Normality and Paracompactness in Fuzzy Topological Spaces

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Abstract

The importance of paracompactness (and the concepts related to it) in the field of Topology is well known.

In this paper we obtain two characterizations of normal fuzzy topological spaces using Luo's and Abd El-Monsef and others' paracompact fuzzy topological spaces. Thus, it shows that normality of fuzzy topological spaces can be considered as a paracompact-type property with several kinds of paracompact fuzzy topological spaces. Indeed, we prove that for a fuzzy Hausdorff fuzzy topological space (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness) there is a characterization using Luo's paracompact fuzzy topological spaces, and also another result with a characterization using definition due to Abd El-Monsef, Zeyada, El-Deeb, and Hanafy. This supposes a stimulus for further investigations, tending to obtain characterizations of other fuzzy separation properties (for example fuzzy complete regularity) as fuzzy covering properties.

Keywords: Topology, Fuzzy sets, Separation properties, Covering properties, Fuzzy paracompactness

1 Introduction

Paracompactness is a concept of great importance in general topology. It also has very interesting applications in fields such as differential geometry (and topology), mathematical analysis,..

Obviously, it is considered a "covering property".

As with many other concepts in ordinary topology, when moving to the fuzzy category there are several possible generalizations. Some are "good extensions" (i.e., the functors passing from the category of topological spaces to the category of fuzzy topological spaces, verify that the fuzzy space satisfies the fuzzy extension of a certain property if and only if, the ordinary topological space satisfies it), and other no.

In general topology, the concepts of covering, refinement, and local finiteness are essential to the concept of paracompactness. Each one of these three concepts gives rise to different extensions in the fuzzy category.

Normality is a very useful separation property of topological spaces. Some authors obtained characterizations of normality as a covering property.

The concept of normality of general topology, analogously gives rise to different extensions in the fuzzy category, some are "good extensions" and others are not. In this paper we work on fuzziness, obtaining two characterizations of fuzzy normality as a fuzzy covering property. Indeed, we show that one can characterize fuzzy normality as a paracompact-type fuzzy property in two senses.

Indeed, the originality of these investigations consists in the fact that it is possible to characterize a fuzzy property of separation as (several) covering properties. This supose a new view of separation properties of fuzzy topological spaces, displayed as covering properties.

There exist various different definitions of paracompactness of fuzzy topological spaces due to Luo [5], and Abd El-Monsef, Zeyada, El-Deeb, and Hanafy [2], On the other hand, various authors (Abdelhay [1] and Boyte [3]) obtained a characterization of normality as a covering property of Hausdorff topological spaces. A notion of a Hausdorff fuzzy property that is a good extension of classical topological property can be seen in [8]. Many other definitions and results on topological fuzziness can be found in the book [4].

This paper continues our previous research on characterizations of fuzzy regularity as a paracompact-type fuzzy property in two senses ([6], [7]).

2 Definitions and main results

Definition 1 [5] *Let* μ *be a set in fts* (X,τ) *and let* $r\in(0,1]$, $s\in[0,1)$*; we define*

$\mu_{[r]}=\chi_{\{x\in X:\mu(x)\geqq r\}}$
$\mu_{(s)}=\chi_{\{x\in X:\mu(x)>s\}}$
$\mu_{<r>}=r\mu_{[r]}$

Definition 2 [5] *Let* $\mathcal{A}$ *be a family of sets and* μ *be a set in fts* (X,τ)*. We say that* $\mathcal{A}$ *is locally finite (resp.* $*-$*locally finite) in* μ *if for each point e in* μ*, there exists a* $\nu\in\mathcal{Q}(e)$ *such that* ν *is quasi-coincident (resp. intersects) with at most a finite number of sets of* $\mathcal{A}$ *; we often omit the word "in* ν*" when* $\nu=X$*.*

Definition 3 [5] *Let be* (X,τ) *a fts, a family of sets, and* μ *be a set in fts* (X,τ)*. We say that* $\mathcal{A}$ *is locally finite (resp.* $*-$*locally finite) with respect to (w.r.t., in brief)* μ *if there exists a open set* ν*, such that* $\mu\leq\nu$*,and is* ν *is quasi-coincident (resp. intersects) with at most a finite number of sets of* $\mathcal{A}$

Definition 4 [5] *A family of sets* $\mathcal{A}$ *is called a Q-cover of a set* μ *if for each* $x\in supp(\mu)$*, there exist an* $\nu\in\mathcal{A}$ *such that* ν *and* μ *are quasi-coincident at x. Let* $r\in(0,1]$*.* $\mathcal{A}$ *is called a* $r-Q-$*cover of* μ *if* $\mathcal{A}$ *is a Q-cover of* $\mu_{<r>}$*.*

Definition 5 [5] *Let $r \in (0,1]$, μ be a set in a fts (X,τ). We say that μ is r-paracompact (resp. r^*-paracompact) if for each $r-$open Q-cover of μ there exists an open refinement of it which is both locally finite (resp.$*$-locally finite) in μ and $r-Q-$cover of μ. The fuzzy set μ is called S-paracompact (resp. S^*-paracompact) if for every $r \in (0,1]$, μ is $r-$paracompact (resp. r^*-paracompact).*

Definition 6 [2] *A family of fuzzy sets $\mathcal{U}$ is called an $L-$cover of a fuzzy set μ if $\vee\{\upsilon \mid \upsilon \in \mathcal{U}\} \geqq \mu$.*

Definition 7 [2] *Let μ be a fuzzy set in a fts (X,τ). We say that μ is fuzzy paracompact (resp. $*-$fuzzy paracompact) if for each open $L-$cover $\mathcal{B}$ of μ and for each $\xi \in (0,1]$, there exists an open refinement $\mathcal{B}^*$ of $\mathcal{B}$ wich is both locally finite (resp. $*-$locally finite) in μ and $L-$cover of $\mu - \xi$. We say that a fts (X,τ) is fuzzy paracompact (resp. $*-$fuzzy paracompact) if each constant set in X is fuzzy paracompact (resp. $*-$fuzzy paracompact).*

Theorem 8 *Let (X,τ) a fuzzy Hausdorff fts (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness). Then (X,τ) is fuzzy normal if and only if for each $r \in (0,1]$, for each $r-$open Q-cover $\mathcal{U}$ of (X,τ), for each fuzzy closed ν of (X,τ), and each $\mu \in \mathcal{U}$ such that $\nu \leq \mu$, there exists an open refinement of $\mathcal{U}$ which is both $*$-locally finite w.r.t. ν and a $r-Q-$cover of (X,τ).*

Proof. ($\Rightarrow$) *For each $r \in (0,1]$, let $\mathcal{U}$ be a $r-$open Q-cover of (X,τ), a fuzzy closed ν of (X,τ),and an open fuzzy set $\mu \in \mathcal{U}$, such that $\nu \leq \mu$. Then, we have that the family of crisp sets $\{U_{(1-r)} \mid U \in \mathcal{U}\}$ is an open cover of $(X,[\tau])$, which is Hausdorff and normal ([8],[4]).Then ([1],[3]), it has an open refinement $\mathcal{V}_\nu \subset [\tau]$ which is a cover of X, and is locally finite w.r.t . $\nu^{-1}([1-a,1])$, because to be ν closed fuzzy is equivalent to be ν' open fuzzy, then if*

$0 \lneqq a \lneqq r$, we have that $\{x \in X \mid \nu'(x) > a\} = \{x \in X \mid 1-a > \nu(x)\}$ is open,

$\{x \in X \mid \nu(x) \geqq 1-a\} = \nu^{-1}([1-a,1])$ is closed, and $\mu(x) \geqq \nu(x) \geqq 1-a > 1-r$, that implies $\nu^{-1}([1-a,1]) \subset \mu_{(1-r)}$. For each $V \in \mathcal{V}_\nu$ we have an $U_V \in \mathcal{U}$ with $V \subset (U_V)_{(1-r)}$.

Let $\mathcal{W}_\nu = \{\chi_V \wedge U_V \mid V \in \mathcal{V}_\nu\}$. Then $\mathcal{W}_\nu \subset \tau$, is both an open refinement of $\mathcal{U}$ and an $r-Q-$cover of (X,τ), and also is $*-$locally finite w.r.t. ν, indeed, as $\mathcal{V}_\nu$ is locally finite w.r.t. $\nu^{-1}([1-a,1])$, we have an open set G containing $\nu^{-1}([1-a,1])$, and G intersects with only finite number of members of $\mathcal{V}_\nu$. Then, since $\nu(x) \geqq 1-a$, is $x \in G$,and $\chi_G \in \mathcal{Q}(\nu)$ intersects with only a finite number of members of $\mathcal{W}_\nu$,(because $\chi_G \wedge (\chi_V \wedge U_V) \neq \varnothing$ implies that exists $x \in X$ such that $x \in G$,and $x \in V$)

($\Longleftarrow$) *Let $\mathcal{U} \subset [\tau]$ be an open cover of $(X,[\tau])$; then $\{\chi_U \mid U \in \mathcal{U}\}$ is an open $r-\mathcal{Q}$-cover of 1_X, for each $r \in (0,1]$ and, for each F closed of $(X,[\tau])$, such that $F \subset U \in \mathcal{U}$ is $r\chi_F \leq \chi_U$.Then, from the hypothesis, there exits an open refinement $\mathcal{V}_F$ which is an $r-\mathcal{Q}$-cover of 1_X and also locally finite w.r.t. $r\chi_F$. Let $\mathcal{W} = \{V_{(1-r)} \mid V \in \mathcal{V}_F\}$; then $\mathcal{W} \subset [\tau]$ is both a refinement of $\mathcal{U}$ and a cover of $(X,[\tau])$. Also,$\mathcal{W}$ is locally finite w.r.t. F. Indeed: we take an open set*

O_1,such that is quasi-coincident with $r\chi_F$ and also quasi-coincident with only a finite number of members $V_1, ..., V_n$, of $\mathcal{V}_F$. Let $(O_1)_{(r)}$, then, we have that $O_1 q r\chi_F$ implies that there is $z \in F$ with $O_1(z) + r > 1$,and $F \subset (O_1)_{(r)} \in [\tau]$. For each $V \in \mathcal{V}_F$, if $(O_1)_{(r)} \cap V_{(1-r)} \neq \varnothing$, we have a point $z \in X$, such that $O_1(z) > r$, $V(z) > l - r$, $O_1(z) + V(z) > 1$, then $O_1 qV$ and $V \in \{V_1, ..., V_n\}$. Hence the neighborhood $(O_1)_{(r)}$ of F intersects with only a finite number of members $(V_1)_{(1-r)}, ...(V_n)_{(1-r)} \in \mathcal{W}$. ■

Theorem 9 *Let (X, τ) a fuzzy Hausdorff fts (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness). Then (X, τ) is fuzzy normal if and only if for each $r \in I$, and for each open $L-$cover $\mathcal{B}$ of r, for each $\xi \in (0, 1]$, and for each fuzzy closed ν of (X, τ), such that $\nu \leq \mu$, for some $\mu \in \mathcal{B}$, there exists an open refinement $\mathcal{B}^*$ of $\mathcal{B}$ which is both $*-$locally finite w.r.t. ν and $L-$cover of $r - \xi$.*

Proof. $(\Rightarrow)$ For each $r \in I$, and for each open $L-$cover $\mathcal{B}$ of r, for each $\xi \in (0, 1]$, and for each fuzzy closed ν of (X, τ), such that $\nu \leq \mu$, for some $\mu \in \mathcal{B}$, we have that the family of crisp sets $\mathcal{U} = \{\mu^{-1}((r - \xi]) \mid \mu \in \mathcal{B}\} \subset [\tau]$ is an open cover of $(X, [\tau])$ which is Hausdorff and normal ([8],[4]). But to be ν closed fuzzy is equivalent to be ν' open fuzzy, then, as above,

$\nu'^{-1}((a, 1]) = \{x \in X \mid \nu'\ (x) > a\} = \{x \in X \mid 1 - a > \nu(x)\}$ is open, and

$\{x \in X \mid \nu\ (x) \geqq 1 - a\} = \nu^{-1}([1 - a, 1])$ is closed in $(X, [\tau])$, for $0 \leq a < 1 - r + \xi$. Then ([1],[3]), $\mathcal{U}$ has an open refinement $\mathcal{V}_{\nu,a} \subset [\tau]$ which is a cover of X and is locally finite w.r.t. $\nu^{-1}([1 - a, 1])$, (because $\nu^{-1}([1 - a, 1]) \subset \mu^{-1}(r - \xi]))$.

For each $V \in \mathcal{V}_{\nu,a}$, there exists $\mu_V \in \mathcal{B}$, such that $V \subset \mu_V^{-1}((r - \xi, 1])$. So $\mathcal{W}_{\nu,a} = \{\chi_V \wedge \mu_V \mid V \in \mathcal{V}_{\nu,a}\} \subset \tau$ is refinement of $\mathcal{B}$, $r - Q$-cover of (X, τ), and $*-$locally finite w.r.t. ν. Indeed, since $\mathcal{V}_{\nu,a}$ is locally finite w.r.t. $\nu^{-1}([1 - a, 1])$, there exists $G \in [\tau]$ such that $\nu^{-1}([1 - a, 1]) \subset G$ and G intersects with at most a finite number of members of $\mathcal{V}_{\nu,a}$. So, $\chi_G q\nu$, and χ_G intersects with a finite number of members of $\mathcal{W}_{\nu,a}$ (because for each $x \in \nu^{-1}([1 - a, 1])$, we have that $\chi_G(x) + \nu(x) \geqslant 1 + 1 - a > 1$, and $\chi_G \wedge (\chi_V \wedge \mu_V) \neq 0$, then there is $x \in X$, $x \in G$, such that $(\chi_V \wedge \mu_V)(x) \neq 0$ and $G \cap V \neq \varnothing$).

$(\Longleftarrow)$ Let $\mathcal{U} \subset [\tau]$ be an open cover of $(X, [\tau])$ and F a closed set contained in some $U \in \mathcal{U}$, then $\mathcal{B} = \{\chi_U \mid U \in \mathcal{U}\}$ is an open L-cover of (X, τ) and for each, $r \in I$ is $\vee\{\chi_U \mid U \in \mathcal{U}\} \geqq r$.For each $\xi \in (0, r]$, since $\chi_F \leq \chi_U \in \mathcal{B}$, there exists an open refinement $\mathcal{B}^*$ of $\mathcal{B}$ which is both locally finite w.r.t. χ_F and $L-$cover of $r - \xi$. This implies that $\vee\{G \mid G \in \mathcal{B}^*\} \geqq r - \xi_1$ for all $\xi_1 > \xi$.

Let $\mathcal{U}^* = \{G^{-1}((r - \xi, 1]) \mid G \in \mathcal{B}^*\}$, then $\mathcal{U}^* \subset [\tau]$ is an open refinement of $\mathcal{U}$ (indeed, for each $G^{-1}((r - \xi, 1]) \in \mathcal{U}^*$ there exists $V_G \in \mathcal{U}$ such that $G^{-1}((r - \xi, 1]) \subset V_G$). Since $\vee\{G \mid G \in \mathcal{B}^*\} \geqq r - \xi_1$ then $\mathcal{U}^*$ is an open refinement of $\mathcal{U}$. And, since $\chi_F \leq \chi_U \in \mathcal{B}$ there exists $A \in \mathcal{Q}(\chi_F)$ which intersects with only $G_1, .., G_n \in \mathcal{B}^*$.Since, there is some $x \in F$ such that $A(x) + \chi_F(x) > 1$, we have $A(x) > 0$, then $F \subset A^{-1}((0, 1]) \in [\tau]$.

If $A^{-1}((0, 1]) \cap G^{-1}((r - \xi, 1]) \neq \varnothing$, there exists some point z, such that $A(z) > 0$ and $G(z) > r - \xi$, so $A \wedge G \neq \varnothing$.Then, $F \subset A^{-1}((0, 1])$ and if $A^{-1}((0, 1])$ intersects with infinite members of $\mathcal{U}^*$, A intersects with infinite members of $\mathcal{B}^*$ .Thus $\mathcal{U}^*$ is locally finite w.r.t. F.

This yields that the Hausdorff topological space $(X,[\tau])$ is normal ([1],[3]) and (X,τ) is fuzzy normal ([4]). ■

3 Usability and applications

This a paper on fuzzy topology, therefore it is a theoretical paper. However, many papers on fuzzy mathematics are having applications to engineering, computing, artificial intelligence, medicine, and other sciences, in particular, some papers about fuzzy topology authored by ours.

In addition, topology, one of the branches of mathematics with a more theoretical aspect, has important applications in experimental sciences, computing and engineering.

4 Conclusion

In this paper, fuzzy normality is characterized as a fuzzy covering property. Future research could obtain characterization of other fuzzy separation properties (fuzzy completely regularity) as fuzzy covering properties. Obviously, these are all theoretical problems, but this is not a limitation, because many theoretical findings have further practical applications.

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