UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE ÓPTICA Y OPTOMETRÍA

TESIS DOCTORAL

Aplicaciones de los dinucleótidos para el diagnóstico y el tratamiento del ojo seco

Applications of dinucleotides for dry eye diagnosis and treatment

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas

TESIS DOCTORAL

Espacios de Sobolev de Funciones con Valores Vectoriales y Métricos

Sobolev Spaces of Vector-valued and Metric-valued Mappings

Memoria para optar al grado de doctor presentada por

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Contents

Abstract

Analysis in metric spaces emerged at the end of the 90's where various areas of Mathematics come together such as Functional Analysis, Differential Geometry, Geometric Measure Theory, Probability, PDEs or Complex Variables. One of the main purposes of this field is to understand what are the strictly necessary elements that make certain definitions and results work to develop analytical and geometric tools that can be applied in spaces that have little structure a priori, in this case, in a purely metric context.

As indicated in the expository article [17], which describes current developments in Analysis in metric spaces, some of the main lines that have been developed so far in this area have been Poincaré inequalities in metric spaces, quasi-conformal maps, non-linear Potential Theory, differentiability of Lipschitz functions, certain bi-Lipschitz immersion theorems, Fractal Dynamics or Geometric Measure Theory in metric spaces, among others.

This thesis is situated within the field of Analysis in Metric Spaces. In this context, the study of Lipschitz, Sobolev and related classes of functions are of particular relevance, as can be seen in the books [51, 56]. We address a range of problems concerning different classes of functions defined on either Euclidean domains or metric measure spaces, which take values in vector or metric spaces. The tackled issues revolve around two primary aspects: firstly, investigating the diverse properties of differentiability exhibited by the considered Lipschitz or Sobolev functions, with a suitable notion of regularity for metric-valued functions, by employing the notion of metric differentiability; and secondly, providing a comprehensive description of spaces comprising vector-valued functions of bounded variation, and examining the regularity properties of such mappings when the target space is a metric space, through an analysis of their points of non-approximate continuity.

On the other hand, the development of Geometric Measure Theory owes much to the advancements in Analysis in Metric Spaces. This branch of Mathematics focuses on understanding the intricate structure and measurement of complex sets. Pioneering mathematicians like Riemann, Weierstrass, and Lebesgue paved the way by extending fundamental concepts such as continuity and measure beyond the confines of Euclidean space. Their efforts laid the groundwork for exploring the geometry of abstract sets within general metric spaces. The emergence of Geometric Measure Theory in the mid-20th century, headed by the work of Federer and Fleming, was prompted by the imperative for a more comprehensive grasp of sets exhibiting irregular or fractal-like configurations, which eluded adequate description through classical methods of Euclidean geometry and Lebesgue measure theory. Central to Geometric Measure Theory are the concepts of Hausdorff measure, which offers a means to quantify the "size" or "dimension" of sets with irregular shapes, and rectifiable sets, which are those sets for which it is still possible to define tangent spaces (a.e.) in a very weak sense. It is worth mentioning that, in this manuscript, these two concepts in the metric context will also play a central role.

To establish the context, Chapter 1 will provide a basic introductory overview of Measure Theory and Functional Analysis, laying the groundwork for the subsequent chapters. We will also cover various key topics in Analysis in metric measure spaces.

Firstly, we will look into the basics of Measure Theory, exploring the concept of metric measure spaces. This will include discussions on the measurability of sets and Bochner integration for functions with values in a Banach space. Additionally, we will address the Radon-Nikodým Property, which plays a fundamental role in Measure Theory. Next, we will explore some essential tools for analysis in metric measure spaces. We will examine the Lebesgue differentiation theorem and its application in this context, as well as curves in metric spaces and the concept of the modulus of a family of curves whenever the metric space is equipped with a measure. We will also discuss upper gradients and Poincaré inequalities, which are crucial for understanding the structure and properties of functions in this setting.

Together, this introduction to the basics of Measure Theory, Functional Analysis, and Geometric Analysis in Chapter 1 will provide readers with a solid understanding of the fundamental concepts necessary to successfully address the topics covered in the subsequent chapters of the thesis.

In Chapter 2, we explore metric measure spaces equipped with a Cheeger differentiable structure, aiming to determine the conditions under which a concept of metric differentiability can be established for mappings with values into an arbitrary metric space. In order to properly set a notion of first order regularity in the lines of Cheeger, but in the context of metric-valued maps, we will explore metric differentiability, employing charts, within the framework of metric measure spaces. To this end, we set (X, d, μ) to be a metric measure space and (Y, d_Y) a metric space, and consider mappings $f : X \to Y$.

Cheeger proved in [23] that, if μ is doubling and X supports a Poincaré inequality, then the space admits a countable atlas $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ of Borel sets $U_i \subset X$, that cover X up to a set of measure zero, and Lipschitz maps $\varphi_i : X \to \mathbb{R}^{N_i}$ for which a Rademacher's Theorem holds for Lipschitz mappings $f : X \to \mathbb{R}$, obtaining a linear differential going through the charts φ_i. This decomposition was initially referred to as a measurable differentiable structure in early literature (see e.g. [67, 68]), but in more recent work, it has become common to refer to it as X being a Lipschitz differentiability space (LDS for short). Recent works have aimed to understand the nature of these spaces, even without assuming μ doubling and a Poincaré inequality on X (see e.g. [11, 13, 15, 64]). Under the assumption that X is an LDS, we will study a notion of metric differentiability using charts (see Definition 2.16), as we are interested in the regularity of Lipschitz maps $f : X \to Y$, where (Y, d_Y) is a metric space. Notice that f mapping into a general metric space, which lacks a linear structure, is an obstruction for a proper definition of a linear differential. This regularity notion was first considered by Kirchheim in [71] for maps $f : \mathbb{R}^N \to Y$, and its close relation with the classical notion of rectifiability was showcased there.

In Chapter 2, we will provide further details about all these definitions, as well as some well-known results, in order to generalize Rademacher's Theorem for Lipschitz mappings from a Lipschitz differentiability space X into a metric space Y. For this purpose, it will be necessary, and also sufficient, to assume that X can be decomposed into rectifiable sets. More precisely, we obtain in Theorem 2.23 that, if every Lipschitz mapping $f : X \to Y$ with values in any metric space Y is metrically differentiable almost everywhere with respect to a *weak* Lipschitz chart (U, φ), then the space $(U, d_{|U}, \mu_{|U})$ is rectifiable. As a consequence, whenever X is a Lipschitz differentiability space, then every Lipschitz mapping $f : X \to Y$ is metrically differentiable a.e. if, and only if, X admits a decomposition into rectifiable charts (see Corollary 2.24). We will also apply this Rademacher-type result in order to obtain a Stepanov Theorem in Theorem 2.25.

The main results of Chapter 2 have been collected in [18].

Transitioning to Chapter 3, our attention turns to Sobolev classes of vector-valued mappings. Originating in the 1930s by Sergei Sobolev, the classical notion of Sobolev space has evolved into an indispensable tool in the realms of partial Differential Equations and Calculus of Variations. These spaces serve as fundamental domains for numerous differential operators, supporting essential concepts in Mathematical Analysis and providing a rich framework for studying diverse phenomena where the classical notions of first order regularity might be too restrictive. Here we consider the Sobolev classes $W^{1,p}(\Omega, V)$ of mappings from an open set $\Omega \subset \mathbb{R}^N$ into a Banach space V. This setting also appears in the field of Partial Differential Equations (see, for example, [1, 29]). It also appears in the field of Probability as seen, for example, in [63].

Parallel developments in Geometric Measure Theory have offered alternative formulations of Sobolev spaces within the framework of metric measure spaces (X, d, μ). N. Shanmugalingam introduced the notion of a Newton-Sobolev space $N^{1,p}(X)$ through the concept of upper gradients [86]. Notably, this approach extends to mappings into a Banach space V, and remains stable under scalarizations up to a μ-a.e. representative, where composing with normalized functionals of V yields mappings in $N^{1,p}(X)$ with uniformly controlled energy. This leads to an equivalent definition, referred to as the Sobolev-Reshetnyak space $R^{1,p}(\Omega, V)$ in the Euclidean setting, initially proposed by Y. Reshetnyak in 1997 [85]. Remarkably, in the case of real-valued functions, $N^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$, rendering scalarization of vector-valued mappings a viable alternative for studying the Newton-Sobolev space using the classical Sobolev space. To elaborate, a mapping $f : \Omega \to V$ belongs to the class $R^{1,p}(\Omega, V)$ if $\langle v^*, f\rangle \in W^{1,p}(\Omega)$ for all $v^* \in V^*$ with $\|v^*\| \leq 1$, and there exists a non-negative function $g \in L^p(\Omega)$ such that $|\nabla\langle v^*, f\rangle| \leq g$ almost everywhere for each $v^* \in V^*$ with $\|v^*\| \leq 1$. This definition facilitates the retrieval of various useful tools from $W^{1,p}(\Omega)$ for $R^{1,p}(\Omega, V)$ through the scalarization.

Chapter 3 aims to investigate the Sobolev-Reshetnyak space $R^{1,p}(\Omega, V)$ alongside its interaction with $W^{1,p}(\Omega, V)$, where $\Omega \subset \mathbb{R}^N$ is an open set and V is a Banach space. In Theorem 3.18, we establish that $W^{1,p}(\Omega, V)$ is a closed subspace of $R^{1,p}(\Omega, V)$. Additionally, Theorem 3.22 reveals that these spaces coincide if and only if the underlying space V has the Radon-Nikodým Property. Moreover, this chapter presents various characterizations of $R^{1,p}(\Omega, V)$. Specifically, for $1 \leq p < \infty$, Theorem 3.23 demonstrates that, up to a representative, a mapping $f : \Omega \to V$ belongs to $R^{1,p}(\Omega, V)$ if and only if it lies in $L^p(\Omega, V)$ and possesses partial metric derivatives $(m\partial_i f)_{i=1}^N$ almost everywhere, in such a way that the maps $x \mapsto m\partial_i f(x)$ belong to $L^p(\Omega)$ for each $i = 1, \ldots, N$. Similarly, an analogous characterization for w^*-derivatives is described in Theorem 3.27, assuming V is the dual of a separable Banach space. Furthermore, for the case $p = \infty$, we establish in Theorem 3.21 that $f \in R^{1,\infty}(\Omega, V)$ if and only if it has a representative which exhibits uniform local Lipschitz continuity.

The main results of Chapter 3 have been collected in [20, 19].

The theory of functions of bounded variation was first developed in order to study existence and regularity properties of minimal surfaces. A nice overview can be found in the book [6], and [41] gives a nice discussion on fine properties of BV functions in Euclidean spaces. Since then, the theory has found applications in other areas as well, including image processing [5, 22], or quasiconformal mappings [70] (see also references therein). The extent of these applications also reach the context of functions of bounded variation in metric measure spaces.

Recent research on mappings of finite distortion and quasisymmetric mappings indicate a need to understand metric space-valued mappings of bounded variation on metric measure spaces, see for instance [3, 76, 57]. Motivated by this, in Chapter 4 we seek to study mappings of bounded variation in metric measure spaces of controlled geometry, that is, spaces where the measure is doubling and supports a 1-Poincaré inequality.

Unlike Sobolev functions, functions of bounded variation exhibit less regularity; classical examples include the Cantor staircase function on the Euclidean unit interval and characteristic functions of smooth Euclidean sets. However, an Euclidean set whose characteristic function is of bounded variation can have non-smooth boundary. As with characteristic functions of such sets, more general functions of bounded variation in Euclidean domains exhibit discontinuous behavior along certain subsets, called *jump sets*. The situation gets more complicated when the function of bounded variation is not real-valued but a map from a Euclidean domain into a metric space, as in [3]. Yet another layer of complication comes from considering functions of bounded variation from a metric measure space into a metric space. The goal of Chapter 4 is to explore regularity properties of such maps. In this case, the lack of smoothness implies that we have no notion of inward normal direction in the sense analogous to [43] or [88, (1.2)], and hence the classical definition of jump points as in [43, 88] (see also [6, Definition 3.67]) is not suitable here.

To begin this exploration, we first need to consider what constitutes a sensible concept of mappings with bounded variation from a metric measure space into a metric space. The idea of real-valued functions with bounded variation on metric measure spaces, particularly those endowed with a doubling measure that supports a 1-Poincaré inequality, was initially introduced by Miranda Jr. in [84]. Since then, this notion has been the subject of extensive research, and the papers [4, 8, 35, 39, 58, 60, 61, 62, 75, 78, 79] contain a small sample of the outcomes from such a study. The papers [4, 84, 8, 78, 39] consider the definition of functions of bounded variation in the metric setting via relaxation of Sobolev functions, while the papers [58, 60, 61, 62] consider functions of bounded variation as those whose local behavior is controlled by a sequence of non-negative Borel functions that serve as a substitute for upper gradients [82].

In Chapter 4, we will start by extending these definitions to accommodate vector-valued mappings $f : X \to V$ from a metric measure space (X, d, μ) into a Banach space V. We will denote by $BV(X, V)$ the space of functions of bounded variation functions using the natural energy seminorm given by an approximation by a sequence of Newton-Sobolev mappings as in [84], and by $BV_{AM}(X, V)$ the space of bounded variation using the relaxed notion of AM-modulus and a sequence of approximating upper gradients following the ideas in [82]. We establish that both definitions coincide when X is doubling, supports a Poincaré inequality, and the target space is a Banach space (see Theorem 4.15). Additionally, both definitions are applicable for mappings into a complete and separable metric space (Y, d_Y). In Section 4.4, we prove that $BV(X, Y) \subset BV_{AM}(X, Y)$, although here we provide some examples that exhibit a strict inclusion, even if X is equipped with a doubling measure and supports a 1-Poincaré inequality.

Subsequently, we opt for the notion of $BV_{AM}(X, Y)$ as our focal class to investigate the regularity properties of mappings of bounded variation with values in a metric space. In Section 4.5, we adapt the concept of a jump set $\mathcal{J}(u)$ for a mapping $u \in BV_{AM}(X, Y)$. This set corresponds to the points of non-approximate continuity of u but offers a geometric perspective better suited to our requirements. We establish that $\mathcal{J}(u)$ is σ-finite with respect to the co-dimension 1 Hausdorff measure $\mathcal{H}^{-1}$, and, when Y is proper, for $\mathcal{H}^{-1}$-almost every $x \in \mathcal{J}(u)$, the quantity of infinitesimal values in Y that approximate $u(x)$ is uniformly bounded, with the bound depending solely on the data of X and not on the selection of u. These properties are consolidated in Theorem 4.24.

The main results in Chapter 4 have been collected in [21].

Resumen

El Análisis en Espacios Métricos es un campo que surgió a finales de los años 90 donde convergen varias áreas de las Matemáticas como el Análisis Funcional, la Geometría Diferencial, la Teoría Geométrica de la Medida, la Probabilidad, las Ecuaciones en Derivadas Parciales o la Variables Compleja. Uno de los principales propósitos de este campo es comprender cuáles son los elementos estrictamente necesarios que hacen que ciertas definiciones y resultados funcionen para desarrollar herramientas analíticas y geométricas que puedan aplicarse en espacios que tienen poca estructura a priori, en este caso, en un contexto puramente métrico.

Como se indica en el artículo expositivo [17], que describe los desarrollos actuales en el Análisis en Espacios Métricos, algunas de las principales líneas que se han desarrollado hasta ahora en esta área han sido desigualdades de Poincaré en espacios métricos, funciones cuasi-conformes, Teoría del Potencial no lineal, diferenciabilidad de funciones Lipschitz, ciertos teoremas de inmersión bi-Lipschitz, Dinámica Fractal o Teoría Geométrica de la Medida en espacios métricos, entre otros.

Esta tesis se sitúa dentro del campo del Análisis en Espacios Métricos. En este contexto, el estudio de las clases de funciones Lipschitz, Sobolev y relacionadas tiene una relevancia particular, como se puede ver en los libros de [51, 56]. Abordamos una serie de problemas relacionados con diferentes clases de funciones definidas en dominios euclídeos o en espacios métricos de medida, que toman valores en espacios vectoriales o métricos. Los problemas abordados giran en torno a dos aspectos principales: en primer lugar, investigar las diversas propiedades de diferenciabilidad exhibidas por las funciones Lipschitz o Sobolev consideradas, con una noción adecuada de regularidad para funciones con valores en espacios métricos, empleando la noción de diferenciabilidad métrica; y en segundo lugar, proporcionar una descripción detallada de los espacios que comprenden funciones de variación acotada, y examinar las propiedades de regularidad de tales funciones, cuando el espacio de llegada es un espacio métrico, a través de un análisis de sus puntos de no continuidad aproximada.

Por otro lado, el desarrollo de la Teoría Geométrica de la Medida debe mucho a los avances en el Análisis en Espacios Métricos. Esta rama de las matemáticas se centra en comprender la estructura intrincada y la medida de conjuntos complejos. Matemáticos pioneros como Riemann, Weierstrass y Lebesgue allanaron el camino al extender conceptos fundamentales como la continuidad y la medida más allá de los límites del espacio euclidiano. Sus esfuerzos sentaron las bases para explorar la geometría de conjuntos abstractos dentro de espacios métricos generales. La aparición de la Teoría Geométrica de la Medida a mediados del siglo XX, encabezada por el trabajo de Federer y Fleming, fue impulsada por la necesidad de comprender de manera más integral conjuntos que exhiben configuraciones irregulares o de tipo fractal, que escapaban a una descripción adecuada mediante métodos clásicos de geometría euclidiana y Teoría de la Medida de Lebesgue. Elementos centrales de la Teoría Geométrica de la Medida son los conceptos de medida de Hausdorff, que ofrece un medio para cuantificar el ”tamaño” o “dimensión” de conjuntos con formas irregulares, y conjuntos rectificables, que son aquellos conjuntos para los cuales todavía es posible definir espacios

tangentes (en casi todo punto) en un sentido muy débil. Vale la pena mencionar que, en este manuscrito, estos dos conceptos en el contexto métrico también desempeñarán un papel central.

Para establecer el contexto, el Capítulo 1 proporcionará una introducción a los conceptos básicos de la Teoría de la Medida y el Análisis Funcional, sentando las bases para los capítulos posteriores. También cubriremos varios temas clave en el Análisis en Espacios Métricos de medida.

En primer lugar, examinaremos los conceptos básicos de la Teoría de la Medida, explorando el concepto de espacios métricos de medida. Esto incluirá discusiones sobre la medibilidad de conjuntos y la integración de Bochner para funciones con valores en un espacio de Banach. Además, abordaremos la Propiedad de Radon-Nikodým, que desempeña un papel fundamental en la Teoría de la Medida. A continuación, exploraremos algunas herramientas esenciales para el Análisis en Espacios Métricos de medida. Examinaremos el teorema de diferenciación de Lebesgue y su aplicación en este contexto, así como las curvas en espacios métricos y el concepto del módulo de una familia de curvas cuando el espacio métrico de interés esté equipado con una medida. También discutiremos los conceptos de gradientes superiores e desigualdades de Poincaré, que constituyen una herramienta esencial para la comprensión de la estructura de un espacio métrico de medida y las propiedades de funciones definidas en dichos entornos.

Con esto, el Capítulo 1 tiene como objetivo ofrecer al lector una breve introducción a los campos de la Teoría de la Medida, el Análisis Funcional y el Análisis Geométrico, con el fin de proporcionar una comprensión sólida de los conceptos fundamentales necesarios para abordar con éxito los temas tratados en los capítulos posteriores de la tesis.

En el Capítulo 2 centraremos el estudio en aquellos espacios métricos de medida que presentan una estructura diferenciable de Cheeger, con el objetivo de determinar las condiciones bajo las cuales se puede establecer un concepto de diferenciabilidad métrica para funciones que toman valores en un espacio métrico arbitrario. Para abordar esta cuestión, estudiaremos en detalle el concepto de diferenciabilidad métrica definida para funciones en espacios métricos de medida, mediante el uso de cartas. Establecemos entonces, como objetos de estudio, un espacio métrico de medida (X, d, μ), un espacio métrico (Y, d_Y), y funciones $f : X \to Y$.

Cheeger demostró en [23] que, si μ es doblante y X admite una desigualdad de Poincaré, entonces el espacio admite un atlas numerable $(U_i, \varphi_i)_{i\in\mathbb{N}}$ de conjuntos de Borel $U_i \subset X$, que cubren X salvo un conjunto de medida nula, y funciones Lipschitz $\varphi_i : X \to \mathbb{R}^{N_i}$ de manera que se cumple el Teorema de Rademacher para funciones Lipschitz $f : X \to \mathbb{R}$, obteniendo una diferencial lineal definida a través de las cartas φ_i. Esta descomposición recibió el nombre de estructura diferenciable medible en la literatura temprana (ver por ejemplo [67, 68]), pero en trabajos más recientes es más común encontrar la terminología de que X es un espacio de diferenciabilidad Lipschitz (LDS por sus siglas en inglés). Trabajos recientes han apuntado a comprender la naturaleza de estos espacios, incluso sin asumir que μ sea doblante y que X posea una desigualdad de Poincaré (ver por ejemplo [11, 13, 15, 64]). Bajo la suposición de que X es un LDS, estudiaremos una noción de diferenciabilidad métrica utilizando cartas (ver Definición 2.16), ya que estamos interesados en la regularidad de las funciones Lipschitz $f : X \to Y$, donde (Y, d_Y) es un espacio métrico, y por tanto la carencia de una estructura lineal es una obstrucción para definir adecuadamente una diferencial lineal. El concepto de diferenciabilidad métrica fue considerada por primera vez por Kirchheim en [71] para funciones $f : \mathbb{R}^N \to Y$, y su estrecha relación con la noción clásica de rectificabilidad fue destacada en dicho trabajo.

En el Capítulo 2, proporcionaremos más detalles sobre todas estas definiciones, así como algunos resultados bien conocidos, con el fin de generalizar el Teorema de Rademacher para funciones Lipschitz que van de un espacio de diferenciabilidad Lipschitz X a un espacio métrico Y. Para ello, será necesario, y también suficiente, asumir que X puede descomponerse en conjuntos rectificables. Siendo más precisos, obtendremos en el Teorema 2.23 que, si cada función Lipschitz $f : X \to Y$ con valores en cualquier espacio métrico Y es métricamente diferenciable en casi todo punto con respecto a una carta Lipschitz *débil* (U, φ), entonces el espacio $(U, d_{|U}, \mu_{|U})$ es rectificable. Como consecuencia, si X es un espacio de diferenciabilidad Lipschitz, entonces cada función Lipschitz $f : X \to Y$ es métricamente diferenciable en casi todo punto si, y solo si, X admite una descomposición en cartas rectificables (ver Corolario 2.24). También aplicaremos este resultado de tipo Rademacher para obtener un Teorema de Stepanov en el Teorema 2.25.

Los resultados principales del Capítulo 2 han sido recopilados en [18].

En el Capítulo 3, nuestra atención se centra en las clases de Sobolev de funciones con valores vectoriales. En la década de 1930 Sergei Sobolev introdujo la noción clásica del espacio de Sobolev, que ha evolucionado hasta convertirse en una herramienta indispensable en los ámbitos de las Ecuaciones en Derivadas Parciales y el Cálculo de Variaciones. Estos espacios sirven como dominios para numerosos operadores diferenciales, respaldando conceptos esenciales en el Análisis Matemático y proporcionando un contexto adecuado para el estudio de diversos fenómenos en los que los conceptos clásicos de regularidad de primer orden son demasiado restrictivos. Aquí consideramos las clases de Sobolev $W^{1,p}(\Omega, V)$ de funciones definidas en un conjunto abierto $\Omega \subset \mathbb{R}^N$ y que toman valores en un espacio de Banach V. Este contexto también aparece en el campo de las Ecuaciones en Derivadas Parciales (ver, por ejemplo, [1, 29]). También aparece en el campo de la Probabilidad, como se ve, por ejemplo, en [63].

Paralelamente, el desarrollo en el campo de la Teoría Geométrica de la Medida ha propuesto formulaciones alternativas de los espacios de Sobolev dentro del marco de funciones definidas en un espacio métrico de medida (X, d, μ). N. Shanmugalingam introdujo la noción de espacio de Newton-Sobolev $N^{1,p}(X)$ a través del concepto de gradientes superiores [86]. Además, este enfoque se extiende también a funciones con valores en un espacio de Banach V, y permanece estable, salvo quizá en un conjunto nulo, bajo escalarizaciones, de forma que al componer con funcionales normalizados de V se obtienen funciones en $N^{1,p}(X)$ con energía uniformemente acotada. Esto conduce a una definición alternativa que fue considerada por primera vez para funciones definidas en un conjunto abierto $\Omega \subset \mathbb{R}^N$ por Y. Reshetnyak en 1997 [85] y que recibe el nombre de espacio de Sobolev-Reshetnyak $R^{1,p}(\Omega, V)$. En el caso de funciones de valores reales, $N^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$, lo que hace que la escalarización de funciones con valores vectoriales sea una alternativa viable para estudiar el espacio de Newton-Sobolev usando el espacio de Sobolev clásico. Una función $f : \Omega \to V$ pertenece a la clase $R^{1,p}(\Omega, V)$ si $\langle v^*, f\rangle \in W^{1,p}(\Omega)$ para todo $v^* \in V^*$ con $\|v^*\| \leq 1$, y existe una función no negativa $g \in L^p(\Omega)$ tal que $|\nabla\langle v^*, f\rangle| \leq g$ en casi todo punto para cada $v^* \in V^*$ con $\|v^*\| \leq 1$. Esta definición facilita la recuperación, a través de la escalarización, de varias herramientas y propiedades conocidas para el espacio $W^{1,p}(\Omega)$, y esto nos permitirá abordar el estudio de las funciones en $R^{1,p}(\Omega, V)$.

El Capítulo 3 tiene como objetivo investigar el espacio de Sobolev-Reshetnyak $R^{1,p}(\Omega, V)$, así como relacionarlo con el espacio $W^{1,p}(\Omega, V)$, donde $\Omega \subset \mathbb{R}^N$ es un conjunto abierto y V es un espacio de Banach. En el Teorema 3.18 establecemos que $W^{1,p}(\Omega, V)$ es un subespacio cerrado de $R^{1,p}(\Omega, V)$. Además, el Teorema 3.22 revela que estos espacios coinciden si y solo si V tiene la Propiedad de Radon-Nikodým.

Además, este capítulo presentará varias caracterizaciones de $R^{1,p}(\Omega, V)$. Para ser más concretos, en el caso $1 \leq p < \infty$ el Teorema 3.23 demuestra que, cambiando de representante si es necesario, una función $f : \Omega \to V$ pertenece a $R^{1,p}(\Omega, V)$ si y solo si está en $L^p(\Omega, V)$ y posee derivadas parciales métricas $(m\partial_i f)_{i=1}^N$ en casi todo punto, de tal manera que las funciones $x \mapsto m\partial_i f(x)$ pertenecen a $L^p(\Omega)$ para cada $i = 1, \ldots, N$. De manera similar, una caracterización análoga se describe en el Teorema 3.27, donde usaremos el concepto de derivadas w^* cuando V sea el dual de un espacio de Banach separable. Además, para el caso $p = \infty$, establecemos en el Teorema 3.21 que $f \in R^{1,\infty}(\Omega, V)$ si y solo si admite un representante uniformemente localmente Lipschitz.

Los principales resultados del Capítulo 3 han sido recopilados en [20, 19].

La teoría de funciones de variación acotada se desarrolló inicialmente para estudiar propiedades de existencia y regularidad de superficies mínimales. Proponemos el libro [6] para una visión general del campo, así como [41], que ofrece una buena discusión sobre las propiedades finas de las funciones de variación acotada en espacios euclídeos. Desde entonces, la teoría ha encontrado aplicaciones en otras áreas, incluyendo el procesamiento de imágenes [5, 22], o funciones cuasiconformes [70] (ver también las referencias en ellos). La extensión de estas aplicaciones también alcanza el contexto de funciones de variación acotada en espacios métricos de medida.

Investigaciones recientes sobre funciones de distorsión finita y funciones cuasisimétricas indican la necesidad de comprender el comportamiento de las funciones de variación acotada en espacios métricos de medida que toman valores en un espacio métrico, ver por ejemplo [3, 76, 57]. Motivados por esto, en el Capítulo 4 buscamos estudiar funciones de variación acotada en espacios métricos de medida de geometría controlada, es decir, espacios donde la medida es doblante y admite una 1-desigualdad de Poincaré.

A diferencia de las funciones de Sobolev, las funciones de variación acotada exhiben menos regularidad; ejemplos clásicos incluyen la función escalonada de Cantor en el intervalo unitario euclidiano y funciones características de conjuntos euclidianos suaves. Sin embargo, un conjunto euclidiano cuya función característica es de variación acotada puede tener una frontera no suave. Como con las funciones características de tales conjuntos, las funciones de variación acotada más generales en dominios euclidianos exhiben comportamiento discontinuo a lo largo de ciertos subconjuntos, llamados conjuntos de *salto*. La situación se complica más cuando la función de variación acotada no es de valores reales sino una función definida en un dominio euclidiano y que toma valores en un espacio métrico, como en [3]. Otra capa de complejidad se añade si consideramos, además, que las funciones no son escalares, si no que toman valores en un espacio métrico más general. El objetivo del Capítulo 4 es explorar propiedades de regularidad de tales funciones. En este caso, se carece de la noción de dirección normal en el sentido análogo a [43] o [88, (1.2)], y por lo tanto, la definición clásica de puntos de salto como en [43, 88] (ver también [6, Definición 3.67]) no es adecuada aquí.

Para comenzar esta exploración, primero necesitamos considerar qué constituye un concepto sensato de funciones con variación acotada desde un espacio métrico de medida en un espacio métrico. La idea de funciones de valores reales con variación acotada en espacios métricos de medida, particularmente aquellos dotados con una medida doblante que admite una 1-desigualdad de Poincaré, fue introducida inicialmente por Miranda Jr. en [84]. Desde entonces, esta noción ha sido objeto de una extensa investigación, y los artículos [4, 8, 35, 39, 58, 60, 61, 62, 75, 78, 79] contienen una pequeña muestra de los resultados de dicho estudio. Los artículos [4, 84, 8, 78, 39] consideran la definición de funciones de variación acotada en el contexto métrico mediante la relajación de funciones de Sobolev, mientras que los artículos [58, 60, 61, 62] consideran funciones de variación acotada como aquellas cuyo comportamiento local está controlado por una secuencia de funciones de Borel no negativas que sirven como sustituto de los gradientes superiores [82].

En el Capítulo 4, comenzaremos extendiendo estas definiciones para el caso de funciones $f : X \to V$ definidas en un espacio métrico de medida (X, d, μ) con valores en un espacio de Banach V. Denotaremos por $BV(X, V)$ al espacio de funciones de variación acotada utilizando la seminorma de energía dada por una aproximación mediante una sucesión de funciones de Newton-Sobolev como en [84], y por $BV_{AM}(X, V)$ al espacio de funciones de variación acotada utilizando la noción relajada de AM-módulo y una sucesión de gradientes superiores aproximantes, siguiendo las ideas en [82]. Estableceremos que ambas definiciones coinciden cuando X es doblante, admite una 1-desigualdad de Poincaré, y el espacio de destino es un espacio de Banach (ver Teorema

4.15). Además, ambas definiciones son aplicables para funciones con valores en un espacio métrico completo y separable (Y, d_Y). En la Sección 4.4, demostraremos que $BV(X,Y) \subset BV_{AM}(X,Y)$, pero también destacaremos algunos ejemplos que muestran una inclusión estricta, incluso si X está equipado con una medida doblante y admite una 1-desigualdad de Poincaré.

Finalmente, optamos por la noción de $BV_{AM}(X,Y)$ como nuestra clase de interés para investigar las propiedades de regularidad de funciones de variación acotada con valores en un espacio métrico. En la Sección 4.5, adaptamos el concepto de conjunto de salto $\mathcal{J}(u)$ para una función $u \in BV_{AM}(X,Y)$. Este conjunto corresponde a los puntos de no continuidad aproximada de u, pero ofrece una perspectiva geométrica mejor adaptada a nuestros objetivos. Estableceremos que $\mathcal{J}(u)$ es σ-finito con respecto a la medida de Hausdorff de co-dimensión 1 $\mathcal{H}^{-1}$, y, cuando Y es propio, para $\mathcal{H}^{-1}$-casi todo $x \in \mathcal{J}(u)$, la cantidad de valores infinitesimales en Y que aproximan $u(x)$ está uniformemente acotada, con la cota dependiendo únicamente de la información implícita en X, y no de la elección de u. Todas estas propiedades de regularidad se consolidan en el Teorema 4.24.

Los principales resultados en el Capítulo 4 se han recopilado en [21].

Chapter 1

Preliminaries

In order to provide some background in the field of Measure Theory and Analysis in Metric Spaces we lay out in this chapter the concepts and results that are well known in the literature and will be involved throughout the different chapters of this thesis.

In Section 1.1 we first recall the general notions of measurable spaces and measures, and afterwards develop this theory for measures defined on the Borel sets of a metric space, using the topology generated by open balls. This will lead to the definition of a metric measure space which will be present in the underlying context of Chapters 2 and 4. We refer to [41] for an extensive study on general Measure Theory in the Euclidean setting, and [56, Section 3.3] for the topic developed in order to approach the theory in metric measure spaces.

As one of the main topics of this thesis is the study of vector-valued mappings, we introduce integration theory via the Bochner integral in Section 1.2. For a more comprehensive study on this topic we refer to [32].

Section 1.3 is devoted to Measure Theory of general vector measures and the Radon-Nikodým property. See [16] and [32] for more details about the field.

To end this chapter, in Section 1.4.1 we will explain some of the tools used in metric measure spaces in order to study regularity of mappings. Since the Lebesgue differentiation Theorem provides a rich theory of zero order calculus, in Section 1.4.1 we will focus on some conditions that provide the validity of this result in a metric measure space. For first order regularity, the lack of linearity in a general metric space complicates a proper definition of a differential in the classical way, and here we present one of the alternatives which involves the use of curves. For that reason we provide in Section 1.4.2 some geometric concepts (p-Modulus, upper gradients and Poincaré inequalities) that rely on the behavior of mappings along curves, and that will be relevant in Chapter 4, and though Chapter 3 is set on Euclidean domains, we will use there the more general ideas and techniques presented in this section. This topic has produced a lot of literature recently (see [17] for a brief overview about its history and related topics) and we will provide specific references whenever needed, although we propose here [52] and [56] for further study.

1.1 Metric measure spaces.

Measures and outer measures.

Let Ω be a nonempty set and $\mathcal{P}(\Omega)$ the collection of all subsets of Ω. A *σ-algebra* $\Sigma \subset \mathcal{P}(\Omega)$ in Ω is a collection sets that contains $\Omega \in \Sigma$, $\Omega \setminus E \in \Sigma$ for each $E \in \Sigma$ and countable unions of sets in Σ are also in Σ. We call the pair (Ω, Σ) a *measurable space*.

Combining that a σ-algebra is closed under complement and contable unions, using Morgan's law, i.e. for any set of indices I

$$\bigcap_{i\in I}(\Omega \setminus E_i) = \Omega \setminus \bigcup_{i\in I} E_i$$

for any collection of sets $E_i \subset \Omega$, then countable intersections of sets in Σ also belong to Σ.

A function $\mu : \Sigma \to [0, +\infty]$ is called a *countably subadditive measure* on (Ω, Σ) if

(i) $\mu(\emptyset) = 0$.

(ii) $\mu\left(\bigcup_{i\in I} E_i\right) = \sum_{i\in I} \mu(E_i)$ for all countable collections $\{E_i\}_{i\in I}$ of pairwise disjoint sets in Σ.

On the other hand one can define an outer measure, which is defined for all subsets of Ω. We call $\mu : \mathcal{P}(\Omega) \to [0, \infty]$ an *outer measure* if it satisfies

(i) $\mu(\emptyset) = 0$.

(ii) $\mu(E) \leq \sum_{i\in I} \mu(E_i)$ whenever $E \subset \bigcup_{i\in I} E_i$, $\{E_i\}_{i\in I} \subset \mathcal{P}(\Omega), I \subset \mathbb{N}$.

The σ-algebra of an outer measure is given by the sets $E \subset \Omega$ such that

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E) \text{ for all } A \subset \Omega.$$

and any measure μ in a measurable space (Ω, Σ) can be extended to an outer measure $\overline{\mu}$ by considering

$$\overline{\mu}(E) := \inf\{\mu(F) : E \subset F \subset \Sigma\}.$$

In what follows, by a measure we mean an outer measure. For further details on properties of outer measures we refer to [41, Section1.1.1].

For μ a measure on (Ω, Σ) we say that a set $E \subset \Omega$ is *μ-measurable* if $E \in \Sigma$ (although we will usually call it measurable if the measure used is understood from the context). In particular we will say that E is *negligible* or that it is a *null set* if $\mu(E) = 0$. We say that a property holds *μ-almost everywhere* (or μ-a.e.) in Ω if there exists a negligible set $N \in \Sigma$ so that the property holds in $\Omega \setminus N$.

We call (Ω, Σ, μ) a *measure space* and we will always assume it to be non-trivial, that is, $\mu(\Omega) > 0$. Moreover, we say that (Ω, Σ, μ) is a *finite measure space* if $\mu(\Omega) < \infty$, and it is *σ-finite* if Ω can be decomposed as a union of countably many sets of finite measure.

Borel, regular and Radon measures.

A set function $d : X \to [0, \infty)$ defined on a set X is a *metric* if

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

and (X, d) is called a *metric space*. A metric always induces a natural topology, where the open sets consist on all open balls defined as

$$B(x, r) := \{y \in X : d(x, y) < r\}, \quad x \in X, r > 0.$$

Whenever the center or the radius of a ball is not relevant we will denote B for the ball and $\mathrm{rad}(B)$ for its radius. On the other hand we denote a *closed ball* as $\overline{B}(x,r) := \{y \in X : d(x,y) \le r\}$, which is in general potentially larger than the topological closure of $B(x,r)$.

We also define the distance between two sets $E, F \subset X$ as

$$\mathrm{dist}(E,F) := \inf\{d(x,y) : x \in E, y \in F\},$$

and the diameter of a set $E \subset X$ as

$$\mathrm{diam}(E) := \sup\{d(x,y) : x, y \in E\}.$$

Given (X, d_X) and (Y, d_Y) two metric spaces, we denote by $\mathrm{LIP}(X,Y)$ the set of Lipschitz mappings, that is, mappings $f : X \to Y$ such that

$$\mathrm{LIP}(f) := \sup_{x,y\in X} \frac{d_Y(f(x), f(y))}{d_X(x,y)} < \infty.$$

Moreover, given a point $x \in X$ we consider the pointwise Lipschitz constant of f at x to be as follows:

$$\mathrm{Lip} f(x) := \limsup_{y\to x} \frac{d_Y(f(x), f(y))}{d_X(x,y)}.$$

We also denote $S(f) := \{x \in X : \mathrm{Lip} f(x) < \infty\}$.

Lipschitz maps are good candidates for a suitable notion of first order regularity of mappings defined on spaces without a linear structure, and they will play a very important role throughout the entire exposition of this thesis. The following result due to Rademacher shows the importance of Lipschitz maps when studying first order regularity.

Lemma 1.1. (Rademacher's Theorem [41, Theorem 3.2]) *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ a Lipschitz function. Then f is differentiable $\mathcal{L}^n$-a.e. in U.*

Similar results have been developed in more general contexts and this will be one of the main topics of Chapter 2. We also present here an extension result of Lipschitz maps that follows from McShane's extension theorem and will be useful later on.

Lemma 1.2. [56, Corollary 4.1.7] *Let (X,d) a metric space. Given a Lipschitz mapping $f : U \subset X \to \ell^\infty$ then there exists a Lipschitz extension $\hat{f} : X \to \ell^\infty$, that is, $f(x) = \hat{f}(x)$ for every $x \in U$.*

Last, we present the classical Kuratowski embedding Theorem, which is very useful to study metric-valued mappings by embedding the target space into a Banach space, where the structure there allows to use several tools from Functional Analysis.

Lemma 1.3. (Kuratowski embedding Theorem, [74]) *Any metric space (X,d) admits an isometric embedding into the Banach space $\ell^\infty(X)$.*

A metric space can always be equipped with the *Borel σ-algebra* $\mathcal{B}$ generated by the topology of open balls (that is, the smallest σ-algebra containing all open balls). Any set $E \in \mathcal{B}$ is called a *Borel set* of X. On the other hand, a measure μ on X is called a *Borel measure* if Borel sets are measurable.

An additional requirement for a measure that is often needed is the fact that any set can be approximated by measurable sets. We say that μ is *regular* if for every set $A \subset X$ there exists a measurable set $E \subset X$ such that $\mu(A) = \mu(E)$. If μ is also Borel we call it *Borel regular*. Furthermore, we say that μ is *Radon* if it is Borel regular, $\mu(K) < \infty$ for every compact set $K \subset X$, it is *outer regular*, that is,

$$\mu(E) = \inf\{\mu(O) : O \text{ open}, E \subset O\}$$

for each measurable $E \subset X$, and

$$\mu(O) = \sup\{\mu(K) : K \text{ compact}, K \subset O\}$$

for each open set $O \subset X$.

We gather all the concepts introduced to define metric measure spaces, which will play a very important role from now on, although further discussions on their geometry will be considered in later sections.

Definition 1.4 A *metric measure space* (X, d, μ) is a metric space (X, d) equipped with a measure μ that is Borel regular and such that, for all $x \in X$, there exists $r > 0$ for which $0 < \mu(B(x, r)) < \infty$.

Proposition 1.5. [**56, Lemma 3.3.28**] *Let (X, d, μ) be a separable metric measure space. There exists a countable collection of open balls with finite measure covering X. In particular, X is a σ-finite measure space.*

Last, we point out the outer and inner regularity of the measure of a metric measure space.

Proposition 1.6. [**56, Propositions 3.3.37 and 3.3.41**] *Let (X, d, μ) be a metric measure space.*

(i) *The measure μ is outer regular, that is, for every measurable $E \subset X$*

$$\mu(E) = \inf\{\mu(U) : U \textit{ open}, E \subset U\}$$

and then for every $\varepsilon > 0$ there exists an open set U, with $E \subset U$ and $\mu(U \setminus E) < \varepsilon$.

(ii) *The measure μ is inner regular, that is, for every measurable $E \subset X$*

$$\mu(E) = \sup\{\mu(C) : C \textit{ closed}, C \subset E\}$$

and then for every $\varepsilon > 0$ there exists a closed set C, with $C \subset E$ and $\mu(E \setminus C) < \varepsilon$.

(iii) *If μ is a Radon measure then for every measurable set $E \subset X$*

$$\mu(E) = \sup\{\mu(K) : K \textit{ compact}, K \subset E\}.$$

and then for every $\varepsilon > 0$ there exists a compact set K, with $K \subset E$ and $\mu(E \setminus K) < \varepsilon$.

We remind now the definition of the push forward measure through a mapping. Let $f : X \to Y$ be a mapping with X and Y metric spaces and μ a measure on X. We define the *push-forward measure* $f_\# \mu$ on Y as

$$f_\#\mu(E) := \mu(f^{-1}(E)) \quad \text{for all } E \subset Y.$$

On the other hand for a given set $E \subset X$ one can define the *restriction of μ to E* as the standard mapping restriction $\mu_{|_E} : E \to [0, \infty)$.

Hausdorff measure.

In any metric space (X, d) one can always define a natural measure that comes from the metric d. In addition, the construction of this measure depends on a parameter that acts as a notion of dimension, allowing to study a variety of objects with their respective measures. As an easy example think of $\mathbb{R}^N$ with $N \geq 2$ and any hyperplane in it. Then the standard Lebesgue measure in $\mathbb{R}^N$

will not provide information about the size of a subset in the hyperplane, but a lower dimension measure (the $(N-1)$-Lebesgue measure) will do. In a general metric space the analogue to this is the Hausdorff measure. For $s > 0$, first fix $\delta > 0$ and consider for any set $E \subset X$

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i)^s : E \subset \bigcup_{i \in I} E_i, \operatorname{diam}(E_i) \leq \delta \right\}.$$

The *s-dimensional Hausdorff measure* is defined as

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

Remark 1.7 The definition of the Hausdorff measure is often considered with an additional normalization factor in order to obtain that $\mathcal{H}^N = \mathcal{L}^N$ in $\mathbb{R}^N$, as in [56, Section 4.3], but we are not interested in specific computation for the measure so we will not add this constant, as in [12] (page 8). However, keep in mind that, with our definition of Hausdorff measure, $\mathcal{L}^N$ and $\mathcal{H}^N$ are comparable, that is, there exists $C > 0$ such that for every set $E \subset X$, $C^{-1}\mathcal{L}^N(E) \leq \mathcal{H}^N(E) \leq C\mathcal{L}^N(E)$.

Lemma 1.8. *Let (X, d_X) and (Y, d_Y) be metric spaces and consider a Lipschitz mapping $f : X \to Y$. Then for each $A \subset X$ such that $\mathcal{H}^s(A) = 0$ one also has $\mathcal{H}^s(f(A)) = 0$.*

On the other hand, for a given set $A \subset X$, the *co-dimension* 1 Hausdorff measure of A is defined as

$$\mathcal{H}^{-1}(A) := \lim_{\delta \to 0^+} \inf \left\{ \sum_{i \in I \subset \mathbb{N}} \frac{\mu(B_i)}{\operatorname{rad}(B_i)} : A \subset \bigcup_{i \in I} B_i, \operatorname{rad}(B_i) \leq \delta \right\}.$$

If one thinks of μ as an Ahlfors Q-regular measure, that is, the measure of any ball B comparable to $\operatorname{rad}(B)^Q$, the co-dimension 1 Hausdorff measure is the $(Q-1)$-Hausdorff measure. In more general metric measure spaces this notion of co-dimensional measure provides a natural candidate for the study of boundaries of *somehow regular* sets.

1.2 The Bochner integral.

Along this section, (Ω, Σ, μ) will denote a σ-finite measure space and $(V, \|\cdot\|)$ a Banach space. We recall that the *dual space* of V, denoted by V^*, is the Banach space of all bounded linear mappings $v^* : V \to \mathbb{R}$, also called *functionals* of V. We will denote

$$\langle v^*, v \rangle := v^*(v) \quad \text{for } v \in V, v^* \in V^*.$$

Whenever we write $\|v^*\|$ we mean the operator norm $\|v^*\|_{V^*} = \sup\{|\langle v^*, v\rangle| : \|v\| \leq 1\}$. A sequence $\{v_i\}_{i=1}^\infty \subset V$ with $\|v_i\| \leq 1$ for all i is said to be a norming sequence for V^* if $\|v^*\| = \sup_{i \in \mathbb{N}} \langle v^*, v_i \rangle$. It follows from Hahn-Banach Theorem that for every $v \in V$ one can find $v^* \in V^*$ with $\|v^*\| \leq 1$ such that $\langle v^*, v \rangle = \|v\|$.

Measurability of vector-valued mappings.

A function $s : \Omega \to V$ is said to be a *measurable simple function* if there exist vectors $v_1, \dots, v_m \in V$ and disjoint measurable subsets $E_1, \dots, E_m$ of Ω such that

$$s = \sum_{i=1}^{m} v_i \chi_{E_i}.$$

A mapping $f : \Omega \to V$ is said to be *measurable* if there exists a sequence of measurable simple functions $(s_n : \Omega \to V)_{n=1}^{\infty}$ converging to f μ-almost everywhere on Ω. A first result concerning measurability is the classical Egoroff's Theorem that still holds true for vector-valued mappings and allows to strengthen pointwise convergence of sequences of measurable mappings. We refer to [41, Theorem 1.16] for the Euclidean setting, while the vector-valued setting in measure spaces can be dealt in a similar way, see for example [56, Page 41].

Theorem 1.9. (Egoroff's Theorem) *Suppose $\mu(X) < \infty$ and let $f_i : \Omega \to V$, $i \in \mathbb{N}$, be a sequence of measurable functions converging pointwise μ-almost everywhere to a measurable map $f : \Omega \to V$. Then for all $\varepsilon > 0$ there exists $N_\varepsilon \subset \Omega$ measurable with $\mu(N_\varepsilon) < \varepsilon$ and $f_i \to f$ uniformly in $\Omega \setminus N_\varepsilon$.*

As a consequence of Egoroff's Theorem one can prove that measurability is closed under pointwise convergence:

Corollary 1.10. [56, Corollary 3.1.5] *If $f_i : X \to V$ are measurable and converge pointwise μ-almost everywhere to $f : X \to V$, then f is also measurable.*

The previous results play an important role in the proof of the following criterion for measurability of Banach space-valued mappings (see e.g. Theorem 2 in [32, Section III.3] or [56, Section 3.1]).

Theorem 1.11. (Pettis measurability Theorem) *Consider a σ-finite measure space (Ω, Σ, μ) and a Banach space V. A function $f : \Omega \to V$ is measurable if and only if it satisfies the following two conditions:*

1. *f is weakly-measurable, i.e., for each $v^* \in V^*$, we have that $\langle v^*, f\rangle : \Omega \to \mathbb{R}$ is measurable.*
2. *f is essentially separably-valued, i.e., there exists $Z \subset \Omega$ with $\mu(Z) = 0$ such that $f(\Omega \setminus Z)$ is a separable subset of V.*

Note that, if $f : \Omega \to V$ is measurable, Pettis measurability Theorem yields the measurability of the norm $\|f\| : \Omega \to \mathbb{R}$.

For a separable Banach space V the notion of measurability can also be considered in the usual Borel sense.

Corollary 1.12. [56, Corollary 3.1.2] *Consider a σ-finite measure space (Ω, Σ, μ) and a separable Banach space V. A function $f : \Omega \to V$ is measurable if and only if $f^{-1}(O)$ is Borel for every open set $O \subset V$.*

Integrability of vector-valued mappings.

Once measurability is defined via simple functions, a natural way of defining integration raises, yielding the so called *Bochner integral* when these notions are considered for vector-valued mappings. Suppose first that $S = \sum_{i=1}^m v_i \chi_{E_i}$ is a measurable simple function as before, where $E_1, \dots, E_m$ are measurable and pairwise disjoint. If, in addition, $\mu(E_i) < \infty$ for each $i \in \{1, \dots, m\}$ we then say that S is *Bochner integrable*, and we denote the integral of S over Ω by

$$\int_\Omega S \, d\mu := \sum_{i=1}^m \mu(E_i) v_i.$$

Now consider an arbitrary measurable mapping $f : \Omega \to V$. We say that f is *Bochner integrable* if there exists a sequence $(S_n)_{n=1}^{\infty}$ of integrable simple functions such that

$$\lim_{n\to\infty} \int_{\Omega} \|S_n - f\|\, d\mu = 0,$$

where the integral is considered in the Lebesgue sense. In this case, the *Bochner integral* of f is defined as:

$$\int_{\Omega} f\, d\mu := \lim_{n\to\infty} \int_{\Omega} S_n\, d\mu.$$

It can be seen that this limit exists as an element of V, and it does not depend on the choice of the sequence $(S_n)_{n=1}^{\infty}$. Also, for a measurable subset $E \subset \Omega$, we say that f is integrable on E if $f\,\chi_E$ is integrable on Ω, and we denote $\int_E f\, d\mu := \int_\Omega f\,\chi_E\, d\mu$. Furthermore we recall the standard notations for the mean value of f over E,

$$f_E = -\!\!\!\!\!\!\int_E f\, d\mu = \frac{1}{\mu(E)} \int_E f\, d\mu.$$

From now on we will skip the term Bochner when referring to the Bochner integral. Notice also that in the real-valued case the notion of Bochner integral coincides with the Lebesgue integral.

The following characterization will be very useful to check integrability of a mapping (see e.g. [56, Proposition 2.7]).

Proposition 1.13. *Let (Ω, Σ, μ) be a σ-finite measure space and V a Banach space. A function $f : \Omega \to V$ is Bochner-integrable if, and only if, f is measurable and $\int_\Omega \|f\|\, d\mu < \infty$.*

Furthermore, basic properties of the Lebesgue integral hold true for the Bochner integral. This is not surprising as the definitions follow the same line of reasoning and V has a linear structure, which is the only tool required for most of the properties in the case of the Lebesgue integral.

Proposition 1.14. **[32, pp. 46–48] Properties of the Bochner integral.**

(*i*) *The operator $f \mapsto \int_\Omega f d\mu$ is linear.*

(*ii*) *For each integrable function $f : \Omega \to V$ and $E \subset \Sigma$ $\left\|\int_E f d\mu\right\| \le \int_E \|f\| d\mu$.*

(*iii*) *If $f : \Omega \to V$ is integrable, then* $\displaystyle\lim_{\mu(E)\to 0} \int_E f d\mu = 0.$

(*iv*) *Let $f : \Omega \to V$ and $\{E_k\}_{k=1}^{\infty} \subset \Sigma$ be a family of disjoint measurable sets so that f is integrable on E_k for all k and*

$$\sum_{k=1}^{\infty} \int_{E_k} f d\mu < \infty.$$

Then

$$\int_{\bigcup_{k=1}^{\infty} E_k} f d\mu = \sum_{k=1}^{\infty} \int_{E_k} f d\mu.$$

(*v*) *Let W be a Banach space and $T : V \to W$ a continuous linear operator. If $f : \Omega \to V$ and $T \circ f$ are integrable then*

$$T\left(\int_\Omega f d\mu\right) = \int_\Omega T \circ f d\mu.$$

Remark 1.15 Property (v) will be particularly useful when $W = \mathbb{R}$ and $T = v^* \in V^*$. As long as f is integrable, so it is $\langle v^*, f\rangle$ and hence

$$\left\langle v^*, \int_\Omega f\, d\mu \right\rangle = \int_\Omega \langle v^*, f\rangle\, d\mu.$$

L^p spaces of vector-valued mappings.

We introduce now the classes of vector-valued *p-integrable* functions on (Ω, Σ, μ) in the usual way. First consider the equivalence relation so that two mappings $f, g : \Omega \to V$ are *equivalent* if they coincide almost everywhere, that is, $\mu(\{x \in \Omega : f(x) \neq g(x)\}) = 0$. We say that f is a *μ-representative* of g if they are equivalent.

Fix $1 \leq p < \infty$. Then $L^p(\Omega, V)$ is defined as the space of all equivalence classes of measurable mappings $f : \Omega \to V$ for which

$$\int_\Omega \|f\|^p \, d\mu < \infty.$$

It can be seen that the space $L^p(\Omega, V)$ is a Banach space endowed with the natural norm

$$\|f\|_p := \left(\int_\Omega \|f\|^p d\mu\right)^{\frac{1}{p}}.$$

For the case $p = \infty$ the space $L^\infty(\Omega, V)$ consists on all classes of measurable mappings $f : \Omega \to V$ such that

$$\|f\|_\infty := \inf\{\alpha > 0 : \|f\| \leq \alpha \ \mu\text{-a.e.}\} < \infty.$$

Notice that $\|\cdot\|_\infty$ is also a complete norm, and thus $(L^\infty(\Omega, V), \|\cdot\|_\infty)$ is a Banach space.

Consider now any $1 \leq p \leq \infty$. As customary, for scalar-valued functions we denote $L^p(\Omega) = L^p(\Omega, \mathbb{R})$. We will also consider the corresponding spaces $L^p_{\text{loc}}(\Omega, V)$ of vector-valued *locally p-integrable* functions. We say that a measurable function $f : \Omega \to V$ belongs to $L^p_{\text{loc}}(\Omega, V)$ if every point in Ω has a neighborhood U so that $f_{|U} \in L^p(U, V)$.

One can also define the L^p spaces for mappings $f : \Omega \to (Y, d_Y)$ when (Y, d_Y) is a metric space (see Section 1.1 for the notion of a metric space). In this case we start by considering classes of Borel measurable mappings in the sense of Corollary 1.12, that is, $f^{-1}(O)$ is Borel for every open set $O \subset Y$. Then we fix a point $y_0 \in Y$ and define $L^p(\Omega, Y, y_0)$ as all classes of Borel measurable mappings such that

$$\int_\Omega d_Y(f(x), y_0)^p \, d\mu(x) < \infty,$$

for $1 \leq p < \infty$, and $L^\infty(\Omega, Y, y_0)$ as all classes of Borel measurable mappings such that

$$\inf\{\alpha > 0 : d_Y(f(x), y_0) \leq \alpha \ \text{ for a.e. } x \in \Omega\} < \infty.$$

We refer to [45, Section 2.1] for further details on this topic.

1.3 The Radon-Nikodým property.

Consider (Ω, Σ) a measurable space. In Section 1.1 we presented the definition of a measure as a non-negative real-valued function. However, it is also of interest to consider cases where the measure takes values in a vector space. In particular we will consider measures with values in a Banach space, thus we provide the following definition.

Definition 1.16 Let (Ω, Σ) be a measurable space, and let V be a Banach space. We will say that $\mu : \Sigma \to V$ is a *vector measure* if it satisfies:

(i) $\mu(\emptyset) = 0$.

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ for every pair of disjoint measurable sets $A, B \in \Sigma$.

If property (ii) holds for every countable infinite family of pairwise disjoint measurable sets, we will say that μ is *countably additive*. We will also denote by $|\mu|$ the *total variation* of μ, given by

$$|\mu|(\Omega) = \sup \left\{ \sum_{i=1}^{m} \|\mu(A_i)\| : m \in \mathbb{N}, \Omega = \bigcup_{i=1}^{m} A_i \text{ with } A_i \in \Sigma \text{ for all } i = 1, \dots, m \right\}.$$

We will then say that μ is a measure of *bounded variation* if $|\mu|(\Omega) < \infty$.

Definition 1.17 Given two measures ν and μ on a measurable space (Ω, Σ), where ν is a vector measure of bounded variation, and μ is a scalar σ-finite measure, we will say that ν is *absolutely continuous* with respect to μ, denoted as $\nu \ll \mu$, if for all $A \in \Sigma$ $\mu(A) = 0$ implies $|\nu|(A) = 0$. Note that the definition is equivalent to $|\nu| \ll \mu$.

An important classical result in Measure Theory is the so called Radon-Nikodým Theorem, that we now state in the context of this section.

Theorem 1.18. (Radon-Nikodým Theorem) *Let (Ω, Σ, μ) be a measure space with μ a σ-finite measure on Ω. Let $\nu : \Sigma \to \mathbb{R}$ be a measure of bounded variation which is absolutely continuous with respect to μ, then there exists a measurable function $F \in L^1(\Omega)$ so that*

$$\nu(E) = \int_E F \, d\mu \quad \textit{for all } E \in \Sigma, \tag{1.1}$$

and F is unique up to μ-representatives.

Notice that in the above theorem ν takes values in $\mathbb{R}$ so it could take negative values and thus here we use the concept of measure given in Definition 1.16. In the general case of vector measures this theorem might not hold, as one can see in [16, p. 103], where it is considered measure ν with values in c_0, that is, the space of sequences converging to zero, given by

$$\nu(E) := \left(\int_E 1 \, dt, \int_E e^{it} dt, \dots, \int_E e^{int} dt, \dots \right)$$

which verifies $|\nu| \ll \mathcal{L}^1$, but (1.1) does not hold. A similar example can also be found in [32, p. 50 Example 10]. This example serves as motivation for the property that we are now introducing, which has been extensively studied in the literature of Measure Theory.

Definition 1.19 We will say that a Banach space V has the *Radon-Nikodým property* with respect to a measure space (Ω, Σ, μ) if for every vector measure $\nu : \Sigma \to V$ of bounded variation which is absolutely continuous with respect to μ, it is possible to find an integrable mapping $F \in L^1(\Omega, V)$ such that

$$\nu(E) = \int_E F d\mu, \text{ for all } E \in \Sigma.$$

and we then call $F := \frac{d\nu}{d\mu}$ the *Radon-Nikodým derivative* of ν with respect to μ. Additionally, we will say that V has the Radon-Nikodým property (RNP for short) if it has it with respect to all finite measure spaces.

Remark 1.20 This property takes its name from Theorem 1.18 which implies that $V = \mathbb{R}$ is a Banach space with the Radon-Nikodým property. Thus, given a measure space (Ω, Σ, μ), for any measure $\nu : \Sigma \to \mathbb{R}$, if $\nu \ll \mu$, the Radon-Nikodým derivative $\frac{d\nu}{d\mu}$ exists.

The Radon-Nikodým property has several equivalent characterizations, and some of them are well suited for the study of first order regularity of certain mappings.

Theorem 1.21. [16, **Theorem 5.21**] *Given a Banach space V, the following conditions are equivalent:*

(*i*) *V has the Radon-Nikodým property.*

(*ii*) *Every Lipschitz function $f : [0,1] \to V$ is almost everywhere differentiable.*

(*iii*) *Every absolutely continuous function $f : [0,1] \to V$ is almost everywhere differentiable.*

In the literature, one might encounter (*iii*) being referred to as the definition of a Gelfand space, see for example [32, page 107, Theorem 2]. However, the term "Gelfand space" is seldom employed nowadays, and conditions (*ii*) or (*iii*) are used as alternative definitions of the Radon-Nikodým property.

1.4 Analysis in metric measure spaces.

1.4.1 Doubling measures.

A measure μ is *doubling* if there exists some constant $C_d \geq 1$ such that whenever $x \in X$, $r > 0$ we have

$$0 < \mu(B(x, 2r)) \leq C_d \, \mu(B(x, r)).$$

We say that (X, d, μ) is doubling if μ is a doubling measure on X. This condition implies that there exists a constant $Q > 0$, called the *upper mass bound*, so that for all $0 < r < R$ and all $x, y \in X$, with $x \in B(y, R)$ we have

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \geq \frac{1}{4^Q} \left(\frac{r}{R}\right)^Q .$$

Some of the classical results in Analysis that do not involve derivatives were extended in the late 1970s to metric spaces endowed with a doubling measure (called in [25] spaces of homogeneous type). In particular, one of the most important consequences of a doubling measure is the Lebesgue differentiation Theorem.

Theorem 1.22. [51, **Theorem 1.8**] *Let (X, d, μ) be a metric measure space with μ doubling, V a Banach space and $f \in L^1_{loc}(X, V)$. Then*

$$\lim_{r \to 0} \fint_{B(x,r)} \|f(y) - f(x)\| \, d\mu(y) = 0$$

for almost every $x \in X$. In particular

$$\lim_{r \to 0} \fint_{B(x,r)} f \, d\mu = f(x)$$

for almost every $x \in X$.

From a broader perspective, one can consider the notion of Lebesgue points of a mapping $f \in L^1_{lox}(X, V)$. We say that $x \in X$ is a *Lebesgue point* of f if

$$\lim_{r \to 0^+} \fint_{B(x,r)} \|f(y) - f^*(x)\| \, d\mu(y) = 0,$$

where $f^*(x) := \lim_{r \to 0^+} f_{B(x,r)}$. Then, as a consequence of Theorem 1.22, given a mapping $f \in L^1_{loc}(X, V)$, almost every point in X is a Lebesgue point of f.

Remark 1.23 If we consider mappings that take values into a metric space (Y, d_Y) one could use the above theorem through an isometric embedding of Y into a Banach space to obtain that for $f \in L^1_{loc}(X, Y)$ almost every $x \in X$ satisfies

$$\limsup_{r \to 0^+} \fint_{B(x,r)} d_Y(f(y), f(x))\, d\mu(y) = 0.$$

Then for such x, if we set $E_\varepsilon(x) := \{y \in X \,:\, d_Y(f(y), f(x)) > \varepsilon\}$, we have that

$$\begin{aligned}
\limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap E_\varepsilon(x))}{\mu(B(x,r))} &= \limsup_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \chi_{E_\varepsilon(x)}(y)\, d\mu(y) \\
\leq \limsup_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \frac{d_Y(f(x), f(y))}{\varepsilon}\, d\mu(y) & \\
= \limsup_{r \to 0^+} \frac{1}{\varepsilon} \fint_{B(x,r)} d_Y(f(x), f(x))\, d\mu(y) = 0. &
\end{aligned}$$

The above property is usually referred to as $E_\varepsilon(x)$ having density zero at x. In general, we define the *density* of a set $E \subset X$ at a point $x \in E$ as

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))},$$

and we say that x is a *density point* of E if such limit equals 1.

Definition 1.24 Let (X, d, μ) be a metric measure space and (Y, d_Y) be a metric space. We say that $z_0 \in Y$ is the *approximate limit* of a mapping $f : X \to Y$ at a point $x \in X$, and denote it by

$$z_0 = \operatorname{ap}\lim_{y \to x} f(y),$$

if for every $\varepsilon > 0$ the set $\{y \in X : d_Y(f(y), z_0) \geq \varepsilon\}$ has density zero at x. Moreover, we say that a point $x \in X$ is *a point of approximate continuity* of f if

$$\operatorname{ap}\lim_{y \to x} f(y) = f(x),$$

or, equivalently, if for every $\varepsilon > 0$ we have

$$\limsup_{r \to 0^+} \frac{\mu(\{y \in B(x,r) \,:\, d_Y(f(x), f(y)) \geq \varepsilon\})}{\mu(B(x,r))} = 0.$$

Remark 1.23 establishes that for $f \in L^1_{loc}(X, Y)$ almost every point in X is a point of approximate continuity of f.

The global doubling condition in Theorem 1.22 can be relaxed to the following pointwise infinitesimal condition a.e. in X: we say that μ is *pointwise doubling at* $x \in X$ if

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x,r))} < \infty.$$

Under the assumption that μ is pointwise doubling at almost every point, Lebesgue differentiation Theorem still holds, see for example [56, Theorem 3.4.3 and page 81].

1.4.2 Curves in a metric measure space.

By a *curve* in a metric space (X, d) we mean a continuous function $\gamma : I \to X$, where $I \subset \mathbb{R}$ is an interval, and by a subcurve of a curve $\gamma : I \to X$ we mean the curve $\gamma|_J$ for some $J \subset I$. We say that γ is a *compact curve* if $I = [a, b] \subset \mathbb{R}$ is a compact interval. The *length* of a compact curve $\gamma : I \to X$ is given by

$$\ell(\gamma) := \sup \left\{ \sum_{j=1}^{n} d(\gamma(t_{j-1}), \gamma(t_j)) : a = t_0 < \cdots < t_n = b \right\}.$$

We say that γ is *rectifiable* if its length is finite. Every rectifiable curve γ can be re-parametrized so that it is *arc-length parametrized* via the length function $s_\gamma : [a, b] \to [0, \ell(\gamma)]$ given by

$$s_\gamma(t) = \ell(\gamma|_{[a,t]}).$$

We also define the speed of a compact rectifiable curve by

$$|\gamma'|(t) := \lim_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|}$$

if it exists. Due to Rademacher's Theorem (see Lemma 1.1), if γ is Lipschitz (or absolutely continuous) this limit exists a.e. and

$$\ell(\gamma) = \int_a^b |\gamma'|(t)\, dt.$$

As a consequence we have the following result.

Corollary 1.25. *The arc-length reparametrization $\tilde{\gamma}$ of a compact rectifiable curve γ in X is 1-Lipschitz and $|\tilde{\gamma}'|(t) = 1$ for $\mathcal{L}^1$- almost every $t \in [0, \ell(\gamma)]$.*

The integral of a Borel function $\rho : X \to [0, \infty]$ over a rectifiable compact curve $\gamma : [a, b] \to X$ is defined as

$$\int_\gamma \rho\, ds := \int_0^{\ell(\gamma)} \rho(\tilde{\gamma}(t))\, dt$$

and if γ is Lipschitz (or absolutely continuous) this coincides by a change of variable with

$$\int_a^b \rho(\gamma(t))|\gamma'|(t)\, dt$$

In what follows, let $\mathcal{M}$ denote the family of all non-constant compact rectifiable curves in X. Also, when defining a curve $\gamma : [0, \ell(\gamma)] \to X$ we will assume that it is arc-length parametrized without writing $\tilde{\gamma}$, otherwise we will specify $\gamma : [a, b] \to X$.

1.4.3 Modulus of a family of curves.

In non-smooth calculus the notion of p-modulus of a family of curves has proven to be very useful along the literature. Originally, this concept was introduced in the context of complex analysis for the study of quasiconformal mappings, although a deeper study has shown that quasiconformality has a strong nature suited for analysis in more general metric spaces. A short review of the history of this topic can be found in [53].

For each subset $\Gamma \subset \mathcal{M}$, we denote by $F(\Gamma)$ the set of so-called *admissible functions* for Γ, that is, the family of all Borel functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \, ds \geq 1$$

for all $\gamma \in \Gamma$. Then, for each $1 \leq p < \infty$, the *p-modulus of* Γ is defined as follows:

$$\mathrm{Mod}_p(\Gamma) := \inf_{\rho \in F(\Gamma)} \int_X \rho^p \, d\mu.$$

We say that a property holds for *p-almost every curve* $\gamma \in \mathcal{M}$ if the p-modulus of the family of curves failing the property is zero. The basic properties of p-modulus are given in the next proposition (see e.g. [48, Theorem 5.2] or [56, Chapter 5]).

Proposition 1.26. *The p-modulus is an outer measure on* $\mathcal{M}$*, that is:*

(i) $\mathrm{Mod}_p(\emptyset) = 0$.

(ii) If $\Gamma_1 \subset \Gamma_2$ *then* $\mathrm{Mod}_p(\Gamma_1) \leq \mathrm{Mod}_p(\Gamma_2)$.

(iii) $\mathrm{Mod}_p\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) \leq \sum_{n=1}^{\infty} \mathrm{Mod}_p(\Gamma_n)$.

Normally we just want to check whether a family of curves has zero p-modulus or not, and so the following result gives a very useful characterization for this fact. We refer to [48, Theorem 5.5] or [56, Lemma 5.2.8] for its proof.

Lemma 1.27. *Let* $\Gamma \subset \mathcal{M}$*. Then* $\mathrm{Mod}_p(\Gamma) = 0$ *if, and only if, there exists a non-negative Borel function* $g \in L^p(X)$ *such that*

$$\int_\gamma g \, ds = \infty$$

for all $\gamma \in \Gamma$.

We will also use the following fact (see, e.g. [56, Lemma 5.2.15]).

Lemma 1.28. *Suppose that* $E \subset X$ *has zero-measure and denote*

$$\Gamma_E^+ := \{\gamma \in \mathcal{M} : \mathcal{L}^1(\{t \in [0, \ell(\gamma)] : \gamma(t) \in E\}) > 0\}.$$

Then, for every $1 \leq p < \infty$, $\mathrm{Mod}_p(\Gamma_E^+) = 0$.

We finish this discussion with the classical Fuglede's Lemma (for a proof, see e.g. [48, Theorem 5.7] or [56, Chapter 5]).

Lemma 1.29 (Fuglede's Lemma). *Let* $(g_n)_{n=1}^{\infty}$ *be a sequence of Borel functions* $g_n : X \to [-\infty, \infty]$ *that converges in* $L^p(X)$ *to some Borel function* $g : X \to [-\infty, \infty]$*. Then there is a subsequence* $(g_{n_k})_{k=1}^{\infty}$ *such that*

$$\lim_{k \to \infty} \int_\gamma |g_{n_k} - g| ds = 0$$

for p-almost every curve $\gamma \in \mathcal{M}$.

1.4.4 Upper gradients.

Given a metric measure space (X, d, μ) and a Banach space V, an *upper gradient* of a mapping $f : X \to V$ is a non-negative Borel function $g : X \to [0, \infty]$ such that for each non-constant compact rectifiable curve $\gamma : [a, b] \to X$

$$\|f(\gamma(b)) - f(\gamma(a))\| \leq \int_\gamma g\, ds \tag{1.2}$$

and for $1 \leq p \leq \infty$ it is called *p-weak upper gradient* if (1.2) holds for p-almost every non-constant compact rectifiable curve in X. This notion was originally studied in [54], where they use the term "very weak gradient", and extensively studied in the real-valued setting in [86]. For a detailed discussion about upper gradients of vector-valued mappings we refer to [56, Chapter 6], from where we now mention some properties that will be used later on.

A locally Lipschitz mapping $f : X \to V$ has the lower pointwise Lipschitz constant

$$\mathrm{lip} f(x) := \liminf_{r \to 0^+} \sup_{y \in B(x,r)} \frac{\|f(x) - f(y)\|}{r}$$

as an upper gradient (see [56, Lemma 6.2.6]). As a consequence the upper pointwise Lipschitz constant $\mathrm{Lip} f$ is also an upper gradient of f.

It is important to notice that, if X is separable and a map $f : X \to V$ has a p-weak upper gradient in $L^p(X)$ for some $1 \leq p < \infty$, then [56, Theorem 6.3.20] yields the existence of a *minimal p-weak upper gradient*, that is, a p-weak upper gradient of minimal L^p-norm.

1.4.5 Poincaré inequality.

Let $1 \leq p < \infty$. We say that a metric measure space (X, d, μ) supports a *p-Poincaré inequality* if there are constants $C > 0, \lambda \geq 1$ such that for each $u, g \in L^1_{loc}(X)$, with g an upper gradient of u, we have

$$⨍_B |u - u_B| d\mu \leq C \mathrm{diam}(B) \left(⨍_{\lambda B} g^p d\mu \right)^{\frac{1}{p}}$$

for each ball $B \subset X$.

This inequality is not necessarily achieved in a general metric measure space, and the relevance of the intrinsic impact that supporting a Poincaré inequality has over the underlying geometry of the metric space was founded in [54]. Since then this property has seen a lot of protagonism in the literature of analysis in metric measure spaces. A good overview of Poincaré inequalities can be found in [49] and [56, Chapters 8 and 9].

As we just mentioned, if X supports a Poincaré inequality this implies a richer geometry on the space. Namely, a metric measure space with a Poincaré inequality is also *quasiconvex*, that is, there exists a constant $C > 0$ such that for every pair of points $x, y \in X$ there exists a curve $\gamma \in \mathcal{M}$ joining x and y. with $\ell(\gamma) \leq C d(x, y)$.

Proposition 1.30. **[56, Sec. 8.3]** *Let (X, d, μ) be a metric measure space supporting a Poincaré inequality. Then X is quasiconvex.*

For further geometric implications of p-Poincaré inequalities see for example [54, 66, 37, 40].

Chapter 2

Metric differentiability in Metric Measure Spaces

Throughout this chapter (X, d, μ) is a metric measure space with a complete and separable metric d. Moreover, by the completeness assumption we also have that μ is Radon (see [56, Corollary 3.3.47]). The main topic of the chapter is the study of Lipschitz maps from (X, d, μ) into a metric space Y and their differentiability properties. One might think of the classical results of Rademacher and Stepanov as an starting point for the line of reasoning along this chapter.

Since metric measure spaces lack a linear structure, the notion of a differential is not clear. However, by employing an atlas of charts taking values into an Euclidean space allows to consider linearity through the charts. So, in Section 2.1, we will introduce some background about Lipschitz differentiability spaces, where an atlas is chosen so that Rademacher's Theorem holds for real-valued Lipschitz maps by using the linearity given by the chart (see Definition 2.4). Moving on, with this structure in place, one can follow a similar idea in order to define metric differentials of maps into a given metric space target. For that, in Section 2.3 we use the charts in order to adapt the notion of metric differentiability introduced in the following result due to Kirchheim.

Theorem 2.1. **[71, Theorem 2]** *Let $U \subset \mathbb{R}^N$ be a measurable set, (Y, d_Y) a metric space and $f : U \to Y$ a Lipschitz mapping. Then f is metrically differentiable $\mathcal{L}^N$-almost everywhere, that is, for a.e. $x \in U$, there is a unique seminorm $\mathrm{md}_x f$ on $\mathbb{R}^N$ such that*

$$\lim_{\substack{y,z\to x\\ y,z\in U}} \frac{|d_Y(f(y), f(z)) - \mathrm{md}_x f(y-z)|}{|y-x| + |z-x|} = 0.$$

The condition obtained in Kirchheim's Theorem is called metric differentiability, and we will study the nature of this result when the domain is not necessarily Euclidean but a Lipschitz differentiability space. Namely we will see that the result is true in this context if, and only if, the space admits a rectifiable decomposition. This later condition and its relationship with Lipschitz differentiability spaces will be explained in Section 2.2 and then we will provide the main result of this chapter, a characterization of Kirchheim's theorem in metric measure spaces in terms of rectifiability, in Section 2.3.1.

Following the ideas in [80], it is natural to continue with a Stepanov-type result once we have a Rademacher-type theorem as the one we present in Section 2.3.1 and thus in Section 2.3.2 we will do so by following the techniques in [65].

2.1 Lipschitz Differentiability Spaces.

Here we discuss some definitions of structures in metric spaces where one can define a differential, namely, Lipschitz and rectifiable charts. We seek some results generalizing Rademacher and

Stepanov Theorems in this context and study the nature of the metric spaces where this kind of results hold. To this end, we will decompose our space into charts with certain properties. We say that (U, φ) is a *chart of dimension* k in a metric measure space (X, d, μ) if $U \subset X$ is Borel and $\varphi : X \to \mathbb{R}^k$ is Lipschitz. Furthermore we say that a collection of charts $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ is an *atlas* of X if

$$\mu\left(X \setminus \bigcup_{i\in\mathbb{N}} U_i\right) = 0.$$

Roughly speaking, we will focus on spaces that admit an atlas satisfying the Rademacher Theorem for real-valued functions. This characteristic designates them as *Lipschitz differentiability spaces*, a concept initially introduced by Cheeger [23], with further exploration conducted by Keith in [68]. It's worth noting that, in early literature, this property was referred to as having a *measurable differentiable structure*. The updated terminology of Lipschitz differentiability spaces was established in [11], where a comprehensive analysis of these spaces was conducted, utilizing Alberti representations. Additional contributions to this field can be found, for example, in [24, 47, 15, 13, 14, 40].

Lipschitz charts and weak Lipschitz charts.

This section will introduce some background on Lipschitz differentiability spaces, and for that we first present the various notions of charts that we need to consider in order to establish a structure on a metric measure space, enabling us to accurately define a differential.

Lipschitz charts.

Definition 2.2 *We say that* (U, φ) *is a* Lipschitz chart *in* X *if* $U \subset X$ *is Borel and* $\varphi : X \to \mathbb{R}^N$ *is a Lipschitz mapping such that for every function* $f \in \mathrm{LIP}(X)$ *and almost every* $x \in U$ *there exists a unique linear map* $d_x f : \mathbb{R}^N \to \mathbb{R}$ *such that*

$$\mathrm{Lip}(f - d_x f \circ \varphi)\,(x) = 0,$$

that is,

$$\limsup_{y\to x} \frac{|f(x) - f(y) - d_x f(\varphi(x) - \varphi(y))|}{d(x,y)} = 0. \tag{2.1}$$

The notion of Lipschitz chart is often referred to in the literature as a *Cheeger chart*, since it was first considered in [23].

Weak Lipschitz charts.

Definition 2.3 *We say that* (U, φ) *is a* weak Lipschitz chart *in* X *if* $U \subset X$ *is Borel and* $\varphi : X \to \mathbb{R}^N$ *is a Lipschitz mapping such that for every function* $f \in \mathrm{LIP}(X)$ *and almost every* $x \in U$ *there exists a unique linear map* $d_x f : \mathbb{R}^N \to \mathbb{R}$ *such that*

$$\mathrm{Lip}(f - d_x f \circ \varphi)_{|U}\,(x) = 0,$$

that is,

$$\limsup_{U\ni y\to x} \frac{|f(x) - f(y) - d_x f(\varphi(x) - \varphi(y))|}{d(x,y)} = 0. \tag{2.2}$$

In the literature, another notion is often considered instead of that of a weak Lipschitz chart, and that is an *approximate Lipschitz chart*, where for every Lipschitz function $f \in \mathrm{LIP}(X)$ and a.e. $x \in U$ there exists a unique linear map $d_x f : \mathbb{R}^N \to \mathbb{R}$ so that

$$\operatorname{ap}\lim_{y\to x}\frac{|f(x)-f(y)-d_xf(\varphi(x)-\varphi(y))|}{d(x,y)}=0. \tag{2.3}$$

We will not employ this notion here, but some of the references we provide may utilize it, so we provide an easy argument that will allow us to make use of the results present in the literature that are stated for approximate Lipschitz charts.

Notice that, from Definition 1.24, one can see that L is the approximate limit of $f : X \to \mathbb{R}$ at a point $x \in X$ if there exists a Borel set $A \subset X$ such that x is a density point of A and

$$\lim_{A\ni y\to x} f(y) = L.$$

This implies that, whenever (X, d, μ) satisfies the condition that almost every point of a Borel subset is a density point, then for a chart (U, φ) in X, choosing $A = U$ as above, condition (2.2) yields (2.3) for almost every $x \in U$. Consequently, the notion of a weak Lipschitz chart is stronger than that of an approximate Lipschitz chart. It is worth noting that the assumption of density points representing almost every point for any Borel subset of X is needed here and will be addressed in the subsequent discussion concerning porous sets. We will observe there that this assumption holds true whenever porous sets are null. Therefore, references utilizing the concept of an approximate Lipschitz chart can be considered whenever porous sets are null.

Definition 2.4 We say that a metric measure space (X, d, μ) is a *Lipschitz differentiability space*, LDS in short, (resp. weak LDS) if there exists a countable family $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ of Lipschitz charts (resp. weak Lipschitz charts) of arbitrary dimension and such that $\mu\left(X\setminus \bigcup_i U_i\right) = 0$.

Given a N-dimensional Lipschitz chart (U_i, φ_i) of an LDS, a result from De Philippis, Marchese and Rindler [30, Theorem 4.1.1] concludes that

$$(\varphi_i)_\#(\mu_{|U_i}) \ll \mathcal{L}^N_{|\varphi_i(U_i)}.$$

A related result was provided by Kell and Mondino in [69, Theorem 1.3]. Namely, if in addition to X being an LDS, it also admits an atlas of *bi-Lipschitz charts*, that is, a chart (U_i, φ_i) where $\varphi_i : U_i \to \mathbb{R}^{N_i}$ is bi-Lipschitz, then there exists a collection of Borel sets $\{E_j\}_{j\in\mathbb{N}}$ covering X up to a measure zero set, bi-Lipschitz equivalent to Borel sets in $\mathbb{R}^{N_j}$ and

$$\mu_{|E_j} \ll \mathcal{H}^{N_j}.$$

Porous sets.

It inmediately follows that an LDS is, in particular, a weak LDS. However, the converse might fail. An additional assumption was considered in [16] to characterize when the notions of weak LDS and LDS coincide. Here we briefly go through the ideas presented in that paper in order to provide some of the results that will be used later on.

Definition 2.5 Let (X, d, μ) be a metric measure space. Given $S \subset X$ and $x \in S$, S is called *η-porous* at x, for $\eta > 0$, if there exists a sequence $x_j \to x$ with

$$d(x_j, S) := \inf\{d(x_j, y) : y \in S\} \geq \eta d(x_j, x) \quad \text{for all } j \in \mathbb{N}.$$

We say that S is *porous* if every $x \in S$ is η-porous for some $\eta > 0$.

In [15, Section 2] or [11, Section 4] it was shown that, given a porous set $S \subset X$, one can construct a Lipschitz function in X that is nowhere differentiable in S, and thus any porous set must have measure zero whenever X is a Lipschitz differentiability space, see [15, Theorem 2.4].

Remark 2.6 The condition of porous sets having measure zero is intermediate between the doubling and the pointwise doubling conditions defined in Section 1.4.1, that is, if the measure is doubling, then all porous sets are null, as mentioned in [28, Remark 2.9]. Moreover, according to [83, Theorem 3.6 (iv)], if all porous sets in X have measure zero then μ is pointwise doubling almost everywhere.

It also holds that porous sets being null is a sufficient condition to self-improve a weak-Lipschitz chart into a Lipschitz chart (see [14, Lemma 2.6], or [15, Proposition 2.8]; in this last reference, the authors prove self-improvement of an approximate Lipschitz chart into a Lipschitz chart). To show this, we first give the following lemma.

Lemma 2.7. *Let (X, d, μ) be a metric measure space such that all porous sets have zero measure and $U \subset X$ Borel. Then for almost all $x \in U$ and for every $\varepsilon > 0$ there exists $r > 0$ such that for $y \in B(x, r)$ there exists $z(y) \in U$ with $d(y, z(y)) \leq \varepsilon d(x, y)$.*

The ideas of this lemma follow those of the same claim found in the proof of [67, Theorem 3.5] or [10, Proposition 2.9], where the measure is assumed to be doubling. But as in the proof of [15, Proposition 2.8], the claim still follows.

Proof of Lemma 2.7 Let $x \in U$ be a density point. Assume by contradiction that there exists $\varepsilon > 0$ and a sequence $y_k \in X$ converging to x such that $B(y_k, \varepsilon d(x, y_k)) \cap U) = \emptyset$ for all $k \in \mathbb{N}$. We can assume that $\varepsilon \leq 1$ and then by the density condition of x where the radius $r_k \to 0$ is chosen as the sequence $r_k = (1+\varepsilon)d(x, y_k)$

$$\begin{aligned} 0 &= \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus U)}{\mu(B(x,r))} = \limsup_{k\to\infty} \frac{\mu(B(x,(1+\varepsilon)d(x,y_k))\setminus U)}{\mu(B(x,(1+\varepsilon)d(x,y_k)))} \\ &\geq \limsup_{k\to\infty} \frac{\mu(B(y_k,\varepsilon d(x,y_k))\setminus U)}{\mu(B(x,(1+\varepsilon)d(x,y_k)))} = \limsup_{k\to\infty} \frac{\mu(B(y_k,\varepsilon d(x,y_k)))}{\mu(B(x,(1+\varepsilon)d(x,y_k)))} \\ &\geq \limsup_{k\to\infty} \frac{\mu(B(y_k,\varepsilon d(x,y_k)))}{\mu(B(x,2d(x,y_k)))}. \end{aligned} \tag{2.4}$$

On the other hand, as explained in the proof of [15, Proposition 2.8], using [83, Theorem 3.6] we have that, for almost every $x \in U$ and for all $\varepsilon > 0$

$$\liminf_{y\to x} \frac{\mu(B(y,\varepsilon d(x,y))}{\mu(B(x,2d(x,y))} > 0,$$

which leads to a contradiction with (2.4) for almost all density points of U. Finally, the claim follows since the hypothesis of porous sets being null implies that μ is pointwise doubling almost everywhere (Remark 2.6), and thus almost all points in U are density points by [56, Theorem 3.4.3 and p. 81]. ■

We present now a direct conclusion from Lemma 2.7 that adds comfort to the technicalities involving this result. First, as in the hypotheses of Lemma 2.7 assume that porous sets in X are null and let $U \subset X$ Borel. Given $x \in U$ that satisfies the conclusion of Lemma 2.7 and a sequence $\{y_k\}_{k\in\mathbb{N}} \subset X$ converging to x, there exists a sequence $\varepsilon_k > 0$ converging to 0 such that $d(x, y_k) < \varepsilon_k$ for all $k \in \mathbb{N}$. Now applying Lemma 2.7 to x and ε_k for each, k there exist radii $\{r_k\}_{k\in\mathbb{N}}$ such that for each $y \in B(x, r_k)$ there exists $z(y) \in U$ with $d(y, z(y)) < \varepsilon d(x, y)$. Going back to the sequence $\{y_k\}_{k\in\mathbb{N}}$, we can choose a subsequence $\{y_{k_j}\}_{j\in\mathbb{N}}$ such that $y_{k_j} \in B(x, r_j)$ for all $j \in \mathbb{N}$. Then, denoting $z_j := z(y_{k_j})$ we have

$$\limsup_{j\to\infty} \frac{d(y_{k_j}, z_j)}{d(x, y_{k_j})} = 0. \tag{2.5}$$

This yields that $\operatorname{Lip} f_{|U} = \operatorname{Lip} f$ almost everywhere in U for any Lipschitz mapping $f : X \to \mathbb{R}$. Indeed, choosing $z_j \in U$ as in (2.5)

$$\begin{aligned}
\operatorname{Lip} f(x) &= \limsup_{j\to\infty} \frac{|f(x)-f(y_{k_j})|}{d(x,y_{k_j})} \\
&\leq \limsup_{j\to\infty} \left(\frac{|f(x)-f(z_j)|}{d(x,z_j)} \frac{d(x,z_j)}{d(x,y_{k_j})} + \frac{|f(y_{k_j})-f(z_j)|}{d(x,y_{k_j})} \right) \\
&\leq \limsup_{j\to\infty} \left(\frac{|f(x)-f(z_j)|}{d(x,z_j)} + \operatorname{LIP}(f) \frac{d(y_{k_j},z_j)}{d(x,y_{k_j})} \right) \\
&= \operatorname{Lip} f_{|U}(x).
\end{aligned}$$

Here we used (2.5) together with the triangle inequality to obtain $\limsup_{j\to\infty} \frac{d(x,z_j)}{d(x,y_{k_j})} \leq 1$. Since the inequality $\operatorname{Lip} f_{|U} \leq \operatorname{Lip} f$ holds in general this proves that $\operatorname{Lip} f(x) = \operatorname{Lip} f_{|U}(x)$ at almost every point $x \in U$. This fact yields the self-improvement of a weak Lipschitz chart into a Lipschitz chart, giving the following characterization.

Corollary 2.8. **[15, Theorem 2.4 and Proposition 2.8]** *Let (X, d, μ) be a metric measure space. Then X is a Lipschitz differentiability space if, and only if, it is a weak Lipschitz differentiability space and porous sets are null.*

As a consequence, since in general a countable union of weak Lipschitz differentiabilty spaces is also a weak Lipschitz differentiability space, whenever the porous sets have measure zero, then this property also applies to the concept of Lipschitz differentiability spaces. However, this may not hold true in general as was noticed by D. Bate and S. Li in [13, Page 5] where they consider the set of $\mathbb{R}^2$ given by

$$(\{0\} \times [0,1]) \cup \{(x, p/2^n) : n \in \mathbb{N}, 1 \leq p < 2^n \text{ odd}, \pm x \in [2^{-n} - 4^{-n}, 2^{-1}]\},$$

equipped with the $\mathcal{H}^1$-measure, which despite being a countable union of intervals, the map $|x|$ is nowhere differentiable at $\{0\} \times [0,1]$. In fact, this vertical segment is a porous set of positive $\mathcal{H}^1$-measure.

2.2 Rectifiability.

We present two different ways, but very related, of decomposing a metric measure space into a family of sets Lipschitz equivalent to Euclidean sets. This notion is known as rectifiability and we first present the most classical definition, given by Federer [43].

Definition 2.9 A metric measure space (X, d, μ) is *k-rectifiable* if there exists a countable family of Lipschitz mappings $\psi_i : E_i \subset \mathbb{R}^k \to X$ defined on measurable sets E_i such that

$$\mu\left(X \backslash \bigcup_{i=1}^{\infty} \psi_i(E_i) \right) = 0 \quad \text{and} \quad \mu \ll \mathcal{H}^k.$$

Remark 2.10 According to a result of Kirchheim [71, Lemma 4], Lipschitz functions that parametrize a k-rectifiable set can be chosen to be bi-Lipschitz, that is, if X is k-rectifiable with decomposition $\psi_i : E_i \subset \mathbb{R}^k \to X$, then there exist $\{E_{i,j}\}_{j\in\mathbb{N}} \subset E_i$ for all i so that

(*i*) $E_{i,j} \subset E_i$ is Borel for each $j \in \mathbb{N}$ and $\mu\left(\psi_i(E_i) \backslash \bigcup_{j\in\mathbb{N}} \psi_i(E_{i,j}) \right) = 0$ for each $i \in \mathbb{N}$ and

(*ii*) $\psi_{i,j} := \psi_{i|_{E_{i,j}}}$ is bi-Lipschitz.

In the Euclidean setting, this follows from [43, Lemma 3.2.2]. Observe that a $k-$rectifiable metric measure space has a natural decomposition into charts. Namely, using the notation in Definition 2.9, if (X, d, μ) is $k-$rectifiable, it can be written as a countable collection of $k-$dimensional charts $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ where

$$U_i = \psi_i(E_i) \text{ and } \varphi_i = \psi_i^{-1} \text{ are bi-Lipschitz.}$$

This fact inspires the following definition, see for example [64, Section 5].

Definition 2.11 We say that a $k-$dimensional chart (U, φ) is $k-$*rectifiable* if φ is bi-Lipschitz on U and

$$\varphi_{\#}(\mu_{|U}) \ll \mathcal{L}^k_{|\varphi(U)}. \tag{2.6}$$

We say that a metric measure space (X, d, μ) *admits a rectifiable decomposition* if there exists a countable family $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ of rectifiable charts such that $\mu\left(X\backslash \bigcup_i U_i\right) = 0$.

Lemma 2.12. *Let (X, d, μ) be a metric measure space and let (U, φ) be a rectifiable chart of dimension k in X. Then there exist a countable collection of Borel sets $V_i \subset U$ with $\mu\left(U\backslash \bigcup_i V_i\right) = 0$ and constants $C_i > 0$ so that*

$$\frac{1}{C_i}\mathcal{L}^k_{|\varphi_i(V_i)} \leq \varphi_{i\,\#}(\mu_{|V_i}) \leq C_i \mathcal{L}^k_{|\varphi_i(V_i)} \tag{2.7}$$

where $\varphi_i = \varphi_{|V_i} : V_i \to \mathbb{R}^k$.

Proof. The refinement of the chart is given by

$$V_i := \varphi^{-1}\left(\left\{\frac{1}{i} < \frac{d[\varphi_{\#}(\mu_{|U})]}{d\mathcal{L}^k_{|\varphi(U)}} \leq i\right\}\right).$$

By the Radon-Nykodym Theorem, see Theorem 1.18, we have

$$\varphi_{\#}(\mu_{|U})(E) = \int_E \frac{d[\varphi_{\#}(\mu_{|U})]}{d\mathcal{L}^k_{|\varphi(U)}} d\mathcal{L}^k_{|\varphi(U)}.$$

Therefore, (2.7) holds and the set $E = \left\{\frac{d[\varphi_{\#}(\mu_{|U})]}{d\mathcal{L}^k_{|\varphi(U)}} = 0\right\}$ satisfies

$$\mu_{|U}\left(\varphi^{-1}(E)\right) = \varphi_{\#}(\mu_{|U})(E) = 0,$$

proving that $\{V_i\}_{i\in\mathbb{N}}$ covers U up to a set of measure zero. ∎

Due to Remark 2.10, it follows that if X is k-rectifiable with maps $\psi_i : E_i \to X$ and (2.6) holds for $\varphi_i = \psi_i^{-1}$, then X admits a rectifiable decomposition. Conversely, by Remark 1.7 and Lemma 1.8 the condition (2.6) is stronger than the absolute continuity condition in Definition 2.9, so a rectifiable chart is, in particular, a rectifiable space. However, X admitting a rectifiable decomposition does not necessarily imply that it is a rectifiable space since the charts of the decomposition are of arbitrary dimension. Then the next result follows naturally.

Lemma 2.13. *Let (X, d, μ) be a metric measure space. Then X admits a rectifiable decomposition if, and only if, there exists a disjoint family of k_i-rectifiable spaces $\{U_i\}_{i=1}^{\infty}$ such that $X = Z \cup \bigcup_i U_i$, with $\mu(Z) = 0$, and*

$$\frac{1}{C_{i,j}}\mathcal{L}^{k_i}_{|E_{i,j}} \leq (\psi_{i,j}^{-1})_{\#}(\mu_{|\psi_{i,j}(E_{i,j})}) \leq C_{i,j}\mathcal{L}^{k_i}_{|E_{i,j}}$$

for some constants $C_{i,j} > 0$, where $\{(E_{i,j}, \psi_{i,j})\}_{j=1}^{\infty}$ are given by the k-rectifiability of U_i.

By [30, Theorem 4.1.1] it follows that an LDS such that its chart maps are bi-Lipschitz also admits a rectifiable decomposition, just by considering as the rectifiable charts the ones given by the LDS condition. However, the converse is not true [13, p. 5]. Instead, what one has due to Lemma 2.12 and [64, Lemma 4.1] is a decomposition into weak Lipschitz charts with bi-Lipschitz maps. Given the hypothesis that porous sets in X are null, we know that weak Lipschitz charts self-improve into Lipschitz charts (see Corollary 2.8). Therefore, we can deduce the following direct consequence of Lemma 2.12.

Corollary 2.14. *Let (X, d, μ) be a metric measure space such that all porous sets are null and consider a chart (U, φ) of dimension k in X. Then (U, φ) is a rectifiable chart if and only if it admits a decomposition into Lipschitz charts with bi-Lipschitz chart maps.*

Corollary 2.15. *Let (X, d, μ) be a metric measure space. Then the following are equivalent:*

(i) X admits a rectifiable decomposition and all its porous sets are of measure zero.

(ii) X is an LDS and admits a rectifiable decomposition.

(iii) X is an LDS with bi-Lipschitz chart maps.

The notion of rectifiability in Definition 2.11 is slightly more restrictive than the classical definition of a k-rectifiable set given in Definition 2.9, primarily due to the absolute continuity condition. However, when dealing with an LDS, due to [30, Theorem 4.1.1], the discussion above allows to treat both definitions equally, up to a certain decomposition if necessary. Also in what follows we will make use of the bi-Lipschitz behaviour of the charts whenever X admits a rectifiable decomposition.

2.3 The metric differential.

In this section we will follow the ideas of [44] to define a metric differential for mappings in metric measure space with an atlas, and study the relationship of the generalization of Kirchheim's result [71, Theorem 2] and the rectifiability of a chart. Furthermore, we discuss a Stepanov result for spaces admitting a rectifiable decomposition.

We recall that a seminorm is a mapping $\mathrm{n} : \mathbb{R}^N \to \mathbb{R}^+$ which is non-negative, subadditive and absolutely homogeneous. We will denote by sn^N to the space of all seminorms in $\mathbb{R}^N$ endowed with the following metric

$$D(\mathrm{n}_1, \mathrm{n}_2) := \sup_{|v|\leq 1} |\mathrm{n}_1(v) - \mathrm{n}_2(v)| = \mathrm{Lip}(\mathrm{n}_1 - \mathrm{n}_2)(0).$$

We also denote

$$|||\mathrm{n}||| := D(\mathrm{n}, 0) = \mathrm{Lip}(\mathrm{n})(0).$$

Definition 2.16 Given a metric measure space (X, d, μ) and a weak Lipschitz chart (U, φ) of dimension N in X, we say that a mapping $f : X \to Y$ with values in a metric space (Y, d_Y) is *weakly metrically differentiable* at a given point $x \in U$ with respect to the chart (U, φ) if there exists $\mathrm{md}_x f \in \mathrm{sn}^N$ such that

$$\lim_{\substack{y \to x \\ y \in U}} \frac{|d_Y(f(x), f(y)) - \mathrm{md}_x f(\varphi(x) - \varphi(y))|}{d(x, y)} = 0. \tag{2.8}$$

If (U, φ) is a Lipschitz chart then we say that f is *metrically differentiable* at $x \in U$ with respect to the chart (U, φ) provided (2.8) holds for $y \to x$ with $y \in X$ arbitrary.

If $X = \mathbb{R}^N$ then $(\mathbb{R}^N, id_{\mathbb{R}^N})$ is a Lipschitz chart in X due to Rademacher's Theorem (Lemma 1.1). Hence, for $f : \mathbb{R}^N \to Y$, our definition yields the classical definition of metric differentiability.

Example 2.17 *In a general setting, Definition 2.16 is quite restrictive. For example, let us consider the Heisenberg group $\mathbb{H}$ equipped with the Carnot-Carathéodory metric and the Haar measure. We refer to [72] for the fact that $\mathbb{H}$ is an LDS with a single chart φ, defined by $\varphi(x,y,z) = (x,y)$. In this case one can prove that the identity map from $\mathbb{H}$ to itself is nowhere metrically differentiable. Indeed, if we fix a point $(x_0, y_0, z_0) \in \mathbb{H}$, then for any seminorm $\mathrm{n} \in \mathrm{sn}^2$*

$$\mathrm{n}(\varphi(x_0, y_0, z) - \varphi(x_0, y_0, z_0)) = 0.$$

Now, for $\mu-$almost every $p = (x,y,z) \in \mathbb{H}$, there exist $z_n \to z$ such that $p_n = (x, y, z_n) \to (x, y, z)$ but $\varphi((x, y, z_n)) = \varphi((x, y, z))$ for every $n \in \mathbb{N}$. Let f be the identity map in $\mathbb{H}$. Then

$$0 < d_{\mathbb{H}}(p, p_n) = d_{\mathbb{H}}(f(p), f(p_n)) \neq \mathrm{md}_p f(\varphi(p) - \varphi(p_n)) = 0,$$

so f is not metrically differentiable at $\mu-$a.e. $p \in \mathbb{H}$.

As we will see in this section, the difficulties of obtaining metric differentiability in the Heisenberg group are closely related to its non-rectifiability. Recall that the Heisenberg group is a purely 2−unrectifiable space. In order to study the nature of this problem in a general metric measure space we first introduce some properties of the metric differential.

Lemma 2.18. *Let (X, d, μ) be a metric measure space with (U, φ) a rectifiable chart of dimension N in X, and (Y, d_Y) a metric space. Then there exists a constant $C > 0$ depending only on φ such that, for any mapping $f : X \to Y$ metrically differentiable at $x \in U$ with respect to (U, φ), one has*

$$\frac{1}{C}\mathrm{Lip}f(x) \leq |||\mathrm{md}_x f||| \leq C\mathrm{Lip}f(x). \tag{2.9}$$

Proof. Let $x \in U$. As φ is Lipschitz there exists $C > 0$ such that

$$\frac{\mathrm{md}_x f(\varphi(y) - \varphi(x))}{d(x,y)} \leq C \frac{\mathrm{md}_x f(\varphi(y) - \varphi(x))}{|\varphi(y) - \varphi(x)|}. \tag{2.10}$$

On the other hand, for any vector $z \in \mathbb{R}^N$

$$\mathrm{Lip}(\mathrm{md}_x f)(0) = \limsup_{z \to 0} \frac{\mathrm{md}_x f(z)}{|z|}.$$

Considering $z = \varphi(y) - \varphi(x)$, $y \to x$ implies $z \to 0$ as φ is continuous, and then

$$\lim_{y \to x} \frac{\mathrm{md}_x f(\varphi(y) - \varphi(x))}{|\varphi(y) - \varphi(x)|} = \lim_{z \to 0} \frac{\mathrm{md}_x f(z)}{z} = \mathrm{Lip}(\mathrm{md}_x f)(0) = |||\mathrm{md}_x f|||.$$

This, together with (2.10) and the definition of metric differential at x, concludes that

$$\mathrm{Lip}f(x) = \lim_{y \to x} \frac{d_Y(f(x), f(y))}{d(x,y)} = \lim_{y \to x} \frac{\mathrm{md}_x f(\varphi(y) - \varphi(x))}{d(x,y)} \leq C \cdot |||\mathrm{md}_x f|||.$$

For the first inequality in (2.9) we use the Lipschitz condition of φ^{-1} to obtain

$$\frac{\mathrm{md}_x f(\varphi(y) - \varphi(x))}{d(x,y)} \geq \frac{1}{C} \frac{\mathrm{md}_x f(\varphi(y) - \varphi(x))}{|\varphi(y) - \varphi(x)|}. \tag{2.11}$$

Then we proceed analogously as we did in the second inequality, but using (2.11) instead of (2.10).
■

Lemma 2.18 provides a nice estimate of the metric differential for points in the set $S(f) := \{x \in X : \operatorname{Lip} f(x) < \infty\}$ of a measurable mapping $f : X \to Y$. Moreover, we can decompose $S(f)$ as the union of the sets

$$E_k := \left\{ x \in S(f) : \frac{d_Y(f(x), f(y))}{d(x,y)} \leq k \text{ if } d(x,y) < \frac{1}{k} \right\}, \tag{2.12}$$

allowing to locally study f as a Lipschitz map in order to apply a Rademacher-type result to $f_{|E_k}$ and obtain metric differentiability almost everywhere in E_k. However, to recover the notion of metric differentiability for f and not just for $f_{|E_k}$ one must go through similar arguments as we did for Corollary 2.8. We will do so in Theorem 2.20 below, but for that we first proof that E_k is measurable for each k, in order to have that a.e. point is a density point.

Lemma 2.19. *Let (X, d, μ) be a metric measure space and (Y, d_Y) a metric space. Let $f : X \to Y$ and $L, \delta > 0$. Then the following set is closed:*

$$E := \left\{ x \in X : \frac{d_Y(f(x), f(y))}{d(x,y)} \leq L \text{ if } d(x,y) < \delta \right\}.$$

Proof. Consider a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that $d(x_n, x) \to 0$. Let $y \in X$ be such that $d(x,y) < \delta$ and $0 < \varepsilon < \delta - d(x,y)$, then there exists $n_0 \in \mathbb{N}$ such that $d(x, x_{n_0}) < \varepsilon$. Hence

$$d(y, x_{n_0}) \leq d(x, x_{n_0}) + d(x,y) < \varepsilon + d(x,y) < \delta.$$

Then we have

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f(x_{n_0})) + d_Y(f(x_{n_0}), f(y)) \leq L(d(x, x_{n_0}) + d(x_{n_0}, y)) \\ &\leq L(2d(x, x_{n_0}) + d(x,y)) < 2L\varepsilon + Ld(x,y) \end{aligned}$$

and, as ε is arbitrarily small, we have $d_Y(f(x), f(y)) \leq Ld(x,y)$, proving that $x \in E$, and thus, E is closed. ■

Theorem 2.20. *Suppose (X, d, μ) is a metric measure space such that porous sets are null and let (U, φ) be a rectifiable chart of dimension N in X. Let (Y, d_Y) be a metric space and $f : X \to Y$, then f is metrically differentiable at almost every point $x \in U \cap S(f)$ such that there exists $\operatorname{md}_x f \in \operatorname{sn}^N$ satisfying*

$$\lim_{\substack{y \to x \\ y \in U}} \frac{|d_Y(f(x), f(y)) - \operatorname{md}_x f(\varphi(x) - \varphi(y))|}{d(x,y)} = 0. \tag{2.13}$$

In particular, weak metric differentiability a.e. yields metric differentiability a.e. with respect to (U, φ).

Proof. First recall that $S(f) = \bigcup_{k \in \mathbb{N}} E_k$ where E_k are defined as in (2.12), and they are Borel for every $k \in \mathbb{N}$ due to Lemma 2.19.

Fix $k_0 \in \mathbb{N}$ and $x \in U \cap E_{k_0}$ satisfying the conclusion of Lemma 2.7 for the Borel set $U \cap E_{k_0}$ and such that (2.13) holds at x. Let $\{y_j\}_{j \in \mathbb{N}}$ be a sequence in X converging to x, then by Lemma 2.7, and particularly by the discusion leading to (2.5), passing to a subsequence if necessary (that we still denote by y_j), there exists $\{z_j\}_{j \in \mathbb{N}}$ in $U \cap E_{k_0}$ such that

$$\limsup_{j \to \infty} \frac{d(y_j, z_j)}{d(x, y_j)} = 0. \tag{2.14}$$

In particular, $z_j \to x$. Now for each $j \in \mathbb{N}$ we have

$$
\begin{aligned}
\frac{|d_Y(f(x), f(y_j)) - \mathrm{md}_x f(\varphi(x) - \varphi(y_j))|}{d(x, y_j)} &\le \frac{|d_Y(f(x), f(y_j)) - d_Y(f(x), f(z_j))|}{d(x, y_j)} \\
&+ \frac{|d_Y(f(x), f(z_j)) - \mathrm{md}_x f(\varphi(x) - \varphi(z_j))|}{d(x, y_j)} \\
&+ \frac{|\mathrm{md}_x f(\varphi(y_j) - \varphi(z_j))|}{d(x, y_j)} \\
&=: \mathrm{I}_j + \mathrm{II}_j + \mathrm{III}_j
\end{aligned}
$$

The proof follows by checking

$$
\limsup_{j\to\infty} \mathrm{I}_j = \limsup_{j\to\infty} \mathrm{II}_j = \limsup_{j\to\infty} \mathrm{III}_j = 0.
$$

We now estimate each term separately.
First, recall that $z_j \subset E_{k_0}$, and since $d(y_j, z_j) \to 0$ as $j \to \infty$, for sufficiently large j we can assume $d(y_j, z_j) < 1/k_0$. Then, by the definition of E_{k_0} we have

$$
\begin{aligned}
\limsup_{j\to\infty} \mathrm{I}_j &:= \limsup_{j\to\infty} \frac{|d_Y(f(x), f(y_j)) - d_Y(f(x), f(z_j))|}{d(x, y_j)} \\
&\le \limsup_{j\to\infty} \frac{d_Y(f(y_j), f(z_j))}{d(x, y_j)} \le k_0 \limsup_{j\to\infty} \frac{d(y_j, z_j)}{d(x, y_j)} = 0.
\end{aligned}
$$

To estimate II_j notice that using triangle inequality in (2.14) yields

$$
\lim_{j\to\infty} \frac{d(x, z_j)}{d(x, y_j)} \le 1,
$$

and then by the hypothesis (2.13), since $z_j \in U$ for every $k \in \mathbb{N}$

$$
\begin{aligned}
\limsup_{j\to\infty} \mathrm{II}_j &:= \limsup_{j\to\infty} \frac{|d_Y(f(x), f(z_j)) - \mathrm{md}_x f(\varphi(x) - \varphi(z_j))|}{d(x, y_j)} \\
&= \limsup_{j\to\infty} \frac{|d_Y(f(x), f(z_j)) - \mathrm{md}_x f(\varphi(x) - \varphi(z_j))|}{d(x, z_j)} \frac{d(x, z_j)}{d(x, y_j)} \le 0.
\end{aligned}
$$

Last, by the Lipschitz condition of φ, Lemma 2.18 applied to $f_{|U \cap S(f)}$ and (2.14) we have

$$
\begin{aligned}
\limsup_{j\to\infty} \mathrm{III}_j &:= \limsup_{j\to\infty} \frac{|\mathrm{md}_x f(\varphi(y_j) - \varphi(z_j))|}{d(x,_k)} \\
&\le \limsup_{j\to\infty} \frac{\mathrm{LIP}(\varphi)|||\mathrm{md}_x f|||d(y_j, z_j)}{d(x, y_j)} \\
&\le \limsup_{j\to\infty} \frac{C\mathrm{LIP}(\varphi)\mathrm{Lip} f(x) d(y_j, z_j)}{d(x, y_j)} = 0.
\end{aligned}
$$

We conclude that f is metrically differentiable at x. Now for each $k \in \mathbb{N}$ we denote by N_k the set of all points in $U \cap E_k$ such that the conclusion of Lemma 2.7 does not hold. Then for each $k \in \mathbb{N}$, since porous sets in X are null and $U \cap E_k$ is measurable by Lemma 2.19, $\mu(N_k) = 0$. Thus $\mu(S(f) \setminus N) = 0$, where $N = \bigcup_k N_k$. We have then proved that, for almost every point $x \in S(f) \cap U$ such that (2.13) holds, f is metrically differentiable at x. ■

2.3.1 Metric differentiability in rectifiable spaces.

We now generalize Kirchheim's Theorem [71, Theorem 2] to metric measure spaces, where the metric differential is obtained with respect to a rectifiable chart.

Corollary 2.21. *Let (X, d, μ) be a metric measure space and let (U, φ) be a $k-$rectifiable chart. Then every Lipschitz mapping $f : X \to Y$ into a metric space (Y, d_Y) is weakly metrically differentiable almost everywhere with respect to (U, φ).*

Proof. Let $g = f \circ \varphi^{-1} : \varphi(U) \to Y$. Notice that g is a composition of Lipschitz mappings, so it is also Lipschitz. By Kirchheim's Theorem [71, Theorem 2], for $\mathcal{H}^k$-almost every $z \in \varphi(U)$ there exists a unique seminorm $\mathrm{md}_z g$ on $\mathbb{R}^k$ such that

$$\lim_{\substack{y\to z\\ y\in\varphi(U)}} \frac{|d_Y(g(z), g(y)) - \mathrm{md}_z g(y-z)|}{|y-z|} = 0. \tag{2.15}$$

On the other hand, $g(\varphi(x)) = f(x)$ for each $x \in U$. Fix $x_0 \in U$ such that for $z_0 := \varphi(x_0)$ there exists a unique seminorm $\mathrm{md}_{z_0} g$ on $\mathbb{R}^k$ such that (2.15) holds. As φ is continuous, if $x \in U$ and $x \to x_0$, then $\varphi(x) \to z_0$. Therefore

$$\begin{aligned}
&\lim_{\substack{x\to x_0\\ x\in U}} \frac{|d_Y(f(x_0), f(x)) - \mathrm{md}_z g(\varphi(x) - \varphi(x_0))|}{d(x, x_0)} \\
\le\ &\lim_{\substack{z\to z_0\\ z=\varphi(x)}} \frac{|d_Y(g(z_0), g(z)) - \mathrm{md}_{z_0} g(z-z_0)|}{\frac{1}{C}|z-z_0|} = 0.
\end{aligned}$$

where C is the Lipschitz constant of φ. Then $\mathrm{md}_{x_0} f := \mathrm{md}_{\varphi(x_0)} g$ is the metric differential of f at $x_0 \in U$. We finish the proof by noticing that, because $\varphi_\#(\mu_{|U}) \ll \mathcal{L}^k_{|\varphi(U)}$ and

$$\mathcal{L}^k(\{z_0 \in \varphi(U) : \mathrm{md}_{z_0} g \text{ does not exist}\}) = 0,$$

we conclude that

$$\mu(\{x_0 \in U : \mathrm{md}_{x_0} f \text{ does not exist}\}) = 0.$$

■

We now prove the necessity of a chart admitting a rectifiable decomposition, when the chart is considered as a metric measure space with the restricted metric and measure, in order to obtain that every Lipschitz map is weakly metrically differentiable almost everywhere with respect to the chart. First we recall the following technical lemma that will be useful in the sequel.

Lemma 2.22. **[68, 3.1.1]** *Let (X, μ) be a σ-finite measure space. Suppose there is a property P defined for the measurable sets of X which satisfies the following. Every measurable set $A \subset X$ with $\mu(A) > 0$, contains a measurable set V with $\mu(V) > 0$ such that V satisfies property P. Then there exists a countable decomposition*

$$X = Z \cup \bigcup_i V_i,$$

where $\mu(Z) = 0$ and $\{V_i\}$ is a collection of mutually disjoint measurable sets each of which satisfies property P.

Theorem 2.23. *Let (X, d, μ) be a metric measure space and (U, φ) a k-dimensional weak Lipschitz chart. Suppose that for every metric space (Y, d_Y) and every Lipschitz mapping $f : X \to Y$, f is weakly metrically differentiable with respect to (U, φ) at almost every $x \in U$. Then $(U, \mathrm{d}_{|U}, \mu_{|U})$ admits a rectifiable decomposition.*

Proof. In order to see that $(U, d|_U, \mu|_U)$ admits a rectifiable decomposition we just need to prove the existence of countably many subsets $V_i \subset U$ so that

$$\mu\left(U \setminus \bigcup_{i\in\mathbb{N}} V_i\right) = 0$$

and $\varphi|_{V_i}$ is bi-Lipschitz for all i. Indeed, since the condition $\varphi_{\#}(\mu_{|U}) \ll \mathcal{L}^k_{|\varphi(U)}$ given by [30, Theorem 1.1] would transfer to any subset of U, then $(V_i, \varphi_{|V_i})$ would be a rectifiable chart for all $i \in \mathbb{N}$.

Let us construct this decomposition. Let $f : X \to \ell^\infty(K)$ be a Lipschitz extension (see Lemma 1.2) of the identity $\kappa \circ \mathrm{id}_U : U \to \ell^\infty(U)$, where κ denotes the Kuratowski embedding (see Lemma 1.3). As f is a Lipschitz mapping then by our assumption there exists a seminorm $\mathrm{md}_x f : \mathbb{R}^k \to \mathbb{R}$ such that

$$\lim_{U\ni y\to x} \frac{|d(f(x), f(y)) - \mathrm{md}_x f(\varphi(y) - \varphi(x))|}{d(x, y)} = 0 \tag{2.16}$$

for $\mu-$almost every $x \in U$. Let $N \subset U$ be a null set such that (2.16) holds everywhere in $U\setminus N$ and rewrite the limit in (2.16) as follows:

$$\lim_{n\to\infty} F_n(x) = 0 \quad \text{where} \quad F_n(x) := \sup_{y\in B(x,\frac{1}{n})\cap U} \frac{|d(f(x), f(y)) - \mathrm{md}_x f(\varphi(x) - \varphi(y))|}{d(x, y)}.$$

By Egoroff's Theorem there is a set $K \subset U \setminus N$ of almost full measure such that $F_n \to 0$ uniformly in K. Moreover, since μ is Radon, we can assume that K is compact and of positive measure. By the uniform convergence there exists $n_0 \in \mathbb{N}$ so that

$$\sup_{y\in B(x,\frac{1}{n_0})\cap K} \frac{|d_Y(f(x), f(y)) - \mathrm{md}_x f(\varphi(x) - \varphi(y))|}{d(x, y)} \le F_{n_0}(x) \le \frac{1}{2}, \quad x \in K.$$

In particular, for any $x \in K$ and any $y \in B(x, \frac{1}{n_0}) \cap K$, since $d_Y(f(x), f(y)) = d(x, y)$ then

$$d(x, y) \le 2\mathrm{md}_x f(\varphi(x) - \varphi(y)) \le 2|||\mathrm{md}_x f|||\, |\varphi(x) - \varphi(y)|.$$

Consider a covering of K by balls $\left\{B(x, \frac{1}{2n_0})\right\}_{x\in K}$. By compactness, there exist $x_1, x_2, \cdots, x_m$ such that

$$K \subset \bigcup_{i=1}^{m} B\left(x_i, \frac{1}{2n_0}\right).$$

Choose $x_0 \in \{x_1, x_2, \cdots, x_m\}$ such that $\mu\left(K \cap B(x_0, \frac{1}{2n_0})\right) > 0$. Define

$$A_k := \left\{x \in K : \mathrm{md}_x f \text{ exists and } \frac{1}{k} \le |||\mathrm{md}_x f||| \le k\right\}.$$

Recall that f is the identity mapping on U, so at every point $x \in K$ we have that $|||\mathrm{md}_x f||| \neq 0$. As $\bigcup_{k=1}^\infty A_k \supset K$, there exists $k_0 \ge 1$ such that $\mu\left(A_{k_0} \cap K \cap B(x, \frac{1}{2n_0})\right) > 0$. Now, let $x, y \in A_{k_0} \cap K \cap B(x_0, \frac{1}{2n_0})$. Since $y \in A_{k_0} \cap K \cap B(x, \frac{1}{n_0})$ we have that

$$d(x, y) \le 2k_0|\varphi(x) - \varphi(y)|,$$

that is, φ is injective on $A_{k_0}\cap K\cap B(x_0, \frac{1}{2n_0})$ and so φ^{-1} is $2k_0$-Lipschitz on $\varphi(A_{k_0}\cap K\cap B(x_0, \frac{1}{2n_0}))$. In particular, φ is bi-Lipschitz on $A_{k_0} \cap K \cap B(x_0, \frac{1}{2n_0})$. By Lemma 2.22, there exists a countable decomposition

$$U = Z \cup \bigcup_i V_i,$$

where $\mu(Z) = 0$ and $\{V_i\}$ is a collection of mutually disjoint measurable sets each of which satisfies that φ is bi-Lipschitz on V_i. ■

Consider a weak Lipschitz chart (U, φ). Applying Corollary 2.21 to a chart (V_i, φ_i) the rectifiable decomposition of (U, φ) given by Theorem 2.23 would yield weak metric differentiability almost everywhere with respect to the chart (V_i, φ_i), but not necessarily with respect to the chart (U, φ). However, if we assume that all porous sets in X have measure zero, since by the construction in the proof of Theorem 2.23 $\varphi_i = \varphi_{|V_i}$ then by Theorem 2.20 we can extend the weak metric differentiability with respect to (V_i, φ_i) to metric differentiability with respect to (U, φ) for almost every point in U. This discussion leads to the following result as a consequence of Theorem 2.20, Corollary 2.21 and Theorem 2.23.

Corollary 2.24. *Let (X, d, μ) be a Lipschitz differentiability space. Then X admits a rectifiable decomposition if, and only if, for every metric space (Y, d_Y) and every Lipschitz mapping $f : X \to Y$, f is metrically differentiable almost everywhere with respect to the atlas of Lipschitz charts in X.*

2.3.2 Stepanov Theorem.

Theorem 2.25. *Let (X, d, μ) be a metric measure space whose porous sets are null and (U, φ) a rectifiable chart in X. Then any mapping $f : X \to Y$, where (Y, d_Y) is a metric space, is metrically differentiable with respect to (U, φ) almost everywhere on $U \cap S(f)$.*

Proof. By Kuratowski's embedding we can suppose without loss of generality that $f : X \to \ell^\infty(Y)$, as differentiability is invariant under isometries. The goal is to prove that f is metrically differentiable at almost every point of $M := U \cap S(f)$.

For each $k \in \mathbb{N}$ consider the set

$$E_k := \left\{ x \in S(f) : \frac{\|f(x) - f(y)\|_{\ell^\infty}}{d(x, y)} \leq k \text{ if } d(x, y) < \frac{1}{k} \right\},$$

and let $\{B_{k,j}\}_{j=1}^\infty$ be a covering by open sets of X such that $\operatorname{diam}(B_{k,j}) \leq \frac{1}{k}$. For each $k, j \in \mathbb{N}$ denote $M_{k,j} = M \cap E_k \cap B_{k,j}$.

Clearly $S(f) = \bigcup_k E_k$, and then it is measurable by Lemma 2.19. Hence, M is measurable as so it is U by definition, concluding that $M_{k,j}$ is a measurable set for each $k, j \in \mathbb{N}$.

On the other hand $S(f) = \bigcup_{k,j} E_k \cap B_{k,j}$ as $\{B_{k,j}\}$ is a covering of X, and then we have

$$M = \bigcup_{k,j=1}^\infty M_{k,j},$$

so it suffices to prove that, for each $k, j \in \mathbb{N}$, f is metrically differentiable almost everywhere in $M_{k,j}$.

Fix $j, k \in \mathbb{N}$. First, notice that $f|_{M_{k,j}}$ is Lipschitz. Indeed, if $x, y \in M_{k,j}$ then $d(x, y) \leq \frac{1}{k}$ since $x, y \in B_{k,j}$, and then by the definition of E_k one has

$$\|f(x) - f(y)\|_{\ell^\infty} \leq k d(x, y).$$

Then there exists a Lipschitz mapping $\hat{f} : X \to \ell^\infty(Y)$ such that $\hat{f}|_{M_{k,j}} = f|_{M_{k,j}}$, and by Corollary 2.21 and Theorem 2.20 $\hat{f}$ is metrically differentiable with respect to (U, φ) at almost every point $x \in U$.

Let $x \in M_{k,j}$ such that $\hat{f}$ is metrically differentiable at x. Then there exists a seminorm $\mathrm{md}_x \hat{f} \in \mathrm{sn}^{\dim(\varphi(U))}$ such that

$$\lim_{y \to x} \frac{|\|\hat{f}(x) - \hat{f}(y)\|_{\ell^\infty} - \mathrm{md}_x \hat{f}(\varphi(x) - \varphi(y))|}{d(x,y)} = 0.$$

If $y \in M_{k,j}$ then $\hat{f}(y) = f(y)$, so we have

$$\lim_{\substack{y \to x \\ y \in M_{k,j}}} \frac{|\|f(x) - f(y)\|_{\ell^\infty} - \mathrm{md}_x \hat{f}(\varphi(x) - \varphi(y))|}{d(x,y)} = 0. \tag{2.17}$$

We conclude that f is metrically differentiable at x by Theorem 2.20. ■

Chapter 3

Vector-valued Sobolev spaces

This chapter is devoted to the study of first order vector-valued Sobolev spaces, and its regularity properties, such as Lipschitz conditions, absolute continuity, existence of derivatives and differentiability.

In the past decades several articles about alternative definitions of Sobolev mappings have been published all of which are suitable for general metric spaces as domains. For us it is specially interesting the one approach via upper gradients, the so called Newton Sobolev space introduced by Shanmugalingam in [86]. For a mapping $f \in L^p(X, V)$ we define the seminorm

$$\|f\|_{N^{1,p}(X,V)} := \|f\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p-upper gradients $g \in L^p(X)$ of f. The *Newton-Sobolev space* $N^{1,p}(X, V)$ is the collection of equivalence classes of measurable mappings $f : X \to V$ such that $\|f\|_{N^{1,p}(X,V)} < \infty$ under the equivalence relationship

$$f \sim g \iff \|f - g\|_{N^{1,p}} = 0.$$

In [56, Theorem 7.1.20] it is shown that, up to a change of representative, the vector-valued Newtonian space $N^{1,p}(X, V)$ coincides with a scalarization if itself, that is, that $\langle v^*, f\rangle \in N^{1,p}(X)$ for all $v^* \in V^*$ with $\|v^*\| \leq 1$ with a uniform L^p bound for all its minimal upper gradients. In fact, this scalarization can also be done by composing with each 1-Lipschitz maps or by the map $\|f - v\|$ for every $v \in V$.

On the other hand, it is also interesting to study vector-valued Sobolev spaces in Euclidean domains, since here we can properly define the classical notion of Sobolev space $W^{1,p}(\Omega, V)$ for an open set $\Omega \subset \mathbb{R}^N$ (notice that in a metric setting the lack of directions is an obstacle to define weak partial derivatives). For real-valued functions it is known that $N^{1,p}(\Omega) = W^{1,p}(\Omega)$, so the scalarization mentioned above can be considered in $W^{1,p}(\Omega)$ in this setting. This was considered by Reshetnyak in 1997 [85] for mappings with values in a complete and separable metric space using $x \mapsto d(f(x), z)$ for every z in the target space as the scalarization condition. Later on, what Reshetnyak introduced was studied in the literature as the Sobolev-Reshetnyak space, with special interest for mappings with values in a Banach space.

In this chapter, we will focus on the Sobolev-Reshetnyak space $R^{1,p}(\Omega, V)$ of mappings from an open set $\Omega \subset \mathbb{R}^n$ into a Banach space V. Although, as we mentioned, we have that $R^{1,p}(\Omega, V) = N^{1,p}(\Omega, V)$ (with different equivalence classes) the use of the Sobolev-Reshetnyak description has a much shorter history in the literature than the one of a Newton-Sobolev map. Some references of interest for the study of $R^{1,p}(\Omega, V)$ include [50, 42, 26, 27, 65]. We showcase in this chapter that the description via scalarization of a Sobolev space given by $R^{1,p}(\Omega, V)$ gives a rich theory in the Euclidean setting for the study of vector-valued mappings. It is also worth noticing that thanks to the Kuratowski embedding the results here can be extended to mappings with values in a metric space.

In Sections 3.2 and 3.3 we give the formal definitions and some properties of $W^{1,p}(\Omega, V)$ and $R^{1,p}(\Omega, V)$ in order to compare them, and see that in general $W^{1,p}(\Omega, V)$ is a closed subset of $R^{1,p}(\Omega, V)$ (see 3.18, but the reverse inclusion is not true in general. This was studied in [50] where the authors conclude that both spaces are the same as long as V is dual to a separable Banach space, however there is a subtlety regarding measurability of weak*-derivatives that must be considered as we will see in Corollary 3.30.

We will make use of the ideas behind the Beppo-Levi Characterization of $W^{1,p}(\Omega, V)$ in order to understand the regularity subtleties that differ from those of $R^{1,p}(\Omega, V)$.

Theorem 3.1. (Beppo-Levi Characterization [81, Theorem 1.41]) *$f \in W^{1,p}(\Omega, V)$ if, and only if, it is absolutely continuous and differentiable almost everywhere along almost every line parallel to a coordinate axis.*

Unfortunately, this characterization does not hold for Sobolev-Reshetnyak spaces, but the absolute continuity along lines does, so it is easy to foresee that the Radon-Nikodým property suffices in order to have $W^{1,p}(\Omega, V) = R^{1,p}(\Omega)$. We will see the details in Section 3.3 and proof in Theorem 3.22 that the RNP is, not only sufficient, but necessary to obtain the equality.

This discussion also leads to a natural development of a Beppo-Levi type characterization of $R^{1,p}(\Omega, V)$ using weaker notions of differentiability such as metric or weak*-derivatives that follow from absolute continuity even if V does not have the Radon-Nikodým property. To this end, we study absolutely continuous mappings in Section 3.1 and provide a construction via Fubini's Theorem of metric and weak* partial derivatives for an absolutely continuous along almost every line map (see Theorem 3.9). We then end this chapter by providing some characterizations of $R^{1,p}(\Omega, V)$ using these notions of derivatives in Section 3.4.

3.1 Derivatives for absolutely continuous mappings.

Recall that, if (X, d) is a metric space, a mapping $f : [a, b] \longrightarrow X$ is said to be *absolutely continuous* if, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{m} |b_i - a_i| < \delta \Longrightarrow \sum_{i=1}^{m} d(f(b_i), f(a_i)) < \varepsilon,$$

for any disjoint intervals $[a_1, b_1], \ldots, [a_m, b_m] \subset [a, b]$. With standard arguments one can prove that an absolutely continuous mapping $f : [a, b] \to V$ is a compact rectifiable curve and its length function $s_f(t) = \ell(f|_{[a,t]})$ is also absolutely continuous.

In the case of functions $f : \Omega \to X$ defined on an open subset of $\mathbb{R}^N$, we will consider the property of being absolutely continuous when restricted to each compact subinterval of $\ell \cap \Omega$, for almost every line ℓ parallel to a coordinate axis. The precise definition is as follows. Here, as usual, $\{e_1, \ldots, e_N\}$ denotes the unit vector basis of $\mathbb{R}^N$.

Definition 3.2 Let $\Omega \subset \mathbb{R}^N$ be an open set and X a metric space. We say that $f : \Omega \to X$ is *absolutely continuous on a.e. line* (in short *ACL*) if, for each $i = 1, \ldots, N$ there exists a $\mathcal{L}^{N-1}$-null subset Z_i of the hyperplane $H_i = \{(x_1, \ldots, x_N) : x_i = 0\}$ such that for every $u \in H_i \setminus Z_i$, the function $t \mapsto f(u + te_i)$ is absolutely continuous on every compact interval $[a, b]$ such that $u + te_i \in \Omega$ for $a \leq t \leq b$.

In this section we will work on the subtleties of applying Fubini's Theorem to the derivatives that naturally arise from the ACL property of a mapping. Recall that under the assumption of

V having the Radon-Nikodým property the absolutely continuity yields the standard notion of differentiation, but when this property is not present weaker notions can be consider as we now inroduce.

3.1.1 Metric derivatives.

Along this section we will study mappings $f : \Omega \to V$ from an open set $\Omega \subset \mathbb{R}^N$ to a Banach space V, We first recall the definition of metric derivatives, considered by Kirchheim in [71] (see also [34]).

Definition 3.3 Let $\Omega \subset \mathbb{R}^N$ be an open set and V a Banach space. For each $i = 1, \ldots, N$, we define the *i-th partial metric derivative* of a function $f : \Omega \to V$ at $x \in \Omega$ as the following limit, if it exists:

$$m\partial_i f(x) := \lim_{h \to 0} \frac{\|f(x + he_i) - f(x)\|}{|h|}.$$

In the case that $N = 1$, we denote by $\mathrm{md}\, f(x)$ its *metric derivative*, that is, the only partial derivative.

The following result is due to Ambrosio [3] (see also [56, Theorem 4.4.8] and [34, Theorem 2.1 (ii)]).

Theorem 3.4. *Let V be a Banach space and $f : [a, b] \to V$ be an absolutely continuous function. Then f admits a metric derivative almost everywhere.*

The following lemma extends this previous result to functions with N-dimensional domain $\Omega \subset \mathbb{R}^N$. Note that if $f : \Omega \to V$ is absolutely continuous on a.e. line, then given a direction e_i with $1 \le i \le N$, for almost every line ℓ parallel to e_i we have from the previous theorem that the set E_ℓ of points in $\ell \cap \Omega$ for which the i-th partial metric derivative of f does not exist is $\mathcal{L}^1$-null. In order to see that the set E of points in Ω for which the i-th partial metric derivative of f does not exist is $\mathcal{L}^N$-null, we are going to apply Fubini's Theorem, but this requires to show first that the set E is measurable in $\mathbb{R}^N$, which is not obvious, see [33] for an example where measurability of the set points of differentiation fails and thus Fubini's Theorem fails. Thus the proof of the lemma goes through proving this measurability, and for that we follow the ideas of [38, Corollary 3.4].

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set and V a Banach space. If $f : \Omega \to V$ is measurable and absolutely continuous on a.e. line, then it admits partial metric derivatives almost everywhere in Ω.*

Proof. The proof is independent for each direction $i = 1, \ldots, N$, so without loss of generality we show it for the last coordinate x_N. Consider $D \subset \Omega$ the set of all points $x \in \Omega$ such that $m\partial_N f(x)$ exists. Hence if D is measurable then Fubini's Theorem yields that $\Omega \backslash D$ is a null set. As we just need to prove that D is measurable and Ω can be expressed as a countable union of open N-cubes, we can assume that Ω is an open cube and then denote

$$\Omega = I_1 \times \cdots \times I_N.$$

Consider also the set

$$\widetilde{\Omega} := \{x = u + te_N \in \Omega \,:\, u \in I_1 \times \cdots \times I_{N-1}, t \mapsto f(u + te_N) \text{ is continuous on } I_N\}.$$

By the ACL property of f, we have that $\Omega\backslash\widetilde{\Omega}$ is a $\mathcal{L}^N$-null set. Then, it will be sufficient to prove the measurability of the set

$$\widetilde{D} := \{x \in \widetilde{\Omega} : m\partial_N f(x) \text{ exists } \}.$$

Now fix $x \in \widetilde{\Omega}$. Using the continuity of the map $t \mapsto f(x + te_N)$, we obtain that $\alpha = m\partial_N f(x)$ if, and only if, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that, for every $t \in \mathbb{Q}$ with $0 < |t| < \delta$, we have that

$$\left| \frac{\|f(x + te_N) - f(x)\|}{|h|} - \alpha \right| \leq \varepsilon.$$

From here, we see that $\alpha = m\partial_N f(x)$ if, and only if, for every sequence (r_k) of non-zero rational numbers converging to zero, we have that

$$\alpha = \lim_{k\to\infty} \frac{\|f(x + r_k e_N) - f(x)\|}{|r_k|}.$$

Therefore, it is easy to check that $x \in \widetilde{D}$ if, and only if, for every sequence (r_k) of non-zero rational numbers converging to zero, the sequence

$$\left\{ \frac{\|f(x + r_k e_N) - f(x)\|}{|r_k|} \right\}$$

is a Cauchy sequence in $\mathbb{R}$.

Now denote $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$ and define, for $r, s \neq 0$ and $\varepsilon > 0$:

$$D(r, s, \varepsilon) := \left\{ x \in \widetilde{\Omega} : \left| \frac{\|f(x + re_N) - f(x)\|}{|r|} - \frac{\|f(x + se_N) - f(x)\|}{|s|} \right| \leq \varepsilon \right\},$$

As a consequence of our previous observations, we obtain that

$$\widetilde{D} = \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{\delta\in\mathbb{Q}^+} \bigcap_{r,s\in(-\delta,\delta)\cap\mathbb{Q}^*} D(r, s, \varepsilon).$$

Since the sets $D(r, s, \varepsilon)$ are measurable, this gives the measurability of $\widetilde{D}$. ■

3.1.2 Weak-derivatives.*

When the target space is the dual of a separable Banach space, the notion of partial weak*-derivatives turns out to be specially appropriate for the results that we will address in Section 3.4. The partial weak*-derivatives were considered by Hajłasz and Tyson in [50]. We also refer to [7], where weak*-differentiability is developed in the context of dual Banach space valued Lipschitz functions. We proceed with the same scheme followed for the metric derivatives.

Definition 3.6 Let $\Omega \subset \mathbb{R}^N$ be an open set, $V = Y^*$ be the dual of a separable Banach space Y, and $f : \Omega \to V$. For each $i = 1, \ldots, N$, we define the *i-th partial weak*-derivative* of f at a point $x \in \Omega$ as the following w^*-limit, if it exists:

$$w^*\partial_i f(x) = w^* - \lim_{h\to 0} \frac{f(x + he_i) - f(x)}{h}$$

In the case that $N = 1$, we denote $w^*df(x)$ the corresponding *weak*-derivative.*

The next result and its proof is contained in [50, Lemma 2.8].

Lemma 3.7. *Let $V = Y^*$ be the dual of a separable Banach space and let $f : [a, b] \to V$ be an absolutely continuous function. Then f is weak*-derivable almost everywhere in $[a, b]$. Furthermore, if $F \subset Y$ is a dense and countable vector space over $\mathbb{Q}$, we have that f is weak*-differentiable at a point x if, and only if, for each $y \in F$ the following limit exists:*

$$\lim_{h\to 0} \left\langle y, \frac{f(x+h) - f(x)}{h} \right\rangle .$$

We now give an analog to Lemma 3.5, concerning the almost everywhere existence of partial weak*-derivatives. Recall that, if $V = Y^*$ is a dual space, a function $f : \Omega \to V$ is said to be *weak*-measurable* if $\langle y, f\rangle$ is measurable for each $y \in Y$.

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $V = Y^*$ the dual of a separable Banach space. If $f : \Omega \to V$ is weak*-measurable and absolutely continuous on a.e. line, then f admits partial weak*-derivatives almost everywhere in Ω.*

Proof. The proof follows the same lines of Lemma 3.5. As in that case, we consider the direction e_N, and it will suffice to prove that the set D of all points $x \in \Omega$ such that $w^*\partial_N f(x)$ exists is measurable. We may also assume that $\Omega = I_1 \times \cdots \times I_N$ is an open cube, and we consider the set

$$\widetilde{\Omega} := \{x = u + te_N \in \Omega \,:\, u \in I_1 \times \cdots \times I_{N-1}, t \mapsto f(u + te_N) \text{ is continuous on } I_N\} .$$

By the ACL property of f, we have that $\Omega\backslash\widetilde{\Omega}$ is a $\mathcal{L}^N$-null set. Then, it will be sufficient to prove the measurability of the set

$$\widetilde{D} := \{x \in \widetilde{\Omega} : w^*\partial_N f(x) \text{ exists }\}.$$

Let $F \subset Y$ be a dense and countable vector space over $\mathbb{Q}$. Given $x \in \widetilde{\Omega}$, by Lemma 3.7 we have that $x \in \widetilde{D}$ if, and only if, for each $y \in F$ the following limit exists:

$$\lim_{h\to 0} \left\langle y, \frac{f(x+he_N) - f(x)}{h} \right\rangle .$$

Now for each $y \in F$, each $r, s \neq 0$ and each $\varepsilon > 0$, denote

$$D(y, r, s, \varepsilon) := \left\{ x \in \widetilde{\Omega} : \left| \left\langle y, \frac{f(x+re_N) - f(x)}{r} - \frac{f(x+se_N) - f(x)}{s} \right\rangle \right| \leq \varepsilon \right\} .$$

Then reasoning as in Lemma 3.5 we see that

$$\widetilde{D} = \bigcap_{y\in F} \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{\delta\in\mathbb{Q}^+} \bigcap_{r,s\in(-\delta,\delta)\cap\mathbb{Q}^*} D(y, r, s, \varepsilon).$$

Since the sets $D(y, r, s, \varepsilon)$ are measurable, the result follows. ■

The following theorem collects the information above and relates both types of partial derivatives as the main result of this section.

Theorem 3.9. *Let Ω be an open subset of $\mathbb{R}^N$ and $V = Y^*$ the dual of a separable Banach space. If $f : \Omega \to V$ is measurable and absolutely continuous on a.e. line, then f admits partial metric derivatives and partial weak*-derivatives almost everywhere. For each $i = 1, \ldots, N$ the functions $m\partial_i f$ and $\|w^*\partial_i f\|$ are measurable, and*

$$m\partial_i f(x) = \|w^*\partial_i f(x)\|$$

at almost every $x \in \Omega$.

Proof. Fix $i \in \{1, \dots, N\}$. Lemma 3.5 and Lemma 3.8 yield, respectively the existence of $m\partial_i f$ and $w^*\partial_i f$ almost everywhere in Ω. Moreover $m\partial_i f$ is measurable, since f is measurable and $m\partial_i f$ is the limit a.e. of the sequence of measurable functions:

$$m\partial_i f(x) = \lim_{k\to\infty} \left\{ k \left\| f\left(x + \frac{1}{k}e_i\right) - f(x) \right\| \right\}.$$

In the same way, for each $y \in Y$ we have that the function $\langle y, w^*\partial_i f\rangle$ is measurable since

$$\langle y, w^*\partial_i f(x)\rangle = \lim_{k\to\infty} \left\langle y,\, k\left(f(x + \frac{1}{k}e_i) - f(x)\right)\right\rangle.$$

In addition, $\|w^*\partial_i f\|$ is also measurable, since

$$\|w^*\partial_i f(x)\| = \sup_{y\in D} |\langle y, w^*\partial_i f(x)\rangle|,$$

where D is any countable dense subset of the unit ball of Y, which exists due to the separability of Y.

Note that, for every $x \in \Omega$ and every $y \in Y$, we have that

$$\left| \left\langle y,\, \frac{f(x+he_i) - f(x)}{h} \right\rangle \right| \le \|y\| \cdot \left\| \frac{f(x+he_i) - f(x)}{h} \right\|.$$

Taking limits when h goes to zero we obtain that, for almost every $x \in \Omega$ and every $y \in Y$ with $\|y\| \le 1$,

$$|\langle y, w^*\partial_i f(x)\rangle| \le m\partial_i f(x).$$

Then

$$\|w^*\partial_i f(x)\| = \sup_{\|y\|\le 1} |\langle y, w^*\partial_i f(x)| \le m\partial_i f(x)$$

for almost every $x \in \Omega$.

On the other hand, f is absolutely continuous on almost every compact segment parallel to a coordinate axis and contained in Ω. Let $\sigma : [a,b] \to \Omega$ be one of such segments on which f is absolutely continuous, and suppose that $\sigma(t) = x + te_i$ for $a \le t \le b$, where $x \in \Omega$ and $i \in \{e_1, \dots, e_N\}$. Thus for each $y \in Y$ with $\|y\| \le 1$ we have that $\langle y, f \circ \sigma\rangle$ is also absolutely continuous on $[a,b]$. Therefore for every $a \le s < t \le b$:

$$|\langle y, f(x+te_i) - f(x+se_i)\rangle| \le \int_s^t |\langle y, w^*\partial_i f(x+\tau e_i)\rangle|\, d\tau \le \int_s^t \|w^*\partial_i f(x+\tau e_i)\|\, d\tau.$$

As a consequence we obtain that

$$\|f(x+te_i) - f(x+se_i)\| = \sup_{\|y\|\le 1} |\langle y, f(x+te_i) - f(x+se_i)\rangle| \le \int_s^t \|w^*\partial_i f(x+\tau e_i)\|\, d\tau.$$

Again by the absolute continuity of $f\circ\sigma$ on $[a,b]$, we know from [50, Lemma 2.7] that the function $\tau \mapsto m\partial_i f(x+\tau e_i)$ belongs to $L^1([a,b])$, so it does the function $\tau \mapsto \|w^*\partial_i f(x+\tau e_i)\|$. Therefore, using the Lebesgue differentiation Theorem we conclude that

$$m\partial_i f(z) = \lim_{h\to 0^+} \frac{\|f(z+he_i) - f(z)\|}{|h|} \le \lim_{h\to 0^+} \frac{1}{h}\int_0^h \|w^*\partial_i f(z+\tau e_i)\|\, d\tau = \|w^*\partial_i f(z)\|$$

holds for $\mathcal{L}^1$-almost every point z of almost every line parallel to the i-th coordinate axis. In this way, using Fubini's Theorem we see that $m\partial_i f \le \|w^*\partial_i f\|$ almost everywhere in Ω. ■

Notice that for the proofs of Lemmas 3.5 and 3.8 we used the differentiability property given by absolutely continuous mappings in an interval. We recall (see Section 1.3) that if V has the Radon-Nikodým property this differentiation can be considered in the standard Fréchet sense, leading to a very similar result of which we give now a sketch of the proof, since it follows similarly of those of Lemmas 3.5 and 3.8.

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $f : \Omega \to V$ be measurable and absolutely continuous on a.e. line. Then, the set of points in Ω where f admits classical partial derivatives is measurable. Furthermore, if V has Radon-Nikodým Property, then f admits classical partial derivatives almost everywhere in Ω.*

Proof. The proof can be carried out as in Lemma 3.5. We may assume that $\Omega = I_1 \times \cdots \times I_N$ is an open cube, we fix the direction e_N, and we consider the set

$$\widetilde{\Omega} := \{x = u + te_N \in \Omega \, : \, u \in I_1 \times \cdots \times I_{N-1}, t \mapsto f(u + te_N) \text{ is continuous on } I_N\}.$$

Since $\mathcal{L}^N(\Omega \backslash \widetilde{\Omega}) = 0$ by he ACL property of f, it is sufficient to prove the measurability of the set $\widetilde{D}$ of all points $x \in \widetilde{\Omega}$ such that the classical partial derivative $\partial f(x)/\partial x_N$ exists. Reasoning as in Lemma 3.5, this follows from the fact that

$$\widetilde{D} = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{\delta \in \mathbb{Q}^+} \bigcap_{r,s \in (-\delta,\delta) \cap \mathbb{Q}^*} \left\{ x \in \widetilde{\Omega} : \left\| \frac{f(x + re_N) - f(x)}{r} - \frac{f(x + se_N) - f(x)}{s} \right\| \leq \varepsilon \right\}.$$

If, in addition, V has the Radon-Nikodým Property, by Theorem 1.21 we obtain that f admits classical partial derivatives at $\mathcal{L}^1$-almost every point of almost every line parallel to a coordinate axis. Then Fubini's Theorem gives the result. ∎

3.2 Sobolev spaces $W^{1,p}(\Omega, V)$.

In the context of mappings from an open set $\Omega \subset \mathbb{R}^N$ to a Banach space V one can consider the classical definition of a Sobolev space via distributional derivatives, and thus we will start our discussion about Sobolev spaces with a brief overview on this definition.

Consider then $\Omega \subset \mathbb{R}^N$ open and V a Banach space, and let $1 \leq p \leq \infty$. We denote by $C_0^\infty(\Omega)$ the space of all real-valued functions that are infinitely differentiable and have compact support in Ω. This class of functions allows us to apply the integration by parts formula against functions in $L^p(\Omega, V)$. In this way we can define weak derivatives as follows. Given $f \in L^p(\Omega, V)$ and $i \in \{1, \ldots, N\}$, a function $f_i \in L^1_{\text{loc}}(\Omega, V)$ is said to be the *i-th weak partial derivative* of f if

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} f \, d\mathcal{L}^N = -\int_\Omega \varphi \, f_i \, d\mathcal{L}^N$$

for every $\varphi \in C_0^\infty(\Omega)$. As defined, it is easy to see that partial derivatives are unique, so we denote $f_i = \partial f / \partial x_i$. If f admits all weak partial derivatives, we define its *weak gradient* as the vector $\nabla f = (f_1, \ldots, f_N)$, and the *length* of the gradient is

$$|\nabla f| := \left(\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|^2 \right)^{\frac{1}{2}}.$$

Using this, the classical first-order Sobolev spaces of vector-valued functions are defined as follows.

Definition 3.11 Let $1 \leq p < \infty$, Ω be an open subset of $\mathbb{R}^N$ and let V be a Banach space. We define the Sobolev space $W^{1,p}(\Omega, V)$ as the set of all classes of functions $f \in L^p(\Omega, V)$ that admit a weak gradient satisfying $\partial f/\partial x_i \in L^p(\Omega, V)$ for all $i \in \{1, \ldots, N\}$. This space is equipped with the natural norm

$$\|f\|_{W^{1,p}} := \left(\int_\Omega \|f\|^p \, d\mathcal{L}^N\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla f|^p \, d\mathcal{L}^N\right)^{\frac{1}{p}}.$$

We denote by $W^{1,p}(\Omega) = W^{1,p}(\Omega, \mathbb{R})$.

It can be shown that the space $W^{1,p}(\Omega, V)$, endowed with this norm, is a Banach space. Furthermore, the Meyers-Serrin Theorem also holds in the context of vector-valued Sobolev functions, so in particular the space $C^1(\Omega, V) \cap W^{1,p}(\Omega, V)$ is dense in $W^{1,p}(\Omega, V)$. We refer to [73, Theorem 4.11] for a proof of this fact.

It is well known that every function in $W^{1,p}(\Omega, V)$ admits a representative which is absolutely continuous and almost everywhere differentiable along almost every line parallel to a coordinate axis (see [73, Theorem 4.16] or [9, Theorem 3.2]), where differentiability is understood in the usual Fréchet sense. In order to obtain more generality and approach the techniques used for metric measure spaces we give an analogous result considering curves instead of lines parallel to the coordinate axes. For that, we will use the concept of modulus introduced in Section 1.4.3 and take advantage of the nice properties in $\mathbb{R}^N$ to relate it to the concept of almost every line as we see in the next lemma.

Lemma 3.12. *Let $N > 1$ be a natural number, let $w \in \mathbb{R}^N$ be a vector with $|w| = 1$ and let H be a hyperplane orthogonal to w, on which we consider the corresponding $(N-1)$-dimensional Lebesgue measure $\mathcal{L}^{N-1}$. For each Borel subset $E \subset H$ consider the family $\Gamma(E)$ of all non-trivial straight segments parallel to w and contained in a line passing through E. Then, for a fixed $1 \leq p < \infty$, we have that $\mathrm{Mod}_p(\Gamma(E)) = 0$ if, and only if, $\mathcal{L}^{N-1}(E) = 0$.*

Proof. Each curve in $\Gamma(E)$ is of the form $\gamma_x(t) = x + tw$, for some $x \in E$, and is defined on some interval $a \leq t \leq b$. For each $q, r \in \mathbb{Q}$ with $q < r$, let $\Gamma_{q,r}$ denote the family of all such paths γ_x, where $x \in E$, which are defined on the fixed interval $[q, r]$. According to [56, Equation (5.3.12)], we have that

$$\mathrm{Mod}_p(\Gamma_{q,r}) = \frac{\mathcal{L}^{N-1}(E)}{(r-q)^p}.$$

Suppose first that $\mathcal{L}^{N-1}(E) = 0$. Then $\mathrm{Mod}_p(\Gamma_{q,r}) = 0$ for all $q, r \in \mathbb{Q}$ with $q < r$. Thus by subadditivity we have that $\mathrm{Mod}_p(\bigcup_{q,r} \Gamma_{q,r}) = 0$. Now each segment $\gamma_x \in \Gamma(E)$ contains a sub-segment in some $\Gamma_{q,r}$. This implies that the corresponding admissible functions satisfy $F(\bigcup_{q,r} \Gamma_{q,r}) \subset F(\Gamma(E))$, and therefore

$$\mathrm{Mod}_p(\Gamma(E)) \leq \mathrm{Mod}_p\Big(\bigcup_{q,r} \Gamma_{q,r}\Big) = 0.$$

Conversely, if $\mathrm{Mod}_p(\Gamma(E)) = 0$ then $\mathrm{Mod}_p(\Gamma_{q,r}) = 0$ for any $q, r \in \mathbb{Q}$ with $q < r$, and therefore $\mathcal{L}^{N-1}(E) = 0$. ∎

First we study the case of $C^1(\Omega, V)$ mappings in order to use it through density in $W^{1,p}(\Omega, V)$.

Lemma 3.13. *Let Ω be an open subset of $\mathbb{R}^N$ and let V be a Banach space. If $f \in C^1(\Omega, V)$ and γ is a rectifiable curve in Ω, parametrized by arc-length, then $f \circ \gamma$ is absolutely continuous and differentiable almost everywhere. Moreover, the derivative of $f \circ \gamma$ belongs to $L^1([0, \ell(\gamma)], V)$ and*

$$(f \circ \gamma)(t) - (f \circ \gamma)(0) = \int_0^t (f \circ \gamma)'(\tau) \, d\tau.$$

for each $t \in [0, \ell(\gamma)]$.

Proof. Since $\gamma : [0, \ell(\gamma)] \to \Omega$ is a rectifiable curve parametrized by arc-length, in particular it is 1-Lipschitz, so it is differentiable almost everywhere. Furthermore, the derivative $\gamma'(\tau)$ has Euclidean norm $|\gamma'(\tau)| = 1$ whenever it exists. Additionally $f \in C^1(\Omega, V)$, so the chain rule yields that $f \circ \gamma$ is differentiable almost everywhere. Now denote $h = f \circ \gamma$. Since

$$h'(t) = \lim_{n\to\infty} \frac{h(t+1/n) - h(t)}{1/n}$$

we see that h' is limit of a sequence of measurable functions, and hence measurable. Furthermore, as $f \in C^1(\Omega, V)$ and $\gamma([0, \ell(\gamma)])$ is compact, there exists $K > 0$ such that $|\nabla f(\gamma(\tau))| \leq K$ for all $\tau \in [0, \ell(\gamma)])$. Then

$$\begin{aligned}\|h'\|_1 &= \int_0^{\ell(\gamma)} \|(\nabla f(\gamma(\tau))) \cdot \gamma'(\tau)\| \, d\tau = \int_0^{\ell(\gamma)} \left\| \sum_{i=1}^N \frac{\partial f(\gamma(\tau))}{\partial x_i} \cdot \gamma_i'(\tau) \right\| d\tau \\ &\leq \int_0^{\ell(\gamma)} \sum_{i=1}^N \left\| \frac{\partial f(\gamma(\tau))}{\partial x_i} \right\| |\gamma_i'(\tau)| \, d\tau \leq \int_0^{\ell(\gamma)} \|\nabla f(\gamma(\tau))\| \, |\gamma'(\tau)| \, d\tau \leq K\ell(\gamma),\end{aligned}$$

concluding that $h' \in L^1([0, \ell(\gamma)], V)$. Now for each $v^* \in V^*$, applying the Fundamental Theorem of Calculus to the scalar function $\langle v^*, h\rangle$ we see that for each $t \in [0, \ell(\gamma)]$ we have that

$$\langle v^*, h\rangle(t) - \langle v^*, h\rangle(0) = \int_0^t \langle v^*, h'(\tau)\rangle \, d\tau = \left\langle v^*, \int_0^t h'(\tau) \, d\tau \right\rangle.$$

As a consequence, $h(t) - h(0) = \int_0^t h'(\tau) \, d\tau$ for every $t \in [0, \ell(\gamma)]$. ∎

Theorem 3.14. *Let $1 \leq p < \infty$, let Ω be an open subset of $\mathbb{R}^N$ and let V be a Banach space. Then every $f \in W^{1,p}(\Omega, V)$ admits a representative which is absolutely continuous and differentiable almost everywhere over p-almost every non-constant compact rectifiable curve γ in Ω.*

Proof. Let $\mathcal{M}$ denote the family of all non-constant rectifiable curves in Ω which, without loss of generality, we can assume to be parametrized by arc-length. By the Meyers-Serrin density Theorem, there exists a sequence $(f_n)_{n=1}^\infty$ of functions in $C^1(\Omega, V)$ converging to f in $W^{1,p}(\Omega, V)$-norm. In particular, f_n converges to f in $L^p(\Omega, V)$, and then there exists a subsequence of $(f_n)_{n=1}^\infty$, still denoted by f_n, converging almost everywhere to f. Choose a null subset $\Omega_0 \subset \Omega$ such that $f_n \to f$ pointwise on $\Omega \setminus \Omega_0$. Now consider

$$\Gamma^+_{\Omega_0} := \{\gamma : [0, \ell(\gamma)] \to \Omega \in \mathcal{M} : \mathcal{L}^1(\{t \in [0, \ell(\gamma)] : \gamma(t) \in \Omega_0\}) > 0\}.$$

By Lemma 1.28, $\mathrm{Mod}_p(\Gamma^+_{\Omega_0}) = 0$. In addition, for every curve $\gamma \in \mathcal{M} \setminus \Gamma^+_{\Omega_0}$ the set $E := \{t \in [0, \ell(\gamma)] : \gamma(t) \in \Omega_0\}$ has zero measure, and therefore $f_n \circ \gamma \to f \circ \gamma$ almost everywhere on $[0, \ell(\gamma)]$.

On the other hand, as $f_n \to f$ in $W^{1,p}(\Omega, V)$, we also have that $|\nabla f_n - \nabla f| \to 0$ in $L^p(\Omega)$. Then we can apply Fuglede's Lemma 1.29 and we obtain a subsequence of $(f_n)_{n=1}^\infty$, that we keep denoting by f_n, such that

$$\lim_{n\to\infty} \int_\gamma |\nabla f_n - \nabla f| \, ds = 0 \tag{3.1}$$

for every curve $\gamma \in \mathcal{M} \setminus \Gamma_1$, where $\mathrm{Mod}_p(\Gamma_1) = 0$. Notice that for every curve $\gamma \in \mathcal{M} \setminus \Gamma_1$ the Fuglede identity (3.1) will also hold for any subcurve of γ, since $\int_{\gamma|_{[s,t]}} |\nabla f_n - \nabla f| \, ds \leq \int_\gamma |\nabla f_n - \nabla f| \, ds$ for each $0 \leq s \leq t \leq \ell(\gamma)$.

Furthermore, by Lemma 1.27, the family of curves Γ_2 satisfying that $\int_\gamma |\nabla f| ds = \infty$ or $\int_\gamma |\nabla f_n| ds = \infty$ for some n has null p-modulus. Finally, we consider the family $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma^+_{\Omega_0}$ and note that, by subadditivity, $\mathrm{Mod}_p(\Gamma) = 0$.

Now fix a curve $\gamma \in \mathcal{M}\setminus\Gamma$. For each $n \in \mathbb{N}$ define $g_n := (\nabla f_n \circ \gamma)\cdot\gamma'$, where $\nabla f_n \circ \gamma = \left(\frac{\partial f_n}{\partial x_i} \circ \gamma\right)_{i=1}^N$ and $\gamma' = (\gamma_i')_{i=1}^N$ are N-tuples of V^N and $\mathbb{R}^N$ respectively, and thus the product here denotes the natural inner product

$$(\nabla f_n \circ \gamma)\cdot\gamma' = \sum_{i=1}^N \left(\frac{\partial f_n}{\partial x_i} \circ \gamma\right)\cdot\gamma_i'$$

By Lemma 3.13 the function $f_n \circ \gamma$ is almost everywhere differentiable, its derivative $(f_n \circ \gamma)' = g_n$ belongs to $L^1([0,\ell(\gamma)],V)$ and satisfies

$$f_n \circ \gamma(t) - f_n \circ \gamma(s) = \int_s^t g_n \, d\mathcal{L}^1 \tag{3.2}$$

for each $s,t \in [0,\ell(\gamma)]$. Moreover, taking into account that γ is parametrized by arc-length, we see that $|\gamma'| = 1$ almost everywhere in $[0,\ell(\gamma)]$, and we obtain that, for every function $u \in W^{1,p}(\Omega,V)$,

$$\begin{aligned}\|(\nabla u \circ \gamma)\cdot\gamma'\| &= \left\|\sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\circ\gamma\right)\cdot\gamma_i'\right\| \le \sum_{i=1}^N \left\|\left(\frac{\partial u}{\partial x_i}\circ\gamma\right)\cdot\gamma_i'\right\| \\ &= \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\circ\gamma\right\|\cdot|\gamma_i'| \le |\nabla u \circ \gamma|\cdot|\gamma'| = |\nabla u\circ\gamma|.\end{aligned}$$

Then for any $0 \le s \le t \le \ell(\gamma)$ we have that

$$\begin{aligned}\left\|\int_s^t g_n \, d\mathcal{L}^1 - \int_s^t (\nabla f \circ \gamma)\cdot\gamma' \, d\mathcal{L}^1\right\| &\le \int_s^t \|g_n - (\nabla f\circ\gamma)\cdot\gamma'\| \, d\mathcal{L}^1 \\ &= \int_s^t \left\|(\nabla f_n \circ\gamma - \nabla f\circ\gamma)\cdot\gamma'\right\| d\mathcal{L}^1 \\ &\le \int_s^t |\nabla f_n - \nabla f|\circ\gamma \, d\mathcal{L}^1 \\ &\le \int_\gamma |\nabla f_n - \nabla f| \, ds \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Hence $(\nabla f\circ\gamma)\cdot\gamma' \in L^1([0,\ell(\gamma)],V)$ and

$$\lim_{n\to\infty}\int_s^t g_n \, d\mathcal{L}^1 = \int_s^t (\nabla f\circ\gamma)\cdot\gamma' \, d\mathcal{L}^1. \tag{3.3}$$

Next we are going to see that the sequence $(f_n\circ\gamma)_{n=1}^\infty$ is equicontinuous. This will follow from the fact that $(|\nabla f_n \circ\gamma|)_{n=1}^\infty$ is equiintegrable, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{n\ge1}\int_A |\nabla f_n\circ\gamma| \, d\mathcal{L}^1 \le \varepsilon \text{ if } A \subset [0,\ell(\gamma)] \text{ and } \mathcal{L}^1(A) < \delta.$$

Fix $\varepsilon > 0$. Then by (3.1) there exists $n_0 \in \mathbb{N}$ such that

$$\int_0^{\ell(\gamma)} \left||\nabla f_n\circ\gamma| - |\nabla f\circ\gamma|\right| d\mathcal{L}^1 < \frac{\varepsilon}{2} \tag{3.4}$$

for all $n \ge n_0$. Now notice that as $\gamma \notin \Gamma_2$ then $|\nabla f_n \circ \gamma|$ and $|\nabla f\circ\gamma|$ are integrable on $[0,\ell(\gamma)]$, hence by the absolute continuity of the integral we can choose a $\delta > 0$ such that for any $A \subset [0,\ell(\gamma)]$ with $\mathcal{L}^1(A) < \delta$

$$\int_A |\nabla f_n\circ\gamma| \, d\mathcal{L}^1 < \frac{\varepsilon}{2}, \tag{3.5}$$

for all $n \in \{1, \dots, n_0\}$ and

$$\int_A |\nabla f \circ \gamma| \, d\mathcal{L}^1 < \frac{\varepsilon}{2}. \tag{3.6}$$

Then for $n \geq n_0$ by (3.4) and (3.6)

$$\int_A |\nabla f_n \circ \gamma| \, d\mathcal{L}^1 \leq \int_A |\nabla f \circ \gamma| \, d\mathcal{L}^1 + \int_0^{\ell(\gamma)} ||\nabla f_n \circ \gamma| - |\nabla f \circ \gamma|| \, d\mathcal{L}^1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This, together with (3.5), gives that $\int_A |\nabla f_n \circ \gamma| \, d\mathcal{L}^1 < \varepsilon$ for every $n \in \mathbb{N}$, as we wanted to prove. Hence by (3.2) we have that, if $s, t \in [0, \ell(\gamma)]$ are such that $|s - t| < \delta$, then

$$\|f_n \circ \gamma(s) - f_n \circ \gamma(t)\| \leq \int_s^t |\nabla f_n \circ \gamma| \, d\mathcal{L}^1 < \varepsilon.$$

This yields that $(f_n \circ \gamma)_{n=1}^{\infty}$ is an equicontinuous sequence. Since in addition $(f_n \circ \gamma)_{n=1}^{\infty}$ converges on a dense subset of $[0, \ell(\gamma)]$ we obtain that, in fact, $(f_n \circ \gamma)_{n=1}^{\infty}$ converges uniformly on $[0, \ell(\gamma)]$.

Now we choose a representative of f defined as follows:

$$f(x) := \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

With this definition we obtain that, for every curve $\gamma \in \mathcal{M} \setminus \Gamma$ and every $t \in [0, \ell(\gamma)]$, the sequence $\{(f_n \circ \gamma)(t)\}_{n=1}^{\infty}$ converges to $f \circ \gamma(t)$. Therefore, using (3.2) and (3.3) we see that, for every $s, t \in [0, \ell(\gamma)]$,

$$\begin{aligned} (f \circ \gamma)(t) - (f \circ \gamma)(s) &= \lim_{n\to\infty} ((f_n \circ \gamma)(t) - (f_n \circ \gamma)(s)) \\ &= \lim_{n\to\infty} \int_s^t g_n \, d\mathcal{L}^1 = \int_s^t (\nabla f \circ \gamma) \cdot \gamma' \, d\mathcal{L}^1. \end{aligned}$$

From here we deduce that $f \circ \gamma$ is absolutely continuous and almost everywhere differentiable on $[0, \ell(\gamma)]$. ■

3.3 Sobolev-Reshetnyak spaces $R^{1,p}(\Omega, V)$

These Sobolev-Reshetnyak spaces have been considered in [55] and [50]. We give a definition taken from [50], which is slightly different, but equivalent, to the original definition in [85]. We will discuss later on about the definition proposed by Reshetnyak, while considering mappings in a metric measure space and not only in an Euclidean open set.

Definition 3.15 Let Ω be an open subset of $\mathbb{R}^N$ and let V be a Banach space. Given $1 \leq p \leq \infty$, the Sobolev-Reshetnyak space $R^{1,p}(\Omega, V)$ is defined as the space of all classes of functions $f \in L^p(\Omega, V)$ satisfying

(1) for every $v^* \in V^*$ such that $\|v^*\| \leq 1$, $\langle v^*, f \rangle \in W^{1,p}(\Omega)$;

(2) there is a non-negative function $g \in L^p(\Omega)$ such that the inequality $|\nabla \langle v^*, f \rangle| \leq g$ holds almost everywhere, for all $v^* \in V^*$ satisfying $\|v^*\| \leq 1$.

We now define the norm

$$\|f\|_{R^{1,p}} := \|f\|_p + \inf_{g \in \mathcal{R}(f)} \|g\|_p,$$

where $\mathcal{R}(f)$ denotes the family of all non-negative functions $g \in L^p(\Omega)$ satisfying (2).

Theorem 3.16. *Let $\Omega \subset \mathbb{R}^n$ be an open set, V a Banach space, and $1 \le p < \infty$. Then $(R^{1,p}(\Omega, V), \|\cdot\|_{1,p})$ is a Banach space.*

Proof. Clearly, the Sobolev-Reshetnyak space is a normed vector space. Next, we need to show that it is complete.

Consider a Cauchy sequence $(f_n)_{n=1}^{\infty}$ in $R^{1,p}(\Omega, V)$. Define $f_0 \equiv 0$, and by taking a subsequence (also denoted as f_n), we can assume that

$$\|f_n - f_{n-1}\|_{R^{1,p}} < \frac{1}{2^n},\ \forall n \ge 1.$$

This implies that for each $n \ge 1$,

$$\inf\{\|g\|_p : g \in \mathcal{R}(f_n - f_{n-1})\} < \frac{1}{2^n},$$

and thus, we can choose a sequence $(g_n)_{n=1}^{\infty} \subset L^p(\Omega)$ such that for each $n = 1, 2, \ldots,$

$$g_n \in \mathcal{R}(f_n - f_{n-1}) \text{ and } \|g_n\|_p < \frac{1}{2^n}.$$

Define $g := \sum_{n=1}^{\infty} g_n$, which satisfies $\|g\|_p \le \sum_{n=1}^{\infty} \|g_n\|_p < \sum_{n=1}^{\infty} \|g_n\|_p < \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$, then $g \in L^p(\Omega)$. On the other hand, $\|f_n - f_{n-1}\|_p \le \|f_n - f_{n-1}\|_{R^{1,p}}$, so $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\Omega, V)$, and therefore, since $L^p(\Omega, V)$ is complete, there exists $f \in L^p(\Omega, V)$ such that $f_n \to f$ in the norm $\|\cdot\|_p$.

Furthermore, it is also true that for every $v^* \in V^*$ with $\|v^*\| \le 1$ and for each $n \in \mathbb{N}$, we have $\langle v^*, f_n\rangle \in W^{1,p}(\Omega)$, and for $n \ge 2$, $|\nabla(\langle v^*, f_n\rangle - \langle v^*, f_{n-1}\rangle)| \le g_k$, and also

$$\|\langle v^*, f_n\rangle - \langle v^*, f_{n-1}\rangle\|_p \le \|f_n - f_{n-1}\|_p < \tfrac{1}{2^n}$$
$$\|\nabla\langle v^*, f_n\rangle - \nabla\langle v^*, f_{n-1}\rangle\|_p \le \|g_n\|_p < \tfrac{1}{2^n}.$$

From this, it follows that the sequence $(\langle v^*, f_n\rangle)_{n=1}^{\infty}$ is a Cauchy sequence in $W^{1,p}(\Omega)$, so by completeness, there will exist a function $h_{v^*} \in W^{1,p}(\Omega)$ to which it converges in the norm of $W^{1,p}(\Omega)$, and in particular, it will converge in the norm of $L^p(\Omega)$. Also, since $\langle v^*, f_n\rangle$ converges to $\langle v^*, f\rangle$ in the norm $L^p(\Omega)$, we have

$$\langle v^*, f\rangle = h_{v^*} \in W^{1,p}(\Omega),\ \forall v^* \in V^* : \|v^*\| \le 1.$$

On the other hand, we have that $|\nabla\langle v^*, f_n\rangle| \to |\nabla\langle v^*, f\rangle|$ in $L^p(\Omega)$, so we can take a convergent subsequence almost everywhere. Since, for all $n \in \mathbb{N}$

$$|\nabla\langle v^*, f_n\rangle| = \left|\nabla\left\langle v^*, \sum_{i=1}^{n}(f_i - f_{i-1})\right\rangle\right| \le \sum_{i=1}^{n}|\nabla\langle v^*, f_i - f_{i-1}\rangle| \le \sum_{i=1}^{n} g_i \le g,$$

we conclude that $|\nabla\langle v^*, f\rangle| \le g$. With this, we have proven that $f \in R^{1,p}(\Omega, V)$ and that $g \in \mathcal{R}(f)$.

To finish, we need to show that $\|f - f_n\|_{R^{1,p}} \to 0$. For each $m \in \mathbb{N}$, we define $G_m := \sum_{i=1}^{m} g_i$. For $n > m$, we have

$$\begin{aligned}|\nabla(\langle v^*, f_n\rangle - \langle v^*, f_m\rangle)| &= |\nabla\langle v^*, f_n\rangle - \nabla\langle v^*, f_m\rangle| \\ &\le \sum_{i=m+1}^{n} |\nabla\langle v^*, f_i\rangle - \nabla\langle v^*, f_{i-1}\rangle| \\ &\le \sum_{i=m+1}^{n} g_i \le \sum_{i=m+1}^{\infty} = g - G_m.\end{aligned}$$

Taking $n \to \infty$ in the inequality, we have, almost everywhere,

$$|\nabla\langle v^*, f\rangle - \nabla\langle v^*, f_m\rangle| \leq g - G_m, \ \forall m \geq 1,$$

and therefore $g - G_m \in \mathcal{R}(f - f_m)$, which implies

$$\|f - f_m\|_{R^{1,p}} \leq \|f - f_m\|_p + \|g - G_m\|_p \to 0,$$

so the sequence $(f_n)_{n=1}^{\infty}$ converges in $R^{1,p}(\Omega, V)$. ■

Lemma 3.17. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let V be a Banach space. If $f : \Omega \to V$ is Lipschitz and has bounded support, then $f \in R^{1,p}(\Omega, V)$ for each $p \geq 1$.*

Proof. From [41, Theorem 4.5] or [52, Theorem 4.1] we know that a real valued Lipschitz function belong to $W^{1,\infty}_{loc}(\Omega)$ and having bounded support implies that it belongs to $W^{1,p}(\Omega)$ for all $1 \leq p \leq \infty$ as well. Thus, if $f : \Omega \to V$ is Lipschitz with bounded support then so it is $\langle v^*, f\rangle$ for all $v^* \in V^*$ such that $\|v^*\| \leq 1$, with Lipschitz constant $\mathrm{LIP}(f)$, which acts as an upper bound in the sense of Definition 3.15(2), thus $f \in R^{1,p}(\Omega, V)$ for all $1 \leq p \leq \infty$. ■

As we have mentioned, our main goal in this note is to compare Sobolev and Sobolev-Reshetnyak spaces. We first give a general result.

Theorem 3.18. *Let Ω be an open subset of $\mathbb{R}^N$ and let V be a Banach space. For each $1 \leq p \leq \infty$, the space $W^{1,p}(\Omega, V)$ is a closed subspace of $R^{1,p}(\Omega, V)$ and furthermore, for every $f \in W^{1,p}(\Omega, V)$, we have*

$$\|f\|_{R^{1,p}} \leq \|f\|_{W^{1,p}} \leq \sqrt{N}\,\|f\|_{R^{1,p}}.$$

Proof. That $W^{1,p}(\Omega, V) \subset R^{1,p}(\Omega, V)$ and $\|f\|_{R^{1,p}} \leq \|f\|_{W^{1,p}}$ for all $f \in W^{1,p}(\Omega, V)$ was proved in [50, Proposition 2.3].

Now we will show the second inequality. Consider $f \in W^{1,p}(\Omega, V)$, let $g \in \mathcal{R}(f)$, and choose a vector $w \in \mathbb{R}^N$ with $|w| = 1$. Taking into account Theorem 3.14 and Lemma 3.12 we see that, for almost all $x \in \Omega$, there exists the directional derivative

$$D_w f(x) = \lim_{t\to 0} \frac{1}{t}(f(x + tw) - f(x)) \in V.$$

For each $v^* \in V^*$ with $\|v^*\| \leq 1$ we then have, for almost al $x \in \Omega$,

$$|D_w\langle v^*, f\rangle(x)| = |\nabla\langle v^*, f\rangle(x) \cdot w| \leq |\nabla\langle v^*, f\rangle(x)| \leq g(x).$$

Thus, again for almost all $x \in \Omega$,

$$\|D_w f(x)\| = \sup_{\|v^*\|\leq 1} |\langle v^*, D_w f(x)\rangle| = \sup_{\|v^*\|\leq 1} |D_w\langle v^*, f\rangle(x)| \leq g(x).$$

In this way we see that the weak partial derivatives of f are such that $\|(\partial f/\partial x_i)(x)\| \leq g(x)$ for every $i \in \{1, \ldots, N\}$ and almost all $x \in \Omega$. From here, the desired inequality follows. Finally, from the equivalence of the norms on $W^{1,p}(\Omega, V)$ we see that it is a closed subspace. ■

However, the following simple example shows that the opposite inclusion does not hold in general.

Example 3.19 *Consider the interval $I = (0,1)$ and let $f : I \to \ell^\infty$ be the function given by*

$$f(t) = \left(\frac{\sin(nt)}{n}\right)_{n=1}^{\infty}$$

for all $t \in I$. Then $f \in R^{1,p}(I, \ell^\infty)$ but $f \notin W^{1,p}(I, \ell^\infty)$ for every $1 \le p \le \infty$.

Proof. Since f is Lipschitz, we see from Lemma 3.17 that $f \in R^{1,p}(I, \ell^\infty)$ for all $1 \le p \le \infty$. Suppose now that $f \in W^{1,p}(I, \ell^\infty)$. From Theorem 3.14 we have that f is almost everywhere differentiable on p-almost every rectifiable curve in I. Since, by Lemma 1.27, the family consisting of a single non-trivial segment $[a,b] \subset I$ has positive p-modulus, we obtain that f is almost everywhere differentiable on I. But this is a contradiction, since in fact f is nowhere differentiable. Indeed, for each $t \in I$, the limit

$$\lim_{h \to 0} \frac{1}{h}(f(t+h) - f(t))$$

does not exist in ℓ^∞. This can be seen taking into account that $f(I)$ is contained in the space c_0 of sequences convergent to zero, which is a closed subspace of ℓ^∞, while the coordinatewise limit is $(\cos(nt))_{n=1}^{\infty}$, which does not belong to c_0, but the ℓ^∞ norm and the pointwise convergence norm are equivalent, leading to a contradiction. ■

Before going further, we give the following result, which parallels Theorem 3.14, and whose proof is based on [56, Theorem 7.1.20].

Theorem 3.20. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let V be a Banach space and suppose $1 \le p < \infty$. Then, every $f \in R^{1,p}(\Omega, V)$ admits a representative such that, for p-almost every rectifiable curve γ in Ω, the composition $f \circ \gamma$ is absolutely continuous.*

Proof. Consider $f \in R^{1,p}(\Omega, V)$. In particular, f is measurable, hence there exists a null set $E_0 \subset \Omega$ such that $f(\Omega \setminus E_0)$ is a separable subset of V. Then we can choose a countable set $\{v_i\}_{i=1}^{\infty} \subset V$ whose closure in V contains the set

$$f(\Omega \setminus E_0) - f(\Omega \setminus E_0) := \{f(x) - f(y) : x, y \in \Omega \setminus E_0\} \subset V.$$

Additionally, we can apply the Hahn-Banach Theorem to select a countable set $\{v_i^*\}_{i=1}^{\infty} \subset V^*$ such that $\langle v_i^*, v_i \rangle = \|v_i\|$ and $\|v_i^*\| = 1$ for each $i \in \mathbb{N}$. As before, let $\mathcal{M}$ denote the family of all non-constant rectifiable curves in Ω. From Theorem 3.14 we obtain that, for each $i \in \mathbb{N}$, there is a representative f_i of $\langle v_i^*, f \rangle$ in $W^{1,p}(\Omega)$ such that f_i is absolutely continuous on p-almost every curve $\gamma \in \mathcal{M}$. Let E_i denote the set where f_i differs from $\langle v_i^*, f \rangle$, and define $\Omega_0 = \bigcup_i E_i \cup E_0$, which is also a null set. Now let $g \in \mathcal{R}(f)$ and define

$$g^*(x) := \sup_i |\nabla \langle v_i^*, f(x) \rangle|.$$

We may also assume that g and g^* are Borel functions and $g^*(x) \le g(x)$ for each $x \in \Omega$. In particular, $g^* \in L^p(\Omega)$. For a curve $\gamma : [a,b] \to \Omega$ in $\mathcal{M}$, consider the following properties:

(*i*) the function g^* is integrable on γ;

(*ii*) $\mathcal{L}^1(\{t \in [a,b] : t \in \gamma^{-1}(\Omega_0)\}) = 0$;

(*iii*) for each $i \in \mathbb{N}$ and every $a \le s \le t \le b$,

$$|f_i(\gamma(t)) - f_i(\gamma(s))| \le \int_s^t |\nabla \langle v_i^*, f \rangle (\gamma(\tau))| \, d\tau \le \int_{\gamma|_{[s,t]}} g^* \, ds.$$

By Lemma 1.27 and Lemma 1.28, respectively, we have that properties (i) and (ii) are satisfied by p-almost every curve $\gamma \in \mathcal{M}$. From Theorem 3.14 we obtain that property (iii) is also satisfied by p-almost every curve $\gamma \in \mathcal{M}$. Thus the family Γ of all curves $\gamma \in \mathcal{M}$ satisfying simultaneously (i), (ii) and (iii) represents p-almost every non-constant rectifiable curve on Ω. Now we distinguish two cases.

First, suppose that $\gamma : [a, b] \to \Omega$ is a curve in Γ whose endpoints satisfy $\gamma(a), \gamma(b) \notin \Omega_0$. Hence we can choose a subsequence $\{v_{i_j}\}_{j=1}^{\infty}$ converging to $f(\gamma(b)) - f(\gamma(a))$, and then

$$\begin{aligned}\|f(\gamma(b)) - f(\gamma(a))\| &= \lim_{j\to\infty} \|v_{i_j}\| = \lim_{j\to\infty} |\langle v_{i_j}^*, v_{i_j}\rangle| \\ &\le \limsup_{j\to\infty} \Big(|\langle v_{i_j}^*, v_{i_j} - f(\gamma(a)) + f(\gamma(b))\rangle| + |\langle v_{i_j}^*, f(\gamma(a)) - f(\gamma(b))\rangle| \Big) \\ &\le \limsup_{j\to\infty} \Big(\|v_{i_j} - f(\gamma(a)) + f(\gamma(b))\| + |\langle v_{i_j}^*, f(\gamma(a))\rangle - \langle v_{i_j}^*, f(\gamma(b))\rangle| \Big) \\ &= \limsup_{j\to\infty} |\langle v_{i_j}^*, f(\gamma(a))\rangle - \langle v_{i_j}^*, f(\gamma(b))\rangle| \\ &= \limsup_{j\to\infty} |f_{i_j}(\gamma(a)) - f_{i_j}(\gamma(b))| \le \int_\gamma g^*\, ds.\end{aligned}$$

Suppose now that $\gamma : [a, b] \to \Omega$ is a curve in Γ with at least one endpoint in Ω_0. In fact, we can suppose that $\gamma(a) \in \Omega_0$. By property (ii), we can choose a sequence $\{t_k\}_{k=1}^{\infty} \subset [a, b]$ converging to a and such that $\gamma(t_k) \notin \Omega_0$. Then by the previous case

$$\|f(\gamma(t_k)) - f(\gamma(t_l))\| \le \int_{\gamma|_{[t_k,t_l]}} g^*\, ds$$

for any $k, l \in \mathbb{N}$, and hence, as g^* is integrable on γ, then $\{f(\gamma(t_k))\}_{k=1}^{\infty}$ is convergent. Suppose now that $\sigma : [c, d] \to \Omega$ is another curve in Γ satisfying $\sigma(c) = \gamma(a)$, and let $\{s_m\}_{m=1}^{\infty} \subset [c, d]$ be a sequence converging to a such that $\sigma(s_m) \notin \Omega_0$ for every $m \in \mathbb{N}$. Then

$$\|f(\gamma(t_k)) - f(\sigma(s_m))\| \le \int_{\sigma|_{[c,s_m]}} g^* ds + \int_{\gamma|_{[a,t_k]}} g^*\, ds \overset{k,m\to\infty}{\longrightarrow} 0.$$

This proves that the limit of $f(\gamma(t_k))$ as $k \to \infty$ is independent of the curve γ and the sequence $\{t_k\}_{k=1}^{\infty}$. Now we choose a representative f_0 of f defined in the following way:

1. If $x \in \Omega \setminus \Omega_0$ we set $f_0(x) = f(x)$.
2. If $x \in \Omega_0$ and there exists $\gamma : [a, b] \to \Omega$ in Γ such that $\gamma(a) = x$, we set $f_0(x) = \lim_{k\to\infty} f(\gamma(t_k))$ where $\{t_k\}_{k=1}^{\infty} \subset [a, b]$ is a sequence converging to a such that $\gamma(t_k) \notin \Omega_0$ for each k.
3. Otherwise, we set $f_0(x) = 0$.

By definition, $f_0 = f$ almost everywhere and, for every $\gamma : [a, b] \to \Omega$ in Γ,

$$\|f_0(\gamma(b)) - f_0(\gamma(a))\| \le \int_\gamma g^*\, ds \le \int_\gamma g\, ds.$$

Furthermore, as this also holds for any subcurve of γ by the definition of Γ, we also have that for every $a \le s \le t \le b$

$$\|f_0 \circ \gamma(t) - f_0 \circ \gamma(s)\| \le \int_{\gamma|_{[s,t]}} g\, ds. \tag{3.7}$$

Therefore, the integrability of g on γ gives that $f \circ \gamma$ is absolutely continuous. ∎

For the case $p = \infty$, we give a more precise characterization of $R^{1,\infty}(\Omega, V)$ using uniformly locally Lipschitz mappings, which parallels the case of the real-valued Sobolev space $W^{1,\infty}(\Omega)$ (see [52, Theorem 4.1]). Recall that given $K \geq 0$, a function $f : \Omega \to V$ is said to be *locally K-Lipschitz* if each point $x \in \Omega$ has a neighborhood U_x such that $f|_{U_x}$ is K-Lipschitz. It is easy to see, using the convexity of the balls, that f is locally K-Lipschitz on Ω if, and only if, it is K-Lipschitz on each open ball contained in Ω.

Theorem 3.21. *Let $\Omega \subset \mathbb{R}^N$ be an open set and V a Banach space. Consider a function $f \in L^\infty(\Omega, V)$. Then $f \in R^{1,\infty}(\Omega, V)$ if and only if f has a representative which is locally K-Lipschitz, for some $K \geq 0$. Furthermore, in this case, the optimal local Lipschitz constant is*

$$K = \inf_{g \in \mathcal{R}(f)} \|g\|_\infty.$$

Proof. First suppose that $f \in R^{1,\infty}(\Omega, V)$. As f is measurable, it is essentially separably valued, so there exists a null set $E_0 \subset \Omega$ such that the difference set $f(\Omega \backslash E_0) \subset V$ is separable. Choose then a countable sequence $\{v_i\}_{i=1}^\infty \subset V$ whose closure contains the set $f(\Omega\backslash E_0) - f(\Omega\backslash E_0)$. Consider now, by Hahn-Banach Theorem, a sequence $\{v_i^*\}_{i=1}^\infty \subset V^*$ such that

$$\|v_i^*\| = 1 \text{ and } |\langle v_i^*, v_i\rangle| = \|v_i\| \ \text{ for all } i \geq 1.$$

By the definition of $R^{1,\infty}(\Omega, V)$, for each $i \geq 1$ we know that $\langle v_i^*, f\rangle \in W^{1,\infty}(\Omega)$, and then there exists a locally Lipschitz representative f_i of $\langle v_i^*, f\rangle$ (see, e.g., [52, Theorem 4.1]). For each $i \geq 1$, choose a null set $E_i \subset \Omega$ such that

$$f_i(x) = \langle v_i^*, f(x)\rangle \ \text{ for all } x \in \Omega\backslash E_i,$$

and a constant $K_i \geq 0$ such that f_i is locally K_i-Lipschitz on Ω. In fact, we can choose the optimal Lipschitz constant $K_i = \|\nabla\langle v_i^*, f\rangle\|_\infty$ (see [52, Remark 4.2]). Then, for each open ball $B \subset \Omega$

$$\|f_i(p) - f_i(q)\| \leq K_i|p - q| \ \text{ for all } p, q \in B.$$

Now let $g \in \mathcal{R}(f)$. Then $g \in L^\infty(\Omega)$ and

$$\sup_{i\geq 1} K_i = \sup_{i\geq 1} \|\nabla\langle v_i^*, f\rangle\|_\infty \leq \sup_{i\geq 1} |\nabla\langle v_i^*, f\rangle| \leq \|g\|_\infty < \infty.$$

If we define $K = \|g\|_\infty$, we only need to check that there exists a representative f_0 of f so that, if $B \subset \Omega$ is an arbitrary open ball, then $\|f_0(p) - f_0(q)\| \leq K|p - q|$, for all $p, q \in B$.

Suppose first that $\Omega = B$. Consider the null set $E = \bigcup_{i=1}^\infty E_i \cup E_0$, and let $p, q \in B\backslash E$. Then there exists a subsequence $\{v_{i_j}\}_{j=1}^\infty$ of $\{v_i\}$ such that

$$\lim_{j\to\infty} v_{i_j} = f(p) - f(q).$$

Therefore

$$\begin{aligned}
\|f(p) - f(q)\| &= \lim_{j\to\infty} \|v_{i_j}\| = \lim_{j\to\infty} |\langle v_{i_j}^*, v_{i_j}\rangle| \\
&\leq \limsup_{j\to\infty} \left(\left|\langle v_{i_j}^*, v_{i_j} + f(q) - f(p)\rangle\right| + \left|\langle v_{i_j}^*, f(p) - f(q)\rangle\right| \right) \\
&\leq \limsup_{j\to\infty} \left(\|v_{i_j} - f(p) + f(q)\| + \left|\langle v_{i_j}^*, f(p) - f(q)\rangle\right| \right) \\
&= \limsup_{j\to\infty} \left|\langle v_{i_j}^*, f(p) - f(q)\rangle\right| = \limsup_{j\to\infty} |f_{i_j}(p) - f_{i_j}(q)| \leq K|p - q| .
\end{aligned}$$

This means that the restriction $f|_{B\setminus E}$ is K-Lipschitz on B. Since $B\setminus E$ is dense in B, this implies that $f|_{B\setminus E}$ has a unique K-Lipschitz extension f_0 to B, and f_0 is the required representative of f. In general, as the open set Ω can be covered by a sequence of balls, the direct implication follows from the previous case.

Conversely, suppose that f is locally K-Lipschitz. Then for each $v^* \in V^*$ with $\|v^*\| \le 1$ we have that $\langle v^*, f\rangle$ is also locally K-Lipschitz. Thus $\langle v^*, f\rangle \in W^{1,\infty}(\Omega)$ with $|\nabla\langle v^*, f\rangle| \le K$. Therefore, the constant function $g = K$ belongs to $\mathcal{R}(f)$ and $f \in R^{1,\infty}(\Omega, V)$. ■

Note that in the previous theorems, in contrast with Theorem 3.14, for p-almost every curve γ the composition $f \circ \gamma$ is absolutely continuous but, in general, it need not be differentiable almost everywhere unless the space V satisfies the Radon-Nikodým Property, naturally yielding our next result. We refer to Section 1.3 for an introduction to the Radon-Nikodým property and related bibliography.

Theorem 3.22. *Let Ω be an open subset of $\mathbb{R}^N$, let V be a Banach space and $1 \le p \le \infty$. Then $W^{1,p}(\Omega, V) = R^{1,p}(\Omega, V)$ if, and only if, the space V has the Radon-Nikodým property.*

Proof. Suppose first that V has the Radon-Nikodým Property and let $1 \le p < \infty$. Consider $f \in R^{1,p}(\Omega, V)$ and let $g \in \mathcal{R}(f)$. Fix a direction e_i parallel to the x_i-axis for any $i \in \{1, \dots, N\}$. From Theorem 3.20 we obtain a suitable representative of f such that, over p-almost every segment parallel to some e_i, f is absolutely continuous and, because of the Radon-Nikodým Property, almost everywhere differentiable. Therefore, by Lemma 3.12 and Fubini's Theorem we have that, for almost every $x \in \Omega$ and every $i \in \{1, \dots, N\}$, there exists the directional derivative

$$D_{e_i} f(x) = \lim_{h\to 0} \frac{f(x + he_i) - f(x)}{h}.$$

Note that each $D_{e_i} f$ is measurable, and that from Equation (3.7) above it follows that $\|D_{e_i} f(x)\| \le g(x)$ for almost every $x \in \Omega$. Thus $D_{e_i} f \in L^p(\Omega, V)$ for each $i \in \{1, \dots, N\}$. In addition, for every $v^* \in V^*$ we have that $\langle v^*, D_{e_i} f\rangle$ is the weak derivative $\langle v^*, f\rangle$. Then for every $\varphi \in C_0^\infty(\Omega)$

$$\left\langle v^*, \int_\Omega \varphi\, D_{e_i} f \right\rangle = \int_\Omega \varphi \langle v^*, D_{e_i} f\rangle = -\int_\Omega \frac{\partial\varphi}{\partial x_i} \langle v^*, f\rangle = \left\langle v^*, -\int_\Omega \frac{\partial\varphi}{\partial x_i} f \right\rangle.$$

Thus for every $i \in \{1, \dots, N\}$ the directional derivative $D_{e_i} f$ is, in fact, the i-th weak derivative of f, that is, $\partial f/\partial x_i = D_{e_i} f \in L^p(\Omega, V)$. It follows that $f \in W^{1,p}(\Omega, V)$. For the case $p = \infty$, if $f \in R^{1,\infty}(\Omega, V)$ cover Ω by a collection of open balls $\{B_i\}_{i\in\mathbb{N}}$ so by Theorem 3.21 there exists $K > 0$ such that $f_{|B_i}$ is K-Lipschitz for all i. In particular $f_{|B_i}$ is ACL and then by Theorem 3.10 there exist null sets $E_i \subset B_i$ such that f admits partial derivatives on each $x \in B_i \setminus E_i$. Let $E_0 = \bigcup_{i\ge 1} E_i$ so then f admits partial derivatives in $\Omega \setminus E$, while E has measure zero. The argument follows as in the case $1 \le p < \infty$.

For the converse, suppose that V does not have the Radon-Nikodým Property. Then there exists a Lipschitz function $h : [a, b] \to V$ which is not differentiable almost everywhere. We may also assume that $[a, b] \times R_0 = R$ is an N-dimensional rectangle contained in Ω, where R_0 is an $(N-1)$-dimensional rectangle. The function $f : [a, b] \times R_0 \to V$ given by $f(x_1, \dots, x_N) = h(x_1)$ is Lipschitz, so it admits an extension $\tilde{f} : \Omega \to V$ which is Lipschitz and has bounded support. Then, as noted in Remark 3.17, we have that $\tilde{f} \in R^{1,p}(\Omega, V)$. On the other hand, $\tilde{f}$ is not almost everywhere differentiable along any horizontal segment contained in $[a, b] \times R_0 = R$. From Lemma 3.12 and Theorem 3.14, we deduce that $\tilde{f} \notin W^{1,p}(\Omega, V)$. ■

3.4 Characterizations of $R^{1,p}(\Omega, V)$.

In light of Theorem 3.22 one could ask about the Sobolev-Reshetnyak space in the case where V is a general Banach space. It becomes evident that without the condition of almost everywhere differentiability along curves as stated in Theorem 3.20 and the Radon-Nikodym property, one might contemplate the weaker notions we discussed in Section 3.1, that is, metric and weak*-derivatives.

3.4.1 $R^{1,p}(\Omega, V)$ and metric derivatives.

Theorem 3.23. *Let V be a Banach space and $\Omega \subset \mathbb{R}^N$ open. Given $1 \leq p \leq \infty$ and $f \in L^p(\Omega, V)$ the following are equivalent:*

(i) $f \in R^{1,p}(\Omega, V)$.

(ii) f admits a representative that is absolutely continuous on a.e. line, and admits metric partial derivatives almost everywhere, satisfying that $m\partial_i f \in L^p(\Omega)$ for all $i = 1, \dots, N$.

Proof. $(i) \Rightarrow (ii)$ Consider first the case $1 \leq p < \infty$. Let $f \in R^{1,p}(\Omega, V)$ and consider $g \in \mathcal{R}(f)$. From the proof of Theorem 3.20 we obtain that f admits a representative f_0 which is absolutely continuous on every compact segment $\sigma : [a, b] \to \Omega$ parallel to a coordinate axis and, furthermore, for every such segment $\sigma(t) = a + te_i$ for $a \leq t \leq b$ with $x \in \Omega$ and $i \in \{1, \dots, N\}$, we have that

$$\|f(x + be_i) - f(x + ae_i)\| \leq \int_a^b g(x + \tau e_i)\, d\tau. \tag{3.8}$$

Additionally, since $g \in L^p(\Omega)$, from Fubini's Theorem we also obtain that $g \circ \sigma$ is integrable on $[a, b]$ for almost every such compact segment $\sigma : [a, b] \to \Omega$ parallel to a coordinate axis.

By Lemma 3.5, the ACL property of f_0 yields the existence of all partial metric derivatives at almost every $x \in \Omega$, and moreover for each $i \in \{1, \dots, N\}$ the partial metric derivative $m\partial_i f_0$ is measurable. From (3.8), we see that

$$m\partial_i f_0(z) = \lim_{h \to 0^+} \frac{\|f_0(z + he_i) - f_0(z)\|}{h} \leq \lim_{h \to 0^+} \frac{1}{h} \int_0^h g(z + \tau e_i)\, d\tau = g(z)$$

holds for $\mathcal{L}^1$-almost every point z of almost every line parallel to the i-th coordinate axis. Thus we obtain that $m\partial_i f_0(x) \leq g(x)$ at almost every $x \in \Omega$. In this way, we conclude that $m\partial_i f_0 \in L^p(\Omega)$ for each $i = 1, \dots, N$.

On the other hand, the case $p = \infty$ follows combining the characterization given in Theorem 3.21 with the almost everywhere metric differentiability of Lipschitz functions obtained by Kirchheim in [71].

$(ii) \Rightarrow (i)$ Let $1 \leq p \leq \infty$ and $f \in R^{1,p}(\Omega, V)$. Keep denoting by f the given representative which is absolutely continuous on a.e. line. Then for every $v^* \in V^*$ we obtain that $\langle v^*, f\rangle$ is also absolutely continuous on a.e. line. In addition, for each $i = 1, \dots, N$ the corresponding partial derivative satisfies

$$\left|\frac{\partial\langle v^*, f\rangle}{\partial x_i}(x)\right| = \lim_{h \to 0} \left|\left\langle v^*, \frac{f(x + he_i) - f(x)}{h}\right\rangle\right| \leq \|v^*\| \cdot m\partial_i f(x).$$

whenever $m\partial_i f(x)$ exists. Since this happens almost everywhere and $m\partial_i f$ is a non-negative function in $L^p(\Omega)$ then

$$\frac{\partial\langle v^*, f\rangle}{\partial x_i} \in L^p(\Omega).$$

In this way, from the Beppo-Levi characterization of Sobolev functions (see, e.g., [56, Theorem 6.1.13] or [81, Theorem 1.41]) it follows that $\langle v^*, f\rangle \in W^{1,p}(\Omega)$. Consider now

$$g := \sum_{i=1}^{N} m\partial_i f \in L^p(\Omega).$$

Then we obtain that, for every $v^* \in V^*$ with $\|v^*\| \leq 1$, and almost everywhere on Ω,

$$|\nabla\langle v^*, f\rangle| = \left(\sum_{i=1}^{N}\left|\frac{\partial\langle v^*, f\rangle}{\partial x_i}\right|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{N}(m\partial_i f)^2\right)^{\frac{1}{2}} \leq g\,.$$

This shows that $g \in \mathcal{R}(f)$, and $f \in R^{1,p}(\Omega, V)$. ∎

Remark 3.24 Let $f \in R^{1,p}(\Omega, V)$ as before. By the previous theorem, we can choose a suitable representative satisfying (ii). The above proof gives that, for this representative:

$$\|f\|_p + \sup_{1\leq i\leq N} \|m\partial_i f\|_p \leq \|f\|_{R^{1,p}} \leq \|f\|_p + \sum_{i=1}^{N} \|m\partial_i f\|_p.$$

In what follows consider the notation $\nabla_m f := \big(m\partial_i f\big)_{i=1}^{N}$ and

$$|\nabla_m f| := \left(\sum_{i=1}^{N} \|m\partial_i f\|^2\right)^{\frac{1}{2}}.$$

Thanks to the metric derivatives given by Theorem 3.23 we can take advantage of Morrey's inequality of the classical Sobolev space and get an analogous embedding result for the Sobolev-Reshetnyak space using the *metric gradient* $|\nabla_m f|$.

Theorem 3.25. *Let $\Omega \subset \mathbb{R}^N$ be open, and V a Banach space, and let $f \in R^{1,p}(\Omega, V)$ with $p > N$. The following inequality holds:*

$$\|f(x) - f(y)\| \leq C|x - y|^{1-\frac{N}{p}} \||\nabla_m f|\|_{L^p}$$

for almost every $x, y \in \Omega$, where C depends only on N and p.

Proof. First, we use the measurability of f to consider a sequence $\{v_i\}_{i=1}^{\infty} \subset V$ dense in the set $f(\Omega\backslash E_0) - f(\Omega\backslash E_0)$, where E_0 is the null set given by the condition that f is essentially separably evaluated. We also consider the functionals $\{v_i^*\}_{i=1}^{\infty} \subset V^*$ such that $\|v_i^*\| = 1$ and $\langle v_i^*, v_i\rangle = \|v_i\|$.

Since $f \in R^{1,p}(\Omega, V)$, it follows in particular that $\langle v_i^*, f\rangle \in W^{1,p}(\Omega)$ for each $i \in N$. Therefore, we can apply Morrey's inequality for the classical case and obtain a representative f_i of $\langle v_i^*, f\rangle$ for each i such that:

$$|f_i(x) - f_i(y)| \leq C|x - y|^{1-\frac{N}{p}} \|\nabla f_i\|_{L^p} = C|x - y|^{1-\frac{N}{p}} \||\nabla\langle v_i^*, f\rangle|\|_{L^p}.$$

Taking into account also that for each $i = 1, \ldots, N$ we have:

$$\left|\frac{\partial\langle v^*, f\rangle(x)}{\partial x_i}\right| = \lim_{h\to 0} \frac{|\langle v^*, f\rangle(x + he_i) - \langle v^*, f\rangle(x)|}{h}$$
$$\leq \lim_{h\to 0} \|v^*\| \cdot \frac{\|f(x + he_i) - f(x)\|}{|h|} \leq m\partial_i f(x)$$

we would have that

$$|f_i(x) - f_i(y)| \le C|x-y|^{1-\frac{N}{p}} \||\nabla_m f|\|_{L^p}.$$

Let us denote by Z_0 the union of the null sets generated by the representatives f_i. Let $x, y \in \Omega \backslash (Z_0 \cup E_0)$. Due to the density of the sequence $\{v_i\}_{i=1}^{\infty}$, we can consider a subsequence $\{v_{i_j}\}_{j=1}^{\infty}$ such that:

$$\lim_{j\to\infty} v_{i_j} = f(x) - f(y),$$

and therefore:

$$\begin{aligned}
\|f(x) - f(y)\| &= \lim_{j\to\infty} \|v_{i_j}\| = \lim_{j\to\infty} |\langle v_{i_j}^*, v_{i_j}\rangle| \\
&\le \limsup_{j\to\infty} \left(\left|\langle v_{i_j}^*, v_{i_j} + f(y) - f(x)\rangle\right| + \left|\langle v_{i_j}^*, f(x) - f(y)\rangle\right| \right) \\
&\le \limsup_{j\to\infty} \left(\|\langle v_{i_j}^*, v_{i_j} + f(y) - f(x)\rangle\| + \left|\langle v_{i_j}^*, f(x) - f(y)\rangle\right| \right) \\
&= \limsup_{j\to\infty} \left|\langle v_{i_j}^*, f(x) - f(y)\rangle\right| = \limsup_{j\to\infty} |f_{i_j}(x) - f_{i_j}(y)| \\
&\le C|x-y|^{1-\frac{N}{p}} \||\nabla_m f|\|_{L^p}.
\end{aligned}$$

Since $Z_0 \cup E_0$ is a null set, we have established the Morrey inequality for almost every point in Ω, as desired. ■

Remember that, in the context of vector-valued functions, a mapping may not be almost everywhere differentiable but almost everywhere metrically differentiable in the set $S(f)$, that is, the set of points where the pointwise Lipschitz constant is finite. This is due to Stepanov's Theorem (see 2.25), and thus Morrey's inequality yields the following metric differentiability result.

Theorem 3.26. *Let $\Omega \subset \mathbb{R}^N$ open and let V be a Banach space. Consider $f \in R^{1,p}(\Omega, V)$ with $p > N$. Then f is metrically differentiable almost everywhere.*

Proof. By Theorem 3.25 for almost every $x, y \in \Omega$ we have

$$\begin{aligned}
\|f(x) - f(y)\| &\le C_1|x-y|^{1-\frac{N}{p}} \left(\int_{B(x,|x-y|)} |\nabla_m f|^p \, d\mathcal{L}^N \right)^{\frac{1}{p}} \\
&\le C_1|x-y| \frac{1}{(\omega(N)^{-1}\mu(B(x,|x-y|)))^{\frac{1}{p}}} \left(\int_{B(x,|x-y|)} |\nabla_m f|^p \, d\mathcal{L}^N \right)^{\frac{1}{p}} \\
&= C_2|x-y| \left(\fint_{B(x,|x-y|)} |\nabla_m f|^p \, d\mathcal{L}^N \right)^{\frac{1}{p}},
\end{aligned}$$

where $\omega(N) = \mathcal{L}^N(B(0,1))$. If in addition we choose x to be a Lebesgue point of $|\nabla_m f|^p$ we have

$$\limsup_{y\to x} \frac{\|f(x) - f(y)\|}{|x-y|} \le \limsup_{y\to x} C \left(\fint_{B(x,|x-y|} |\nabla_m f|^p \, d\mathcal{L}^N \right)^{\frac{1}{p}} = |\nabla_m f(x)| < \infty.$$

By Stepanov's Theorem we conclude that f is metrically differentiable at almost every point. ■

3.4.2 $R^{1,p}(\Omega, V)$ and weak*-derivatives.

We now consider the case where $V = Y^*$ is the dual of a separable Banach space Y, in order to obtain a characterization of $R^{1,p}(\Omega, V)$ involving the partial weak*-derivatives. Note that a countable subset $\{y_k\}_{k=1}^{\infty}$ of the unit ball of Y is said to be *norming for* V if, for each $v \in V$, we have that

$$\|v\| = \sup_{k \in \mathbb{N}} |\langle y_k, v\rangle|.$$

Theorem 3.27. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $V = Y^*$ be the dual of a separable Banach space. Given $1 \le p \le \infty$ and $f \in L^p(\Omega, V)$, the following conditions are equivalent:*

(i) $f \in R^{1,p}(\Omega, V)$.

(ii) There exists a representative of f that is absolutely continuous on a.e. line and admits partial weak-derivatives almost everywhere, satisfying that $\|w^*\partial_i f\| \in L^p(\Omega)$ for all $i = 1, \ldots, N$.*

(iii) For each $i = 1, \ldots, N$ there exists a weak-measurable function $g_i : \Omega \to V$, with $\|g_i\| \in L^p(\Omega)$, and such that for every $y \in Y$:*

$$\int_\Omega \langle y, f(x)\rangle \frac{\partial \varphi(x)}{\partial x_i}\, dx = -\int_\Omega \langle y, g_i(x)\rangle \varphi(x)\, dx \quad \textit{for all} \quad \varphi \in C_0^\infty(\Omega)\,.$$

(iv) There exists a non-negative function $g \in L^p(\Omega)$ such that, for every $y \in Y$ with $\|y\| \le 1$ we have that $\langle y, f\rangle \in W^{1,p}(\Omega)$ and $|\nabla \langle y, f\rangle| \le g$ almost everywhere on Ω.

(v) There exist a countable set $\{y_k\}_{k=1}^{\infty} \subset Y$ which is norming for V, and a non-negative function $g \in L^p(\Omega)$ such that, for every $k \in \mathbb{N}$ we have that $\langle y_k, f\rangle \in W^{1,p}(\Omega)$ and $|\nabla\langle y_k, f\rangle| \le g$ almost everywhere on Ω.

Proof. *(ii)* $\Leftrightarrow$ *(i)* This follows at once from Theorem 3.9 and Theorem 3.23.

(ii) $\Rightarrow$ *(iii)* Consider a representative f as in *(ii)* and for each $i = 1, \ldots, N$ define $g_i = w^*\partial_i f$. As we have noted in Theorem 3.9, each $g_i : \Omega \to V$ is weak*-measurable and $\|g_i\| \in L^p(\Omega)$. Also, for each $y \in Y$, the function $\langle y, f\rangle$ is absolutely continuous on a.e. line, and therefore its i-th classical partial derivative exists almost everywhere on Ω and coincides with its i-th weak (or distributional) derivative. That is, for each $i = 1, \ldots, N$ we have that

$$\langle y, w^*\partial_i f(x)\rangle = \frac{\partial\langle y, f\rangle}{\partial x_i}(x) \quad \text{a.e. } x \in \Omega.$$

Then *(iii)* follows.

(iii) $\Rightarrow$ *(iv)* For each $y \in Y$ with $\|y\| \le 1$ it is clear that the function $\langle y, f\rangle \in L^p(\Omega)$ and also for each $i = 1, \ldots, N$ the function $\langle y, g_i\rangle \in L^p(\Omega)$. The integral equality in the statement of *(iii)* gives that $\langle y, g_i\rangle$ is the i-th weak (or distributional) partial derivative of $\langle y, f\rangle$. In this way, by the Beppo-Levi characterization of Sobolev functions, we see that $\langle f, y\rangle \in W^{1,p}(\Omega)$. Additionally, if we define

$$g = \sum_{i=1}^{N} \|g_i\| \in L^p(\Omega)$$

we obtain that

$$|\nabla\langle y, f\rangle| \le g \quad \text{a.e. for all} \quad y \in Y \quad \text{such that} \quad \|y\| \le 1.$$

(iv) ⇒ *(v)* Let $\{y_k\}_{k=1}^{\infty}$ be a dense countable subset of the unit ball of Y. It is clear that $\{y_k\}_{k=1}^{\infty}$ is norming for V, so the implication follows.

(v) ⇒ *(ii)* Let $\{y_k\}_{k=1}^{\infty}$ be a countable set in the unit ball of Y, which is norming for V. For each $k \in \mathbb{N}$ we have that $\langle y_k, f\rangle \in W^{1,p}(\Omega)$, and therefore it admits a representative f_k that is absolutely continuous on a.e. line. Let E_k denote the set where f_k differs from $\langle y_k, f\rangle$ and define $\Omega_0 = \bigcup_k E_k$, which is a null set.

Consider the family Σ of all compact segments $\sigma : [a,b] \to \Omega$ which are parallel to a coordinate axis, and satisfy the following properties:

(1) The composition $g \circ \sigma$ is integrable on $[a,b]$.

(1) The length of σ in Ω_0 is zero, that is, $\mathcal{L}^1(\{t \in [a,b] : \sigma(t) \in \Omega_0\}) = 0$.

(3) For each $k \in \mathbb{N}$, the composition $f_k \circ \sigma$ is absolutely continuous on $[a,b]$.

We claim that the family Σ represents almost all compact segments in Ω which are parallel to a coordinate axis. This follows taking into account for property (3) that each f_k is absolutely continuous on a.e. line, and on the other hand using Fubini's Theorem for properties (1) and (2). Note that every segment $\sigma : [a,b] \to \Omega$ in Σ is of the form $\sigma(t) = x + te$ for $a \le t \le b$, where $x \in \Omega$ and e is a unit vector $e \in \{\pm e_1, \dots, \pm e_N\}$. Also, if $\sigma : [a,b] \to \Omega$ is a segment in Σ, it is clear that for all $a \le s < t \le b$ the restriction $\sigma|_{[s,t]}$ also belongs to Σ. Now we distinguish two cases:

Suppose first that $\sigma : [a,b] \to \Omega$ is a segment in Σ whose endpoints satisfy $\sigma(a), \sigma(b) \notin \Omega_0$. Then for each $k \in \mathbb{N}$ we have that

$$|\langle y_k, f(\sigma(b))\rangle - \langle y_k, f(\sigma(a))\rangle| = |f_k(\sigma(b)) - f_k(\sigma(a))| \le \int_a^b |\nabla\langle y_k, f(\sigma(\tau))\rangle|\, d\tau \le \int_a^b g(\sigma(\tau))\, d\tau.$$

As a consequence,

$$\|f(\sigma(b)) - f(\sigma(a))\| = \sup_{k\in\mathbb{N}} |\langle y_k, f(\sigma(b)) - f(\sigma(a))\rangle| \le \int_a^b g(\sigma(\tau))\, d\tau.$$

Suppose now that $\sigma : [a,b] \to \Omega$ is a segment in Σ with at least one endpoint in Ω_0. In fact, we can suppose that $\sigma(a) \in \Omega_0$. By property (2), we can choose a sequence $\{t_k\}_{k=1}^{\infty} \subset [a,b]$ converging to a and such that $\sigma(t_k) \notin \Omega_0$. Then by the previous case

$$\|f(\sigma(t_k)) - f(\sigma(t_l))\| \le \left|\int_{t_k}^{t_l} g(\sigma(\tau))\, d\tau\right|.$$

for any $k, l \in \mathbb{N}$, and hence, as g is integrable on $[a,b]$, we see that $\{f(\sigma(t_k))\}_{k=1}^{\infty}$ is convergent. Suppose now that $\gamma : [c,d] \to \Omega$ is another segment in Σ satisfying $\sigma(a) = \gamma(c)$, and let $\{s_m\}_{m=1}^{\infty} \subset [c,d]$ be a sequence converging to c such that $\gamma(s_m) \notin \Omega_0$ for every $m \in \mathbb{N}$. Then

$$\|f(\sigma(t_k)) - f(\gamma(s_m))\| \le \int_c^{s_m} g(\sigma(\tau))\, d\tau + \int_a^{t_k} g(\sigma(\tau))\, d\tau \stackrel{k,m\to\infty}{\longrightarrow} 0.$$

This proves that the limit of $f(\sigma(t_k))$ as $k \to \infty$ is independent of the choice of the segment σ and the sequence $\{t_k\}_{k=1}^{\infty}$.

Now we define a representative f_0 of f in the following way:

1. If $x \in \Omega \setminus \Omega_0$ we set $f_0(x) = f(x)$.
2. If $x \in \Omega_0$ and there exists a segment $\sigma : [a,b] \to \Omega$ in Σ such that $\sigma(a) = x$, we set $f_0(x) = \lim_{k\to\infty} f(\sigma(t_k))$ where $\{t_k\}_{k=1}^{\infty} \subset [a,b]$ is a sequence converging to a such that $\sigma(t_k) \notin \Omega_0$ for each k.
3. Otherwise, we set $f_0(x) = 0$.

By definition, $f_0 = f$ almost everywhere and, for every segment $\sigma : [a,b] \to \Omega$ in Σ and every $a \le s < t \le b$, we have:

$$\|f_0(\sigma(t)) - f_0(\sigma(s))\| \le \int_s^t g(\sigma(\tau))\, d\tau.$$

Since $g \circ \sigma \in L^1([a,b])$, this implies that f_0 is absolutely continuous on a.e. line. Then using Theorem 3.9 and the Lebesgue differentiation Theorem we deduce that

$$\|w^*\partial_i f(z)\| = \lim_{h\to 0^+} \frac{\|f(z+he_i) - f(z)\|}{h} \le \lim_{h\to 0^+} \frac{1}{h}\int_0^h g(z+\tau e_i)\, d\tau = g(z),$$

holds for $\mathcal{L}^1$-almost every point z of almost every line parallel to the i-th coordinate axis. In this way we see that $\|w^*\partial_i f\| \in L^p(\Omega)$ for all $i = 1, \dots, N$. ■

Remark 3.28 Equivalences $(i) \Leftrightarrow (iii)$ and $(i) \Leftrightarrow (iv)$ of the above Theorem have been independently obtained by Creutz and Evseev in [26]. In fact, [26, Theorem 1.4] states a slightly more general fact than $(i) \Leftrightarrow (iv)$ in our result, since they do not assume separability of Y.

Remark 3.29 Combining Lemma 3.9 and Morrey's inequality in Theorem 3.25 an analogous inequality holds for the weak*-derivatives, that is, if $V = Y^*$ is dual to a separable Banach space and $f \in R^{1,p}(\Omega, V)$ then

$$\|f(x) - f(y)\| \le C|x-y|^{1-\frac{N}{p}} \|\nabla_{w^*} f\|_{L^p}$$

for almost every $x, y \in \Omega$, where $\nabla_{w^*} f = \left(\|w^*\partial_i f\|\right)_{i=1}^N$.

The following result provides a simple criterion for deciding whether a member of $R^{1,p}(\Omega, V)$ belongs to $W^{1,p}(\Omega, V)$ or not, involving only the measurability of the partial weak*-derivatives.

Corollary 3.30. *Suppose that $\Omega \subset \mathbb{R}^N$ is open, $V = Y^*$ is the dual of a separable Banach space Y, $1 \le p \le \infty$, and $f \in R^{1,p}(\Omega, V)$. Then, $f \in W^{1,p}(\Omega, V)$ if and only if $w^*\partial_i f$ is measurable for all $i = 1, \dots, N$.*

Proof. The direct implication is clear. For the converse, let $i \in \{1, \dots, N\}$ and suppose that $w^*\partial_i f$ is measurable. Since by Theorem 3.27 we know that $\|w^*\partial_i f\| \in L^p(\Omega)$, we obtain that $w^*\partial_i f \in L^p(\Omega, V)$. We are going to check that, in fact, $w^*\partial_i f$ is the i-th weak (or distributional) partial derivative of f. To this end, fix $\varphi \in C_0^\infty(\Omega)$. Note that, for each $y \in Y$ with $\|y\| \le 1$, by the proof of Theorem 3.27, we see that

$$\begin{aligned}\left\langle y, \int_\Omega f(x)\frac{\partial\varphi(x)}{\partial x_i}\, dx \right\rangle &= \int_\Omega \langle y, f(x)\rangle \frac{\partial\varphi(x)}{\partial x_i}\, dx \\ &= -\int_\Omega \langle y, w^*\partial_i f(x)\rangle \varphi(x)\, dx \\ &= -\left\langle y, \int_\Omega w^*\partial_i f(x)\, \varphi(x)\, dx \right\rangle.\end{aligned}$$

As a consequence,

$$\int_{\Omega} f(x) \frac{\partial \varphi(x)}{\partial x_i} \, dx = - \int_{\Omega} w^* \partial_i f(x) \varphi(x) \, dx.$$

Therefore, $f \in W^{1,p}(\Omega, V)$. ■

As a consequence we now provide a nice description of the Sobolev-Reshetnyak space when $V = \ell^\infty$, since $\ell^\infty = (\ell^1)^*$ is the dual of the separable space ℓ^1 of absolutely summable sequences. In addition, the unit vector basis $\{e_k\}_{k=1}^\infty$ of ℓ^1 is clearly a norming set for ℓ^∞. For an open set $\Omega \subset \mathbb{R}^N$, a function $f : \Omega \to \ell^\infty$ has components $f = (f_k)_{k=1}^\infty$, where $f_k = \langle e_k, f \rangle$ are real-valued functions and for each $x \in \Omega$, $\sup\limits_{k \geq 1} |f_k(x)| < \infty$. Then a direct application of Theorem 3.27 (v) gives the following.

Corollary 3.31. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $1 \leq p \leq \infty$, and $f \in L^p(\Omega, \ell^\infty)$, with components $f = (f_k)_{k=1}^\infty$. Then $f \in R^{1,p}(\Omega, \ell^\infty)$ if and only the following two conditions are satisfied:*

(i) $f_k \in W^{1,p}(\Omega)$ for all $k \in \mathbb{N}$.

(ii) There exists a non-negative function $g \in L^p(\Omega)$ such that $|\nabla f_k| \leq g$ a.e. for all $k \in \mathbb{N}$.

Finally, we use the previous results to reconsider Example 3.19, but now with a different approach. Recall that ℓ^∞ lacks the Radon-Nikodým Property, and therefore, for any open set $\Omega \subset \mathbb{R}^N$ the classical Sobolev space $W^{1,p}(\Omega, \ell^\infty)$ is a proper subspace of $R^{1,p}(\Omega, \ell^\infty)$. The function f given in Example 3.19 belongs to $R^{1,p}((0,1), \ell^\infty) \setminus W^{1,p}((0,1), \ell^\infty)$ but we now justify that f cannot be in $W^{1,p}((0,1), \ell^\infty)$ by proving that its weak*-derivative is not essentially separably-valued.

Example 3.32 *Consider the function $f : (0,1) \to \ell^\infty$ given by*

$$f(t) = \left(\frac{\sin(2\pi n t)}{2\pi n} \right)_{n=1}^\infty .$$

Then f belongs to $R^{1,p}((0,1), \ell^\infty)$ and $g(t) := (\cos(2\pi n t))_{n=1}^\infty$ is the weak-derivative of f. Furthermore, g is not essentially separably-valued and so f does not belong to $W^{1,p}((0,1), \ell^\infty)$.*

Indeed, by Corollary 3.31, f belongs to $R^{1,p}((0,1), V)$. It is easy to see that g is the weak*-derivative of f. We use Corollary 3.30 to prove that f does not belong to $W^{1,p}(\Omega, V)$. We claim that g is not measurable. By Pettis Theorem, it is enough to see that g is not essentially separably-valued. To this end, given a null set $Z \subset (0,1)$, we will find a non-countable set $B \subset (0,1) \setminus Z$ such that for any pair of different elements $x, y \in B$ the inequality $\|g(x) - g(y)\|_\infty > \frac{1}{3}$ holds. Notice that the function $h(x) = \cos(2\pi x)$ is uniformly continuous on $\mathbb{R}$, so there exists $\delta > 0$ such that $|\cos(2\pi x) - \cos(2\pi y)| < \frac{1}{3}$ for each $x, y \in \mathbb{R}$ such that $|x - y| < \delta$. It is known (see e.g. [87], page 108) that there exists a non-measurable Hamel basis of $\mathbb{R}$ as a $\mathbb{Q}$-vector space, and hence we can find a non-measurable set $A \subset (0,1) \setminus \mathbb{Q}$ such that if $x \neq y$ in A, then the set $\{1, x, y\}$ is $\mathbb{Q}$-linearly independent. Given any null set $Z \subset (0,1)$, consider the set $B = A \setminus Z$, that is not measurable, and hence non-countable. Take $x \neq y$ in B. Since $\{1, x, y\}$ are $\mathbb{Q}$-linearly independent, by the Kronecker's Theorem on Diophantic Approximation (see [46], page 507) there exist $n, p, q \in \mathbb{Z}$ such that

$$\left| p + nx - \frac{1}{4} \right| < \delta \quad \text{and} \quad |q + ny| < \delta.$$

If $n \in \mathbb{N}$,

$$\begin{aligned}|g_n(x) - g_n(y)| &= |\cos(2\pi nx) - \cos(2\pi ny)| = |\cos(2\pi(p + nx)) - \cos(2\pi(q + ny))| \\ &\geq |\cos(2\pi 0) - \cos\left(2\pi\frac{1}{4}\right)| - |\cos(2\pi(p + nx)) - \cos\left(2\pi\frac{1}{4}\right)| \\ &- |\cos(2\pi(q + ny)) - \cos(2\pi 0)| \\ &> 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}.\end{aligned}$$

For $n \leq -1$, consider the component g_m with $m = -n$ and the proof is analogous. Hence,

$$\|g(x) - g(y)\|_\infty > \frac{1}{3} \text{ for all different } x, y \in B$$

and there exists a non-countable set in $g((0,1)\backslash Z)$ with that property, proving $g((0,1)\backslash Z)$ is not separable.

Chapter 4
Mappings of Bounded Variation

In the present chapter we consider two definitions of mappings of bounded variation from a metric measure space into a metric space or, in particular, a Banach space, and for that we adapt the definitions given by Miranda Jr. and O. Martio for the scalar-valued case, that is: one by relaxation of Newton-Sobolev maps in the same spirit of [84] and the other by assuming that the local behavior of the map is controlled by a sequence of non-negative Borel functions that serve as a substitute for upper gradients, as in [82]. When the domain is a complete metric measure space, doubling, and supports a 1-Poincaré inequality and the target is a Banach space, both notions yield the same class of maps. However, when the target metric space is not a Banach space, the two approaches do not in general yield the same class of functions, with the notion of relaxation of Sobolev functions yielding a *strictly smaller* subclass of maps. Thus, in the setting of general metric space targets, it is more appropriate to study mappings of bounded variation based on the sequence of upper gradients as first proposed by Martio in [82].

Having made the choice of the definition of mappings of bounded variation, in the second part of the chapter we explore the fine properties of mappings of bounded variation, from a complete doubling metric measure space supporting a 1-Poincaré inequality, into a proper metric space. We determine Hausdorff co-dimensional measure properties of sets of jump discontinuity points of such mappings.

4.1 Two notions of mappings of bounded variation.

Along this chapter (X, d, μ) will denote a metric measure space, where (X, d) is a complete metric space and μ a Borel doubling measure, and V is a Banach space.

We now introduce two well-known notions of mappings of bounded variation. The first one, $BV(X, V)$, was widely studied in [84], while the other one, $BV_{AM}(X, V)$, was first introduced in [82] and turned out to be equal to $BV(X, V)$ when $V = \mathbb{R}$ and the measure on X is doubling and supports a 1-Poincaré inequality (see [35]). A natural problem is to study the relation of these two spaces when V is a general Banach space.

4.1.1 Vector-valued mappings of bounded variation via relaxation of Newton-Sobolev mappings

Let $u \in L^1(X, V)$ and define its BV energy seminorm as

$$\|Du\|(X) := \inf\left\{\liminf_{i\to\infty}\int_X g_{u_i}\,d\mu\right\},$$

where the infimum is taken over all sequences $(u_i)_{i\in\mathbb{N}}$ of mappings in $N^{1,1}(X,V)$ with minimal upper gradients g_{u_i} that converge to u in $L^1(X,V)$. The following definition extends the notion of real-valued BV mappings introduced by Miranda in [84] to the vector valued setting.

Definition 4.1 Let (X,d,μ) be a metric measure space and V a Banach space. We define $BV(X,V)$ to be the class of mappings $u\in L^1(X,V)$ such that $\|Du\|(X)<\infty$. We denote $BV(X):=BV(X,\mathbb{R})$.

It was shown in [84] that for a real-valued function $u\in BV(X)$ the map $U\mapsto\|Du\|(U)$ for open sets $U\subset X$ can be extended via a Carathéodory construction to a Radon outer measure on X, which is also denoted by $\|Du\|$. This construction follows analogously for vector-valued mappings as in our context, so in particular for $u\in BV(X,V)$ and a Borel set $A\subset X$ we have

$$\|Du\|(A) := \inf\{\|Du\|(U)\,:\,U \text{ is open in } X \text{ and } A\subset U\}.$$

If $E\subset X$ is a measurable set, we say that E is of *finite perimeter* if $\chi_E\in BV(X)$ and we denote the perimeter measure by

$$P(E,A) := \|D\chi_E\|(A),\quad A\subset X \text{ Borel}.$$

For functions in the class $BV(X)$ the following co-area formula is known.

Lemma 4.2. (Coarea formula, [84, Proposition 4.2]) *Let $E\subset X$ be a Borel set and $u\in BV(X)$. Then*

$$\|Du\|(E) = \int_{-\infty}^{\infty} P(\{u>t\},E)dt.$$

Thanks to the work of Ambrosio [4], we know the structure of sets of finite perimeter. In order to showcase the properties of such sets we first describe the measure-theoretic and reduced boundaries of subsets of X. For $E\subset X$ we say that a point $x\in X$ belongs to the *measure-theoretic boundary* ∂_*E of E if

$$\limsup_{r\to 0^+}\frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))}>0 \quad\text{and}\quad \limsup_{r\to 0^+}\frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))}>0. \tag{4.1}$$

For a real number $\beta>0$ we say that $x\in X$ belongs to the *reduced boundary* $\Sigma_\beta E$ of E if

$$\liminf_{r\to 0^+}\frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))}\ge\beta \quad\text{and}\quad \liminf_{r\to 0^+}\frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))}\ge\beta. \tag{4.2}$$

Lemma 4.3. *Suppose that X is complete and that μ is doubling and supports a 1-Poincaré inequality. Then there is a positive real number $\gamma\le 1/2$, depending only on the doubling constant and the constants associated with the Poincaré inequality, such that for each set E of finite perimeter,*

$$\mathcal{H}^{-1}(\partial_*E\setminus\Sigma_\gamma E)=0 \quad\textit{and}\quad P(E,X)\approx\mathcal{H}^{-1}(\Sigma_\gamma E).$$

We also point out that in fact, the property of a measurable set being of finite perimeter is characterized by the property that $\mathcal{H}^{-1}(\Sigma_\gamma E)$ is finite; this result was first proved by Lahti [75], and is new even in the Euclidean setting, refining Federer's characterization of Euclidean sets of finite perimeter.

4.1.2 AM-modulus of a family of curves.

We now introduce a notion of modulus of a family of curves that it is weaker than the 1-modulus defined in Section 1.4.3, but it turns out to be better suited for the study of BV maps as we will see in 4.4. This notion, called AM-modulus, was first proposed by Martio in [82]. Following [82] we define the *AM-modulus* to be

$$\mathrm{AM}(\Gamma) := \inf_{(\rho_i)_{i\in\mathbb{N}}} \liminf_{i\to\infty} \int_X \rho_i d\mu,$$

where the infimum is taken over all sequences of AM-admissible functions, that is, sequences $(\rho_i)_{i\in\mathbb{N}}$ of non-negative Borel functions $\rho_i : X \to \mathbb{R}$ such that for each $\gamma \in \Gamma$ we have

$$\liminf_{i\to\infty} \int_\gamma \rho_i ds \geq 1.$$

We say that a property holds for 1-almost every curve (respectively AM-almost every curve) on X if it holds outside a family of curves of zero 1-modulus (resp. AM-modulus).
For a curve family Γ we always have $\mathrm{AM}(\Gamma) \leq \mathrm{Mod}_1(\Gamma)$, and examples of the strict inequality can be found in [35] and [59].

The following lemma provides a useful characterization of null families with respect to the AM-modulus in the same spirit of Lemma 1.27 and it follows from [59, Theorem 25]. However we provide here a proof in our context for simplicity.

Lemma 4.4. *Let Γ be a family of curves in X. Then $\mathrm{AM}(\Gamma) = 0$ if and only if there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ of non-negative Borel functions with $\sup_i \int_X \rho_i\, d\mu < \infty$ such that for each $\gamma \in \Gamma$ we have*

$$\liminf_{i\to\infty} \int_\gamma \rho_i\, ds = \infty.$$

Proof. Suppose first that $\mathrm{AM}(\Gamma) = 0$. Then for each positive integer k we can find a sequence $(\rho_{k,i})_{i\in\mathbb{N}}$ of non-negative Borel functions on X with $\liminf_{i\to\infty} \int_\gamma \rho_{k,i}\, ds \geq 1$ for each $\gamma \in \Gamma$, such that $\sup_i \int_X \rho_{k,i}\, d\mu < 2^{-k}$. For each positive integer i we set $\rho_i = \sum_{k=1}^\infty \rho_{k,i}$. By the monotone convergence Theorem we know that for each $\gamma \in \Gamma$,

$$\int_\gamma \rho_i\, ds = \sum_{k=1}^\infty \int_\gamma \rho_{k,i}\, ds,$$

and so

$$\liminf_i \int_\gamma \rho_i\, ds \geq \sum_{k=1}^\infty \liminf_i \int_\gamma \rho_{k,i}\, ds = \infty,$$

and at the same time, for each positive integer i we have

$$\int_X \rho_i\, d\mu = \sum_{k=1}^\infty \int_X \rho_{k,i}\, d\mu \leq \sum_{k=1}^\infty 2^{-k} = 1.$$

The desired conclusion follows.

Now suppose that Γ is such that there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ of non-negative Borel functions on X such that $\sup_i \int_X \rho_i\, d\mu =: \alpha < \infty$ and for each $\gamma \in \Gamma$ we have $\liminf_{i\to\infty} \int_\gamma \rho_i\, ds = \infty$. Then for each $\varepsilon > 0$ the sequence $(\varepsilon\rho_i)_{i\in\mathbb{N}}$ is a sequence of AM-admissible functions for Γ, with $\limsup_i \int_X \varepsilon\rho_i\, d\mu = \varepsilon\alpha$. Thus $\mathrm{AM}(\Gamma) \leq \varepsilon\alpha$ for each $\varepsilon > 0$. Thus we have that $\mathrm{AM}(\Gamma) = 0$. ■

From the above lemma, it follows that if Γ is a family of curves with $\mathrm{AM}(\Gamma) = 0$, then for each $\varepsilon > 0$ there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ such that $\sup_i \int_X \rho_i\, d\mu < \varepsilon$ and for each $\gamma \in \Gamma$ we have $\liminf_{i\to\infty} \int_\gamma \rho_i\, ds = \infty$.

4.1.3 The notion of $BV_{AM}(X, V)$

Now we turn our attention to the definition of $BV_{AM}(X, V)$. The notion of $BV_{AM}(X, \mathbb{R})$ was first proposed by Honzlová-Exnerová, Malý, and Martio in a series of papers [60, 61, 62] using the notion of AM-modulus, see also [58]. This notion was adopted by Lahti in [76] to study metric space-valued BV mappings. In this section we focus on this notion of BV maps when the target space is a Banach space.

Definition 4.5 Let (X, d, μ) be a metric measure space and V a Banach space. Let $(\rho_i)_{i\in\mathbb{N}}$ be a sequence of non-negative Borel functions on X. We say that this sequence is an *AM-bounding sequence* for a function $u : X \to V$ if for AM-a.e. curve $\gamma : [a, b] \to X$ there is a null set $N_\gamma \subset [a, b]$ (that is, $\mathcal{H}^1(N_\gamma) = 0$) such that for every $\tau, t \in [a, b] \setminus N_\gamma$ with $\tau < t$, we have

$$\|u(\gamma(\tau)) - u(\gamma(t))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i ds. \tag{4.3}$$

We say that a mapping $u \in L^1(X, V)$ is in the class $BV_{AM}(X, V)$ if there is an AM-bounding sequence $(\rho_i)_{i\in\mathbb{N}}$ for u such that

$$\liminf_{i\to\infty} \int_X \rho_i \, d\mu < \infty.$$

We also define the BV-AM energy seminorm of u as

$$\|D_{AM}u\|(X) := \inf_{(\rho_i)_i} \liminf_{i\to\infty} \int_X \rho_i \, d\mu,$$

where the infimum is taken over all AM-bounding sequences $(\rho_i)_{i\in\mathbb{N}}$ of u.

As in the case of the object $\|Du\|$, the above $\|D_{AM}u\|$ can be extended to be a Radon measure on X via a Carathéodory construction, see for example [82, Theorem 4.1] for the real-valued case, whose arguments can be adapted to the context of vector-valued mappings.

Note that in considering an AM-bounding sequence for u, we discount a family Γ of curves in X such that $\mathrm{AM}(\Gamma) = 0$. If the AM-bounding sequence $(\rho_i)_{i\in\mathbb{N}}$ is such that the exceptional family Γ is empty, then we say that $(\rho_i)_{i\in\mathbb{N}}$ is a *strong bounding sequence* for u.

Lemma 4.6. *Let $u \in BV_{AM}(X, V)$ and $v : X \to V$. Suppose that there is a set $N \subset X$ with $\mu(N) = 0$ such that for each $x \in X \setminus N$ we have $u(x) = v(x)$. Then a sequence $(\rho_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u if and only if it is an AM-bounding sequence for v; hence $v \in BV_{AM}(X, V)$ with $\|D_{AM}u\|(X) = \|D_{AM}v\|(X)$ and $\|D_{AM}(u - v)\|(X) = 0$.*

Proof. Since N is a null-set, by enlarging it if needed (recall that μ is a Borel measure), we can also assume that it is a Borel set as well. It follows that with Γ_N^+ the collection of all non-constant compact rectifiable curves $\gamma : [a, b] \to X$ for which $\mathcal{H}^1(\gamma^{-1}(N)) > 0$, we have $\mathrm{AM}(\Gamma_N^+) \le \mathrm{Mod}_1(\Gamma_N^+) = 0$. Thus, for each AM-bounding sequence $(\rho_i)_{i\in\mathbb{N}}$ of the original function u and for each $\gamma \notin \Gamma_0 \cup \Gamma_N^+$, we can replace N_γ with $N_\gamma \cup \gamma^{-1}(N)$ to see that this is an AM-bounding sequence for v as well. Here Γ_0 is the exceptional family associated with the bounding sequence; so $\mathrm{AM}(\Gamma_0) = 0$.

Since $u - v = 0$ μ-a.e. in X, the final claim follows from noting that $\mathrm{AM}(\Gamma_N^+) = 0$ and so the sequence $(g_i)_{i\in\mathbb{N}}$, with each g_i the zero function, is an AM-bounding sequence for $u - v$. ■

Lemma 4.7. *Suppose that $(\rho_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for a map u from X to V such that $\sup_i \int_X \rho_i \, d\mu < \infty$. Then for each $\varepsilon > 0$ we can find a strong bounding sequence $(g_i)_{i\in\mathbb{N}}$ of u such that for each $i \in \mathbb{N}$ we have $\int_X |g_i - \rho_i| \, d\mu < \varepsilon$.*

Proof. Let $u \in BV_{AM}(X,V)$ and $(\rho_i)_{i\in\mathbb{N}}$. Then there exists Γ with $\mathrm{AM}(\Gamma)=0$ such that for each non constant compact rectifiable curve $\gamma \notin \Gamma$, the relation (4.3) holds for $s,t \in \mathrm{dom}(\gamma)\backslash\gamma^{-1}(N_\gamma)$ where $\mathcal{H}^1(N_\gamma)=0$. Since $\mathrm{AM}(\Gamma)=0$, by Lemma 4.4 there exists a sequence of non-negative Borel functions $(g_i)_i$ such that

$$\liminf_{i\to\infty}\int_X g_i\,d\mu<\infty \quad \text{and} \quad \liminf_{i\to\infty}\int_\gamma g_i\,ds=\infty \ \ \forall\gamma\in\Gamma.$$

Now let Γ_0 be the family of all non-constant compact rectifiable curves γ in X for which we have $\liminf_{i\to\infty}\int_\gamma g_i ds=\infty$. Then $\mathrm{AM}(\Gamma_0)=0$ and each subcurve of a curve that is not in Γ_0 is also not in Γ_0. For each $\varepsilon>0$, since $\Gamma\subset\Gamma_0$, we have that for each $\gamma\notin\Gamma_0$,

$$\|u(\gamma(\tau))-u(\gamma(t))\|\le\liminf_{i\to\infty}\int_{\gamma|_{[\tau,t]}}\rho_i ds\le\liminf_{i\to\infty}\int_{\gamma|_{[\tau,t]}}\rho_i+\varepsilon g_i ds \tag{4.4}$$

for every $\tau,t\notin\gamma^{-1}(N_\gamma)$.

If $\gamma\in\Gamma_0$ is such that every subcurve of γ also belongs to Γ_0, then for each $\tau,t\in[a,b]$ with $\tau<t$ we have $\liminf_{i\to\infty}\int_{\gamma|_{[\tau,t]}} g_i\,ds=\infty$, and so the choice of $N_\gamma=\emptyset$ works. If it is not the case that every subcurve of γ also belongs to Γ_0, then let $\mathcal{C}_0(\gamma)$ be the collection of all non-degenerate (that is, containing more than one point) intervals $I\subset[a,b]$ for which, whenever J is a compact subinterval of I we must have $\gamma|_J\notin\Gamma_0$, and whenever J is a compact subinterval of $[a,b]$ containing I in its interior, we must have $\gamma|_J\in\Gamma_0$. By the maximality of the intervals in the collection $\mathcal{C}_0(\gamma)$, two intervals in this collection are either disjoint or are equal as intervals. Moreover, these intervals have non-empty interior. It follows that as $\mathbb{Q}$ is dense in $\mathbb{R}$, the collection $\mathcal{C}_0(\gamma)$ is countable.

With $\gamma:[a,b]\to X$, consider all $a_0,b_0\in[a,b]\cap\mathbb{Q}$ with $a_0<b_0$ for which $\gamma|_{[a_0,b_0]}\notin\Gamma_0$, that is, $\liminf_{i\to\infty}\int_{\gamma|_{[a_0,b_0]}} g_i\,ds<\infty$; hence there is a corresponding null set $N[a_0,b_0]\subset[a_0,b_0]$ with $\mathcal{H}^1(N[a_0,b_0])=0$, so that for each $\tau,t\in[a_0,b_0]\setminus N[a_0,b_0]$ we have (4.4) holding true. Let $\mathcal{C}(\gamma)$ be the collection of all such $[a_0,b_0]\subset[a,b]$, and set

$$N_\gamma:=\bigcup_{[a_0,b_0]\in\mathcal{C}(\gamma)}N[a_0,b_0]\cup\bigcup_{J\in\mathcal{C}_0(\gamma)}\{\inf J,\sup J\}.$$

Note that as $a_0,b_0\in\mathbb{Q}$, the collection $\mathcal{C}(\gamma)$ is a countable collection. Hence $\mathcal{H}^1(N_\gamma)=0$ by the subadditivity of $\mathcal{H}^1$ on $[a,b]$. Now let $\tau,t\in[a,b]\setminus N_\gamma$ with $\tau<t$. If $[\tau,t]\subset[a_0,b_0]$ for some $[a_0,b_0]\in\mathcal{C}(\gamma)$, then (4.4) holds. If there is no $[a_0,b_0]\in\mathcal{C}(\gamma)$ for which $[\tau,t]\subset[a_0,b_0]$, then we must have that $\liminf_{i\to\infty}\int_{\gamma|_{[\tau,t]}} g_i\,ds=\infty$, and so we now have

$$\|u(\gamma(\tau))-u(\gamma(t))\|\le\liminf_{i\to\infty}\int_{\gamma|_{[\tau,t]}}(\rho_i+\varepsilon g_i)\,ds=\infty.$$

Therefore $(\rho_i+\varepsilon g_i)_i$ satisfies (4.3) for every non constant compact rectifiable curve. Moreover, since ε is arbitrary, one can approach the energy $\|D_{AM}u\|(X)$ just by taking the infimum over the upper bounds of u that verify (4.3) for every non-constant compact rectifiable curve. ∎

Lemma 4.8. *Suppose that $(\rho_i)_{i\in\mathbb{N}}$ is an AM–bounding sequence for $u:X\to V$ and that η is a non-negative L-Lipschitz function with support in a bounded set U; moreover, suppose that η is constant on an open set $V\Subset U$. Then $(\eta\rho_i+L\,|u|\,\chi_{U\setminus V})_i$ is an AM–bounding sequence for ηu.*

Proof. For each $i\in\mathbb{N}$ we set $g_i:=\eta\rho_i+L\,|u|\,\chi_{U\setminus V}$.

Let Γ be the exceptional family for the AM–bounding sequence $(\rho_i)_i$ with respect to u; so $\mathrm{AM}(\Gamma)=0$, and for each non-constant compact rectifiable curve $\gamma:[a,b]\to X$ with $\gamma\notin\Gamma$, we have a null set $N_\gamma \subset [a,b]$ with $\mathcal{H}^1(N_\gamma) = 0$ such that whenever $t, \tau \in [a,b] \setminus N_\gamma$ with $t < \tau$, we have

$$\|u(\gamma(t)) - u(\gamma(\tau))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[t,\tau]}} \rho_i \, ds.$$

Fix such $\tau, t \in [a,b]$. Consider a partition $t = t_0 < t_1 < \cdots < t_k = \tau$ of the interval $[t,\tau]$ so that $t_1, \cdots, t_{k-1} \notin N_\gamma$. Then

$$\begin{aligned}
\|u(\gamma(t))\eta(\gamma(t)) - u(\gamma(\tau))\eta(\gamma(\tau))\| &\le \sum_{j=1}^{k} \|u(\gamma(t_{j-1}))\eta(\gamma(t_{j-1})) - u(\gamma(t_j))\eta(\gamma(t_j))\| \\
&\le \sum_{j=1}^{k} \|u(\gamma(t_{j-1}))\eta(\gamma(t_{j-1})) - u(\gamma(t_j))\eta(\gamma(t_{j-1}))\| \\
&\qquad + \sum_{j=1}^{k} \|u(\gamma(t_j))\eta(\gamma(t_{j-1})) - u(\gamma(t_j))\eta(\gamma(t_j))\| \\
&\le \liminf_{i\to\infty} \sum_{j=1}^{k} \int_{\gamma|_{[t_{j-1},t_j]}} [\eta(\gamma(t_{j-1}))\rho_i + \|u(\gamma(t_j))\| \operatorname{Lip}\eta] \, ds.
\end{aligned}$$

The above must be true for all such partitions of the interval $[t,\tau]$. Since $u \circ \gamma$ and $\operatorname{Lip}\eta \circ \gamma$ are Borel functions on $[a,b]$, it follows that

$$\|u(\gamma(t))\eta(\gamma(t)) - u(\gamma(\tau))\eta(\gamma(\tau))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[t,\tau]}} g_i \, ds.$$

■

Lemma 4.9. *Let $u \in BV(X,V)$. Then the upper gradients of an approximating sequence form an AM-bounding sequence for u. In particular $u \in BV_{AM}(X,V)$ with $\|D_{AM}u\|(X) \le \|Du\|(X)$.*

Proof. If $u \in BV(X,V)$, then there exists a sequence $(u_i)_{i\in\mathbb{N}} \in N^{1,1}(X,V)$ with upper gradients g_i such that

$$\limsup_{i\to\infty} \int_X g_i d\mu < \infty,$$

and such that $u_i \to u$ in $L^1(X,V)$. In particular, (passing to a subsequence if necessary) $u_i \to u$ pointwise almost everywhere in X. Then there exists a null set N such that $\lim_{i\to\infty} u_i(x) = u(x)$ for every $x \in X \setminus N$. By enlarging N if necessary, we may also assume that N is a Borel set (recall that μ is Borel regular). Hence, by Lemma 1.28, we know that if Γ_N^+ is the collection of all compact non-constant rectifiable curves γ in X with $\mathcal{H}^1(\gamma^{-1}(N)) > 0$, then $\mathrm{AM}(\Gamma_N^+) \le \mathrm{Mod}_1(\Gamma_N^+) = 0$.

Let $\gamma : [a,b] \to X$ be a non-constant compact rectifiable curve such that $\gamma \notin \Gamma_N^+$. We set $N_\gamma := \gamma^{-1}(N)$. Then for $\tau, t \in [a,b] \setminus N_\gamma$ with $\tau < t$, we have that

$$\lim_{i\to\infty} u_i(\gamma(t)) - u_i(\gamma(\tau)) = u(\gamma(t)) - u(\gamma(\tau)).$$

Moreover, since g_i is an upper gradient of u_i, it follows that

$$\|u_i(\gamma(t)) - u_i(\gamma(\tau))\| \le \int_{\gamma|_{[\tau,t]}} g_i \, ds.$$

Combining the above two, we see that for $\tau, t \in [a,b] \setminus N_\gamma$ with $\tau < t$, we have

$$\|u(\gamma(t)) - u(\gamma(\tau))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} g_i\, ds.$$

Thus we have shown that $(g_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u. Now fix $\varepsilon > 0$ and choose $(u_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ such that

$$\liminf_{i\to\infty} \int_X g_i d\mu \le \|Du\|(X) + \varepsilon.$$

Since the previous argument holds for any choice of $(u_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$, we have that $(g_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u, and hence,

$$\|D_{\mathrm{AM}}u\|(X) \le \liminf_{i\to\infty} \int_X g_i d\mu \le \|Du\|(X) + \varepsilon,$$

and then taking $\varepsilon \to 0$ completes the proof. ■

Remark 4.10 Notice that condition (4.3) holds outside a null set N_γ, which depends on the choice of γ, but as seen in the previous proof, whenever $u \in BV(X,V)$, we can choose a null set $N \subset X$ to be independent of the curve, such that $N_\gamma = \gamma^{-1}(N)$. In the following sections we will prove that $BV(X,V) = BV_{AM}(X,V)$. However the construction of the approximation by Newtonian mappings of a BV_{AM}-mapping will yield a sequence of upper gradients different to the original AM-bounding sequence for u, and so in general we cannot assume that *every* AM-bounding sequence of u comes with a universal null set $N \subset X$ as above.

4.2 Poincaré inequalities and Semmes pencil of curves.

We say that a metric measure space (X, d, μ) supports a *V-valued AM–Poincaré inequality* if there are $C > 0$, $\lambda \ge 1$ so that for each $u \in BV_{AM}(X,V)$ and any AM-upper bound $(\rho_i)_i$ of u we have

$$\fint_B \|u - u_B\| d\mu \le C\mathrm{rad}(B) \liminf_{i\to\infty} \fint_{\lambda B} \rho_i d\mu$$

for each ball $B \subset X$. As with the 1-Poincaré inequality, the AM-Poincaré inequality implies the following version, which involves the AM-BV energy.

Lemma 4.11. *If X supports a V-valued AM–Poincaré inequality, then for $u \in BV_{AM}(X,V)$, we have that*

$$\fint_B \|u - u_B\| d\mu \le C\,\mathrm{rad}(B) \frac{\|D_{AM}u\|(\lambda B)}{\mu(\lambda B)}$$

for each ball $B \subset X$.

Proof. Let $\varepsilon > 0$ and let $(\rho_k)_{k\in\mathbb{N}}$ be an AM-upper bound for u in λB such that

$$\liminf_{k\to\infty} \int_{\lambda B} \rho_k d\mu < \|D_{AM}u\|(\lambda B) + \varepsilon.$$

Let $0 < \delta < 1/2$, and let η be a $1/\delta$-Lipschitz function such that $\eta = 1$ on $\lambda(1-\delta)B$ and $\eta = 0$ on $X \setminus \lambda B$, with $0 \le \eta \le 1$ on X. We then have that $(\eta\rho_k + \delta^{-1}\|u\|\,\chi_{\lambda B\setminus\lambda(1-\delta)B})_{k\in\mathbb{N}}$ is an AM-upper bound for ηu in X, see for example Lemma 4.8 above. Thus, by the AM–Poincaré inequality and by the doubling property of μ, we have that

$$\fint_{(1-\delta)B}\fint_{(1-\delta)B} \|u(y)-u(x)\|\,d\mu(y)\,d\mu(x) \le 2\fint_{(1-\delta)B} \|u-u_{(1-\delta)B}\|d\mu$$
$$\le C(1-\delta)\,\mathrm{rad}(B)\liminf_{k\to\infty}\fint_{\lambda(1-\delta)B}\left(\eta\rho_k + \frac{|u|}{\delta}\chi_{\lambda B\setminus\lambda(1-\delta)B}\right)d\mu$$
$$\le C\,\mathrm{rad}(B)\liminf_{k\to\infty}\fint_{\lambda B}\rho_k d\mu$$
$$< C\,\mathrm{rad}(B)\left(\frac{\|D_{AM}u\|(\lambda B)+\varepsilon}{\mu(\lambda B)}\right).$$

Now letting $\delta\to 0$ and taking $\varepsilon\to 0$, we have that

$$\fint_B \|u-u_B\|d\mu \le C\,\mathrm{rad}(B)\frac{\|D_{AM}u\|(\lambda B)}{\mu(\lambda B)}.$$

■

The goal of this section is to see that if X supports a 1-Poincaré inequality, then it supports a V-valued AM–Poincaré inequality. For that we will use the fact that spaces supporting a 1-Poincaré inequality have a Semmes Pencil of curves (see [35, Theorem 3.10]).

Definition 4.12 We say that the metric measure space (X,d,μ) supports a *Semmes pencil* of curves if there exists $C>0$ so that for each $x,y\in X$ with $x\ne y$ there exists a family $\Gamma_{x,y}$ of non-constant compact rectifiable curves equipped with a probability measure $\sigma_{x,y}$ such that each $\gamma\in\Gamma_{x,y}$ connects x to y, $\ell(\gamma)\le Cd(x,y)$, and for each Borel set $A\subset X$ the map $\gamma\mapsto\ell(\gamma\cap A)$ is $\sigma_{x,y}$-measurable with

$$\int_{\Gamma_{x,y}}\ell(\gamma\cap A)d\sigma_{x,y}(\gamma)\le C\int_{A\cap CB_{x,y}}R_{x,y}(z)d\mu(z), \tag{4.5}$$

where $CB_{x,y}:=B(x,Cd(x,y))\cup B(y,Cd(x,y))$ and

$$R_{x,y}(z):=\frac{d(x,z)}{\mu(B(x,d(x,z))}+\frac{d(y,z)}{\mu(B(y,d(y,z))}.$$

Theorem 4.13. *Suppose that μ is doubling, X has a Semmes pencil of curves, and that for μ-a.e. $x\in X$ we have $\liminf_{r\to 0^+}\mu(B(x,r))/r=0$. Then X supports a Vvalued AM–Poincaré inequality for every Banach space target V.*

This result was proven for the real-valued AM-Poincaré inequality in [35, Proposition 3.9], but as we are now dealing with a vector-valued map, we provide the complete proof here, especially since there seems to be a gap in the details of the proof in [35] which we fixed here. In doing so, we saw that we needed the additional condition that $\liminf_{r\to 0^+}\frac{\mu(B(x,r))}{r}=0$ for almost every $x\in X$. This condition fails for example when $X=\mathbb{R}$, but in spaces that are not one-dimensional in nature this is automatically satisfied via the upper mass bound estimates for the doubling measure μ on the connected space X, when the upper mass bound exponent can be taken to be larger than 1. When $X=\mathbb{R}^2$ is equipped with the measure $d\mu(x)=|x|^{-1}d\mathcal{L}^2$, the point $x=0$ fails the condition $\liminf_{r\to 0^+}\frac{\mu(B(x,r))}{r}=0$, but this condition holds at all other $x\in\mathbb{R}^2$.

Proof. Fix a Banach space V and let $u\in BV_{AM}(X,V)$. Let B be a ball in X and $(\rho_k)_{k=1}^\infty$ be a strong bounding sequence for u in $4CB$ with C the constant associated with the Semmes pencil of curves, that is, condition (4.3) holds for all curves in $4CB$. By Theorem 1.22, we know that $\mu(N)=0$, where N is the set of points $x\in X$ for which either $\liminf_{r\to 0^+}\mu(B(x,r))/r>0$ or

$$\limsup_{r\to 0^+} \fint_{B(x,r)} \|u(x)-u(z)\| d\mu(z) > 0.$$

Let $x, y \in X \setminus N$ be two distinct points, and for each $\varepsilon > 0$ consider the sets

$$E_\varepsilon(x) := \{z \in X : \|u(x)-u(z)\| > \varepsilon\}, \quad E_\varepsilon(y) := \{z \in X : \|u(y)-u(z)\| > \varepsilon\}.$$

Now, since x and y are Lebesgue points of u, by Remark 1.23 we know that

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E_\varepsilon(x))}{\mu(B(x,r))} = 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(y,r)\cap E_\varepsilon(y))}{\mu(B(y,r))} = 0.$$

Let $\{r_i\}_i$ be a decreasing sequence of radii so that $r_1 \le \frac{1}{4}d(x,y)$, $r_{i+1} \le \frac{1}{4}r_i$, $\frac{\mu(B(x,r_i))}{r_i} \le 2^{-i}$, $\frac{\mu(B(y,r_i))}{r_i} \le 2^{-i}$, and in addition

$$\frac{\mu(B(x,r_i)\cap E_\varepsilon(x))}{\mu(B(x,r_i))} < 2^{-i} \quad \text{and} \quad \frac{\mu(B(y,r_i)\cap E_\varepsilon(y))}{\mu(B(y,r_i))} < 2^{-i}.$$

For each i let $\mathrm{R}_i(x) := B(x,r_i)\backslash B(x,r_i/2)$ and denote by $\Gamma_i(x)$ the collection of all curves $\gamma \in \Gamma_{x,y}$ such that

$$\mathcal{H}^1\left(\gamma^{-1}(\mathrm{R}_i(x)\backslash E_\varepsilon(x))\right) = 0$$

and define $\Gamma_i(y)$ analogously, replacing x by y. Now use the fact that μ is doubling with constant C_d and $\Gamma_{x,y}$ is a Semmes family of curves to obtain

$$\begin{aligned}
\frac{r_i}{2}\sigma_{x,y}(\Gamma_i(x)) &\le \int_{\Gamma_{x,y}} \ell(\gamma\cap \mathrm{R}_i(x)\cap E_\varepsilon(x))\sigma_{x,y}(\gamma) \\
&\le \int_{CB_{x,y}\cap E_\varepsilon(x)\cap \mathrm{R}_i(x)} R_{x,y}(z)d\mu(z) \\
&\le \int_{CB_{x,y}\cap E_\varepsilon(x)\cap \mathrm{R}_i(x)} \left(\frac{d(x,z)}{\mu(B(x,d(x,z)))} + \frac{d(y,z)}{\mu(B(y,d(y,z)))}\right) d\mu(z) \\
&\le \int_{CB_{x,y}\cap E_\varepsilon(x)\cap \mathrm{R}_i(x)} \left(\frac{r_i}{\mu(B(x,r_i/2))} + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\right) d\mu(z) \\
&\le \frac{r_i}{\mu(B(x,r_i/2))}\mu(E_\varepsilon(x)\cap B(x,r_i)) + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\mu(B(x,r_i)) \\
&\le r_i C_d 2^{-i} + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\mu(B(x,r_i)).
\end{aligned}$$

We note that the above estimate also fills in the gap found in the proof for the real-valued case in [35, page 243]. Set $C_{x,y} := C_d + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}$. Then from the above argument we see that

$$\sigma_{x,y}(\Gamma_i(x)) \le 2C_d\, 2^{-i} + \frac{4C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\frac{\mu(B(x,r_i))}{r_i} \le 2C_{x,y}\, 2^{-i}.$$

Then for each positive integer n we have

$$\sigma_{x,y}\left(\bigcup_{i=n}^{\infty} \Gamma_i(x)\right) \le C_{x,y}\, 2^{1-n}.$$

Define

$$\Gamma(x) := \bigcap_{n\in\mathbb{N}} \bigcup_{i=n}^{\infty} \Gamma_i(x).$$

It follows that $\sigma_{x,y}(\Gamma(x)) = 0$. When $\gamma \in \Gamma_{x,y}\backslash\Gamma(x)$, there exists a positive integer n_0 such that $\gamma \notin \Gamma_i(x)$ for every $i \ge n_0$. It suffices to have $\gamma \notin \Gamma_i(x)$ for some $i \ge n_0$ to get that there exists

$\hat{x} \in \gamma \backslash (E_\varepsilon(x) \cup N_\gamma)$ for any $\mathcal{H}^1$-null set N_γ in γ. Now consider the same argument replacing x by y in order to construct $\Gamma(y)$. Note that $\sigma_{x,y}(\Gamma(x) \cup \Gamma(y)) = 0$.

Recall that condition (4.3) holds for every non-constant, compact, rectifiable curve because $(\rho_i)_i$ is a strong AM-bounding sequence. Therefore, for every curve $\gamma \in \Gamma_{x,y} \backslash (\Gamma(x) \cup \Gamma(y))$ there exists an $\mathcal{H}^1$-null set N_γ such that

$$\|u(\gamma(\tau)) - u(\gamma(t))\| \le \liminf_{k\to\infty} \int_\gamma \rho_k \, ds$$

whenever $\tau, t \in \operatorname{dom}(\gamma) \backslash \gamma^{-1}(N_\gamma)$. Since $\gamma \notin \Gamma(x) \cup \Gamma(y)$, there exist $\hat{x} \in \gamma \backslash (E_\varepsilon(x) \cup N_\gamma)$ and $\hat{y} \in \gamma \backslash (E_\varepsilon(y) \cup N_\gamma)$ such that

$$\|u(x) - u(y)\| \le \|u(\hat{x}) - u(\hat{y})\| + 2\varepsilon \le \liminf_{k\to\infty} \int_\gamma \rho_k ds + 2\varepsilon. \tag{4.6}$$

(Notice that we can actually get not only such $\hat{x}$ and $\hat{y}$ but two sequences of points $x_i \notin E_\varepsilon(x) \cup N_\gamma$ and $y_i \notin E_\varepsilon(y) \cup N_\gamma$ converging to x and y respectively, but we do not need that here). By the Semmes family inequality (4.5), we have

$$\int_{\Gamma_{x,y}} \int_\gamma \rho_k \, ds \, d\sigma_{x,y}(\gamma) \le C \int_{CB_{x,y}} \rho_k(z) R_{x,y}(z) d\mu(z)$$

for each $x, y \in X \setminus N$, $k \in \mathbb{N}$. Therefore, by $\sigma_{x,y}(\Gamma(x) \cup \Gamma(y)) = 0$ and by (4.6), we see that

$$\|u(x) - u(y)\| = \int_{\Gamma_{x,y}} \|u(x) - u(y)\| d\sigma_{x,y}(\gamma) \le C \liminf_{k\to\infty} \int_{CB_{x,y}} \rho_k(z) R_{x,y}(z) d\mu(z) + 2\varepsilon.$$

Recall that $\mu(N) = 0$. Now, for each ball $B \subset X$,

$$\begin{aligned}
\fint_B \|u - u_B\| d\mu &\le \fint_B \fint_B \|u(x) - u(y)\| d\mu(y) d\mu(x) \\
&\le C \fint_B \fint_B \liminf_{k\to\infty} \int_{CB_{x,y}} \rho_k(z) R_{x,y}(z) d\mu(z) d\mu(y) d\mu(x) + 2\varepsilon \\
&\le \frac{C}{\mu(B)^2} \liminf_{k\to\infty} \int_B \int_B \int_{4CB} \rho_k(z) R_{x,y}(z) d\mu(z) d\mu(y) d\mu(x) + 2\varepsilon \\
&= \frac{C}{\mu(B)^2} \liminf_{k\to\infty} \int_{4CB} \rho_k(z) \int_B \int_B R_{x,y}(z) d\mu(y) d\mu(x) d\mu(z) + 2\varepsilon,
\end{aligned} \tag{4.7}$$

where we have used Tonelli's Theorem in the last equality.

Now, to obtain an estimate for the inner two integrals above, we fix $z \in 4CB$. Let $R := \operatorname{rad}(B)$. By the doubling property of μ, we have with $B_i = B(z, 5C\,2^{-i}R)$ for $i = 0, 1, \ldots,$

$$\begin{aligned}
\int_B \int_B \frac{d(x,z)}{\mu(B(x, d(x,z)))} \, d\mu(y) \, d\mu(x) &= \mu(B) \int_B \frac{d(x,z)}{\mu(B(x, d(x,z)))} \, d\mu(x) \\
&\le \mu(B) \int_{B(z,5CR)} \frac{d(x,z)}{\mu(B(x, d(x,z)))} \, d\mu(x) \\
&\lesssim \mu(B) \sum_{i=0}^{\infty} \int_{B_i \setminus B_{i+1}} \frac{2^{-i}R}{\mu(B(z, 5C\,2^{-i}R))} \, d\mu(x) \\
&\lesssim \mu(B) \sum_{i=0}^{\infty} 2^{-i}R \\
&\lesssim \mu(B) \, R,
\end{aligned}$$

where we have implicitly used the fact that $\mu(\{w\}) = 0$ for each $w \in X$. Recall that $\lesssim$ denotes an inequality with a comparison factor, that is, $a \lesssim b$ if there exists a constant C such that $a \le Cb$. The comparison constants above depend solely on the doubling constant of μ and the constant C. A similar estimate also gives

$$\int_B \int_B \frac{d(y,z)}{\mu(B(y,d(y,z)))}\, d\mu(y)\, d\mu(x) \lesssim \mu(B)\, R.$$

Now from (4.7) we see that

$$\fint_B \|u - u_B\|\, d\mu \lesssim \frac{C\ \mathrm{rad}(B)}{\mu(B)} \liminf_{k\to\infty} \int_{4CB} \rho_k(z)\, d\mu(z) \,+\, 2\varepsilon.$$

Taking $\varepsilon \to 0$, we have that

$$\fint_B \|u - u_B\| d\mu \le C\, \mathrm{rad}(B) \liminf_{k\to\infty} \fint_{4CB} \rho_k d\mu.$$

Now, let $(\rho_k)_{k=1}^\infty$ be an AM-upper bound for u. That is, (4.3) holds for AM-a.e. curve. Then by Lemma 4.7, we know that there exists a sequence of non-negative Borel functions $(g_k)_{k=1}^\infty$ with $\limsup_{k\to\infty} \int_X g_k\, d\mu < \infty$ such that for all $\varepsilon > 0$, $(\rho_k + \varepsilon g_k)_{k=1}^\infty$ is a strong bounding sequence for u, that is, (4.3) holds for all curves. Applying the above result, we obtain

$$\begin{aligned}\fint_B \|u - u_B\| d\mu &\le C\, \mathrm{rad}(B) \liminf_{k\to\infty} \fint_{4CB} (\rho_k + \varepsilon g_k) d\mu \\ &\le C\, \mathrm{rad}(B) \left(\liminf_{k\to\infty} \fint_{4CB} \rho_k d\mu + \varepsilon \limsup_{k\to\infty} \fint_{4CB} g_k d\mu \right) \\ &\le C\, \mathrm{rad}(B) \left(\liminf_{k\to\infty} \fint_{4CB} \rho_k d\mu + \frac{\varepsilon}{\mu(4CB)} \limsup_{k\to\infty} \int_X g_k d\mu \right).\end{aligned}$$

Taking $\varepsilon \to 0^+$ yields the desired result. ■

Corollary 4.14. *The following are equivalent whenever (X, d, μ) is a metric measure space with μ a doubling measure satisfying $\limsup_{r\to 0^+} \mu(B(x,r))/r = 0$ for each $x \in X$.*

(i) X supports a V-valued AM–Poincaré inequality for every Banach space V.

(ii) X supports a V-valued AM–Poincaré inequality for some Banach space V.

(iii) X supports an AM–Poincaré inequality for real-valued functions.

(iv) X supports a Semmes pencil of curves.

(v) X supports a 1-Poincaré inequality.

Proof. $(i) \Rightarrow (ii)$ is immediate. If (ii) holds, then in particular X supports a 1-Poincaré inequality for Banach space-valued functions, and hence supports a 1-Poincaré inequality for real-valued functions, as $\mathbb{R}$ can be isometrically embedded into that Banach space, that is, $(ii) \Rightarrow (v)$. From [35, Theorem 3.10] we know that (v), (iv), and (iii) are equivalent. Finally, $(iv) \Rightarrow (i)$ follows from Theorem 4.13. ■

The condition $\liminf_{r\to 0^+} \mu(B(x,r))/r = 0$ for μ-a.e. $x \in X$ precludes us from considering spaces that have components that are one-dimensional in nature, as for example in $\mathbb{R}$ and graphs. It is perhaps possible to handle this situation separately, as was shown for real-valued BV functions in [79].

4.3 The equality $BV_{AM}(X,V) = BV(X,V)$.

We now finish the discussion of comparing both definitions of mappings of bounded variation that we introduced in Section 4.1.

Theorem 4.15. *Let (X,d,μ) be a complete doubling metric measure space supporting a 1-Poincaré inequality, and let V be a Banach space. Suppose also that for μ-a.e. $x \in X$ we have that* $\liminf_{r\to 0^+} \mu(B(x,r))/r = 0$. *Then $BV(X,V) = BV_{AM}(X,V)$, with comparable BV energy seminorms.*

Proof. By Lemma 4.9 we have seen that $BV(X,V) \subset BV_{AM}(X,V)$ with the energy seminorm control $\|D_{\mathrm{AM}}u\|(X) \le \|Du\|(X)$. Thus it only remains to show the reverse inequality. To this end, let $u \in BV_{AM}(X,V)$. We will make use of the version of Poincaré inequality identified in Lemma 4.11 above.

Since the measure μ is doubling, for each $\varepsilon > 0$ there is a countable covering $\{B_i\}_i$ of X by balls of radius ε such that for each $T \ge 1$ there is a constant $C_T > 0$, depending solely on T and the doubling constant associated with μ, such that $\sum_i \chi_{TB_i} \le C_T$ on X. Moreover, for each i there is a non-negative C/ε-Lipschitz function φ_i, with support in $2B_i$, so that $\sum_i \varphi_i = 1$ on X; see for example the discussion at the beginning of [56, Section 9.2]. Such a collection of functions $(\varphi_i)_{i\in\mathbb{N}}$ is called a Lipschitz partition of unity in X. Using this Lipschitz partition of unity, we now construct a locally Lipschitz continuous approximation of u as follows:

$$u_\varepsilon := \sum_i u_{B_i}\,\varphi_i, \qquad \text{where} \qquad u_{B_i} := \fint_{B_i} u\,d\mu.$$

Let $x \in X$ and fix an index j such that $x \in B_j$. Then it follows that whenever $\varphi_i(x) \neq 0$, necessarily $x \in 2B_i$ and so $2B_i \cap B_j$ is non-empty; in this case, $2B_i \subset 5B_j$. Hence, using also the fact that $u(x) = \sum_i u(x)\varphi_i(x)$, we obtain

$$\begin{aligned} u_\varepsilon(x) - u(x) = \sum_i [u_{B_i} - u(x)]\varphi_i(x) &= \sum_{i;2B_i\cap B_j\neq\emptyset} [u_{B_i} - u(x)]\varphi_i(x) \\ &= \sum_{i;2B_i\cap B_j\neq\emptyset} \varphi_i(x) \fint_{B_i} [u - u(x)]\,d\mu. \end{aligned}$$

Thus by the doubling property of μ and the bounded overlap property of the balls $\{5B_j\}_j$, we obtain

$$\begin{aligned} \|u_\varepsilon(x) - u(x)\| &\le \sum_{i;2B_i\cap B_j\neq\emptyset} \fint_{B_i} \|u - u(x)\|\,d\mu \\ &\le \sum_{i;2B_i\cap B_j\neq\emptyset} \fint_{B_i} \|u - u_{5B_j}\|\,d\mu + \|u_{5B_j} - u(x)\| \\ &\lesssim \fint_{5B_j} \|u - u_{5B_j}\|\,d\mu + \|u_{5B_j} - u(x)\|, \end{aligned}$$

and integrating over B_j and summing up over j, and using the fact that $\{B_j\}_j$ is a cover of X, we obtain

$$
\begin{aligned}
\int_X \|u_\varepsilon(x) - u(x)\|\, d\mu(x) &\le \sum_j \int_{B_j} \|u_\varepsilon(x) - u(x)\|\, d\mu(x) \\
&\lesssim \sum_j \int_{B_j} \left(\fint_{5B_j} \|u - u_{5B_j}\|\, d\mu + \|u_{5B_j} - u(x)\| \right) d\mu(x) \\
&\lesssim \sum_j \left(\mu(B_j) \fint_{5B_j} \|u - u_{5B_j}\|\, d\mu + \mu(B_j) \fint_{5B_j} \|u_{5B_j} - u(x)\|\, d\mu(x) \right) \\
&\lesssim \sum_j \mu(B_j) \fint_{5B_j} \|u - u_{5B_j}\|\, d\mu \\
&\lesssim \sum_j \varepsilon \, \|D_{AM} u\|(5\lambda B_j) \\
&\lesssim \varepsilon \, \|D_{AM} u\|(X).
\end{aligned}
$$

In obtaining the penultimate inequality above, we used the AM-Poincaré inequality, and in obtaining the last inequality above, we relied on the bounded overlap of the collection $\{5\lambda B_j\}_j$. Thus $u_\varepsilon \to u$ in $L^1(X,V)$ as $\varepsilon \to 0^+$. As u_ε is locally Lipschitz continuous (as we will show next) on the separable metric space X, it follows that u_ε is measurable, and so the convergence holds in $L^1(X,V)$.

To show that $u_\varepsilon \in N^{1,1}(X,V)$, it suffices to show that u_ε is locally Lipschitz continuous on X with its local Lipschitz constant function $\mathrm{Lip}(u_\varepsilon) \in L^1(X)$. To do so, we fix $x \in X$ and choose an index j such that $x \in B_j$. Considering $y \in B_j$ as well, we see that

$$
u_\varepsilon(y) - u_\varepsilon(x) = \sum_{i;2B_i \cap B_j \neq \emptyset} u_{B_i} \left(\varphi_i(x) - \varphi_i(y) \right) = \sum_{i;2B_i \cap B_j \neq \emptyset} \left(u_{B_i} - u_{5B_j} \right) \left(\varphi_i(x) - \varphi_i(y) \right).
$$

Using the Lipschitz property of the functions φ_i, we now see by the Poincaré inequality that

$$
\begin{aligned}
\|u_\varepsilon(y) - u_\varepsilon(x)\| \lesssim \frac{d(x,y)}{\varepsilon} \sum_{i;2B_i \cap B_j \neq \emptyset} \|u_{B_i} - u_{5B_j}\| &\lesssim \frac{d(x,y)}{\varepsilon} \sum_{i;2B_i \cap B_j \neq \emptyset} \fint_{B_i} \|u - u_{5B_j}\|\, d\mu \\
&\lesssim \frac{d(x,y)}{\varepsilon} \fint_{5B_j} \|u - u_{5B_j}\|\, d\mu \\
&\lesssim d(x,y) \, \frac{\|D_{AM} u\|(5\lambda B_j)}{\mu(B_j)}.
\end{aligned}
$$

It follows that

$$
\mathrm{Lip}(u_\varepsilon)(x) \lesssim \inf_{j\,: x \in B_j} \frac{\|D_{AM} u\|(5\lambda B_j)}{\mu(B_j)}.
$$

Thus u_ε is locally Lipschtz continuous on X by Proposition 1.30 and [36, Corollary 2.4], and it only remains to show that $\mathrm{Lip}(u_\varepsilon) \in L^1(X)$. Using the fact that $\{B_j\}_j$ covers X, we see that

$$
\int_X \mathrm{Lip}(u_\varepsilon)\, d\mu \lesssim \sum_j \int_{B_j} \frac{\|D_{AM} u\|(5\lambda B_j)}{\mu(B_j)}\, d\mu = \sum_j \|D_{AM} u\|(5\lambda B_j) \lesssim \|D_{AM} u\|(X) < \infty,
$$

and this completes the proof that $u_\varepsilon \in N^{1,1}(X)$. As $u_\varepsilon \to u$ in $L^1(X,V)$ and as $\sup_\varepsilon \int_X \mathrm{Lip}(u_\varepsilon)\, d\mu \lesssim \|D_{AM} u\|(X) < \infty$, it follows that $u \in BV(X,V)$ with $\|Du\|(X) \lesssim \|D_{AM} u\|(X)$, completing the proof of Theorem 4.15. Note that the comparison constant in the above inequality depends solely on the doubling constant of the measure μ and the constants from the Poincaré inequality. ■

4.4 Metric space-valued functions of bounded variation

In the previous sections we considered mappings from the metric space X into a Banach space V; in this section we consider the case of mappings into a metric space (Y, d_Y). We will assume here that Y is complete and separable.

We begin with the intrinsic definitions, analogous to the definitions of $BV(X, V)$ and $BV_{AM}(X, V)$ given above.

Lemma 4.16. *Let $u : X \to Y$ be a measurable function. Then the following are equivalent:*

(a) *There is a family Γ_0 of curves in X with $\mathrm{AM}(\Gamma_0) = 0$ and a sequence $(\rho_i)_{i\in\mathbb{N}}$ of non-negative Borel measurable functions on X such that for each non-constant compact rectifiable curve $\gamma : [a, b] \to X$ with $\gamma \notin \Gamma_0$ there is a set $N_\gamma \subset [a, b]$ with $\mathcal{H}^{-1}(N_\gamma) = 0$ such that for each $\tau, t \in [a, b] \setminus N_\gamma$ with $\tau < t$ we have*

$$d_Y(u(\gamma(\tau)), u(\gamma(t))) \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i \, ds,$$

with

$$\sup_i \int_X \rho_i \, d\mu < \infty.$$

(b) *For every Banach space V and isometric embedding $\Phi : Y \to V$ we have that $\Phi \circ u \in BV_{AM}(X, V)$.*

(c) *There is a Banach space V and an isometric embedding $\Phi : Y \to V$ such that $\Phi \circ u \in BV_{AM}(X, V)$.*

In any (and hence all) of the cases above, we have that

$$\|D_{AM}\Phi \circ u\|(X) = \inf_{(\rho_i)_{i\in\mathbb{N}}} \liminf_{i\to\infty} \int_X \rho_i \, d\mu$$

where the infimum is over all sequences $(\rho_i)_{i\in\mathbb{N}}$ satisfying (a) above.

Proof. If V is a Banach space and Φ is an isometric embedding of Y into V, then for each $x, z \in X$ we have that $d_Y(u(x), u(z)) = \|\Phi(u(x)) - \Phi(u(z))\|$, and so we know that (a) implies (b) and that (b) implies (c). Indeed, every complete separable metric space can be isometrically embedded in the Banach space ℓ^∞ by the Kuratowski embedding Theorem.

Thus it only remains to show that (c) implies (a). To this end, suppose that V is a Banach space, Φ an isometric embedding of Y into V, and that $\Phi \circ u \in BV_{AM}(X, V)$. Then there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ that is an AM-bounding sequence for $\Phi \circ u$, and a family Γ_0 with $\mathrm{AM}(\Gamma_0) = 0$ such that whenever $\gamma : [a, b] \to X$ with $\gamma \notin \Gamma_0$ there is a set $N_\gamma \subset [a, b]$ with $\mathcal{H}^{-1}(N_\gamma) = 0$ such that for each $\tau, t \in [a, b] \setminus N_\gamma$ with $\tau < t$ we have

$$\|\Phi \circ u(\gamma(\tau)) - \Phi \circ u(\gamma(t))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i \, ds,$$

with

$$\sup_i \int_X \rho_i \, d\mu < \infty.$$

As Φ is an isometric embedding of Y into V, it follows that $d_Y(u(\gamma(\tau)), u(\gamma(t))) = \|\Phi \circ u(\gamma(\tau)) - \Phi \circ u(\gamma(t))\|$. The condition (a) follows.

The above argument shows that the sequence $(\rho_i)_{i\in\mathbb{N}}$ satisfies (a) if and only if it is an AM-bounding sequence for $\Phi \circ u$. The final claim of the above lemma now follows. ∎

Definition 4.17 We say that a map $u : X \to Y$ is in $BV_{AM}(X,Y)$ if $u \in L^1(X,Y)$ and there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ satisfying any of the hypothesis of Lemma 4.16 (a).

Lemma 4.18. *Let $\Phi : Y \to V$ be an isometric embedding of the metric space Y into a Banach space V. Suppose that*

(a) *$u \in BV_{AM}(X,V)$, and*

(b) *there is a set $N \subset X$ with $\mu(N) = 0$ such that for each $x \in X \setminus N$ we have that $u(x) \in \Phi(Y)$.*

Then $u \circ \Phi^{-1} \in BV_{AM}(X,Y)$. Here Φ^{-1} stands in for the inverse map of the bijective map $\Phi : Y \to \Phi(Y)$.

Proof. Since any modification of u on a set of μ-measure zero results in the same equivalence class of u in $BV_{AM}(X,V)$ (see Lemma 4.6), we can modify u on N by setting $u(x)$ to be some fixed point in $\Phi(Y)$ if $x \in N$. The conclusion now follows from the previous lemma. ∎

Unlike $BV_{AM}(X,Y)$, the situation for $BV(X,Y)$ is more complicated.

Definition 4.19 We say that a map $u : X \to Y$ is in $BV(X,Y)$ if there is a sequence $(u_k)_{k\in\mathbb{N}}$ from $N^{1,1}(X,Y)$ such that $u_k \to u$ in $L^1(X,Y)$ and $\sup_{k\in\mathbb{N}} \int_X g_{u_k}\, d\mu < \infty$.

We have seen that $BV_{AM}(X,V) = BV(X,V)$ whenever X is doubling and supports a 1-Poincaré inequality and V is a Banach space (see Theorem 4.15), and as in the proof of Lemma 4.9, we can see that $BV(X,Y) \subset BV_{AM}(X,Y)$. However, $BV(X,Y)$ is in general a strictly smaller subset of $BV_{AM}(X,Y)$. This supports the choice of $BV_{AM}(X,Y)$ as the space of mappings of bounded variation in [76, 77].

Example 4.20 *Consider $X = [-1,1]$ and $Y = \{0,1\}$. Let $u := \chi_{[0,1]}$ and $\rho_i := i\chi_{[-1/i,0]}$. Then for each $x,y \in [-1,1]$, if $x,y < 0$ or $x,y \geq 0$ then $|u(x) - u(y)| = 0$ so it is immediate that $(\rho_i)_i$ satisfies the upper bound inequality. If $x < 0$ and $y \geq 0$ then*

$$\liminf_{i\to\infty} \int_x^y i\chi_{[-1/i,0]} d\mathcal{L}^1 = 1 = |u(x) - u(y)|.$$

Thus the sequence $(\rho_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u (and indeed, it is a strong bounding sequence for u). Therefore we know that $u \in BV_{AM}(X,Y)$. However Newtonian mappings are absolutely continuous on $[-1,1]$, and so, since $Y = \{0,1\}$, they must be constant; hence it is not possible to approximate (in L^1 norm) u by Newtonian mappings, proving that $u \notin BV(X,Y)$.

While the above example shows how topological obstructions can prevent approximations by $N^{1,1}$-maps, the next example provides a more analytical obstruction.

Example 4.21 *Let $X = [-1,1]$ and $Y = (\{0\} \times [0,1]) \cup ([0,1] \times \{1\}) \cup (\{1\} \times [0,1])$ be both equipped with the Euclidean metric, and X be also equipped with the Lebesgue measure $\mathcal{L}^1$. Let $u \in BV_{AM}(X,Y)$ be given by $u(x) = (0,0)$ when $-1 \leq x \leq 0$ and $u(x) = (1,0)$ when $0 < x \leq 1$. If $(u_k)_k$ is a sequence of functions in $N^{1,1}(X,Y)$ such that $u_k \to u$ in $L^1(X)$, then for sufficiently large k we know that there are points $x_k, y_k \in X$ with $|x_k + 1| < \frac{1}{10}$ and $|y_k - 1| < \frac{1}{10}$ such that*

$|u(x_k)-(0,0)| < \frac{1}{10}$ and $|u(y_k)-(1,0)| < \frac{1}{10}$. On the other hand, by the absolute continuity on $[-1,1]$ of functions in $N^{1,1}(X,Y)$, we have that $\int_X g_{u_k}\, d\mathcal{L}^1 = length(u_k) \geq 1 + 2\times\frac{9}{10} = \frac{14}{5}$. In inferring the above, we used the fact that u_k is also a path in Y. On the other hand, $\|D_{AM}u\|(X) = 1$, and so it is not possible to have $\|D_{AM}u\|(X) = \inf_{(u_k)\subset N^{1,1}(X,Y)} \liminf_{k\to\infty}\int_X g_{u_k}\, d\mathcal{L}^1$.

The above example does not preclude an L^1-approximation of the AM–BV function by a sequence of Newton-Sobolev functions. The next example gives a metric obstruction to the existence of even such an approximation.

Example 4.22 *Let $X = [-1,1]$ and $Y = (\{0\}\times[-1,1]) \cup \{(x,\sin(1/x)) : 0 < x \leq 1\}$ be both equipped with the Euclidean metrics, and X also be equipped with the 1-dimensional Lebesgue measure. Let $u : X \to Y$ be given by $u(x) = (0,0)$ if $-1 \leq x \leq 0$, and $u(x) = (1,\sin(1))$ when $0 < x \leq 1$. Then $\|D_{AM}u\|(X) = \sqrt{1+\sin^2(1)}$, but due to the absolute continuity of functions in $N^{1,1}(X,Y)$ and the lack of paths in Y connecting $u(-1)$ to $u(1)$, there can be no sequence of functions in $N^{1,1}(X,Y)$ that gives an L^1-approximation of u.*

By the ACL (absolute continuity on lines) property of functions in $N^{1,1}(X,Y)$, the above examples have higher dimensional analogs, but we will not go into details here.

Remark 4.23 While $BV(X,Y)$ need not equal $BV_{AM}(X,Y)$ in general, we do have a relationship between the two notions. Thanks to the Kuratowski embedding Theorem, we can isometrically embed any separable metric space (Y,d_Y) into the Banach space ℓ^∞. Thanks to Theorem 4.15 and Lemma 4.18, we know that with V any Banach space and $\Phi : Y \to V$ an isometric embedding, whenever Y is complete, we have $BV_{AM}(X,Y)$ is the same as the class

$$\left\{ u \in BV(X,V) \,:\, \mu(\{x \in X \,:\, u(x) \notin \Phi(Y)\}) = 0 \right\}.$$

4.5 Approximate continuity and jump sets.

Throughout this section, in addition to the measure μ being doubling and supporting a 1-Poincaré inequality, we will also assume that X is complete. In this section we consider the regularity properties of functions in the class $BV_{AM}(X,Y)$, with (Y,d_Y) a separable and proper metric space (that is, closed and bounded subsets of Y are compact). As seen from the examples in Section 4.4, when Y is not a Banach space, it is more natural to consider the class $BV_{AM}(X,Y)$ rather than $BV(X,Y)$. More specifically we will study the set of points where a mapping of bounded variation is approximately continuous and its complement (called the jump set) and that $\mathcal{H}^{-1}$-almost every point in the jump set cannot take an arbitrarily large ammount of values (in an infinitesimal sense) under the mapping. The precise statement that we seek to proof in this section is as follows.

Theorem 4.24. *Let (X,d,μ) be a complete doubling metric measure space supporting a 1-Poincaré inequality, and let (Y,d_Y) be a proper metric space. Then for each $u \in BV_{AM}(X:Y)$ there is a set $\mathcal{J}(u) \subset X$ such that $\mathcal{J}(u)$ is σ-finite with respect to the co-dimension 1 Hausdorff measure $\mathcal{H}^{-1}$ on X and there exists a set $N \subset \mathcal{J}(u)$ with $\mathcal{H}^{-1}(N) = 0$ such that the following hold:*

(a) Every point in $X \setminus \mathcal{J}(u)$ is a point of approximate continuity of u.

(b) For each $x_0 \in \mathcal{J}(u)\setminus N$ there are at least two, and at most k_0, number of points $y_1, y_2, \cdots, y_k \in Y$ such that for each $\varepsilon > 0$ and $i = 1, 2, \cdots, k$ we have

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} \geq \gamma,$$

and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus \bigcup_{i=1}^k u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} = 0.$$

In the above, both k_0 and γ are constants that depend solely on the doubling and Poincaré constants of the space X, and in particular are independent of Y, u and ε.

4.5.1 Definition of Jump sets.

We recall that almost every point in X is a point of approximate continuity of $u \in L^1(X,Y)$ (see Remark 1.23). However, for functions $u \in BV_{AM}(X,Y)$ we would like a better control. We may broaden the definition of approximate continuity given in Definition 1.24 by saying that u is *approximately continuous* at x if there is some $y_0 \in Y$ such that for every $\varepsilon > 0$ we have

$$\limsup_{r\to 0^+} \frac{\mu(\{y\in B(x,r) \,:\, d_Y(y_0,u(y))\geq \varepsilon\})}{\mu(B(x,r))} = 0.$$

Since μ-almost every point in X is a point of approximate continuity of u, if $x \in X$ such that there is some y_0 satisfying the above density condition, then we can re-define u at x by setting $u(x) := y_0$; such a modification is a better representative of u; moreover, such a modification needs to be done only on a set of μ-measure zero, thanks to the Lebesgue differentiation Theorem (see Theorem 1.22 and Remark 1.23).

Note that if x is a point of approximate continuity in the above sense with y_0 the corresponding value, and if Y is bounded, then for each $\varepsilon > 0$ we have

$$\limsup_{r\to 0^+} \fint_{B(x,r)} d_Y(u(z),y_0)\, d\mu(z) \leq \varepsilon + \operatorname{diam}(Y) \limsup_{r\to 0^+} \frac{\mu(\{y\in B(x,r) \,:\, d_Y(y_0,u(y))\geq \varepsilon\})}{\mu(B(x,r))} = \varepsilon,$$

and so necessarily x is a Lebesgue point of u as well.

The notions of approximate continuity and jump values as considered in [2, page 294] are somewhat different than ours in that it is required there that for every continuous function $g : Y \to \mathbb{R}$, the map $g\circ u$ is approximately continuous at x with the approximate limit being $g(y_0)$. Such an indirect definition seems to be not needed here, and we take the definition of approximate continuity proposed by Ambrosio in [3, Definition 1.1].

Let us consider points $x \in X$ that are not points of approximate continuity in the above, more expanded, sense. We would like to call such points x jump points of u. For this reason, we define the *jump set* $\mathcal{J}(u)$ of u to be the collection of all points in X that are not points of approximate continuity, in the above broader sense, of u. We would like to know that for $x \in \mathcal{J}(u)$ there are finitely many points $y_1, y_2, \cdots, y_k$, with $k \leq k_0$ where k_0 is independent of u and x, that act as jump values of u at x. This may not be possible at all $x \in \mathcal{J}(u)$, but we would like to ensure that this is possible for $\mathcal{H}^{-1}$-a.e. $x \in \mathcal{J}(u)$.

We first make the simplifying reduction that Y is a compact metric space and later on we will discuss the case of a non compact proper Y. Now, if $x \in \mathcal{J}(u)$, then for every $y \in Y$ there is some $\varepsilon_y > 0$ such that

$$\limsup_{r\to 0^+} \frac{\mu(\{z\in B(x,r) \,:\, d_Y(y,u(z))\geq \varepsilon_y\})}{\mu(B(x,r))} > 0.$$

For each $y \in Y$ and $\varepsilon > 0$ set

$$F(y,\varepsilon) := \{z \in X : d_Y(u(z), y) \geq \varepsilon\}. \tag{4.8}$$

Note that then for every $0 < \varepsilon \leq \varepsilon_y$, we have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap F(y,\varepsilon))}{\mu(B(x,r))} > 0.$$

We fix $\varepsilon > 0$, and cover the compact set Y by finitely many balls $B(y_i, \varepsilon)$, $i = 1, \cdots, N_\varepsilon$. Note that as $B(x,r) = \bigcup_{i=1}^{N_\varepsilon} u^{-1}(B(y_i,\varepsilon))$, necessarily there is some $y_1 \in Y$, relabeled if necessary, such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_1,\varepsilon)))}{\mu(B(x,r))} > 0.$$

If we also have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(Y\setminus B(y_1,3\varepsilon)))}{\mu(B(x,r))} > 0, \tag{4.9}$$

then we have two sets $E_1, E_2 \subset X$ with $E_1 = u^{-1}(B(y_1,\varepsilon))$ and $E_2 = u^{-1}(B(w_1,\varepsilon))$, $d_Y(y_1,w_1) \geq 3\varepsilon$, such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} > 0, \text{ and } \limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E_2)}{\mu(B(x,r))} > 0. \tag{4.10}$$

If (4.9) fails, then we know that

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\setminus u^{-1}(B(y_1,3\varepsilon)))}{\mu(B(x,r))} = 0,$$

and so

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(\overline{B}(y_1,3\varepsilon)))}{\mu(B(x,r))} = 1.$$

In this case, we can cover the compact set $\overline{B}(y_1, 3\varepsilon)$ by balls of radii $\varepsilon/6^2$, and obtain a point $y_2 \in \overline{B}(y_1, 3\varepsilon)$ so that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_2,\varepsilon/6^2)))}{\mu(B(x,r))} > 0.$$

If we know that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus u^{-1}(B(y_2,3\varepsilon/6^2)))}{\mu(B(x,r))} > 0,$$

then we can set $E_1 = u^{-1}(B(y_2,\varepsilon/6^2))$ and $E_2 = u^{-1}(B(w_2,\varepsilon/6^2))$, with $6\varepsilon \geq d_Y(y_2,w_2) \geq 3\varepsilon/6^2$ such that (4.10) holds. If the above analog of (4.9) fails, then we know that

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(\overline{B}(y_2,3\varepsilon/6^2)))}{\mu(B(x,r))} = 1,$$

and the process inductively continues. Thus we obtain a sequence of points $y_1, y_2, \cdots$ with $d_Y(y_i, y_{i+1}) \leq 3\varepsilon/6^i$ and so that

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(\overline{B}(y_i,3\varepsilon/6^i)))}{\mu(B(x,r))} = 1.$$

If this process continues ad infinitum, then we obtain a Cauchy sequence $\{y_i\}_i$ in Y which, by the completeness of Y, must converge to a point y_∞ for which we would have that for each $\tau > 0$,

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_\infty,\tau)))}{\mu(B(x,r))} = 1,$$

and so re-setting $u(x) = y_\infty$ would show that $x \notin \mathcal{J}(u)$. Therefore the inductive process above must terminate at some index k, and so we know that there is some $y_k, w_k \in Y$ such that $d_Y(y_k, w_k) \geq 3\varepsilon/6^k$, and with $E_1 = u^{-1}(B(y_k, \varepsilon/6^k))$ and $E_2 = u^{-1}(B(w_k, \varepsilon/6^k))$, condition (4.10) holds. Note that $\operatorname{dist}(B(y_k, \varepsilon/6^k), B(w_k, \varepsilon/6^k)) \geq \varepsilon/6^k > 0$. Hence, the following is an equivalent definition of a jump point of $u \in BV_{AM}(X, Y)$. We classify the following as the definition of jump points as this is the definition most useful in the proof of Theorem 4.24.

Definition 4.25 Let $u : X \to Y$. We say that $x_0 \in X$ is a *jump point* of u if there exist sets $E_1, E_2 \subset X$ such that

$$\limsup_{r \to 0} \frac{\mu(B(x_0, r) \cap E_i)}{\mu(B(x_0, r))} > 0 \quad \text{for} \quad i = 1, 2, \tag{4.11}$$

and there exist balls $B_1, B_2 \subset Y$ with $\operatorname{dist}(B_1, B_2) \geq \operatorname{rad}(B_1)$ such that $u(E_i) \subset B_i$ for $i = 1, 2$.

From the discussion preceding the above definition, we know that $x \in \mathcal{J}(u)$ if and only if x is a jump point in the sense of Definition 4.25 above.

Since $Y \ni w \mapsto d_Y(w, y_0)$ is a 1-Lipschitz map for each $y_0 \in Y$, the next lemma follows by an easy verification of (4.3) with the aid of triangle inequality.

Lemma 4.26. *Let $u \in BV_{AM}(X, Y)$, and $y_0 \in Y$. Then $v : X \to \mathbb{R}$ given by $v(x) = d_Y(u(x), y_0)$ belongs to the class $BV_{AM}(X) = BV(X)$.*

As a corollary to the above lemma, the co-area formula from Lemma 4.2 yields the following, from which we obtain σ-finiteness of the jump set with respect to $\mathcal{H}^{-1}$.

Corollary 4.27. *Let $u \in BV_{AM}(X, Y)$. For each $y \in Y$ and $\rho > 0$ set $E(y, \rho) := u^{-1}(B(y, \rho))$. Then for each $y \in Y$ there is a set $D_y \subset [0, \infty)$ with $\mathcal{L}^1(D_y) = 0$ such that for each $\rho \in (0, \infty) \setminus D_y$ we have that $E(y, \rho)$ is of finite perimeter in X.*

The next result proves that the set $\mathcal{J}(u)$, as constructed above, satisfies its σ-finiteness with respect to the co-dimensional measure $\mathcal{H}^{-1}$ claimed in the statement of Theorem 4.24.

Corollary 4.28. *For each $u \in BV_{AM}(X, Y)$, the jump set $\mathcal{J}(u)$ is σ-finite with respect to the co-dimension 1 Hausdorff measure $\mathcal{H}^{-1}$ on X.*

Proof. As Y is separable, there exists a countable dense subset Y_0 of Y, and for each $y \in Y_0$, let

$$\mathcal{R}(y) := \{\rho > 0 : P(E(y, \rho), X) < \infty\}.$$

By Corollary 4.27, we have that $\mathcal{L}^1((0, \infty) \setminus \mathcal{R}(y)) = 0$, and so there exists a countable subset $\mathcal{R}_0(y) \subset \mathcal{R}(y)$ dense in $(0, \infty)$. By Lemma 4.3, it follows that $\mathcal{H}^{-1}(\partial_*(E(y, \rho)) < \infty$ for each $\rho \in \mathcal{R}(y)$, where $\partial_* E(y, \rho)$ is the measure-theoretic boundary of $E(y, \rho) = u^{-1}(B(y, \rho))$, as given by (4.1).

Now, for each $x \in \mathcal{J}(u)$, we have by Definition 4.25 and the density of Y_0 in Y, that there exist $y_1, y_2 \in Y_0$, $\rho_1 \in \mathcal{R}_0(y_1)$, and $\rho_2 \in \mathcal{R}_0(y_2)$ such that $u(E_1) \subset B_1 \subset B(y_1, \rho_1)$, $u(E_2) \subset B_2 \subset B(y_2, \rho_2)$, and $\operatorname{dist}(B(y_1, \rho_1), B(y_2, \rho_2)) > 0$. Here E_1, E_2, B_1, and B_2 are as given in Definition 4.25. Then, we have that $x \in \partial_* E(y_1, \rho_1)$, and so it follows that

$$\mathcal{J}(u) \subset \bigcup_{y \in Y_0} \bigcup_{\rho \in \mathcal{R}_0(y)} \partial_* E(y, \rho).$$

■

Jump points for real-valued functions.

The notion of real-valued functions of bounded variation in metric measure spaces was first proposed by Miranda Jr. in [84], and its fine properties were studied in [8, 75]; an elegant account of real-valued functions of bounded variation and their fine properties in the Euclidean setting can be found in [41, Definition 5.9 and Theorem 5.17].

The fine properties of real-valued BV functions studied there include approximate continuity and jump points. The notion of approximate continuity, as proposed in Section 4.5, is the same as that found in real analysis texts and in [4, 8]. We now verify that the notion of jump points, as given in Section 4.5, agrees with the corresponding notion as given in [6].

As considered in [4], given $u : X \to \mathbb{R}$ $x_0 \in X$ is a jump point of u if $u^\wedge(x_0) < u^\vee(x_0)$, where

$$u^\wedge(x_0) = \sup\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \le t\})}{\mu(B(x_0,r))} = 0\right\},$$
$$u^\vee(x_0) = \inf\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \ge t\})}{\mu(B(x_0,r))} = 0\right\}.$$

See the discussion in the beginning of this chapter for further detail on alternate nomenclature used in literature on Euclidean BV functions.

Lemma 4.29. *Let $u : X \to \mathbb{R}$. Then $x_0 \in \mathcal{J}(u)$ if and only if $u^\wedge(x_0) < u^\vee(x_0)$.*

Proof. Suppose first that $u^\wedge(x_0) = u^\vee(x_0) =: \beta$. It then follows that for each $\varepsilon > 0$,

$$\lim_{r\to 0^+} \frac{\mu(B(x_0,r)\cap\{|u-\beta| \ge \varepsilon\})}{\mu(B(x_0,r))} = 0,$$

and so by Definition 1.24 the point x_0 is a point of approximate continuity of u, that is, $x_0 \notin \mathcal{J}(u)$.

For the converse suppose that $-\infty < u^\wedge(x_0) < u^\vee(x_0) < \infty$, and choose t_1^-, t_1^+, t_2^-, and t_2^+ such that $t_1^- < u^\wedge(x_0) < t_1^+ < t_2^- < u^\vee(x_0) < t_2^+$. Then

$$\lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \le t_1^-\})}{\mu(B(x_0,r))} = 0 \quad\text{and}\quad \lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \ge t_2^+\})}{\mu(B(x_0,r))} = 0.$$

Since $u^\wedge(x_0) < t_1^+$ and $u^\vee(x_0) > t_2^-$ we also have

$$\limsup_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \le t_1^+\})}{\mu(B(x_0,r))} > 0 \quad\text{and}\quad \limsup_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \ge t_2^-\})}{\mu(B(x_0,r))} > 0,$$

and so if we set $E_i = \{t_i^- \le u \le t_i^+\}$ then

$$\lim_{r\to 0} \frac{\mu(B(x_0,r)\cap E_i)}{\mu(B(x_0,r))} > 0 \quad\text{for}\quad i = 1, 2,$$

with $u(E_i) \subset B_i := (t_i^-, t_i^+)$. Since $t_1^+ < t_2^-$, $\operatorname{dist}(B_1, B_2) > 0$. Thus x_0 is not a point of approximate continuity of u, that is, $x_0 \in \mathcal{J}(u)$. If $u^\wedge(x_0) = -\infty$ or if $u^\vee(x_0) = \infty$, then we replace u with $\chi_{K_n} \cdot u$ and proceed as above, with $K_n = \{|u| \le n\}$. ■

4.5.2 Jump values and its uniform finiteness $\mathcal{H}^{-1}$-a.e.

As pointed out in [78], a BV function can take on infinitely many values near the jump point, but such a bad behavior cannot happen on a large set. To demonstrate a similar behavior of metric space-valued BV functions, we first consider what it means for a point in the target metric space to be a jump value near a jump point of the BV function.

Definition 4.30 With $u \in BV_{AM}(X,Y)$ and $x \in \mathcal{J}(u)$, we say that a point $y_0 \in Y$ is a *jump value* of u at x if for every $\varepsilon > 0$ we have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_0,\varepsilon)))}{\mu(B(x,r))} > 0.$$

The next proposition verifies the claim (b) of Theorem 4.24 whenever Y is compact. We will deal with the case of Y being a proper but non-compact metric space at the end of this section.

Proposition 4.31. *There exists $k_0 \in \mathbb{N}$ so that for every $u \in BV_{AM}(X,Y)$ there is a set $N \subset X$ with $\mathcal{H}^{-1}(N) = 0$ such that for each $x \in \mathcal{J}(u)\setminus N$ there are at least two and at most k_0 jump values $y_1, \cdots, y_k \in Y$ of u at x. Furthermore, for every $\varepsilon > 0$ and $i = 1, 2, \cdots, k$, we have*

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} \geq \gamma,$$

and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus \bigcup_{i=1}^{k} u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} = 0.$$

Here k_0 and γ are constants depending only on the doubling constant and the Poincaré constants of X, and in particular are independent of Y, u, and ε.

Proof. Since Y is compact, it is separable. As above, let Y_0 be a countable dense subset of Y, and for each $y \in Y_0$ let

$$\mathcal{R}(y) = \{\rho > 0 : P(E(y,\rho), X) < \infty\}.$$

Note from Corollary 4.27 that $\mathcal{L}^1((0,\infty)\setminus \mathcal{R}(y)) = 0$. Let $\mathcal{R}_0(y)$ be a countable dense subset of $\mathcal{R}(y)$. For each $y \in Y_0$ and $\rho \in \mathcal{R}(y)$ we know that $\mathcal{H}^{-1}(\partial_* E(y,\rho)\setminus \Sigma_\gamma(E(y,\rho))) = 0$, where $\partial_* E(y,\rho)$ is the measure-theoretic boundary of $E(y,\rho)$, as given by (4.1), and $\Sigma_\gamma(E(y,\rho))$ is the reduced boundary of $E(y,\rho)$, as given by (4.2). Here $0 < \gamma \leq \frac{1}{2}$ is a number that depends solely on the constants associated with the doubling property of μ and the Poincaré inequality (see for example [4, Theorem 5.3]). Let

$$N := \bigcup_{y\in Y_0} \bigcup_{\rho\in\mathcal{R}_0(y)} \partial_*(E(y,\rho))\setminus \Sigma_\gamma(E(y,\rho)).$$

Then, by the countability of the collections, we have that $\mathcal{H}^{-1}(N) = 0$. We now fix $x \in \mathcal{J}(u)\setminus N$. We proceed in an inductive fashion hinted at in the discussion preceding Definition 4.25.

Let E_1 be one of the two sets identified in Definition 4.25, associated with the jump point x, and let B_1 be the corresponding ball in Y such that $u(E_1) \subset B_1$ and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} > 0 \text{ and } \limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus E_1)}{\mu(B(x,r))} > 0.$$

Since the distance between the balls B_1 and B_2 in Definition 4.25 is positive, we are free to choose the center y_1 of B_1 to be in Y_0 and then, by increasing the radius slightly if necessary, have the radius of B_1 be in the set $\mathcal{R}_0(y_1)$. A similar modification can be made to the ball B_2. We can now replace E_1 with $u^{-1}(B_1)$ and E_2 with $u^{-1}(B_2)$; hence from now on, $E_1 = u^{-1}(B_1)$. Thus we have that $x \in \partial_* E_1$ because of the existence of B_2, and as $x \notin N$, we see that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} \geq \gamma \text{ and } \liminf_{r\to 0^+} \frac{\mu(B(x,r)\setminus E_1)}{\mu(B(x,r))} \geq \gamma.$$

Let $\rho > 0$ be the radius of the ball B_1, and note that the distance between B_1 and B_2 is at least ρ. Covering the closed ball $\overline{B_1}$ by balls $B(y_{2,1}, \rho/12), \cdots, B(y_{2,N_2}, \rho/12)$, with $y_{2,i} \in Y_0$ for $i = 1, \cdots, N_2$, and $B(y_{2,i}, \rho/12)$ intersects $\overline{B_1}$. By doing so, we can find a point $y_2 \in \frac{13}{12}B_1$ such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_2,\rho/12)))}{\mu(B(x,r))} > 0.$$

We can then find $\rho_2 \in \mathcal{R}_0(y_2)$ such that $\rho/12 \leq \rho_2 < \rho/11$, so that with $E_{2,1} = u^{-1}(B(y_2,\rho_2))$, we have by the fact that $x \notin N$,

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E_{2,1})}{\mu(B(x,r))} \geq \gamma.$$

In the above, we have used the fact that B_2 does not intersect $\overline{B_1} \cup \overline{B}(y_2,\rho_2)$ to know that $x \in \partial_* E_{2,1}$. We proceed inductively as follows. Once $y_i \in Y_0$ and $\rho_i \in \mathcal{R}_0(y_i)$, $i = 1, \cdots, k$, are selected such that $d_Y(y_i, y_{i+1}) < 2\rho_i$ and $\rho_{i+1} < \rho_i/11$, and with $E_{i,1} = u^{-1}(B(y_i,\rho_i))$, we have

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E_{i,1})}{\mu(B(x,r))} \geq \gamma$$

for $i = 2, \cdots, k$. We cover $\overline{B}(y_k,\rho_k)$ by balls $B(y_{k+1,1},\rho_k/12), \cdots, B(y_{k+1,N_{k+1}},\rho_k/12)$, each intersecting $\overline{B}(y_k,\rho_k)$ with $y_{k+1,i} \in Y_0$ for $i = 1, \cdots, N_{k+1}$, and hence find $y_{k+1} \in Y_0$ so that $d(y_k, y_{k+1}) < 2\rho_k$, and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_{k+1},\rho_k/12)))}{\mu(B(x,r))} > 0.$$

We then find $\rho_{k+1} \in \mathcal{R}_0(y_{k+1})$ such that $\rho_k/12 \leq \rho_{k+1} < \rho_k/11$, and hence, with $E_{k+1,1} = u^{-1}(B(y_{k+1},\rho_{k+1}))$, we have that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E_{k+1,1})}{\mu(B(x,r))} \geq \gamma.$$

Note that as for each j we have $\rho_j < (11)^{-j}\rho$, and as $\operatorname{dist}(B_1,B_2) > \rho$, necessarily $B(y_k,\rho_k)\cap B_2 = \varnothing$. Moreover, as $d_Y(y_k,y_{k+1}) < (11)^{k-1}\rho$, we also have that the sequence $\{y_j\}_j$ is a Cauchy sequence in Y, and as Y is complete, converges to some $y_\infty \in Y$. We now show that y_∞ is a jump value of u at x. Let $\varepsilon > 0$; then there is some positive integer k so that $B(y_k,\rho_k) \subset B(y_\infty,\varepsilon)$. It follows that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_\infty,\varepsilon)))}{\mu(B(x,r))} \geq \liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E_{k,1})}{\mu(B(x,r))} \geq \gamma > 0.$$

Thus u has at least one jump value at x, and moreover, we also have that for each $\varepsilon > 0$,

$$\liminf_{r\to 0} \frac{\mu(B(x,r)\cap u^{-1}(B(y_\infty,\varepsilon)))}{\mu(B(x,r))} \geq \gamma.$$

Note also, from switching the roles of the sets E_1 and E_2, we obtain a second jump value of u at x.

Now, if $z \in Y$ is any other jump value of u at x, then for each $\varepsilon > 0$ with $\varepsilon < d_Y(z, y_\infty)/20$, we can find $z_1 \in B(z,\varepsilon/2)\cap Y_0$ and $0 < \tau < \varepsilon/4$ such that $\tau \in \mathcal{R}_0(z_1)$ and note that $u^{-1}(B(z_1,\tau)) \subset u^{-1}(B(z,\varepsilon))$ with

$$\liminf_{r\to 0} \frac{\mu(B(x,r)\cap u^{-1}(B(z,\varepsilon)))}{\mu(B(x,r))} \geq \liminf_{r\to 0} \frac{\mu(B(x,r)\cap u^{-1}(B(z_1,\tau)))}{\mu(B(x,r))} \geq \gamma; \tag{4.12}$$

that is, *each* jump value of u at x satisfies the above lower density at least γ at x_0. As $\gamma > 0$, there are at most $k_0 := \lceil 1/\gamma \rceil$ number of such jump values for x.

Now suppose that we have identified all the jump values $y_1, \cdots, y_k$ of u at x, with $2 \le k \le k_0$. We claim that for each $\tau > 0$, the set $E(\tau) := \bigcup_{j=1}^k E(y_j, \tau_j)$ has density 1 at x, that is,

$$\liminf_{r \to 0^+} \frac{\mu(B(x,r) \cap E(\tau))}{\mu(B(x,r))} = 1.$$

Here $\tau_i \in \mathcal{R}_0(y_i)$ is such that $\frac{2}{3}\tau < \tau_i \le \tau$. It suffices to know this for all sufficiently small $\tau > 0$, and so we consider $\tau > 0$ for which the closed balls $\overline{B}(y_i, \tau_i)$ are pairwise disjoint. If the claim does not hold, then we would have that

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \setminus E(\tau))}{\mu(B(x,r))} > 0.$$

Then let $0 < \varepsilon < \tau/20$ such that for each i, j with $i \ne j$ we have that $\operatorname{dist}(\overline{B}(y_i, \tau_i), \overline{B}(y_j, \tau_j)) > 20\varepsilon$. Now setting $K(\tau) = X \setminus E(\tau)$, we have from (4.12) that

$$\liminf_{r \to 0^+} \frac{\mu(B(x,r) \setminus K(\tau))}{\mu(B(x,r))} \ge \gamma > 0 \text{ and simultaneously, } \limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap K(\tau))}{\mu(B(x,r))} > 0.$$

Now, repeating the covering argument employed in the first part of this proof, we cover $Y \setminus \bigcup_{j=1}^k B(y_j, \tau_j)$ by finitely many balls of radii ε, and so find a ball B_1, centered at $w_1 \in Y \setminus \bigcup_{j=1}^k B(y_j, \tau_j)$, such that

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap K(\tau) \cap E(w_1, \varepsilon))}{\mu(B(x,r))} > 0.$$

Then by modifying w_1 if necessary, we can ensure that $w_1 \in Y_0$, and then find $\rho_1 \in \mathcal{R}_0(w_1)$ so that $\varepsilon \le \rho_1 < \frac{13}{12}\varepsilon$. Note that $B(w_1, \rho_1)$ is necessarily disjoint from $u(E(\tau/2))$ by this choice. Therefore we must have $E(w_1, \rho_1) \subset K(\tau/2)$, and so

$$\liminf_{r \to 0^+} \frac{\mu(B(x,r) \cap K(\tau/2) \cap E(w_1, \rho_1))}{\mu(B(x,r))} = \liminf_{r \to 0^+} \frac{\mu(B(x,r) \cap E(w_1, \rho_1))}{\mu(B(x,r))} \ge \gamma.$$

At this point, we can repeat the preceding argument that established the existence of the jump values to conclude that there must be a jump value in Y attained by u along $K(\tau)$, violating the maximality of the collection of jump values considered above. It follows that $K(\tau)$ has density 0 at x. ∎

We now consider the case that the metric space (Y, d_Y) is a proper metric space that is not compact. Recall that the proofs and discussions in Section 4.5 dealt with the case that Y is compact as then we can focus on covering Y by *finitely many* balls of radius $\varepsilon > 0$ and hence find a ball whose pre-image has positive density at a point $x \in \mathcal{J}(u)$. If we had instead a countably infinite many balls needed to cover Y, then we do not know that there must be one ball whose pre-image has positive density at x. When Y is not compact, this is because Y is not bounded; hence we cannot cover Y by finitely many balls of fixed radius $\varepsilon > 0$. In this section we point out how to deal with this situation.

As in the proof of Proposition 4.31, let Y_0 be a countable dense subset of Y, and for each $y \in Y_0$ let $\mathcal{R}_0(y)$ be a countable dense subset of $\mathcal{R}(y)$. If there is some $R > 0$ and $a \in Y$ such that $\mu(u^{-1}(Y \setminus B(a, R))) = 0$, then we can replace Y with $\overline{B}(a, R)$ and the proof of Proposition 4.31 identifies the jump values of u at points in $\mathcal{J}(u) \setminus N$. Hence we may assume without loss of generality that no such a, R, exists. In this case, we fix a point $a \in Y_0$ and note by the co-area formula Lemma 4.2 applied to the real-valued function $d_a \circ u$ of bounded variation given by $x \mapsto d_Y(a, u(x))$, that

$$\int_0^\infty P(u^{-1}(B(a,t)), X)\, dt = \|D\,(d_a \circ u)\|(X) \le \|D_{AM} u\|(X) < \infty.$$

It follows that for each positive integer n we can find $R_n > n$ such that $P(u^{-1}(B(a,R_n)), X) < 1/n$. We now enlarge the null set N, chosen in the proof of Proposition 4.31, by replacing N with

$$N \cup \bigcup_{k\in\mathbb{N}} \partial_* u^{-1}(B(a,R_k)) \setminus \Sigma_\gamma u^{-1}(B(a,R_k)).$$

We now fix $x \in \mathcal{J}(u) \setminus N$. Then, with $x \in \mathcal{J}(u) \setminus N$ as in the proof of Proposition 4.31, we have one of two cases:

(a) For each positive integer n we have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(\overline{B}(a,R_n)))}{\mu(B(x,r))} = 0.$$

(b) There is some positive integer n_0 such that for each $n \ge n_0$ we have

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(\overline{B}(a,R_n)))}{\mu(B(x,r))} > 0.$$

Should Case (a) happen, we say that u is approximately continuous at x with approximate limit ∞. Such points form a μ-measure null subset of X because, by embedding Y into a Banach space and using Bochner integrals, we know that μ-a.e. point in X is a Lebesgue point of u as $u \in L^1(X,V)$; note that the value of the function at a Lebesgue point must necessarily be a point in the Banach space and hence cannot be infinite in nature. We can include them in the set of approximately continuous points of u. Thus it suffices to take care of Case (b). In this case, we focus on covering the compact set $\overline{B}(a,R_n)$ for some fixed $n \ge n_0$ by balls $B(y_i,\varepsilon)$, $i = 1,\cdots,N_\varepsilon$, where implicitly N_ε now depends on the choice of R_n as well, but as n is fixed, this dependence is suppressed. Here we ensure that $0 < \varepsilon < R_n/10$. In so doing, we find one point, say y_1, such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(B(y_1,\varepsilon)))}{\mu(B(x,r))} > 0.$$

Thus we can choose $E_1 = u^{-1}(B(y_1,\varepsilon))$, and as x is not a point of approximate continuity of u, we also know that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \setminus u^{-1}(B(y_1,\varepsilon)))}{\mu(B(x,r))} > 0.$$

If we also have

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \setminus u^{-1}(B(a,R_{2n})))}{\mu(B(x,r))} > 0,$$

then necessarily $x \in \partial_* u^{-1}B(a,R_{2n})$ and so as $x \notin N$, we must have that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r) \setminus u^{-1}(B(a,R_{2n})))}{\mu(B(x,r))} \ge \gamma.$$

If for all positive integers n the above density property holds for $u^{-1}(B(a,R_{2n}))$, then we can consider ∞ to be one of the jump values of u at x. Continuing the argument found in the proof of Proposition 4.31 by covering $\overline{B}(a, \frac{11}{10}R_n)$ by balls of radius $\varepsilon/6^2$ to find y_2, and proceeding from there to find a sequence $y_i \in Y$ that converges to $y_\infty \in Y$, we see that y_∞ must also be a jump value of u at x. The rest of the argument as found in the proof of Proposition 4.31 holds, as long as we consider ∞ to be one of the jump values if necessary.

If ∞ is a jump value of u at x, then we must necessarily have that $x \in \Sigma_\gamma u^{-1}(B(a, R_k))$ for each k. As

$$1/k > P(u^{-1}(B(a, R_k)), X) \approx \mathcal{H}^{-1}(\Sigma_\gamma u^{-1}(B(a, R_k))),$$

we must have that

$$\mathcal{H}^{-1}\left(\bigcap_k \Sigma_\gamma u^{-1}(B(a, R_k))\right) = 0.$$

That is, the collection of all points $x \in \mathcal{J}(u) \setminus N$ that have ∞ as a jump value must be of $\mathcal{H}^{-1}$-measure zero as well. All other points in $\mathcal{J}(u)$ can be handled by the proof of Proposition 4.31 by using covering arguments only for the compact set $\overline{B}(a, R_j)$ for sufficiently large j.

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