immersion $f : S \to M$ we have*

$$\bigcup_{c \subset S} \mathcal{C}(f \circ c) \subset \mathcal{C}_{RS}(f, S).$$

Proof. It is enough to prove the theorem for minimal solenoids, since each leaf $c \subset S$ is contained in a minimal solenoid $S_0 \subset S$, and

$$\mathcal{C}(f \circ c) \subset \mathcal{C}_{RS}(f, S_0) \subset \mathcal{C}_{RS}(f, S).$$

The last inclusion holds because if μ is a transversal measure for S_0, then it defines a transversal measure μ' for S, which is clearly invariant by holonomy. Now the generalized currents coincide, $(f, S_{\mu'}) = (f, S_{0,\mu})$, as can be seen by in a fixed flow-box covering of S. Therefore, the Ruelle-Sullivan homology classes are the same, $[f, S_{\mu'}] = [f, S_{0,\mu}]$.

The statement for minimal solenoids follows from theorem 6.9 below. □

Theorem 6.9. *Let S be a minimal 1-solenoid. For any immersion $f : S \to M$ we have*

$$\mathcal{C}(f, S) \subset \mathcal{C}_{RS}(f, S).$$

Proof. Consider an element $a \in \mathcal{C}(f, S)$ obtained as limit of a sequence $([(f \circ c_n)_{s_n, t_n}])$, where c_n is an positively oriented parametrized leaf of S and $s_n < t_n$, $s_n \to -\infty$, $t_n \to \infty$. The points $(c_n(t_n))$ must accumulate a point $x \in S$, and taking a subsequence, we can assume they converge to it. Choose a small local transversal T of S at this point, such that $f(T) \subset B$ where $B \subset M$ is a contractible ball in M. By minimality, the return map $R_T : T \to T$ is well defined.

Note that we may assume that $\bar{T} \subset T'$, where T' is also a local transversal. By compactness of $\bar{T}$, the return time for $R_{T'} : T' \to T'$ of any leaf, measured with the arc-length parametrization, for any $x \in \bar{T}$, is universally bounded. Therefore we can adjust the sequences (s_n) and (t_n) such that $c_n(s_n) \in \bar{T}$ and $c_n(t_n) \in \bar{T}$, by changing each term by an amount $O(1)$. Now, after further taking a subsequence, we can arrange that $c_n(s_n), c_n(t_n) \in T$.

Taking again a subsequence if necessary we can assume that we have a Schwartzman limit of the measures μ_n which correspond to the arc-length on $c_n([s_n, t_n])$ normalized with total mass 1. The limit measure μ desintegrates on leaves because of theorem 3.6, so it defines a trasnversal measure μ.

The transversal measures corresponding to μ_n are atomic, supported on $T \cap c_n([s_n, t_n])$, assigning the weight $l([x, R_T(x)])$ to each point in $T \cap c_n([s_n, t_n])$. The transversal measure corresponding to μ is its normalized limit. For each 1-cohomology class, we may choose a closed 1-form ω representing it and vanishing on B (this is so because $H^1(M, B) = H^1(M)$, since B is contractible). Assume that we have constructed $[(f \circ c_n)_{s_n, t_n}]$ by using γ_{n, s_n, t_n} inside B. So

$$\langle [f, S_{\mu_n}], \omega \rangle = \int_S f^* \omega \, d\mu_n = \int_{f \circ c_n([s_n, t_n])} \omega = \langle [(f \circ c_n)_{s_n, t_n}], [\omega] \rangle \,,$$

thus

$$\langle [f, S_\mu], [\omega] \rangle = \lim_{n \to \infty} \frac{1}{t_n - s_n} \langle [f, S_{\mu_n}], \omega \rangle = \lim_{n \to \infty} \langle \frac{[(f \circ c_n)_{s_n, t_n}]}{t_n - s_n}, [\omega] \rangle = \langle a, [\omega] \rangle \,.$$

Thus the generalized current of the limit measure coincides with the Schwartzman limit.   □

We use the notation $\partial^* C$ for the extremal points of a compact convex set C. For the converse result, we have:

Theorem 6.10. *Let S be a minimal solenoid and an immersion $f : S \to M$. We have*

$$\partial^* \mathcal{C}_{RS}(f, S) \subset \bigcup_{c \subset S} \mathcal{C}(f \circ c) \subset \mathcal{C}(f, S) \,.$$

Proof. We have seen that the points in $\partial^* \mathcal{C}_{RS}(f, S)$ come from ergodic measures in $\mathcal{M}_{\mathcal{L}}(S)$ by the Ruelle-Sullivan map. Therefore it is enough to prove the following theorem that shows that the Schwartzman cluster of almost all leaves is reduced to the generalized current for an ergodic 1-solenoid.   □

Theorem 6.11. *Let S be a minimal 1-solenoid endowed with an ergodic measure $\mu \in \mathcal{M}_{\mathcal{L}}(S)$. Consider an immersion $f : S \to M$. Then for μ-almost all leaves $c \subset S$ we have that $f \circ c$ is a Schwartzman asymptotic 1-cycle and*

$$[f \circ c] = [f, S_\mu] \in H_1(M, \mathbb{R}) \,.$$

Therefore the immersion $f : S_\mu \to M$ represents its Ruelle-Sullivan homology class.

In particular, this homology class is independent of the metric g on M up to a scalar factor.

Proof. The proof is an application of Birkhoff's ergodic theorem. Choose a small local transversal T such that $f(T) \subset B$, where $B \subset M$ is a small contractible ball. Consider the associated Poincaré first return map $R_T : T \to T$. Denote by μ_T the transversal measure supported on T.

For each $x \in T$ we consider $\varphi_T(x)$ to be the homology class in M of the loop image by f of the leaf $[x, R_T(x)]$ closed by a segment in B joining x with $R_T(x)$. In this way we have defined a measurable map

$$\varphi_T : T \to H_1(M, \mathbb{Z}) \,.$$

Also for $x \in S$, we denote by $l_T(x)$ the length of the leaf joining x with its first impact on T (which is $R_T(x)$ for $x \in T$). We have then an upper semi-continuous map

$$l_T : S \to \mathbb{R}_+ \,.$$

Therefore l_T is bounded by compactness of S. In particular, l_T is bounded on T and thefore in $L^1(T, \mu_T)$. The boundedness of l_T implies also the boundedness of φ_T by lemma 4.1.

Consider $x_0 \in T$ and its return points $x_i = R_T^i(x_0)$. Let $0 < t_1 < t_2 < t_3 < \ldots$ be the times of return for the positive arc-length parametrization. We have

$$t_{i+1} - t_i = l_T(x_i).$$

Therefore

$$t_n = \sum_{i=0}^{n-1}(t_{i+1} - t_i) = \sum_{i=0}^{n-1} l_T \circ R_T^i(x_0),$$

and by Birkhoff's ergodic theorem

$$\lim_{n \to +\infty} \frac{1}{n} t_n = \int_T l_T(x) \, d\mu_T(x) = \mu(S) = 1.$$

Now observe that, by contracting B, we have

$$[f \circ c_{0,t_n}] = [f \circ c_{0,t_1}] + [f \circ c_{t_1,t_2}] + \ldots + [f \circ c_{t_{n-1},t_n}]$$
$$= \varphi_T(x_0) + \varphi_T \circ R_T(x_0) + \ldots + \varphi_T \circ R_T^{n-1}(x_0).$$

We recognize a Birkhoff's sum and by Birkhoff's ergodic theorem we get the limit

$$\lim_{n \to +\infty} \frac{1}{n}[f \circ c_{0,t_n}] = \int_T \varphi_T(x) \, d\mu_T(x) \in H_1(M, \mathbb{R}).$$

Finally, putting these results together,

$$\lim_{n \to +\infty} \frac{1}{t_n}[f \circ c_{0,t_n}] = \lim_{n \to +\infty} \frac{[f \circ c_{0,t_n}]/n}{t_n/n} = \frac{\int_T \varphi_T(x) \, d\mu_T(x)}{\int_T l_T(x) \, d\mu_T(x)} = \int_T \varphi_T(x) \, d\mu_T(x).$$

Let us see that this equals the generalized current. Take a closed 1-form $\omega \in \Omega^1(M)$, which we can assume to vanish on B. Then

$$\langle [f, S_\mu], \omega \rangle = \int_T \left(\int_{[x, R_T(x)]} f^* \omega \right) d\mu_T(x) = \int_T \langle \varphi_T(x), \omega \rangle d\mu_T(x),$$

and so

$$[f, S_\mu] = \int_T \varphi_T(x) \, d\mu_T(x).$$

Observe that so far we have only proved that $\mathcal{C}_+^g(f \circ c) = \{[f, S_\mu]\}$ for almost all leaves $c \subset S$. Considering the reverse orientation, the result follows for the negative clusters, and finally for the whole cluster of almost all leaves.

The last statement follows since $[f, S_\mu]$ only depends on $\mu \in \mathcal{M}_T(S)$, which is independent of the metric up to scalar factor, thanks to the isomorphism of theorem 3.2. $\qquad \square$

Therefore for a minimal oriented ergodic 1-solenoid, the generalized current coincides with the Schwartzman asymptotic homology class of almost all leaves. It is natural to ask when this holds for all leaves, i.e. when the solenoid fully represents the generalized current. This indeed happens when the solenoid S is uniquely ergodic (unique ergodicity for a 1-solenoid implies that all orbits are dense and therefore minimality, by proposition 5.8 in [1]).

Theorem 6.12. *Let S be a uniquely ergodic oriented 1-solenoid, and let $\mathcal{M}_\mathcal{L}(S) = \{\mu\}$. Let $f : S \to M$ be an immersion. Then for each leaf $c \subset S$ we have that $f \circ c$ is a Schwartzman asymptotic cycle with*

$$[f \circ c] = [f, S_\mu] \in H_1(M, \mathbb{R}),$$

and we have

$$\mathcal{C}^g(f \circ c) = \mathcal{C}^g(f, S) = \mathbb{P}\mathcal{C}_{RS}(f, S) = \{[f, S_\mu]\} \subset H_1(M, \mathbb{R}).$$

Therefore $f : S \to M$ fully represents its Ruelle-Sullivan homology class $[f, S_\mu]$.

7. SCHWARTZMAN k-DIMENSIONAL CYCLES

We study in this section how to extend Schwartzman theory to k-dimensional submanifolds of M. We assume that M is a compact C^∞ Riemannian manifold.

Given an immersion $c : N \to M$ from an oriented smooth manifold N of dimension $k \geq 1$, it is natural to consider exhaustions (U_n) of N with $U_n \subset N$ being k-dimensional compact submanifolds with boundary ∂U_n. We close U_n with a k-dimensional oriented manifold Γ_n with boundary $\partial \Gamma_n = -\partial U_n$ (that is, ∂U_n with opposite orientation, so that $N_n = U_n \cup \Gamma_n$ is a k-dimensional compact oriented manifold without boundary), in such a way that $c_{|U_n}$ extends to a piecewise smooth map $c_n : N_n \to M$. We may consider the associated homology class $[c_n(N_n)] \in H_k(M, \mathbb{Z})$. By analogy with section 4, we consider

$$(9) \qquad \frac{1}{t_n}[c_n(N_n)] \in H_k(M, \mathbb{R}),$$

for increasing sequences (t_n), $t_n > 0$, and $t_n \to +\infty$, and look for sufficient conditions for (9) to have limits in $H_k(M, \mathbb{R})$. Lemma 4.1 extends to higher dimension to show that, as long as we keep control of the k-volume of $c_n(\Gamma_n)$, the limit is independent of the closing procedure.

Lemma 7.1. *Let (Γ_n) be a sequence of closed (i.e. compact without boundary) oriented k-dimensional manifolds with piecewise smooth maps $c_n : \Gamma_n \to M$, and let (t_n) be a sequence with $t_n > 0$ and $t_n \to +\infty$. If*

$$\lim_{n \to +\infty} \frac{\mathrm{Vol}_k(c_n(\Gamma_n))}{t_n} = 0,$$

then in $H_k(M, \mathbb{R})$ we have

$$\lim_{n \to +\infty} \frac{[c_n(\Gamma_n)]}{t_n} = 0.$$

The proof follows the same lines as the proof of lemma 4.1. We define now k-dimensional Schwartzman asymptotic cycles.

Definition 7.2. (Schwartzman asymptotic k-cycles and clusters) *Let $c : N \to M$ be an immersion from a k-dimensional oriented manifold N into M. For all increasing sequences (t_n), $t_n \to +\infty$, and exhaustions (U_n) of N by k-dimensional compact submanifolds with boundary, we consider all possible Schwartzman limits*

$$\lim_{n \to +\infty} \frac{[c_n(N_n)]}{t_n} \in H_k(M, \mathbb{R}),$$

where $N_n = U_n \cup \Gamma_n$ is a closed oriented manifold with

$$(10) \qquad \frac{\mathrm{Vol}_k(c_n(\Gamma_n))}{t_n} \to 0.$$

Each such limit is called a Schwartzman asymptotic k-cycle. These limits form the Schwartzman cluster $\mathcal{C}(c, N) \subset H_k(M, \mathbb{R})$ of N.

Observe that a Schwartzman limit does not depend on the choice of the sequence (Γ_n), as long as it satisfies (10). Note that this condition is independent of the particular Riemannian metric chosen for M.

As in dimension 1 we have

Proposition 7.3. *The Schwartzman cluster $\mathcal{C}(c, N)$ is a closed cone of $H_k(M, \mathbb{R})$.*

The Riemannian structure on M induces a Riemannian structure on N by pulling back by c. We define the Riemannian exhaustions (U_n) of N as exhaustions of the form

$$U_n = \bar{B}(x_0, R_n),$$

i.e. the U_n are Riemannian (closed) balls in N centered at a base point $x_0 \in N$ and $R_n \to +\infty$. If the R_n are generic, then the boundary of U_n is smooth

We define the Riemannian Schwartzman cluster of N as follows. It plays the role of the balanced Riemannian cluster of section 4 for dimension 1.

Definition 7.4. *The Riemann-Schwartzman cluster of $c : N \to M$, $\mathcal{C}^g(c, N)$, is the set of all limits, for all Riemannian exhaustions (U_n),*

$$\lim_{n \to +\infty} \frac{1}{\mathrm{Vol}_k(c_n(N_n))}[c_n(N_n)] \in H_k(M, \mathbb{R}),$$

such that $N_n = U_n \cup \Gamma_n$ and

(11)
$$\frac{\mathrm{Vol}_k(c_n(\Gamma_n))}{\mathrm{Vol}_k(c_n(N_n))} \to 0.$$

All such limits are called Riemann-Schwartzman asymptotic k-cycles.

Definition 7.5. *The immersed manifold $c : N \to M$ represents a homology class $a \in H_k(M, \mathbb{R})$ if the Riemann-Schwartzman cluster $\mathcal{C}^g(c, N)$ contains only a,*

$$\mathcal{C}^g(c, N) = \{a\}.$$

We denote $[c, N] = a$, and call it the Schwartzman homology class of (c, N).

Now we can define the notion of representation of homology classes by immersed solenoids extending definition 6.1 to higher dimension.

Definition 7.6. (Schwartzman representation of homology classes) *Let $f : S \to M$ be an immersion in M of an oriented k-solenoid S. Then S is a Riemannian solenoid with the pull-back metric f^*g.*

    (1) *If S is endowed with a transversal measure $\mu = (\mu_T) \in \mathcal{M}_T(S)$, the immersed solenoid $f : S_\mu \to M$ represents a homology class $a \in H_1(M, \mathbb{R})$ if for (μ_T)-almost all leaves $l \subset S$, we have that (f, l) is a Riemann-Schwartzman asymptotic k-cycle with $[f, l] = a$.*

    (2) *The immersed solenoid $f : S \to M$ fully represents a homology class $a \in H_1(M, \mathbb{R})$ if for all leaves $l \subset S$, we have that (f, l) is a Riemann-Schwartzman asymptotic k-cycle with $[f, l] = a$.*

Definition 7.7. (Equivalent exhaustions) *Two exhaustions (U_n) and $(\hat{U}_n)$ are equivalent if*

$$\frac{\mathrm{Vol}_k(U_n - \hat{U}_n) + \mathrm{Vol}_k(\hat{U}_n - U_n)}{\mathrm{Vol}_k(U_n)} \to 0.$$

Note that if two exhaustions (U_n) and $(\hat{U}_n)$ are equivalent, then

$$\frac{\mathrm{Vol}_k(\hat{U}_n)}{\mathrm{Vol}_k(U_n)} \to 1.$$

Moreover, if $N_n = U_n \cup \Gamma_n$ are closings satisfying (11), then we may close $\hat{U}_n$ as follows: after slightly modifying $\hat{U}_n$ so that U_n and $\hat{U}_n$ have boundaries intersecting transversally, we glue

$F_1 = U_n - \hat{U}_n$ to $\hat{U}_n$ along $F_1 \cap \partial \hat{U}_n$, then we glue a copy of $F_2 = \hat{U}_n - U_n$ (with reversed orientation) to $\hat{U}_n$ along $F_2 \cap \partial \hat{U}_n$. The boundary of $\hat{U}_n \cup F_1 \cup F_2$ is homeomorphic to ∂U_n, so we may glue Γ_n to it, to get $\hat{N}_n = \hat{U}_n \cup F_1 \cup F_2 \cup \Gamma_n$. Note that

$$\mathrm{Vol}_k(\hat{N}_n) = \mathrm{Vol}_k(N_n) + 2\,\mathrm{Vol}_k(\hat{U}_n - U_n) \approx \mathrm{Vol}_k(N_n)\,.$$

Define $\hat{c}_n$ by $\hat{c}_{n|F_1} = c_{|(U_n - \hat{U}_n)}$, $\hat{c}_{n|F_2} = c_{|(\hat{U}_n - U_n)}$ and $\hat{c}_{n|\Gamma_n} = c_{n|\Gamma_n}$. Then

$$[c_n(N_n)] = [\hat{c}_n(\hat{N}_n)]\,,$$

so both exhaustions define the same Schwartzman asymptotic k-cycles.

Definition 7.8. (Controlled solenoid) *Let $V \subset S$ be an open subset of a solenoid S. We say that S is controlled by V if for any Riemann exhaustion (U_n) of any leaf of S there is an equivalent exhaustion $(\hat{U}_n)$ such that for all n we have $\partial \hat{U}_n \subset V$.*

Definition 7.9. (Trapping region) *An open subset $W \subset S$ of a solenoid S is a trapping region if there exists a continuous map $\pi : S \to \mathbb{T}$ such that*

(1) *For some $0 < \epsilon_0 < 1/2$, $W = \pi^{-1}((-\epsilon_0, \epsilon_0))$.*

(2) *There is a global transversal $T \subset \pi^{-1}(\{0\})$.*

(3) *Each connected component of $\pi^{-1}(\{0\})$ intersects T in exactly one point.*

(4) *0 is a regular value for π, that is, π is smooth in a neighborhood of $\pi^{-1}(\{0\})$ and it $d\pi$ is surjective at each point of $\pi^{-1}(\{0\})$ (the differential $d\pi$ is understood leaf-wise).*

(5) *For each connected component L of $\pi^{-1}(\mathbb{T} - \{0\})$ we have $\overline{L} \cap T = \{x, y\}$, where $\{x\} \in \overline{L} \cap T \cap \pi^{-1}((-\epsilon_0, 0])$ and $\{y\} \in \overline{L} \cap T \cap \pi^{-1}([0, \epsilon_0))$. We define $R_T : T \to T$ by $R_T(x) = y$.*

Let C_x be the (unique) component of $\pi^{-1}(\{0\})$ through $x \in T$. By (4), C_x is a smooth $(k-1)$-dimensional manifold. By (5), there is no holonomy in $\pi^{-1}((-\epsilon_0, \epsilon_0))$, so C_x is a compact submanifold. Let L_x be the connected component of $\pi^{-1}(\mathbb{T} - \{0\})$ with $\overline{L}_x \cap T = \{x, y\}$. This is a compact manifold with boundary

$$\partial \overline{L}_x = C_x \cup C_y = C_x \cup C_{R_T(x)}\,. \tag{12}$$

Proposition 7.10. *If S has a trapping region W with global transversal T, then holonomy group of T is generated by the map R_T.*

Proof. If γ is a path with endpoints in T, we may homotop it so that each time it traverses $\pi^{-1}(\{0\})$, it does it through T. Then we may split γ into sub-paths such that each path has endpoints in T and no other points in $\pi^{-1}(\{0\})$. Each of this sub-paths therefore lies in some $\overline{L}_x$ and has holonomy R_T, R_T^{-1} or the identity. The result follows. $\square$

Theorem 7.11. *A solenoid S with a trapping region W is controlled by W.*

Proof. Fix a base point $y_0 \in S$ and a exhaustion (U_n) of the leaf l through y_0 of the form $U_n = \bar{B}(y_0, R_n)$, $R_n \to +\infty$. Consider $x_0 \in T$ so that $y_0 \in \overline{L}_{x_0}$. The leaf l is the infinite union

$$l = \bigcup_{n \in \mathbb{Z}} \overline{L}_{R_T^n(x_0)}\,.$$

If $R_T^n(x_0) = x_0$ for some $n \geq 1$ then l is a compact manifold. Then for some N, we have $U_N = l$, so the controlled condition of definition 7.8 is satisfied for l.

Assume that $R_T(x_0) \neq x_0$. Then l is a non-compact manifold. For integers $a < b$, denote

(13)
$$\hat{U}_{a,b} := \bigcup_{k=a}^{b-1} \overline{L}_{R_T^k(x_0)} \,.$$

This is a manifold with boundary

$$\partial \hat{U}_{a,b} = C_{R_T^a(x_0)} \cup C_{R_T^b(x_0)} \,.$$

Given U_n, pick the maximum $b \geq 1$ and minimum $a \leq 0$ such that $\hat{U}_{a,b} \subset U_n$, and denote $\hat{U}_n = \hat{U}_{a,b}$ for such a and b. Clearly $\partial \hat{U}_n \subset W$. Let us see that (U_n) and $(\hat{U}_n)$ are equivalent exhaustions, i.e. that

$$\frac{\mathrm{Vol}_k(U_n - \hat{U}_n)}{\mathrm{Vol}_k(U_n)} \to 0 \,.$$

Let $b' \geq 1$ the minimum and $a' \leq 0$ the maximum such that $U_n \subset \hat{U}_{a',b'}$. Let us prove that

$$\mathrm{Vol}_k(\hat{U}_{a',b'} - \hat{U}_{a,b})$$

is bounded. This clearly implies the result.

Take $y \in \overline{L}_{R_T^{b'-1}(x_0)} \cap U_n$. Then $d(y_0, y) \leq R_n$. By compactness of T, there is a lower bound $c_0 > 0$ for the distance from C_x to $C_{R_T(x)}$ in L_x, for all $x \in T$. Taking the geodesic path from y_0 to y, we see that there are points in $y_i \in \overline{L}_{R_T^{b'-i}(x_0)}$ with $d(y_0, y_i) \leq R_n - (i-2)c_0$, for $2 \leq i \leq b'$.

As $\overline{L}_{R_T^b(x_0)}$ is not totally contained in U_n, we may take $z \in \overline{L}_{R_T^b(x_0)} - U_n$, so $d(y_0, z) > R_n$. Both z and $y_{b'-b}$ are on the same leaf $\overline{L}'_{R_T^b(x_0)}$. By compactness of T, the diameter for a leaf $\overline{L}_x$ is bounded above by some $c_1 > 0$, for all $x \in T$. So

$$R_n - (b' - b - 2)c_0 \geq d(y_0, y_{b'-b}) \geq d(y_0, z) - d(y_{b'-b}, z) > R_n - c_1 \,,$$

hence

$$b' - b < \frac{c_1}{c_0} + 2 \,.$$

Analogously,

$$a - a' < \frac{c_1}{c_0} + 2 \,.$$

Again by compactness of T, the k-volumes of $\overline{L}_x$ are uniformly bounded by some $c_2 > 0$, for all $x \in T$. So

$$\mathrm{Vol}_k(\hat{U}_{a',b'} - \hat{U}_{a,b}) \leq (b' - b + a - a')c_2 < 2\left(\frac{c_1}{c_0} + 2\right)c_2 \,,$$

concluding the proof.   □

Theorem 7.12. *Let S be a minimal oriented k-solenoid endowed with a transversal ergodic measure $\mu \in \mathcal{M}_{\mathcal{L}}(S)$ and with a trapping region $W \subset S$. Consider an immersion $f : S \to M$ such that $f(W)$ is contained in a contractible ball in M. Then $f : S_\mu \to M$ represents its Ruelle-Sullivan homology class $[f, S_\mu]$, i.e. for μ_T-almost all leaves $l \subset S$,*

$$[f, l] = [f, S_\mu] \in H_k(M, \mathbb{R}) \,.$$

If S_μ is uniquely ergodic, then $f : S_\mu \to M$ fully represents its Ruelle-Sullivan homology class.

In particular, this homology class is independent of the metric g on M up to a scalar factor.

Proof. We define a map $\varphi_T : T \to H_k(M,\mathbb{Z})$ as follows: given $x \in T$, consider $f(\overline{L}_x)$. Since $\partial f(\overline{L}_x)$ is contained in a contractible ball B of M, we can close $f(L_x)$ locally as $N_x = f(\overline{L}_x) \cup \Gamma_x$ and define an homology class $\varphi_T(x) = [N_x] \in H_k(M,\mathbb{Z})$. This is independent of the choice of the closing. This map φ_T is measurable and bounded in $H_k(M,\mathbb{Z})$ since the k-volume of Γ_x may be chosen uniformly bounded. Also we can define a map $l_T : T \to \mathbb{R}_+$ by $l_T(x) = \mathrm{Vol}_k(\overline{L}_x)$. It is also a measurable and bounded map.

We have seen that every Riemann exhaustion (U_n) is equivalent to an exhaustion $(\hat{U}_n)$ with $\partial \hat{U}_n \subset W$. Note also that we can saturate the exhaustion $(\hat{U}_n)$ into $(\hat{U}_{n,m})_{n \le 0 \le m}$, with $\hat{U}_{n,m}$ defined in (13), where $\partial \hat{U}_{n,m} = C_{R_T^n(x_0)} \cup C_{R_T^m(x_0)}$, and $x_0 \in T$ is a base point. Since $f(W)$ is contained in a contractible ball B of M, we can always close $f(\hat{U}_{n,m})$, with a closing inside B, to get $N_{n,m}$ defining an homology class $[N_{n,m}] \in H_k(M,\mathbb{Z})$. Moreover we have

$$[N_{n,m}] = \sum_{i=n}^{m-1} \varphi_T(R_T^i(x_0)) \,.$$

Thus by ergodicity of μ and Birkhoff's ergodic theorem, we have that for μ_T-almost all $x_0 \in T$,

$$\frac{1}{m-n}[N_{n,m}] \to \int_T \varphi_T \; d\mu_T \,.$$

Also

$$\mathrm{Vol}_k(\hat{U}_{n,m}) = \sum_{i=n}^{m-1} l_T(R_T^i(x_0)) \,,$$

where $\mathrm{Vol}_k(N_{n,m})$ differs from $\mathrm{Vol}_k(\hat{U}_{n,m})$ by a bounded quantity due to the closings. By Birkhoff's ergodic theorem, for μ_T-almost all $x_0 \in T$,

$$\frac{1}{m-n} \mathrm{Vol}_k f(\hat{U}_{n,m}) \to \int_T l_T \; d\mu_T = \mu(S) = 1 \,.$$

Thus we conclude that for μ_T-almost $x_0 \in T$,

$$\frac{1}{\mathrm{Vol}_k(N_{n,m})}[N_{n,m}] \to \int_T \varphi_T \; d\mu_T \,,$$

It is easy to see as in theorem 6.11 that $\int_T \varphi_T \; d\mu_T$ is the Rulle-Sullivan homology class $[f,S_\mu]$. $\qquad \square$

Actually, when $f : S \to M$ is an immersed oriented uniquely ergodic k-solenoid with a trapping region which is mapped to a contractible ball in M, we may prove that $f : S_\mu \to M$ fully represents the Ruelle-Sullivan homology class $[f,S_\mu]$ by checking that the exhaustion $\hat{U}_n$ satisfies the controlled growth condition (see definition 3.3) and using corollary 3.7 which guarantees that the normalized measures μ_n supported on $\hat{U}_n$ converge to the unique Schwartzman limit μ.

Appendix. NORM ON THE HOMOLOGY

Let M be a compact C^∞ Riemannian manifold. For each $a \in H_1(M,\mathbb{Z})$ we define

$$l(a) = \inf_{[\gamma]=a} l(\gamma) \,,$$

where γ runs over all closed loops in M with homology class a and $l(\gamma)$ is the length of γ,

$$l(\gamma) = \int_\gamma \; ds_g \,.$$

By application of Ascoli-Arzela it is classical to get

Proposition A.1. *For each $a \in H_1(M, \mathbb{Z})$ there exists a minimizing geodesic loop γ with $[\gamma] = a$ such that*

$$l(\gamma) = l(a) \,.$$

Note that the minimizing property implies the geodesic character of the loop. We also have

Proposition A.2. *There exists a universal constant $C_0 = C_0(M) > 0$ only depending on M, such that for $a, b \in H_1(M, \mathbb{Z})$ and $n \in \mathbb{Z}$, we have*

$$l(n \cdot a) \le |n| \, l(a) \,,$$

and

$$l(a + b) \le l(a) + l(b) + C_0 \,.$$

(We can take for C_0 twice the diameter of M.)

Proof. Given a loop γ, the loop $n\gamma$ obtained from γ running through it n times (in the direction compatible the sign of n) satisfies

$$[n\gamma] = n \, [\gamma],$$

and

$$l(n\gamma) = |n| \, l(\gamma) \,.$$

Therefore

$$l(n \cdot a) \le l(n\gamma) = |n| \, l(\gamma) \,,$$

and we get the first inequality taking the infimum over γ.

Let C_0 be twice the diameter of M. Any two points of M can be joined by an arc of length smaller than or equal to $C_0/2$. Given two loops α and β with $[\alpha] = a$ and $[\beta] = b$, we can construct a loop γ with $[\gamma] = a + b$ by picking a point in α and another point in β and joining them by a minimizing arc which pastes together α and β running through it back and forth. This new loop satisfies

$$l(\gamma) = l(\alpha) + l(\beta) + C_0 \,,$$

therefore

$$l(a + b) \le l(\alpha) + l(\beta) + C_0 \,.$$

and the second inequality follows. $\square$

Remark A.3. It is not true that $l(n \cdot a) = n \, l(\gamma)$ if $l(a) = l(\gamma)$. To see this take a surface M of genus $g \ge 2$ and two elements $e_1, e_2 \in H_1(M, \mathbb{Z})$ such that

$$l(e_1) + l(e_2) < l(e_1 + e_2) \,.$$

(For instance we can take M to be the connected sum of a large sphere with two small 2-tori at antipodal points, and let e_1, e_2 be simple closed curves, non-trivial in homology, inside each of the two tori.) Let $a = e_1 + e_2$. Then

$$l(n \cdot a) = l(n \cdot (e_1 + e_2)) \le n \, l(e_1) + n \, l(e_2) + C_0 \,,$$

we get for n large

$$l(n \cdot a) < n \, l(a) \,.$$

Theorem A.4. (Norm in homology) *Let $a \in H_1(M, \mathbb{Z})$. The limit*

$$||a|| = \lim_{n \to +\infty} \frac{l(n \cdot a)}{n} \,,$$

exists and is finite. It satisfies the properties

(i) *For $a \in H_1(M, \mathbb{Z})$, we have $||a|| = 0$ if and only if a is torsion.*

(ii) *For $a \in H_1(M, \mathbb{Z})$ and $n \in \mathbb{Z}$, we have $||n \cdot a|| = |n| \, ||a||$.*

(iii) *For $a, b \in H_1(M, \mathbb{Z})$, we have*

$$||a + b|| \leq ||a|| + ||b|| \, .$$

(iv) $||a|| \leq l(a)$.

Proof. Let $u_n = l(n \cdot a) + C_0$. By the properties proved before, the sequence (u_n) is sub-additive

$$u_{n+m} \leq u_n + u_m \, ,$$

therefore

$$\limsup_{n \to +\infty} \frac{u_n}{n} = \liminf_{n \to +\infty} \frac{u_n}{n} \, .$$

Moreover, we have also

$$\frac{u_n}{n} \leq l(a) < +\infty \, ,$$

thus the limit exists and is finite. Property (iv) holds.

Property (ii) follows from

$$||n \cdot a|| = \lim_{m \to \infty} \frac{l(mn \cdot a)}{m} = |n| \lim_{m \to \infty} \frac{l(m|n| \cdot a)}{m|n|} = |n| \, ||a|| \, .$$

Property (iii) follows from

$$l(n \cdot (a + b)) \leq l(n \cdot a) + l(n \cdot b) + C_0 \leq n \, l(a) + n \, l(b) + C_0 \, ,$$

dividing by n and passing to the limit.

Let us check property (i). If a is torsion then $n \cdot a = 0$, so $||a|| = \frac{1}{n}||n \cdot a|| = 0$. If a is not torsion, then there exists a smooth map $\phi : M \to S^1$ which corresponds to an element $[\phi] \in H^1(M, \mathbb{Z})$ with $m = \langle [\phi], a \rangle > 0$. Then for any loop $\gamma : [0, 1] \to M$ representing $n \cdot a$, $n > 0$, we take $\phi \circ \gamma$ and lift it to a map $\tilde{\gamma} : [0, 1] \to \mathbb{R}$. Thus

$$\tilde{\gamma}(1) - \tilde{\gamma}(0) = \langle [\phi], n \cdot a \rangle = m \, n \, .$$

Now let C be an upper bound for $|d\phi|$. Then

$$m \, n = |\tilde{\gamma}(1) - \tilde{\gamma}(0)| = l(\phi \circ \gamma) \leq C \, l(\gamma) \, ,$$

so $l(\gamma) \geq m \, n / C$, hence $l(n \cdot a) \geq m \, n / C$ and $||a|| \geq m/C$. $\qquad\square$

Now we can define a norm in $H_1(M, \mathbb{Q}) = \mathbb{Q} \otimes H_1(M, \mathbb{Z})$ by

$$||\lambda \otimes a|| = |\lambda| \cdot ||a|| \, ,$$

and extend it by continuity to $H_1(M, \mathbb{R}) = \mathbb{R} \otimes H_1(M, \mathbb{Z})$.

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