

Structure preserving transformations in hyperkähler Euclidean spaces

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The definition and structure of hyperkähler structure preserving transformations (invariance group) for quaternionic structures have been recently studied and some preliminary results on the Euclidean case discussed. In this work we present the whole structure of the invariance Lie algebra in the Euclidean case for any dimension.

I. INTRODUCTION

Hyperkähler manifolds are a traditional field of study for geometers [1, 4, 8, 9] but it has become also of interest to Theoretical Physics, in particular after the pioneering work of Atiyah, Hitchin and collaborators [3, 5, 22]; see also the bibliography in [11] and [21, 26] for more recent contributions.

Motivated mainly by our study of hyperhamiltonian dynamics [14, 27] and its applications in Physics [17, 18] (and integrable systems [13, 15]), we are interested in canonical transformations in hyperhamiltonian dynamics; this also led to investigating transformations in hyperkähler manifolds which preserve the hyperkähler structure; this is a topic which of course has been already considered by Differential Geometry – albeit from a geometric rather than dynamical point of view [23]. The requirement of strictly preserving each of the complex structures (which leads to tri-holomorphic maps) is exceedingly restrictive, and one should instead focus on more relaxed requirements.

In standard Hamilton dynamics or symplectic geometry one requires the preservation of the symplectic structure (i.e. of the symplectic form ω). In the framework of hyperkähler manifolds we are in the presence of three symplectic structures ω_α ($\alpha = 1, 2, 3$), associated via the Kähler relation $\omega_\alpha = (J_\alpha \cdot, \cdot)$ to the complex structures J_α (and to the Riemannian metric g) defined on the manifold M . Under many aspects (and in particular for hyperhamiltonian dynamics) it is natural to consider a set $\{\tilde{\omega}_\alpha\}$ obtained as $\tilde{\omega}_\alpha = R_{\alpha\beta}\omega_\beta$, with R a matrix in $SO(3)$, as *equivalent* to the set $\{\omega_\alpha\}$. Thus one considers transformations in M which preserve the metric and which map the set ω_α to an equivalent one; these are called *hypersymplectic* or *quaternionic*. We stress these are of interest not only for hyperhamiltonian dynamics but for hyperkähler geometry as well: the invariance group of quaternionic structures has been studied and identified in the Differential Geometry literature devoted to hyperkähler and quaternionic manifolds [23], based on a rather abstract approach; on the other hand, our discussion will be based on very explicit linear algebra construction and some standard (classification) theory of Lie algebras. We trust this approach may be more familiar to physicists, and we believe it is worth having a completely explicit discussion.

In this paper, we investigate the (connected component of the) group of quaternionic transformations for Euclidean spaces \mathbf{R}^{4n} of arbitrary dimension $4n$; this is of course much simpler than the general case but is a necessary first step before dealing with more complex situations. It turns out that in this case one is able to provide a fairly complete characterization of the Lie algebra of this group, also called the *invariance algebra* \mathfrak{L}_n below, in arbitrary dimension.

We show (Theorem 3) by a completely explicit procedure (based on standard linear algebra and in which the main difficulty is that of having a convenient notation, plus some general results from the theory of Lie algebras), that $\mathfrak{L}_n = \mathfrak{su}(2) \oplus \mathfrak{sp}(n)$. We also show (Theorem 5), again in a fully transparent way, that the “strong invariance algebra”, leaving each of the ω_α invariant, is $\mathfrak{g}_n = \mathfrak{sp}(n)$.

As already mentioned, these results are not new *in se*, being known since some time in the differential geometric literature [23]; but they were obtained in a rather abstract way, while the derivation we provide here is fully explicit and based on standard linear algebra.

These results are of course also in agreement with Berger’s list of holonomy groups for Riemannian manifolds [6] and further research on this topic (see [23] for a comprehensive exposition of this subject; see also [21, 26]). In fact, the structure group of the manifolds under study contains the holonomy groups, and – as it should be expected since there are no other additional structure involved – it turns out they coincide.

The work presented in this paper contains a detailed description of the representations of these groups appearing in the structure of the holonomy groups, using the standard representations of the quaternionic structure in \mathbf{R}^4 .

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Note that some ambiguity is present in the literature concerning the notation for symplectic groups; for us $\mathrm{Sp}(n)$ will be the set of $(2n \times 2n)$ (complex) unitary symplectic matrices (thus with real representation of dimension $4n$), with Lie algebra $\mathfrak{sp}(n) \subset \mathrm{Mat}(2n, \mathbf{C}) \simeq \mathrm{Mat}(4n, \mathbf{R})$.

Finally, we would like to briefly stress the *physical* relevance of the flat case. This is due not only to the fact the Dirac equation is set in flat Minkowski space (which would maybe suffice by itself), but also to the fact that most of the physically relevant nontrivial hyperkähler manifolds are obtained from higher dimensional Euclidean \mathbf{R}^{4n} manifolds (with standard hyperkähler structure) via the moment map construction pioneered by Hitchin *et al.* [22]; thus e.g. the hyperkähler structure (beside of course the metric) for the Taub-NUT manifold [29, 30] can be built explicitly starting from those in \mathbf{R}^8 [18]. Thus, albeit maybe not so interesting for Geometry, the Euclidean case has a substantial relevance for Physics, and we believe it is worth having a fully explicit discussion of the invariance group for hyperkähler structures in Euclidean \mathbf{R}^{4n} spaces.

The *plan of the paper* is as follows. In Section II we will recall some basic facts about hyperkähler manifolds and maps defined on it, also to set our general notation; here we will also discuss some facts about orientation in hyperkähler manifolds, to be used in the following. In Section III we will specialize to the simplest hyperkähler manifolds, i.e. spaces \mathbf{R}^{4n} with Euclidean metric and three complex structures satisfying the quaternionic relations (these define symplectic structures via the Kähler relation); we argue that in this framework we can reduce to a simple setting, i.e. to *standard hyperkähler structures*, and we can thus reduce to a problem defined in terms of these standard structures (which may still have different orientations). In this section we will also set the stage to discuss hypersymplectic maps at the infinitesimal level. After this general discussion, we will tackle the simplest case $n = 1$ in Section IV, showing how the basic problem can be solved in very explicit terms. Higher dimensional cases are considerably more involved; in order to help the reader familiarize with the tools needed to deal with the general case we give, in Section V, a separate discussion of the $n = 2$ case: this presents the difficulties of the general case, but it is still possible to follow the problem, and its combinatorial aspects, in a nearly explicit manner. Here four different orientations are possible for the hyperkähler structure (including standard ones), and the discussion of Section II comes to our help in order not to have to discuss each of them separately. In Section VI we discuss the general case, using the tools developed for $n = 1$ and $n = 2$ as well as some general results from Lie algebra theory, and, finally, we draw our conclusions in Section VII. Some technical details of the $n = 2$ and general n case discussion and computations are confined to the two appendices. The symbols \triangle and \odot signal, respectively, the end of proofs and remarks.

II. BASIC NOTIONS

We will start by recalling the basic notions needed for our discussion; that is, we will recall what are hyperkähler manifolds and which are the characteristics required to a map on a hyperkähler manifold to consider it as preserving the structure on it.

In the following I_k will denote the k -dimensional identity matrix; we will also denote by \mathcal{M}_k the set of k -dimensional real matrices, i.e. $\mathcal{M}_k := \mathrm{Mat}(k, \mathbf{R})$.

A. Hyperkähler manifolds

We will give only the basic definition of hyperkähler manifold. For details on hyperkähler manifolds, see e.g. [1, 3–5, 11].

Definition 1. A hyperkähler manifold $(V, g; \mathbf{J})$ is a real smooth orientable Riemannian manifold (V, g) of dimension $m = 4n$ equipped with an ordered triple $\mathbf{J} = \{J_1, J_2, J_3\}$ of orthogonal almost-complex structures which are covariantly constant under the Levi-Civita connection, $\nabla J_\alpha = 0$; and satisfy the quaternionic relations

$$J_\alpha J_\beta = \epsilon_{\alpha\beta\gamma} J_\gamma - \delta_{\alpha\beta} I. \quad (1)$$

The requirement $\nabla J_\alpha = 0$ implies that the J_α are actually complex structures on (V, g) , due to the Newlander-Nirenberg theorem [28].

The ordered triple $\mathbf{J} = \{J_1, J_2, J_3\}$ will also be called a *hyperkähler structure* on V ; thus a hyperkähler manifold is an orientable smooth manifold V equipped with a Riemannian metric g and with a hyperkähler structure invariant under the associated Levi-Civita connection ∇ .

Definition 2. Let \mathbf{J} and $\widehat{\mathbf{J}}$ be different hyperkähler structures on the same Riemannian manifold (V, g) ; if each of them can be expressed in terms of the other,

$$\widehat{J}_\alpha = \sum_{\beta=1}^3 r_{\alpha\beta} J_\beta , \quad (2)$$

the two structures are said to be equivalent. An equivalence class of hyperkähler structures on (V, g) is said to be a quaternionic structure on (V, g) .

It should be stressed that since both the J and the \widehat{J} satisfy the quaternionic relations (1), necessarily the matrix R with entries $r_{\alpha\beta}$ in (2) belongs to the Lie group $\mathrm{SO}(3)$. Moreover, as both J and \widehat{J} are covariantly constant, it follows that $\nabla R = 0$ as well.

Remark 1. Hyperkähler structures related by linear transformations such as those considered in Definition 2 should in many aspects be seen as substantially equivalent (hence the notion of equivalent structures). In this sense, the relevant structure is the quaternionic one; we will take this into account when looking for structure-preserving transformations on V . \odot

If we define local coordinates in V , the $(1,1)$ tensors J_α are represented by matrices (which we denote again by J_α with a standard abuse of notation), and the quaternionic relation (1) holds between such matrices.

Associated to the metric and each of the complex structures J_α we can construct three symplectic forms ω_α by the Kähler relation; then (V, g, ω_α) is a Kähler manifold for any α . We also say that $(V, g; \omega_1, \omega_2, \omega_3)$ is a *hypersymplectic manifold*.

Remark 2. Thus a hypersymplectic manifold is an orientable smooth manifold V of dimension $4n$, equipped with a Riemannian metric g and an ordered triple of covariantly constant symplectic forms ω_α , such that the complex structures J_α obtained from these via the Kähler relation obey the quaternionic relations (1). Note that here, differently from the standard symplectic case, the metric plays a key role through the Kähler relation. \odot

Remark 3. The notion of equivalent hyperkähler structures induces naturally a notion of equivalent hypersymplectic structures: two hypersymplectic structures are equivalent if the complex structures \mathbf{J} and $\widehat{\mathbf{J}}$ they induce via the Kähler relation are equivalent, i.e. satisfy (2). \odot

In local coordinates on V , these symplectic forms are written as

$$\omega_\alpha = \frac{1}{2} (K_\alpha)_{ij} \mathfrak{x}^i \wedge \mathfrak{x}^j , \quad (3)$$

with $K_\alpha = gJ_\alpha$; it follows from Definition 1 that the K_α are covariantly constant under the Levi-Civita connection and (see (1) above) that they satisfy

$$K_\alpha g^{-1} K_\beta = \epsilon_{\alpha\beta\gamma} K_\gamma - \delta_{\alpha\beta} I . \quad (4)$$

B. Maps on hyperkähler manifolds

We would now like to characterize the maps $\varphi : V \rightarrow V$ which leave invariant the hyperkähler structure, or at least the equivalence class of hyperkähler structures discussed above, i.e. the quaternionic structure on (V, g) .

If $\varphi : V \rightarrow V$ is an arbitrary smooth map in V , the hyperkähler structure will change according to the rule of transformations of $(1,1)$ tensors (which amounts to a conjugation), i.e.

$$J_\alpha \rightarrow \widetilde{J}_\alpha := \Lambda J_\alpha \Lambda^{-1} ,$$

where Λ is the Jacobian of φ .

Definition 4. The map $\varphi : V \rightarrow V$ is strongly hyperkähler for (V, g, \mathbf{J}) if it preserves both the metric g and the hyperkähler structure \mathbf{J} .

It is obvious that the set of strongly hyperkähler maps for (V, g, \mathbf{J}) is a group. Such maps are also called *tri-holomorphic*, as they are holomorphic for each of the three complex structures J_α . The group of strongly hyperkähler

maps on $(V, g; \mathbf{J})$ will be called the strong hyperkähler group on $(V, g; \mathbf{J})$; or for short the *strong invariance group* of V . Correspondingly, its elements will be called, with an abuse of language, *strong invariance maps*.

Definition 5 *The map $\varphi : V \rightarrow V$ is hyperkähler for (V, g, \mathbf{J}) if it preserves the metric and maps the hyperkähler structure into an equivalent one. In this case it is also said to be quaternionic, as it preserves the quaternionic structure on (V, g) .*

Here again it is obvious that the set of hyperkähler maps for (V, g, \mathbf{J}) is a group. This will be called the hyperkähler group on $(V, g; \mathbf{J})$; or for short the *invariance group* of V . Correspondingly, its elements will be called, with an abuse of language, *invariance maps*. Any strong invariance map is also an invariance map; the set of maps which are hyperkähler but not strongly hyperkähler will also be denoted as *regular invariance maps*.

Remark 4. It is clear that, by the correspondence mentioned at the end of the previous subsection (and based simply on the Kähler relation), the maps preserving the hyperkähler structure will also preserve the hypersymplectic one, and those mapping the hyperkähler structure into an equivalent one will also map the hypersymplectic structure into an equivalent one. Thus it would also be legitimate to denote the maps and groups identified in the Definitions 4 and respectively 5 above as strongly hypersymplectic and respectively hypersymplectic ones; the groups will correspondingly be called the *strong hypersymplectic group* and the *hypersymplectic group*. \odot

C. Orientation

As recalled above, a hyperkähler manifold is orientable; we can thus consider in particular orientation-switching maps \mathcal{P} , obviously satisfying $\mathcal{P}^2 = I$. Under such a map the hyperkähler structure will not be preserved, but will be mapped to a (non-equivalent) *dual* one [19]; note that the Riemannian metric can instead be invariant under such an orientation-switching map (this will in particular be the case for the Euclidean metric for any orthogonal \mathcal{P}). From now on we will only consider maps \mathcal{P} preserving the metric.

It is quite obvious that hyperkähler structures which are dual to each other (we refer to these as a *dual pair*) are strongly related; it also turns out that in some physical applications of hyperhamiltonian dynamics (in particular, in the description of the Dirac equation in hyperhamiltonian terms [17]) one needs both elements of a dual pair.

When we work in the symplectic framework, so that the hyperkähler structure corresponds to a triple of symplectic structures $\{\omega_\alpha\}$, the action of the map \mathcal{P} on these is simply given by the pull-back. This induces a conjugation between the ω_α and the dual ones, $\tilde{\omega}_\alpha = \mathcal{P}^* \omega_\alpha$, and hence between a hyperkähler structure and its dual one. This shows that hyperkähler structures related by such an orientation switch are conjugated. (Representing the forms ω_α (and the complex structures J_α) in coordinates, the conjugation is described by the action of a matrix P which is orthogonal with respect to the metric.)

We conclude that hyperkähler structures related by such a map will be invariant under isomorphic groups of transformations.

In the following we will have to consider spaces \mathbf{R}^{4n} and the possibility to independently switch orientation in the \mathbf{R}^4 subspaces on which the $\omega_\alpha \wedge \omega_\alpha$ give a volume form; the same considerations presented above will also hold for the restriction of the hyperkähler structure to each of these subspaces, and this will be rather useful to simplify our computations.

III. EUCLIDEAN SPACES

In this note we are concerned with the simplest occurrence of hyperkähler manifolds, i.e. Euclidean spaces. We will thus specialize our general notions and discussion to this specific case.

A. Hyperkähler structures

The simplest example of a hyperkähler manifold is \mathbf{R}^{4n} with the Euclidean metric, equipped with the *standard hyperkähler structures* detailed below (these will play a role in our general discussion). It should be noted that, since the metric is here Euclidean, the covariant derivative is the usual derivative (the Levi-Civita connection is trivial). Then $\partial_{x_i} J(x) = 0$ for $i = 1, \dots, 4n$, and the hyperkähler structure is actually constant.

The standard positively or negatively oriented standard hyperkähler structures in \mathbf{R}^4 are given respectively by

$$Y_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (5)$$

$$\hat{Y}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{Y}_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{Y}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (6)$$

Remark 5. The qualification on orientation follows from considering the associated symplectic structures ω_α and $\hat{\omega}_\alpha$; with Ω the standard volume form on \mathbf{R}^4 , one has $(1/2)(\omega_\alpha \wedge \omega_\alpha) = \Omega$ and $(1/2)(\hat{\omega}_\alpha \wedge \hat{\omega}_\alpha) = -\Omega$, for all α . \odot

In the four-dimensional case ($n = 1$), it is easily checked that any constant hyperkähler structure J_α , i.e. any set of constant skew-symmetric matrices satisfying the quaternionic relations (1), can be transformed through a conjugation with a matrix $P \in \text{SO}(4)$ into one of the two inequivalent (under $\text{SO}(4)$ conjugation) sets of matrices Y_α and \hat{Y}_α , $\alpha = 1, 2, 3$. (Note that indeed any skew-symmetric matrix in \mathbf{R}^4 is written as a sum of the Y_α and of the \hat{Y}_α .) This corresponds to the $\mathfrak{su}(2)$ algebra having two irreducible representations in \mathbf{R}^4 .

A similar result holds in \mathbf{R}^{4n} : in this case one acts with $G = \text{SO}(4n)$, and the hyperkähler structures can be transformed into some direct sum of the above standard ones; we stress in this sum there will in general be blocks of each orientation. More precisely we have the following

Lemma 1. *Given any quaternionic structure $\{J_\alpha\}$ in \mathbf{R}^{4n} , there exists a conjugation given by a regular matrix $P \in \text{SO}(4n)$, such that $\tilde{J}_\alpha := PJ_\alpha P^{-1}$ are diagonal 4×4 block matrices, and the blocks in the diagonal are equal to either Y_α or \hat{Y}_α .*

Proof. As we are in Euclidean spaces, $\nabla J_\alpha = 0$ means the J_α are actually constant; thus they provide a (real, quaternionic) representation of $\text{SU}(2)$. This can be decomposed as the sum of real quaternionic irreducible representations, which are well known (see e.g. chap.8 of [24]) to be four dimensional. See also the discussion below. \triangle

B. Hyperkähler and strongly hyperkähler maps

Let us now consider (strongly) hyperkähler maps in Euclidean spaces; in this case we can be quite more specific, due to the specially simple metric and the triviality of the Levi-Civita connection.

If g is the matrix associated to the metric and Λ is the Jacobian of a transformation φ in V , the change in the metric is

$$g \rightarrow \tilde{g} = \Lambda g \Lambda^T; \quad (7)$$

for the Euclidean metric $g = I_4$ and hence $\tilde{g} = \Lambda \Lambda^T$; thus $\tilde{g} = g$ requires

$$\Lambda \Lambda^T \equiv I_4 \quad (8)$$

and it should be $\Lambda(x) \in \text{O}(4n)$ (or $\Lambda(x) \in \text{SO}(4n)$ if we want to keep the orientation) for any $x \in V$.

Moreover, as both the original and the transformed complex structures should be covariantly constant, it should also be $\nabla \Lambda = 0$: but since here ∇ is the trivial connection, this means $\partial_i \Lambda(x) = 0$ for all $i = 1, \dots, 4n$, hence Λ is constant.

This shows at once that hyperkähler and strongly hyperkähler maps should be constant orthogonal (or special orthogonal if we want to preserve orientation) ones, in full generality.

C. Strongly hyperkähler maps

Let us now consider in detail strongly hyperkähler maps, and represent the J_α by means of the corresponding matrices in coordinates. Regarding the preservation of the hyperkähler structure, we should impose in this (strong invariance) case

$$\Lambda J_\alpha \Lambda^{-1} = J_\alpha; \quad (9)$$

this means that Λ should commute with each of the J_α . Since these matrices are a representation of $\mathfrak{su}(2)$, we could apply representation theory to this problem.

Indeed, let us consider a set of three $m \times m$ real matrices, J_α , $\alpha = 1, 2, 3$ satisfying $J_\alpha J_\beta = \epsilon_{\alpha\beta\gamma} J_\gamma - \delta_{\alpha\beta} I_m$. This relation implies $[J_\alpha, J_\beta] = 2\epsilon_{\alpha\beta\gamma} J_\gamma$, stating that the matrices $\Gamma_\alpha = (1/2)J_\alpha$, $\alpha = 1, 2, 3$ form a representation \mathcal{R} of $\mathfrak{su}(2)$. But it also yields $J_\alpha^2 = -I_m$, $\Gamma_\alpha^2 = -(1/4)I_m$. This property implies that the eigenvalues of the Casimir $-\sum_\alpha \Gamma_\alpha^2$ are $(3/4)$ and then the representation is (complex) reducible into a direct sum of a certain number of $(\frac{1}{2})$ representations:

$$\mathcal{R} = \left(\frac{1}{2}\right) \oplus \cdots \oplus \left(\frac{1}{2}\right) .$$

Since we have real matrices, the representation is real reducible to a diagonal block form, with 4×4 blocks, each of them associated to a $(\frac{1}{2}) \oplus (\frac{1}{2})$ representation of $\mathfrak{su}(2)$ with real matrices and the dimension is $m = 4n$.

D. Quaternionic maps

Let us now consider quaternionic (or hyperkähler, see Definition 5) maps. The requirement to map \mathbf{J} into a possibly different, but equivalent, structure $\tilde{\mathbf{J}}$ implies that

$$\tilde{J}_\alpha = \Lambda J_\alpha \Lambda^{-1} = \sum_{\beta=1}^3 R_{\alpha\beta} J_\beta, \quad \sum_{\beta=1}^3 R_{\alpha\beta}^2 = 1, \quad \alpha = 1, 2, 3. \quad (10)$$

In the same way as in the strong version, the first condition implies that the matrices Λ should be in $O(4n)$, and in $SO(4n)$ if we want to leave invariant the orientation. As for the second condition, it yields the following constraint. The new matrices \tilde{J}_α should satisfy the quaternionic relations (1), i.e.

$$\tilde{J}_\alpha \tilde{J}_\beta = \epsilon_{\alpha\beta\gamma} \tilde{J}_\gamma - \delta_{\alpha\beta} I. \quad (11)$$

Substituting (sum over repeated indices is assumed)

$$R_{\alpha\mu} R_{\beta\nu} J_\mu J_\nu = \epsilon_{\mu\nu\rho} R_{\alpha\mu} R_{\beta\nu} J_\rho + \delta_{\mu\nu} R_{\alpha\mu} R_{\beta\nu} I = \epsilon_{\alpha\beta\gamma} R_{\gamma\rho} J_\rho - \delta_{\alpha\beta} I, \quad (12)$$

we obtain

$$R_{\alpha\mu} R_{\beta\mu} = \delta_{\alpha\beta}, \quad \epsilon_{\mu\nu\rho} R_{\alpha\mu} R_{\beta\nu} = \epsilon_{\alpha\beta\gamma} R_{\gamma\rho}. \quad (13)$$

The first condition means that the matrix R is an element of $O(3)$. The second one means that the vector product of its first and second column is the third one, which yields $R \in SO(3)$. Then, in the end we obtain the equation

$$\Lambda J_\alpha \Lambda^{-1} = \sum_{\beta=1}^3 R_{\alpha\beta} J_\beta, \quad \alpha = 1, 2, 3, \quad \Lambda \in SO(4n), \quad R \in SO(3). \quad (14)$$

Thus our problem is to determine which $\Lambda \in SO(4n)$ will satisfy equation (14) for a certain $R \in SO(3)$ (which is fixed by Λ). The equation (14) will also be called the *finite invariance equation*.

E. The infinitesimal approach

It will be convenient, in particular in the high-dimensional case, to approach this problem from the infinitesimal point of view.

At first order in a certain parameter ε , we have

$$\Lambda = I_{4n} + \varepsilon X, \quad X + X^T = 0, \quad (15)$$

and, in terms of the same parameter,

$$R = I_3 + \varepsilon L, \quad L + L^T = 0. \quad (16)$$

Equation (14), for any quaternionic structure J_α , is then written at the infinitesimal level as

$$(I_{4n} + \varepsilon X)J_\alpha(I_{4n} - \varepsilon X) = \sum_{\beta=1}^3 (\delta_{\alpha\beta} + \varepsilon \mathbf{L}_{\alpha\beta})J_\beta, \quad \alpha = 1, 2, 3, \quad (17)$$

with $X \in \mathcal{M}_{4n}(\mathbf{R})$, $\mathbf{L} \in \mathcal{M}_3$; $X + X^T = 0$, $\mathbf{L} + \mathbf{L}^T = 0$. That is, at first order in ε

$$[X, J_\alpha] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} J_\beta, \quad \alpha = 1, 2, 3. \quad (18)$$

The equation (18) will also be called the *infinitesimal invariance equation*, or shortly (as we will mainly work in the infinitesimal approach) the *invariance equation*.

The main result of this note is the solution of these equations for Euclidean spaces, i.e. for $V = \mathbf{R}^{4n}$ with the Euclidean metric $g = I_{4n}$.

Note that (18) should be seen as an equation for X and \mathbf{L} ; on the other hand if we fix X , i.e. if we consider a given hyperkähler transformation, we can easily find \mathbf{L} in terms of X .

As mentioned above, our main task is to characterize the group of invariance maps for (V, g, \mathbf{J}) when (V, g) is \mathbf{R}^{4n} with the Euclidean metric. In the infinitesimal approach, we will of course look for the Lie algebra of this group; this will be called the *invariance algebra* and denoted as \mathfrak{L} . More specifically, we will denote by \mathfrak{L}_n the invariance algebra for the hyperkähler structures in the Euclidean space \mathbf{R}^{4n} .

In order to grasp the problem and the approach to its solution, we find convenient to first consider the simplest (and somehow degenerate) case $n = 1$, which we do in the next section; and then the first non-degenerate case $n = 2$ in Section V, before tackling the general case in Section VI.

In the following we will systematically use standard cartesian coordinates on the manifold \mathbf{R}^{4n} , and represent the tensors J_α by real $4n$ -dimensional matrices in the chosen coordinate system without further notice.

IV. THE SPACE \mathbf{R}^4

When the manifold is \mathbf{R}^4 , the explicit characterization of quaternionic maps can be obtained in a simple way via either the infinitesimal approach sketched above, or directly working at the finite level. The arguments used in the discussion and the proof below are well known, but keeping them in mind will help in the study of higher dimensional cases.

A. The infinitesimal approach

Any skew symmetric 4×4 matrix X is necessarily a linear combination of the two sets Y_α and \hat{Y}_α , $\alpha = 1, 2, 3$, given above; we recall these satisfy $[Y_\alpha, \hat{Y}_\beta] = 0$. Thus we have

$$X = \frac{1}{2} \sum_{\beta=1}^3 c_\beta Y_\beta + \frac{1}{2} \sum_{\beta=1}^3 \hat{c}_\beta \hat{Y}_\beta, \quad (19)$$

and the invariance equation for the positively oriented standard structure is

$$\left[\frac{1}{2} \sum_{\beta=1}^3 c_\beta Y_\beta + \frac{1}{2} \sum_{\beta=1}^3 \hat{c}_\beta \hat{Y}_\beta, Y_\alpha \right] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} Y_\beta, \quad \alpha = 1, 2, 3. \quad (20)$$

Then, for $\alpha = 1, 2, 3$,

$$\frac{1}{2} \sum_{\beta=1}^3 c_\beta [Y_\beta, Y_\alpha] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} Y_\beta, \quad \sum_{\beta=1}^3 c_\beta \sum_{\gamma=1}^3 \epsilon_{\beta\alpha\gamma} Y_\gamma = \sum_{\gamma=1}^3 \mathbf{L}_{\alpha\gamma} Y_\gamma, \quad (21)$$

and finally,

$$\mathbf{L}_{\alpha\beta} = \sum_{\gamma=1}^3 \epsilon_{\alpha\beta\gamma} c_\gamma, \quad \alpha, \beta = 1, 2, 3. \quad (22)$$

It follows from this that we have the

Theorem 1. *The invariance algebra for any hyperkähler structure in $(V, g) = (\mathbf{R}^4, I_4)$ is $\mathfrak{L}_1 = \mathfrak{so}(4) \simeq \mathfrak{su}(2) \times \mathfrak{su}(2)$.*

Proof. As seen above, any hyperkähler structure can be reduced to either the positively or the negatively oriented standard structure; so it suffices to consider these. We will consider the positively oriented structure; the discussion for the negatively oriented one is exactly the same, upon interchanging the role of the Y_α and of the \hat{Y}_α . As we have noted above, the effective group, rotating Y_α , is a $\text{SO}(3)$ subgroup, with a Lie algebra generated by the matrices Y_α . The other $\text{SO}(3)$ subgroup, which is generated by the matrices \hat{Y}_α , leaves the matrices J_α invariant. The result can be understood in terms of pure group or Lie algebra theory. In fact, the group $\text{SO}(4)$ is not a simple group but the direct product of two $\text{SO}(3)$ groups. Its Lie algebra has a real representation given by 4×4 matrices which splits into the direct sum of two $\mathfrak{su}(2)$ algebras. Since they commute, the action of the whole algebra through the adjoint action is reduced to the action of one of the subalgebras on itself. This is the reason why L , see equation (22), is in fact in the 3-dimensional representation of $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, the action of the other algebra being trivial. \triangle

B. The finite approach

In this simple case, we could actually solve the problem using finite transformations and the real version of Schur lemma (see e.g. [24], chapter 8). From a practical point of view, we can directly compute the matrices commuting with this representation. The solution of equation (9) is

$$\Lambda = \begin{pmatrix} a & -d & b & -c \\ d & a & c & b \\ -b & -c & a & d \\ c & -b & -d & a \end{pmatrix} = aI_4 + b \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} + d \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \quad (23)$$

Here the σ_i are the standard (complex, two dimensional) Pauli matrices [25]; in particular

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}.$$

This real matrix (with $\det \Lambda = (a^2 + b^2 + c^2 + d^2)^2$) must be in $\text{O}(4)$, and (as a, b, c, d are real) the only condition to be satisfied is

$$a^2 + b^2 + c^2 + d^2 = 1, \quad (24)$$

and then, it is in $\text{SO}(4)$. The matrices Λ form a subgroup of $\text{SO}(4)$, in fact, the subgroup generated by the negatively oriented standard form of the quaternionic structure. The coefficients a, b, c, d could depend on the point $x \in \mathbf{R}^4$. Then, the set of matrices leaving invariant the positively oriented standard quaternionic structure is, in each point of \mathbf{R}^4 , the quaternion group (the set of quaternion units) generated by the negatively oriented standard quaternionic structure, which is a subgroup of $\text{SO}(4)$, in fact $\text{SO}(3)$.

Remark 6. It may be useful to show explicitly (in a simple case) the interrelation between the matrices R and Λ , also to illustrate the situation in the finite approach. Every element $\Lambda \in \text{SO}(4)$ can be written in a 2×2 block-diagonal form (after a conjugation with an appropriate $P \in \text{SO}(4)$). Let us consider those elements (pure rotations in the 2-dimensional planes (x_1, x_2) and (x_3, x_4)) of the form

$$\Lambda_0 = \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 & 0 & 0 \\ \sin \phi_1 & \cos \phi_1 & 0 & 0 \\ 0 & 0 & \cos \phi_2 & -\sin \phi_2 \\ 0 & 0 & \sin \phi_2 & \cos \phi_2 \end{pmatrix}, \quad \phi_1, \phi_2 \in \mathbf{R}, \quad \Lambda = P\Lambda_0 P^{-1}. \quad (25)$$

If in this basis the quaternionic structure is the positively oriented standard one, the matrices Y_α are transformed under this matrix Λ_0 as

$$Y_1 \rightarrow Y_1 \quad (26)$$

$$Y_2 \rightarrow Y_2 \cos(\phi_1 + \phi_2) - Y_3 \sin(\phi_1 + \phi_2) \quad (27)$$

$$Y_3 \rightarrow Y_2 \sin(\phi_1 + \phi_2) + Y_3 \cos(\phi_1 + \phi_2), \quad (28)$$

that is, the matrix R is

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_1 + \phi_2) & -\sin(\phi_1 + \phi_2) \\ 0 & \sin(\phi_1 + \phi_2) & \cos(\phi_1 + \phi_2) \end{pmatrix}. \quad (29)$$

The generic situation will be

$$\Lambda Y_\alpha \Lambda^{-1} = P(\Lambda_0(P^{-1}Y_\alpha P)\Lambda_0^{-1})P^{-1} = \sum_{\beta=1}^3 R_{\alpha\beta}Y_\beta,$$

$$\Lambda_0(P^{-1}Y_\alpha P)\Lambda_0^{-1} = \sum_{\beta=1}^3 R_{\alpha\beta}P^{-1}Y_\beta P;$$

or,

$$\Lambda_0 J_\alpha \Lambda_0^{-1} = \sum_{\beta=1}^3 R_{\alpha\beta} J_\beta, \quad J_\alpha = P^{-1}Y_\alpha P; \quad (30)$$

it is not easy to write R in terms of Λ in an explicit way.

Finally, note that the set $\{I_4, Y_1, Y_2, Y_3\}$ provides a real representation of the quaternion units $1, i, j, k$. An analysis using quaternion algebra would yield the same results we have got above. \odot

V. THE SPACE \mathbf{R}^8

In the previous Section we discussed the case $n = 1$, which is somewhat degenerate in that the hyperkähler structures in standard form consist of a single block. In this section we will tackle the first non-degenerate case, i.e. $n = 2$ or \mathbf{R}^8 ; this will present the difficulties met in the general one \mathbf{R}^{4n} , but for it the identification of the invariance algebra \mathfrak{L} is still rather straightforward.

First of all, we note that albeit Lemma 1 would lead us to deal with four different types of hyperkähler structures, the invariance groups (and algebras) for them are isomorphic; this follows from different classes of standard hyperkähler structures being conjugated by the action of matrices in $O(8)$.

Lemma 2. *All hyperkähler structures in Euclidean \mathbf{R}^8 are conjugated under $O(8)$.*

Proof. Using Lemma 1, the quaternionic structure J_α (which we recall is necessarily constant) in \mathbf{R}^8 endowed with the Euclidean metric, can be reduced to one of the following types (in the third case the order of the blocks can be reversed):

$$Y_\alpha^{(1)} = \begin{pmatrix} Y_\alpha & \\ & Y_\alpha \end{pmatrix}, \quad Y_\alpha^{(2)} = \begin{pmatrix} \hat{Y}_\alpha & \\ & \hat{Y}_\alpha \end{pmatrix}, \quad Y_\alpha^{(3)} = \begin{pmatrix} Y_\alpha & \\ & \hat{Y}_\alpha \end{pmatrix}. \quad (31)$$

In fact there exist a four dimensional matrix Q which satisfies $QQ^T = \lambda I_4$ and can be chosen in $O(4) \setminus SO(4)$, i.e. in the elements of $O(4)$ with determinant equal to -1 , such that $Y_\alpha = Q^{-1}\hat{Y}_\alpha Q$ for $\alpha = 1, 2, 3$; here $Q^{-1} = Q^T$, $\det Q = -1$. Then, if $Q_2 = \text{diag}(Q, Q) \in O(8)$ and $Q_3 = \text{diag}(I_4, Q) \in O(8)$, we get

$$Q_2^{-1}Y_\alpha^{(2)}Q_2 = Y_\alpha^{(1)}, \quad Q_3^{-1}Y_\alpha^{(3)}Q_3 = Y_\alpha^{(1)}. \quad (32)$$

This shows that all the standard quaternionic structures – and hence, in view of Lemma 1, all the quaternionic structures – in \mathbf{R}^8 are conjugated, as stated. \triangle

Theorem 2. *For any hyperkähler structure in $(V, g) = (\mathbf{R}^8, I_8)$, the invariance algebra is $\mathfrak{L}_2 = \mathfrak{su}(2) \otimes \mathfrak{sp}(2)$.*

Proof. In view of Lemma 2, we can just deal with $Y_\alpha^{(1)}$. An orthogonal transformation, leaving invariant the Euclidean metric I_8 is an element of $O(8)$, satisfying

$$\Lambda \Lambda^T = I_8, \quad \det \Lambda = 1; \quad (33)$$

at the infinitesimal level the invariance of the quaternionic relations imposes, as above,

$$[X, J_\alpha] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} J_\beta, \quad \alpha = 1, 2, 3, \quad (34)$$

where $X = -X^T \in \mathcal{M}_8$ and $\mathbf{L} = -\mathbf{L}^T \in \mathcal{M}_3$; note that here $\Lambda \in \mathbf{O}(8)$ implies actually $X \in \mathfrak{so}(8)$.

As in the previous case, the three matrices J_α generate an $\mathfrak{su}(2)$ algebra which is contained in the algebra $\mathfrak{so}(8)$ (of dimension 28) of $\mathbf{SO}(8)$. However, in this case, the situation is not so simple, because the whole $\mathfrak{so}(8)$ cannot be generated by the quaternionic matrices (even considering both orientations and their combinations in the 8×8 matrices). A basis of $\mathfrak{so}(8)$ is:

$$\begin{pmatrix} Y_\alpha & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & Y_\alpha \end{pmatrix}, \begin{pmatrix} \hat{Y}_\alpha & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \hat{Y}_\alpha \end{pmatrix}, \begin{pmatrix} 0 & Y_\alpha \\ Y_\alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & \hat{Y}_\alpha \\ \hat{Y}_\alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & S_i \\ -S_i & 0 \end{pmatrix}. \quad (35)$$

with $\alpha = 1, 2, 3$, and S_i , $i = 1, \dots, 10$, the set of 4×4 elementary symmetric matrices (that is, E_{jj} and $E_{jk} + E_{kj}$, where E_{jk} is the elementary matrix with 1 in the position jk and 0 elsewhere).

Let us first consider the structure $Y_\alpha^{(1)}$ and the equation (18). If we write

$$X = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}, \quad A + A^T = 0, \quad C + C^T = 0 \quad (36)$$

we get

$$\left[\begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}, \begin{pmatrix} Y_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix} \right] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} \begin{pmatrix} Y_\beta & 0 \\ 0 & Y_\beta \end{pmatrix} \quad (37)$$

and the relations

$$[A, Y_\alpha] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} Y_\beta, \quad [C, Y_\alpha] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} Y_\beta, \quad [B, Y_\alpha] = 0. \quad (38)$$

Using the previous results in dimension 4, we obtain, from the first relation

$$A = \frac{1}{2} \sum_{\beta=1}^3 a_\beta Y_\beta + \frac{1}{2} \sum_{\beta=1}^3 \hat{a}_\beta \hat{Y}_\beta, \quad C = \frac{1}{2} \sum_{\beta=1}^3 c_\beta Y_\beta + \frac{1}{2} \sum_{\beta=1}^3 \hat{c}_\beta \hat{Y}_\beta, \quad \mathbf{L}_{\alpha\beta} = \sum_{\gamma=1}^3 \epsilon_{\alpha\beta\gamma} a_\gamma, \quad (39)$$

and then $a_\beta = c_\beta$, while \hat{a}_β and \hat{c}_β are arbitrary constants.

As for B , we get (here B_S is the symmetric part of B)

$$B = \frac{1}{2} \sum_{\beta=1}^3 b_\beta Y_\beta + \frac{1}{2} \sum_{\beta=1}^3 \hat{b}_\beta \hat{Y}_\beta + B_S, \quad [B, Y_\alpha] = 0 \Rightarrow b_\alpha = 0, \quad B_S = \lambda I_4, \quad (40)$$

and \hat{b}_β are arbitrary constants.

These results provide a subalgebra of $\mathfrak{so}(8)$ with basis

$$\begin{pmatrix} Y_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix}, \begin{pmatrix} \hat{Y}_\alpha & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \hat{Y}_\alpha \end{pmatrix}, \begin{pmatrix} 0 & \hat{Y}_\alpha \\ \hat{Y}_\alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}. \quad (41)$$

In fact, the matrix \mathbf{L} is different from zero only for the generators

$$\begin{pmatrix} Y_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix}, \quad (42)$$

and the corresponding algebra is $\mathfrak{su}(2)$. The other matrices, which generate a Lie algebra of dimension 10 which commutes with the algebra $\mathfrak{su}(2)$ generated by the matrices (42), leave invariant each of the matrices J_α , $\alpha = 1, 2, 3$. The commutation table appears in Appendix A, see Table I.

It is an easy task to construct the adjoint representation and the Killing form and this allows to identify the algebra as a real compact semisimple Lie algebra, $\mathfrak{sp}(2)$, whose complex extension is isomorphic to the Lie algebra C_2 (or B_2) in the Cartan classification. The details of the computations are contained in Appendix A.

If we consider the structure $Y_\alpha^{(i)}$, $i = 2, 3$, the equation to be solved is

$$[X^{(i)}, Y_\alpha^{(i)}] = \sum_{\beta=1}^3 L_{\alpha\beta} Y_\beta^{(i)}, \quad \alpha = 1, 2, 3, \quad i = 2, 3, \quad (43)$$

and then

$$[X^{(i)}, Q_i Y_\alpha^{(1)} Q_i^{-1}] = \sum_{\beta=1}^3 L_{\alpha\beta} Q_i Y_\beta^{(1)} Q_i^{-1} \quad (44)$$

or

$$[Q_i^{-1} X^{(i)} Q_i, Y_\alpha^{(1)}] = \sum_{\beta=1}^3 L_{\alpha\beta} Y_\beta^{(1)}, \quad (45)$$

and the algebra formed by the matrices $X^{(i)}$ is now conjugated to the one we get for the first structure $Y_\alpha^{(1)}$. In fact, in the case $Y_\alpha^{(2)}$, the roles of Y_α and \hat{Y}_α are simply exchanged. The basis for the subalgebra is

$$\begin{pmatrix} \hat{Y}_\alpha & 0 \\ 0 & \hat{Y}_\alpha \end{pmatrix}, \quad \begin{pmatrix} Y_\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & Y_\alpha \end{pmatrix}, \quad \begin{pmatrix} 0 & Y_\alpha \\ Y_\alpha & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}, \quad (46)$$

and the invariance algebra is again $\mathfrak{sp}(2)$.

Finally, for the third possible structure $Y_\alpha^{(3)}$, we also get $\mathfrak{sp}(2)$ and a basis is:

$$\begin{pmatrix} Y_\alpha & 0 \\ 0 & \hat{Y}_\alpha \end{pmatrix}, \quad \begin{pmatrix} \hat{Y}_\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & Y_\alpha \end{pmatrix}, \quad \begin{pmatrix} 0 & Z_i \\ -Z_i^T & 0 \end{pmatrix}, \quad (47)$$

where Z_i , $i = 1, \dots, 4$ are the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (48)$$

which correspond to matrices Q intertwining the two sets Y_α and \hat{Y}_α . The invariance algebra is still the same. This concludes the proof. \triangle

Remark 7. We have obtained above $\mathfrak{L}_1 = \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The first $\mathfrak{su}(2)$ corresponds to the strong invariance algebra and the second one to the regular invariance algebra. In the \mathbf{R}^8 case, as we have seen, the invariance algebra is $\mathfrak{L}_2 = \mathfrak{su}(2) \oplus \mathfrak{sp}(2)$. In fact, the case \mathbf{R}^4 has exactly the same structure, since $\mathfrak{sp}(1) \approx \mathfrak{su}(2)$. \odot

VI. THE GENERAL CASE: \mathbf{R}^{4n}

We are now ready to tackle the general case, i.e. the Euclidean space \mathbf{R}^{4n} . We will face two difficulties: the notation, which is necessarily rather cumbersome, and more substantially the identification of the invariance algebra \mathfrak{L} . In our discussion we will again rely on Lemma 1, and hence consider hyperkähler structures in standard form; and will make use of the notation introduced in the discussion of the \mathbf{R}^8 case.

Given two square matrices A and B , respectively of dimension m and n , we choose the basis of their tensorial product in such a way that the $m \times m$ block matrix (with $n \times n$ blocks) is

$$A \otimes B = (a_{ij} B), \quad A = (a_{ij}), \quad (49)$$

and denote by E_{ij} the usual elementary matrix:

$$(E_{ij})_{kl} := \delta_{ik} \delta_{jl}, \quad E_{ij} E_{kl} = \delta_{jk} E_{il}. \quad (50)$$

Then $E_{ij} \otimes B$ is a $nm \times nm$ matrix formed by $n \times n$ blocks, where the ij block is equal to B , and all elements elsewhere are zero.

We recall that the invariance condition for a quaternionic structure in \mathbf{R}^{4n} can be expressed by (18); recall also that $X = -X^T \in \mathcal{M}_{4n}$, $\mathbf{L} = -\mathbf{L}^T \in \mathcal{M}_3$.

Using Lemma 1, any quaternionic structure in \mathbf{R}^{4n} is conjugated to

$$J_\alpha = \sum_{i=1}^n E_{ii} \otimes J_\alpha^{(i)}, \quad \alpha = 1, 2, 3, \quad (51)$$

where $J_\alpha^{(i)}$ is any of the two nonequivalent quaternionic structures in \mathbf{R}^4 , i.e. either Y_α or \hat{Y}_α , and we can reduce our problem to the simplest case of block-diagonal quaternionic structures.

However, we can have in the diagonal both kinds of orientations, a difficulty which can be easily surmounted using the fact, already used in the case \mathbf{R}^8 , that there exist an orthogonal matrix Q , with $\det Q = -1$, such that $Y_\alpha = Q^{-1} \hat{Y}_\alpha Q$, for $\alpha = 1, 2, 3$; see Lemma 2 above. We actually extend this to a higher dimensional setting; the proof of this is straightforward and hence omitted.

Lemma 3. *Let J_α be a quaternionic structure in \mathbf{R}^{4n} set in a 4×4 block-diagonal form, $J_\alpha = \sum_{i=1}^n E_{ii} \otimes J_\alpha^{(i)}$. Then, the block-diagonal matrix*

$$\mathbf{Q} = \text{diag}(Q_1, Q_2, \dots, Q_n), \quad Q_i = \begin{cases} I_4 & \text{if } J_\alpha^{(i)} = Y_\alpha, \\ Q & \text{if } J_\alpha^{(i)} = \hat{Y}_\alpha, \end{cases} \quad (52)$$

satisfies

$$\mathbf{Q}^{-1} J_\alpha \mathbf{Q} = \sum_{i=1}^n E_{ii} \otimes Y_\alpha. \quad (53)$$

We need another preliminary result before going into the explicit computation of the invariance algebra, generalizing Lemma 2 and Lemma 3; this will allow us to reduce all the different orientation cases to the positively oriented one.

Lemma 4. *The invariance algebras of all the quaternionic structures in \mathbf{R}^{4n} are isomorphic.*

Proof. We follow, with obvious modifications, the argument used in the case \mathbf{R}^8 . Thanks to Lemma 1, we can just consider structures in standard form. The invariance equation is

$$[X, J_\alpha] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} J_\beta, \quad \alpha = 1, 2, 3. \quad (54)$$

If we conjugate the quaternionic structure, $\tilde{J}_\alpha = U J_\alpha U^{-1}$, we get

$$[X, U^{-1} \tilde{J}_\alpha U] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} U^{-1} \tilde{J}_\beta U, \quad \alpha = 1, 2, 3; \quad (55)$$

that is,

$$[UXU^{-1}, \tilde{J}_\alpha] = \sum_{\beta=1}^3 \mathbf{L}_{\alpha\beta} \tilde{J}_\beta. \quad (56)$$

Then the algebra generated by the matrices X is isomorphic via a conjugation to the algebra generated by $\tilde{X} = UXU^{-1}$.

The two operations we make to pass from any quaternionic structure to the positively oriented block-diagonal one, via a preliminary reduction to a block-diagonal one with arbitrary orientation, are conjugations; thus the statement is proved. \triangle

We are now ready to identify the structure of the invariance algebra \mathfrak{L}_n ; this is the main result of the present work; we will split its proof into several lemmas.

Theorem 3. *The invariance algebra for $(\mathbf{R}^{4n}, I_{4n}, \mathbf{J})$ is $\mathfrak{L}_n = \mathfrak{su}(2) \oplus \mathfrak{sp}(n)$.*

Proof. We note preliminarily that in order to preserve the Euclidean metric in \mathbf{R}^{4n} , necessarily the infinitesimal transformation $X = -X^T \in \mathcal{M}_{4n}$ belongs to $\mathfrak{so}(4n)$. Moreover, Lemma 4 guarantees the invariance algebra for different hyperkähler structures on $(V, g) = (\mathbf{R}^{4n}, I_{4n})$ are isomorphic; thus we only have to study the positively oriented standard one.

That is, we consider the hyperkähler structure given by $\{J_1, J_2, J_3\}$ with

$$J_\alpha = \sum_{i=1}^n E_{ii} \otimes Y_\alpha, \quad \alpha = 1, 2, 3. \quad (57)$$

In this case, the infinitesimal invariance condition is expressed by equation (18) (where $\mathbf{L} = -\mathbf{L}^T \in \mathcal{M}_3$ is the infinitesimal transformation corresponding to a rotation in \mathbf{R}^3), see Section III E above.

Lemma 5. *The subalgebra of $\mathfrak{L}_n \subset \mathfrak{so}(4n)$ of the X satisfying the infinitesimal invariance equation (18) (and hence leaving invariant the quaternionic structure) has dimension $n(2n+1)+3$, and a basis of it is provided by*

$$\begin{aligned} & \begin{pmatrix} Y_\alpha & & & \\ & Y_\alpha & & \\ & & \ddots & \\ & & & Y_\alpha \end{pmatrix}, \quad \begin{pmatrix} \hat{Y}_\alpha & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \hat{Y}_\alpha \end{pmatrix}, \\ & \begin{pmatrix} 0 & \hat{Y}_\alpha & 0 & \\ \hat{Y}_\alpha & 0 & 0 & \\ 0 & 0 & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \hat{Y}_\alpha & \\ 0 & 0 & 0 & \\ \hat{Y}_\alpha & 0 & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & \hat{Y}_\alpha \\ & & 0 & \hat{Y}_\alpha & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & I_4 & 0 & \\ -I_4 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & I_4 & \\ 0 & 0 & 0 & \\ -I_4 & 0 & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & I_4 \\ & & 0 & -I_4 & 0 \end{pmatrix}. \end{aligned}$$

Proof. The equation (18) can be read in the following way. The matrices J_α generate a $\mathfrak{so}(3)$ algebra. The matrices X obviously form also a subalgebra \mathfrak{L} of $\mathfrak{so}(4n)$, and

$$\begin{aligned} [[X, \tilde{X}], J_\alpha] &= [[X, J_\alpha], \tilde{X}] - [[\tilde{X}, J_\alpha], X] = \sum_{\beta} \mathbf{L}_{\alpha\beta} [J_\beta, \tilde{X}] - \sum_{\beta} \tilde{\mathbf{L}}_{\alpha\beta} [J_\beta, X] \\ &= \sum_{\gamma} [\tilde{\mathbf{L}}, \mathbf{L}]_{\alpha\gamma} J_\gamma. \end{aligned} \quad (58)$$

Note that equation (18) simply states the fact that the subalgebra $\mathfrak{su}(2)$ generated by the J_α is an ideal of the algebra \mathfrak{L} .

We can construct a basis of $\mathfrak{so}(4n)$, which has dimension $2n(4n-1)$, using the matrices Y_α, \hat{Y}_α , their products and the tensor products with the matrices E_{ij} .

More explicitly, such a basis is provided by the matrices

$$\begin{aligned} A_{ij\alpha} &= \frac{1}{2}(E_{ij} + E_{ji}) \otimes Y_\alpha, \quad i, j = 1, \dots, n, \quad \alpha = 1, 2, 3 \\ \hat{A}_{ij\alpha} &= \frac{1}{2}(E_{ij} + E_{ji}) \otimes \hat{Y}_\alpha, \quad i, j = 1, \dots, n, \quad \alpha = 1, 2, 3 \\ B_{ij\alpha\beta} &= \frac{1}{2}(E_{ij} - E_{ji}) \otimes Y_\alpha \hat{Y}_\beta, \quad i < j = 1, \dots, n, \quad \alpha, \beta = 1, 2, 3 \\ C_{ij} &= \frac{1}{2}(E_{ij} - E_{ji}) \otimes I_4, \quad i < j = 1, \dots, n. \end{aligned}$$

It will be notationally convenient to use unconstrained indices i, j with the conventions

$$B_{ij\alpha\beta} = -B_{ji\alpha\beta}, \quad C_{ij} = -C_{ji}, \quad i > j, \quad B_{ii\alpha\beta} = 0, \quad C_{ii} = 0.$$

Some of the commutation relations among these elements will be used in the sequel and can be computed explicitly:

$$\begin{aligned}
[A_{ij\alpha}, A_{kk\gamma}] &= \sum_{\nu} \epsilon_{\alpha\gamma\nu} (\delta_{jk} + \delta_{ik}) A_{ij\nu} - \delta_{\alpha\gamma} (\delta_{jk} - \delta_{ik}) C_{ij} \\
[\hat{A}_{ij\alpha}, A_{kk\gamma}] &= (\delta_{jk} - \delta_{ik}) B_{ij\gamma\alpha} \\
[B_{ij\alpha\beta}, A_{kk\gamma}] &= \sum_{\nu} \epsilon_{\alpha\gamma\nu} (\delta_{jk} + \delta_{ik}) B_{ij\nu\beta} - \delta_{\alpha\gamma} (\delta_{jk} - \delta_{ik}) \hat{A}_{ij\beta} \\
[C_{ij}, A_{kk\gamma}] &= (\delta_{jk} - \delta_{ik}) A_{ij\gamma}.
\end{aligned} \tag{59}$$

Note that the matrices in the positively oriented quaternionic structure are in this notation written as

$$J_{\alpha} = \sum_i A_{ii\alpha} = \sum_i E_{ii} \otimes Y_{\alpha}. \tag{60}$$

With the convention that indices in the coefficients are also unconstrained, and $b_{ij\alpha\beta} = -b_{ji\alpha\beta}$, $c_{ij} = -c_{ji}$, the invariance condition for this structure is thus written in terms of the basis (59) as

$$\begin{aligned}
& \sum_{i,j,k,\alpha} a_{ij\alpha} [A_{ij\alpha}, A_{kk\gamma}] + \sum_{i,j,k,\alpha} \hat{a}_{ij\alpha} [\hat{A}_{ij\alpha}, A_{kk\gamma}] + \sum_{i,j,k,\alpha,\beta} b_{ij\alpha\beta} [B_{ij\alpha\beta}, A_{kk\gamma}] + \sum_{i,j,k} c_{ij} [C_{ij}, A_{kk\gamma}] \\
&= \sum_{\nu,i} L_{\gamma\nu} A_{ii\nu}, \quad \gamma = 1, 2, 3.
\end{aligned} \tag{61}$$

Substituting in this the commutators computed above (and understanding all equations are for $\gamma = 1, 2, 3$) we get

$$\begin{aligned}
& \sum_{i,j,k,\alpha} a_{ij\alpha,\nu} \epsilon_{\alpha\gamma\nu} (\delta_{jk} + \delta_{ik}) A_{ij\nu} - \sum_{i,j,k,\alpha} a_{ij\alpha} \delta_{\alpha\gamma} (\delta_{jk} - \delta_{ik}) C_{ij} + \sum_{i,j,k,\alpha} \hat{a}_{ij\alpha} (\delta_{jk} - \delta_{ik}) B_{ij\gamma\alpha} \\
&+ \sum_{i,j,k,\alpha,\beta,\nu} b_{ij\alpha\beta} \epsilon_{\alpha\gamma\nu} (\delta_{jk} + \delta_{ik}) B_{ij\nu\beta} - \sum_{i,j,k,\alpha,\beta} b_{ij\alpha\beta} \delta_{\alpha\gamma} (\delta_{jk} - \delta_{ik}) \hat{A}_{ij\beta} \\
&+ \sum_{i,j,k} c_{ij} (\delta_{jk} - \delta_{ik}) A_{ij\gamma} = \sum_{\nu,i} L_{\gamma\nu} A_{ii\nu};
\end{aligned}$$

upon standard simplification this reduces to

$$2 \sum_{i,j,\alpha,\nu} \epsilon_{\alpha\gamma\nu} a_{ij\alpha} A_{ij\nu} + 2 \sum_{i,j,\alpha,\beta,\nu} \epsilon_{\alpha\gamma\nu} b_{ij\alpha\beta} B_{ij\nu\beta} = \sum_{\nu,i} L_{\gamma\nu} A_{ii\nu}. \tag{62}$$

This should be seen as a matrix equation, i.e. a set of scalar equations, for the coefficients a, b (note the c_{ij} cancelled out) and for the matrix elements L_{ij} . As for the a_{ijk} and the L_{ij} , the solution is

$$a_{ij\alpha} = 0, \quad i \neq j \tag{63}$$

$$L_{12} = 2a_{ii3}, \quad L_{13} = -2a_{ii2}, \quad L_{23} = 2a_{ii1}. \tag{64}$$

On the other hand the equations for $b_{ij\alpha\beta}$ have the unique solution

$$b_{ij\alpha\beta} = 0, \quad i, j = 1, \dots, n, \quad \alpha, \beta = 1, 2, 3. \tag{65}$$

The invariance algebra is then formed by the elements of the form

$$\begin{aligned}
X &= \sum_{i,j,\alpha} a_{ij\alpha} A_{ij\alpha} + \sum_{i,j,\alpha} \hat{a}_{ij\alpha} \hat{A}_{ij\alpha} + \sum_{i,j} c_{ij} C_{ij} \\
&= \frac{1}{2} L_{23} J_1 + \frac{1}{2} L_{31} J_2 + \frac{1}{2} L_{12} J_3 \\
&\quad + \frac{1}{2} \sum_{i,j,\alpha} \hat{a}_{ij\alpha} (E_{ij} + E_{ji}) \otimes \hat{Y}_{\alpha} + \frac{1}{2} \sum_{i,j} c_{ij} (E_{ij} - E_{ji}) \otimes I_4.
\end{aligned} \tag{66}$$

One can check explicitly that these elements satisfy the invariance equation.

It seems at first sight that this would leave open the possibility that other elements are also in the invariance algebra. But actually the algebra spanned by the elements thus identified is a maximal subalgebra of $\mathfrak{so}(4n)$; given that obviously not all elements of $\mathfrak{so}(4n)$ preserve the quaternionic structure, one is guaranteed to have indeed identified the full invariance algebra. This concludes the proof of Lemma 5. \triangle

Theorem 4. *The invariance algebra \mathfrak{L}_n is the direct sum of two mutually commuting subalgebras,*

$$\mathfrak{L} = \mathfrak{su}(2) \oplus \mathfrak{g}, \quad (67)$$

one of them being the $\mathfrak{su}(2)$ algebra generated by the J_α , and the other being a Lie algebra of dimension $n(2n+1)$.

Proof. Obviously there is a subalgebra generated by the three first diagonal elements in (66), i.e. generated by $X_\alpha = \text{diag}(Y_\alpha, \dots, Y_\alpha)$; this is precisely the $\mathfrak{su}(2)$ subalgebra. (This fact corresponds to the one, already remarked, that (18) means that the subalgebra generated by the J_α is an ideal in \mathfrak{L} .) It is obvious from (66) that all other elements also form a subalgebra, and that the two subalgebras commute due to $[Y_\alpha, \hat{Y}_\beta] = 0$. The statement on the dimension of \mathfrak{g} follows by direct inspection. \triangle

Remark 8. It also follows easily from $[Y_\alpha, \hat{Y}_\beta] = 0$ that \mathfrak{g} is actually the strong invariance algebra for \mathbf{J} . \odot

We are left with the task of identifying the Lie algebra \mathfrak{g} ; this is not immediate and will require some Lie algebra theory. We actually know that \mathfrak{g} is equal to $\mathfrak{sp}(n)$ when $n = 1, 2$, see Theorem 1 and Theorem 2 (and Remark 7). It turns out that this is always the case.

Theorem 5. *The strong invariance algebra for the standard positively oriented hyperkähler structure on Euclidean \mathbf{R}^{4n} is $\mathfrak{g} = \mathfrak{sp}(n)$.*

Proof. Let us consider the complex extension of \mathfrak{g} . We can construct a Chevalley basis following the same procedure as in the case \mathbf{R}^8 (see A). We first define

$$H = i\hat{Y}_3, \quad E_+ = \frac{1}{2}(\hat{Y}_1 + i\hat{Y}_2), \quad E_- = \frac{1}{2}(-\hat{Y}_1 + i\hat{Y}_2) \quad (68)$$

with commutation relations

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H. \quad (69)$$

Using these, we define new matrices, which are linear combinations of the elements defined above:

$$\begin{aligned} \mathcal{H}_i &= (E_{ii} - E_{i+1, i+1}) \otimes H, \quad i = 1, \dots, n-1 \\ \mathcal{H}_n &= E_{nn} \otimes H \\ \mathcal{E}_{\pm, j}^\ell &= E_{jj} \otimes E_\pm, \quad j = 1, \dots, n \\ \mathcal{E}_{\pm, jk}^{s,1} &= \pm \frac{1}{2}(E_{jk} - E_{kj}) \otimes I_4 + \frac{1}{2}(E_{jk} + E_{kj}) \otimes H, \quad 1 \leq j < k \leq n \\ \mathcal{E}_{\pm, jk}^{s,2} &= (E_{jk} + E_{kj}) \otimes E_\pm, \quad 1 \leq j < k \leq n. \end{aligned} \quad (70)$$

It turns out that, as can be checked by an explicit computation, the matrices

$$\mathcal{H}_i, \quad \mathcal{E}_{\pm, j}^\ell, \quad \mathcal{E}_{\pm, jk}^{s,1}, \quad \mathcal{E}_{\pm, jk}^{s,2} \quad (71)$$

form a basis of the Lie algebra C_n (in the Cartan notation) in a $4n$ -dimensional representation. The matrices \mathcal{H}_i ($i = 1, \dots, n$) are a basis of a Cartan subalgebra, which we will denote as \mathfrak{h} ; the matrices $\mathcal{E}_{\pm, j}^\ell$ are the root vectors corresponding to the long roots, and the $\mathcal{E}_{\pm, jk}^{s,r}$ to the short ones. The details of the computation of the commutation relations, and in particular the determination of the root system, which show that this is the Lie algebra C_n , are given in Appendix B. This concludes the proof of Theorem 4. \triangle

Proof of Theorem 3 (conclusion). We can now conclude the proof of Theorem 3; for this it is necessary to come back considering $\mathfrak{L} = \mathfrak{su}(2) \oplus \mathfrak{g} \subset \mathfrak{so}(4n)$. The complex extension of the orthogonal algebra $\mathfrak{so}(4n)$, is D_{2n} in the Cartan notation. The maximal subgroups of the classical groups were classified by Dynkin in [12], and the result we need is that $A_1 \oplus C_n$ is a maximal subalgebra of the Lie algebra D_{2n} , which is in agreement with our results in Theorems 4 and 5.

In fact, the fundamental representation $(10 \cdots 0)$, in the highest weight notation, of D_{2n} is irreducible when restricted to the subalgebra $A_1 \oplus C_n$, as

$$D_{2n} \rightarrow A_1 \oplus C_n, \quad (1 \overbrace{0 \cdots 0}^{2n-1}) \rightarrow (1) \oplus (1 \overbrace{0 \cdots 0}^{n-1}), \quad (72)$$

and decomposes, when restricted to A_1 , into $2n$ copies of the spin $1/2$ representation (i.e. (1) in the highest weight notation),

$$D_{2n} \rightarrow A_1, \quad (1 \overbrace{0 \cdots 0}^{2n-1}) \rightarrow 2n(1); \quad (73)$$

and, when restricted to C_n , into the sum of two copies of the fundamental representation of C_n (we showed this fact by an explicit computation in Appendix A for the case \mathbf{R}^8):

$$D_{2n} \rightarrow C_n, \quad (1 \overbrace{0 \cdots 0}^{2n-1}) \rightarrow 2(1 \overbrace{0 \cdots 0}^{n-1}). \quad (74)$$

Since \mathfrak{g} is a semisimple compact Lie algebra, we finally have the complete structure of the algebra \mathfrak{L} , i.e. $\mathfrak{L} = \mathfrak{su}(2) \oplus \mathfrak{sp}(n)$, as claimed. \triangle

Remark 9. Our discussion shows, as mentioned in passing, that the invariance algebra $\mathfrak{L}_n = \mathfrak{su}(2) \oplus \mathfrak{g} \subset \mathfrak{so}(4n)$ is actually a maximal subalgebra of $\mathfrak{so}(4n)$. The elements in $\mathfrak{su}(2)$ correspond to regular invariance transformations, and those in $\mathfrak{g} = \mathfrak{sp}(n)$ to the strong invariance ones. \odot

Remark 10. We should note that although the invariance algebras are isomorphic for different quaternionic structures on \mathbf{R}^{4n} , their realizations are not the same and depend on the different quaternionic structures (in particular, on the different orientations they may have). We have seen in the case \mathbf{R}^8 how they can be constructed and the differences among them. The construction follows essentially the same lines in the general case. \odot

Remark 11. The representation of $\mathfrak{g} = \mathfrak{sp}(n)$ is complex reducible, and then, there exists a matrix P which transforms the $4n \times 4n$ matrices into a $2n \times 2n$ block diagonal form. Each block corresponds to the fundamental representation (of dimension $2n$) of C_n . The explicit computation is made for \mathbf{R}^8 in Appendix A, but it cannot be easily generalized. \odot

Remark 12. The appearance of the (compact) symplectic algebra $\mathfrak{sp}(n)$ is not a surprising result in this context. In fact, it can be identified with the Lie algebra $\mathfrak{sl}(n, \mathbf{H})$ of quaternionic $n \times n$ matrices with purely imaginary trace [7]. The symplectic group $\mathrm{Sp}(2n)$ can be realized in terms of quaternions as a subgroup of the general linear group $\mathrm{GL}(n, \mathbf{H})$. \odot

VII. CONCLUSIONS

There is a natural notion of *equivalent hyperkähler structures* on a hyperkähler manifold $(V, g; J_1, J_2, J_3)$; the quaternionic (or hypersymplectic) transformations $\Phi : V \rightarrow V$ are those which preserve the Riemannian metric g and which map the hypercomplex structure $\{J_\alpha\}$, and hence the associated hypersymplectic structure $\{\omega_\alpha\}$, into an equivalent one. In the case where the complex structures J_α , and hence the associated symplectic structures ω_α , are individually invariant under the transformation Φ , one says that Φ is strongly hypersymplectic. It is clear that hypersymplectic (and strongly hypersymplectic) maps form a Lie group.

In this paper we have investigated adopting a fully explicit approach the continuous group of quaternionic (hypersymplectic) transformations for the real spaces \mathbf{R}^{4n} equipped with the Euclidean metric $g = I_{4n}$, obtaining a complete classification for their Lie algebra and hence the connected component of the identity in the group (the full group is then recovered by taking into account transformations which permute the different four dimensional blocks, and possibly other discrete maps).

In particular, we have shown (Theorem 5) that the algebra of strongly hypersymplectic maps, also called the strong invariance algebra, is $\mathfrak{g}_n = \mathfrak{sp}(n)$. As for the full invariance algebra \mathfrak{L}_n , this is always the direct sum of the strong invariance one and of the regular invariance algebra (the algebra of transformation mapping the hyperkähler structure into an equivalent one, different from the original one); we have shown that the regular invariance algebra is given by $\mathfrak{su}(2)$, hence (Theorem 3) $\mathfrak{L}_n = \mathfrak{su}(2) \oplus \mathfrak{sp}(n)$.

As already mentioned in the Introduction, these results are not new, being known in the differential geometric literature devoted to hyperkähler and quaternionic manifolds [23]. In this context they were obtained by rather

abstract methods, so that the contribution of this paper lies in that the proofs are fully explicit and make use of elementary linear algebra plus Cartan's classification of simple Lie algebras and Dynkin's classification of maximal subgroups of simple Lie groups.

Our proofs used the standard real quaternionic representation for $SU(2)$, see Lemma 1, as well as some other Lemmas (see Lemmas 2, 3 and 4 here). In the Euclidean case it turns out we have to deal with the possibility of different orientations. The Lemmas proved in this paper show that hyperkähler structures in $(\mathbf{R}^{4n}, I_{4n})$ corresponding to different orientations are conjugated in $O(4n)$ and hence lead to isomorphic groups and algebras; thus one has to deal with a single case (say with fully positive orientation) for each dimension. As for the study of this single case, the identification of the regular invariance algebra has been rather straightforward, while for identifying the strong invariance algebra we resorted to some general results from the theory of Lie algebras.

The result discussed here should be seen as an equivalent in hyperkähler geometry of the familiar identification of the symplectic group in standard symplectic geometry. These results also have a relevance in connection with hyperhamiltonian dynamics [14, 27] and hence of its physical applications [17, 18] (as well as its applications in the theory of integrable systems [13, 15]).

The present results only apply to Euclidean spaces; it should be stressed again that this setting suffices to describe physically relevant cases and equations, such as the Pauli and the Dirac ones [17]; also, most of the physically relevant hyperkähler manifolds are obtained as quotients (via a momentum map-like construction due to Hitchin *et al.* [22]) of standard \mathbf{R}^{4n} hyperkähler manifolds. Moreover, the fact we were able to fully classify the invariance algebra in this case is encouraging in view of the treatment of more general cases. In particular, we have recently been able to fully describe the hyperkähler structure in Taub-NUT spaces [18]; it would be quite natural to attempt a classification of quaternionic maps for these, and hence a classification of Taub-NUT manifolds up to equivalence of hyperkähler structures.

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Appendix A: Identification of the strong invariance algebra in the 8-dimensional Euclidean space.

We describe the complex Lie algebra C_2 using the fundamental 4-dimensional representation. After an appropriated choice of a basis of this algebra, we will check that the commutation table is the same as the one we can compute for the 10-dimensional subalgebra of the algebra (41). We finally identify the corresponding real form.

The Lie algebra C_2 .

The 4-dimensional fundamental representation of C_2 can be defined by the matrices:

$$\begin{aligned}
 X_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & X_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
 X_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & X_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
 X_7 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & X_9 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

satisfying

$$X_i^T J_4 + J_4 X_i = 0, \quad i = 1, \dots, 10, \quad J_4 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}. \quad (\text{A1})$$

The two first matrices X_1, X_2 , are a basis of a Cartan subalgebra related to the two simple roots α_1 and α_2 . The rest of matrices corresponds to the root spaces: $\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2$ and $-2\alpha_1 - \alpha_2$. For our purposes it is more convenient to use the basis

$$\begin{aligned} \tilde{X}_1 &= \frac{1}{2}X_1, & \tilde{X}_2 &= \frac{1}{2}X_1 + X_2, & \tilde{X}_3 &= -X_9, & \tilde{X}_4 &= X_{10}, & \tilde{X}_5 &= X_5, & \tilde{X}_6 &= X_6, \\ \tilde{X}_7 &= \frac{1}{\sqrt{2}}X_8, & \tilde{X}_8 &= \frac{1}{\sqrt{2}}X_4, & \tilde{X}_9 &= -\frac{1}{\sqrt{2}}X_7, & \tilde{X}_{10} &= -\frac{1}{\sqrt{2}}X_3. \end{aligned}$$

The commutation table is written in Table I.

	\tilde{X}_1	\tilde{X}_2	\tilde{X}_3	\tilde{X}_4	\tilde{X}_5	\tilde{X}_6	\tilde{X}_7	\tilde{X}_8	\tilde{X}_9	\tilde{X}_{10}
\tilde{X}_1	0	0	\tilde{X}_3	$-\tilde{X}_4$	$-\tilde{X}_5$	\tilde{X}_6	0	$-\tilde{X}_8$	0	\tilde{X}_{10}
\tilde{X}_2	0	0	\tilde{X}_3	$-\tilde{X}_4$	\tilde{X}_5	$-\tilde{X}_6$	$-\tilde{X}_7$	0	\tilde{X}_9	0
\tilde{X}_3	$-\tilde{X}_3$	$-\tilde{X}_3$	0	$-\tilde{X}_1 - \tilde{X}_2$	0	0	$-\tilde{X}_{10}$	\tilde{X}_9	0	0
\tilde{X}_4	\tilde{X}_4	\tilde{X}_4	$\tilde{X}_1 + \tilde{X}_2$	0	0	0	0	0	$-\tilde{X}_8$	\tilde{X}_7
\tilde{X}_5	\tilde{X}_5	$-\tilde{X}_5$	0	0	0	$-\tilde{X}_1 + \tilde{X}_2$	$-\tilde{X}_8$	0	0	\tilde{X}_9
\tilde{X}_6	$-\tilde{X}_6$	\tilde{X}_6	0	0	$\tilde{X}_1 - \tilde{X}_2$	0	0	$-\tilde{X}_7$	\tilde{X}_{10}	0
\tilde{X}_7	0	\tilde{X}_7	\tilde{X}_{10}	0	\tilde{X}_8	0	0	$-\tilde{X}_4$	\tilde{X}_2	\tilde{X}_6
\tilde{X}_8	\tilde{X}_8	0	$-\tilde{X}_9$	0	0	\tilde{X}_7	\tilde{X}_4	0	\tilde{X}_5	\tilde{X}_1
\tilde{X}_9	0	$-\tilde{X}_9$	0	\tilde{X}_8	0	$-\tilde{X}_{10}$	$-\tilde{X}_2$	$-\tilde{X}_5$	0	$-\tilde{X}_3$
\tilde{X}_{10}	$-\tilde{X}_{10}$	0	0	$-\tilde{X}_7$	$-\tilde{X}_9$	0	$-\tilde{X}_6$	$-\tilde{X}_1$	\tilde{X}_3	0

TABLE I: Commutation table of C_2 in the basis \tilde{X}_i .

The strong invariance algebra of the positively oriented standard quaternionic structure.

If we consider the 10-dimensional subalgebra in (41) with basis ($W_i = Q_{i+3}$)

$$Q_{1,2,3} = \begin{pmatrix} \hat{Y}_\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_{4,5,6} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{Y}_\alpha \end{pmatrix}, \quad Q_{7,8,9} = \begin{pmatrix} 0 & \hat{Y}_\alpha \\ \hat{Y}_\alpha & 0 \end{pmatrix}, \quad Q_{10} = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}, \quad (\text{A2})$$

we can easily compute the adjoint representation and its Killing form:

$$B = \begin{pmatrix} -12I_6 & \\ & -24I_4 \end{pmatrix}. \quad (\text{A3})$$

Since this quadratic form is non degenerated, it corresponds to a semisimple subalgebra and, since it is negative definite, it is compact. We construct its complex extension and change the basis to:

$$\begin{aligned} \tilde{Q}_1 &= \frac{i}{2}(Q_3 - Q_6), & \tilde{Q}_2 &= \frac{i}{2}(Q_3 + Q_6), & \tilde{Q}_3 &= \frac{1}{2}(Q_1 + iQ_2), & \tilde{Q}_4 &= \frac{1}{2}(Q_1 - iQ_2), \\ \tilde{Q}_5 &= -\frac{1}{2}(Q_4 + iQ_5), & \tilde{Q}_6 &= \frac{1}{2}(Q_4 - iQ_5), & \tilde{Q}_7 &= -\frac{1}{2\sqrt{2}}(Q_7 - iQ_8), \\ \tilde{Q}_8 &= \frac{1}{2\sqrt{2}}(iQ_9 - Q_{10}), & \tilde{Q}_9 &= -\frac{1}{2\sqrt{2}}(Q_7 + iQ_8), & \tilde{Q}_{10} &= -\frac{1}{2\sqrt{2}}(iQ_9 + Q_{10}). \end{aligned}$$

It is straightforward to check that the commutation table in this basis is exactly the same as in Table I, and then both algebras are isomorphic, that is the complex extension of the algebra constructed with the matrices (A2) is C_2 .

We can easily construct a Chevalley basis in terms of the quaternionic expressions:

$$\begin{aligned}
H_1 &= i(Q_3 - Q_6), & H_2 &= iQ_6, \\
E_{\alpha_1} &= \frac{1}{2}(Q_{10} + iQ_9), & E_{-\alpha_1} &= \frac{1}{2}(-Q_{10} + iQ_9), \\
E_{\alpha_2} &= \frac{1}{2}(Q_4 + iQ_5), & E_{-\alpha_2} &= \frac{1}{2}(-Q_4 + iQ_5), \\
E_{\alpha_1+\alpha_2} &= \frac{1}{2}(Q_7 + iQ_8), & E_{-\alpha_1-\alpha_2} &= \frac{1}{2}(-Q_7 + iQ_8), \\
E_{2\alpha_1+\alpha_2} &= \frac{1}{2}(Q_1 + iQ_2), & E_{-2\alpha_1-\alpha_2} &= \frac{1}{2}(-Q_1 + iQ_2),
\end{aligned}$$

that is

$$\begin{aligned}
H_1 &= i \begin{pmatrix} \hat{Y}_3 & 0 \\ 0 & -\hat{Y}_3 \end{pmatrix}, & H_2 &= i \begin{pmatrix} 0 & 0 \\ 0 & \hat{Y}_3 \end{pmatrix}, \\
E_{\alpha_1} &= \frac{1}{2} \begin{pmatrix} 0 & I_4 + i\hat{Y}_3 \\ -I_4 + i\hat{Y}_3 & 0 \end{pmatrix}, & E_{-\alpha_1} &= \frac{1}{2} \begin{pmatrix} 0 & -I_4 + i\hat{Y}_3 \\ I_4 + i\hat{Y}_3 & 0 \end{pmatrix}, \\
E_{\alpha_2} &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \hat{Y}_1 + i\hat{Y}_2 \end{pmatrix}, & E_{-\alpha_2} &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & -\hat{Y}_1 + i\hat{Y}_2 \end{pmatrix}, \\
E_{\alpha_1+\alpha_2} &= \frac{1}{2} \begin{pmatrix} 0 & \hat{Y}_1 + i\hat{Y}_2 \\ \hat{Y}_1 + i\hat{Y}_2 & 0 \end{pmatrix}, & E_{-\alpha_1-\alpha_2} &= \frac{1}{2} \begin{pmatrix} 0 & -\hat{Y}_1 + i\hat{Y}_2 \\ -\hat{Y}_1 + i\hat{Y}_2 & 0 \end{pmatrix}, \\
E_{2\alpha_1+\alpha_2} &= \frac{1}{2} \begin{pmatrix} \hat{Y}_1 + i\hat{Y}_2 & 0 \\ 0 & 0 \end{pmatrix}, & E_{-2\alpha_1-\alpha_2} &= \frac{1}{2} \begin{pmatrix} -\hat{Y}_1 + i\hat{Y}_2 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

which will be generalized to the general case \mathbf{R}^{4n} (70). Finally, the Lie algebra C_2 has no irreducible 8-dimensional representation. However, the matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & i & 0 & 0 \end{pmatrix} \quad (\text{A4})$$

converts the matrices Φ_i of (A2) into a block-diagonal form:

$$P^{-1}\Phi_i P = \begin{pmatrix} X_i & 0 \\ 0 & X_i \end{pmatrix}, \quad (\text{A5})$$

showing in an explicit way that they are a reducible representation of C_2 , which can be decompose into the sum of two copies of the fundamental representation of C_2 with dimension 4.

Among the real forms of C_2 there is only one which is compact, $\mathfrak{sp}(2)$, which is the algebra we were looking for.

Appendix B: Commutation relations of the Lie algebra C_n

It is straightforward to check the commutation relations proving that the basis

$$\mathcal{H}_i, \quad \mathcal{E}_{\pm,j}^\ell, \quad \mathcal{E}_{\pm,jk}^{s,1}, \quad \mathcal{E}_{\pm,jk}^{s,2} \quad (\text{B1})$$

is a Chevalley basis corresponding to C_n . First, the elements in $\mathfrak{h} = \{\mathcal{H}_1, \dots, \mathcal{H}_n\}$ clearly commute (they are block diagonal matrices and the blocks are equal to 0 or to H):

$$[\mathcal{H}_i, \mathcal{H}_j] = 0, \quad i, j = 1, \dots, n. \quad (\text{B2})$$

Second, the matrices $\mathcal{E}_{\pm,j}^\ell, \mathcal{E}_{\pm,jk}^{s,1}, \mathcal{E}_{\pm,jk}^{s,2}$ have the correct commutation relations, that is, they are eigenvectors of the elements in the Cartan subalgebra:

$$\begin{aligned} [\mathcal{H}_i, \mathcal{E}_{\pm,j}^\ell] &= \pm 2(\delta_{ij} - \delta_{i+1,j})\mathcal{E}_{\pm,j}^\ell, \quad i = 1, \dots, n-1, \quad j = 1, \dots, n \\ [\mathcal{H}_n, \mathcal{E}_{\pm,j}^\ell] &= \pm 2\delta_{nj}\mathcal{E}_{\pm,j}^\ell, \quad j = 1, \dots, n \\ [\mathcal{H}_i, \mathcal{E}_{\pm,jk}^{s,1}] &= \pm(\delta_{ij} - \delta_{ik} - \delta_{i+1,j} + \delta_{i+1,k})\mathcal{E}_{\pm,jk}^{s,1}, \quad i = 1, \dots, n-1, \quad 1 \leq j < k \leq n \\ [\mathcal{H}_n, \mathcal{E}_{\pm,jk}^{s,1}] &= \pm(\delta_{nj} - \delta_{nk})\mathcal{E}_{\pm,jk}^{s,1}, \quad 1 \leq j < k \leq n \\ [\mathcal{H}_i, \mathcal{E}_{\pm,jk}^{s,2}] &= \pm(\delta_{ij} + \delta_{ik} - \delta_{i+1,j} - \delta_{i+1,k})\mathcal{E}_{\pm,jk}^{s,2}, \quad i = 1, \dots, n-1, \quad 1 \leq j < k \leq n \\ [\mathcal{H}_n, \mathcal{E}_{\pm,jk}^{s,2}] &= \pm(\delta_{nj} + \delta_{nk})\mathcal{E}_{\pm,jk}^{s,2}, \quad 1 \leq j < k \leq n. \end{aligned} \quad (\text{B3})$$

Let us consider the vector space of the Cartan subalgebra \mathfrak{h} and the forms defined by:

$$e_j : \mathfrak{h} \rightarrow \mathfrak{h}, \quad e_j(E_{ii} \otimes H) = \delta_{ij} \quad (\text{B4})$$

and then

$$e_j(\mathcal{H}_i) = e_j(E_{ii} \otimes H - E_{i+1,i+1} \otimes H) = \delta_{ij} - \delta_{i+1,j}, \quad e_j(\mathcal{H}_n) = e_j(E_{nn} \otimes H) = \delta_{nj}. \quad (\text{B5})$$

Using these roots, the commutation relations (B3) can be written as:

$$\begin{aligned} [\mathcal{H}_i, \mathcal{E}_{\pm,j}^\ell] &= \pm 2e_j(\mathcal{H}_i)\mathcal{E}_{\pm,j}^\ell, \quad i = 1, \dots, n-1, \quad j = 1, \dots, n \\ [\mathcal{H}_n, \mathcal{E}_{\pm,j}^\ell] &= \pm 2e_j(\mathcal{H}_n)\mathcal{E}_{\pm,j}^\ell, \quad j = 1, \dots, n \\ [\mathcal{H}_i, \mathcal{E}_{\pm,jk}^{s,1}] &= \pm(e_j - e_k)(\mathcal{H}_i)\mathcal{E}_{\pm,jk}^{s,1}, \quad i = 1, \dots, n-1, \quad 1 \leq j < k \leq n \\ [\mathcal{H}_n, \mathcal{E}_{\pm,jk}^{s,1}] &= \pm(e_j - e_k)(\mathcal{H}_n)\mathcal{E}_{\pm,jk}^{s,1}, \quad 1 \leq j < k \leq n \\ [\mathcal{H}_i, \mathcal{E}_{\pm,jk}^{s,2}] &= \pm(e_j + e_k)(\mathcal{H}_i)\mathcal{E}_{\pm,jk}^{s,2}, \quad i = 1, \dots, n-1, \quad 1 \leq j < k \leq n \\ [\mathcal{H}_n, \mathcal{E}_{\pm,jk}^{s,2}] &= \pm(e_j + e_k)(\mathcal{H}_n)\mathcal{E}_{\pm,jk}^{s,2}, \quad 1 \leq j < k \leq n. \end{aligned}$$

That is, the root system is

$$2e_i, \quad \pm e_j \pm e_k, \quad i, j, k = 1, \dots, n \quad (\text{B6})$$

which is the root system of the Lie algebra C_n . This ends the proof of Theorem 5.2.

The commutation relations between a root vector and that associated to the opposite root are:

$$\begin{aligned} [\mathcal{E}_{+,i}^\ell, \mathcal{E}_{-,i}^\ell] &= E_{ii} \otimes H = \mathcal{H}_i + \mathcal{H}_{i+1} + \dots + \mathcal{H}_n, \quad n = 1, \dots, n \\ [\mathcal{E}_{+,jk}^{s,1}, \mathcal{E}_{-,jk}^{s,1}] &= (E_{jj} - E_{kk}) \otimes H = \mathcal{H}_j + \mathcal{H}_{j+1} + \dots + \mathcal{H}_{k-1}, \quad 1 \leq j < k \leq n \\ [\mathcal{E}_{+,jk}^{s,2}, \mathcal{E}_{-,jk}^{s,2}] &= (E_{jj} + E_{kk}) \otimes H = \mathcal{H}_j + \mathcal{H}_{j+1} + \dots + \mathcal{H}_{k-1} + 2\mathcal{H}_k + \dots + 2\mathcal{H}_n, \\ &\quad 1 \leq j < k \leq n. \end{aligned}$$

We can easily identified a set of simple roots. The root vectors associated to the simple short roots are:

$$\mathcal{E}_{+,12}^{s,1}, \dots, \mathcal{E}_{+,n-1,n}^{s,1}, \quad [\mathcal{E}_{+,i,i+1}^{s,1}, \mathcal{E}_{-,i,i+1}^{s,1}] = \mathcal{H}_i, \quad i = 1, \dots, n-1 \quad (\text{B7})$$

and the root vector associated to the simple long root is

$$\mathcal{E}_{+,n}^\ell, \quad [\mathcal{E}_{+,n}^\ell, \mathcal{E}_{-,n}^\ell] = \mathcal{H}_n. \quad (\text{B8})$$

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