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Cuerdas y no conmutatividad

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Guillermo Horcajada Reales

Director

Fernando Ruiz Ruiz

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Cuerdas y no conmutatividad

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Guillermo Horcajada Reales

Dirigida por el Doctor

Fernando Ruiz Ruiz

Departamento de Física Teórica I
Facultad de Ciencias Físicas
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A Sara,
TODO

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Índice

Resumen

Los conceptos geométricos clásicos que usan las ecuaciones de Einstein para describir el espacio, el tiempo y la gravedad no son compatibles con los principios de la mecánica cuántica. A distancias muy cortas, cercanas o por debajo de la longitud de Planck $l_p \approx 10^{-36}$m, se espera que la estructura del espacio-tiempo se haga difusa, con un principio de incertidumbre asociado a las propias coordenadas espacio temporales. Tal principio de incertidumbre podría derivarse de relaciones de conmutación no triviales $[x^\mu, x^\nu] \neq 0$ entre operadores asociados a la posición en el espacio-tiempo. Conmutadores de este tipo aparecen de manera natural en teoría de cuerdas, que, por otra parte, contiene y generaliza las ecuaciones de Einstein. Constituye por ello un escenario idóneo para el estudio de la naturaleza no determinista del espacio-tiempo.

En efecto, Chu y Ho [4] demostraron que la cuantización canónica de la cuerda abierta en espacio-tiempo de Minkowski con 2-forma B y dilatón constantes conduce a conmutadores no triviales entre los operadores de posición de los extremos de la cuerda. Este hecho sugiere la interpreción de la D-brana sobre la que pueden moverse dichos extremos como un espacio no conmutativo. Seiberg y Witten [5] dieron un paso más y encontraron un límite de baja energía bien definido en el cual la dinámica de los extremos de la cuerda se desacopla de la de los modos internos y se describe como una teoría de Yang-Mills no conmutativa sobre la D-brana. Es díficil trasladar estos resultados a backgrounds más generales para la cuerda. Los modelos de Wess-Zumino-Witten (WZW) [6] constituyen los backgrounds no triviales mejor conocidos, pero no se conoce una caracterización completa de las D-branas en estos modelos. El objetivo de esta tesis es mejorar la comprensión del origen de no conmutatividad a partir de (i) el estudio de la cuerda abierta en backgrounds no triviales y (ii) la caracterización de las D-branas sobre las que pueden moverse sus extremos. El trabajo de esta tesis ha dado lugar a las publicaciones [I], [II] y [III].

En la ref. [I] se cuantiza canónicamente la cuerda abierta en un background con métrica tipo onda *pp* con 2-forma y dilatón constantes. Para ello, se resuelven las ecuaciones clásicas de movimiento y condiciones de contorno en el gauge cono de luz. La solución general hallada para la posición de la

cuerda, escrita como sumas de modos de oscilación, se usa para calcular los momentos canónicos conjugados. A continuación, se calcula la forma simpléctica, que resulta ser independiente del tiempo y no singular. Su inversa determina las relaciones de conmutación para los modos de oscilación, promovidos a operadores en el proceso de cuantización canónica. A partir de las reglas de conmutación que resultan para los modos, se calculan las de los operadores posición y momento de la cuerda. Se obtiene que los operadores de posición no conmutan, no sólo en los extremos de la cuerda sino también en los puntos intermedios. De esta manera se muestra que la no conmutatividad no se restringe a la D-brana sobre la que se mueven los extremos, sino que se extiende a todo el espacio-tiempo. En el límite en que la métrica *pp* considerada se reduce a la de Minkowski, se recuperan los resultados de la literatura y la no conmutatividad se limita a los extremos.

Los modelos de WZW cerrados pueden describirse convenientemente en términos de las corrientes quirales $J_{\pm}$. Para extender esta descripción a los modelos de WZW abiertos es necesario escribir las condiciones de contorno para la cuerda abierta también en términos de $J_{\pm}$. Para ello se ha propuesto la condición $J_+\big|_{\partial\Sigma} = F(g)J_-\big|_{\partial\Sigma}$, conocida como condición de pegado, que relaciona las corrientes quirales a través de una aplicación lineal F que actúa sobre el álgebra de Lie del modelo. Esta aplicación F puede en general depender del punto g del espacio-tiempo donde se evalúa la condición, escribiéndose por ello $F(g)$. Para que una condición de pegado defina una D-brana $\mathcal{N}$, debe ser equivalente a una condición de contorno. La equivalencia entre estos dos tipos de condiciones se estudia en detalle en la ref. [II]. El análisis allí efectuado muestra que una condición de contorno con una 2-forma ω puede escribirse como una de pegado para algún F si y sólo si $G\big|_{\mathcal{N}} - \omega$ es invertible, donde $G\big|_{\mathcal{N}}$ es la restricción de la métrica espaciotemporal a la D-brana. Por el contrario, se demuestra que una condición de pegado con un F dado es equivalente a una condición de contorno si F es una isometría y el conjunto de movimientos posibles $\partial_\tau X\big|_{\partial\Sigma}$ de los extremos de la cuerda, permitidos por las condiciones de pegado, genera una distribución integrable. La 2-forma ω que aparece en la condición de contorno resultante queda determinada por la isometría F, y debe ser tal que su derivada exterior sea igual a la 3-forma H de WZW, $d\omega = H\big|_{\mathcal{N}}$.

Estos resultados se aplican a condiciones de pegado dadas por isometrías F de la forma $F = R$, con R un automorfismo constante del álgebra de Lie, y se obtienen como D-branas las llamadas clases de conjugación R-twined. Este resultado es bien conocido [7, 8], si bien demostraciones previas, al contrario de la nuestra, dejaban al margen D-branas con métrica de signatura degenerada. En cambio, cuando la isometría es de la forma $F = -R$ es resultado no es tan nítido. Se encuentra que si el álgebra de Lie del modelo es semisimple, la condición de integrabilidad no se satisface y por tanto las correspondientes condiciones de pegado no pueden describir

D-branas.

En la publicación [III], se usan estas técnicas para encontrar D-branas a partir de condiciones de pegado en el modelo de Nappi-Witten [9], arquetipo de modelo de WZW no semisimple. Se consideran isometrías F, tanto constantes como no constantes, y se encuentran D-branas de todas las dimensiones, algunas de signatura lorentziana, otras euclídea y otras degeneradas.

Abstract

The geometric ideas underlying the description of space and time given by Einstein's equations are known not to be compatible with the principles of quantum mechanics. At distances of the order of the Planck length $l_p \approx 10^{-36}$m, the structure of spacetime is expected to become fuzzy and some uncertainty principle associated to spacetime coordinates is believed to take place. A plausible realization of this idea is through nontrivial commutation relations $[x^\mu, x^\nu] \neq 0$ for position operators. String theory provides a natural scenario for commutators of this type to take place and contains the Einstein equations in its low energy limit. It thus seems to include all the necessary inputs to study the nature of space time fuzziness.

In fact, it has been shown by Chu and Ho [4] that the position operators for the endpoints of an open string in Minkowski spacetime, with constant 2-form and dilaton fields, satisfy commutation relations of this type. This fact suggests the interpretation of the D-brane on which the endpoints may move as a noncommutative spacetime. Seiberg and Witten [5] went further and found a precise prescription for a low energy limit in which the string endpoints dynamics decouple from the bulk and is described by a noncommutative Yang-Mills theory on the D-brane. Translating these results to more general string backgrounds is difficult. Wess-Zumino-Witten (WZW) models [6] are the best known and more widely studied nontrivial backgrounds for the string. Yet a characterization of D-branes in these models is lacking. The goal of this thesis is to gain insight into the origin of noncommutativity through (i) the study of the open string on nontrivial backgrounds and (ii) the characterization of D-branes on which its endpoints may move. It is based on results collected in refs. [I], [II] and [III].

In ref. [I], canonical quantization of the open string on the Penrose limit of $dS_n \times S^n$ with constant background 2-form and dilaton fields is performed. This is done in several steps. The equations of motion for the classical string coordinates subject to the boundary conditions specified by the background 2-form are first solved. The solutions are given as sums over oscillation modes and are used to compute canonical momenta. The symplectic form is next calculated and it turns out to be time independent and nonsingular, hence ready for canonical quantization. Commutation relations for the os-

cillation mode operators are read from the inverse of the symplectic form. It remains to find the commutators of the string position and canonical momentum operators. Noticeably the string position operators at equal τ do not commute for all values of the string worldsheet parameter σ, thus realizing noncommutativity in the bulk. If a smooth limit is taken to reduce the *pp* background considered to Minkowski spacetime, the results in the literature are recovered and noncommutativity gets confined to the endpoints.

Closed WZW models are best described in terms of the chiral currents $J_\pm$. Extending this description to open WZW models requires writing the boundary conditions for the open string also in terms of the currents. A condition to achieve this has been proposed under the name of gluing condition. It takes the form $J_+\big|_{\partial\Sigma} = F(g)J_-\big|_{\partial\Sigma}$ and matches the chiral currents at string endpoints through a linear map F acting on the WZW Lie algebra. This map F may generally depend on the spacetime point g where the condition is formulated and is so written $F(g)$. For this gluing condition to describe a D-brane $\mathcal{N}$ it must be equivalent to a boundary condition. The equivalence of these two types of condition is studied in detail in ref. [II]. The analysis shows that a boundary condition with a 2-form ω can be rewritten as a gluing condition for some F if and only if $G\big|_{\mathcal{N}} - \omega$ is invertible, where $G\big|_{\mathcal{N}}$ is the restriction of the full spacetime metric to the D-brane. Conversely, it is shown that, for a gluing condition with a given F to be equivalent to a boundary condition, F has to be an isometry and the set of possible motions $\partial_\tau X\big|_{\partial\Sigma}$ of the string endpoints allowed by the gluing condition must generate an integrable distribution. Such an F determines the 2-form ω in the resulting boundary condition, which must additionally have the WZW 3-form H as exterior differential, $d\omega = H\big|_{\mathcal{N}}$.

This approach, when applied to gluing conditions with isometries of the form $F = R$ with R a constant Lie algebra automorphism, gives as D-branes the so called R-twined conjugacy classes, as already known [7, 8]. The proof given here extends however to metrically degenerate D-branes. If the isometry is of the form $F = -R$, the result is not so neat. In particular, for a WZW model with semisimple Lie algebra the integrability condition is not satisfied and thus the corresponding gluing condition does not describe a D-brane.

In ref. [III], the techniques explained above are used to find D-branes in the Nappi-Witten model [9], the archetype of non-semisimple WZW model. Both constant and nonconstant isometries F are considered, and Lorentzian signature D-branes of all dimensions are found, as well as some euclidean signature and metrically degenerate ones.

Capítulo 1

Introducción

El carácter no determinista de la mecánica cuántica hace esperable que a distancias muy pequeñas la estructura del espacio-tiempo se vuelva difusa. Una idea recurrente en la literatura para justificar y describir este carácter difuso del espacio-tiempo es un principio de incertidumbre asociado a una no conmutatividad entre operadores posición.

Esta idea puede elaborarse en términos más precisos. Supongamos que se desea localizar un suceso con una cierta precisión Δx. Esta medida requiere introducir una cantidad de energía en una región con tamaño del orden de Δx mediante un fotón o alguna otra partícula. Por el principio de incertidumbre de Heisenberg, la imprecisión en el momento de la partícula es inversamente proporcional a Δx y se corresponde con la energía mínima necesaria para la medida. De acuerdo con las ecuaciones de Einstein, esta energía curva el espacio-tiempo alrededor de la región, hasta el punto, si la región es muy pequeña y la energía muy grande, de formar un horizonte de sucesos que encierra la región e impide la salida de señales de la misma, con lo que la medida no es posible. La longitud de Planck $l_p \approx 10^{-36}$m marca la escala de longitudes en la que los efectos gravitacionales se hacen tan fuertes que impiden la medida, proporcionando así una cota a la precisión de las medidas de posición. Una posible realización de esta idea es considerar operadores posición que no conmuten entre sí y que, de acuerdo con los principios de la mecánica cuántica, conduzcan a unos relaciones de incertidumbre posición-posición en las que l_p aparezca como cota inferior a la incertidumbre espacio-temporal. Doplicher, Fredenhagen y Roberts [10] dieron un primer paso en este sentido en 1995 y mostraron que relaciones de conmutación

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,,$$

con $\theta^{\mu\nu}$ una matriz real antisimétrica constante adecuada con dimensiones de longitud al cuadrado, conducen a relaciones de incertidumbre del tipo

$$c\Delta t\left(\Delta x^1+\Delta x^2+\Delta x^3\right)\geq l_p^2$$
$$\Delta x^1\Delta x^2+\Delta x^2\Delta x^3+\Delta x^3\Delta x^1\geq l_p^2\,.$$

En cualquier caso, se realice mediante conmutadores no triviales para operadores posición o mediante cualquier otro mecanismo, la compatibilidad de la observación a grandes distancias con la estabilidad del campo gravitatorio generado por la energía necesaria para efectuar la observación es un requisito razonable a exigir a cualquier teoría cuántica de la gravedad.

Por su naturaleza, la teoría de cuerdas parece especialmente adecuada para incorporar un análisis de no conmutatividad espacio-temporal y sus implicaciones sobre la estructura del espacio-tiempo. Proporciona además un escenario en el que las ecuaciones de Einstein se recuperan como límite de bajas energías. En esta memoria se estudian algunos aspectos de no conmutatividad dentro del ámbito de teoría de cuerdas.

1.1. No conmutatividad espacio-temporal

Históricamente la primera realización de un espacio-tiempo no conmutativo se debe a Snyder [11] en 1947, si bien su motivación no era entender la naturaleza del espacio-tiempo a distancias muy pequeñas, sino encontrar un argumento que justificase la introducción de un "cutoff" ultravioleta. De forma general, por espacio-tiempo no conmutativo se entiende el álgebra generada por operadores hermíticos posición x^μ que satisfacen reglas de conmutación no triviales

$$[x^\mu,x^\nu]=i\theta^{\mu\nu}\,, \tag{1.1}$$

donde $\theta^{\mu\nu}$ es antisimétrico en $\{\mu,\nu\}$, tiene dimensiones de longitud al cuadrado y puede ser constante o una función de los operadores posición.

De acuerdo con la mecánica cuántica, para $\theta^{\mu\nu}$ constante, unas relaciones de conmutación del tipo (1.1) dan lugar a un principio de incertidumbre generalizado

$$\Delta x^\mu\Delta x^\nu>=\frac{1}{2}\left|\theta^{\mu\nu}\right|\,. \tag{1.2}$$

El marco natural para el estudio de teorías con conmutadores arbitrarios de sus operadores posición y momento es la llamada cuantización por deformación, o cuantización en el espacio de fases; ver [12] para una descripción completa, incluyendo una selección de artículos y un listado de referencias. En ella, los observables son funciones de las variables canónicas posición y momento, con un álgebra de funciones asociativa pero no conmutativa que se suele llamar deformada.

En la geometría convencional la dualidad de Gelfand asegura que se puede caracterizar e incluso identificar un espacio topológico Hausdorff localmente compacto $\mathcal{M}$ mediante el álgebra $C^\star$ de funciones continuas definidas sobre $\mathcal{M}$. La geometría no conmutativa [13] generaliza esta dualidad para describir espacios no conmutativos mediante álgebras deformadas de funciones con productos no conmutativos. Una de estos productos es el producto estrella de Moyal-Groenewold [14], que Kontsevich generalizó a variedades de Poisson generales [15]. Para funciones $f(x)$ y $g(x)$ de las coordenadas y matriz de no conmutatividad $\theta^{\mu\nu}$ constante, el producto estrella se puede escribir

$$f(x) \star g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\zeta}} f(x+\xi)g(x+\zeta)\big|_{\xi=\zeta=0} \,. \tag{1.3}$$

La correspondencia de Weyl-Wigner [16, 17] entre funciones en el espacio de fases y operadores sobre el espacio de Hilbert asociado al sistema cuántico en cuestión, asegura la equivalencia entre la descripción cuántica en el espacio de fases y la descripción de Schrödinger. El corchete de Moyal-Groenewold, definido a partir del producto $\star$ como

$$[f, g]_\star = f \star g - g \star f \,, \tag{1.4}$$

es el equivalente al conmutador de operadores en el espacio de Hilbert. En particular, para las funciones x^μ y x^ν se tiene $[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}$.

En la formulación de teorías cuánticas de campos no conmutativas, es decir, en espacios-tiempo no conmutativos, el producto $\star$ sustituye al producto ordinario. Las teorías de Yang-Mills no conmutativas son particularmente interesantes. En ellas se mezclan las simetrías gauge internas con las simetrías espacio-temporales y puede pensarse en utilizarlas como teorías gauge de gravedad [18, 19]

1.2. No conmutatividad en teoría de cuerdas y D-branas

En 1998 Chu y Ho [4] demostraron que en la cuantización de ciertos modelos sigma no lineales para cuerdas abiertas y D-branas aparecen de forma natural relaciones de conmutación del tipo (1.1). Ello abre paso a la realización de teorías de campos no conmutativas dentro del ámbito de teoría de cuerdas. Poco después, Seiberg y Witten formularon teorías no conmutativas tomando el límite de bajas energías de la cuerda abierta [5]. En esta sección repasamos brevemente estos resultados, lo que nos sirve también para introducir la notación que usaremos en esta memoria.

Una D-brana clásica puede definirse como una subvariedad del espacio-tiempo sobre la que los extremos de una cuerda abierta pueden moverse libremente. Si la dimensión de la D-brana es $p+1$, es frecuente llamarla

Dp-brana. Sobre la D-brana actúa un potencial A de tipo electromagnético, bajo el cual los extremos de la cuerda están cargados.

Recordemos que la acción de Polyakov para la cuerda abierta bosónica en un espacio-tiempo $\mathcal{M}$ con coordenadas locales $\{X^\mu\}$, métrica $G_{\mu\nu}$, 2-forma $B_{\mu\nu}$ y dilatón Φ, cuyos extremos se mueven sobre una Dp-brana $\mathcal{N}$ está dada por

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \Big(\sqrt{-\gamma}\,\gamma^{rs}\, G_{\mu\nu}\,\partial_r X^\mu\,\partial_s X^\nu + \epsilon^{rs} B_{\mu\nu}\,\partial_r X^\mu\,\partial_s X^\nu \\ + \alpha'\sqrt{-\gamma}\,R\,\Phi\Big) - \frac{1}{2\pi\alpha'}\oint_{\partial\Sigma} d\tau\, A_i\,\partial_\tau x^i\ . \end{aligned} \tag{1.5}$$

En su movimiento la cuerda describe una superficie Σ de dimensión dos llamada hoja del universo. Si se eligen en Σ coordenadas locales $\sigma^r = (\tau, \sigma)$, el embedding de Σ en $\mathcal{M}$ está dado por los campos clásicos $X^\mu(\tau,\sigma)$. La invariancia bajo reparametrizaciones nos permite tomar $\sigma \in [0,\pi]$, correspondiendo 0 y π a los extremos de la cuerda. En estas coordenadas, γ_{rs} denota la métrica sobre Σ, γ su determinante y R su escalar de Ricci. La métrica γ_{rs} es en realidad un campo gauge de la acción, que eligiendo parámetros τ y σ adecuados, y empleando la simetría conforme, puede tomarse igual a la métrica de Minkowski. El último término de la ec. (1.5) describe la interacción de los extremos de la cuerda con la Dp-brana $\mathcal{N}$ y está escrito en términos de las coordenadas $\{x^i\}$ sobre $\mathcal{N}$ [1]. Si escribimos el embedding de $\mathcal{N}$ en $\mathcal{M}$ como funciones $X^\mu = f^\mu\left(x^0, \ldots, x^p\right)$, el extremo $\sigma = 0$ de la cuerda, restringido a moverse sobre la D-brana, seguirá una trayectoria espacio-temporal

$$X^\mu(\tau, 0) = f^\mu\left(x^0(\tau), \ldots, x^p(\tau)\right)\ ,$$

y análogamente para $X^\mu(\tau,\pi)$. De forma compacta podemos escribir

$$\left(X^\mu\left(\tau,\sigma\right) - f^\mu\left(x(\tau)\right)\right)\Big|_{\partial\Sigma} = 0\,, \tag{1.6}$$

donde se ha abusado ligeramente de la notación al denotar el movimiento de los dos extremos de la cuerda por $x(\tau) \equiv \{x^i(\tau)\}$, cuando en realidad son funciones distintas para cada extremo.

Las condiciones de contorno para la cuerda son condiciones mixtas de tipo Dirichlet y Neumann. En efecto, por un lado la ec. (1.6), que limita el movimiento de los extremos a la D-brana, se interpreta como condiciones de contorno de tipo Dirichlet para las coordenadas ortogonales a la D-brana. Por otro, de la acción (1.5) surgen las condiciones

$$\left(\partial_i f^\mu G_{\mu\nu}\partial_\sigma X^\nu - \omega_{ij}\partial_\tau x^j\right)\Big|_{\partial\Sigma} = 0\,, \tag{1.7}$$

[1] Nótese que los campos clásicos que dan la posición de la cuerda se denotan con mayúsculas, mientras que se reservan las minúsculas para las coordenadas sobre la D-brana

que son de de tipo Neumann y relacionan los valores de $\partial_\sigma X$ en los extremos de la cuerda con $\partial_\tau x$. En la ec. (1.7) aparece la 2-forma $\omega_{ij} = B_{ij} - F_{ij}$, donde $F_{ij} = \partial_i A_j - \partial_j A_i$ es el campo asociado al potencial A y, abusando de la notación, B_{ij} denota la restricción o pullback de $B_{\mu\nu}$ sobre la D-brana.

Recordemos que los campos G, B y Φ que conforman el background de la cuerda surgen de los estados sin masa del espectro al cuantizar la **cuerda cerrada** bosónica en espacio-tiempo de Minkowski. Son desde el primer momento objetos dinámicos. La cancelación de la anomalía de Weyl requiere que las funciones beta se anulen. Para bajas energías, primer orden en α', las funciones beta asociadas a los campos son

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa}\,, \tag{1.8}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi\, H_{\lambda\mu\nu}\,, \tag{1.9}$$

$$\beta^\Phi = -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\mu \Phi\, \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}\,, \tag{1.10}$$

y las ecuaciones

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0 \tag{1.11}$$

gobiernan la dinámica del background de la cuerda. Estas ecuaciones incluyen o generalizan las ecuaciones de Einstein y se pueden deducir mediante un principio variacional para la acción efectiva

$$S = \frac{1}{2\kappa_0} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left(R_G - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 \partial_\mu \Phi\, \partial^\mu \Phi \right)\,. \tag{1.12}$$

Aquí $H = dB$ es la 3-forma derivada de B y R_G el tensor de Ricci asociado a la métrica G.

Análogamente, la cuantización de la **cuerda abierta** en espacio-tiempo plano con condiciones de contorno mixtas Dirichlet y Neumann, lleva a la aparición de la D-brana como objeto dinámico sobre el que actúa un potencial A que surge de los estados sin masa del espectro. Ahora la conservación de la simetría conforme en la teoría cuántica lleva a la acción de Dirac-Born-Infeld

$$S_{DBI} = -T_p \int d^{p+1} x\; e^{-\Phi} \sqrt{-\det(\gamma_{ij} + \omega_{ij})} \tag{1.13}$$

como acción efectiva para la D-brana [20]. En esta acción vemos de nuevo la combinación $\omega_{ij} = B_{ij} - F_{ij}$. La acción (1.5) es invariante bajo transformaciones gauge $B \to B + d\Lambda$, $A \to A + \Lambda$, donde Λ es una 1-forma arbitraria. Esta simetría gauge combina el campo A definido sobre la D-brana con el campo B definido en todo el espacio-tiempo. La 2-forma ω es precisamente el field strength correspondiente a esta invariancia gauge.

Mostremos ahora el ejemplo más sencillo en el que aparece no conmutatividad en los extremos de la cuerda abierta. Consideremos una cuerda abierta y una Dp-brana en espacio-tiempo de Minkowski de dimensión D con coordenadas $\{X^\mu\} = \{X^0, \ldots, X^{D-1}\}$. Supongamos que el campo ω_{ij} es constante sobre la Dp-brana y que ésta es un hiperplano definido por el embedding

$$\begin{aligned} X^i &= x^i \qquad & i &= 0, \cdots, p \\ X^a &= c^a \qquad & a &= p+1, \cdots, D-1\,, \end{aligned}$$

donde c^a son constantes y x^i son las coordenadas de la D-brana. Con esta configuración las condiciones de contorno 1.7 se reducen a

$$\left(\partial_\sigma X^i - \omega^i{}_j \partial_\tau X^j\right)\Big|_{\sigma=0,\pi} = 0\,.$$

Utilizando que los momentos canónicos conjugados son

$$P^i = \frac{1}{2\pi\alpha'}\left(\partial_\tau X^i - \omega^i{}_j \partial_\sigma X^j\right)$$

e introduciendo la matriz $M_{ij} = \eta_{ij} - \omega_{ik}\omega^k{}_j$, con η_{ij} la métrica de Minkowski, las condiciones de contorno pueden escribirse

$$\left(2\pi\alpha'\omega^i{}_j P^j - M^i{}_j \partial_\sigma X^j\right)\Big|_{\sigma=0,\pi} = 0\,.$$

Conmutando con $X^k(\tau, \sigma')$ se llega a la expresión

$$\left(2\pi\alpha'\omega^i{}_j \left[P^j(\tau,\sigma), X^k(\tau,\sigma')\right] - M^i{}_j \partial_\sigma \left[X^j(\tau,\sigma), X^k(\tau,\sigma')\right]\right)\Bigg|_{\sigma=0,\pi} = 0\,.$$

Esta ecuación es claramente incompatible con las dos primeras de las relaciones de conmutación canónicas

$$[\, X^i(\tau,\sigma), P_j(\tau,\sigma')\,] = i\,\delta^i_j\,\delta(\sigma-\sigma')\,, \tag{1.14}$$

$$[\, X^i(\tau,\sigma), X^j(\tau,\sigma')\,] = 0\,, \tag{1.15}$$

$$[\, P_i(\tau,\sigma), P_j(\tau,\sigma')\,] = 0\,. \tag{1.16}$$

Tomando esta observación como punto de partida, Chu y Ho [4] cuantizaron canónicamente la cuerda abierta en este background ($G_{\mu\nu} = \eta_{\mu\nu}$, $\omega_{ij} = $ cte., $\Phi = $ cte.). Demostraron que,

i) si bien (1.14) y (1.16) siguen siendo válidas,

ii) el conmutador (1.15) sólo es cierto en los puntos intermedios de la cuerda, mientras que en sus extremos se tiene

$$\left[X^i(\tau,0), X^j(\tau,0)\right] = 2\pi i\alpha'(M^{-1}\omega)^{ij}\,, \tag{1.17}$$

$$\left[X^i(\tau,\pi), X^j(\tau,\pi)\right] = -2\pi i\alpha'(M^{-1}\omega)^{ij}\,. \tag{1.18}$$

Desde el punto de vista de los extremos, la D-brana sobre la que se mueven es un espacio-tiempo no conmutativo.

Posteriormente, Chu y Ho [21] aplicaron los mismos métodos a una espacio-tiempo de tipo onda *pp* con 2-forma B constante, correspondiente al límite de Penrose de $AdS_5 \times S^5$. Aparte de estos y algunos otros casos concretos [22] y resultados aproximados [23, 24], para espacios-tiempo curvos generales no se conocen los detalles de las D-branas y su no conmutatividad. Mención especial merece una clase de espacios-tiempo, conocidos como modelos de Wess-Zumino-Witten (WZW) [6], sobre los que volveremos en la siguiente sección, para los que sí se han podido obtener algunos resultados generales gracias a sus propiedades algebraicas [25].

El siguiente hito en el estudio de no conmutatividad en el ámbito de teoría de cuerdas son los resultados de Seiberg y Witten [5]. Estos autores formularon explícitamente una teoría de campos no conmutativa tomando el límite de baja energía $G_{ij} \sim {\alpha'}^2 \sim \epsilon \to 0$ en el que se desacoplan los modos internos de la cuerda de los modos de sus extremos. El límite mantiene los conmutadores (1.17) y (1.18) pero elimina el primer término de la acción (1.5), con lo que ésta se reduce a la suma de una acción topológica para los modos internos de la cuerda y de la acción

$$S = -\frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau \, A_i \partial_\tau x^i \,, \tag{1.19}$$

para los grados de libertad de los extremos, donde $A_i = \omega_{ji} x^j$. Aunque clásicamente la acción (1.19) es invariante bajo transformaciones gauge $\delta A_i = \partial_i \lambda$, Seiberg y Witten mostraron que en la teoría cuántica la invariancia es en realidad bajo transformaciones "no conmutativas"

$$\delta A_i = \partial_i \lambda + i\lambda \star A_i - iA_i \star \lambda \,,$$

con $\star$ el producto de Moyal-Groenewold. De esta forma se obtiene una teoría de Yang-Mills no conmutativa sobre la D-brana.

En todos estos resultados la no conmutatividad se limita a los extremos de la cuerda y está ligada al campo $\omega = B - F$ sobre la D-brana. Sin embargo, la contribución a ω del campo B, definido en principio en todo el espacio-tiempo, nos hace plantearnos en esta tesis si la restricción de la no conmutatividad a los extremos de la cuerda es una propiedad universal, o si es una característica de los modelos estudiados.

1.3. Modelos de Wess-Zumino-Witten

Un modelo de WZW es un modelo sigma no lineal definido sobre un grupo de Lie G con álgebra de Lie $\mathfrak{g}$ para la que exista una métrica invariante Ω,

$$\Omega\left(U, [V, W]\right) = \Omega\left([U, V], W\right) \qquad \forall\, U, V, W \in \mathfrak{g} \,.$$

A partir de la métrica Ω puede definirse una métrica bi-invariante sobre la variedad G cuyas componentes en el punto $g(X)$ en un sistema de coordenadas local $\{X^\mu\}$ están dadas por

$$G_{\mu\nu} = \Omega\left(g^{-1}\partial_\mu g, g^{-1}\partial_\nu g\right) . \tag{1.20}$$

Análogamente se define una 3-forma H como

$$H_{\mu\nu\lambda} = \Omega\left(\left[g^{-1}\partial_\mu g, g^{-1}\partial_\nu g\right], g^{-1}\partial_\lambda g\right) . \tag{1.21}$$

El modelo de WZW cerrado tiene acción clásica

$$S = \frac{k}{4\pi}\int_\Sigma d^2\sigma\,\Omega\left(g^{-1}\partial_r g, g^{-1}\partial^r g\right) + \frac{k}{4\pi}\int_{M_3} H\,, \tag{1.22}$$

donde M_3 es una subvariedad de G de dimensión tres con frontera ∂M_3 igual a la inmersión en G de la 2-variedad Σ. Para cualquier 2-forma B tal que $H = dB$ localmente, la acción (1.22) reproduce la acción de Polyakov para la cuerda cerrada sin dilatón[2]. A pesar de que B no existe globalmente, es bien conocido que la acción (1.22) es consistente con una integral de camino bien definida [6].

Para cuerdas abiertas en backgrounds $(G_{\mu\nu}, B_{\mu\nu}, \Phi = 0)$ con B tal que $H = dB$ localmente, también es posible escribir la acción de Polyakov (1.5) como un modelo de WZW, que ahora recibe el nombre de abierto. En este caso, el último término de (1.22) se sustituye por

$$\frac{k}{4\pi}\left(\int_\Sigma g^*B + \int_{\partial\Sigma} g^*A\right) , \tag{1.23}$$

donde g^*B es el pullback de B y g^*A el de A. Al igual que ocurría con el caso cerrado, el término (1.23) es consistente con una integral de camino para el modelo sigma correspondiente [26].

Tanto en el caso abierto como en el cerrado, las ecuaciones de movimiento son

$$\partial_+ J_- = \partial_- J_+ = 0\,, \tag{1.24}$$

donde $\partial_\pm$ indica derivada parcial con respecto a $\sigma^\pm = \tau \pm \sigma$ y $J_\pm$ son las corrientes quirales, definidas por

$$J_- = g^{-1}\partial_- g \qquad J_+ = -\,\partial_+ g\, g^{-1} .$$

Las ecuaciones de movimiento (1.24) son triviales de resolver, módulo condiciones de contorno, sobre las que enseguida volveremos. Esto hace que los modelos de WZW sean más tratables que otros modelos sigma y les confiere un estatus especial dentro de la teoría de cuerdas. De hecho muchos de los

[2] Dada por la ec. (1.5) tras tomar $\Phi = 0$ y $\partial\Sigma = 0$

backgrounds mejor estudiados en teoría de cuerdas son productos directos de modelos de WZW, tal es el caso de $AdS_3 \times S^3 \times T^4$ [27, 28], de los espacios de la familia $AdS_n \times S^n$ [29, 30], o del modelo de Nappi-Witten [9].

Para el estudio de la cuerda cerrada, las condiciones de contorno son periódicas y no presentan mayor dificultad. El caso de la cuerda abierta es más complicado. Dada la forma tan sencilla de las ecuaciones de movimiento (1.24), es natural intentar expresar las condiciones de contorno del modelo sigma en términos de las corrientes J_+ y J_- del modelo de WZW. A las condiciones de este tipo que resultan para las corrientes se les llama condiciones de pagado. Se inicia así un programa encaminado a la caracterización y clasificación de D-branas en modelos de WZW.

La condición de pegado más simple que cabe considerar es

$$J_+\Big|_{\partial\Sigma} = J_-\Big|_{\partial\Sigma}\,, \tag{1.25}$$

Puede demostrarse que las soluciones en g a esta ecuación son las clases de conjugación del grupo y que todas ellas reproducen condiciones de contorno admisibles para la cuerda abierta [7, 31]. Es decir $\mathcal{C}(g_0) = \left\{e^V g_0\, e^{-V}\colon\ V \in \mathfrak{g}\right\}$ es una D-brana para todo $g_0 \in$ G. El ansatz (1.25) puede complicarse y considerarse en su lugar

$$(J_+ - RJ_-)\Big|_{\partial\Sigma} = 0\,, \tag{1.26}$$

donde R es un automorfismo del álgebra de Lie $\mathfrak{g}$. Este caso también describe condiciones de contorno aceptables para la cuerda abierta. Ahora las soluciones son de la forma $\mathcal{C}^R(g_0) = \left\{e^{RV} g_0\, e^{-V}\colon\ V \in \mathfrak{g}\right\}$, llamada clase de conjugación R-twined [8, 32, 33].

En esta tesis se plantea la caracterización de D-branas en el modelo de WZW más allá de las condiciones de pegado (1.25) y (1.26).

1.4. Problemas a tratar

En esta memoria se pretende profundizar en los orígenes de la no conmutatividad entre coordenadas espacio-temporales a partir del estudio de la cuerda abierta y de las D-branas en las que se mueven sus extremos. Se estudian los siguientes dos problemas:

(i) La formulación de modelos sigma no lineales para la cuerda abierta en los que la no conmutatividad no se restrinja a la D-brana sobre la que se mueven los extremos. En concreto, consideraremos un espacio-tiempo con métrica de tipo onda *pp* y 2-forma B constante, que resuelve las ecuaciones (1.11) y se corresponde con el límite de Penrose de una geometría $dS_n \times S^n$.

(ii) La clasificación y caracterización geométrica de D-branas en modelos de WZW mediante condiciones de pegado

$$J_+\Big|_{\partial\Sigma} = F(g)J_-\Big|_{\partial\Sigma}\,, \tag{1.27}$$

donde $F(g)$ es ahora un operador lineal sobre el álgebra de Lie del modelo, generalizando así la condición (1.26) que se limita a automorfismos constantes R.

Capítulo 2

Resultados y discusión

Los resultados que se presentan en esta memoria proporcionan soluciones a los dos problemas planteados en el apartado 1.4. La no conmutatividad de los operadores posición de la cuerda sugerida en el primero de ellos se estudia en la sección 2.1 y sus resultados se recogen en la publicación [I]. El segundo, la caracterización de D-branas en modelos de WZW mediante condiciones de pegado, se aborda en las secciones 2.2 y 2.3, de corte más matemático, y se encuentra desarrollado con detalle en las refs. [II] y [III].

2.1. Cuerdas no conmutativas en un espacio-tiempo de tipo *pp*

Nuestro punto de partida es un background formado por (i) una métrica $G_{\mu\nu}$ dada por

$$ds^2 = -\,dx^+dx^- + m^2\left[(x^1)^2 - (x^2)^2\right](dx^+)^2 + \sum_{i=1}^{2}(dx^i)^2 + \sum_{a=3}^{D-2}(dx^a)^2\,,$$

con m un parámetro de masa, (ii) una 2-forma $B_{\mu\nu}$

$$B_{ij} = \epsilon_{ij}B\,, \qquad B_{ab} = 0\,,$$

constante y (iii) un dilatón $\Phi = \Phi_0$ también constante. Estos campos satisfacen trivialmente las ecuaciones (1.11), por lo que constituyen un background para la cuerda bosónica a bajas energías. El background considerado puede obtenerse como límite de Penrose de una geometría $dS_n \times S^n$, de la que hereda la imposibilidad de definir globalmente un hamiltoniano definido positivo. Este hecho lo distingue de otros backgrounds estudiados en la literatura [4, 21, 22], que presentan no conmutatividad espacio-temporal en los extremos de la cuerda. El objetivo es cuantizar la cuerda abierta en este background y estudiar si la no conmutatividad entre los operadores posición $X^\mu(\tau,\sigma)$ se limita a los extremos de la cuerda o se extiende también a otros valores de σ. Para ello trabajaremos en el gauge cono de luz

$$X^+ = \kappa\tau\,, \tag{2.1}$$

y cuantizaremos canónicamente las soluciones clásicas a las ecuaciones de movimiento.

En el gauge (2.1) las ecuaciones de movimiento que se siguen de la acción de Polyakov para las coordenadas de la cuerda son

$$\Box X^1 + m^2\kappa^2 X^1 = 0\,, \tag{2.2}$$
$$\Box X^2 - m^2\kappa^2 X^2 = 0\,, \tag{2.3}$$
$$\Box X^a = 0\,, \tag{2.4}$$

donde $\Box = -\partial_\tau^2 + \partial_\sigma^2$ es el D'Alambertiano en dos dimensiones. A su vez, las condiciones de contorno toman la forma

$$\partial_\sigma X^1 - B\,\partial_\tau X^2\Big|_{\sigma=0,\pi} = 0\,, \tag{2.5}$$
$$\partial_\sigma X^2 + B\,\partial_\tau X^1\Big|_{\sigma=0,\pi} = 0\,, \tag{2.6}$$
$$\partial_\sigma X^a\Big|_{\sigma=0,\pi} = 0\,. \tag{2.7}$$

A partir de aquí procedemos de la siguiente manera:

1. Resolvemos las ecuaciones de movimiento clásicas (2.2)-(2.4) con las condiciones de contorno (2.5)-(2.7). Para las coordenadas X^a la solución es la misma que para la cuerda en un background trivial plano y no requiere mayor atención. Por contra la forma de las condiciones de contorno para X^1 y X^2 implica una dependencia no trivial de los correspondientes modos de oscilación con sus frecuencias, dada por las ecs. (2.16)-(2.19), (2.22)-(2.23) y (2.27)-(2.28) de la ref. [I]. Usando los resultados obtenidos para la posición X^a y X^i como suma de modos de Fourier, calculamos el desarrollo en modos de los momentos canónicos conjugados correspondientes

$$P_i = \frac{1}{2\pi\alpha'}\,\left(\partial_\tau X^i - B\,\epsilon_{ij}\partial_\sigma X^j\right), \tag{2.8}$$
$$P_a = \frac{1}{2\pi\alpha'}\,\partial_\tau X^a\,. \tag{2.9}$$

2. Sustituimos las expresiones resultantes para los operadores posición y momento en la definición de la forma simpléctica

$$\Omega = \int_0^\pi d\sigma\,\left(dP_i \wedge dX^i + dP_a \wedge dX^a\right)$$

y calculamos ésta explícitamente. La solución se presenta en la ec. (3.7) de la publicación [I].

3. Al cuantizar canónicamente, las amplitudes de los modos de oscilación se convierten en operadores, cuyas reglas de conmutación vienen dadas por las componentes de la inversa de Ω. Las ecs. (3.15)-(3.18) de la ref. [I] reúnen los conmutadores no triviales. Finalmente se calcula los conmutadores entre los operadores X^i a partir de los de sus modos.

4. Para poder comparar nuestros resultados con la literatura tomamos la aproximación $m\kappa \ll 1$ y desarrollamos en serie de potencias de $m\kappa$.

Esta forma de proceder lleva al resultado para $[X^1, X^2]$ presentado en la sección 4 de la ref. [I], que también puede escribirse como

$$[X^1(\tau,\sigma), X^2(\tau,\sigma')] = i\left[\Theta_0(\sigma,\sigma') + (m\kappa)^2\,\Theta_2(\sigma,\sigma') + O(m^4\kappa^4)\right] .$$

La función $\Theta_0(\sigma,\sigma')$ es distinta de cero sólo en los extremos $\sigma = \sigma' = 0, \pi$ de la cuerda y reproduce los resultados de Chu y Ho [4] para espacio-tiempo de Minkowski. Para valores arbitrarios de σ y σ' la función $\Theta_2(\sigma,\sigma')$ es distinta de cero, lo que demuestra que no conmutatividad espacio-temporal no está limitada a la D-brana, tal y como se pretendía argumentar. En este sentido, cabe hablar de cuerdas no conmutativas.

Análogamente pueden calcularse las reglas de conmutación entre los operadores posición y momento, obteniéndose al orden en $m\kappa$ que estamos trabajando las canónicas

$$[X^i(\tau,\sigma), P_k(\tau,\sigma')] = i\,\delta^i_j\,\delta(\sigma-\sigma') + \mathcal{O}(m^4\kappa^4)\,.$$

En lo que se refiere al uso de cuantización canónica [34] en un background no minkowskiano con condiciones de contorno no triviales, como es nuestro caso, debemos resaltar dos aspectos. Primero, que la forma simpléctica Ω ha resultado ser independiente de τ, a pesar de la complicada dependencia de los modos de oscilación de sus frecuencias. Y segundo, que Ω es invertible, lo que indica un correcto tratamiento de los grados de libertad de la cuerda.

2.2. Construcción geométrica de D-branas en modelos WZW

En la ref. [II] se analiza la equivalencia entre (i) las condiciones de contorno (1.7) que se obtienen de la la acción de WZW para la cuerda abierta y que reproducimos aquí por comodidad

$$\left(\partial_i f^\mu G_{\mu\nu}\partial_\sigma X^\nu - \omega_{ij}\partial_\tau x^j\right)\Big|_{\partial\Sigma} = 0\,, \tag{2.10}$$

y (ii) las condiciones de pegado

$$J_+\Big|_{\sigma^+=\sigma^-} = F(g)J_-\Big|_{\sigma^+=\sigma^-} \tag{2.11}$$

que puedan escribirse en términos de las corrientes quirales $J_- = g^{-1}\partial_- g$ y $J_+ = -\,\partial_+ g\, g^{-1}$. Dado que $J_\pm$ toman valores en el álgebra de Lie $\mathfrak{g}$ del modelo, $F(g)$ es en cada punto g del grupo G un operador lineal que actúa sobre $\mathfrak{g}$. Aquí consideramos el caso general en que $F(g)$ no es constante sobre G.

Concretamente nos planteamos:

(a) Encontrar los requisitos que debe satisfacer $F(g)$ para que las condiciones de pegado sean equivalentes a las de contorno que definen una D-brana.

(b) Resolver el problema inverso, o lo que es lo mismo, encontrar las condiciones de contorno de una D-brana que pueden escribirse como condiciones de pegado.

Para contestar estas cuestiones, es conveniente plantearlas en términos de coordenadas locales $\{X^\mu\}$ sobre la variedad G. Para ello se define

$$\mathcal{F}(X) = -\,\bar{e}^{\,-1} F(g)\, e\,, \tag{2.12}$$

donde $e^A{}_\mu$ y $\bar{e}^A{}_\mu$ son respectivamente los vielbeins invariantes por la izquierda y por la derecha que relacionan las coordenadas locales con los generadores $\{T_A\}$ del álgebra de Lie $\mathfrak{g}$,

$$g^{-1}\, dg = T_A\, e^A{}_\mu\, dX^\mu \qquad dg\, g^{-1} = T_A\, \bar{e}^A{}_\mu\, dX^\mu\,.$$

Con ello las condiciones de pegado (2.11) toman la forma

$$(\mathcal{F} - 1)\, \partial_\tau X\big|_{\partial\Sigma} = (\mathcal{F} + 1)\, \partial_\sigma X\big|_{\partial\Sigma}\,. \tag{2.13}$$

A su vez, las condiciones de contorno (2.10) para una D-brana $\mathcal{N}$ pueden escribirse más sucintamente como

$$G\Big(z\,,\, \partial_\sigma X\big|_{\partial\Sigma}\Big) = \omega\Big(z\,,\, \partial_\tau X\big|_{\partial\Sigma}\Big) \quad \text{para todo} \quad z \in T_g\mathcal{N}\,, \tag{2.14}$$

donde recordemos que G es la métrica sobre el espacio-tiempo G, ω es la 2-forma $B - dF$, y $T_g\mathcal{N}$ es el espacio-tangente a la D-brana en un punto g de la misma.

2.2.1. El problema directo

Para responder al problema **(a)** damos los siguientes pasos:

1. El movimiento de un extremo de la cuerda que se encuentra en un punto g de la D-brana viene descrito por un vector $t^\mu(g) := \partial_\tau X^\mu\big|_{\partial\Sigma}$ tangente a la D-brana. Para que este movimiento sea compatible con las condiciones de pegado debe existir un vector $u = \partial_\sigma X\big|_{\partial\Sigma}$ tal que se satisfaga (2.13). Llamemos Π_g al conjunto de todos los posibles movimientos para los extremos de la cuerda en un punto g, es decir

$$\Pi_g = \Big\{ \, t \in T_g(G) \colon \;\; [\mathcal{F}(g) - 1]t = [\mathcal{F}(g) + 1]u \;\; \text{para algún} \;\; u \in T_g(G) \, \Big\} \, .$$

En la sección 3 de la ref. [II] se demuestra que Π_g es la imagen del operador $\mathcal{F} + 1$,

$$\Pi_g = \mathrm{Im}\,(\mathcal{F} + 1) \, . \tag{2.15}$$

2. Para que una condición de pegado sea equivalente a una condición de contorno, la D-brana $\mathcal{N}$ debe incluir todos los posibles movimientos Π_g de los extremos de la cuerda. En términos matemáticos esto significa que debe existir un subvariedad $\mathcal{M}$ de G que contenga a $\mathcal{N}$ tal que

$$\Pi^{\mathcal{M}} = \{ (g, \Pi_g) : \;\; g \in \mathcal{M} \, \} \tag{2.16}$$

sea una distribución integrable. Para que $\Pi^{\mathcal{M}}$ sea distribución el rango de Π_g debe ser constante para todo g en $\mathcal{M}$. A su vez, la integrabilidad de $\Pi^{\mathcal{M}}$ requiere, según el teorema de Frobenius, que la distribución sea involutiva, o lo que es lo mismo

$$[\Pi_g, \Pi_g] \subset \Pi_g \, . \tag{2.17}$$

En la subsección 3.2 de la ref. [II], se demuestra que en términos de $F(g)$ la condición (2.17) implica que para todo U, V en $\mathfrak{g}$ existe W en $\mathfrak{g}$ tal que

$$[\, F(g)\, U g - g\, U \, , \, F(g)\, V\, g - gV \,] = F(g)\, W g - gW \, . \tag{2.18}$$

3. Supongamos que la condición de involutividad (2.17) se satisface. Entonces se tiene una subvariedad $\mathcal{N}$ sobre la que se mueven los extremos de la cuerda, con $T_g\mathcal{N} = \mathrm{Im}\,(\mathcal{F} + 1)$. Esto no significa que $\mathcal{N}$ sea una D-brana. Para ello, es además necesario que las condiciones de pegado (2.13) puedan escribirse como condiciones de contorno (2.14) para un cierto ω. En la subsección 3.3 de la ref. [II] se demuestra que éste es el caso si y sólo si

$$G(\mathcal{F}u \, , \mathcal{F}v) = G(u \, , v) \quad \text{para todo} \quad u, v \in T_g\mathrm{G} \, , \tag{2.19}$$

y que entonces la 2-forma ω está definida por su acción sobre campos vectoriales $t_A = FT_A g - gT_A = t^\mu{}_A \partial_\mu$ de la siguiente forma

$$\omega(t_A, t_B) = \Omega\, (\, \mathrm{Ad}_{g^{-1}} FT_A - T_A \, , \, \mathrm{Ad}_{g^{-1}} FT_B + T_B \,) \, . \tag{2.20}$$

Nótese que la condición (2.19) establece que $F(g)$ es una isometría de la métrica Ω sobre $\mathfrak{g}$.

4. No toda isometría $F(g)$ para la que se satisface involutividad es aceptable como condición de pegado. Debemos restringirnos a aquellas que determinan a través de la ecuación (2.20) 2-formas ω tales que

$$d\omega = H\big|_{\mathcal{N}} \, , \tag{2.21}$$

donde H es la 3-forma del modelo de WZW dada en ec. (1.21).

A modo de resumen, para que una condición de pegado describa una D-brana es necesario que el operador $F(g)$ que “pega” las corrientes quirales en la frontera satisfaga las condiciones (2.18), (2.19) y (2.21).

En el caso particular de isometrías $F = R$, con R un automorfismo de $\mathfrak{g}$ constante, no es difícil demostrar —ver subsección 5.1 de la ref. [II]— que las condiciones de pegado definen D-branas dadas por las llamadas clases de conjugación R-twined $\mathcal{C}(R, g_0) = \{e^{RV} g_0\, e^{-V}\colon\ V \in \mathfrak{g}\}$. Nuestra demostración no impone ninguna restricción sobre la signatura de la métrica en la D-brana, generalizando así resultados previos [7, 8, 32]. Por contra, en la subsección 5.2 de la ref. [II] se demuestra, si bien en este caso es más complicado, que para álgebras $\mathfrak{g}$ semisimples e isometrías de la forma $F = -R$ las condiciones de pegado no dan lugar a ninguna D-brana, lo que resuelve ciertas disputas al respecto en la literatura [32].

2.2.2. El problema inverso

En el problema inverso planteado en **(b)** la condición de contorno se supone conocida y quiere escribirse como una condición de pegado. Esto significa determinar $F(g)$ a partir de la 2-forma ω. Para ello procedemos de la siguiente manera:

1. Definimos una aplicación $\mathcal{K}\colon T_g\mathcal{N} \to T_g\mathrm{G}/(T_g\mathcal{N}^{\perp})$ cuya acción sobre w en $T_g\mathcal{N}$ está dada por

$$G(z\,, \mathcal{K}w) = \frac{1}{2}\left[\,G(z,w) - \omega(z,w)\,\right] \quad \text{para todo} \quad z \in T_g\mathcal{N}\,. \tag{2.22}$$

En la sección 4 de la ref. [II] se demuestra que la aplicación $\mathcal{K}$ es biyectiva si y sólo si

$$det\,(G|_{\mathcal{N}} - \omega) \neq 0\,. \tag{2.23}$$

2. En lo que sigue suponemos que se da la condición (2.23). Entonces la aplicación inversa $\mathcal{K}^{-1}\colon T_g\mathrm{G}/(T_g\mathcal{N}^{\perp}) \to T_g\mathcal{N}$ existe y es posible definir en términos de ella $(\mathcal{F}+1)\colon T_g\mathrm{G} \to T_g\mathcal{N}$ como $(\mathcal{F}+1)v = \mathcal{K}^{-1}(v + T_g\mathcal{N}^{\perp})$ con v en $T_g\mathrm{G}$. Es fácil ahora comprobar que $T_g\mathcal{N} = \mathrm{Im}\,(\mathcal{F}+1)$ y que

$$\omega(\,(\mathcal{F}+1)\,u\,,\,(\mathcal{F}+1)\,v\,) = G(\,(\mathcal{F}+1)\,u\,,\,(\mathcal{F}-1)\,v\,) \tag{2.24}$$

para todo u, v en T_gG

3. Como $\partial_\tau X\big|_{\partial\Sigma}$ pertenece a $T_g\mathcal{N}$, existe un v en $T_g\mathrm{G}$ tal que $\partial_\tau X\big|_{\partial\Sigma} = (\mathcal{F}+1)v$. Combinando esta observación con la ec. (2.24), en la sección 4 de la publicación [II] se demuestra que las ecuaciones de contorno (2.14) siempre pueden escribirse como condiciones de pegado.

Así pues, una condición de contorno puede escribirse como una de pegado si y sólo si se satisface la ecuación (2.23). Esta condición siempre se cumple si la métrica sobre la D-brana tiene signatura euclídea, pero puede dejar fuera del análisis mediante condiciones de pegado D-branas de signatura lorentziana o degenerada.

2.3. D-branas con signatura Lorentziana en el modelo de Nappi-Witten

En la publicación [III], que resumimos a continuación, se aplican los resultados de la sección anterior al modelo de Nappi-Witten [9].

Se trata de un modelo de WZW que describe un background exacto para la cuerda en cuatro dimensiones. El álgebra del modelo, conocida como de Nappi-Witten, es no semisimple, tiene dimensión cuatro y generadores $\{P_1, P_2, J, T\}$ que satisfacen las relaciones de conmutación

$$[J, P_M] = \epsilon_{MN} P_N , \qquad [P_M, P_N] = \epsilon_{MN} T , \qquad [T, P_M] = [T, J] = 0 ,$$

con $M, N = 1, 2$. Al no ser semisimple, la forma de Killing del álgebra es degenerada. No obstante, el álgebra admite una métrica invariante

$$\Omega = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{2.25}$$

donde k y b son parámetros reales arbitrarios. Para estas reglas de conmutación y esta Ω es fácil encontrar de forma explícita los automorfismos R del álgebra y las isometrías F de Ω, ver sección 3 de la ref. [III]. Conviene destacar que no toda isometría es un automorfismo, ni todo automorfismo una isometría. Llamaremos R_+ a los automorfismos inner y R_- a los outer. Particularmente interesantes para nosotros son los automorfismos que además son isometrías, y que denotaremos $R^{\Omega}_{\pm}$.

El álgebra de Nappi-Witten define mediante exponenciación el espacio-tiempo del modelo sigma como un grupo de Lie, cuyos elementos pueden parametrizarse mediante coordenadas $\{x_1, x_2, u, v\}$,

$$g(x_M, u, v) = e^{x_M P_M} \, e^{uJ} \, e^{vT} \, .$$

En estas coordenadas la métrica G y la 3-forma H del modelo se escriben

$$ds^2 = dx_1^2 + dx_2^2 + (x_2 \, dx_1 - x_1 \, dx_2) \, du + 2 \, du \, dv \tag{2.26}$$

$$H = dx_1 \wedge dx_2 \wedge du \, . \tag{2.27}$$

Sentados estos preliminares, pasamos a buscar D-branas en el modelo de Nappi-Witten descritas por condiciones de pegado $J_+ = F(g) J_-$. Consideramos el caso general de isometrías $F(g)$ no constantes. Recordemos que, según los resultados de la sección 2.2, para que $F(g)$ operador lineal que actúa sobre el álgebra de Lie del modelo defina una D-brana se tienen que dar las siguientes tres condiciones: (i) que $F(g)$ sea isometría de Ω, (ii) que genere una distribución integrable de campos vectoriales

$$t_a(x) = FT_a g - gT_a = t^{\mu}{}_a(x)\,\partial_\mu\,, \tag{2.28}$$

y (iii) que se satisfaga $d\omega = H$. Puesto que se han calculado de forma explícita las isometrías, basta comprobar para cuáles de ellas se satisfacen las condiciones (ii) y (iii).

Para isometrías constantes de la forma $F = R^{\Omega}_{\pm}$ las D-branas son las clases de conjugación $R^{\Omega}_{\pm}$-twined [35, 36]. Sin embargo, si $F = -R^{\Omega}_{\pm}$, los campos vectoriales (2.28) no definen una distribución por no tener rango constante, lo que implica, al contrario de lo propuesto en [35], que en este caso no existen D-branas.

Si la isometrías es no constante de la forma $F(g) = -R^{\Omega}_{+}(g)$, encontramos D3-branas y D1-branas con métricas lorentzianas. Cuando $F(g) = -R^{\Omega}_{-}(g)$ encontramos D2-branas con métrica degenerada y D0-branas con métrica euclídea. Ver ref. [III], secciones 4 y 6, para la forma explícita de las D-branas obtenidas y los cálculos que conducen a ellas.

También existen D-branas para isometrías que no sean de la forma $\pm R(g)$, con R automorfismo. En la sección 5 de la ref. [III] se presentan algunas D2-branas y D0-branas de este tipo con métrica lorentziana.

Capítulo 3

Conclusiones

En esta memoria se han estudiado algunos aspectos de D-branas y no conmutatividad en el ámbito de teoría de cuerdas. Las principales conclusiones obtenidas son las siguientes.

(i) Hemos probado que la cuantización de la cuerda abierta en un background no trivial con 2-forma B no nula da lugar a relaciones de conmutación no triviales entre los operadores de posición de la cuerda. La realización de no conmutatividad en los extremos de la cuerda para algunos backgrounds sencillos era bien conocida. Aquí hemos encontrado un background de tipo onda *pp*, descrito en la sección 2.1, en el que no conmutatividad espacio-temporal se extiende a toda la cuerda.

(ii) Es sabido que en los modelos de WZW resulta muy conveniente escribir las condiciones de contorno para los extremos de la cuerda que se mueven sobre una D-brana como condiciones de pegado. Estas últimas relacionan o "pegan" las corrientes quirales del modelo mediante un operador lineal F que actúa sobre su álgebra de Lie. Hemos demostrado que toda D-brana con métrica de signatura euclídea descrita por una condición de contorno admite una condición de pegado equivalente. Si la D-brana tiene signatura lorentziana o es degenerada, la condición necesaria y suficiente para que tal condición de pegado equivalente exista se da en la sección 2.2.2.

(iii) En la sección 2.2.1 hemos determinado un procedimiento que permite decidir si una condición de pegado para las corrientes de un modelo de WZW describe una D-brana. Este procedimiento valida con generalidad el caso particular considerado en la literatura en que el operador F es un automorfismo constante del álgebra de Lie del modelo.

(iv) En la sección 2.3 hemos aplicado la caracterización de D-branas mediante condiciones de pegado discutida en los puntos anteriores al modelo de Nappi-Witten. Esto nos ha permitido encontrar D-branas de todas las dimensiones, algunas con signatura lorentziana, otras con signatura euclídea y otras de métrica degenerada, ninguna de las cuales había sido considerada antes en la literatura.

(v) El procedimiento referido en el punto **(iii)**. pone de manifiesto que no toda condición de pegado describe una D-brana. En particular, permite dilucidar si condiciones de pegado con operador $F = -R$, donde R es automorfismo constante, que han suscitado interés en la literatura especializada, dan lugar a D-branas. Hemos probado que éste no es el caso, tanto para modelos de WZW con álgebra de Lie semisimple como para el modelo de Nappi-Witten.

Bibliografía

[I] G. Horcajada and F. Ruiz Ruiz, "Quantization of the open string on plane-wave limits of $dS_n \times S^n$ and non-commutativity outside branes", *Nucl. Phys.* **B799** (2008) 110–135, `arXiv:0711.2991`.

[II] G. Horcajada and F. Ruiz Ruiz, "Geometric construction of D-branes in WZW models", *JHEP* **09** (2011) 020, `arXiv:1104.4722`.

[III] R. Hernandez, G. Horcajada, and F. Ruiz Ruiz, "D-branes with Lorentzian signature in the Nappi-Witten model", *JHEP* **08** (2011) 047, `arXiv:1104.4730`.

[4] C.-S. Chu and P.-M. Ho, "Noncommutative open string and D-brane", *Nucl. Phys.* **B550** (1999) 151–168, `arXiv:hep-th/9812219`.

[5] N. Seiberg and E. Witten, "String theory and noncommutative geometry", *JHEP* **09** (1999) 032, `arXiv:hep-th/9908142`.

[6] E. Witten, "Nonabelian bosonization in two dimensions", *Comm. Math. Phys.* **92** (1984), no. 4, 455–472.

[7] A. Yu. Alekseev and V. Schomerus, "D-branes in the WZW model", *Phys. Rev.* **D60** (1999) 061901, `arXiv:hep-th/9812193`.

[8] S. Stanciu, "D-branes in group manifolds", *JHEP* **01** (2000) 025, `arXiv:hep-th/9909163`.

[9] C. R. Nappi and E. Witten, "A WZW model based on a nonsemisimple group", *Phys. Rev. Lett.* **71** (1993) 3751–3753, `arXiv:hep-th/9310112`.

[10] S. Doplicher, K. Fredenhagen, and J. E. Roberts, "The Quantum structure of space-time at the Planck scale and quantum fields", *Commun. Math. Phys.* **172** (1995) 187–220, `arXiv:hep-th/0303037`.

[11] H. S. Snyder, "Quantized space-time", *Phys. Rev.* **71** Jan (1947) 38–41.

[12] C. Zachos, D. Fairlie, and T. Curtright, "Quantum mechanics in phase space: An overview with selected papers", World Scientific, 2005.

[13] J. Várilly, "An introduction to noncommutative geometry", European Mathematical Society, 2006.

[14] H. Groenewold, "On the principles of elementary quantum mechanics", *Physica* **12** (1946), no. 7, 405 – 460.

[15] M. Kontsevich, "Deformation quantization of poisson manifolds", *Letters in Mathematical Physics* **66** (2003), no. 3, 157–216, arXiv:q-alg/9709040.

[16] H. Weyl, "Quantenmechanik und gruppentheorie", *Z. Phys.* **46** (1927) 1–33.

[17] E. Wigner, "On the quantum correction for thermodynamic equilibrium", *Phys. Rev.* **40** Jun (1932) 749–759.

[18] M. R. Douglas and N. A. Nekrasov, "Noncommutative field theory", *Rev. Mod. Phys.* **73** (2001) 977–1029, arXiv:hep-th/0106048.

[19] R. J. Szabo, "Symmetry, gravity and noncommutativity", *Class. Quant. Grav.* **23** (2006) R199–R242, arXiv:hep-th/0606233.

[20] D. Tong, "String Theory", arXiv:0908.0333.

[21] C.-S. Chu and P.-M. Ho, "Noncommutative D-brane and open string in pp wave background with B field", *Nucl. Phys.* **B636** (2002) 141–158, arXiv:hep-th/0203186.

[22] L. Dolan and C. R. Nappi, "Noncommutativity in a time dependent background", *Phys. Lett.* **B551** (2003) 369–377, arXiv:hep-th/0210030.

[23] P.-M. Ho and Y.-T. Yeh, "Noncommutative D-brane in nonconstant NS NS B field background", *Phys. Rev. Lett.* **85** (2000) 5523–5526, arXiv:hep-th/0005159.

[24] L. Davidovic and B. Sazdovic, "Noncommutativity in weakly curved background by canonical methods", *Phys. Rev.* **D83** (2011) 066014, arXiv:1004.4483.

[25] V. Schomerus, "Lectures on branes in curved backgrounds", *Class. Quant. Grav.* **19** (2002) 5781–5847, arXiv:hep-th/0209241.

[26] C. Klimcik and P. Severa, "Open strings and D-branes in WZNW model", *Nucl. Phys.* **B488** (1997) 653–676, arXiv:hep-th/9609112.

[27] A. Giveon, D. Kutasov, and N. Seiberg, "Comments on string theory on AdS(3)", *Adv. Theor. Math. Phys.* **2** (1998) 733–780, `arXiv:hep-th/9806194`.

[28] I. Pesando, "The GS type IIB superstring action on AdS(3) x S(3) x T**4", *JHEP* **02** (1999) 007, `arXiv:hep-th/9809145`.

[29] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in AdS(5) x S**5 background", *Nucl. Phys.* **B533** (1998) 109–126, `arXiv:hep-th/9805028`.

[30] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov, and B. Zwiebach, "Superstring theory on AdS(2) x S**2 as a coset supermanifold", *Nucl. Phys.* **B567** (2000) 61–86, `arXiv:hep-th/9907200`.

[31] A. Yu. Alekseev, A. Recknagel, and V. Schomerus, "Noncommutative world volume geometries: Branes on SU(2) and fuzzy spheres", *JHEP* **09** (1999) 023, `arXiv:hep-th/9908040`.

[32] S. Stanciu, "D-branes in an AdS(3) background", *JHEP* **09** (1999) 028, `arXiv:hep-th/9901122`.

[33] G. Felder, J. Frohlich, J. Fuchs, and C. Schweigert, "The Geometry of WZW branes", *J. Geom. Phys.* **34** (2000) 162–190, `arXiv:hep-th/9909030`.

[34] L. D. Faddeev and R. Jackiw, "Hamiltonian Reduction of Unconstrained and Constrained Systems", *Phys. Rev. Lett.* **60** (1988) 1692–1694.

[35] S. Stanciu and A. A. Tseytlin, "D-branes in curved space-time: Nappi-Witten background", *JHEP* **06** (1998) 010, `arXiv:hep-th/9805006`.

[36] J. M. Figueroa-O'Farrill and S. Stanciu, "More D-branes in the Nappi-Witten background", *JHEP* **01** (2000) 024, `arXiv:hep-th/9909164`.

PUBLICACIONES

[I] G. Horcajada and F. Ruiz Ruiz, "Quantization of the open string on plane-wave limits of $dS_n \times S^n$ and non-commutativity outside branes", *Nucl. Phys.* **B799** (2008) 110–135, `arXiv:0711.2991`.

[II] G. Horcajada and F. Ruiz Ruiz, "Geometric construction of D-branes in WZW models", *JHEP* **09** (2011) 020, `arXiv:1104.4722`.

[III] R. Hernandez, G. Horcajada, and F. Ruiz Ruiz, "D-branes with Lorentzian signature in the Nappi-Witten model", *JHEP* **08** (2011) 047, `arXiv:1104.4730`.

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Quantization of the open string on plane-wave limits of $dS_n \times S^n$ and non-commutativity outside branes

G. Horcajada, F. Ruiz Ruiz *

Departamento de Física Teórica I, Universidad Complutense de Madrid, 28040 Madrid, Spain

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Abstract

The open string on the plane-wave limit of $dS_n \times S^n$ with constant B_2 and dilaton background fields is canonically quantized. This entails solving the classical equations of motion for the string, computing the symplectic form, and defining from its inverse the canonical commutation relations. Canonical quantization is proved to be perfectly suited for this task, since the symplectic form is unambiguously defined and non-singular. The string position and the string momentum operators are shown to satisfy equal-time canonical commutation relations. Noticeably the string position operators define non-commutative spaces for all values of the string world-sheet parameter σ, thus extending non-commutativity outside the branes on which the string endpoints may be assumed to move. The Minkowski space–time limit is smooth and reproduces the results in the literature, in particular non-commutativity gets confined to the endpoints.

1. Introduction

Solutions to the Einstein equations in general relativity have been known for a long time to have plane waves as limits [1]. These limits, known as Penrose limits, give a plane wave space–time approximation for the full space–time along a null geodesic. This observation led in the sixties and seventies to a detailed study of the geometric properties of plane-wave metrics and of matter fields defined on them [2]. Already within string theory, it soon became clear that higher-dimensional plane waves give exact solutions to string theory, provided the Kalb–Ramond and dilaton fields satisfy certain conditions [3,4]. The generalization of the Penrose

* Corresponding author.
E-mail address: ferruiz@fis.ucm.es (F. Ruiz Ruiz).

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limiting procedure relating higher-dimensional plane waves with more complicated solutions to string theory [5] further triggered the interest in such space–times.

By now, there is a very extensive literature on plane waves in string theory. Motivated by the fact that $AdS_4 \times S^7$ and $AdS_7 \times S^4$ are solutions to M-theory and $AdS_5 \times S^5$ is a solution of IIB supergravity, and by the AdS/CFT correspondence, special attention has been given the Penrose limit [6–8]

$$AdS_k \times S^n \quad pp\text{-limit:} \quad ds^2 = -dx^+ dx^- - m^2 \mathbf{x}_{k+n-2}^2 (dx^+)^2 + d\mathbf{x}_{k+n-2}^2 \tag{1.1}$$

of $AdS_k \times S^n$ spaces. Two milestones in this regard are (i) the quantization [9] of the R–R sector of the closed superstring on this background for $k = n = 5$, and (ii) the derivation of its spectrum from that of $U(N)$ $\mathcal{N} = 4$ super-Yang–Mills theory [10]. The interest has extended also to type IIB superstring models in 6 dimensions [11] describing generalizations of the Nappi–Witten model. As a matter of fact, the Nappi–Witten model [12] is itself the Penrose limit of $AdS_2 \times S^2$. There has been as well interest on strings on 4-dimensional homogenous plane-wave backgrounds [13]. These have the form (1.1) with m^2 replaced by a function $C|x^+|^{-2}$ and, for different values of the constant C, occur as the Penrose limit of FRW metrics, near horizon regions of Dp-brane backgrounds and fundamental strings backgrounds [8,14].

In this paper we consider quantization of the open string on the Penrose limit of $dS_n \times S^n$ with non-zero constant 2-form B_2. To date, no background p-forms have been found that support $dS_n \times S^n$ as a solution to IIB supergravity. Yet there are indications that de Sitter space may occur in type IIA theories [15]. In any case, the Penrose limit of $dS_n \times S^n$ is an exact solution of string theory in the critical dimension [4]. There are other motivations for taking de Sitter space–time: its “apparent” simplicity when it comes to quantum gravity [16], the dS/CFT correspondence [17] and the fact that the non-existence of a positive conserved energy indicates that there cannot be unbroken supersymmetry, so it seems a good starting point to go down in the number of supersymmetries. The motivation for taking $B_2 \neq 0$ comes from an interest in understanding non-commutativity in relation with gravity. As is well known, string theory gives explicit realizations of non-commutative spaces. The simplest example is provided by an open string in Minkowski space–time with endpoints moving on a D-brane on which a magnetic field is defined: upon quantization, the string position operators generate a non-commutative space along the brane [18–20]. Since non-commutativity is postulated as a candidate to reconcile quantum mechanics with general relativity [21], and the low energy limit of string theory includes general relativity, it seems natural to explore the non-commutativity/gravity connection within string theory. One way to push forward this approach is to examine non-commutativity for plane wave backgrounds. As a matter of fact, this program has already started for the open string on plane-wave limits of $AdS_n \times S^n$. In 10 dimensions with a constant non-zero B_2 in Ref. [22], and in 4 dimensions with a Nappi–Witten 2-form in [23]. In both instances, the string endpoints define non-commutative spaces. Here we investigate non-commutativity for the Penrose limit of $dS_n \times S^n$.

More precisely, we will quantize the open string interacting through a plane-wave metric

$$ds^2 = -dx^+ dx^- + m^2 \big[(x^1)^2 - (x^2)^2\big](dx^+)^2 + \sum_{i=1}^{2} (dx^i)^2 + \sum_{a=3}^{D-2} (dx^a)^2 \tag{1.2}$$

and constant antisymmetric and dilaton fields

$$B_{ij} = \epsilon_{ij} B, \qquad B_{ab} = 0, \qquad \Phi = \Phi_0. \tag{1.3}$$

It will come out that the string position operators $X^1(\tau,\sigma)$ and $X^2(\tau,\sigma')$ do not commute for arbitrary σ and σ'. This is in contrast with the results available so far for open strings on $AdS_n \times S^n$ plane-wave limits supported by a non-zero B_2 [19,22,23], for which non-commutativity is restricted to the brane manifold on which the string endpoints move. Our results are consistent with those in the literature for Minkowski space–time [19], since the latter are recovered in the limit $m \to 0$; in particular non-commutativity gets confined to the string endpoints.

We will work in light-cone and conformal gauges. The paper is organized as follows. In Section 2 we derive the equations of motion for the classical string and solve them. The solution turns out to be an infinite sum over modes, with a highly non-trivial dependence on the parameter m. As compared to the open string in Minkowski space–time, two important differences are encountered. The first one is that the string has a finite number of non-oscillating degrees of freedom associated to modes exponentially growing and decaying in τ. The second one is that the string total momentum is not an independent degree of freedom but receives contributions from all the modes. In Section 3 the string is canonically quantized. This is done by calculating the symplectic form and then using it to find the commutation relations for the operators associated to all the string modes. The symplectic form is unambiguous and non-singular, not being necessary to provide additional constraints or to modify its definition so as to fix the commutators. As a check it is shown that the string momentum and the string position operators satisfy equal-time canonical commutation relations. Section 4 shows that the string position operators X^1 and X^2 do not commute for arbitrary values of σ and σ', thus defining non-commutative waves fronts. In Section 5 we find the eigenstates and the spectrum of the Hamiltonian. Section 6 contains our conclusions. We have included Appendices A and B with some of the details of the calculations of Sections 2 and 4.

2. The classical string

Due to its length, this section is divided into five parts. In the first one, we study the background metric (1.2). Section 2.2 contains the derivation of the equations of motion and of the boundary conditions for the classical open string in the background (1.2)–(1.3). The equations of motion are solved in Section 2.3, where expressions for the string coordinates as sums over modes ready to be quantized are found. Section 2.4 presents a brief discussion of the string center of mass coordinates and the string total momentum. Finally, in Section 2.5 we discuss the case $m^2\kappa^2 \ll 1$, with $X^+ = \kappa\tau$ the light-cone gauge condition.

2.1. The background as the Penrose limit of $dS_n \times S^n$

The metric (1.2) is the Penrose limit of $dS_2 \times S^2 \times \mathrm{E}^{D-4}$, with E^{D-4} Euclidean space in $D-4$ dimensions. Although well known, let us very briefly check this point. Consider k-dimensional de Sitter space–time times an n-sphere, $dS_k \times S^n$, both of radius ℓ. Its metric can be written as

$$ds^2 = \ell^2\left[-\left(1-\rho^2\right)dt^2 + \frac{d\rho^2}{1-\rho^2} + \rho^2\, d\Omega_{k-2}^2 + \left(1-r^2\right)d\chi^2 + \frac{dr^2}{1-r^2} + r^2\, d\Omega_{n-2}'^2\right], \tag{2.1}$$

where $d\Omega_{k-2}^2$ and $d\Omega_{n-2}'^2$ are the round metrics on the unit $(k-2)$ and $(n-2)$-spheres. Consider now, as in the anti-de Sitter case [10], the trajectory along χ in the vicinity of $\rho = r = 0$. Making the changes $u^{\pm} = t \pm \chi$, rescaling

$$u^+ = x^+, \quad u^- = \frac{x^-}{\ell^2}, \quad \rho = \frac{\bar{\rho}}{\ell}, \quad r = \frac{\bar{r}}{\ell} \quad \text{with } \ell \to \infty, \tag{2.2}$$

and introducing a mass scale $x^+ \to 2mx^+$, $x^- \to x^-/2m$, one arrives at

$$dS_k \times S^n \quad pp\text{-limit:} \quad ds^2_{pp} = -dx^+\,dx^- + m^2\big(\mathbf{x}^2_{k-1} - \mathbf{y}^2_{n-1}\big)\big(dx^+\big)^2 + d\mathbf{x}^2_{k-1} + d\mathbf{y}^2_{n-1}. \tag{2.3}$$

Here Cartesian coordinates $\mathbf{x}_{k-1} = (\bar{\rho}, \Omega_{k-2})$ and $\mathbf{y}_{n-1} = (\bar{r}, \Omega_{n-2})$ have been introduced. Backgrounds

$$ds^2 = ds^2_{pp} + ds^2\left(\mathrm{E}^{D-n-k}\right), \qquad H_3 = dB_2 = A_{ij}\big(x^+\big)\,dx^+ \wedge dx^i \wedge dy^j$$

are solutions to all orders in α' for the bosonic/fermionic string in $D = 26/10$ provided A_{ij} satisfies the condition [4]

$$4m^2(n-k) = A_{ij}A^{ij}.$$

H_3 vanishes for $k = n$, in which case one may take $B_2 = B_{ij}\,dx^i \wedge dy^j$, with B_{ij} constant. The metric (1.2) is recovered for $k = n = 2$ and is non-singular, meaning it is geodesically complete. The results in this paper are trivially extended to the case $k = n = 5$.

It is important to note the positive sign in front of $\mathbf{x}^2_{k-1}$ in the metric coefficient g_{++} in Eq. (2.3). This has its origin in the fact that we have started with de Sitter space–time, rather than anti-de Sitter, and implies that the metric (2.3) does not admit a conserved positive energy. To understand this we recall that in de Sitter space there is no positive conserved energy since there is no generator of its isometry group, $SO(1,d)$, which is timelike everywhere. In the coordinates (2.1), the generator $\partial/\partial t$ is timelike for $\rho < 1$, but vanishes at the event horizon $\rho = 1$. Hence, $\partial/\partial t$ and its associated Hamiltonian can only be used to define time evolution in the region $0 \leqslant \rho \leqslant 1$ within the event horizon. Upon forming $dS_n \times S^n$ and taking the Penrose limit, this implies that for the metric (2.3) the sign of the energy depends on the sign of $\mathbf{x}^2_{k-1} - \mathbf{y}^2_{n-1}$. This is a property of the background considered.

2.2. *Classical action, field equations and momenta*

Our starting point is the bosonic part of the classical action

$$S = \frac{1}{4\pi\alpha'}\int d\tau\,d\sigma\,\big(\sqrt{-\gamma}\gamma^{rs}G_{\mu\nu}\partial_r X^\mu \partial_s X^\nu + \epsilon^{rs}B_{\mu\nu}\partial_r X^\mu \partial_s X^\nu + \alpha'\sqrt{-\gamma}R\Phi\big)$$

for the open string on the D-dimensional background $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, $\Phi(X)$ in Eqs. (1.2)–(1.3). Greek letters $\mu, \nu, \ldots$ denote space–time indices, while lower case letters $r, s, \ldots$ from the end of the Roman alphabet denote world-sheet indices. Here γ_{rs} is the metric on the string world-sheet, R its scalar curvature and ϵ^{rs} is defined by $\epsilon^{01} = 1$. As usual the world-sheet coordinates τ and σ take values on the intervals $-\infty < \tau < \infty$ and $0 \leqslant \sigma \leqslant \pi$. We are using units in which string coordinates have dimensions of length and τ, σ are dimensionless. From now on we will use capital case letters X's for the string coordinates.

If wished, the string endpoints may be assumed to lie on a Dp-brane on which a magnetic field F_{ij} lives.[1] This amounts to adding to the action a term

$$\delta S = \frac{1}{2\pi\alpha'} \int d\tau\, A_i \partial_\tau X^i \Big|_{\sigma=0}^{\sigma=\pi},$$

with $A_i(X)$ the $U(1)$ gauge field on the brane. If this term is included in the action, the analysis in this paper goes through with the only difference that the field B_{ij} must be replaced by the Born–Infeld field strength $\mathcal{B}_{ij} = B_{ij} - F_{ij}$, where F_{ij} is the $U(1)$ field strength on the brane.

The string action has three world-sheet symmetries. We will fix one of them by working in light-cone gauge [24]

$$X^+ = \kappa\tau,$$

with κ a parameter with dimensions of length. The other two will be fixed by choosing conformal gauge

$$h^{rs} = \sqrt{-\gamma}\gamma^{rs} = \mathrm{diag}(-1, +1).$$

In this gauge, the classical action becomes

$$S = \int d\tau\, L,$$

where the Lagrangian L is given by

$$\begin{aligned} L = p_- \partial_\tau x^- - \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \big\{ & m^2\kappa^2 \big[\big(X^1\big)^2 - \big(X^2\big)^2 \big] + \big(\partial_\tau X^i\big)^2 + \big(\partial_\tau X^a\big)^2 \\ & - \big(\partial_\sigma X^i\big)^2 - \big(\partial_\sigma X^a\big)^2 - 2B\big[\partial_\tau X^1 \partial_\sigma X^2 - \partial_\sigma X^1 \partial_\tau X^2\big]\big\}, \end{aligned}$$

with

$$p_- = -\frac{\kappa}{4\alpha'}$$

the momentum conjugate to $x^-(\tau)$, defined [25] as the average over σ at a given τ of $X^-(\tau, \sigma)$

$$x^-(\tau) = \frac{1}{\pi} \int_0^\pi d\sigma\, X^-(\tau, \sigma).$$

Here we have reserved the subscript i for the 1 and 2 directions, while a runs from 3 to $D-2$, a convention that we will follow from now on.

The field equations and boundary conditions are obtained by varying the action with respect to X^i and X^a. They take the form

$$\Box X^1 + m^2\kappa^2 X^1 = 0, \tag{2.4}$$

$$\Box X^2 - m^2\kappa^2 X^2 = 0, \tag{2.5}$$

$$\Box X^a = 0, \tag{2.6}$$

[1] p is 1 for $k = n = 2$ in (2.3) and 4 for $k = n = 5$.

with $\square = -\partial_\tau^2 + \partial_\sigma^2$ the 2-dimensional d'Alambertian, and

$$\partial_\sigma X^1 - B\partial_\tau X^2\big|_{\sigma=0,\pi} = 0, \tag{2.7}$$

$$\partial_\sigma X^2 + B\partial_\tau X^1\big|_{\sigma=0,\pi} = 0, \tag{2.8}$$

$$\partial_\sigma X^a\big|_{\sigma=0,\pi} = 0. \tag{2.9}$$

To quantize the theory we will need the momenta. In our case, these are given by

$$p_- = -\frac{\kappa}{4\alpha'}, \tag{2.10}$$

$$P_i = \frac{1}{2\pi\alpha'}\big(\partial_\tau X^i - B\epsilon_{ij}\partial_\sigma X^j\big), \tag{2.11}$$

$$P_a = \frac{1}{2\pi\alpha'}\partial_\tau X^a. \tag{2.12}$$

In terms of them, the Lagrangian L can be written as

$$L = -p_-\partial_\tau x^- + \int_0^\pi d\sigma\,\big[-\big(P_i\partial_\tau X^i + P_a\partial_\tau X^a\big) + \mathcal{H}\big],$$

where the Hamiltonian density $\mathcal{H}$ has the form

$$\begin{aligned} 4\pi\alpha'\mathcal{H} = {} & \big(2\pi\alpha' P_i + B\epsilon_{ij}\partial_\sigma X^j\big)^2 + (2\pi\alpha' P_a)^2 + \big(\partial_\sigma X^i\big)^2 + \big(\partial_\sigma X^a\big)^2 \\ & - m^2\kappa^2\big[\big(X^1\big)^2 - \big(X^2\big)^2\big]. \end{aligned} \tag{2.13}$$

We note that $\mathcal{H}$ is not positive definite because of the negative sign in front of $(X^1)^2$. As explained in Section 2.1, this originates in the fact that in de Sitter space–time there is no positive conserved energy and implies that $\mathcal{H}$ can only be used to account for time evolution in the region where it is non-negative.

2.3. Solution to the classical equations of motion

The solution for X^a is the well-known mode sum

$$X^a(\tau,\sigma) = c_0^a + d_0^a\tau + \sum_{n\neq 0}^{\infty} i\frac{c_n^a}{n}\cos n\sigma\, e^{-in\tau}, \tag{2.14}$$

where c_n^a are complex constants of integration (mode amplitudes). Reality of X^a implies that c_0^a and d_0^a are real and that $(c_n^a)^\star = c_{-n}^a$.

The solution for X^1 and X^2 is more involved. To find it we use separation of variables $X^i(\tau,\sigma) = T_i(\tau)S_i(\sigma)$. This gives

$$\frac{\ddot{T}_1}{T_1} = \frac{S_1''}{S_1} - m^2\kappa^2 = -\lambda_1^2,$$

$$\frac{\ddot{T}_2}{T_2} = \frac{S_1''}{S_1} + m^2\kappa^2 = -\lambda_2^2,$$

where the dot and prime indicate differentiation with respect to τ and σ respectively. The boundary conditions (2.7) and (2.8) imply that non-trivial solutions are only possible for $\lambda_1 = \lambda_2$. We therefore set $\lambda := \lambda_1 = \lambda_2$, introduce

$$\alpha = \sqrt{\lambda^2 - m^2\kappa^2}, \qquad \beta = \sqrt{\lambda^2 + m^2\kappa^2} \tag{2.15}$$

and distinguish several cases.

Case 1.: $\lambda = 0$. It is straightforward to see that non-trivial solutions only exist if $m\kappa$ is an integer. In particular, for $m\kappa$ an odd integer the solution reads

$$X^1_{\mathrm{o}}(\tau,\sigma) = \left[a_{\mathrm{o}} + b_{\mathrm{o}}\tau \sinh\left(\frac{m\kappa\pi}{2}\right)\right]\cos(m\kappa\sigma), \tag{2.16}$$

$$X^2_{\mathrm{o}}(\tau,\sigma) = \frac{B}{m\kappa} b_{\mathrm{o}} \cosh\left[m\kappa\left(\frac{\pi}{2} - \sigma\right)\right], \tag{2.17}$$

whereas for $m\kappa$ an even integer the solution takes the form

$$X^1_{\mathrm{e}}(\tau,\sigma) = \left[a_{\mathrm{e}} + b_{\mathrm{e}}\tau \cosh\left(\frac{m\kappa\pi}{2}\right)\right]\cos(m\kappa\sigma), \tag{2.18}$$

$$X^2_{\mathrm{e}}(\tau,\sigma) = \frac{B}{m\kappa} b_{\mathrm{e}} \sinh\left[m\kappa\left(\frac{\pi}{2} - \sigma\right)\right], \tag{2.19}$$

with a_{o}, b_{o} and a_{e}, b_{e} arbitrary constants of integration in every instance.

Case 2.: $\lambda^2 = \pm m^2\kappa^2$. This corresponds to either α or β zero and it is very easy to show that the only solution for X^1 and X^2 is the trivial one.

Case 3.: $\lambda^2 \neq 0, \pm m^2\kappa^2$. Solving then for T_i and S_i and imposing the boundary conditions, it follows that the eigenvalues λ must satisfy the equation

$$\left(\lambda^4 B^4 + \alpha^2\beta^2\right)\sin\alpha\pi\,\sin\beta\pi - 2\lambda^2 B^2\alpha\beta(\cos\alpha\pi\,\cos\beta\pi - 1) = 0. \tag{2.20}$$

Solutions to this equation may occur either because both its terms vanish or because none of them vanishes but their sum does. We therefore consider two subcases:

Subcase 3.1. Both terms in Eq. (2.20) vanish. Since α and β are non-zero, we must have

$$\sin\alpha\pi\,\sin\beta\pi = \cos\alpha\pi\,\cos\beta\pi - 1 = 0. \tag{2.21}$$

It is very easy to see then that the modes for X^1 and X^2 have the form

$$X^1_{(k,l)}(\tau,\sigma) = \frac{i}{\lambda}\left(a_{\lambda(k,l)}\frac{\alpha}{B}\cos\beta\sigma + b_{\lambda(k,l)}\sin\beta\sigma\right)e^{-i\lambda\tau}, \tag{2.22}$$

$$X^2_{(k,l)}(\tau,\sigma) = -\left(b_{\lambda(k,l)}\frac{\beta}{\lambda^2 B}\cos\alpha\sigma + a_{\lambda(k,l)}\sin\alpha\sigma\right)e^{-i\lambda\tau}, \tag{2.23}$$

where $a_{\lambda(k,l)}$ and $b_{\lambda(k,l)}$ are arbitrary constants of integration. It follows from Eqs. (2.21) that α and β must be integers and that their difference must be an even integer. Hence we write

$$\alpha = k, \qquad \beta = k + 2l, \tag{2.24}$$

with k and l arbitrary positive integers since $\beta \geqslant \alpha$ and α and β are defined as positive. With this, Eqs. (2.15) imply

$$m^2\kappa^2 = 2l(k+l) > 0, \qquad \lambda = \pm\sqrt{l^2 + (l+k)^2}. \tag{2.25}$$

The first one of these equations states that $m^2\kappa^2$ is an even integer. We thus conclude that for $m^2\kappa^2$ an even integer, there are as many modes of type (2.22)–(2.23) as pairs (k, l) of positive integers solving the equation $m^2\kappa^2 = 2l(k+l)$, which is clearly a finite number.

Subcase 3.2. We now look at solutions λ to Eq. (2.20) such that

$$\sin\alpha\pi\,\sin\beta\pi \neq 0. \tag{2.26}$$

In this case the modes for X^1 and X^2 read

$$X^1_\lambda(\tau,\sigma) = i\,\frac{c_\lambda}{\lambda B}\left(\alpha\cos\beta\sigma + \frac{K_\lambda}{\beta}\sin\beta\sigma\right)e^{-i\lambda\tau}, \tag{2.27}$$

$$X^2_\lambda(\tau,\sigma) = -\left(\frac{K_\lambda}{\lambda^2 B^2}\cos\alpha\sigma + \sin\alpha\sigma\right)e^{-i\lambda\tau}, \tag{2.28}$$

where c_λ is an arbitrary constant of integration and K_λ is given by

$$K_\lambda = \frac{\lambda^2 B^2\sin\alpha\pi + \alpha\beta\sin\beta\pi}{\cos\beta\pi - \cos\alpha\pi}. \tag{2.29}$$

Let us study the solutions of Eq. (2.20) under condition (2.26). Eq. (2.20) is an equation in λ^2, so its solutions come in pairs $(\lambda, -\lambda)$. Solutions with $\lambda^2 > 0$ provide real λ and oscillating degrees of freedom. By contrast, solutions with $\lambda^2 < 0$ correspond to imaginary λ, for which the τ-exponentials are real.

For $\lambda^2 > 0$ and sufficiently large, the left-hand side of Eq. (2.20) can be expanded in powers of $x = m^2\kappa^2/\lambda^2 \ll 1$, with result

$$\begin{aligned}&\left(1+B^2\right)^2\sin^2\lambda\pi - x^2\left[\frac{\lambda^2\pi^2}{4}\left(1-B^2\right)^2\right.\\ &\quad\left.+\left(1+B^2\right)\left(\frac{\lambda\pi}{8}\sin 2\lambda\pi + \sin^2\lambda\pi\right)\right] + \mathcal{O}\left(x^3\right) = 0.\end{aligned} \tag{2.30}$$

The left-hand side is, up to order x^3, negative for integer λ and positive for non-integer λ. It follows that the left-hand side of Eq. (2.20), to which (2.30) is an approximation for large λ, must change its sign twice in the vicinity of every integer $n \gg |m\kappa|$, thus proving the existence of two solutions around n. These solutions can be found as power series in $m\kappa/n$ by making for λ in the neighborhood of n the ansatz

$$\lambda_n = n\sum_{k=0}^{\infty} a_k\left(\frac{m\kappa}{n}\right)^k, \qquad a_0 = 1,$$

where the coefficient a_0 has been taken equal to 1 since $\lambda = n$ solves Eq. (2.20) to lowest order. Substituting this ansatz in Eq. (2.20) and solving order by order in $m\kappa/n$, one obtains two different sets of solutions for the coefficients $\{a_k\}$, leading to

$$\lambda_n^{(1,2)} = n\left[1 \pm \frac{m^2\kappa^2}{2n^2}\,\frac{1-B^2}{1+B^2} + \mathcal{O}\left(\frac{m^4\kappa^4}{n^4}\right)\right].$$

This confirms the existence of two real eigenvalues for every large enough integer n, thus showing that there are infinitely many real solutions with $|\lambda| > |m\kappa|$.

By contrast, there is only a finite number of real solutions with $|\lambda| < |m\kappa|$ and this number depends on the value of $m\kappa$. This can be seen as follows. Assume, without loss of generality, that $m\kappa$ is in between two consecutive integers, so that $N \leqslant |m\kappa| < N+1$, with N a positive integer. Denote by N' the integer such that $N' < \sqrt{2}|m\kappa| \leqslant N'+1$. Study the sign of the right-hand side of Eq. (2.20) as a function of β by dividing the interval for β in subintervals $[0,1], [1,2], \ldots, [N', N'+1]$. It is not then very difficult to prove that

(i) for N' even there are $2(N'-N+1)$ real solutions, and

(ii) for N' odd the number of solutions is also $2(N'-N+1)$ if

$$\frac{2\sqrt{2}}{|m\kappa|\pi B^2}\sin\bigl(\sqrt{2}|m\kappa|\pi\bigr)+\cos\bigl(\sqrt{2}|m\kappa|\pi\bigr)+1>0$$

and $2(N'-N)$ otherwise.

We come now to imaginary solutions. For $\lambda^2 < 0$, with $|\lambda| > m\kappa$, the left-hand side of Eq. (2.20) is positive definite and never vanishes. Hence imaginary solutions must have $|\lambda| < m\kappa$. Using similar arguments to those employed for real λ, it can be seen that in this case the number of solution for a given $m\kappa$ is $2(N+1)$, with N the integer such that $N < |m\kappa| \leqslant N+1$. We note that imaginary λ's occur due to the different signs with which $(X^1)^2$ and $(X^2)^2$ enter the background metric (1.2) and account for exponential growth of X^1 and X^2 at $\tau \to \pm\infty$. This is reminiscent of de Sitter space, for which space expands so fast that light rays cannot follow.

This analysis shows that there are infinitely many modes of type (2.27)–(2.28), of which a finite number of them have imaginary λ with $|\lambda| < |m\kappa|$, a finite number have real λ with $|\lambda| < |m\kappa|$, and infinitely many of them have real λ with $|\lambda| > |m\kappa|$. It is important to emphasize that this is so for arbitrary values of $m\kappa$, since Eq. (2.20) and condition (2.26) do not place any limitation on $m\kappa$. These modes can also be written in the following way, which will be very useful in some parts of this paper. The eigenvalue equation (2.20) can be recast as

$$F_+(\lambda)F_-(\lambda)=0,$$

with $F_\pm(\lambda)$ functions given by

$$F_\pm(\lambda)=\frac{\alpha\beta}{\lambda^2B^2}-\frac{(\cos\alpha\pi\pm 1)(\cos\beta\pi\mp 1)}{\sin\alpha\pi\,\sin\beta\pi}. \tag{2.31}$$

Condition (2.26) and the observation that $F_+(\lambda)$ and $F_-(\lambda)$ do not have common zeros imply that the set of solutions to the eigenvalue equation (2.20) is the union of the disjoint sets $\Lambda_+ = \{\lambda_+\}$ and $\Lambda_- = \{\lambda_-\}$ of solutions of the equations

$$F_\pm(\lambda_\pm)=0. \tag{2.32}$$

It is then a matter of algebra to write X^1 and X^2 as

$$X^i_\lambda(\tau,\sigma)=\begin{cases} X^i_+(\tau,\sigma) & \text{if } \lambda\in\Lambda_+,\\ X^i_-(\tau,\sigma) & \text{if } \lambda\in\Lambda_-,\end{cases}\qquad i=1,2, \tag{2.33}$$

with $X^i_\pm$ given by

$$X^1_\pm(\tau,\sigma)=ic_\lambda\frac{\alpha}{\lambda B}\left(\cos\beta\sigma+\frac{\sin\beta\pi}{\cos\beta\pi\mp 1}\sin\beta\sigma\right)e^{-i\lambda\tau}, \tag{2.34}$$

$$X^2_\pm(\tau,\sigma)=-c_\lambda\left(\frac{\cos\alpha\pi\pm 1}{\sin\alpha\pi}\cos\alpha\sigma+\sin\alpha\sigma\right)e^{-i\lambda\tau}. \tag{2.35}$$

Putting all cases together, we conclude that the solution for the boundary problem for X^1, X^2 is:

(1) If $m\kappa$ is not an integer and its square is not an even integer, the only modes that occur are those in Eqs. (2.34)–(2.35), corresponding to $\lambda \in \Lambda_\pm$.

(2) If $m\kappa$ is not an integer but its square is an even integer, one has in addition the modes (k,l) in (2.22)–(2.23).

(3) If $m\kappa$ is an even integer, there is one additional mode, $X^1_{\rm e}$, $X^2_{\rm e}$ in (2.18)–(2.19).

(4) Finally, if $m\kappa$ is an odd integer, the only modes that occur are those in (1) and $X^1_{\rm o}$, $X^2_{\rm o}$ in (2.16)–(2.17).

We summarize all these situations by writing

$$X^i(\tau,\sigma) = \sum_{\lambda\in\Lambda_\pm} X^i_\lambda + \delta_{m^2\kappa^2,\text{even}} \sum_{(k,l)} X^i_{(k,l)} + \delta_{m\kappa,\text{even}} X^i_{\rm e} + \delta_{m\kappa,\text{odd}} X^i_{\rm o}. \tag{2.36}$$

The mode expansions for the momenta P_i, P_a follow from their expressions (2.11)–(2.12) in terms of string coordinates and the mode expansions for the string coordinates. For the flat a-directions it is trivial to arrive at

$$2\pi\alpha' P_a = d^a_0 + \sum_{n\neq 0}^{\infty} c^a_n \cos n\sigma\, e^{-in\tau}.$$

For the i-directions we have

$$P_i(\tau,\sigma) = \sum_{\lambda\in\Lambda_\pm} P_{i,\lambda} + \delta_{m^2\kappa^2,\text{even}} \sum_{(k,l)} P_{i,(k,l)} + \delta_{m\kappa,\text{even}} P_{i,\rm e} + \delta_{m\kappa,\text{odd}} P_{i,\rm o}, \tag{2.37}$$

where the explicit expressions for the various contributions to the right-hand side can be found in Appendix A.

2.4. The string center of mass coordinates and the string total momentum

The string center of mass coordinates

$$x^{i,a}_{\rm cm}(\tau) = \frac{1}{\pi} \int_0^\pi d\sigma\, X^{i,a}(\tau,\sigma)$$

and the string total momentum

$$p_{i,a}(\tau) = \int_0^\pi d\sigma\, P_{i,a}(\tau,\sigma)$$

are straightforward to calculate from the mode expansions in the previous subsection. Let us consider for instance the total momentum. For the flat a-directions integration over $d\sigma$ gives the standard result $p_a = d^a_0/2\alpha'$. The a-component of the total string momentum is thus given by one of the string modes in that direction. The situation for the 1 and 2-component is very different. Indeed, integration over $d\sigma$ of the equations in Appendix A yields

$$\begin{aligned} p_1(\tau) &= \frac{\delta_{m\kappa,\mathrm{even}}}{\pi\alpha'}\frac{B^2}{m\kappa}b_{\mathrm{e}}\sinh\left(\frac{mk\pi}{2}\right) \\ &\quad - \frac{m^2\kappa^2}{\pi\alpha'}\left[\delta_{m^2\kappa^2,\mathrm{even}}\sum_{\substack{(k,l)\\ k\ \mathrm{odd}}}\frac{b_{\lambda(k,l)}}{\beta\lambda^2}e^{-i\lambda\tau} + \sum_{\lambda\in\Lambda_-}\frac{Bc_\lambda}{\beta^2}\frac{\cos\alpha\pi - 1}{\sin\alpha\pi}e^{-i\lambda\tau}\right] \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} p_2(\tau) &= \frac{\delta_{m\kappa,\mathrm{odd}}}{\pi\alpha'}\left[Bb_{\mathrm{o}}\tau\sinh\left(\frac{mk\pi}{2}\right) - a_{\mathrm{o}}B\right] \\ &\quad + \frac{im^2\kappa^2}{\pi\alpha'}\left[\delta_{m^2\kappa^2,\mathrm{even}}\sum_{\substack{(k,l)\\ k\ \mathrm{odd}}}\frac{\lambda a_{\lambda(k,l)}}{\alpha}e^{-i\lambda\tau} + \sum_{\lambda\in\Lambda_+}\frac{c_\lambda}{\lambda\alpha}e^{-i\lambda\tau}\right]. \end{aligned} \tag{2.39}$$

The components p_1 and p_2 receive contributions from all the string modes in those directions. More importantly, p_1 and p_2 are not conserved since their derivatives with respect to τ do not vanish. This is not a surprise, for the plane-wave metric (1.2) is not invariant under translations in the 1 and 2-directions. Upon quantization, we therefore do not expect the eigenvalues of the corresponding operators to play a significant rôle. It is trivial to convince oneself that this collective nature of p_i is also true for the string center of mass coordinates, whose explicit expression can be trivially obtained through integration over $d\sigma$.

2.5. *Case $|m\kappa| \ll 1$*

We finish by considering the regime $|m\kappa| \ll 1$. Since $m\kappa$ is not an integer, nor $m^2\kappa^2$ is an even integer, the only modes that exist in this case are those in (2.27)–(2.28), or equivalently (2.34)–(2.35). Furthermore, the mode eigenvalues λ can be explicitly found as formal power series in $m\kappa$ by making the ansatz $\lambda = \sum_0^\infty b_k(m\kappa)^k$ and solving Eq. (2.20) for the coefficients b_k order by order. Proceeding in this way we obtain:

(i) *Imaginary eigenvalues*. As already mentioned, they have $|\lambda| < |m\kappa|$. The algebra shows that there are only two of them, $\Lambda^{\mathrm{I}} = \{\pm i\lambda^{\mathrm{I}}\}$, given by

$$\lambda^{\mathrm{I}} = \frac{m\kappa}{\sqrt{1+B^2}}\left[1 + \frac{(m\kappa)^2}{12}\frac{\pi^2B^2}{1+B^2} + \frac{(m\kappa)^4}{1440}\frac{\pi^4B^2(5B^2-24)}{(1+B^2)^2} + \mathcal{O}\left(m^6\kappa^6\right)\right]. \tag{2.40}$$

In terms of Eqs. (2.32), they happen to solve $F_-(\lambda) = 0$, thus belong to Λ_-.

(ii) *Real eigenvalues with $|\lambda| < |m\kappa|$*. There are also two of them, $\Lambda^{\mathrm{R}} = \{\pm\lambda^{\mathrm{R}}\}$, where

$$\lambda^{\mathrm{R}} = \frac{m\kappa}{\sqrt{1+B^2}}\left[1 - \frac{(m\kappa)^2}{12}\frac{\pi^2B^2}{1+B^2} + \frac{(m\kappa)^4}{1440}\frac{\pi^4B^2(5B^2-24)}{(1+B^2)^2} + \mathcal{O}\left(m^6\kappa^6\right)\right]. \tag{2.41}$$

They are now solutions of $F_+(\lambda) = 0$, thus are in Λ_+.

(iii) *Real eigenvalues with $|\lambda| > |m\kappa|$*. They read

$$\left\{\begin{matrix}\lambda_n\\ \tilde{\lambda}_n\end{matrix}\right\} = n\left[1 \pm \frac{m^2\kappa^2}{2n^2}\frac{1-B^2}{1+B^2} - \frac{m^4\kappa^4}{8n^4}\frac{B^4-6B^2+1}{(1+B^2)^2} + \mathcal{O}\left(m^6\kappa^6\right)\right], \tag{2.42}$$

where n is a non-zero integer and the $+/-$ signs on the right-hand side correspond to $\lambda_n/\tilde{\lambda}_n$ on the left side. We will use the notation $\Lambda := \{\lambda_n\}$ and $\tilde{\Lambda} := \{\tilde{\lambda}_n\}$. These eigenvalues can be reorganized in terms of solutions of Eqs. (2.32) as

$$\lambda_+^{(n)} = \begin{cases} \lambda_n & \text{if } n \text{ even,} \\ \tilde{\lambda}_n & \text{if } n \text{ odd,} \end{cases} \qquad \lambda_-^{(n)} = \begin{cases} \tilde{\lambda}_n & \text{if } n \text{ even,} \\ \lambda_n & \text{if } n \text{ odd.} \end{cases} \tag{2.43}$$

It is instructive to compare the mode eigenvalues with those for the open string in flat space–time and zero antisymmetric field, i.e., with $m = B = 0$. In that case, X^1 and X^2 have the same expansion as in (2.14) and the mode eigenvalues are the integers. The flat zero mode $\lambda_{\text{flat}} = 0$ has multiplicity four in the 1, 2-directions, for there are four arbitrary constants of integration, which in our notation would be denoted c_0^1, c_0^2, d_0^1, d_0^2. Every pair $(n, -n)$ of non-zero flat modes is also 4-degenerate in these directions, for in each direction there are two complex coefficients c_{-n}^i and c_n^i and one complex constraint $(c^i)_n^\star = c_{-n}^i$. If m and B are switched on, the flat zero mode unfolds into two non-zero imaginary modes $(i\lambda^{\text{I}}, -i\lambda^{\text{I}})$ and two non-zero real modes $(\lambda^{\text{R}}, -\lambda^{\text{R}})$, and every pair of flat modes $(n, -n)$ unfolds into four modes $(\lambda_n, \tilde{\lambda}_n, \lambda_{-n}, \tilde{\lambda}_{-n})$. Whereas in Minkowski space–time, the string center of mass and string total momentum are independent degrees of freedom associated to the 4-degenerate zero mode, in our plane-wave background they are collective quantities.

3. Quantization

There is a discussion in the literature for $m = 0$ as for how to quantize the open string with non-trivial boundary conditions like those in (2.7) and (2.8). It seems to be a widespread believe that these boundary conditions impeach the use of canonical quantization. In fact, for $m = 0$, Dirac quantization, with the boundary conditions regarded as constraints, has been used as an alternative. The problem that arises then is whether the boundary conditions should be regarded as first or second class, and this is not a trivial choice for they lead to different results [26,27].

We will use plain canonical quantization and show that there is nothing wrong with it. Our approach consists of two steps. In the first one we compute the symplectic form in terms of the modes. This is straightforward, since the action is first order in time derivatives and it is well known how to proceed in these cases [28,29]. The resulting symplectic form will be non-singular, so it has an inverse. Its inverse defines, upon standard canonical quantization [28], the commutation relations for the quantum theory. We emphasize that the calculation of the symplectic form may be involved but, as pointed out in Refs. [28,29], as far as it is non-singular there is nothing wrong with canonical quantization and there is no need to introduce constraints of any type. It is also worth noting in this respect that the boundary conditions have already been taken into account in solving the classical equations of motions, so one would naïvely expect the symplectic form to already account for them. We will see that this quantization method is consistent with the equal-time commutation relations

$$\left[X^i(\tau,\sigma), P_j(\tau,\sigma')\right] = i\delta^i_j\delta(\sigma-\sigma'). \tag{3.1}$$

In Section 5 we will explicitly construct the Fock–Hilbert space for the theory and find the Hamiltonian spectrum.

3.1. Symplectic form and canonical quantization

The symplectic form

$$\Omega = \int_0^\pi d\sigma \left(\mathbf{d}P_i \wedge \mathbf{d}X^i + \mathbf{d}P_a \wedge \mathbf{d}X^a\right)$$

is the sum of two contributions, which we will call Ω_{pp} and Ω_{flat}. They respectively arise from the modes in the i-directions and the flat a-directions. Since they do not mix, the symplectic form can be studied by separately looking at each one of these two sectors.

Let us first look at Ω_{flat}. Recalling the mode expansions for X^a and P_a, one easily arrives at

$$\Omega_{\text{flat}} = \int_0^\pi d\sigma\, \mathbf{d}P_a \wedge \mathbf{d}X^a = \frac{1}{2\alpha'} \sum_{a=3}^{D-2} \left(\mathbf{d}d_0^a \wedge \mathbf{d}c_0^a - \sum_{n\neq 0} \frac{i}{2n} \mathbf{d}c_n^a \wedge \mathbf{d}c_{-n}^a\right). \tag{3.2}$$

This can be written as

$$\Omega_{\text{flat}} = \frac{1}{2} \Omega_{MM'}\, \mathbf{d}A_M \wedge \mathbf{d}A_{M'}, \tag{3.3}$$

where $\{A_M\} = \{d_0^a, c_0^a, c_n^a\}$ and a summation over indices $M = (a, n)$ and $M' = (a', n')$ is understood. The form Ω_{flat} is non-singular and can be inverted. Upon quantization, the amplitudes $\{A_M\}$ become operators with commutation relations given by the inverse of Ω as

$$[A_M, A_{M'}] = i\left(\Omega^{-1}\right)_{MM'}. \tag{3.4}$$

This yields the standard commutation relations

$$\left[c_0^a, d_0^b\right] = 2i\alpha'\delta^{ab}, \qquad \left[c_n^a, c_m^b\right] = 2\alpha' n \delta^{ab} \delta_{n+m,0}.$$

Reality of the field operators X^a imply that c_0^a and d_0^a are Hermitean and that $c_{-n}^a = (c_n^a)^\dagger$. So far, this is the same analysis as for Minkowski space–time.

To compute Ω_{pp} it is most convenient to use Eq. (2.11) and write P_i in terms of derivatives of X^i with respect to τ and σ. This gives

$$\Omega_{pp} = \int_0^\pi d\sigma\, \mathbf{d}P_i \wedge \mathbf{d}X^i = \tilde{\Omega}_{pp} + \bar{\Omega}_{pp},$$

where $\tilde{\Omega}_{pp}$ and $\bar{\Omega}_{pp}$ read

$$\tilde{\Omega}_{pp} = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma\, \mathbf{d}\left(\partial_\tau X^i\right) \wedge \mathbf{d}X^i \tag{3.5}$$

and

$$\bar{\Omega}_{pp} = \frac{B}{2\pi\alpha'} \mathbf{d}X^1 \wedge \mathbf{d}X^2 \Big|_{\sigma=0}^{\sigma=\pi}. \tag{3.6}$$

As compared to the flat a-directions, for which $2\pi\alpha' P_a = \partial_\tau X^a$, the non-trivial boundary conditions not only modify the modes but also add a boundary term $\bar{\Omega}_{pp}$ to the symplectic form.

Computation of the boundary piece $\bar{\Omega}_{pp}$ is straightforward. To calculate $\tilde{\Omega}_{pp}$, we use the mode expansion (2.36), integrate over $d\sigma$, rearrange the mode sums and employ that the eigenvalues $\lambda \in \Lambda_\pm$ are solutions of Eqs. (2.32). After some very long, but also very straightforward algebra, we obtain that

$$\Omega_{pp} = \Omega_{\Lambda_\pm} + \delta_{m^2\kappa^2,\text{even}}\Omega_{\{(k,l)\}} + \delta_{m\kappa,\text{even}}\Omega_{\text{e}} + \delta_{m\kappa,\text{odd}}\Omega_{\text{o}}. \tag{3.7}$$

The various contributions in this equation are given by

$$\Omega_{\Lambda_\pm} = \frac{i}{2\pi\alpha'}\sum_{\lambda\in\Lambda_\pm} f(\lambda)\,\mathbf{d}c_\lambda \wedge \mathbf{d}c_{-\lambda}, \tag{3.8}$$

$$\Omega_{\{(k,l)\}} = -\frac{i}{4\alpha' B}\sum_{(k,l)}\big[f_a(\lambda)\,\mathbf{d}a_{\lambda(k,l)} \wedge \mathbf{d}a_{-\lambda(k,l)} + f_b(\lambda)\mathbf{d}b_{\lambda(k,l)} \wedge \mathbf{d}b_{-\lambda(k,l)}\big], \tag{3.9}$$

$$\Omega_{\text{e}} = -\frac{1}{4\alpha'}\cosh\left(\frac{m\kappa\pi}{2}\right)\mathbf{d}a_{\text{e}} \wedge \mathbf{d}b_{\text{e}}, \tag{3.10}$$

$$\Omega_{\text{o}} = -\frac{1}{4\alpha'}\sinh\left(\frac{m\kappa\pi}{2}\right)\mathbf{d}a_{\text{o}} \wedge \mathbf{d}b_{\text{o}}, \tag{3.11}$$

where $f(\lambda)$, $f_a(\lambda)$ and $f_b(\lambda)$ read

$$f(\lambda) = -\frac{\lambda\alpha(\cos\alpha\pi \pm 1)}{\sin\alpha\pi}\left[\frac{2(m\kappa)^4}{\lambda^2\alpha^2\beta^2} \pm \frac{\pi}{\alpha\sin\alpha\pi} \mp \frac{\pi}{\beta\sin\beta\pi}\right], \tag{3.12}$$

$$f_a(\lambda) = \frac{\lambda}{B^2}\left(1 + B^2 - \frac{m^2\kappa^2}{\lambda^2}\right), \tag{3.13}$$

$$f_b(\lambda) = \frac{1}{\lambda B^2}\left(1 + B^2 + \frac{m^2\kappa^2}{\lambda^2}\right). \tag{3.14}$$

In accordance with the notation that we are using, the double signs $\pm$ on the right of the equation for $f(\lambda)$ apply, respectively, to the eigenvalues $\lambda_\pm$ solving Eqs. (2.32).

We make at this point two comments concerning the computation of Ω_{pp}. The first one is that the only non-zero components $\Omega_{MM'}$ of the symplectic form have $M + M' = 0$, where M labels all the existing mode $\{A_M\} = \{a_{\text{o}}, b_{\text{o}}, a_{\text{e}}, b_{\text{e}}, a_{\lambda(k,l)}, b_{\lambda(k,l)}, c_\lambda\}$. Some authors call this orthogonality of modes. Note in particular that there is not any mixing of the modes for $m^2\kappa^2 = \text{even}$, $m\kappa = \text{even}$ and $m\kappa = \text{odd}$ among themselves, nor with modes $\lambda \in \Lambda_\pm$. The second comment is to emphasize that the result above for Ω_{pp} follows straightforwardly from Eqs. (3.5)–(3.6) after plain integration over $d\sigma$, without any assumption whatsoever.

The form Ω_{pp} is non-singular and has an inverse Ω_{pp}^{-1}. Canonical quantization is then straightforward. The amplitudes $\{A_M\}$ become operators. Hermiticity of X^i implies that a_{o}, b_{o}, a_{e}, b_{e} and c_λ ($\lambda \in \Lambda_\pm$ imaginary) are Hermitean and that

$$a^\dagger_{\lambda(k,l)} = a_{-\lambda(k,l)}, \qquad b^\dagger_{\lambda(k,l)} = b_{-\lambda(k,l)}, \qquad c^\dagger_\lambda = c_{-\lambda}(\lambda \in \Lambda_\pm\text{real}).$$

The commutation rules are obtained from the inverse of Ω_{pp} as in (3.3)–(3.4), the only non-trivial commutation relations being

$$\big[c_\lambda, c^\dagger_\lambda\big] = -\frac{\pi\alpha'}{f(\lambda)}, \tag{3.15}$$

$$\big[a_{\lambda(k,l)}, a^{\dagger}_{\lambda(k,l)}\big] = \frac{2\alpha'}{f_a(\lambda)}, \qquad \big[b_{\lambda(k,l)}, b^{\dagger}_{\lambda(k,l)}\big] = \frac{2\alpha'}{f_b(\lambda)}, \tag{3.16}$$

$$[a_{\rm e}, b_{\rm e}] = -4i\alpha' \operatorname{cosech}\left(\frac{m\kappa\pi}{2}\right), \tag{3.17}$$

$$[a_{\rm o}, b_{\rm o}] = -4i\alpha' \operatorname{sech}\left(\frac{m\kappa\pi}{2}\right). \tag{3.18}$$

We note that $f(\lambda)$ is real for λ real and imaginary for λ imaginary. The space of states on which these operators act and their action is given in Section 5. Let us move on to study the consistency of this quantization with the canonical commutation relations (3.1).

3.2. Canonical commutation relations

The commutator $[X^i(\tau,\sigma), P_j(\tau,\sigma')]$ can be computed by replacing X^i and P_j with their mode expansions and using the relations (3.15)–(3.18) for the mode operators in them. In doing so, the τ-dependence of the commutator is removed and a mode sum is left. This sum involves in particular an infinite sum over mode eigenvalues $\lambda \in \Lambda_{\pm}$ whose terms are products of sines and cosines at $\alpha\sigma$, $\beta\sigma$, $\alpha\sigma'$, $\beta\sigma'$ with complicated coefficients involving the function $f(\lambda)$. We do not see a way to perform this sum in closed form and obtain a compact expression for the commutator. We will instead expand the commutator in powers of $m\kappa$ and perform the mode sums order by order in $m\kappa$. We do this in the sequel.

If $|m\kappa| \ll 1$, the only modes that exist are those in Eqs. (2.34)–(2.35). We recall from Section 2.4 that in this case the mode eigenvalues are given by $\Lambda_{\rm I} = \{\pm i\lambda^{\rm I}\}$, $\Lambda_{\rm R} = \{\pm\lambda^{\rm R}\}$, $\Lambda = \{\lambda_n\}$ and $\tilde{\Lambda} = \{\tilde{\lambda}_n\}$ in Eqs. (2.40)–(2.42), with $n = \pm 1, \pm 2, \ldots$. We denote by $\{c^{\rm I}_{\pm}\}$, $\{c^{\rm R}_{\pm}\}$, $\{c_n\}$ and $\{\tilde{c}_n\}$ the corresponding annihilation and creation operators, for which hermiticity of the string position operators implies

$$\big(c^{\rm I}_{\pm}\big)^{\dagger} = c^{\rm I}_{\pm}, \qquad \big(c^{\rm R}_{+}\big)^{\dagger} = c^{\rm R}_{-}, \qquad (c_n)^{\dagger} = c_{-n}, \qquad (\tilde{c}_n)^{\dagger} = \tilde{c}_{-n}.$$

Expanding the right-hand side of Eq. (3.15) in powers of $m\kappa$, we obtain the following commutations relations for them:

$$\big[c^{\rm I}_{+}, c^{\rm I}_{-}\big] = -\frac{i\alpha' B^2}{2(2+B^2)(1+B^2)^{1/2}} \frac{1}{m\kappa}\left[1 + \frac{\pi^2 B^2 (m\kappa)^2}{6(2+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right], \tag{3.19}$$

$$\big[c^{\rm R}_{+}, c^{\rm R}_{-}\big] = -\frac{\alpha'\pi^2 B^2}{8(1+B^2)^{3/2}} m\kappa\left[1 + \frac{\pi^2(1-B^2)(m\kappa)^2}{6(1+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right], \tag{3.20}$$

$$[c_n, c_k] = \frac{\alpha'\pi^2 B^4}{4n^3(1+B^2)^3}(m\kappa)^4\left[1 - \frac{(3-5B^2)(m\kappa)^2}{2n^2(1+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right]\delta_{n+k,0}, \tag{3.21}$$

$$[\tilde{c}_n, \tilde{c}_k] = \frac{\alpha' B^2}{n(1+B^2)}\left[1 + \frac{3(m\kappa)^2}{2n^2(1+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right]\delta_{n+k,0}, \tag{3.22}$$

all other commutators being zero. The commutator $[X^i(\tau,\sigma), P_j(\tau,\sigma')]$ can then be written as a sum

$$\big[X^i(\tau,\sigma), P_j(\tau,\sigma')\big] = \sum_{\omega=I,R,n,\tilde{n}} C^i{}_j(\omega;\sigma,\sigma')$$

of four contributions $C^i{}_j(\omega;\sigma,\sigma')$ arising from the four sets in which the modes have been organized. Each one of these contributions is a power series in $m\kappa$, depends on σ and σ' and can be computed with relative ease order by order. To illustrate this, let us take as an example $i=j=2$. After some algebra we obtain

$$C^2{}_2(\mathrm{I};\sigma,\sigma') = -\frac{i\alpha' B^2(m\kappa)^2}{2(1+B^2)}\left(\frac{\pi^2}{2} - \pi\sigma - \pi\sigma' + 2\sigma\sigma'\right) + \mathcal{O}\big(m^4\kappa^4\big), \tag{3.23}$$

$$C^2{}_2(\mathrm{R};\sigma,\sigma') = i\alpha' - \frac{i\alpha' B^2(m\kappa)^2}{2(1+B^2)}\left(\frac{\pi^2}{6} + \pi\sigma - \sigma^2 - \pi\sigma' + \sigma'^2\right) + \mathcal{O}\big(m^4\kappa^4\big), \tag{3.24}$$

$$\begin{aligned} C^2{}_2(\Lambda;\sigma,\sigma') = 2i\alpha'\sum_{n=1}^{\infty}\cos n\sigma'\bigg[\cos n\sigma + \frac{B^2(m\kappa)^2}{1+B^2}\bigg(\frac{\cos n\sigma}{n^2} + \frac{\sigma\sin n\sigma}{n} \\ - \frac{\pi}{2}\frac{\sin n\sigma}{n}\bigg)\bigg] + \mathcal{O}\big(m^4\kappa^4\big), \end{aligned} \tag{3.25}$$

$$C^2{}_2(\tilde{\Lambda};\sigma,\sigma') = -\frac{i\alpha' B^2(m\kappa)^2}{1+B^2}\sum_{n=1}^{\infty}\big(2\sigma' - \pi\big)\frac{\sin n\sigma\cos n\sigma'}{n} + \mathcal{O}\big(m^4\kappa^4\big). \tag{3.26}$$

It follows from inspection of these formuli that only $C^2{}_2(\mathrm{R})$ and $C^2{}_2(\Lambda)$ carry contributions of order zero in $m\kappa$. These are easily summed by recalling that, for functions defined on $[0,\pi]$ with vanishing derivatives at the boundary, Dirac's delta function has the representation

$$\pi\delta(\sigma-\sigma') = 1 + 2\sum_{n=1}^{\infty}\cos n\sigma\cos n\sigma'.$$

Hence

$$\big[X^2(\tau,\sigma), P_2(\tau,\sigma')\big]_0 = i\alpha'\delta(\sigma-\sigma'),$$

where the subscript 0 refers to the order in $m\kappa$. To sum the order-two in $m\kappa$ contributions, it is convenient to introduce variables $\sigma_\pm = \sigma \pm \sigma'$, which take values $\sigma_- \in [-\pi,\pi]$ and $\sigma_+ \in [0,2\pi]$. In terms of these, we have

$$\big[C^2{}_2(\Lambda) + C^2{}_2(\tilde{\Lambda})\big]_2 = \frac{i\alpha' B^2(m\kappa)^2}{2(1+B^2)}\big[F_2(\sigma_-) + F_2(\sigma_+) + \sigma_- F_1(\sigma_-) + \sigma_- F_1(\sigma_+)\big],$$

where F_1 and F_2 stand for the Fourier series

$$F_1(\sigma_-) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_-}{n} = \begin{cases} \frac{\pi|\sigma_-|}{\sigma_-} - \sigma_- & \text{if } 0<|\sigma_-|<\pi, \\ 0 & \text{if } \sigma_- = 0, \pm\pi, \end{cases} \tag{3.27}$$

$$F_1(\sigma_+) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_+}{n} = \begin{cases} \pi - \sigma_+ & \text{if } 0<\sigma_+<2\pi, \\ 0 & \text{if } \sigma_+ = 0, 2\pi, \end{cases} \tag{3.28}$$

$$F_2(\sigma_-) := 2\sum_{n=1}^{\infty}\frac{\cos n\sigma_-}{n^2} = \frac{\sigma_-^2}{2} - \pi|\sigma_-| + \frac{\pi^2}{3}, \tag{3.29}$$

$$F_2(\sigma_+) := 2\sum_{n=1}^{\infty}\frac{\cos n\sigma_+}{n^2} = \frac{\sigma_+^2}{2} - \pi\sigma_+ + \frac{\pi^2}{3}. \tag{3.30}$$

Putting together all contributions of order two in Eqs. (3.23)–(3.26), we obtain

$$\left[X^2(\tau,\sigma),\, P_2(\tau,\sigma')\right]_2 = 0,$$

in agreement with (3.1). Proceeding in the same way, it is straightforward to see that the commutation relations in (3.1) also hold for other values of i and j, so we can write

$$\left[X^i(\tau,\sigma),\, P_k(\tau,\sigma')\right] = i\alpha'\delta^i{}_j\delta(\sigma-\sigma') + \mathcal{O}\big(m^4\kappa^4\big).$$

This proves the consistency of the quantization procedure used here with equal-time canonical commutation relations, at least up to order $m^4\kappa^4$.

We find quite surprising the asymmetric rôle that each type of mode plays in this analysis, yet all combine to produce the desired result. It is also worth noting that $C^i{}_j(\Lambda)$ and $C^i{}_j(\tilde{\Lambda})$ will involve to any order in $m\kappa$ polynomials in $\sigma_\pm$ multiplied with convergent Fourier series of $\sigma_\pm$, thus becoming a question of algebra force to go to higher orders in $m\kappa$. It is by now clear that canonical quantization works and that it does because the symplectic form is non-singular.

4. Non-commutative wave fronts

The plane-wave metric (1.2) foliates space–time by null surfaces $X^+ = \text{const}$. We show next that these spaces are non-commutative. The commutator $[X^1(\tau,\sigma), X^2(\tau,\sigma')]$ can be computed by replacing X^1 and X^2 with their mode expansions and using the commutation relations (3.15)–(3.18) for the mode operators. This results in

$$\begin{aligned}\left[X^1(\tau,\sigma),\, X^2(\tau,\sigma')\right] = i\big[&\Theta_{\Lambda_\pm}(\sigma,\sigma') + \delta_{m^2\kappa^2,\text{even}}\Theta_{\{(k,l)\}}(\sigma,\sigma') \\ &+ \delta_{m\kappa,\text{even}}\Theta_{\text{e}}(\sigma,\sigma') + \delta_{m\kappa,\text{odd}}\Theta_{\text{o}}(\sigma,\sigma')\big],\end{aligned} \tag{4.1}$$

where the contribution $\Theta_{\Lambda_\pm}(\sigma,\sigma')$ is given by

$$\begin{aligned}\Theta_{\Lambda_\pm}(\sigma,\sigma') = \frac{1}{2B}\sum_{\lambda\in\Lambda_\pm}&\frac{\alpha}{\lambda f(\lambda)}\left(\cos\beta\sigma + \frac{\sin\beta\pi}{\cos\beta\pi \mp 1}\sin\beta\sigma\right) \\ &\times\left(\frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi}\cos\alpha\sigma' + \sin\alpha\sigma'\right)\end{aligned} \tag{4.2}$$

and $\Theta_{\{(k,l)\}}(\sigma,\sigma')$, $\Theta_{\text{e}}(\sigma,\sigma')$ and $\Theta_{\text{o}}(\sigma,\sigma')$ read

$$\Theta_{\{(k,l)\}}(\sigma,\sigma') = -4\alpha' B\sum_{(k,l)}\left[\frac{\alpha\cos\beta\sigma\sin\alpha\sigma'}{\lambda^2(1+B^2)-m^2\kappa^2} + \frac{\beta\sin\beta\sigma\cos\alpha\sigma'}{\lambda^2(1+B^2)+m^2\kappa^2}\right], \tag{4.3}$$

$$\Theta_{\text{e}}(\sigma,\sigma') = \frac{4\alpha' B}{m\kappa}\,\text{cosech}\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma)\sinh\left[m\kappa\left(\frac{\pi}{2}-\sigma'\right)\right], \tag{4.4}$$

$$\Theta_{\text{o}}(\sigma,\sigma') = \frac{4\alpha' B}{m\kappa}\,\text{sech}\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma)\cosh\left[m\kappa\left(\frac{\pi}{2}-\sigma'\right)\right]. \tag{4.5}$$

We recall that the sum in $\Theta_{\{(k,l)\}}(\sigma,\sigma')$ is over the finite number of solutions (k,l) of Eq. (2.25) and that α and β in this sum are as in (2.24), so the contributions (4.3)–(4.5) do not pose any problems.

The most complicated piece to understand is the contribution $\Theta_{\Lambda_\pm}(\sigma,\sigma')$. We may proceed as in Section 3 and consider $|m\kappa| \ll 1$. In this case only $\Theta_{\Lambda_\pm}(\sigma,\sigma')$ contributes to the commutator $[X^1, X^2]$. Expanding the right-hand side of Eq. (4.2) in powers of $m\kappa$, the sum over modes can then be performed order by order in $m\kappa$, so that $\Theta(\sigma,\sigma')$ becomes a power series

$$\Theta(\sigma,\sigma')=\sum_{k=0}^{\infty}\Theta_{2k}(\sigma,\sigma')(m\kappa)^{2k}$$

whose coefficients are explicit functions of σ and σ'. The first two terms of this series are calculated in Appendix B. We exhibit here the result. At the string endpoints we obtain

$$\Theta(0,0)=-\Theta(\pi,\pi)=\frac{\alpha'\pi B}{1+B^2}\left[1+\frac{\pi^2(m\kappa)^2}{6(1+B^2)}+\mathcal{O}\big(m^4\kappa^4\big)\right], \tag{4.6}$$

whereas at $\sigma+\sigma'\neq 0, 2\pi$ we have

$$\begin{aligned}\Theta(\sigma,\sigma')=\frac{\alpha' B(m\kappa)^2}{(1+B^2)^2}\bigg\{&B^2\Big[-\frac{\sigma}{6}\big(\sigma^2-3\sigma'^2\big)+\frac{\pi}{4}\big(\sigma^2-\sigma'^2-2\sigma\sigma'\big)-\frac{\pi}{12}(\sigma-3\sigma')\Big]\\&-\frac{\sigma}{12}\big(7\sigma^2+9\sigma'^2\big)+\frac{\pi}{8}\big(7\sigma^2+3\sigma'^2+6\sigma\sigma'\big)-\frac{\pi^2}{4}(3\sigma+\sigma')+\frac{\pi^3}{6}\\&+\frac{\pi}{8}|\sigma-\sigma'|\big[2B^2\big(\sigma^2+\sigma'-\pi\big)+5\sigma-\sigma'-2\pi\big]\bigg\}+\mathcal{O}\big(m^4\kappa^4\big).\end{aligned} \tag{4.7}$$

The limit $m\to 0$ is smooth and reproduces the results in the literature. In fact, as $m\to 0$, that is, as Minkowski space–time is approached, only the first term in (4.6) survives and the results in Ref. [19] are recovered. For $m\neq 0$, two novelties are found: non-commutativity at the string endpoints receives m-dependent corrections, and non-commutativity occurs for arbitrary values of σ and σ', so that it extends all along the string. Even for $\sigma=\sigma'\neq 0,\pi$ non-commutativity pervades, since in that case

$$\Theta(\sigma,\sigma)=\frac{\alpha' B(m\kappa)^2}{6(1+B^2)^2}(2\sigma-\pi)\big[B^2\sigma(\sigma-\pi)-(2\sigma-\pi)^2\big]+\mathcal{O}\big(m^4\kappa^4\big)\neq 0.$$

At the string midpoint $\sigma=\sigma'=\pi/2$ one has $\Theta=0$, not only for $m^2\kappa^2\ll 1$ but also for arbitrary $m\kappa$ since the right-hand side of (4.2) vanishes. Note also that commutativity is recovered as $B\to 0$.

The results in this section may be viewed from two perspectives. The first one is to assume a constant background field $B_{12}=B$ and that the string endpoints move freely, except for the boundary conditions imposed by the presence of the B field. The endpoints are then not distinguished by non-commutativity. The second one is to assume that B_{ij} vanishes but that the string endpoints are constrained to move on a D1-brane located at x_0^a on which a constant magnetic field $F_{12}=B$ is defined. The boundary conditions for X^1 and X^2 then remain unchanged while those for X^a become $X^a|_{\sigma=0,\pi}=x_0^a$. The only difference with the situation discussed here is that the mode expansion for X^a is no longer (2.14) but rather

$$X^a(\tau,\sigma)=x_0^a+\sum_{n\neq 0} i\frac{c_a^a}{n}\sin n\sigma\, e^{-in\sigma}.$$

This only introduces some trivial modifications in the analysis of the flat a-directions [19]. From this point of view, the plane-wave metric extends non-commutativity outside the D1-brane.

5. The Fock–Hilbert space and the spectrum

We want to solve the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle,$$

where the Hamiltonian is the integral over σ of the Hamiltonian density $\mathcal{H}$ in Eq. (2.13). As discussed in Sections 2.1 and 2.3, the classical Hamiltonian is not positive. This translates, upon quantization, into an unbounded Hamiltonian operator from below. It will become explicit below that it is precisely the modes with imaginary λ that make the Hamiltonian unbounded, as otherwise was to be expected. Hence not all the states to be constructed in this section are within reach for an observer but only those with positive eigenenergies.

It is convenient to split H as the sum

$$H = H_{\text{flat}} + H_{pp}$$

of a contribution

$$H_{\text{flat}} = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\,\left[\left(\partial_\tau X^a\right)^2 + \left(\partial_\sigma X^a\right)^2\right]$$

from the flat a-directions and a contribution

$$H_{pp} = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\,\left\{\left(\partial_\tau X^i\right)^2 + \left(\partial_\sigma X^i\right)^2 - m^2\kappa^2\left[\left(X^1\right)^2 - \left(X^2\right)^2\right]\right\}$$

from the 1, 2-directions. The eigenstates of H are then of the form $|\psi\rangle = |\psi_{\text{flat}}\rangle \otimes |\psi_{pp}\rangle$ and the eigenenergies read $E = E_{\text{flat}} + E_{pp}$, with $\{|\psi_{\text{flat}}\rangle, E_{\text{flat}}\}$ and $\{|\psi_{pp}\rangle, E_{pp}\}$ the solutions to the eigenvalue problems

$$H_{\text{flat}}|\psi_{\text{flat}}\rangle = E_{\text{flat}}|\psi_{\text{flat}}\rangle,$$
$$H_{pp}|\psi_{pp}\rangle = E_{pp}|\psi_{pp}\rangle.$$

5.1. Eigenvalue problem for H_{flat}

Apart from the number of dimensions, it is the same problem as for the open string in Minkowski space–time. Using the mode expansions for X^a, one obtains for H_{flat} a sum

$$H_{\text{flat}} = \frac{1}{2\alpha'}\sum_{n=1}^{\infty} :c_n^{a\dagger} c_n^a: + \alpha' p_a^2 + \frac{D-4}{24}$$

of harmonic oscillator Hamiltonians, one for every frequency $n > 0$ in every direction a. As usual, $:AB:$ denotes normal ordering of AB and the sum $\sum_{n>0} n$ entering the normal ordering constant has been regulated using ζ-regularization, so that it takes the value $\zeta(-1) = -1/12$. The solution for H_{flat} is well known. The Fock space is formed by states

$$|\psi_{\text{flat}}\rangle = |\psi_{\{k_n^a\}}\rangle = \bigotimes_{a=3}^{D-2}\left|\left\{k_n^a\right\}_{n=1}^{\infty}, p_a\right\rangle, \quad k_n^a = 0, 1, 2, \ldots, \tag{5.1}$$

with k_n^a the occupancy number of the harmonic oscillator of frequency n in the a-direction. The energies of these states are

$$E_{\text{flat}} = E_{\{k_n^a\}} = \sum_{a=3}^{D-2} \sum_{n=1}^{\infty} n k_n^a + \alpha' p_a^2 + \frac{D-4}{24}.$$

We note that the sum over n is actually finite, since for every eigenstate there is a finite number of non-zero occupancy numbers k_n^a. The action of $c_n^{b\dagger}$ and c_n^b on $|\{k_r^a\}, p_a\rangle$ is $\sqrt{2\alpha' n}$ times the usual one of creation and annihilation harmonic oscillator operators.

5.2. *Eigenvalue problem for H_{pp}*

Employing the mode expansions for X^i, we obtain after some work that

$$H_{pp} = H_{\Lambda_\pm} + \delta_{m^2\kappa^2,\text{even}} H_{\{(k,l)\}} + \delta_{m\kappa,\text{even}} H_{\text{e}} + \delta_{m\kappa,\text{odd}} H_{\text{o}}, \tag{5.2}$$

where $H_{\Lambda_\pm}$ is given by

$$H_{\Lambda_\pm} = \frac{1}{2\pi\alpha'} \sum_{\lambda\in\Lambda_\pm} \lambda f(\lambda) c_\lambda c_{-\lambda} \tag{5.3}$$

and $H_{\{(k,l)\}} H_{\text{e}}$ and H_{o} take the form

$$H_{\{(k,l)\}} = \frac{1}{4\alpha'} \sum_{\lambda(k,l)} \lambda \big[f_a(\lambda) a_\lambda a_{-\lambda} + f_b(\lambda) b_\lambda b_{-\lambda} \big], \tag{5.4}$$

$$H_{\text{e,o}} = \frac{1}{4\pi\alpha'} \left[\frac{\pi}{4} \cosh(m\kappa\pi) + \frac{B^2}{m\kappa} \sinh(m\kappa\pi) - 1 \right] b_{\text{e,o}}^2. \tag{5.5}$$

We first study the problem for $H_{\Lambda_\pm}$ and postpone the solution for the pathological modes $m^2\kappa^2 =$ even, $m\kappa =$ even, $m\kappa =$ odd.

We recall that an infinite number of the modes $\lambda \in \Lambda_\pm$ have real λ and that a finite number of them have imaginary λ. We separate their contributions H_{R} and H_{I} to $H_{\Lambda_\pm}$ and write

$$H_{\Lambda_\pm} = H_{\text{R}} + H_{\text{I}}.$$

The eigenstates and eigenvalues of $H_{\Lambda_\pm}$ are $|\psi_{\Lambda_\pm}\rangle = |\psi_{\text{R}}\rangle \otimes |\psi_{\text{I}}\rangle$ and $E_{\Lambda_\pm} = E_{\text{R}} + E_{\text{I}}$, with $\{|\psi_{\text{R}}\rangle, E_{\text{R}}\}$ and $\{|\psi_{\text{I}}\rangle, E_{\text{I}}\}$ solutions to the problems

$$H_{\text{R}}|\psi_{\text{R}}\rangle = E_{\text{R}}|\psi_{\text{R}}\rangle,$$
$$H_{\text{I}}|\psi_{\text{I}}\rangle = E_{\text{I}}|\psi_{\text{I}}\rangle.$$

5.2.1. *Solution for H_{R}*

The commutation relations (3.15) for the operators c_λ, yield for H_{R}

$$H_{\text{R}} = \frac{1}{\pi\alpha'} \sum_{\substack{\lambda\in\Lambda_\pm \\ \text{Re}\,\lambda>0}} \lambda f(\lambda) {:}c_\lambda^\dagger c_\lambda{:} + K_{\text{R}},$$

where K_{R} is the normal ordering constant

$$K_{\mathrm{R}}=-\frac{1}{2}\sum_{\substack{\lambda\in\Lambda_\pm\\ \mathrm{Re}\,\lambda>0}}\lambda. \tag{5.6}$$

The Hamiltonian H_{R} is a sum of harmonic oscillators, one for every real $\lambda>0$. The eigenstates of H_{R} are then harmonic oscillator states

$$|\psi_{\mathrm{R}}\rangle=\big|\{k_\lambda\}_{\mathrm{Re}\,\lambda>0}\big\rangle,\qquad k_\lambda=0,1,2,\ldots, \tag{5.7}$$

with k_λ the occupancy number for the harmonic oscillator of frequency λ, while the eigenenergies read

$$E_{\mathrm{R}}=E_{\{k_\lambda,\mathrm{Re}\,\lambda>0\}}=\sum_{\mathrm{Re}\,\lambda>0}\lambda k_\lambda+K_{\mathrm{R}}.$$

The action of $c_\lambda^\dagger$ and c_λ on the states $|\{k_{\lambda'}\}\rangle$ is $\sqrt{\pi\alpha'/f(\lambda)}$ times the usual action of annihilation and creation harmonic oscillator operators.

Since there are infinitely many positive real λ with no accumulation point, the normal ordering constant K_{R} needs regularization. For every $m\kappa$ we can always take a sufficiently large integer N such that $m^2\kappa^2\ll N^2$ and split the sum for K_{R} into two sums: one over $0<\lambda<N$ and one over $N<\lambda$. Since $(m\kappa/N)^2\ll 1$, the λ's in the second sum are given by Eq. (2.42), so that K_{R} can be written as

$$K_{\mathrm{R}}=-\frac{1}{2}\sum_{\mathrm{Re}\,\lambda<N}\lambda+\frac{1}{2}\sum_{n=1}^{N}(\lambda_n+\tilde{\lambda}_n)-\frac{1}{2}\sum_{n=1}^{\infty}(\lambda_n+\tilde{\lambda}_n).$$

The first two terms in this equation are finite, while accordingly to (2.42) the third one contains the divergent sum $\sum_{n>0}n$. Regularizing this in the same way as for the flat a-directions we arrive at

$$K_{\mathrm{R}}=\frac{1}{12}+\Delta K(m),$$

where $\Delta K(m)$ collects all m-dependent contributions to K_{R}. For example, for $m^2\kappa^2\ll 1$ the integer N can be taken equal to 1 and from Section 2 it is straightforward to see that

$$\Delta K\big(m^2\kappa^2\ll 1\big)=-\frac{m\kappa}{2\sqrt{1+B^2}}\left[1-\frac{(m\kappa)^2}{12}\frac{\pi^2B^2}{1+B^2}+\mathcal{O}\big(m^3\kappa^3\big)\right].$$

5.2.2. Solution for H_{I}

It is convenient to introduce for every imaginary λ operators $\hat{q}_\lambda$ and $\hat{p}_\lambda$ defined by

$$c_{\pm\lambda}=\sqrt{\frac{\pi\alpha'}{2|\lambda f(\lambda)|}}(\hat{q}_\lambda\pm\hat{p}_\lambda),\qquad \mathrm{Im}\,\lambda>0. \tag{5.8}$$

They are Hermitean and satisfy commutation relations $[\hat{q}_\lambda,\hat{p}_\lambda]=i\,\mathrm{sign}[\lambda f(\lambda)]$. In terms of them, H_{I} takes the form

$$H_{\mathrm{I}}=\sum_{\substack{\lambda\in\Lambda_\pm\\ \mathrm{Im}\,\lambda>0}}\mathrm{sign}\big[\lambda f(\lambda)\big]\big(\hat{p}_\lambda^2-\hat{q}_\lambda^2\big).$$

It is clear that the $H_{\rm I}$ is not bounded from below. Let us forget for a moment about this and formally solve the eigenvalue problem for $H_{\rm I}$. The solution is given by $|\psi_{\rm I}\rangle = \prod |\varphi_\lambda\rangle$ and $E_{\rm I} = \sum E_\lambda$, with the product and the sum extended over all imaginary λ with $\operatorname{Im}\lambda > 0$, and $\{|\psi_\lambda\rangle, E_\lambda\}$ being solutions of

$$\big(\hat{p}_\lambda^2 - \hat{q}_\lambda^2\big)|\psi_\lambda\rangle = E_\lambda|\psi_\lambda\rangle, \quad \operatorname{Im}\lambda > 0. \tag{5.9}$$

To solve (5.9) we work in a position representation, in which the wave function for $|\psi_\lambda\rangle$ is $\psi_\lambda(q_\lambda)$ and the operators $\hat{q}_\lambda$ and $\hat{p}_\lambda$ act on it through multiplication and derivation, i.e., $\hat{q}_\lambda \to q_\lambda$ and $\hat{p}_\lambda \to i\frac{d}{dq_\lambda}$. Eq. (5.9) then becomes

$$\left(\frac{d^2}{dq_\lambda^2} + q_\lambda^2 + E_\lambda\right)\psi_\lambda(q_\lambda) = 0, \quad \operatorname{Im}\lambda > 0.$$

This is the time-independent Schrödinger equation for a particle in an inverted harmonic potential. Such equation does not have bound states and for every real E_λ admits

$$\psi_{\lambda,1}(q_\lambda) = e^{-iq_\lambda^2/2} q_\lambda \Phi\left(\frac{3}{4} + \frac{iE}{4}, \frac{3}{2}; iq_\lambda^2\right)$$

and

$$\psi_{\lambda,2}(q_\lambda) = e^{-iq_\lambda^2/2} \Phi\left(\frac{1}{4} + \frac{iE}{4}, \frac{1}{2}; iq_\lambda^2\right)$$

as two linearly independent solutions, $\Phi(\mu, \nu; z)$ being the degenerate hypergeometric function. Both $\psi_{\lambda,1}(q_\lambda)$ and $\psi_{\lambda,2}(q_\lambda)$ are regular at $q_\lambda = 0$, while at $|q_\lambda| \to \infty$ are superpositions of the oscillating exponentials

$$\frac{1}{\sqrt{|q_\lambda|}} \exp\left[\pm\frac{i}{4}\big(E_\lambda \ln q_\lambda^2 + 2q_\lambda^2\big)\right].$$

The most general solution for $\psi_\lambda(q_\lambda)$ is then an arbitrary linear combination

$$\psi_\lambda(q_\lambda) = C_1\psi_{\lambda,1}(q_\lambda) + C_2\psi_{\lambda,2}(q_\lambda).$$

The state $\psi_\lambda(q_\lambda)e^{iE_\lambda\tau}$ is a scattering state which in this position representation is asymptotically formed by one incoming and one outgoing traveling wave. It is worth noting that these waves are not plane and that the effect of the inverted harmonic potential is felt at $|q_\lambda| \to \infty$. The eigenstates of $H_{\rm I}$ are then

$$|\psi_{\rm I}\rangle = \big|\{E_\lambda\}_{\operatorname{Im}\lambda>0}\big\rangle \to \prod_{\operatorname{Im}\lambda>0} \psi_\lambda(q_\lambda), \quad E_\lambda \text{ real and arbitrary,} \tag{5.10}$$

and the energies read

$$E_{\rm I} = \sum_{\operatorname{Im}\lambda>0} \operatorname{sign}\big[\lambda f(\lambda)\big]E_\lambda.$$

The action of $c_{\pm\lambda}$ on $\psi_\lambda(q_\lambda)$ is through (5.8) and multiplication and derivation. The states $|\psi_{\rm I}\rangle$ play in the 1 and 2-directions the equivalent rôle to that of the plane wave states $|p_a\rangle$ in the flat a-directions. One way to ensure that the eigenenergies are non-negative is to restrict to scattering states with $E_\lambda = \operatorname{sign}[\lambda f(\lambda)]|E_\lambda|$ for every imaginary λ.

Putting everything together, the eigenstates and eigenvalues of $H_{\Lambda_\pm}$ are

$$|\psi_{\Lambda_\pm}\rangle = |\psi_{\{k_n^a\}}\rangle \otimes \big|\{k_\lambda\}_{\mathrm{Re}\,\lambda>0}\big\rangle \otimes \big|\{E_\lambda\}_{\mathrm{Im}\,\lambda>0}\big\rangle,$$

$$E_{\Lambda_\pm} = \sum_{a=3}^{D-2}\sum_{n=1}^{\infty} n k_n^a + \sum_{\mathrm{Re}\,\lambda>0} \lambda k_\lambda + \alpha' p_a^2 + \sum_{\mathrm{Im}\,\lambda>0} \mathrm{sign}\big[\lambda f(\lambda)\big] E_\lambda + \frac{D-2}{24} + \Delta K(m).$$

5.2.3. Contributions from $H_{\{(k,l)\}}$, H_{e}, H_{o}

If $m\kappa$ is such that it squares to an even integer, or is itself an even or odd integer, the Hamiltonian also receives the contributions $H_{\{(k,l)\}}$, H_{e} and H_{o} in (5.4)–(5.5). In case $m^2\kappa^2 = \text{even}$, it is trivial to see that

$$H_{\{(k,l)\}} = \frac{1}{2\alpha'} \sum_{\lambda(k,l)>0} \lambda\big[f_a(\lambda) a_\lambda^\dagger a_\lambda + f_a(\lambda) b_\lambda^\dagger b_\lambda\big] + \sum_{\lambda(k,l)>0} \lambda.$$

This only adds to the total Hamiltonian two harmonic oscillators for every $\lambda(k,l)$, one for the a_λ-mode and one for the b_λ-mode, and contributes to the normal ordering constant with a finite quantity. The eigenstates and eigenenergies are trivial to write. Assume for example $m^2\kappa^2 = 6$. There is then only one solution for (k,l), namely $k = 2, l = 1$ and $\lambda = \sqrt{10}$. This adds two oscillators to the total Hamiltonian and $\sqrt{10}$ to the normal ordering constant.

For $m\kappa = \text{even}$ and $m\kappa = \text{odd}$, the Hamiltonian adds an m-dependent momentum-like contribution to the energy.

6. Conclusion and outlook

In this paper we have canonically quantized the open string on the Penrose limit of $dS_n \times S^n$ supported by constant antisymmetric B_2 and a constant dilaton. Canonical quantization has proved perfectly suited for the task, thus making unnecessary to resort to Dirac quantization and avoiding the problem of whether the boundary conditions for the string endpoints should be regarded as first or second class constraints. The position operators for the quantized string define non-commutative spaces, the wave fronts, for all values of the string parameter σ. Noticeably non-commutativity is not restricted to the string endpoints but extends outside the brane on which the endpoints may be assumed to move. The Minkowski limit is smooth and reproduces the results in the literature [19].

We think that further investigation of strings on plane-wave backgrounds is worth to understand non-commutativity in relation with gravity. The low-energy field-theory limit looks particularly interesting since it may shed light on an effective theory for non-commutative gravity. It must be mentioned in this regard that there is a vast literature [30] on the formulation of Seiberg–Witten maps for gravity and effective non-commutative corrections to general relativity solutions, plane waves among them [31].

From a purely string theory point of view, the strings considered here may be thought of as “in” or “out” states to study string scattering on more complicated spaces, which in turn will have a Penrose limit, and strings near space–time singularities [32].

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Appendix A. Explicit expression for the string momentum

We collect here the contributions to the string momentum components $P_i(\tau,\sigma)$ in Eq. (2.37) of the various existing modes. They are obtained by using (2.11) for X^i_{odd}, X^i_{even}, $X^i_{(k,l)}$ and X^i_λ. For the modes X^i_{o} and X^i_{e}, in (2.16)–(2.17) and (2.18)–(2.19), we have

$$2\pi\alpha' P_{1,\text{o}} = -\frac{m\kappa}{B} b_{\text{o}} \left\{ \sinh\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma) - B^2 \sinh\left[m\kappa\left(\frac{\pi}{2}-\sigma\right)\right]\right\},$$

$$2\pi\alpha' P_{2,\text{o}} = -m\kappa B\left[a_{\text{o}} - \frac{m\kappa}{B} b_{\text{o}}\tau \sinh\left(\frac{m\kappa\pi}{2}\right)\right]\sin(m\kappa\sigma)$$

and

$$2\pi\alpha' P_{1,\text{e}} = \frac{m\kappa}{B} b_{\text{e}} \left\{ \cosh\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma) + B^2 \cosh\left[m\kappa\left(\frac{\pi}{2}-\sigma\right)\right]\right\},$$

$$2\pi\alpha' P_{2,\text{e}} = -m\kappa B\left[a_{\text{e}} + \frac{m\kappa}{B} b_{\text{e}}\tau \cosh\left(\frac{m\kappa\pi}{2}\right)\right]\sin(m\kappa\sigma).$$

The contribution of the modes $X^i_{(k,l)}$ in (2.22)–(2.23) in turn reads

$$2\pi\alpha' P_{1,(k,l)} = \left[\frac{\alpha}{B} a_{\lambda(k,l)}\left(\cos\beta\sigma + B^2\cos\alpha\sigma\right) + b_{\lambda(k,l)}\left(\sin\beta\sigma - \frac{\alpha\beta}{\lambda^2}\sin\alpha\sigma\right)\right]e^{-i\lambda\tau},$$

$$\begin{aligned}2\pi\alpha' P_{2,(k,l)} = i\lambda\bigg[& a_{\lambda(k,l)}\left(\sin\alpha\sigma - \frac{\alpha\beta}{\lambda^2}\sin\beta\sigma\right) \\ & + \frac{\beta}{B\lambda^2} b_{\lambda(k,l)}\left(\cos\alpha\sigma + B^2\cos\beta\sigma\right)\bigg]e^{-i\lambda\tau}.\end{aligned}$$

Finally, the modes X^i_λ in (2.33)–(2.35) yield the contributions

$$\begin{aligned}2\pi\alpha' P_{1,\pm}(\tau,\sigma) = c_\lambda \frac{\alpha}{B}\bigg[& \cos\beta\sigma + \frac{\sin\beta\pi}{\cos\beta\pi \mp 1}\sin\beta\sigma \\ & - B^2\left(\frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi}\sin\alpha\sigma - \cos\alpha\sigma\right)\bigg]e^{-i\lambda\tau}\end{aligned}$$

and

$$\begin{aligned}2\pi\alpha' P_{2,\pm}(\tau,\sigma) = ic_\lambda\lambda\bigg[& \frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi}\cos\alpha\sigma + \sin\alpha\sigma \\ & - \frac{\alpha\beta}{\lambda^2}\left(\sin\beta\sigma - \frac{\sin\beta\pi}{\cos\beta\pi \mp 1}\cos\beta\sigma\right)\bigg]e^{-i\lambda\tau}.\end{aligned} \tag{A.1}$$

Appendix B. Derivation of Eqs. (4.6)–(4.7)

Organizing the modes in the four sets Λ^{I}, Λ^{R}, Λ, $\tilde\Lambda$ introduced in Section 3 and expanding (4.2) in powers of $m^2\kappa^2 \ll 1$, the function $\Theta(\sigma,\sigma')$ becomes a sum

$$i\Theta(\sigma,\sigma') = \sum_{k=I,R,\Lambda,\tilde\Lambda} i\Theta(k;\sigma,\sigma')$$

of four contributions $\Theta(k;\sigma,\sigma')$, each one of which is a power series in $m\kappa$. Up to order four in $m\kappa$, these contributions read

$$i\Theta(\mathrm{I};\sigma,\sigma') = \frac{i\alpha' B}{2(1+B^2)}(\pi-2\sigma) - \frac{i\alpha' B(m\kappa)^2}{12(1+B^2)^2}\big[\sigma(\sigma-\pi)\big(2+B^2\big) - 3B^2\sigma'(\sigma'-\pi) - \pi^2\big] + \mathcal{O}\big(m^4\kappa^4\big), \tag{B.1}$$

$$i\Theta(\mathrm{R};\sigma,\sigma') = \frac{i\alpha' B}{2(1+B^2)}(\pi-2\sigma') + \frac{i\alpha' B(m\kappa)^2}{12(1+B^2)^2}\big[\sigma'(\sigma'-\pi)\big(2+B^2\big) - 3B^2\sigma(\sigma-\pi) - \pi^2\big] + \mathcal{O}\big(m^4\kappa^4\big), \tag{B.2}$$

$$i\Theta(\Lambda;\sigma,\sigma') = -\frac{2i\alpha' B}{1+B^2}\sum_{n=1}^{\infty}\frac{\cos n\sigma\,\sin n\sigma'}{n} + \frac{i\alpha' B(m\kappa)^2}{(1+B^2)^2} \times \sum_{n=1}^{\infty}\bigg[B^2(2\sigma-\pi)\frac{\sin n\sigma\,\sin n\sigma'}{n^2} + (2\sigma'-\pi)\frac{\cos n\sigma\,\cos n\sigma'}{n^2} - 2\big(1-B^2\big)\frac{\cos n\sigma\,\sin n\sigma'}{n^3}\bigg] + \mathcal{O}\big(m^4\kappa^4\big) \tag{B.3}$$

and

$$i\Theta(\tilde{\Lambda};\sigma,\sigma') = -\frac{2i\alpha' B}{1+B^2}\sum_{n=1}^{\infty}\frac{\sin n\sigma\,\cos n\sigma'}{n} - \frac{3i\alpha' B(m\kappa)^2}{(1+B^2)^2}\sum_{n=1}^{\infty}\frac{\sin n\sigma\,\cos n\sigma'}{n^3} + \mathcal{O}\big(m^4\kappa^4\big). \tag{B.4}$$

Summing all the contributions of order zero in $m\kappa$ in these equations, we have

$$i\Theta_0(\sigma,\sigma') = \frac{i\alpha' B}{1+B^2}\big[\pi - \sigma_+ - F_1(\sigma_+)\big],$$

where $F_1(\sigma_+)$ is the Fourier series (3.28). This trivially leads to the order zero contributions in Eqs. (4.6) and (4.7). To sum the order two contributions, we first note that

$$\big[i\Theta(\Lambda) + i\Theta(\tilde{\Lambda})\big]_2 = -\frac{i\alpha' B}{4(1+B^2)^2}\big\{\big[B^2(\sigma_+ + \sigma_- - \pi) + (\sigma_+ - \sigma_- - \pi)\big]F_2(\sigma_-) - \big[B^2(\sigma_+ + \sigma_- - \pi) - (\sigma_+ - \sigma_- - \pi)\big]F_2(\sigma_+) + \big(2B^2+1\big)F_3(\sigma_+) + \big(5-2B^2\big)F_3(\sigma_-)\big\}, \tag{B.5}$$

where the Fourier series $F_2(\sigma_\pm)$ are as in (3.29)–(3.30) and $F_3(\sigma_\pm)$ read

$$F_3(\sigma_-) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_-}{n^3} = \frac{\sigma_-^3}{6} - \frac{\pi}{2}\sigma_-|\sigma_-| + \frac{\pi^2}{3}\sigma_-,$$

$$F_3(\sigma_+) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_+}{n^3} = \frac{\sigma_+^3}{6} - \frac{\pi}{2}\sigma_+^2 + \frac{\pi^2}{3}\sigma_+.$$

Eqs. (B.1), (B.2) and (B.5) then lead to the second order contributions in Section 5.

References

[1] R. Penrose, in: M. Cahen, M. Flato (Eds.), Differential Geometry and Relativity, Reidel, Dordrecht, 1976, p. 271.
[2] R. Penrose, Rev. Mod. Phys. 37 (1965) 215;
For a summary of the geometric properties, see H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers, E. Herlt, Exact Solutions to Einstein's Field Equations, second ed., Cambridge Univ. Press, Cambridge, 2003;
G.W. Gibbons, Commun. Math. Phys. 45 (1975) 191.
[3] D. Amati, C. Klimcik, Phys. Lett. B 219 (1989) 443.
[4] G.T. Horowitz, A.R. Steif, Phys. Rev. Lett. 64 (1990) 260.
[5] R. Güven, Phys. Lett. B 482 (2000) 255, hep-th/0005061.
[6] J. Kowalski-Glikman, Phys. Lett. B 134 (1984) 194.
[7] M. Blau, J. Figueroa-O'Farrill, C. Hull, G. Papadopoulos, JHEP 0201 (2002) 047, hep-th/0110242.
[8] M. Blau, J. Figueroa-O'Farrill, G. Papadopoulos, Class. Quantum Grav. 19 (2002) 4573, hep-th/0202111.
[9] R.R. Metsaev, Nucl. Phys. B 625 (2002) 70, hep-th/0112044;
R.R. Metsaev, A.A. Tseytlin, Phys. Rev. D 65 (2002) 126004, hep-th/0202109.
[10] D.E. Berenstein, J.M. Maldacena, H.S. Nastase, JHEP 0204 (2002) 013, hep-th/0201093.
[11] J.G. Russo, A.A. Tseytlin, JHEP 0204 (2002) 021, hep-th/0202179.
[12] C. Nappi, E. Witten, Phys. Rev. Lett. 71 (1993) 3751, hep-th/9310112.
[13] G. Papadopoulos, J.G. Russo, A.A. Tseytlin, Class. Quantum Grav. 20 (2003) 969, hep-th/0211289.
[14] H. Fuji, K. Ito, Y. Sekino, JHEP 0211 (2002) 005, hep-th/0209004.
[15] E. Silverstein, Simple de Sitter solutions, arXiv: 0712.1196 [hep-th].
[16] E. Witten, Quantum gravity in de Sitter space, hep-th/0106109.
[17] A. Strominger, JHEP 0110 (2001) 034, hep-th/0106113.
[18] N. Seiberg, E. Witten, JHEP 9909 (1999) 032, hep-th/9908142.
[19] C.S. Chu, P.M. Ho, Nucl. Phys. B 550 (1999) 151, hep-th/9812219.
[20] F. Ardalan, H. Arfaei, M.M. Sheikh-Jabbari, JHEP 9902 (1999) 016, hep-th/9810072.
[21] S. Doplicher, K. Fredenhagen, J. Roberts, Commun. Math. Phys. 172 (1995) 187, hep-th/0303037.
[22] C.S. Chu, P.M. Ho, Nucl. Phys. B 636 (2002) 219, hep-th/0203186.
[23] L. Dolan, C.R. Nappi, Phys. Lett. B 551 (2003) 369, hep-th/0210030.
[24] G.T. Horowitz, A.R. Steif, Phys. Rev. D 42 (1990) 1950.
[25] J. Polchinski, String Theory, vol. I, Cambridge Univ. Press, Cambridge, 2000.
[26] F. Ardalan, H. Arfaei, M.M. Sheikh-Jabbari, Nucl. Phys. B 576 (2000) 578, hep-th/9906161.
[27] C.S. Chu, P.M. Ho, Nucl. Phys. B 568 (2000) 447, hep-th/9906192.
[28] E. Witten, Commun. Math. Phys. 92 (1984) 455.
[29] L.D. Faddeev, R. Jackiw, Phys. Rev. Lett. 60 (1988) 1692.
[30] R.J. Szabo, Class. Quantum Grav. 23 (2006) R199, hep-th/0606233.
[31] S. Marculescu, F. Ruiz Ruiz, Phys. Rev. D 74 (2006) 105004, hep-th/0607201.
[32] H. de Vega, N. Sánchez, Phys. Rev. D 45 (1992) 2783;
H. de Vega, M. Ramón Medrano, N. Sánchez, Class. Quantum Grav. 10 (1993) 2007.

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Geometric construction of D-branes in WZW models

G. Horcajada and F. Ruiz Ruiz

Departamento de Física Teórica I, Universidad Complutense de Madrid, Ciudad Universitaria s/n, 28040 Madrid, Spain

E-mail: ghr@fis.ucm.es, ferruiz@fis.ucm.es

Abstract: The geometric description of D-branes in WZW models is pushed forward. Our starting point is a gluing condition $J_+ = FJ_-$ that matches the model's chiral currents at the worldsheet boundary through a linear map F acting on the WZW Lie algebra. The equivalence of boundary and gluing conditions of this type is studied in detail. The analysis involves a thorough discussion of Frobenius integrability, shows that F must be an isometry, and applies to both metrically degenerate and nondegenerate D-branes. The isometry F need not be a Lie algebra automorphism nor constantly defined over the brane. This approach, when applied to isometries of the form $F = R$ with R a constant Lie algebra automorphism, validates metrically degenerate R-twined conjugacy classes as D-branes. It also shows that no D-branes exist in semisimple WZW models for constant $F = -R$.

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Contents

1 Introduction

D-branes have become one of the main research topics in the string literature since the mid nineties. There are many reasons for this. Among them, the evidence that D-branes provide soliton and bound states in string backgrounds [1] and the realization that they become upon quantization noncommutative spacetimes [2–7].

Since WZW models are the building blocks of many string backgrounds [8–10], one sensible program to study D-branes and their properties is to consider their occurrence in models of this type. In fact, there are various approaches to D-branes in WZW models. Among them, the geometric approach [11–21], that regards D-branes as spacetime's submanifolds on which the string worldsheet boundary may be embedded, and the algebraic approach [22–29], that makes use of boundary conformal field theory.

In this paper we reexamine the geometric description of D-branes in a WZW model. The definition of D-brane that we will be using is the naïve geometric one; see section 2 for details. A Dp-brane in a string background $(G_{\mu\nu}, H_{\mu\nu\rho})$ is any $(p+1)$-dimensional submanifold N containing all possible motions for the string endpoints. These motions are specified by the boundary conditions for the string, which in turn can be viewed as a system of first order differential equations characterized by a two-form ω globally defined[1] on N such that $d\omega = H\big|_N$. A way to construct D-branes is thus to find all two-forms ω for which the boundary conditions can be integrated.

Our starting point for the geometric characterization of D-branes in a WZW model is a condition $J_+ = FJ_-$, called gluing condition, that matches the model's chiral currents J_- and J_+ at the world sheet boundary through a linear map F that acts on the model's Lie algebra. This matching condition is not a boundary condition, for it is not obtained by setting to zero the boundary term that arises from the variation of the model's classical action. However, it does specify, for every linear map F, vector fields characterizing tangent motions of the string endpoints. If these vector fields define an integrable distribution, they span the tangent bundle of a submanifold N of the spacetime group manifold. The submanifold N is a D-brane if the gluing condition can be written as a boundary condition with a two-form ω globally defined on N such that $d\omega = H\big|_N$.

We cross examine this approach for WZW models with arbitrary real Lie group G. Our only assumption is that the corresponding Lie algebra admits an invariant nondegenerate metric Ω. This includes in particular noncompact group manifolds with Lorentzian signature, for which there exist metrically degenerate submanifolds N such that the tangent space T_gG at any point g in N cannot be written as an orthogonal sum $T_gN \oplus T_gN^\perp$.

If F is a constant Ω-preserving Lie algebra automorphism and the orthogonal decomposition $T_gG = T_gN \oplus T_gN^\perp$ is assumed, the vector fields defined by F are known to be integrable and the two-form ω satisfying $d\omega = H\big|_N$ is well known [11–14]. Our interest is in cases escaping these two assumptions. In this more general setting, the situation is very different. Firstly, because for an arbitrary linear map $F(g)$, the vector fields specified by the gluing condition, call them t_i, do not always define an integrable distribution. And secondly, because even if they do, the corresponding gluing condition cannot always be written as a boundary condition with a two-form ω globally defined on N such that $d\omega = H\big|_N$. In this regard, we prove the following two results. Every boundary condition for a D-brane N can be written as a gluing condition $J_+ = FJ_-$, provided $\det\left(G\big|_N - \omega\right) \neq 0$, where $G\big|_N$ is the induced metric on N. And every gluing condition can be written as a boundary condition if the linear map $F(g)$ is an isometry of Ω, in which case the two-form ω exists globally and is uniquely defined by its action $\omega(t_i, t_j)$ on the vector fields t_i defined by $F(g)$. For a general isometry $F(g)$, the requirement $d\omega = H\big|_N$ however does not hold but it becomes a matter of straightforward algebra to check it in every instance. These two results open some problems, among them studying the applicability of this approach to D-branes for which a full set of gluing conditions is not known [19, 35–37].

The paper is organized as follows. Section 2 poses the problem and reviews the description of D-branes in WZW models in terms of the gluing condition for the chiral currents.

[1] We will use B for two-forms locally defined on the whole group manifold such that $dB = H$, and the Greek letter ω for the two-form globally defined on the submanifold N.

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In section 3, integrability in terms of Frobenius theorem is studied and it is shown that the two-form ω for which the gluing condition becomes a boundary condition exists if and only if $F(g)$ is an isometry of Ω. Section 4 contains a discussion of the limitations of the gluing condition approach. Isometries of the form $F = \pm R$, where R is a constant Lie algebra automorphism, are considered in section 5. The case $F = R$ has been studied by other Authors [11–14] under the hypothesis that $T_g G = T_g N \oplus T_g N^\perp$, the resulting D-branes being R-twined conjugacy classes. It is shown that this result holds even if the latter assumption on $T_g G$ fails. As regards the case $F = -R$, it is proved that, contrarily to some claims, the gluing condition for $F = -R$ does not provide D-branes for semisimple Lie algebras. In section 6, some examples of g-dependent isometries $F(g)$ are considered. It is shown that two different isometries may define the same integrable distribution but different two-forms ω, one of them satisfying $d\omega = H\big|_N$, hence defining a D-brane, and the other one not. We close the paper with our conclusions and three short appendices collecting technical points.

2 Gluing conditions for chiral currents

In the sigma model approach, a Dp-brane in a string background $(G_{\mu\nu}, H_{\mu\nu\lambda})$ is a $(p+1)$-dimensional submanifold N on which the endpoints of an open string may lie. The submanifold N has embedded coordinates $x^\mu(\tau) = X^\mu(\tau,\sigma)\big|_{\partial\Sigma}$ and these must satisfy the boundary conditions

$$\left(\partial_i f^\mu \, G_{\mu\nu} \, \partial_\sigma X^\nu - \omega_{ij} \, \partial_\tau \alpha^j\right)\Big|_{\partial\Sigma} = 0 \qquad i = 1, \dots, p+1 \,. \tag{2.1}$$

Here $\alpha^1, \dots, \alpha^{p+1}$ are local coordinates on the Dp-brane, so that $x^\mu = f^\mu(\alpha^1, \dots, \alpha^{p+1})$, and ω_{ij} are the components of a two-form ω globally defined on the brane such that $d\omega = H\big|_N$.

We are interested in D-branes in string backgrounds described by WZW models [30] with real Lie group G and Lie algebra $\mathfrak{g}$, both of dimension $\mathtt{d}$. The Lie algebra $\mathfrak{g}$ is a vector space over $\mathbf{R}$ and has generators $\{T_A\}$ with commutation relations

$$[\, T_A, T_B \,] = f_{AB}{}^C \, T_C \qquad A, B, C = 1, \dots, \mathtt{d} \,. \tag{2.2}$$

The algebra $\mathfrak{g}$ is assumed to have a nondegenerate invariant metric Ω, of arbitrary signature, with components $\Omega_{AB} = \Omega(T_A, T_B)$, so that

$$\Omega\big(\, [T_A, T_B], \, T_C\big) = \Omega\big(T_A, \, [T_B, T_C]\,\big) \quad \Leftrightarrow \quad f_{AB}{}^D \, \Omega_{DC} = \Omega_{AD} \, f_{BC}{}^D \,. \tag{2.3}$$

The existence of such a metric is the only restriction on $\mathfrak{g}$. The group G is taken as the connected component obtained from $\mathfrak{g}$ through exponentiation.

If X^μ are local coordinates in G, the left-invariant $e^A{}_\mu$ and right-invariant $\bar{e}^A{}_\mu$ vielbeins that map the group G to its tangent space $T_g G$ at g are defined by

$$g^{-1} \, dg = T_A \, e^A{}_\mu \, dX^\mu \qquad dg \, g^{-1} = T_A \, \bar{e}^A{}_\mu \, dX^\mu \,.$$

In terms of them, the adjoint action of the group on the Lie algebra is

$$\mathrm{Ad}_g(T_A) = g \, T_A \, g^{-1} = T_B \, \bar{e}^{\,B}{}_\mu \, (e^{-1})^\mu{}_A \quad \Leftrightarrow \quad \mathrm{Ad}_g = \bar{e} \, e^{-1} \,. \tag{2.4}$$

The string background $(G_{\mu\nu}, H_{\mu\nu\lambda})$ is defined from Ω by

$$\begin{aligned} G_{\mu\nu} &= \Omega\left(g^{-1}\partial_\mu g\,,\, g^{-1}\partial_\nu g\right) \\ H_{\mu\nu\lambda} &= \Omega\left(\left[g^{-1}\partial_\mu g\,,\, g^{-1}\partial_\nu g\right]\,,\, g^{-1}\partial_\lambda g\right). \end{aligned} \tag{2.5}$$

By construction, the metric $G_{\mu\nu}$ is bi-invariant,

$$G_{\mu\nu} = e^A{}_\mu\,\Omega_{AB}\,e^B{}_\nu = \bar{e}^A{}_\mu\,\Omega_{AB}\,\bar{e}^B{}_\nu \quad \Leftrightarrow \quad G = e^{\mathrm{T}}\Omega\, e = \bar{e}^{\mathrm{T}}\Omega\,\bar{e}\,, \tag{2.6}$$

the superscript T denoting transposition. In this paper the standard notation $G(a,b) = G_{\mu\nu}a^\mu b^\nu$ will be used.

The WZW classical action for the open string in the background $(G_{\mu\nu}, H_{\mu\nu\lambda})$ can be written as [31]

$$S_{\mathrm{WZW}} = \frac{k}{4\pi}\int_\Sigma d^2\sigma\;\Omega\left(g^{-1}\partial_a g, g^{-1}\partial^a g\right) + \frac{k}{4\pi}\left(\int_\Sigma g^*B + \int_{\partial\Sigma} g^*A\right), \tag{2.7}$$

with $g = g(X^\mu(\tau,\sigma))$ and $\sigma^a = (\tau,\sigma)$ world sheet indices. Here B is any two-form defined on G such that $H = dB$, and g^*B is its pullback. The form B may not be globally defined, but must exist locally. This is the case, for example, if H is not exact. The one-form A is defined on the D-brane, exists at least locally and is such that $dA = \omega - B\big|_N$. See ref. [31] for details.

In worldsheet coordinates $\sigma^\pm = \tau \pm \sigma$, the field equations read $\partial_+ J_- = \partial_- J_+ = 0$, where the chiral currents J_- and J_+ are given by

$$J_-(\sigma^-) = g^{-1}\partial_- g \qquad J_+(\sigma^+) = -\,\partial_+ g\, g^{-1}\,.$$

Due to the simplicity of the solutions for J_+ and J_-, we are interested in formulating the boundary conditions for a D-brane in terms of J_+ and J_-. We will then assume that there exists a mapping F from $\mathfrak{g}$ to $\mathfrak{g}$ relating the two currents at the world sheet boundary, that is, $J_+ = F(J_-)$ at $\sigma^+ = \sigma^-$. Recalling that D-branes in the sigma model approach are defined by boundary conditions of order one in $\partial_\pm X^\mu\big|_{\partial\Sigma}$ and noting that J_- and J_+ are already order one in $\partial_\pm X^\mu$, we restrict ourselves to linear maps $F(g)$ that may depend on g but not on $J_\pm(g)$. For the chiral currents at a D-brane we thus require

$$J_+\Big|_{\sigma^+=\sigma^-} = F(g)J_-\Big|_{\sigma^+=\sigma^-}\,, \tag{2.8}$$

with $F(g)$, for every g in N, a linear map that acts on $\mathfrak{g}$ as a vector space. The linear map $F(g)$ is represented by a real $\mathtt{d}\times\mathtt{d}$ matrix with entries $F^A{}_B$ given by $F(g)T_B = T_A\,F^A{}_B(g)$. It is important to note that eq. (2.8) is not a boundary condition derived from the classical action above but a working hypothesis formulated ad hoc. To keep this in mind, eq. (2.8) is called gluing condition, rather than boundary condition. We will take it as starting point for the construction of D-branes.

Eq. (2.8) defines a D-brane if it can be written as a sigma model boundary condition (2.1), with ω a two-form globally defined on N such that [31]

$$H\big|_N = d\omega\,. \tag{2.9}$$

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We will show in section 3 that every boundary condition (2.1) can be written as a gluing condition (2.8), with F and isometry of Ω, except for D-branes N such that $\det(G\big|_N - \omega) = 0$, where $G\big|_N$ is the induced metric on N. Apart from these instances, the gluing condition is capable of constructing all D-branes defined by boundary conditions.

It is convenient to write the gluing condition (2.8) in terms of local coordinates X^μ. To do this [13], it is enough to use for the chiral currents their expressions

$$J_- = \partial_- X^\mu \, e^A{}_\mu \, T_A \qquad J_+ = -\, \partial_+ X^\mu \, \bar{e}^A{}_\mu \, T_A$$

in terms of the string coordinates X^μ and the left and right-invariant vielbeins $e^A{}_\mu$ and $\bar{e}^A{}_\mu$. This yields

$$\partial_+ X^\mu \Big|_{\partial\Sigma} = \mathcal{F}^\mu{}_\nu \, \partial_- X^\nu \Big|_{\partial\Sigma} , \tag{2.10}$$

where the matrix $\mathcal{F}$ is defined by

$$\mathcal{F}(x) = -\, \bar{e}^{\,-1} F(g) \, e \, . \tag{2.11}$$

In worldsheet coordinates τ and σ, eq. (2.10) takes the form

$$\big(\mathcal{F} - 1\big) \, \partial_\tau X \big|_{\partial\Sigma} = \big(\mathcal{F} + 1\big) \, \partial_\sigma X \big|_{\partial\Sigma} \, . \tag{2.12}$$

We emphasize that the matrix $\mathcal{F}(x)$ is defined on N and depends on the string endpoints coordinates $x^\mu = X^\mu\big|_{\partial\Sigma}$ through $e(x)$, $\bar{e}(x)$ and $F\big(g(x)\big)$. It however acts on arbitrary tangent vectors in $T_g G$. To ease the notation, whenever there is no confusion we will remove from $F(g)$ and $\mathcal{F}(g)$ the dependence on g.

3 Integration of the gluing condition and D-branes

In this section, the gluing condition (2.12) is explicitly solved for $\partial_\tau X^\mu\big|_{\partial\Sigma}$. The solution happens to be given in terms of vector fields defined by the linear map F. The involutivity requisite that such fields must satisfy to define a foliation of G in terms of a family of submanifolds N is studied in detail. Finally, it is shown that the gluing condition for a linear map F takes the form of a boundary condition if and only if F is an isometry of Ω, and the two-form ω is constructed from F. When the resulting ω satisfies $d\omega = H\big|_N$, the submanifold N is a D-brane.

3.1 Conditions for the existence of a D-brane

The set of possible motions of the string endpoints at an arbitrary spacetime point g is the set Π_g of solutions $t^\mu(x) := \partial_\tau X^\mu\big|_{\partial\Sigma}$ to the gluing condition (2.12) for some $u^\mu(x) := \partial_\sigma X^\mu\big|_{\partial\Sigma}$. Since $t(x) = t^\mu(x)\,\partial_\mu$ and $u(x) = u^\mu(x)\,\partial_\mu$ are tangent vectors to G at g, we may write

$$\Pi_g = \Big\{ t \in T_g(G)\colon \; \big[\mathcal{F}(g) - 1\big] t = \big[\mathcal{F}(g) + 1\big] u \;\; \text{for} \;\; u \in T_g(G) \Big\}.$$

Equivalently,

$$\Pi_g = \big\{ t \in T_g(G)\colon \; (\mathcal{F} - 1)\, t \in \mathrm{Im}\,(\mathcal{F} + 1) \big\} \, . \tag{3.1}$$

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The set Π_g is a linear subspace of the tangent space T_gG and we will often call it the tangent plane at g defined by F.

Consider $v \in \mathrm{Im}\,(\mathcal{F}+1)$, so that there exists w in $T_g(G)$ such that $v = (\mathcal{F}+1)\,w$. It follows that $(\mathcal{F}-1)\,v = (\mathcal{F}+1)\,(\mathcal{F}-1)\,w$ belongs to $\mathrm{Im}\,(\mathcal{F}+1)$. Hence v is in Π_g and

$$\mathrm{Im}\,(\mathcal{F}+1) \subset \Pi_g\,.$$

Consider now v' in Π_g. It then exists w' in T_gG such that $(\mathcal{F}-1)v' = (\mathcal{F}+1)w'$. This implies that $v' = \frac{1}{2}(\mathcal{F}+1)(v'-w')$, so that v' belongs to $\mathrm{Im}\,(\mathcal{F}+1)$ and

$$\Pi_g \subset \mathrm{Im}\,(\mathcal{F}+1)\,.$$

Hence

$$\Pi_g = \mathrm{Im}\,(\mathcal{F}+1)\,. \tag{3.2}$$

An alternative derivation of this result in terms of the eigenvectors of $\mathcal{F}$ can be found in appendix A.

Since the gluing condition (2.12) holds for arbitrary g, the solution $t(x)$ defines a vector field for a given $u(x)$. If M is a submanifold of G, we define

$$\Pi^M = \{(g, \Pi_g):\ g \in M\,\}\,. \tag{3.3}$$

Π^M is a distribution on M if the tangent plane Π_g has the same dimension for all g in M. According to Frobenius theorem [33], a distribution Π^M is integrable if and only if it is involutive. Integrability ensures that Π_g is, for all g in M, not just a tangent plane but the tangent space to a submanifold N of M, that is $\Pi_g = T_gN$. Involutivity states that the commutator of any two vector fields t_1 and t_2 taking values in Π^M also takes values in Π^M,

$$[t_1, t_2]\,(g) \in \Pi_g\,. \tag{3.4}$$

For the manifold N to define a D-brane, it must contain all the points g in G connected by the integral curves of the vector fields t. This condition cannot be relaxed, since one would then leave out from the D-brane points at which the open string may end. See section 4 for examples.

As a practical matter, to determine if a linear map F defines a D-brane, one may proceed in three steps:

Step 1. Study the rank of the matrix $\mathcal{F}(g)+1$ as a function of g. Consider a submanifold $D_n(F)$ formed by the points g in G such that (i) the rank of $\mathcal{F}(g)+1$ is n, and (ii) g is not connected by integral curves of the vector fields t with points g' at which the rank of $\mathcal{F}(g')+1$ is different from n.

Step 2. Check the involutivity condition (3.4) in $D_n(F)$. If it holds, the distribution $\Pi^{D_n(F)}$ is the tangent bundle of a submanifold N of G of dimension n, or more precisely of a family of submanifolds which foliate $D_n(F)$.

Step 3. Find a two-form ω globally defined on N for which the gluing condition for F can be recast as a sigma model boundary condition and such that $d\omega = H\big|_N$. If such a ω exists, the submanifold N is a D-brane of dimension n.

In what follows we further elaborate these three steps.

3.2 Involutivity in detail

The definition of $\mathcal{F}$ in (2.11) and the expression for the group adjoint action in (2.4) imply that $\mathcal{F}+1 = e^{-1}(-\mathrm{Ad}_{g^{-1}}F + 1)\,e$. The space of tangent directions $\Pi_g = \mathrm{Im}(\mathcal{F}+1)$ can then be written as

$$\Pi_g = g\left[\,\mathrm{Ad}_{g^{-1}}F(g) - 1\,\right]\mathfrak{g}\,. \tag{3.5}$$

For every V in the Lie algebra $\mathfrak{g}$,

$$g\left[\mathrm{Ad}_{g^{-1}}F(g) - 1\right]V = F(g)\,V g - g\,V \tag{3.6}$$

is a vector field. It is actually the sum of a right-invariant vector field Yg, with $Y = F(g)V$, and a left-invariant vector field gY, with $Y = -V$.

Right and left-invariant vector fields act on differentiable functions f defined on G and taking values in $\mathbf{R}$ according to

$$Yg\left(f(g)\right) = \frac{d}{dt}\, f\left(e^{tY} g\right)\bigg|_{t=0} \qquad gY\left(f(g)\right) = \frac{d}{dt}\, f\left(g\, e^{tY}\right)\bigg|_{t=0}. \tag{3.7}$$

If g is parameterized by coordinates x^μ, the vector field components of Yg and gY are

$$\begin{aligned} Yg &= Y^A\, T_A\, g = Y^A\,(\bar{e}^{\,-1})^\mu{}_A\,\partial_\mu = (Y g)^\mu\,\partial_\mu \\ gY &= Y^A\, gT_A = Y^A\,(e^{-1})^\mu{}_A\,\partial_\mu = (gY)^\mu\,\partial_\mu\,. \end{aligned}$$

These equations and $V^A = e^A{}_\mu\, v^\mu$ provide $F(g)Vg - gV = -\left[(\mathcal{F}+1)\,v\right]^\mu\partial_\mu$, which again gives for the vector field t the form used in eq. (3.2). Since $\{T_A\}$ is a basis of $\mathfrak{g}$, the vector fields that define Π_g read

$$t_A = FT_Ag - gT_A = \left[\,(\bar{e}^{\,-1})^\mu{}_B F^B{}_A - (e^{-1})^\mu{}_A\,\right]\partial_\mu\,. \tag{3.8}$$

These fields completely determine the motions of the string endpoints solving the gluing condition with linear map F.

The rank of $\mathcal{F}+1$ is obviously equal to the rank of $\mathrm{Ad}_{g^{-1}}F(g) - 1$. Say it takes the value n for all g in a domain $D_n(F)$ in G. Assume further that the integral curves of the fields t_A are in $D_n(F)$. The involutivity condition (3.4) requires that, for all U and V in $\mathfrak{g}$ and for all g in $D_n(F)$, there exist W in $\mathfrak{g}$ such that

$$\left[\,F(g)\,Ug - g\,U\,,\, F(g)\,V\,g - gV\,\right] = F(g)\,Wg - gW\,. \tag{3.9}$$

It is important to keep in mind that W need not be the same for all g. Eq. (3.9) is an equation in $F(g)$, in the sense that W does not exist for every linear map $F(g)$. After expanding its left hand side, it becomes

$$\left[\,F(g)Ug\,,F(g)Vg\right] - \left[\,F(g)Ug\,,gV\,\right] - \left[\,gU\,,F(g)Vg\,\right] + \left[\,gU\,,gV\,\right] = F(g)\,Wg - gW\,. \tag{3.10}$$

Let us understand each one of the terms in this expression. Using eqs. (3.7), the action of first term on an arbitrary function f is

$$\left[F(g)Ug\,,\,F(g)Vg\right]\,\bigl(f(g)\bigr)=\frac{\partial^2}{\partial s\,\partial t}\,f\Bigl(e^{\,tF(e^{sF(g)U}g)\,V}\,e^{sF(g)U}g\Bigr)\,\Big|_{s=t=0}-\,\bigl(U\leftrightarrow V\bigr)\,.$$

After performing the derivatives with respect to s and t and using eqs. (3.7), this reduces to

$$\begin{aligned}\left[F(g)Ug\,,F(g)Vg\right]\,\bigl(f(g)\bigr)&=\left[F(g)V\,,\,F(g)U\right]g\,\bigl(f(g)\bigr)\\&+\Bigl(F(g)Ug\,\bigl(F(g)\bigr)\Bigr)\,Vg\,\bigl(f(g)\bigr)-\Bigl(F(g)Vg\,\bigl(F(g)\bigr)\Bigr)\,Ug\,\bigl(f(g)\bigr)\,.\end{aligned}\tag{3.11}$$

Proceeding similarly for the other commutators in eq. (3.10), we obtain

$$\left[F(g)Ug\,,gV\right]\,\bigl(f(g)\bigr)=-\Bigl(gV\,\bigl(F(g)\bigr)\Bigr)\,Ug\,\bigl(f(g)\bigr)\tag{3.12}$$

and

$$\bigl[gU\,,\,gV\bigr]\,\bigl(f(g)\bigr)=g\bigl[U,V\bigr]\,\bigl(f(g)\bigr)\,.\tag{3.13}$$

Taking eqs. (3.11)–(3.13) to eq. (3.10) and noting that f is arbitrary, we finally arrive at

$$\begin{aligned}-\bigl[FU,FV\bigr]\,g+g\,\bigl[U,V\bigr]&=FWg-gW\\&-\bigl(\,\bigl(FUg-gU\bigr)\,(F)\,\bigr)\,\,Vg+\bigl(\,\bigl(FVg-gV\bigr)\,(F)\,\bigr)\,\,Ug\,.\end{aligned}\tag{3.14}$$

The last two terms in the right hand side carry the action of the vector fields $F(g)Ug-gU$ and $F(g)Vg-gV$ on $F(g)$ as a function of g, the result being two linear operators that act on V and U.

If F does not depend on g, the action of $FUg-gU$ and $FVg-gV$ on F is zero and eq. (3.14) simplifies to

$$-\bigl[FU\,,FV\bigr]\,g+g\,\bigl[U\,,V\bigr]=FWg-gW.\tag{3.15}$$

3.3 Reduction of isometric gluing conditions to boundary conditions

Assume that the linear map $F(g)$ is such that steps 1 and 2 are satisfied. There is then a submanifold N whose tangent bundle Π^N is formed by the tangent spaces $T_gN=\mathrm{Im}(\mathcal{F}{+}1)$ for all g in N. In what follows we show that the necessary and sufficient condition for the gluing condition (2.12) to be equivalent to a boundary condition (2.1) is that $F(g)$ is an isometry of the Lie algebra metric Ω.

Since $\partial_\tau x$ belongs to T_gN, there exists v in T_gG such that $\partial_\tau x=(\mathcal{F}+1)v$ and the boundary condition (2.1) can be recast as

$$G\Bigl(u_0\,,\,\partial_\sigma X\big|_{\partial\Sigma}\Bigr)=\omega\Bigl(u_0\,,\,\bigl(\mathcal{F}+1\bigr)v\Bigr)\quad\text{for all}\quad u_0\in\mathrm{Im}\bigl(\mathcal{F}+1\bigr)\,.\tag{3.16}$$

The gluing condition (2.12) can in turn be written as

$$\bigl(\mathcal{F}+1\bigr)\,\partial_\sigma X\big|_{\partial\Sigma}=\bigl(\mathcal{F}-1\bigr)\bigl(\mathcal{F}+1\bigr)v\,.$$

This can be viewed as an equation in $\partial_\sigma X\big|_{\partial\Sigma}$, whose solutions are of the form

$$\partial_\sigma X\big|_{\partial\Sigma} = (\mathcal{F}-1)v + v_0\,, \tag{3.17}$$

with arbitrary v_0 in $\mathrm{Ker}\big(\mathcal{F}+1\big)$. Eq. (3.17) implies that

$$G\big(u_0\,,\,\partial_\sigma X\big|_{\partial\Sigma}\big) = G\big(u_0\,,\,(\mathcal{F}-1)v\big) + G\big(u_0, v_0\big) \quad \text{for all} \quad u_0 \in \mathrm{Im}\big(\mathcal{F}+1\big)\,. \tag{3.18}$$

Of the two terms on the right hand side, only the first one is linear in v. From this and the linearity in v of the boundary condition (3.16), we conclude that eq. (3.18) is compatible with the boundary condition (3.16) if and only if the following two requisites are met:

(1) $G\big(u_0, v_0\big) = 0$ for all u_0 in $\mathrm{Im}\big(\mathcal{F}+1\big)$ and all v_0 in $\mathrm{Ker}\big(\mathcal{F}+1\big)$, and

(2) the action of the two-form ω on arbitrary $(\mathcal{F}+1)u$ and $(\mathcal{F}+1)v$ in T_gN is given by

$$\omega\big(\,(\mathcal{F}+1)\,u\,,\,(\mathcal{F}+1)\,v\,\big) = G\big(\,(\mathcal{F}+1)\,u\,,\,(\mathcal{F}-1)\,v\,\big)\,. \tag{3.19}$$

For eq. (3.19) to make sense, its right hand side must be antisymmetric,

$$0 = G\big(\,(\mathcal{F}+1)u\,,\,(\mathcal{F}-1)v\,\big) + G\big(\,(\mathcal{F}+1)v\,,\,(\mathcal{F}-1)u\,\big) = 2\,G\big(\mathcal{F}u\,,\mathcal{F}v\big) - 2\,G\big(u\,,v\big)\,.$$

Since u and v are arbitrary in T_gG, the operator $\mathcal{F}$, defined on N, acts isometrically on the whole tangent space T_gG,

$$G\big(\mathcal{F}u\,,\mathcal{F}v\big) = G\big(u\,,v\big)\,. \tag{3.20}$$

This in turn implies that

$$\mathrm{Im}\,(\mathcal{F}\pm 1) = \mathrm{Ker}\,(\mathcal{F}\pm 1)^{\perp} \tag{3.21}$$

and makes condition (1) trivial. Furthermore, given v in T_gG, consider $v' = v + v_0'$ in T_gG, with arbitrary v_0' in $\mathrm{Ker}(\mathcal{F}+1)$. From eq. (3.21) it follows that

$$G\big(\,(\mathcal{F}+1)\,u\,,\,(\mathcal{F}-1)\,v'\,\big) = G\big(\,(\mathcal{F}+1)\,u\,,\,(\mathcal{F}-1)\,v\,\big)\,.$$

In other words, the right hand side in eq. (3.19) does not depend on the choice of v_0 in (3.17) and the two-form ω as defined by eq. (3.19) is single valued. Finally, ω exists globally on N since it is given by eq. (3.19) through its action on arbitrary vectors $(\mathcal{F}+1)u$ and $(\mathcal{F}+1)v$ in T_gN for any g in N.

From the definition (2.11) of $\mathcal{F}$, the bi-invariance property (2.6) of the metric G and eq. (3.20), it is straightforward that

$$\Omega\big(F(g)T_A, F(g)T_B\big) = \Omega(T_A, T_B) \tag{3.22}$$

for all T_A and T_B in the Lie algebra. This shows that the linear map $F(g)$ is an isometry of the Lie algebra metric Ω.

Note that if $\mathcal{F}$ is an isometry on T_gG, eq. (3.18) not only follows from the gluing condition (2.12) but is equivalent to it. All in all we have that the necessary and sufficient condition for the gluing condition to have the form of a boundary condition is that $F(g)$ is an isometry of Ω, the two-form ω being given by eq. (3.19). In what follows whenever we write $F(g)$ we will be thinking of it as an isometry. In terms of the fields $t_A = g\big[\mathrm{Ad}_{g^{-1}}F(g)-1\big]T_A$ in eq. (3.8), the definition of ω in eq. (3.19) can be written as

$$\omega\left(t_A\,,\,t_B\right) = \Omega\left(\left(\mathrm{Ad}_{g^{-1}}F-1\right)T_A\,,\,\left(\mathrm{Ad}_{g^{-1}}F+1\right)T_B\right). \tag{3.23}$$

We remark that the analysis performed here holds for any linear map F, regardless of whether it is constant or g-dependent.

The condition that ω must satisfy for N to be a D-brane is

$$d\omega = H\big|_N\,, \tag{3.24}$$

where the exterior derivative on the left hand side is taken with respect to the directions in T_gN and not with respect to arbitrary directions in T_gG. Condition (3.24) does not hold for every isometry F defining a submanifold N upon integration of the gluing condition. Examples of this are given in section 6 and in ref. [21]. Let us discuss some cases in which ω fulfills eq. (3.24).

For one and two-dimensional submanifolds N, eq. (3.24) trivially holds and the only requirement for the existence of a D-brane for an isometry F is involutivity. Assume now that N has dimension larger than two and that $F=R$ is a g-independent, Ω-preserving Lie algebra automorphism. The exterior derivative of ω on N is a three-form whose action on vector fields t_1, t_2 and t_3 in T_gN is given by

$$\begin{aligned} d\omega\left(t_1,\,t_2,\,t_3\right) =& t_1\big(\,\omega(t_2,t_3)\,\big) - \omega\left(\left[t_1,t_2\right],\,t_3\right) \\ &+ t_2\big(\,\omega(t_3,t_1)\,\big) - \omega\left(\left[t_2,t_3\right],\,t_1\right) \\ &+ t_3\big(\,\omega(t_1,t_2)\,\big) - \omega\left(\left[t_3,t_1\right],\,t_2\right). \end{aligned} \tag{3.25}$$

Since the vector fields $t_A = g\left(\mathrm{Ad}_{g^{-1}}R-1\right)T_A$ span T_gN, it suffices to calculate $d\omega(t_A,t_B,t_C)$. For that, we need to consider terms of the form $t_A\big(\omega(t_B,t_C)\big)$ and $\omega\big([t_A,t_B],t_C\big)$. Since R does not depend on g,

$$t_A\left(\,\omega(t_B,t_C)\,\right) = \Omega\Big(\,t_A\left(\mathrm{Ad}_{g^{-1}}\right)RT_B\,,\,T_C\Big) - \Omega\Big(\,t_A\left(\mathrm{Ad}_{g^{-1}}\right)RT_C\,,\,T_B\Big).$$

Noting that

$$t_A\left(\mathrm{Ad}_{g^{-1}}\right)V = -\big[\left(\mathrm{Ad}_{g^{-1}}R-1\right)T_A\,,\,\mathrm{Ad}_{g^{-1}}V\,\big]$$

for all V in $\mathfrak{g}$, using that R and $\mathrm{Ad}_{g^{-1}}R$ are Lie algebra automorphisms and recalling that Ω is invariant, it is straightforward to see that

$$t_A\left(\,\omega(t_B,t_C)\,\right) = \Omega\left(\mathrm{Ad}_{g^{-1}}\,RT_B\,,\,\left[T_C,T_A\right]\right) - \Omega\left(T_B\,,\,\mathrm{Ad}_{g^{-1}}\,R\left[T_C,T_A\right]\right) - \left(B\leftrightarrow C\right).$$

Proceeding similarly with $\omega\big([t_A,t_B],t_C\big)$, one has

$$\omega\left(\left[t_A,t_B\right],\,t_C\right) = \Omega\left(\left[T_A,T_B\right],\,\mathrm{Ad}_{g^{-1}}\,RT_C\right) - \Omega\left(\mathrm{Ad}_{g^{-1}}\,R\left[T_A,T_B\right],\,T_C\right).$$

Upon substitution in eq. (3.25), this gives

$$d\omega(t_A, t_B, t_C) = \Omega\Big(\big[\,(\mathrm{Ad}_{g^{-1}}R - 1)\,T_A\,,\,(\mathrm{Ad}_{g^{-1}}R - 1)\,T_B\,\big]\,,\,(\mathrm{Ad}_{g^{-1}}R - 1)\,T_C\Big)\,. \tag{3.26}$$

On the other hand, from eq. (2.5) it trivially follows that the right hand side in (3.26) is equal to $H(t_A, t_B, t_C)$. Hence, for any constant isometry that is also a Lie algebra automorphism, $d\omega = H\big|_N$. It is clear that if F depends on g and/or is not a Lie algebra automorphism, this proof does not stand. In these cases, condition (3.24) can always be checked by hand. See ref. [21] for examples.

We end this section by remarking that we have not assumed at any stage that $T_gG = \Pi_g \oplus \Pi_g^\perp$, thus generalizing previous approaches [14] that, under such an assumption, define ω for F a constant Lie algebra automorphism. In this regard, it is worth noting that $T_gG = \Pi_g \oplus \Pi_g^\perp$ holds for Lie groups with Euclidean signature metric $G_{\mu\nu}$. However, if $G_{\mu\nu}$ is Lorentzian, it may occur that, among the vector fields defining the distribution Π^M, one of them is null and orthogonal to all the others. If this is the case, the induced metric on the D-brane is degenerate and the tangent space T_gG cannot be written as a direct sum of Π_g and $\Pi_g^\perp$. In appendix B an explicit construction of such null vector fields in terms of the eigenvectors of $\mathcal{F}$ is presented, and in ref. [21] a family of degenerate D2-branes for the Nappi-Witten [34] model is found.

4 Limitations of the gluing condition approach

In the previous section we have shown that every gluing condition with F an isometry takes the form of a boundary condition with a two-form ω defined by eq. (3.19). It may, however, occur that a boundary condition describing a D-brane cannot be written as a gluing condition. In this section we tackle this problem and show that every boundary condition with two-form ω defining a D-brane N can be written as a gluing condition if and only if $\det\,(G|_N - \omega) \neq 0$.

Let us then consider a D-brane N with tangent space T_gN specified by the boundary condition (2.1), the two-form ω acting on T_gN. It is most convenient for our purposes to write the boundary condition as

$$G\Big(\delta X\,,\,\partial_\sigma X\big|_{\partial\Sigma}\Big) = \omega\Big(\delta X\,,\,\partial_\tau X\big|_{\partial\Sigma}\Big) \quad \text{for all} \quad \delta X \in T_gN\,, \tag{4.1}$$

with $\partial_\sigma X\big|_{\partial\Sigma}$ in T_gG and $\partial_\tau X\big|_{\partial\Sigma}$ in T_gN. We now define a map $\mathcal{K}\colon T_gN \to T_gG/(T_gN^\perp)$ whose action on w in T_gN is given by

$$G(z\,,\mathcal{K}w) = \frac{1}{2}\,\big[\,G(z,w) - \omega(z,w)\,\big] \quad \text{for all} \quad z \in T_gN\,. \tag{4.2}$$

The map $\mathcal{K}$ is trivially linear and takes values in the quotient $T_gG/(T_gN^\perp)$. To see the latter, assume that y in T_gG is such that $G(z,y) = G(z,\mathcal{K}w)$ for all z in T_gN. It follows that $G(z, y - \mathcal{K}w) = 0$, which in turn implies that $y - \mathcal{K}w$ is in $T_gN^\perp$.

Furthermore, $\mathcal{K}$ is injective if and only if $\det\,(G|_N - \omega) \neq 0$. Indeed, for $w' \neq w$ in T_gN such that $\mathcal{K}w' = \mathcal{K}w$, we have, according to eq. (4.2), that

$$G(z, w' - w) = \omega(z, w' - w) \quad \text{for all} \quad z \in T_gN\,. \tag{4.3}$$

A vector $w' - w \neq 0$ satisfying this condition exists if and only if $\det(G|_N - \omega) = 0$, which proves the statement. Actually, since $\dim(T_g N) + \dim(T_g N^{\perp}) = \dim(T_g G)$, the map $\mathcal{K}$ is bijective if it is injective.

For $\det(G|_N - \omega) \neq 0$, the inverse map $\mathcal{K}^{-1} : T_g G/(T_g N^{\perp}) \to T_g N$ hence exists and is bijective. From $\mathcal{K}^{-1}$ we define a linear map $\mathcal{G} : T_g G \to T_g N$ whose action on an arbitrary element v in $T_g G$ is given by $\mathcal{G}v = \mathcal{K}^{-1}(v + T_g N^{\perp})$. Writing $\mathcal{G}$ as $\mathcal{G} = \mathcal{F} + 1$, it is straightforward to check that $\mathcal{F}$ satisfies $T_g N = \mathrm{Im}(\mathcal{F} + 1)$ and

$$\omega\big((\mathcal{F}+1)\, u \,,\, (\mathcal{F}+1)\, v \big) = G\big((\mathcal{F}+1)\, u \,,\, (\mathcal{F}-1)\, v \big) \tag{4.4}$$

for arbitrary u and v in $T_g G$. Proceeding along the same lines as in subsection 3.3, one can see that the isometric character of $\mathcal{F}$ follows from the antisymmetric property of ω.

Since $\partial_\tau X\big|_{\partial\Sigma}$ in eq. (4.1) belongs to $T_g N$, it can be written as $\partial_\tau X\big|_{\partial\Sigma} = (\mathcal{F}+1)v$ for some v in $T_g G$. Upon noting (4.4), the boundary condition (4.1) takes the form

$$G\Big(\delta X \,,\, \partial_\sigma X\big|_{\partial\Sigma}\Big) = G\Big(\delta X \,,\, (\mathcal{F}-1)v\Big) \quad \text{for all} \quad \delta X \in T_g N\,.$$

This is equivalent to

$$\partial_\sigma X\big|_{\partial\Sigma} = (\mathcal{F}-1)v + v_0\,,$$

with arbitrary v_0 in $T_g N^{\perp} = \mathrm{Ker}\big(\mathcal{F}+1\big)$. Acting with $\mathcal{F}+1$ on both sides of this equation we finally have

$$\big(\mathcal{F}+1\big)\, \partial_\sigma X\big|_{\partial\Sigma} = \big(\mathcal{F}-1\big)\big(\mathcal{F}+1\big)v = \big(\mathcal{F}-1\big)\partial_\tau X\big|_{\partial\Sigma}\,,$$

which is nothing but the gluing condition (2.12) written in terms of world sheet coordinates τ and σ.

Let us go back to eq. (4.3). If there exists $w' - w \neq 0$ in $T_g N$ such that the equation holds and z is taken equal to $w' - w$, the right hand side of eq. (4.3) vanishes and it follows that $w' - w$ is a null vector. Since Euclidean D-branes do not have null vectors, such a $w' - w \neq 0$ does not exist and $\det(G|_N - \omega) \neq 0$. The analysis of D-branes based on the gluing condition (2.8) then provides all Euclidean D-branes described by boundary conditions but may miss some Lorentzian or metrically degenerate D-branes for which $\det(G|_N - \omega) = 0$.

5 An application: D-branes from global isometries

From the analysis in subsection 3.2 it is convenient to distinguish two cases. The first one assumes that F does not depend on g. We call such isometries constant or global and will be treated in this section. The second case accounts for g-dependent isometries $F(g)$. We call them local or nonconstant; some examples will be considered in section 5.

If F is a global (or constant) isometry solving involutivity, eq. (3.15) holds. Frobenius theorem ensures that $\Pi_g = g(\mathrm{Ad}_{g^{-1}}F - 1)\mathfrak{g}$ is the tangent space to a submanifold N of G but it does not identify N. This problem we address next.

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Consider the vector field $t_V(g) = g(\mathrm{Ad}_{g^{-1}}F - 1)V$, with V in $\mathfrak{g}$, and let g_0 be a group element. By definition, the integral curve $\gamma_{t_V}(s; g_0)$ of t_V that goes through g_0 is the solution to

$$\gamma_{t_V}(0; g_0) = g_0 \qquad \frac{d}{ds}\gamma_{t_V}(s; g_0) = t_V\big(\gamma_{t_V}(s; g_0)\big)\,,$$

where s is a real parameter along the curve. Simple inspection shows that the solution is

$$\gamma_{t_V}(s; g_0) = e^{\,sFV} g_0\, e^{-sV}\,. \tag{5.1}$$

The set N_{g_0} of all points connected to g_0 by integral curves of vector fields t_V, with V arbitrary, can always be written as

$$N_{g_0} = \big\{e^{\,FV} g_0\, e^{-V}\colon\; V \in \mathfrak{g}\big\}\,. \tag{5.2}$$

The only candidate for a D-brane containing g_0 is then N_{g_0}. According to section 3, however, the fact that N_{g_0} contains the integral curves of all the fields t_V that go through g_0 is not enough to conclude that N_{g_0} is a D-brane. For this to be the case, $\Pi^{N_{g_0}}$ must be an involutive distribution. In summary, D-branes for constant F, if they exist, have the form of N_{g_0} in (5.2).

5.1 Automorphisms and twined conjugacy classes as D-branes

Take $F = R$ with R a Lie algebra automorphism compatible with condition (3.22). Automorphisms of this type form a group, denoted by $Aut_\Omega(\mathfrak{g})$. Being R an automorphism, it satisfies

$$R[U, V] = [\,RU, RV] \tag{5.3}$$

for all U and V in $\mathfrak{g}$. For any such F, the involutivity equation (3.15) is solved by $W = [V, U]$ and the manifold N_{g_0} is the R-twined conjugacy class $\mathcal{C}(R, g_0)$ of g_0,

$$N_{g_0} = \mathcal{C}(R, g_0) = \big\{e^{\,RV} g_0\, e^{-V}\colon\; V \in \mathfrak{g}\big\}\,.$$

In appendix C it is shown that the dimension of $\Pi_g = g(\mathrm{Ad}_{g^{-1}}F - 1)\mathfrak{g}$ is constant for all g in $\mathcal{C}(R, g_0)$. Furthermore, as proved at the end of section 3, the two-form ω given in (3.19) satisfies $H\big|_{N_{g_0}} = d\omega$. Hence, $\mathcal{C}(R, g_0)$ is a D-brane [11, 13–15].

Note that for $R = 1$, the manifold N_{g_0} is a conventional conjugacy class $\mathcal{C}(1, g_0)$ [11]. Consider now $R \neq 1$ and assume that R is an inner automorphism. By definition, it exists an h in G such that $RV = \mathrm{Ad}_h V$ for all V in $\mathfrak{g}$. Since $\mathrm{Ad}_{h_1}\mathrm{Ad}_{h_2} = \mathrm{Ad}_{h_1 h_2}$, inner automorphisms form a subgroup $\mathrm{Inn}_\Omega(\mathfrak{g})$. Automorphisms which are not inner are called outer and form the equivalence classes of the quotient $Aut_\Omega(\mathfrak{g})/Inn_\Omega(\mathfrak{g})$. Any automorphism R can therefore be written as $R = R_1\tilde{R}_2$, with R_1 inner and $\tilde{R}_2$ of the same type as R. Consider the $(R_1\tilde{R}_2)$-twined conjugacy class of g_0. Using that $R_1 = \mathrm{Ad}_h$ for some h in G, and recalling that $e^{\,t\mathrm{Ad}_r U} = \mathrm{Ad}_r e^{tU}$ for all r in G and all U in $\mathfrak{g}$, it follows that

$$\mathcal{C}(R_1\tilde{R}_2, g_0) = h\,\mathcal{C}(\tilde{R}_2, h^{-1}g_0)$$

for some h in G. If R is inner, so that $\tilde{R}_2 = 1$, the D-branes are the left translates by h of the conventional conjugacy classes [13]. For R outer, the D-branes are the translates of $\tilde{R}_2$-twined conjugacy classes [13, 22].

5.2 D-branes for semisimple Lie algebras

Involutivity for isometries that are not Lie algebra automorphisms is more complicated. It has been suggested [12, 13, 32] that isometries of the form $F = -R$, with R a Lie algebra automorphism, define D-branes. In the sequel we investigate this issue and reach an answer in the negative.

Consider eq. (3.15) and make the change $2Y = [U,V] - W$. Involutivity requires that, for all U and V in $\mathfrak{g}$ and for all g, there must exist Y in $\mathfrak{g}$ such that

$$g\,[U,V] = g\big(\mathrm{Ad}_{g^{-1}}R + 1\big)Y\,.$$

After multiplying from the left with g^{-1}, this becomes

$$[\mathfrak{g},\mathfrak{g}] \subset \big(\mathrm{Ad}_{g^{-1}}R + 1\big)\mathfrak{g}\,. \tag{5.4}$$

We restrict ourselves to semisimple Lie algebras $\mathfrak{g}$. Concerning this restriction, we make two comments. The first one is that if for the invariant metric Ω one takes a Killing form and R is an arbitrary Lie algebra automorphism, $\pm R$ are isometries.[2] The second observation is that two of the most relevant semisimple Lie algebras in string theory are $\mathfrak{sl}(2,\mathbf{R})$ and $\mathfrak{su}(2)$, all whose isometries are either of the form $F = R$ or $F = -R$. As already mentioned, $F = R$ define D-branes. Consideration of $F = -R$ then completes the search of D-branes for constant isometries for $\mathfrak{sl}(2,\mathbf{R})$ and $\mathfrak{su}(2)$.

Let $\mathfrak{g}$ be a semisimple Lie algebra of dimension d. Being semisimple, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ and the involutivity requirement (5.4) reads

$$\mathfrak{g} = \big(\mathrm{Ad}_{g^{-1}}R + 1\big)\mathfrak{g}\,.$$

Let D_{d} denote the set of group elements where this condition holds. That is, g belongs to D_{d} if $(\mathrm{Ad}_{g^{-1}}R + 1)V \neq 0$ for all $V \neq 0$ in $\mathfrak{g}$. The tangent plane (3.5) at all g in D_{d} is then $\Pi_g = T_gG$. The distribution $\Pi^{D_{\mathsf{d}}}$ is trivially involutive and is the tangent bundle to D_{d} itself. The only D-brane candidate provided by $F = -R$ is hence D_{d}. According to our discussion in section 3, for D_{d} to be a D-brane, the integral curves of the vector fields $t_V(g) = g(\mathrm{Ad}_{g^{-1}}R + 1)V$ should be contained in D_{d}. In the remaining of this section we show that this is not the case, thus implying that $F = -R$ does not define a D-brane.

The proof consists in **(i)** finding group elements g outside D_{d}, and **(ii)** showing that these g are connected to elements in D_{d} by integral curves of the vector fields $t_V(g)$.

Proof of (i). A group element g is not in D_{d} if there exists a nonzero V in $\mathfrak{g}$ such that $\big(\mathrm{Ad}_{g^{-1}}R + 1\big)V = 0$. Let us call D_{d}^- to the set formed by such g,

$$D_{\mathsf{d}}^- = G - D_{\mathsf{d}} = \big\{ g \in G\colon\ \mathrm{Ker}\big(\mathrm{Ad}_{g^{-1}}R + 1\big) \neq 0 \big\}\,. \tag{5.5}$$

The group $\mathrm{Aut}(\mathfrak{g})$ has in general several connected components, the component containing the identity being the normal subgroup $\mathit{Inn}(\mathfrak{g})$, and the quotient $\mathit{Aut}(\mathfrak{g})/\mathit{Inn}(\mathfrak{g})$ being a

[2] Note in this regard that a non-simple semisimple Lie algebra may have invariant metrics which are not proportional to its Killing form, in which case a Lie algebra automorphism need not be an isometry. The case of simple Lie algebras is more restrictive since all invariant metrics are proportional to the Killing form.

finite group. It follows that in every component, and in particular in that containing R, there is then an automorphism S such that $S^n = 1$ for an integer n. This implies that the eigenvalues of S can only be n-th roots of 1. Furthermore, if a root $e^{-\mathrm{i}\theta}$ is an eigenvalue with multiplicity m, so is $e^{\mathrm{i}\theta}$. Since S and R are in the same component, they are related by an inner automorphism, meaning that there is an h in G such that $R = \mathrm{Ad}_h S$. We now distinguish three cases:

- S has an eigenvalue -1. For $g = h$, the operator $\mathrm{Ad}_{g^{-1}} R$ has then an eigenvalue -1 and g is in $D^-_{\mathtt{d}}$.

- S does not have an eigenvalue -1 but has two complex conjugate eigenvalues $e^{\mp\mathrm{i}\theta}$ with eigenvectors $Z_1 \pm \mathrm{i}Z_2$,

$$S(Z_1 \pm \mathrm{i}Z_2) = e^{\mp\mathrm{i}\theta}(Z_1 \pm \mathrm{i}Z_2)\,.$$

  Using that $\mathfrak{g}$ is semisimple and that S is an automorphism, it is straightforward to show that $X = [Z_1, Z_2]$ is an eigenvector of S with eigenvalue $+1$ and that there exists a real such that $[X, Z_1] = aZ_2$ and $[X, Z_2] = -aZ_1$. The constant a can be eliminated by redefining Z_1, Z_2 and X, so that

$$[Z_1, Z_2] = X \qquad [X, Z_1] = Z_2 \qquad [X, Z_2] = -Z_1\,.$$

  Take now $W = \frac{\pi}{\sqrt{2}}(Z_1 - Z_2)$ and consider $g = he^{-W}$. It follows after some algebra that the automorphism $\mathrm{Ad}_{g^{-1}} R = \mathrm{Ad}_{e^W} S$ has two eigenvectors with eigenvalue -1,

$$\begin{aligned}\mathrm{Ad}_{g^{-1}} R\, X &= -X\\ \mathrm{Ad}_{g^{-1}} R\,\big(\cos\theta Z_1 + Z_2 - \sin\theta Z_2\big) &= -\big(\cos\theta Z_1 + Z_2 - \sin\theta Z_2\big)\,.\end{aligned}$$

  Hence $g = he^{-W}$ belongs to $D^-_{\mathtt{d}}$.

- S only has eigenvalues $+1$. In this case, S is the identity automorphism and R is inner. For $\mathfrak{g}$ semisimple, it is always possible to take X in its Cartan subalgebra and Z_1 and Z_2 in $\mathfrak{g}$ such that

$$[Z_1, Z_2] = X \qquad [X, Z_1] = Z_1 \qquad [X, Z_2] = -Z_2\,.$$

  It is then very simple to check that $\mathrm{Ad}_{g^{-1}} R$, where g is taken as $g = he^{-W}$ with $W = \sqrt{2}\pi(Z_1 - Z_2)$, has two eigenvectors with eigenvalue -1,

$$\begin{aligned}\mathrm{Ad}_{g^{-1}} R\, X &= -X\\ \mathrm{Ad}_{g^{-1}} R\,\big(Z_1 + Z_2\big) &= -\big(Z_1 + Z_2\big)\,.\end{aligned}$$

  So also in this case $D^-_{\mathtt{d}}$ is not empty.

As shown in appendix C, the spectrum of $\mathrm{Ad}_{g^{-1}} R$ is invariant under R-twined conjugation. Hence, if g is in $D^-_{\mathtt{d}}$, the whole R-twined conjugacy class $\mathcal{C}(R, g)$ is in $D^-_{\mathtt{d}}$. As a result, $D^-_{\mathtt{d}}$ is a union of R-twined conjugacy classes. It is clear that $D^-_{\mathtt{d}}$ has dimension less than $\mathtt{d}$.

Proof of (ii). The tangent space $T_{g_0}G = g_0\mathfrak{g}$ at a g_0 in $D_{\mathtt{d}}^-$ is most conveniently written as

$$T_{g_0}G = g_0\,(\mathrm{Ad}_{g_0^{-1}}R - 1)\,\mathfrak{g} \;\cup\; g_0\,(\mathrm{Ad}_{g_0^{-1}}R + 1)\,\mathfrak{g}\,. \tag{5.6}$$

Since the R-twined conjugacy class $\mathcal{C}(R, g_0)$ is contained in $D_{\mathtt{d}}^-$ and the fields $g(\mathrm{Ad}_{g^{-1}}R - 1)\mathfrak{g}$ generate motions inside $\mathcal{C}(R, g_0)$, there must be at least one vector field $t_V(g) = g\,(\mathrm{Ad}_{g^{-1}}R + 1)V$ whose integral curve goes from $D_{\mathtt{d}}^-$ to $D_{\mathtt{d}}$. Such a curve connects points in the D-brane with points outside the D-brane. We thus conclude that $D_{\mathtt{d}}$ cannot be a D-brane.

This proves that there are no D-branes for a semisimple Lie algebra $\mathfrak{g}$ and $F = -R$, with R a constant automorphism. This result contrasts with previous studies on the subject [12]. If the requirement that $D_{\mathtt{d}}$ contain the integral curves of all the fields t_V were relaxed, $D_{\mathtt{d}}$ would be a D-brane of dimension $\mathtt{d}$, provided it exists a suitable two-form ω. This D-brane would not be filling, since $D_{\mathtt{d}}^-$ is not empty. Furthermore, it would exclude allowed motions for the string endpoints, thus contradicting the definition of D-brane.

6 Some considerations on D-branes for local isometries

For local isometries $F(g)$, involutivity takes the form (3.14). Given a local isometry $F(g)$, it is always possible to construct a new isometry

$$F(g) \to F'(g) = \mathrm{Ad}_g\, F^{-1}(g)\, \mathrm{Ad}_g\,. \tag{6.1}$$

It is very easy to convince oneself that, at any point g in G, both F and F' define the same tangent space $\Pi_g = g\,(\mathrm{Ad}_{g^{-1}}F - 1)\,\mathfrak{g}$. They thus define the same distribution. Furthermore, it is straightforward to check that F' satisfies the involutivity condition (3.14) if and only if F does. Assume that this is the case, so that they define the same submanifold N of G.

The gluing conditions (2.8) for F and F' read

$$F:\quad (\mathcal{F} - 1)\,\partial_\tau X\big|_{\partial\Sigma} = (\mathcal{F} + 1)\,\partial_\sigma X\big|_{\partial\Sigma} \tag{6.2}$$

$$F':\quad (\mathcal{F}' - 1)\,\partial_\tau X\big|_{\partial\Sigma} = (\mathcal{F}' + 1)\,\partial_\sigma X\big|_{\partial\Sigma}\,, \tag{6.3}$$

where the matrices $\mathcal{F}$ and $\mathcal{F}'$ are given by $\mathcal{F} = -\,\bar{e}^{\,-1}Fe$ and $\mathcal{F}' = -\,\bar{e}^{\,-1}F'e$. Noting that $\mathcal{F}' = \mathcal{F}^{-1}$, eq. (6.3) can be written, after multiplication from the left with $\mathcal{F}$, as

$$F':\quad -\,(\mathcal{F} - 1)\,\partial_\tau X\big|_{\partial\Sigma} = (\mathcal{F} + 1)\,\partial_\sigma X\big|_{\partial\Sigma}\,. \tag{6.4}$$

The gluing condition (6.4) for F' has a relative negative sign as compared to the gluing condition (6.2) for F. This sign has important consequences for the recasting of the corresponding gluing conditions as boundary conditions. Indeed, the two-forms ω and ω' associated to F and F' are related by $\omega' = -\,\omega$, so the conditions $d\omega = H\big|_N$ and $d\omega' = H\big|_N$ cannot generally hold simultaneously. Let us see an example.

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Example: Filling D-brane. In this case, the sigma model boundary conditions (2.1) become

$$\omega_{\mu\nu}\,\partial_\tau X^\nu\Big|_{\partial\Sigma} = G_{\mu\nu}\,\partial_\sigma X^\nu\Big|_{\partial\Sigma}\,. \tag{6.5}$$

Assume that the D-brane is defined by an isometry F. This requires in particular that the gluing condition (6.2) can be written as in (6.5), with ω such that $d\omega = H$. See ref. [21] for some examples. The gluing condition (6.4) can then be written in the form (6.5), but needs $\omega' = -\omega$, and $d\omega' \neq H$. The isometry F' hence does not define a D-brane.

We close this section by further illustrating that an integrable gluing condition by itself does not define a D-brane. Consider $F(g) = -\mathrm{Ad}_g$. The tangent plane (3.5) at all g in G is $\Pi_g = T_g G$. The isometry F defines trivially an involutive distribution, the submanifold N being the whole group G. Since $\mathcal{F} = 1$, the gluing condition (6.2) becomes $\partial_\sigma X^\mu\big|_{\partial\Sigma} = 0$. This, in turn, cannot be understood as a sigma model boundary condition, since it requires $\omega = 0$ on the whole group manifold and does not account for a nontrivial H.

7 Conclusion

Given a WZW model with real Lie group G, Lie algebra $\mathfrak{g}$ and invariant Lie algebra metric Ω, we have shown that a linear map $F(g)$ acting on $\mathfrak{g}$ defines a D-brane if the following conditions hold:

(i) $F(g)$ is an isometry of Ω.

(ii) The vector fields $t_A = FT_A g - gT_A = t^\mu{}_A \partial_\mu$ defined by $F(g)$ span a distribution. That is, the matrix formed by the coefficients $t^\mu{}_A$ has constant rank on a submanifold N of the group manifold. If this is the case and the rank is $p+1$, there are $p+1$ linearly independent vector fields k_i that are linear combinations $k_i = c_{iA}\, t_A$ of the fields t_A.

(iii) The integral curves of the fields k_i are contained in N.

(iv) The fields k_i are involutive in N.

(v) The two-form ω globally defined on N by its action $\omega(k_i, k_j)$ on the fields $k_i = c_{iA}\, t_A$ through $\omega(t_A, t_B) = \Omega\left(\mathrm{Ad}_{g^{-1}} FT_A - T_A\,,\, \mathrm{Ad}_{g^{-1}} FT_B + T_B\right)$ satisfies $d\omega = H\big|_N$.

The conditions above account for both metrically nondegenerate and degenerate D-branes. They are met by F any constant Ω-preserving Lie algebra automorphism R, so the well known result [11–14] that the R-twined conjugacy classes of the group G are D-branes extends to metrically degenerate classes.

WZW models based on semisimple Lie algebras are of particular interest in string theory, two of the most studied models being $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbf{R})$. It had been claimed that constant $F = -R$ could provide D-branes for such models. This has been disproved in this paper, since condition (iii) above fails.

For more general scenarios, (ii)–(v) must be checked for any given isometry F. This is however straightforward. In ref. [21] the Nappi-Witten model [34] is considered and

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several families of D-branes for g-dependent isometries $F(g)$ are found, some have Euclidean signature, some have Lorentzian and some are metrically degenerate. It would be interesting to study if D-branes defined by g-dependent isometries have a translate in the algebraic framework, since normal ordering ambiguities may occur. Our interest in this paper has been the geometric description of D-branes in WZW string backgrounds taking as starting point a gluing condition $J_+ = F J_-$ that matches the chiral currents at the world sheet boundary. It remains an open problem to study if the geometric approach presented here describes D-branes for which a full set of gluing conditions have not been found, the so-called permutation D-branes [19] among them.

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A Alternative derivation of eq. (3.2)

Here we present an alternative derivation of eq. (3.2). The idea is to solve the gluing condition (2.12) for $\partial_\tau x^\mu$ in terms of the eigenvectors of the matrix $\mathcal{F}$.

The (generalized) eigenvectors of the matrix $\mathcal{F}$ form a basis of linearly independent vectors. An eigenvalue λ with algebraic multiplicity a_λ and geometric multiplicity m_λ has $i = 1, \ldots, m_\lambda$ eigenvectors $v_{(\lambda\,,\,i\,,\,1)}$ and $a_\lambda - m_\lambda$ generalized eigenvectors that can be organized in m_λ chains

$$\big(\mathcal{F} - \lambda\big)\, v_{(\,\lambda\,,\,i\,,\,1)} = 0 \quad \ldots \quad \big(\mathcal{F} - \lambda\big)\, v_{(\lambda\,,\,i\,,\,\ell_i)} = v_{(\lambda\,,\,i\,,\,\ell_i - 1)} \qquad \ell_i = 2,\, \ldots,\, L_i\,. \tag{A.1}$$

The index $\ell_i = 1, \ldots, L_i$ labels the members of the chain (λ, i). Every chain is headed by an eigenvector $v_{(\lambda\,,\,i\,,\,1)}$ and terminates in a highest-ℓ_i generalized eigenvector $v_{\lambda\,,\,i\,,\,L_i}$. Consider two arbitrary (generalized) eigenvectors $v_{(\lambda\,,\,i\,,\,\ell_i)}$ and $v_{(\mu\,,\,j\,,\,m_j)}$ relative to the eigenvalues λ and μ.

Since the (generalized) eigenvectors $\{v_{(\lambda\,,\,i\,,\,\ell_i)}\}$ are linearly independent, $\partial_\tau X\big|_{\partial\Sigma}$ and $\partial_\sigma X\big|_{\partial\Sigma}$ are linear combinations

$$\partial_\tau X\big|_{\partial\Sigma} = \sum_{\lambda, i, \ell_i} \alpha_{(\lambda\,,\,i\,,\,\ell_i)}\, v_{(\lambda\,,\,i\,,\,\ell_i)} \qquad \partial_\sigma X\big|_{\partial\Sigma} = \sum_{\lambda, i, \ell_i} \beta_{(\lambda\,,\,i\,,\,\ell_i)}\, v_{(\lambda\,,\,i\,,\,\ell_i)}\,,$$

with coefficients $\alpha_{(\lambda\,,\,i\,,\,\ell_i)}$ and $\beta_{(\lambda\,,\,i\,,\,\ell_i)}$. Upon substitution in eq. (2.12), the following set of equations follows for every chain (λ, i)

$$\alpha_{(\lambda\,,\,i\,,\,\ell_i)} + (\lambda - 1)\, \alpha_{(\lambda\,,\,i\,,\,\ell_i - 1)} = \beta_{(\lambda\,,\,i\,,\,\ell_i)} + (\lambda + 1)\, \beta_{(\lambda\,,\,i\,,\,\ell_i - 1)} \qquad \ell_i = 2, \ldots, L_i \tag{A.2}$$

$$(\lambda - 1)\, \alpha_{(\lambda\,,\,i\,,\,L_i)} = (\lambda + 1)\, \beta_{(\lambda\,,\,i\,,\,L_i)}\,. \tag{A.3}$$

We must solve eqs. (A.2)–(A.3) for $\alpha_{(\lambda, i, \ell_i)}$ in terms of $\beta_{(\lambda, i, \ell_i)}$. To this end, we consider the cases $\lambda = -1$, $\lambda = 1$ and $\lambda \neq \pm 1$ separately.

- Assume that $\mathcal{F}$ has a chain $\{v_{(-1,i,\ell_i)}\}$ relative to the eigenvalue $\lambda = -1$. Eq. (A.3) implies $\alpha_{(-1,i,L_i)} = 0$, so the vector $v_{(-1,i,L_i)}$ does not occur in $\partial_\tau x$. Eq. (A.2) in turn implies that there are infinitely many solutions for $\alpha_{(-1,i,1)} \dots \alpha_{(-1,i,L_i-1)}$; one for every choice of $\beta_{(-1,i,1)} \dots \beta_{(-1,i,L_i,)}$. The (generalized) eigenvectors $v_{(-1,i,1)} \dots v_{(-1,i,L_i-1)}$ then occur in $\partial_\tau x$.
- Look next at a chain $\{v_{(1,i,\ell_i)}\}$ with eigenvalue $\lambda = 1$. Eq. (A.3) now requires $\beta_{(1,i,L_i)} = 0$ and leaves $\alpha_{(1,i,L_i)}$ arbitrary. This and eq. (A.2) give arbitrary solutions for all $\alpha_{(1,i,\ell_i)}$. In this case, all the vectors in the chain are tangent.
- Consider finally a chain $\{v_{(\lambda,i,\ell_i)}\}$ relative to an eigenvalue $\lambda \neq \pm 1$. Eqs. (A.2)–(A.3) give arbitrary solutions for all $a_{(\lambda,i,\ell_i)}$ and again all the vectors in the chain occur in $\partial_\tau x$.

The space Π_g of tangent directions is then

$$\Pi_g = \mathrm{Span}\left\{v_{(\lambda,i,\ell_i)} \colon\ (\lambda, \ell_i) \neq (-1, L_i)\right\}$$

and has dimension $\mathtt{d} - m_{-1}$, where we recall that $\mathtt{d}$ is the group dimension and m_{-1} the geometric multiplicity of $\lambda = -1$. Since the nontangent vectors $v_{(-1,i,L_i)}$ are removed from the set of all (generalized) eigenvectors through the action of $\mathcal{F} + 1$, one has

$$\Pi_g = (\mathcal{F} + 1)\,\mathrm{Span}\left\{v_{(\lambda,i,\ell_i)}\right\} = \mathrm{Im}\big(\mathcal{F} + 1\big)\,. \tag{A.4}$$

This is precisely eq. (3.2).

B Metrically degenerate tangent planes

Here we explicitely construct tangent vectors that are orthogonal to all tangent vectors, including itself, so they define a metrically degenerate tangent plane Π_g.

To this end, we first note that the isometry property (3.20) and eq. (A.1) imply the orthogonality relation

$$(1 - \lambda\mu)\; G\big(\, v_{(\lambda\,,\,i\,,\,\ell_i)}\,,\, v_{(\mu\,,\,j\,,\,m_j)}\,\big) = 0\,. \tag{B.1}$$

for two arbitrary (generalized) eigenvectors.

Assume for concreteness that there is only one chain $\{v_{-1,1,1} \dots v_{(-1,1,L)}\}$ of $L \geq 2$ generalized eigenvectors relative to the eigenvalue $\lambda = -1$, and let us write $u_\ell := v_{(-1,1,\ell)}$ for its members. As explained in appendix A, the first $L-1$ vectors in this chain define directions in Π_g. Noting that $\mathcal{F}$ is an isometry and recalling eqs. (A.1), we have

$$G\big(u_1\,,\, u_{\ell+1}\big) = G\big(\mathcal{F}u_1\,,\, \mathcal{F}u_{\ell+1}\big) = G\big(u_1\,,\, u_{\ell+1}\,\big) - G\big(u_1\,,\, u_\ell\,\big) \qquad \ell = 1, \dots, L-1\,.$$

It follows that $G\big(u_1, u_\ell\big) = 0$ for $\ell = 1 \dots, L-1$. Since $\{u_\ell\}$ is the only chain with eigenvalue -1, any other direction in Π_g has eigenvalue $\lambda \neq -1$, and thus eq. (B.1) implies that it is orthogonal to u_1. The eigenvector u_1 is thus orthogonal to all (generalized) eigenvectors spanning Π_g, and in particular to itself.

It is trivial to extend these arguments to show that every eigenvector heading a chain with eigenvalue $\lambda = -1$ defines a null direction orthogonal to Π_g.

C Invariance of the spectrum of $\mathrm{Ad}_{g^{-1}}R$

This appendix contains the discussion of the invariance of the spectrum of the operator $\mathrm{Ad}_{g^{-1}}R$ under R-twined conjugation, where R is a Lie algebra automorphism.

The eigenvalue problem for $\mathrm{Ad}_{g^{-1}}R$ takes the form

$$R\, V_{(\lambda,\ell)} g = \lambda g\, V_{(\lambda,\ell)} + g\, V_{(\lambda,\ell-1)}\,, \tag{C.1}$$

where the last term accounts for the occurrence of generalized eigenvectors. Here the chain labeling index i in appendices A and B has been omitted in the notation since it does not play any rôle. An arbitrary R-twined conjugate g' of g can be written as

$$g' = e^{-RU} g\, e^{U}\,,$$

for some U in $\mathfrak{g}$. After some trivial manipulations, eq. (C.1) can be written in terms of g' as

$$e^{-RU}\, RV_{(\lambda,\ell)}\, e^{RU}\, g' = \lambda\, g' e^{-U}\, V_{(\lambda,\ell)}\, e^{U} + g' e^{-U}\, V_{(\lambda,\ell-1)}\, e^{U}\,. \tag{C.2}$$

Being R a Lie algebra automorphism, the left hand side of this equation is $R\big(e^{-U}V_{(\lambda,\ell)}e^{U}\big)g'$. Eq. (C.2) becomes then

$$\mathrm{Ad}_{g'^{-1}}R\, V'_{(\lambda,\ell)} = \lambda V'_{(\lambda,\ell)} + V'_{(\lambda,\ell-1)} \qquad V'_{(\lambda,\ell)} = e^{-U}V_{(\lambda,\ell-1)}e^{U}\,.$$

The eigenvalues thus remain invariant, while the (generalized) eigenvectors change by ordinary conjugation. As a consequence, the dimension of the linear space generated by the eigenvectors associated to a given eigenvalue is constant under R-twined conjugation.

References

[1] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135] [SPIRES].

[2] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219] [SPIRES].

[3] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030 [hep-th/9903205] [SPIRES].

[4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [SPIRES].

[5] C.-S. Chu and P.-M. Ho, *Noncommutative D-brane and open string in pp-wave background with B-field*, *Nucl. Phys.* **B 636** (2002) 141 [hep-th/0203186] [SPIRES].

[6] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *JHEP* **09** (1999) 023 [hep-th/9908040] [SPIRES].

[7] G. Horcajada and F. Ruiz Ruiz, *Quantization of the open string on plane-wave limits of $dS_n \times S^n$ and non-commutativity outside branes*, *Nucl. Phys.* **B 799** (2008) 110 [arXiv:0711.2991] [SPIRES].

[8] J. Rahmfeld and A. Rajaraman, *The GS string action on $AdS_3 \times S^3$ with Ramond-Ramond charge*, *Phys. Rev.* **D 60** (1999) 064014 [hep-th/9809164] [SPIRES].

JHEP09(2011)020

[9] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *JHEP* **03** (1999) 018 [hep-th/9902098] [SPIRES].

[10] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *Superstring theory on $AdS_2 \times S^2$ as a coset supermanifold*, *Nucl. Phys.* **B 567** (2000) 61 [hep-th/9907200] [SPIRES].

[11] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193] [SPIRES].

[12] S. Stanciu, *D-branes in an AdS_3 background*, *JHEP* **09** (1999) 028 [hep-th/9901122] [SPIRES].

[13] S. Stanciu, *D-branes in group manifolds*, *JHEP* **01** (2000) 025 [hep-th/9909163] [SPIRES].

[14] S. Stanciu, *A note on D-branes in group manifolds: flux quantization and D0-charge*, *JHEP* **10** (2000) 015 [hep-th/0006145] [SPIRES].

[15] J.M. Figueroa-O'Farrill and S. Stanciu, *More D-branes in the Nappi-Witten background*, *JHEP* **01** (2000) 024 [hep-th/9909164] [SPIRES].

[16] C. Bachas and M. Petropoulos, *Anti-de-Sitter D-branes*, *JHEP* **02** (2001) 025 [hep-th/0012234] [SPIRES].

[17] S. Ribault and V. Schomerus, *Branes in the 2D black hole*, *JHEP* **02** (2004) 019 [hep-th/0310024] [SPIRES].

[18] Y. Hikida, R.R. Nayak and K.L. Panigrahi, *D-branes in a big bang/big crunch universe: Nappi-Witten gauged WZW model*, *JHEP* **05** (2005) 018 [hep-th/0503148] [SPIRES].

[19] S. Fredenhagen and T. Quella, *Generalised permutation branes*, *JHEP* **11** (2005) 004 [hep-th/0509153] [SPIRES].

[20] Y.-K.E. Cheung and L. Freidel, *Inner brane: a D3-brane in the Nappi-Witten model from an inner group automorphism*, *Phys. Rev.* **D 79** (2009) 126007 [arXiv:0905.0540] [SPIRES].

[21] R. Hernández, G. Horcajada and F.R. Ruiz, *D-branes with Lorentzian signature in the Nappi-Witten model*, *JHEP* **08** (2011) 047 [arXiv:1104.4730] [SPIRES].

[22] L. Birke, J. Fuchs and C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, *Adv. Theor. Math. Phys.* **3** (1999) 671 [hep-th/9905038] [SPIRES].

[23] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries. I: general theory*, *Nucl. Phys.* **B 558** (1999) 419 [hep-th/9902132] [SPIRES].

[24] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030] [SPIRES].

[25] V. Schomerus and H. Saleur, *The $GL(1|1)$ WZW model: from supergeometry to logarithmic CFT*, *Nucl. Phys.* **B 734** (2006) 221 [hep-th/0510032] [SPIRES].

[26] G. Götz, T. Quella and V. Schomerus, *The WZNW model on* $\mathrm{PSU}(1,1|2)$, *JHEP* **03** (2007) 003 [hep-th/0610070] [SPIRES].

[27] H. Saleur and V. Schomerus, *On the* $\mathrm{SU}(2|1)$ *WZNW model and its statistical mechanics applications*, *Nucl. Phys.* **B 775** (2007) 312 [hep-th/0611147] [SPIRES].

[28] T. Quella, V. Schomerus and T. Creutzig, *Boundary spectra in superspace σ-models*, *JHEP* **10** (2008) 024 [arXiv:0712.3549] [SPIRES].

JHEP09(2011)020

[29] T. Creutzig and V. Schomerus, *Boundary correlators in supergroup WZNW models*, *Nucl. Phys.* **B 807** (2009) 471 [arXiv:0804.3469] [SPIRES].

[30] E. Witten, *Nonabelian bosonization in two dimensions*, *Commun. Math. Phys.* **92** (1984) 455 [SPIRES].

[31] C. Klimčík and P. Ševera, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112] [SPIRES].

[32] S. Stanciu and A.A. Tseytlin, *D-branes in curved spacetime: Nappi-Witten background*, *JHEP* **06** (1998) 010 [hep-th/9805006] [SPIRES].

[33] W.M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, Academic Press, New York U.S.A. (2003).

[34] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, *Phys. Rev. Lett.* **71** (1993) 3751 [hep-th/9310112] [SPIRES].

[35] T. Quella and V. Schomerus, *Symmetry breaking boundary states and defect lines*, *JHEP* **06** (2002) 028 [hep-th/0203161] [SPIRES].

[36] T. Quella, *On the hierarchy of symmetry breaking D-branes in group manifolds*, *JHEP* **12** (2002) 009 [hep-th/0209157] [SPIRES].

[37] J.M. Maldacena, G.W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *JHEP* **07** (2001) 046 [hep-th/0105038] [SPIRES].

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D-branes with Lorentzian signature in the Nappi-Witten model

R. Hernández, G. Horcajada and F. Ruiz Ruiz

Departamento de Física Teórica I, Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail: rafael.hernandez@fis.ucm.es, ghr@fis.ucm.es, ferruiz@fis.ucm.es

Abstract: Lorentzian signature D-branes of all dimensions for the Nappi-Witten string are constructed. This is done by rewriting the gluing condition $J_+ = FJ_-$ for the model chiral currents on the brane as a well posed first order differential problem and by solving it for Lie algebra isometries F other than Lie algebra automorphisms. By construction, these D-branes are not twined conjugacy classes. Metrically degenerate D-branes are also obtained.

Keywords: D-branes, Penrose limit and pp-wave background

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Contents

1 Introduction

The approach to understanding D-branes and their properties in terms of the open strings attached to their worldvolumes has provided a new look on D-branes. One of the most remarkable and influential results along this line is the observation that, for flat spacetime and a globally defined constant two-form B, the D-brane world volume becomes noncommutative upon quantization [1–4]. In particular, if B is of magnetic type, space directions do not commute [1–3], whereas if B is of electric type, it is the time direction that does not commute with the space directions [4]. Since the three-form H vanishes for constant B, the field equations for the string are the same as for $B = 0$. The boundary conditions, however, change, since they involve the field B; the annhilation and creation parts of every string mode get coupled and this coupling leads to noncommutativity upon quantization. Noncommutativity should then be a general feature for D-branes in curved backgrounds.

Two important examples of the latter are provided by (i) the family of *pp*-wave geometries [5, 6] with also a globally defined constant B that describe the Penrose limits of $\mathrm{AdS}_n \times \mathrm{S}^m$ and $\mathrm{dS}_n \times \mathrm{S}^m$, and (ii) an S^3 background with nonzero H [7, 8]. In the first case, the three-form H vanishes and noncommutativity at the string endpoints can be established through canonical quantization. In the second case, canonical quantization is not adequate and noncommutativity is proved by resorting to the SU(2) WZW formulation of the string background.

More generally, since WZW models are the building blocks of many string backgrounds, one expects to learn about D-branes and their noncommutative field theories by looking at open strings on group manifolds. This entails as a first step the characterization of D-branes in WZW models [9–18]. Such characterization is well understood in some cases. In particular, it is known that the metrically nondegenerate R-twined conjugacy classes of a WZW group manifold are D-branes for all Lie algebra metric-preserving automorphisms R. These twined conjugacy classes are obtained as the solutions to a gluing condition $J_+ = RJ_-$ that matches the chiral currents J_+ and J_- of the model at the D-brane. Very little is known, however, if in the gluing condition, instead of a Lie algebra automorphism, an arbitrary isometry F of the Lie algebra metric is considered [18]. This is due to the fact that involutivity (required for the solution to the gluing condition to define a submanifold) holds trivially for Lie algebra automorphisms, whereas for general isometries it usually does not. In this latter case, involutivity often requires to consider isometries $F(g)$ that depend on the group point g at which the gluing condition must be solved.

The Nappi-Witten model [19] is a WZW model describing a four-dimensional noncompact string background. The twined conjugacy classes of its group manifold are well understood [14]. They provide two-dimensional Euclidean D-branes for metric-preserving inner automorphisms and three-dimensional Lorentzian D-branes for metric-preserving outer automorphisms. The purpose of this paper is to go beyond and construct Lorentzian D-branes of dimension one, two, three and four by solving the gluing condition for isometries other than Lie algebra automorphisms. Our motivation aims to constructing noncommutative field theories on noncompact curved backgrounds.

In this paper we apply geometric characterization of D-branes in nonsemisimple Lie groups along the lines of ref. [18] to the Nappi-Witten model. There are other ways to approach the study of D-branes in WZW models. In particular, the so called algebraic program, that uses boundary conformal theory. See e.g. ref. [20] and references therein for compact string backgrounds, ref. [21] for noncompact ones and ref. [22] for their use in the Nappi-Witten model.

The paper is organized as follows. In section 2, we review the semiclassical characterization of D-branes in a WZW model. The material presented there can be found elsewhere [11–13], though it emphasizes some points [18] concerning the rôle of Frobenius theorem and involutivity that have gone somewhat unnoticed in the literature. Section 3 contains a brief account of the Nappi-Witten model, including a complete characterization of its Lie algebra isometries. Sections 4 and 5 are dedicated to constructing the D-branes of interest. In particular, worldvolume filling D-branes, metrically degenerate D2-branes and Lorentzian D1-branes are presented in section 4, whereas Lorentzian D2 and D0-branes are exhibited in section 5. The subject of section 6 is to recover the boundary conditions for the string coordinates from the gluing condition for the chiral currents. Finally, section 7 collects our conclusions.

2 Characterization of D-branes in WZW models

Consider a Lie algebra $\mathfrak{g}$ of dimension $\mathtt{d}$ over $\mathbf{R}$ and an invariant Lie algebra metric Ω defined on it. In a basis $\{T_a\}$, with commutation relations

$$[T_a, T_b] = f_{ab}{}^{c}\, T_c\,, \tag{2.1}$$

the metric components $\Omega_{ab} = \Omega(T_a, T_b)$ satisfy

$$f_{ab}{}^{d}\,\Omega_{dc} = \Omega_{ad}\,f_{bc}{}^{d}\,. \tag{2.2}$$

Here we will be interested in isometries of Ω. An isometry of Ω is a linear map F from $\mathfrak{g}$ to $\mathfrak{g}$ such that $\Omega(FT_a, FT_b) = \Omega(T_a, T_b)$. Writing the action of F on a generator T_a as $F(T_a) = T_b\,F^b{}_a$, with $F^b{}_a$ taking values in $\mathbf{R}$, the isometry condition becomes[1]

$$\Omega_{ab} = F^c{}_a\,\Omega_{cd}\,F^d{}_b \quad\Leftrightarrow\quad F^{\mathrm{T}}\,\Omega\,F = \Omega\,. \tag{2.3}$$

The isometries of Ω form a subgroup $Iso(\Omega)$ of the general linear group $G\ell\,(\mathtt{d}, \mathbf{R})$. Our conventions for matrix notation is that the first index, from left to right, labels rows, and the second one labels columns.

The pair $(\mathfrak{g}, \Omega)$ defines a WZW model described by mappings g from the string world-sheet Σ to the group manifold G obtained from $\mathfrak{g}$ through exponentiation. If G is locally parameterized by the string coordinates $X^\mu(\tau,\sigma)$, the left-invariant $e^a{}_\mu$ and right-invariant $\bar{e}\,^a{}_\mu$ vielbeins at $g(X)$ are

$$g^{-1}\,dg = T_a\,e^a{}_\mu\,dX^\mu\,, \qquad dg\,g^{-1} = T_a\,\bar{e}\,^a{}_\mu\,dX^\mu\,. \tag{2.4}$$

The adjoint action of the group G on the algebra $\mathfrak{g}$ is

$$\mathrm{Ad}_g(T_a) = g\,T_a\,g^{-1} = T_b\,\bar{e}\,^b{}_\mu\,(e^{-1})^\mu{}_a \qquad\Leftrightarrow\qquad \mathrm{Ad}_g = \bar{e}\,e^{-1}\,,$$

where $(e^{-1})^\mu{}_a$ is the inverse of $e^a{}_\mu$, defined by $(e^{-1})^\mu{}_a\,e^b{}_\mu = \delta_a{}^b$. The spacetime metric $\mathsf{G}_{\mu\nu}$ and the three-form $\mathsf{H}_{\mu\nu\lambda}$ specifying the string background are given in terms of Ω by

$$\mathsf{G}_{\mu\nu} = \Omega\left(g^{-1}\partial_\mu g\,,\, g^{-1}\partial_\nu g\right) \tag{2.5}$$

$$\mathsf{H}_{\mu\nu\lambda} = \Omega\left(\left[g^{-1}\partial_\mu g\,,\, g^{-1}\partial_\nu g\right]\,,\, g^{-1}\partial_\lambda g\right). \tag{2.6}$$

In world sheet coordinates $\sigma^\pm = \tau \pm \sigma$, the chiral currents of the model read

$$J_-(\sigma^-) = g^{-1}\partial_- g\,, \qquad J_+(\sigma^+) = -\,\partial_+ g\,g^{-1}$$

and satisfy $\partial_+ J_- = \partial_- J_+ = 0$.

A Dp-brane is a $(p+1)$-dimensional submanifold N of G on which an open string may end. Points in N can be parameterized by the string endpoints coordinates $x^\mu(\tau) = X^\mu(\tau,\sigma)\big|_{\partial\Sigma}$, so we will write $g(x)$. The D-brane can be specified [9, 12, 13, 18] by

(i) An isometry F of Ω, that in general may depend on g, and a condition

$$J_+ = F(g)\,J_- \quad \text{at} \quad \partial\Sigma\,. \tag{2.7}$$

This condition must define $i = 1, \ldots, p+1$ integrable vector fields $k_i(x) = k^\mu{}_i(x)\partial_\mu$ that characterize the tangent bundle of the submanifold N. The fields $k_i(x)$ must define a basis of T_gN for all $g(x)$ in N. In what follows we will denote by α^i the local coordinates along the directions defined by k_i, that is, $k_i = \partial/\partial\alpha^i$.

[1] In matrix notation, in $F^a{}_b$ the index a specifies the row and the index b the column.

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(ii) A two-form ω defined on N, with components $\omega_{ij} = \omega(k_i, k_j)$ satisfying the following two requirements. Firstly, eq. (2.7) must reproduce the usual boundary conditions of the sigma model formulation, which in the presence of a D-brane read [9, 18]

$$\left(k^\mu_{\ i}\, G_{\mu\nu}\, \partial_\sigma X^\nu - \omega_{ij}\, \partial_\tau \alpha^j\right)\Big|_{\partial\Sigma} = 0 \qquad i = 1, \ldots, p+1\,. \tag{2.8}$$

And secondly, $d\omega = H\big|_N$. Note that the variations of the D-brane coordinates α^i and the string endpoints coordinates x^μ with τ are related by $\partial_\tau \alpha^i\, k^\mu_{\ i} = \partial_\tau x^\mu$.

We remark that, from the viewpoint of the sigma model, the boundary conditions take the form (2.8). These are the equations that must be recovered from eq. (2.7). To avoid confusion, eq. (2.7) is called gluing condition.

Writing the chiral currents as $J_- = T_a\, e^a_{\ \mu}\, \partial_- X^\mu$ and $J_+ = -T_a\, \bar{e}^a_{\ \mu}\, \partial_+ X^\mu$, multiplying from the left with the right-invariant inverse vielbein, and using world sheet coordinates τ and σ, condition (2.7) is written as

$$\left(\mathcal{F} - 1\right) \partial_\tau X\big|_{\partial\Sigma} = \left(\mathcal{F} + 1\right) \partial_\sigma X\big|_{\partial\Sigma}\,. \tag{2.9}$$

Here $\mathcal{F}$ stands for

$$\mathcal{F}^\mu_{\ \nu} = -\,(\bar{e}^{-1})^\mu_{\ a}\, F^a_{\ b}\, e^b_{\ \nu} \quad \Leftrightarrow \quad \mathcal{F}(x) = -\,\bar{e}^{-1} F(g)\, e \tag{2.10}$$

and is called matrix of boundary conditions. $\mathcal{F}$ is only defined at $\partial\Sigma$ and depends on x^μ through the vielbeins $e(x)$ and $\bar{e}^{-1}(x)$ and the isometry $F\big(g(x)\big)$.

For any g in G, the only motions compatible with conditions (2.9) are along the curves tangent to the vector fields [12, 13, 18]

$$t_U(g) = FUg - gU\,, \quad U \in \mathfrak{g}\,.$$

Since $U = U^a T_a$ for all U in $\mathfrak{g}$ and $\{T_a\}$ is a basis, it is enough to consider the fields

$$t_a(g) = FT_a g - gT_a\,.$$

At every g, the fields $t_a(g)$ define a space of tangent directions

$$\Pi_g = Span\,\{\, t_a(g)\,\}$$

contained in the tangent space $T_g G$ at g to the whole manifold G . If the tangent planes Π_g have dimension $p+1$ for all g in G, their collection defines a $(p+1)$-dimensional distribution on G,

$$\Pi = \{\, \Pi_g \colon \dim \Pi_g = p+1,\ g \in G\,\}\,. \tag{2.11}$$

Multiplication of $t_a(g)$ from the left with g^{-1} gives $g^{-1} t_a = \mathrm{Ad}_{g^{-1}} F T_a - T_a$. The planes Π_g have dimension $p+1$ for all g if and only if $\mathrm{Ad}_{g^{-1}} F - 1$ has rank $p+1$ for all g. It may occur that Π_g does not have constant dimension over G, so that Π is not a distribution on G. In this case, since the open string endpoints flow along the integral curves of t_a, it is enough to have a distribution

$$\Pi' = \{\, \Pi_g \colon \dim \Pi_g = p+1,\ g \in G'\,\}$$

on a submanifold G' of G, provided G' contains all such curves. See ref. [18] for details and sections 4 and 5 for examples.

For Π_g to be at all g in G', not just a tangent plane, but the tangent space to a submanifold N_{p+1} of G', the distribution Π' must be integrable. According to Frobenius theorem, Π' is integrable if and only if the vector fields t_a are involutive. That is, if and only if the commutator $[t_a, t_b]$ of any two fields t_a and t_b taking values in Π' also takes values in Π'. This amounts to the existence of functions $c_{ab}{}^c(g)$ such that

$$[\,t_a(g), t_b(g)\,] = c_{ab}{}^c(g)\; t_c(g) \tag{2.12}$$

for all g in G'. The distribution Π' is the tangent bundle of N_{p+1}.

The field $t_a(g)$ is the sum of a right-invariant vector field $X_{\rm R}g$, with $X_{\rm R} = FT_a$, and a left-invariant one $gX_{\rm L}$, with $X_{\rm L} = -T_a$. Its action on a differentiable function f defined on G is easily computed from the actions of $X_{\rm R}g$ and $gX_{\rm L}$, given by

$$X_{\rm R}g\,\big(f(g)\big) = \frac{d}{dt}\, f\big(e^{tX_{\rm R}}g\big)\Big|_{t=0}\,, \qquad gX_{\rm L}\,\big(f(g)\big) = \frac{d}{dt}\, f\big(g\,e^{tX_{\rm L}}\,\big)\Big|_{t=0}\,.$$

If $g(x)$ is parameterized by coordinates x^μ, the components of $X_{\rm R}\,g$ and $gX_{\rm L}$ are

$$X_{\rm R}\,g = X^a_{\rm R}\,T_a\,g = X^a_{\rm R}\,(\bar{e}^{\,-1})^\mu{}_a\,\partial_\mu \tag{2.13}$$

$$gX_{\rm L} = X^a_{\rm L}\,gT_a = X^a_{\rm L}\,(e^{-1})^\mu{}_a\,\partial_\mu\,. \tag{2.14}$$

This gives

$$t_a(x) = FT_ag - gT_a = \big[\,(\bar{e}^{\,-1})^\mu{}_b F^b{}_a - (e^{-1})^\mu{}_a\,\big]\,\partial_\mu = t^\mu{}_a(x)\,\partial_\mu. \tag{2.15}$$

The rank of the matrix $t^\mu{}_a(x)$ at $g(x)$ is the dimension of Π_g. Note that the a-th column of this matrix is formed by the components of the tangent vector $t_a(x)$.

Let us restrict our attention for the time being to isometries F that are constant over G. In this case, equation (2.12) takes the simpler form [18]

$$-\,\big[FT_a, FT_b\big]\,g + g\,\big[T_a, T_b\big] = c_{ab}{}^c(g)\,\left(FT_cg - gT_c\right). \tag{2.16}$$

A solution to this equation is provided by $F = R^\Omega$ and $c_{ab}{}^c(g) = -f_{ab}{}^c$, with R^Ω a Lie algebra automorphism satisfying eq. (2.3) and $f_{ab}{}^c$ the Lie algebra structure constants. This is trivial since, by definition, Lie algebra automorphisms R satisfy

$$R\,[T_a, T_b] = [RT_a, RT_b]\,. \tag{2.17}$$

The restriction to automorphisms R^Ω complying with eq. (2.3) comes from the observation that for a general Lie algebra not all automorphisms R are isometries. Automorphisms fulfilling (2.3) condition are called Ω-preserving. The vector fields $t_a = R^\Omega T_a g - gT_a$ are very easy to integrate and give for the submanifold N the R^Ω-twined conjugacy classes of the group G [10, 12, 13],

$$N = \mathcal{C}(R, g_0) = \big\{e^{R^\Omega V} g_0\, e^{-V} :\, V \in \mathfrak{g}\big\}$$

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where g_0 is an arbitrary group element that accounts for the integration constants.

There are suggestions [11, 12, 24] that $F = -R^{\Omega}$, with R^{Ω} an Ω-preserving constant automorphism, may solve involutivity and, hence, may lead to D-branes. For semisimple Lie algebras, however, it has been proved [18] that this is not the case. In the following sections we examine this problem for the Nappi-Witten model, a typical example of nonsemisimple WZW model. We find that $F = -R^{\Omega}$ does not define D-branes if R^{Ω} is constant, but it does if R^{Ω} is conveniently taken to depend on g. The choice of the g-dependence of $R^{\Omega}(g)$ is indicated by the requirement of the constancy of the rank of the matrix $t^{\mu}{}_{a}(x)$ In sections 4 to 6 several examples are presented.

3 The Nappi-Witten model: a brief review

The Nappi-Witten model [19] is constructed upon a nonsemisimple Lie algebra $\mathfrak{g}_{\text{NW}}$ whose exponentiation gives a group manifold G_{NW} describing a nontrivial four-dimensional string background of pp-type. The algebra has dimension four and generators $\{P_1, P_2, J, T\}$ with commutation relations

$$[J, P_M] = \epsilon_{MN} P_N, \qquad [P_M, P_N] = \epsilon_{MN} T, \qquad [T, P_i] = [T, J] = 0, \qquad M, N = 1, 2\,. \tag{3.1}$$

It is the central extension of the Euclidean algebra in two dimensions, T being the central charge. We will use the labeling

$$T_1 = P_1, \qquad T_2 = P_2, \qquad T_3 = J, \qquad T_4 = T\,,$$

so that

$$f_{12}{}^{4} = f_{31}{}^{2} = f_{23}{}^{1} = 1\,. \tag{3.2}$$

The most general invariant metric Ω can be found by solving eqs. (2.2). It reads [19]

$$\Omega = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.3}$$

where k and b are arbitrary real parameters. The parameter k can be absorbed in the coupling constant in front of the classical WZW action, so that it can be set equal to one without loss of generality. As concerns b, it can be set to zero by the following redefinition of the Lie algebra generators:

$$P'_M = P_M, \quad J' = J - \frac{b}{2} T, \quad T' = T\,.$$

Indeed, under such transformations, the Lie algebra commutators (3.1) remain unchanged and the metric Ω takes the form in (3.3) with $b = 0$. We thus set $k = 1$ and $b = 0$ without loss of generality.

Lie algebra isometries. The isometries F of Ω are the solutions to equation (2.3). To find them, it is most convenient to write Ω in eq. (3.3) as $\Omega = M^{\mathrm{T}}\,\eta\,M$, with M the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

and $\eta = diag\,(+,+,+,-)$. Eq. (2.3) then becomes

$$\left(MFM^{-1}\right)^{\mathrm{T}}\eta\left(MF\,M^{-1}\right) = \eta\,.$$

This is solved by MFM^{-1} an arbitrary element of $O(3,1)$, so the isometry group is

$$Iso(\Omega) = \left\{M^{-1}\Lambda\,M\colon\ \Lambda \in O(3,1)\right\}.$$

Note that M is not an isometry.

Lie algebra automorphisms. The automorphisms of the Nappi-Witten algebra can be found by solving eqs. (2.17). Using the structure constants $f_{ab}{}^{c}$ in eq. (3.2), it is straightforward to see that eqs. (2.17) only have two solutions, R_+ and R_-, given by

$$R_\pm(\rho_0,\rho,\phi,\theta) = \begin{pmatrix} \rho_0\cos\phi & \mp\rho_0\sin\phi & \mp\frac{\rho}{\rho_0}\cos\theta & 0 \\ \rho_0\sin\phi & \pm\rho_0\cos\phi & -\frac{\rho}{\rho_0}\sin\theta & 0 \\ 0 & 0 & \pm 1 & 0 \\ \rho\cos(\theta\mp\phi) & \rho\sin(\theta\mp\phi) & \zeta & \pm\rho_0^2 \end{pmatrix}. \tag{3.4}$$

The parameters ρ_0, ρ, ϕ, θ and ζ can take any values on the ranges

$$\rho_0 > 0\,, \qquad \rho \geq 0\,, \qquad 0 \leq \phi\,,\,\theta < 2\pi\,, \qquad -\infty < \zeta < \infty\,.$$

As is well known, the automorphisms R_- and R_+ form a group, $\mathrm{Aut}(\mathfrak{g}_{\mathrm{NW}})$. There are a few observations concerning automorphisms and isometries that we find relevant:

- $\mathrm{Aut}(\mathfrak{g}_{\mathrm{NW}})$ is not $O(3,1)$ nor a subgroup of it. This would require $R_\pm$ to satisfy $R_\pm^{\mathrm{T}}\,\eta\,R_\pm = \eta$, but this only occurs for $\rho_0 = 1$, $\zeta = \rho = 0$.

- Not every isometry is a Lie algebra automorphism. Take for example

$$F_0 = M^{-1}\eta M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.5}$$

  By construction, F_0 is an isometry but does not have the form (3.4), hence is not an automorphism.[2]

[2] As a matrix F_0 is equal to Ω, but they have different index structures: $(F_0)^a{}_b$ and Ω_{ab}.

- Conversely, not every automorphism is an isometry. For this to be the case, $R_\pm$ must satisfy $R_\pm^{\rm T}\,\Omega\, R_\pm = \Omega$. Some algebra shows that this is so if and only if $\rho_0 = 1$ and $\zeta = \mp\rho^2/2$. From now on, we denote by $R_\pm^\Omega$ automorphisms of this type,

$$R_\pm^\Omega(\rho,\phi,\theta) = \begin{pmatrix} \cos\phi & \mp\sin\phi & \mp\rho\cos\theta & 0 \\ \sin\phi & \pm\cos\phi & -\rho\sin\theta & 0 \\ 0 & 0 & \pm 1 & 0 \\ \rho\cos(\theta\mp\phi) & \rho\sin(\theta\mp\phi) & \mp\dfrac{\rho^2}{2} & \pm 1 \end{pmatrix}. \tag{3.6}$$

They form the subgroup $\mathrm{Aut}_\Omega(\mathfrak{g}_{\rm NW})$ of Ω-preserving automorphisms.

The spacetime group manifold. A group element g can be parameterized as [19]

$$g(x_M,u,v) = e^{x_M P_M}\, e^{uJ}\, e^{vT}\,, \qquad M = 1,2$$

in terms of real coordinates x_M, u, v. In this parameterization, the identity element is $e = g(0,0,0)$, while the group law takes the form $g(x)\,g(x') = g(x'')$, with

$$\begin{aligned} x''_M &= x_M + \cos u\, x'_M - \sin u\, \epsilon_{MK}\, x'_K \\ u'' &= u + u' \\ v'' &= v + v' + \frac{1}{2}\cos u\, \epsilon_{MK}\, x_M\, x'_K + \frac{1}{2}\sin u\, x_M\, x'_M\,. \end{aligned}$$

The inverse g^{-1} of g reads

$$g(x_M,u,v)^{-1} = g\left(-\cos u\; x_M - \sin u\; \epsilon_{MK}\, x_K,\; -u,\; -v\right).$$

The left and right-invariant vielbeins follow easily from their definition (2.4). They read

$$e^a{}_\mu = \begin{pmatrix} \cos u & \sin u & 0 & 0 \\ -\sin u & \cos u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{x_2}{2} & -\dfrac{x_1}{2} & 0 & 1 \end{pmatrix}, \qquad \bar e^a{}_\mu = \begin{pmatrix} 1 & 0 & x_2 & 0 \\ 0 & 1 & -x_1 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{x_2}{2} & \dfrac{x_1}{2} & -\dfrac{1}{2}\,(x_1^2+x_2^2) & 1 \end{pmatrix}.$$

Some simple algebra gives then for the adjoint action of the group on the Lie algebra

$$\mathrm{Ad}_g = \bar e\, e^{-1} = \begin{pmatrix} \cos u & -\sin u & x_2 & 0 \\ \sin u & \cos u & -x_1 & 0 \\ 0 & 0 & 1 & 0 \\ x_1\sin u - x_2\cos u & x_1\cos u + x_2\sin u & -\dfrac{1}{2}\,(x_1^2+x_2^2) & 1 \end{pmatrix}. \tag{3.7}$$

It follows that $R_+^\Omega U = \mathrm{Ad}_h U$ for all U in the Nappi-Witten algebra, with h a group element with coordinates

$$x_1 = \rho\sin\theta\,, \qquad x_2 = -\rho\cos\theta\,, \qquad u = \phi\,, \qquad v \text{ arbitrary}\,.$$

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This implies that automorphisms of type R_+^Ω are inner. As regards automorphisms of type R_-^Ω, they are outer since there is no group element h such that $R_-^\Omega U = \mathrm{Ad}_h U$ for all U.

The spacetime metric and the WZW three-form are given by eqs. (2.5) and (2.6). In the coordinates that we are using they take the form

$$ds^2 = dx_1^2 + dx_2^2 + (x_2\, dx_1 - x_1\, dx_2)\, du + 2\, du\, dv \tag{3.8}$$

$$\mathsf{H} = dx_1 \wedge dx_2 \wedge du\,. \tag{3.9}$$

4 Filling D-branes and D-strings from general isometries

We are interested in finding if isometries of the form $F = -R^\Omega$ define D-branes. We will consider both constant and g-dependent automorphisms. It is convenient to separately discuss inner and outer automorphisms.

4.1 Case $F = -R^\Omega_{\textbf{inner}}$

Using eqs. (3.6) and (2.15), for $F = -R_+^\Omega$, we obtain the following vector fields t_a:

$$\begin{aligned} t_1(x) = &- (\cos\phi + \cos u)\, \partial_1 - (\sin\phi + \sin u)\, \partial_2 \\ &+ \frac{1}{2} \left[x_1 (\sin\phi - \sin u) - x_2 (\cos\phi - \cos u) - 2\rho \cos(\theta - \phi) \right] \partial_v \end{aligned} \tag{4.1}$$

$$\begin{aligned} t_2(x) = &(\sin\phi + \sin u)\, \partial_1 - (\cos\phi + \cos u)\, \partial_2 \\ &+ \frac{1}{2} \left[x_1 (\cos\phi - \cos u) + x_2 (\sin\phi - \sin u) - 2\rho \sin(\theta - \phi) \right] \partial_v \end{aligned} \tag{4.2}$$

$$t_3(x) = (x_2 + \rho\cos\theta)\, \partial_1 - (x_1 - \rho\sin\theta)\, \partial_2 - 2\, \partial_u - \frac{\rho}{2} (x_1 \sin\theta - x_2 \cos\theta - \rho)\, \partial_v \tag{4.3}$$

$$t_4(x) = -2\, \partial_v\,. \tag{4.4}$$

They involve the four derivatives ∂_1, ∂_2, ∂_u and ∂_v. In particular, ∂_u only enters t_3 with constant coefficient, so the motion defined by t_3 covers the whole range for u. A simple calculation shows that

$$\det\left(t^\mu{}_a\right) = 8 \left[1 + \cos(\phi - u) \right]. \tag{4.5}$$

Nonexistence of D-branes for constant $F = -R_+^\Omega$. Let us first consider that R_+^Ω does not depend on $g(x)$, so the parameters ρ, ϕ, θ are constant. For points $g(x)$ with $u \neq \phi + (2n+1)\pi$, the determinant (4.5) does not vanish and the tangent planes

$$\Pi_{g(x)} = Span\,\{t_1, t_2, t_3, t_4\} \quad \text{for} \quad u \neq \phi + (2n+1)\pi$$

have dimension four. At points $g(x)$ with $u = \phi + (2n+1)\pi$, however, the determinant (4.5) vanishes. In a neighborhood of these points the fields t_1 and t_2 become

$$\begin{aligned} t_1(x) &= \left[x_1 \sin\phi - x_2 \cos\phi - \rho \cos(\theta - \phi) \right] \partial_v \\ t_2(x) &= \left[x_1 \cos\phi + x_2 \sin\phi - \rho \sin(\theta - \phi) \right] \partial_v\,, \end{aligned}$$

while t_3 and t_4 remain as in (4.3) and (4.4). The fields t_1, t_2 and t_4 define then the same tangent direction, namely ∂_v, so the tangent planes are spanned by t_3 and t_4,

$$\Pi_{g(x)} = Span\,\{t_3,\, t_4\} \quad \text{for} \quad u = \phi + (2n+1)\pi\,,$$

and have dimension 2. Hence, the dimension of $\Pi_{g(x)}$ is not the same for all $g(x)$ in $G_{\rm NW}$, the collection of tangent planes $\Pi_{g(x)}$ is not a distribution on $G_{\rm NW}$ and Frobenius theorem does not apply. The same conclusion can be reached by studying the rank of $\mathrm{Ad}_{g^{-1}}F - 1$ (see the appendix).

One may consider the submanifold

$$G'_{\rm NW} = G_{\rm NW} - \{g(x)\colon\, u = \phi + (2n+1)\pi\}$$

that results from removing from $G_{\rm NW}$ the closed set of group elements $g(x)$ with $u = \phi + (2n+1)\pi$. The collection

$$\Pi' = \left\{\Pi_{g(x)}\colon\; g(x) \in G'_{\rm NW}\right\}$$

is now a distribution of dimension four on $G'_{\rm NW}$. Furthermore, having maximal dimension, it is trivially involutive. The manifold $G'_{\rm NW}$ cannot, however, be accepted as a D-brane. The reason is that it does not contain the integral curves of t_3, which connects points $g(x')$ with $u' \neq \phi + (2n+1)\pi$ with points $g(x)$ with $u = \phi + (2n+1)\pi$ that are not in $G'_{\rm NW}$, thus contradicting the idea that the string endpoints lie on the D-brane. The gluing condition (2.7) does not define then a D-brane for constant $F = -R^{\Omega}_{+}$.

Filling D-branes and D-strings for nonconstant $F = -R^{\Omega}_{+}$. The situation is very different if R^{Ω}_{+} depends on $g(x)$. Assume that we take

$$F_4 = R^{\Omega}_{+}(\rho,\phi,\theta)\,, \qquad \phi(u) = u + \phi_0\,, \qquad \phi_0 = const \neq (2n+1)\pi\,. \tag{4.6}$$

The matrix $t^{\mu}{}_{a}$ has now nonvanishing determinant for all $g(x)$, so the collection Π of all the tangent planes $\Pi_{g(x)}$ is a distribution of dimension four on $G_{\rm NW}$. Having maximal dimension, Π is trivially involutive and is thus the tangent bundle of $G_{\rm NW}$ itself. In section 6 we show that the gluing condition (2.9) for F_4 in eq. (4.6) with ρ and θ constant can be written as a boundary condition (2.8) for a two-form ω such that $d\omega = H$ on $G_{\rm NW}$. The gluing condition for such an F_4 hence defines a filling D-brane.

Consider now the isometry

$$F_2 = R^{\Omega}_{+}(\rho,\phi,\theta)\,, \qquad \phi(u) = u - \pi\,. \tag{4.7}$$

The determinant $\det(t^{\mu}{}_{a})$ then vanishes for all $g(x)$. In the neighborhood of any $g(x)$, the fields t_1 and t_2 read

$$\begin{aligned} t_1(x) &= -\left[\,x_1 \sin u - x_2 \cos u - \rho\cos(\theta - u)\,\right]\partial_v \\ t_2(x) &= -\left[\,x_1 \cos u + x_2 \sin u - \rho\sin(\theta - u)\,\right]\partial_v\,, \end{aligned}$$

while t_3 and t_4 remain as in (4.3) and (4.4). The only partial derivative that occurs in t_1, t_2 and t_4 is ∂_v, so they define the same tangent direction. The tangent planes $\Pi_{g(x)}$

have dimension two for all $g(x)$ and are spanned by t_3 and t_4. Their collection Π_2 is hence a distribution of dimension two on $G_{\rm NW}$ and Frobenius theorem can be used. It is trivial that $[t_3, t_4] = 0$, so the distribution is integrable. Π_2 defines a family of two-dimensional submanifolds N_2 whose tangent space at all $g(x)$ is $T_{g(x)}N_2 = \Pi_{g(x)}$. In section 6, we show that the gluing condition can be recast as a boundary condition for a two-form ω defined on N_2. Such form trivially satisfies $d\omega = H\big|_{N_2}$, so the submanifolds N_2 are D1-branes and provide a foliation of $G_{\rm NW}$.

Redefining $v \to v - bu/2$ and using eq. (3.8), we have that

$$\mathsf{G}(t_3, t_3) = -\left[\left(x_1 - \rho\sin\theta\right)^2 + \left(x_2 + \rho\cos\theta\right)^2\right] < 0\,, \quad \mathsf{G}(t_3, t_4) = 8 > 0\,, \quad \mathsf{G}(t_4, t_4) = 0\,.$$

Every submanifold N_2 in the family has then Lorentzian signature and is a D-string. If α^1 and α^2 parameterize the integral curves of $k_1(x) = t_3(x)$ and $k_2(x) = t_4(x)$ in eqs. (4.3), the D-string is formed by points $x^\mu(\alpha^1, \alpha^2)$ such that

$$dx^\mu = k^\mu{}_1(x)\, d\alpha^1 + k^\mu{}_2(x)\, d\alpha^2\,. \tag{4.8}$$

The induced metric on the D-string takes the form

$$ds_2^2 = \mathsf{G}(k_1, k_1)\,(d\alpha^1)^2 + \mathsf{G}(k_1, k_2)\, d\alpha^1\, d\alpha^2\,. \tag{4.9}$$

Assume now that ρ and θ depend on x_1, x_2 and u, but not on v. Noting that eqs. (4.8) imply that x_1, x_2 and u only depend on α^1, we conclude that $\mathsf{G}(k_1, k_1)$ only depends on α^1 and thus eq. (4.9) is a *pp*-wave metric in 1+1 dimensions.

To find the metric coefficient $\mathsf{G}(k_1, k_1)$ as a function of α^1, some further assumptions on ρ and θ are necessary. For example, for ρ and θ constant, integrating eqs. (4.8), we obtain

$$\begin{aligned} x_1 &= \rho\sin\theta + r_0\cos(\alpha^1 + \varphi_0) \\ x_2 &= -\rho\cos\theta - r_0\sin(\alpha^1 + \varphi_0) \\ u &= -2\alpha + u_0 \\ v &= -2\beta + \frac{r_0\rho}{2}\cos(\alpha^1 + \varphi_0 + \theta) + v_0\,, \end{aligned}$$

with r_0, α_0, u_0 and v_0 integration constants. The D-string metric is then $-r_0^2\,(d\alpha^1)^2 + 8\, d\alpha^1\, d\alpha^2$. To the best of our knowledge, the family (4.9) of D-strings has gone unnoticed in the literature.

4.2 Case $F = -R^\Omega_{\text{outer}}$

For $F = -R^\Omega_-$, there are only three nonzero vector fields t_a, given by

$$\begin{aligned} t_1(x) = &- (\cos\phi + \cos u)\,\partial_1 - (\sin\phi + \sin u)\,\partial_2 \\ &+ \frac{1}{2}\left[x_1\,(\sin\phi - \sin u) - x_2\,(\cos\phi - \cos u) - 2\rho\cos(\theta + \phi)\right]\partial_v \end{aligned} \tag{4.10}$$

$$\begin{aligned} t_2(x) = &- (\sin\phi - \sin u)\,\partial_1 + (\cos\phi - \cos u)\,\partial_2 \\ &- \frac{1}{2}\left[x_1\,(\cos\phi + \cos u) + x_2\,(\sin\phi + \sin u) + 2\rho\sin(\theta + \phi)\right]\partial_v \end{aligned} \tag{4.11}$$

$$t_3(x) = -(x_2 + \rho\cos\theta)\,\partial_1 + (x_1 + \rho\sin\theta)\,\partial_2 - \frac{\rho}{2}\,(x_1\sin\theta + x_2\cos\theta + \rho)\,\partial_v\,. \tag{4.12}$$

They involve ∂_1, ∂_2 and ∂_v, but not ∂_u, hence they define motions that leave u constant. The matrix $t^\mu{}_a$ of the coefficients is now 3×3, with $a = 1,2,3$ and $\mu = 1,2,v$. A straightforward calculation gives

$$\det\left(t^\mu{}_a\right) = -2\left[k(x_1,x_2)\right]^2, \tag{4.13}$$

where $k(x_1,x_2)$ is the function of x_1 and x_2

$$k(x_1,x_2) = \left(x_1 + \rho\sin\theta\right)\,\cos\left(\frac{\phi+u}{2}\right) + \left(x_2 + \rho\cos\theta\right)\,\sin\left(\frac{\phi+u}{2}\right). \tag{4.14}$$

Nonexistence of D-branes for constant $F = -R_-^\Omega$. At points $g(x)$ with $k(x_1,x_2)\neq 0$, the determinant (4.13) does not vanish and the fields t_a define three-dimensional tangent planes $\Pi_{g(x)}$. By contrast, for $g(x)$ with $k(x_1,x_2)=0$, the determinant (4.13) vanishes. It is straightforward to see that the rank of the matrix $t^\mu{}_a$ is one in this case, so the corresponding tangent planes $\Pi_{g(x)}$ have dimension one. The collection of all the planes $\Pi_{g(x)}$ is not a distribution on $G_{\rm NW}$ and Frobenius theorem cannot be used. This conclusion can also be reached by studying the rank of $\mathrm{Ad}_{g^{-1}}F - 1$ for $F = -R_-^\Omega$ (see the appendix).

One could think of removing from $G_{\rm NW}$ the locus of points for which $k(x_1,x_2) = 0$. The resulting submanifold $G'_{\rm NW}$ then does not include all the points accessible to the string endpoints, since $k(x_1,x_2) = 0$ can be reached from $k(x_1,x_2)\neq 0$ through the motions defined by the fields t_a. Hence the gluing condition (2.7) does not define a D-brane for constant $F = -R_-^\Omega$.

D2 and D0-branes for nonconstant $F = -R_-^\Omega$. Let us take now ρ, ϕ and θ in $R_-^\Omega(\rho,\phi,\theta)$ functions of x_1 and x_2 such that $k(x_1,x_2) = k_0$, with k_0 a nonzero constant. The fields t_1, t_2 and t_3 define then three-dimensional tangent planes $\Pi_{g(x)}$ for all $g(x)$ in $G_{\rm NW}$, whose collection Π_3 is a three-dimensional distribution on $G_{\rm NW}$. Alternatively, Π_3 is a three-dimensional distribution on any constant $u=u_0$ three-plane $N_{u_0} = \{g(x)\in G_{\rm NW}\colon\ u=u_0\}$. The distribution Π_3 is trivially involutive and defines the tangent bundle of the three-plane $u=u_0$. This plane has two spacelike directions and one lightlike direction, but no timelike direction, so the metric is degenerate. In section 6, the gluing condition is written as the boundary condition for a two-form ω defined on the three-plane $u=u_0$ such that $d\omega = H\big|_{u_0} = 0$, thus proving that the planes $u=u_0$ are degenerate D2-branes.

We next consider ρ, ϕ and θ functions of x_1 and x_2 such that $k(x_1,x_2)=0$ for all x_1 and x_2. In the neighborhood of any point $g(x)$ in $G_{\rm NW}$ the vector fields t_a in (4.10)–(4.12) take the form

$$t_1(x) = -\left(\cos\phi+\cos u\right)\left(\partial_1 + \frac{\rho}{2}\cos\theta\,\partial_v\right) - \left(\sin\phi+\sin u\right)\left(\partial_2 - \frac{\rho}{2}\sin\theta\,\partial_v\right) \tag{4.15}$$

$$t_2(x) = -\left(\sin\phi-\sin u\right)\left(\partial_1 + \frac{\rho}{2}\cos\theta\,\partial_v\right) + \left(\cos\phi-\cos u\right)\left(\partial_2 - \frac{\rho}{2}\sin\theta\,\partial_v\right) \tag{4.16}$$

$$t_3(x) = -\left(x_2+\rho\cos\theta\right)\left(\partial_1 + \frac{\rho}{2}\cos\theta\,\partial_v\right) + \left(x_1+\rho\sin\theta\right)\left(\partial_2 - \frac{\rho}{2}\sin\theta\,\partial_v\right). \tag{4.17}$$

It is very easy to convince oneself that these vectors define a one-dimensional distribution Π_1 on $G_{\rm NW}$. Being one-dimensional, Π_1 is trivially involutive. Its integral curves N_1 are spacelike since

$$\mathsf{G}(t_a,t_a)>0 \quad \text{for} \quad t_a\neq 0, \quad a=1,2,3\,.$$

To give their explicit form, some further assumptions on the dependence of ρ, ϕ and θ on x_1, x_2 and u must be made. Let us give some examples.

Take $x_1+\rho\sin\theta=0$ and $\phi+u=\phi_0\neq 2n\pi$. Condition $k(x_1,x_2)=0$ implies $x_2+\rho\cos\theta=0$. These three equations define ρ and θ in terms of x_1 and x_2, and ϕ in terms of u. The fields t_1, t_2 and t_3 become

$$t_1=-2\cos\left(u-\frac{\phi_0}{2}\right)t_0\,, \qquad t_2=2\sin\left(u-\frac{\phi_0}{2}\right)t_0\,, \qquad t_3=0\,,$$

where t_0 stands for

$$t_0=\cos\left(\frac{\phi_0}{2}\right)\left(\partial_1+\frac{\rho}{2}\cos\theta\,\partial_v\right)+\sin\left(\frac{\phi_0}{2}\right)\left(\partial_2-\frac{\rho}{2}\sin\theta\,\partial_v\right).$$

It is clear that t_1 and t_2 do not vanish simultaneously and specify the same direction at every x^μ. The integral curves are in this case $v=x_1^0\,x_2-x_2^0\,x_1+v_0$, with x_1^0, x_2^0 and v_0 integration constants.

Assume now that $x_1+\rho\sin\theta$ and $x_2+\rho\cos\theta$ do not vanish simultaneously. The field t_3 is then nonvanishing and the integral curves are formed by $x^\mu(\alpha)$, with $u=u_0$ and x_1, x_2 and v the solutions to

$$\frac{dx_1}{d\alpha}=-\left(x_2+\rho\cos\theta\right), \qquad \frac{dx_2}{d\alpha}=x_1+\rho\sin\theta\,, \qquad \frac{dv}{d\alpha}=-\frac{\rho}{2}\left(x_1\sin\theta+x_2\cos\theta+\rho\right),$$

where α is a parameter along the curve. For $\rho=0$, the integral curves are circles $x_1^2+x_2^2=r_0^2$ of arbitrary radius r_0 located on any two-plane $(u=u_0\,,v=v_0)$. A simple solution for $\rho\neq 0$ is provided by $\rho=-x_1/\sin\theta_0$, with $\theta=\theta_0\neq n\pi$ constant and $\phi=-u_0$. In this case, the integral curves are parabolas $v=-\frac{1}{4}\cot\theta_0\,x_1^2+v_0$ on any two-plane $(x_2\!=x_2^0\,,u\!=\!u_0)$.

In any case, being one-dimensional, $d\omega=\mathsf{H}\big|_{N_1}$ is trivially satisfied, and the curves N_1 are D0-branes.

5 D2 and D0-branes with Lorentzian signature

In the previous section we have constructed D3 and D1-branes with Lorentzian signature by integrating the gluing condition for some g-dependent isometries $F=-R^\Omega$. Here we construct D2-branes and D0-branes, also with Lorentzian signature, for g-dependent isometries $F\neq\pm R^\Omega$.

Since the product of two isometries is an isometry and F_0 in eq. (3.5) is an isometry, $F=F_0R^\Omega$, with R^Ω an arbitrary metric-preserving automorphism, is also an isometry.

Isometries of this type do not have the form $\pm R^{\Omega}$. Let us take for R^{Ω} an inner automorphism, so that we will be considering $F = F_0 R^{\Omega}_{+}$. The corresponding vector fields t_a are

$$\begin{aligned} t_1 - \rho\cos(\phi-\theta)\, t_4 &= (\cos\phi - \cos u)\,\partial_1 + (\sin\phi - \sin u)\,\partial_2 \\ &\quad + \frac{1}{2}\left[\,2\rho\,\cos(\phi-\theta) - x_1\,(\sin\phi + \sin u) + x_2\,(\cos\phi + \cos u)\,\right]\partial_v \\ t_2 + \rho\sin(\phi-\theta)\, t_4 &= -\,(\sin\phi - \sin u)\,\partial_1 + (\cos\phi - \cos u)\,\partial_2 \\ &\quad - \frac{1}{2}\left[\,2\rho\,\sin(\phi-\theta) + x_1\,(\cos\phi + \cos u) + x_2\,(\sin\phi + \sin u)\,\right]\partial_v \\ t_3 + \frac{1}{2}\,(\rho^2+2)\, t_4 &= -\,(x_2 + \rho\cos\theta)\,\partial_1 + (x_1 - \rho\sin\theta)\,\partial_2 + \frac{\rho}{2}\,(x_1\sin\theta - x_2\cos\theta - \rho\,)\,\partial_v \\ t_4 &= -\,x_2\,\partial_1 + x_1\,\partial_2 + \partial_u - \partial_v \end{aligned}$$

From these expressions it follows that $\det\,(t^{\mu}{}_a) = 0$. This indicates that there are no filling D-branes for the isometry that we are considering. We look for D-branes of lower dimension.

Let us take $\phi(u) = u$ and $\rho = 0$. Since the parameter θ always occurs in R^{Ω}_{+} through $\rho\sin\theta$ and $\rho\cos\theta$, see eq. (3.6), we can set without loss of generality $\theta = 0$. The isometry F then reads

$$F_3(u) = F_0\, R^{\Omega}_{+}(0,u,0)\,, \tag{5.1}$$

and the fields t_a become

$$t_1 = (-x_1\sin u + x_2\cos u)\,\partial_v \tag{5.2}$$

$$t_2 = (-x_1\cos u - x_2\sin u)\,\partial_v \tag{5.3}$$

$$t_3 = -\partial_u + \partial_v \tag{5.4}$$

$$t_4 = -x_2\,\partial_1 + x_1\,\partial_2 + \partial_u - \partial_v\,. \tag{5.5}$$

The rank of the matrix $t^{\mu}{}_a$ of coefficients is now three for $x_1^2 + x_2^2 \neq 0$, and one for $x_1 = x_2 = 0$. We discuss these two instances separately.

D2-branes. Consider the four-dimensional submanifold

$$G_4 = \left\{\, g(x) \in G_{\rm NW}:\ x_1^2 + x_2^2 \neq 0 \,\right\}.$$

The group elements $g(x)$ that are not in G_4 have $x_1 = x_2 = 0$. As both x_1 and x_2 approach zero, the coefficients of ∂_1 and ∂_2 in eqs. (5.2)–(5.5) vanish, so the fields t_a do not connect points in G_4 with points outside G_4. In other words, the integral curves of t_a stay in G_4. Furthermore, since the rank of the matrix $t^{\mu}{}_a$ is three for all g in G_4, the fields t_a define a three-dimensional distribution Π_3 on G_4 formed by the tangent planes $\Pi_g = Span\,\{t_2, t_3, t_4\}$. We may alternatively take

$$\Pi_{g(x)} = Span\left\{\, k_1 := -x_2\partial_1 + x_1\partial_2\,,\ \ k_2 := \partial_u\,,\ \ k_3 := \partial_v \right\}. \tag{5.6}$$

The commutator of any two fields k_1, k_2, k_3 vanishes, thus implying that they are involutive. According to Frobenius theorem, Π_3 is the tangent bundle of a family of three-dimensional submanifolds N_3 foliating G_4. If α^1, α^2 and α^3 parameterize the integral curves

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of k_1, k_2 and k_3, a manifold N_3 in the family is formed by points $x^\mu(\alpha^1,\alpha^2,\alpha^3)$ such that

$$dx^\mu = k^\mu{}_1\, d\alpha^1 + k^\mu{}_2\, d\alpha^2 + k^\mu{}_3\, d\alpha^3\,.$$

Integrating these equations we obtain

$$N_3: \quad \begin{array}{l} x_1 = r_0\cos(\alpha^1+\varphi_0) \\ x_2 = r_0\sin(\alpha^1+\varphi_0) \end{array}, \quad u = \alpha^2 + u_0\,, \quad v = \alpha^3 + v_0\,, \tag{5.7}$$

with $r_0 > 0$, φ_0, u_0 and v_0 arbitrary integration constants. Note that $r_0 = 0$ corresponds to $x_1 = x_2 = 0$, which is excluded from G_4 and will be discussed below. The induced metric on N_3 is

$$ds_3^2 = r_0^2\, d\alpha^1\,(d\alpha^1 - d\alpha^2) + 2\, d\alpha^2\, d\alpha^3\,. \tag{5.8}$$

For every $r_0^2 > 0$, this is a *pp*-wave in $2+1$ dimensions. In section 6 it is shown that the gluing condition for the isometry F_3 can be cast as a boundary condition with an admissible two-form ω defined on N_3, thus ensuring that N_3 is a D2-brane.

D0-branes. Let us now consider the two-dimensional submanifold

$$G_2 = \left\{\, g(x)\in G_{\rm NW}:\ x_1 = x_2 = 0\,\right\}.$$

For g in G_2, the fields t_1 and t_2 in (5.2) and (5.3) vanish, while t_3 and t_4 in (5.4) and (5.5) are proportional to each other and define a one-dimensional distribution Π_1 on G_2. Having dimension one, Π_1 is trivially involutive. The integral curves of t_4 have $x_1 = x_2 = 0$ and u and v such that

$$\frac{du}{d\alpha} = 1\,, \quad \frac{dv}{d\alpha} = -1\,,$$

with α a parameter along the curve. Integration gives $v + u = c_0$, with c_0 an arbitrary integration constant. These curves are timelike since $\mathsf{G}(t_4,t_4) = -2 < 0$. Furthermore, the induced metric on them is $ds_1^2 = -2\, d\alpha^2$. In section 6, we prove that the gluing condition for F_3 with $x_1 = x_2 = 0$ can be written as a boundary condition with $\omega = 0$, hence trivially satisfying $d\omega = \mathsf{H}$ on $x_1 = x_2 = 0$. These timelike lines are then D0-branes.

6 Comparison with the sigma-model approach

In sections 4 and 5 we have integrated the gluing condition for a variety of isometries. We have anticipated that, in every one of the case considered, the resulting submanifold N was a D-brane since the corresponding gluing condition could be written as a sigma model boundary condition with a two-form ω defined on N such that $d\omega = \mathsf{H}\big|_N$. Let us show this here.

We first note that there always exists a two-form ω defined on N such that any gluing condition can be written as a boundary condition. For all g in N, ω is specified [18] by its action on tangent vectors $t_a = FT_a g - gT_a$ in T_gN as

$$\omega\big(FT_a g - gT_a\,,\, FT_b g - gT_b\big) = \mathsf{G}\big(FT_a g - gT_a\,,\, FT_b g + gT_b\big)\,, \tag{6.1}$$

where we note the sign change in the second argument on the right hand side. Recall that the linearly independent vector fields k_i $(i = 1, \dots, p+1)$ that span $T_g N$ are linear combinations of t_a and that the components of ω are $\omega_{ij} = \omega(k_i, k_j)$, so that the two form ω is completely determined by eq. (6.1). A separate issue is whether ω satisfies $d\omega = \mathsf{H}\big|_N$. For D-branes of dimension one and two, $d\omega = \mathsf{H}\big|_N = 0$ is trivial. Dimension three and larger must be discussed case by case. We concentrate on these cases.

Filling D-brane. We start with the isometry F_4 in eq. (4.6). The submanifold N_4 obtained by integrating the gluing condition was the whole group $G_{\rm NW}$. Computation of the corresponding matrix $\mathcal{F}_4$ in eq. (2.10) and substitution in eq. (2.9) gives after some algebra

$$\partial_\sigma X_1\big|_{\partial\Sigma} = -\tan\left(\frac{\phi_0}{2}\right)\partial_\tau x_2 + \frac{1}{2}\left[\left(x_1 - \rho\sin\theta\right)\tan\left(\frac{\phi_0}{2}\right) - x_2 - \rho\cos\theta\right]\partial_\tau u \qquad (6.2)$$

$$\partial_\sigma X_2\big|_{\partial\Sigma} = \tan\left(\frac{\phi_0}{2}\right)\partial_\tau x_1 + \frac{1}{2}\left[x_1 - \rho\sin\theta + \left(x_2 + \rho\cos\theta\right)\tan\left(\frac{\phi_0}{2}\right)\right]\partial_\tau u \qquad (6.3)$$

$$\partial_\sigma U\big|_{\partial\Sigma} = 0 \qquad (6.4)$$

$$\begin{aligned} 2\,\partial_\sigma V\big|_{\partial\Sigma} = &\left[x_2 + \rho\cos\theta + \rho\sin\theta\tan\left(\frac{\phi_0}{2}\right)\right]\partial_\tau x_1 - \left[x_1 - \rho\sin\theta + \rho\cos\theta\tan\left(\frac{\phi_0}{2}\right)\right]\partial_\tau x_2 \\ &+ \frac{1}{2}\left[x_1^2 + x_2^2 - \rho\,(x_1\sin\theta - x_2\cos\theta) + \rho\tan\left(\frac{\phi_0}{2}\right)(x_1\cos\theta + x_2\sin\theta)\right]\partial_\tau u\,. \end{aligned} \qquad (6.5)$$

These are the gluing conditions for the chiral currents written in terms of $\partial_\tau x^\mu$ and $\partial_\sigma X^\mu\big|_{\partial\Sigma}$. We want to compare them with the sigma model boundary conditions (2.8).

Since $N_4 = G_{\rm NW}$ and $T_g G_{\rm NW}$ is spanned at all g by the four vector fields $k_i = \delta^\mu{}_i \partial_\mu$, the boundary conditions (2.8) can be written as

$$\left(\mathsf{G}_{\mu\nu}\,\partial_\sigma X^\nu - \omega_{\mu\nu}\,\partial_\tau x^\nu\right)\big|_{\partial\Sigma} = 0\,. \qquad (6.6)$$

Using the expression for the metric $\mathsf{G}_{\mu\nu}$ in eq. (3.8) and noting that $\partial_\tau x^\mu$ are arbitrary, it is a matter of simple algebra to check that the gluing conditions (6.2)–(6.5) take the form of the boundary conditions (6.6) for any two-form ω with

$$\omega_{v\mu} = 0 \qquad (6.7)$$

$$\omega_{12} = -\tan\left(\frac{\phi_0}{2}\right) \qquad (6.8)$$

$$\omega_{1u} = \frac{1}{2}\left[\left(x_1 - \rho\sin\theta\right)\tan\left(\frac{\phi_0}{2}\right) - x_2 - \rho\cos\theta\right] \qquad (6.9)$$

$$\omega_{2u} = \frac{1}{2}\left[x_1 - \rho\sin\theta + \left(x_2 + \rho\cos\theta\right)\tan\left(\frac{\phi_0}{2}\right)\right]. \qquad (6.10)$$

It is a question of algebra to check that these equations can as well be obtained by using (6.1). So far no restriction has been placed on ρ and θ in F_4. By taking them such that $d\omega = dx_1 \wedge dx_2 \wedge du = \mathsf{H}$, we conclude that the gluing condition for F_4 defines a filling D-brane. The simplest way to accomplish this is to choose ρ and θ constant.

It is known [18] that different isometries may define the same submanifold N but not all of them admit a two-form ω on N such that $d\omega = H\big|_N$. Let us illustrate this with the filling D-brane at hand. We start by recalling [18] that, given an isometry F, it is always possible to define a new isometry

$$F' = \mathrm{Ad}_g \, F^{-1} \, \mathrm{Ad}_g \, . \tag{6.11}$$

The gluing condition for F' is integrable if and only if it is for F, in which case they both yield the same submanifold N. The gluing condition (2.9) for F' reads

$$\big(\mathcal{F}' - 1\big)\, \partial_\tau X\big|_{\partial\Sigma} = \big(\mathcal{F}' + 1\big)\, \partial_\sigma X\big|_{\partial\Sigma} \, .$$

Noting that $\mathcal{F}' = \mathcal{F}^{-1}$ and multiplying from the left with $\mathcal{F}$, it becomes

$$\big(\mathcal{F} - 1\big)\, \partial_\tau X\big|_{\partial\Sigma} = -\big(\mathcal{F} + 1\big)\, \partial_\sigma X\big|_{\partial\Sigma} \, .$$

This is the same condition as for F, except for a negative sign in front of the partial derivatives $\partial_\sigma X\big|_{\partial\Sigma}$. We now take F_4 and consider the corresponding F_4'. The gluing conditions for F_4' are then as in eqs. (6.2)–(6.5) with a negative sign in front of every $\partial_\sigma X\big|_{\partial\Sigma}$. To recover the sigma model boundary condition (6.6), we must take $\omega' = -\omega$, with ω as in eqs. (6.8)–(6.10). This in turn implies that $d\omega' = -H$. We conclude that the gluing condition for F_4', though integrable, does not define a D-brane.

Degenerate D2-branes. In subsection 4.2, the planes $u = u_0$ were obtained upon integration of the gluing condition for an isometry $F = -R_-^{\Omega}(\rho,\phi,\theta)$ with parameters ρ, ϕ and θ such that $k(x_1,x_2)$ in eq. (4.14) took a constant value $k_0 \neq 0$ for all x_1 and x_2. For simplicity we set $\rho = 0$. The condition $k(x_1,x_2) = k_0$ then reads

$$x_1 \cos\left(\frac{\phi + u_0}{2}\right) + x_2 \sin\left(\frac{\phi + u_0}{2}\right) = k_0 \tag{6.12}$$

and the isometry $F = -R_-^{\Omega}(0,\phi,0)$ becomes a function of x_1 and x_2. Other choices for ρ are treated similarly.

Calculation of the corresponding $\mathcal{F}_3$ and substitution in eq. (2.9) provides the following gluing conditions:

$$0 = \partial_\tau u \tag{6.13}$$

$$\partial_\sigma X_1 + \frac{x_2}{2}\, \partial_\sigma U\Big|_{\partial\Sigma} = \frac{2}{k_0}\, \sin\left(\frac{\phi + u_0}{2}\right) \partial_\tau v \tag{6.14}$$

$$-\,\partial_\sigma X_2 + \frac{x_1}{2}\, \partial_\sigma U\Big|_{\partial\Sigma} = \frac{2}{k_0}\, \cos\left(\frac{\phi + u_0}{2}\right) \partial_\tau v \tag{6.15}$$

$$-\,\partial_\sigma U\Big|_{\partial\Sigma} = \frac{2}{k_0}\left[\sin\left(\frac{\phi + u_0}{2}\right) \partial_\tau x_1 - \cos\left(\frac{\phi + u_0}{2}\right) \partial_\tau x_2\right] . \tag{6.16}$$

It is very easy to check that eqs. (6.14)–(6.16) can be written as the $i = 1,2,3$ boundary conditions that result from taking $k_1 = \partial_1$, $k_2 = \partial_2$ and $k_3 = \partial_v$ in eqs. (2.8) for ω given by

$$\omega_{12} = 0 \, , \qquad \omega_{13} = \frac{2}{k_0}\, \sin\left(\frac{\phi + u_0}{2}\right) , \qquad \omega_{23} = -\,\frac{2}{k_0}\, \cos\left(\frac{\phi + u_0}{2}\right) . \tag{6.17}$$

This expression for ω can also be obtained by taking $F = -R^{\Omega}_{-}(0,\phi,0)$ in eq. (6.1). Eqs. (6.12) and (6.17) imply that $d\omega = 0$, hence $d\omega = H\big|_{u_0} = 0$.

Lorentzian D2-branes. We close by considering the isometry $F_3(u)$ in eq. (5.1). In section 5 we distinguished two cases: $x_1^2 + x_2^2 = r_0^2 \neq 0$ and $x_1 = x_2 = 0$. In the first one, integration of the gluing condition resulted in the three-dimensional *pp*-wave in eqs. (5.7) and (5.8), whose tangent space is spanned by the vector fields k_1, k_2 and k_3 in eq. (5.6). It is straightforward to show, either by direct computation or by using eq. (6.1), that the gluing condition for $\mathcal{F}_3$ can be recast as boundary conditions with a two-form ω given in components, by

$$\omega_{12} = \omega(k_1,k_2) = \frac{r_0^2}{2} - 2\,, \qquad \omega_{13} = \omega(k_1,k_3) = -2\,, \qquad \omega_{23} = \omega(k_2,k_3) = 1\,. \tag{6.18}$$

It is clear that $d\omega = 0$. On the other hand, since $x_1^2 + x_2^2 = r_0^2$ is a nonzero constant, dx_1 and dx_2 are not independent and the three-form H vanishes on N_3. Hence $d\omega = H\big|_{N_3}$ is trivially satisfied.

D-strings. The two-form ω for any 1-dimensional D-brane is trivially zero. Let us for completeness compute ω for the two-dimensional *pp*-wave N_2 in (4.9) obtained from the isometry F_2 in (4.7). Taking $k_1 = t_3$ and $k_2 = t_4$ in eqs. (4.3) and (4.4) and using eq. (6.1), it is straightforward that $\omega_{12} = \omega(k_1,k_2) = 0$.

7 Outlook

In this paper we have found Lorentzian signature D-branes of all dimensions for the Nappi-Witten string background. We have achieved this by formulating the usual gluing condition $J_+ = FJ_-$ for the corresponding WZW chiral currents J_+ and J_- and by finding solutions for Lie algebra isometries F that are not automorphisms, thus generalizing existing results. Our analysis shows that the methods used to obtain D-branes for Lie algebra automorphisms work very neatly for more general cases but require a careful formulation of integrability/involutivity. In particular, the occurrence of metrically degenerate D-branes and coordinate-dependent isometries $F(g)$ are solvable issues. We envisage various problems lying ahead. The most inmediate one is perhaps the study of the low-energy limit of the corresponding effective D-brane actions, somewhat in the way it is performed in ref. [7]. By doing so, we expect to learn about noncommutative field theory on curved Lorentzian D-branes and non-critical strings [4]. This may also provide a way to approach noncommutative solitons as bound states of strings [25].

A Alternative computation of rank $\left[\mathrm{Ad}_{g^{-1}}(-R^{\Omega}_{\pm}) - 1\right]$

This appendix contains an alternative derivation to that given in the main text that the isometries $F = -R^{\Omega}_{\pm}$ considered in subsections 4.1 and 4.2 do not define distributions for constant $R^{\Omega}_{\pm}$.

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Since the adjoint group action $\mathrm{Ad}_{g^{-1}}$ defines for any group element g an inner metric-preserving Lie algebra automorphism, $\mathrm{Ad}_{g^{-1}} R^{\Omega}_{\pm}$ is the product of two automorphisms, hence an automorphism of the same type as $R^{\Omega}_{\pm}$. In fact, eqs. (3.6) and (3.7) imply that

$$\mathrm{Ad}_{g^{-1}}\, R^{\Omega}_{\pm}(\rho,\phi,\theta) = R^{\Omega}_{\pm}(\tilde\rho,\tilde\phi,\tilde\theta)\,, \tag{A.1}$$

where the parameters $\tilde\rho$, $\tilde\phi$, $\tilde\theta$ depend on ρ, ϕ, θ and $g(x)$ through

$$\begin{aligned} \tilde\phi &= \phi \mp u \\ \tilde\rho\,\cos(\tilde\theta \mp \tilde\phi) &= \rho\,\cos(\theta \mp \phi + u) \pm x_2 \\ \tilde\rho\,\sin(\tilde\theta \mp \tilde\phi) &= \rho\,\sin(\theta \mp \phi + u) \mp x_1\,. \end{aligned}$$

From eqs. (A.1) and (3.6) it follows that the rank of $\mathrm{Ad}_{g^{-1}}F - 1$ is

$$\mathrm{rank}\left(\mathrm{Ad}_{g^{-1}} R^{\Omega}_{+} + 1\right) = \begin{cases} 4 & \text{if} \quad \tilde\phi \neq (2n+1)\pi \\[2ex] 2 & \text{if} \quad \tilde\phi = (2n+1)\pi \end{cases}$$

for $F = -R^{\Omega}_{+}$ and

$$\mathrm{rank}\left(\mathrm{Ad}_{g^{-1}} R^{\Omega}_{-} + 1\right) = \begin{cases} 3 & \text{if} \quad \tilde\rho\sin\!\left(\tilde\theta + \frac{\tilde\phi}{2}\right) \neq 0 \\[2ex] 1 & \text{if} \quad \tilde\rho\sin\!\left(\tilde\theta + \frac{\tilde\phi}{2}\right) = 0 \end{cases}$$

for $F = -R^{\Omega}_{-}$. We see that in both cases the rank of $\mathrm{Ad}_{g^{-1}}F - 1$ is not constant over $G_{\rm NW}$. The fields $t_a = (\mathrm{Ad}_{g^{-1}}F - 1)T_a$ hence do not provide a distribution on $G_{\rm NW}$.

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References

[1] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219] [SPIRES].

[2] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030 [hep-th/9903205] [SPIRES].

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [SPIRES].

[4] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *JHEP* **06** (2000) 021 [hep-th/0005040] [SPIRES].

[5] C.-S. Chu and P.-M. Ho, *Noncommutative D-brane and open string in pp-wave background with B-field*, *Nucl. Phys.* **B 636** (2002) 141 [hep-th/0203186] [SPIRES].

[6] G. Horcajada and F. Ruiz Ruiz, *Quantization of the open string on plane-wave limits of $dS_n \times S^n$ and non-commutativity outside branes*, Nucl. Phys. **B 799** (2008) 110 [arXiv:0711.2991] [SPIRES].

[7] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, JHEP **09** (1999) 023 [hep-th/9908040] [SPIRES].

[8] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Brane dynamics in background fluxes and non-commutative geometry*, JHEP **05** (2000) 010 [hep-th/0003187] [SPIRES].

[9] C. Klimčík and P. Ševera, *Open strings and D-branes in WZNW models*, Nucl. Phys. **B 488** (1997) 653 [hep-th/9609112] [SPIRES].

[10] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, Phys. Rev. **D 60** (1999) 061901 [hep-th/9812193] [SPIRES].

[11] S. Stanciu, *D-branes in an AdS_3 background*, JHEP **09** (1999) 028 [hep-th/9901122] [SPIRES].

[12] S. Stanciu, *D-branes in group manifolds*, JHEP **01** (2000) 025 [hep-th/9909163] [SPIRES].

[13] S. Stanciu, *A note on D-branes in group manifolds: flux quantization and* D0*-charge*, JHEP **10** (2000) 015 [hep-th/0006145] [SPIRES].

[14] J.M. Figueroa-O'Farrill and S. Stanciu, *More D-branes in the Nappi-Witten background*, JHEP **01** (2000) 024 [hep-th/9909164] [SPIRES].

[15] C. Bachas and M. Petropoulos, *Anti-de-Sitter D-branes*, JHEP **02** (2001) 025 [hep-th/0012234] [SPIRES].

[16] S. Ribault and V. Schomerus, *Branes in the* 2*D black hole*, JHEP **02** (2004) 019 [hep-th/0310024] [SPIRES].

[17] Y.-K.E. Cheung and L. Freidel, *Inner brane: a D3-brane in the Nappi-Witten model from an inner group automorphism*, Phys. Rev. **D 79** (2009) 126007 [arXiv:0905.0540] [SPIRES].

[18] G. Horcajada and F.R. Ruiz, *Geometric construction of D-branes in WZW models*, arXiv:1104.4722 [SPIRES].

[19] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, Phys. Rev. Lett. **71** (1993) 3751 [hep-th/9310112] [SPIRES].

[20] V. Schomerus, *Lectures on branes in curved backgrounds*, Class. Quant. Grav. **19** (2002) 5781 [hep-th/0209241] [SPIRES].

[21] V. Schomerus, *Non-compact string backgrounds and non-rational CFT*, Phys. Rept. **431** (2006) 39 [hep-th/0509155] [SPIRES].

[22] G. D'Appollonio and E. Kiritsis, *D-branes and BCFT in Hpp-wave backgrounds*, Nucl. Phys. **B 712** (2005) 433 [hep-th/0410269] [SPIRES].

[23] E. Witten, *Nonabelian bosonization in two dimensions*, Commun. Math. Phys. **92** (1984) 455 [SPIRES].

[24] S. Stanciu and A.A. Tseytlin, *D-branes in curved spacetime: Nappi-Witten background*, JHEP **06** (1998) 010 [hep-th/9805006] [SPIRES].

[25] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B 460** (1996) 335 [hep-th/9510135] [SPIRES].

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