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On the propagation of the kurtosis parameter of general beams

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Abstract

The behaviour of the kurtosis parameter of a partially coherent beam that freely propagates is investigated. A general classification scheme of light beams is given in terms of the number of extremals of the kurtosis in free space. Propagation through *ABCD* optical systems is also considered and a number of general properties of the kurtosis parameter are provided concerning the relationship between the extremals of the kurtosis and the position of the beam waist.

1. Introduction

As is well known, a number of global beam parameters have recently been introduced in the literature to characterize the spatial behaviour of arbitrary laser profiles [1–14]. This parameter has revealed to be useful to evaluate in a quantitative way the laser capabilities in a number of laser material processings [14], such as welding, cutting or hole drilling. Among them the degree of flatness (or sharpness) of the beam intensity distribution has been described by means of the so-called kurtosis parameter [9,13]. In the present work attention will be concentrated on this shape parameter and several properties will be derived. Thus in the next section, after defining the kurtosis for an arbitrary partially coherent beam, a general classification scheme of light beams will be given in terms of the kurtosis behaviour under free propagation. Some kinds of beams of special interest will also be considered. In section 3, propagation through *ABCD* optical systems will be analysed and a number of general properties of the kurtosis parameter concerning the relationship between the extremals of the kurtosis and the position of the beam waist will be shown.

2. Definitions and classification scheme

To handle partially coherent beams, the Wigner distribution function (WDF) of the field is specially suitable. As is well known [3], it can be defined in terms of the cross-spectral density function Γ as

$$h(x,u,z) = \int_{-\infty}^{+\infty} \Gamma(x+s/2, x-s/2, z)\exp(\mathrm{i}ksu)\,\mathrm{d}s, \qquad (1)$$

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where, for simplicity, we have considered the bidimensional case. In Eq. (1), x denotes the spatial coordinate transversal to the propagation direction z, and ku is the wavevector component along the x-axis (hence u would represent an angle of propagation, without taking the evanescent waves into account). Integration of function h over the angular variable u gives the beam intensity, and its integral over the spatial variable x, is proportional to the radiant intensity of the field. Averages $\langle x^m u^n \rangle$ of the function h are defined in the form

$$\langle x^m u^n \rangle = \frac{1}{P} \iint_{-\infty}^{+\infty} x^m u^n \, h(x,u,z) \, \mathrm{d}x \, \mathrm{d}u , \tag{2}$$

where $P = \iint_{-\infty}^{+\infty} h(x,u,z) \, \mathrm{d}x \, \mathrm{d}u$ is the total irradiance of the beam. In particular, $\langle x^2 \rangle$ and $\langle u^2 \rangle$ represent, respectively, the (squared) beam width and the (squared) far-field divergence (angular spread). As is well known [1,9–12], these averages propagates through $ABCD$ optical systems according to the following law

$$\langle x^m u^n \rangle_\mathrm{o} = \langle (Ax+Bu)^m (Cx+Du)^n \rangle_\mathrm{i} , \tag{3}$$

where the subscripts 'o' and 'i' refer to the output and input planes of the optical system characterized by the $ABCD$ matrix whose elements are A, B, C and D (within the Fresnel approach). Eq. (3) follows from the corresponding propagation law of the WDF, namely [1,5,8,11,12]

$$h_\mathrm{i}(x,u) = h_\mathrm{o}(Ax+Bu, Cx+Du) . \tag{4}$$

In terms of higher-order averages, the kurtosis parameter, K, is defined as

$$K = \langle x^4 \rangle / \langle x^2 \rangle^2 \quad (\geq 1) . \tag{5}$$

As it was previously pointed out before [9,13], this parameter is closely related with the flatness of the intensity distribution, since the moments that appear in the above equation depend on z, consequently, K will be, in general, a z-function. The evolution of this shape parameter under free propagation (in the Fresnel approximation) would then be characterized by the number of extremals of the function $K(z)$, closely connected with the number of planes in which the beam becomes sharper or flatter.

As we will next show a general classification scheme of light beams follows from the analysis of such extremal values. It can be shown that the condition for the extremals of the kurtosis, namely,

$$\partial K / \partial z = 0 , \tag{6}$$

becomes (see appendix A)

$$m_0 z^4 + p_0 z^3 + q_0 z^2 + r_0 z + s_0 = 0 , \tag{7}$$

where

$$m = \langle u^4 \rangle \langle xu \rangle - \langle u^2 \rangle \langle xu^3 \rangle , \tag{8}$$

$$p = \langle u^4 \rangle \langle x^2 \rangle + 2 \langle xu^3 \rangle \langle xu \rangle - 3 \langle u^2 \rangle \langle x^2 u^2 \rangle , \tag{9}$$

$$q = 3 \langle xu^3 \rangle \langle x^2 \rangle - 3 \langle x^3 u \rangle \langle u^2 \rangle , \tag{10}$$

$$r = 3 \langle x^2 u^2 \rangle \langle x^2 \rangle - 2 \langle x^3 u \rangle \langle xu \rangle - \langle u^2 \rangle \langle x^4 \rangle , \tag{11}$$

$$s = \langle x^2 \rangle \langle x^3 u \rangle - \langle x^4 \rangle \langle xu \rangle , \tag{12}$$

and the subscript '0' in Eq. (7) refers to the values of the moments at the initial plane $z = 0$. By using Eq. (A.2), we can conclude from Eq. (8) that the coefficient m does not depend on the choice of the initial plane (this means that it can be measured at any plane). In an analogous way, the dependence of the remaining coefficients at such a plane is given by the following relations:

$$\partial p/\partial z = 4m, \tag{13}$$

$$\partial q/\partial z = 3p, \tag{14}$$

$$\partial r/\partial z = 2q, \tag{15}$$

$$\partial s/\partial z = r. \tag{16}$$

To classify an arbitrary beam according to the number of extremals of function $K(z)$ it would be useful to define

$$\Delta = 4R^3 - S^2, \tag{17}$$

and

$$\Delta_1 = 9p^2q^2 - 32mq^3 - 27p^3r + 108mpqr - 108m^2r^2, \tag{18}$$

where

$$R = q^2 - 3pr + 12ms, \tag{19}$$

$$S = 2q^3 - 9pqr - 72qms + 27mr^2 + 27p^2s. \tag{20}$$

It can be shown by performing the corresponding derivatives with respect to z that Δ, Δ_1, R, and S remain constant under free propagation. Taking this into account, since Eq. (7) is a fourth-order polynomial, any partially coherent beam can be classified according to the following scheme (see appendix B)

- **Type I:** Those beams whose kurtosis function $K(z)$ exhibits two maxima and two minima belong to this type. This occurs when $m \neq 0$ and $\Delta > 0$. Moreover, there exists at least one plane z_0 such that

  $$K(z_0) = \langle u^4\rangle/\langle u^2\rangle^2 = K(z \to \pm\infty) \equiv K_\infty, \tag{21}$$

  where K_∞ denotes the value of the kurtosis at the far-field. Note that K_∞ also characterizes the behaviour of the kurtosis at the focal plane of any (converging) lens.
- **Type II:** $K(z)$ exhibits one maximum, one minimum and one inflexion point. The simultaneous conditions needed for this to occur are (i) $m \neq 0$, (ii) $\Delta = 0$, and (iii) $\Delta_1 \neq 0$.

  Since $K(z)$ is a continuous function, which exhibits a maximum and a minimum and tends to the asymptotic value K_∞ when $z \to \pm\infty$, there should therefore exist a unique plane z_0 such that $K(z_0) = K_\infty$.
- **Type III:** $K(z)$ has either (a) two maxima and one minimum, or (b) two minima and one maximum.

  The conditions here are $m = 0$, $p \neq 0$ and $\Delta > 0$.
- **Type IV:** $K(z)$ exhibits one maximum and one minimum. This occurs when either (i) $m = p = 0$, $q \neq 0$, or (ii) $m \neq 0$ and $\Delta < 0$, or (iii) $m \neq 0$, $\Delta = 0$ and $\Delta_1 = 0$. Then we can infer in an analogous way to that shown above that there exists a unique plane z_0 such that $K(z_0) = K_\infty$.
- **Type V:** $K(z)$ has either (a) one maximum and one inflexion point, or (b) one minimum and one inflexion point.

  This occurs when $m = 0$, $p \neq 0$, $\Delta = 0$ and $q^2 > 3pr$.
- **Type VI:** $K(z)$ exhibits either a unique maximum or a unique minimum.

  The conditions needed for this to occur are either (i) $m = 0$, $p \neq 0$ and $\Delta < 0$, or (ii) $m = 0$, $p \neq 0$, $\Delta = 0$, and $q^2 = 3pr$, or (iii) $m = p = q = 0$.

  In the last case (iii), the extremal would be placed at the beam waist.
- **Type VII:** $K(z)$ is a constant. This would occur if $m = p = q = r = s = 0$.

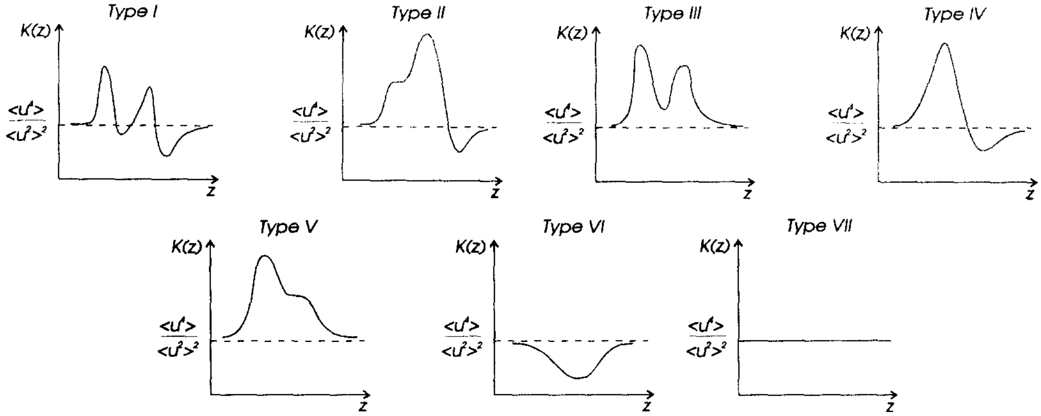

Fig. 1. The different types of the general classification scheme. The dashed line corresponds to the value of the kurtosis at the far-field.

The above seven types of beams cover all the possibilities of behaviour of function $K(z)$, and, consequently, provide a classification scheme for any field according to its degree of flatness (see Fig. 1).

As particular examples of interest we should mention that Hermite-Gauss beam modes and Gauss-Schell mode fields belong to type VII, and any Gaussian beam at the output of a quartic-phase transmittance belongs to type IV [13]. And finally, let us consider a supergaussian field distribution proportional to $\exp[-(x/a)^{2n}]$, $n = 1, 2, 3, \ldots$. It can then be shown that the cases $n = 1$ (Gaussian beam), $n = 2$ and $n \geq 3$ are related with fields belonging to types VII, III and VI, respectively.

3. Propagation through first-order optical systems

To control the sharpness of the beam profile by designing suitable optical systems, it would be useful to know what would be the change (if any) of the kurtosis behaviour of a beam after propagating through a general *ABCD* optical system. Thus we could determine from the measurements of parameters m, p, q, r and s the modification of the type of the beam (according to the above classification scheme) when it travels along an arbitrary first-order system. To do this let us define the following 5×1 matrix,

$$\alpha = \begin{pmatrix} m \\ p \\ q \\ r \\ s \end{pmatrix}, \tag{22}$$

which will be called the kurtosis propagation vector. Since function $K(z)$ can be inferred from the values of parameters m, p, q, r and s at some initial plane, it is clear from Eq. (22) that to analyse the type of the beam after crossing an *ABCD* system it will suffice to know the kurtosis propagation vector at the output of such system. Then, denoting by α_i and α_o these vectors at the input and output planes, respectively, it can be shown after lengthy calculations (by using Eqs. (3) and (8)–(12)) that the relationship between them is

$$\alpha_\text{o} = \mathbf{M}\alpha_\text{i}, \tag{23}$$

where $\mathbf{M}$ is the 5×5 matrix

$$\mathbf{M} = \begin{pmatrix} D^4 & CD^3 & C^2D^2 & C^3D & C^4 \\ 4BD^3 & 3BCD^2 + AD^3 & 2ACD^2 + 2BC^2D & 3AC^2D + BC^3 & 4AC^3 \\ 6B^2D^2 & 3B^2CD + 3ABD^2 & (AD + BC)^2 & 3ABC^2 + 3A^2CD & 6A^2C^2 \\ 4B^3D & B^3C + 3AB^2D & 2AB^2C + 2A^2BD & 3A^2BC + A^3D & 4A^3C \\ B^4 & AB^3 & A^2B^2 & A^3B & A^4 \end{pmatrix} \tag{24}$$

and A, B, C and D are the elements of the $ABCD$ matrix of the system, which, as usual, satisfy the simplecticity condition [11,12,16] $AD - BC = 1$.

A number of general conclusions can be inferred from the above. First, since, as one can check, $\det\mathbf{M} \neq 0$, *a beam of type VII remains of the same type after propagating through any first-order optical system.* In other words, for these kind of fields, $ABCD$ systems are of no use in changing the kurtosis behaviour.

Another interesting consequence refers to the relative position of the extremals of the kurtosis with respect to the beam waist. In this sense, it would be particularly useful that the kurtosis reaches an extremal value at the waist plane, becomes the beam energy is focused on such a plane. From the definitions (8)–(12), it can be shown (see appendix C) that *the kurtosis reaches an extremal at the waist plane if and only if* $s = 0$ *at this plane. This is also equivalent to saying that* $\langle x^3 u\rangle = 0$ *at such plane* (note that beams of type VII should be excluded from this statement). In particular, beams taking real values at the waist plane would fulfil this condition. Moreover (see also appendix C), if $r > 0$ the extremal would be a minimum, whereas if $r < 0$ the kurtosis would reach a maximum.

We then have that those beams for which $\langle x^3 u\rangle \neq 0$ at the waist do not have any extremals of the kurtosis at such plane. This is, in a sense, a somewhat surprising conclusion, because it differs from the usual (but incorrect) idea that any beam is sharper at its waist.

Let us then consider a beam whose kurtosis does not reach an extremal value at the waist. Then we will next show that *such beam can be transformed* (by means of certain $ABCD$ systems) *into another beam now having an extremal of the kurtosis at its new waist if and only if* $m = 0$ *for the input field* (again, beams of type VII are excluded).

To prove this, note first that if we take as input and output waist planes the input and output planes of an optical system, the elements of the $ABCD$ matrix representing such a system should fulfil either (a) $A = D = 0$, $BC = -1$, (this corresponds to an optical Fourier transformer [16]) or (b) $B = C = 0$, $AD = 1$ (this corresponds to a magnifier [16]). But the parameter s at the output is (according to Eqs. (23) and (24))

$$s = m_i B^4 + p_i AB^3 + q_i A^2B^2 + r_i A^3B + A^4 s_i, \tag{25}$$

where the subscript ‘i’ denotes the values at the input (waist) plane. Consequently, to get $s = 0$ (with $s_i \neq 0$) we must use an optical system of the type (a). Eq. (25) then becomes

$$m_i B^4 = 0, \tag{26}$$

which implies $m_i = 0$, and the converse is also true, Q.E.D. In summary, *a beam whose parameter m differs from zero cannot be transformed by means of first-order optical systems into another beam whose kurtosis reaches an extremal value at the waist plane.*

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Appendix A

With respect to Eqs. (7) and (13)–(16), note that the kurtosis parameter can be written out in the form

$$K(z) = \frac{\langle x^4\rangle_0 + 4\langle x^3u\rangle_0 z + 6\langle x^2u^2\rangle_0 z^2 + 4\langle xu^3\rangle_0 z^3 + \langle u^4\rangle_0 z^4}{(\langle x^2\rangle_0 + 2\langle xu\rangle_0 z + \langle u^2\rangle_0 z^2)^2}, \tag{A.1}$$

where the subscript 0 refers to the value of the moments at the initial plane $z = 0$. To write Eq. (A.1) we have applied the *ABCD* law to the moments $\langle x^2\rangle$ and $\langle x^4\rangle$ in the particular case of free propagation ($A = 1$, $B = z$, $C = 0$, $D = 1$). On the other hand, to get Eqs. (7)–(12) we have used the well-known relation [1,9]

$$\partial\langle x^m u^n\rangle_z/\partial z = m\langle x^{(m-1)}u^{(n+1)}\rangle_z. \tag{A.2}$$

Appendix B

In order to obtain the extremal of the kurtosis, we have to find the roots of the following polynomial

$$P(z) = mz^4 + pz^3 + qz^2 + rz + s. \tag{B.1}$$

We first consider that $m \neq 0$ (the case $m = 0$ will be analysed later). Following the standard procedure described, for example, in Ref. [15], (p. 288), let us define

$$A = \begin{vmatrix} m & p & q & r & s & 0 & 0 \\ 0 & m & p & q & r & s & 0 \\ 0 & 0 & m & p & q & r & s \\ 4m & 3p & 2q & r & 0 & 0 & 0 \\ 0 & 4m & 3p & 2q & r & 0 & 0 \\ 0 & 0 & 4m & 3p & 2q & r & 0 \\ 0 & 0 & 0 & 4m & 3p & 2q & r \end{vmatrix}, \tag{B.2}$$

and

$$D = m^6(u-v)^2(u-w)^2(u-t)^2(v-w)^2(v-t)^2(w-t)^2, \tag{B.3}$$

where u, v, w and t denote the roots of the polynomial $P(z)$.

It can be shown [15] that

$$A = mD. \tag{B.4}$$

Moreover,

$$A = (m/27)\Delta, \tag{B.5}$$

where Δ has been defined in Eq. (17). From Eqs. (B.4) and (B.5) it follows that $\mathrm{sgn}(\Delta) = \mathrm{sgn}(D)$. Therefore, from Eq. (B.3) three cases can be distinguised:

(i) $\Delta > 0$: A general fourth-order polynomial would then have either four real roots or four complex roots. But in our case, the second possibility should be excluded because if all the roots were complex then $P(z) \neq 0$ for any z real, and therefore $K(z)$ would be a monotonically increasing (or decreasing) function, which is incompatible with the asymptotic behaviour of K when $z = \pm\infty$ (see Eq. (21)). In conclusion, the four roots of $P(z)$ must be real, which would imply that K exhibits two maxima and two minima.

(ii) $\Delta < 0$: $P(z)$ has two simple real roots (the complex roots can be disregarded because z is a distance and takes real values only). Therefore, in this case $K(z)$ exhibits a minimum and a maximum.

(iii) $\Delta = 0$: $P(z)$ has multiple roots. In this case, since $K'(z) = 4P(z)/\langle x^2 \rangle^3$, the following analysis applies (by using the subsequent derivatives of $K(z)$):

(a) $P(z)$ has one cuadruple root. In this case it can be shown that the first non-zero derivative of the kurtosis at this point is odd, and therefore it must be an inflexion point. But this situation is not compatible with the asymptotic behaviour of K.

(b) $P(z)$ has both a triple and a simple root. $K(z)$ would then exhibit a maximum and a minimum.

(c) $P(z)$ has two double roots, which implies that $K(z)$ exhibits two inflexion points. But again this is not compatible with the far-field behaviour of K.

(d) $P(z)$ has two simple roots (a maximum and a minimum) and a double one (an inflexion point).

Cases (b) and (d) can be distinguished by applying an analogous procedure to that used to analyse the roots of $P(z)$. Thus, it can be shown that the number of roots of the derivative of $P(z)$, namely, $P'(z)$, is given in terms of the sign of Δ_1, defined in Eq. (18). We have then that, in (b) $P'(z)$ becomes

$$P'(z) = (z-u)^2 [3(z-v) + (z-u)], \tag{B.6}$$

which implies that P' has a double root and a simple one. This is fulfilled when $\Delta_1 = 0$. On the other hand, in (d)

$$P'(z) = (z-u)\{2(z-v)(z-w) + (z-u)[(z-v)+(z-w)]\}. \tag{B.7}$$

In this case $P'(z)$ does not have multiple roots. This occurs when $\Delta_1 \neq 0$.

Let us now consider the case $m = 0$ in Eq. (B.1). It can then be shown that the analysis of the roots of $P(z)$ is again given in terms of the sign of Δ after substitution of the value $m = 0$. Accordingly we have that

(i) if $\Delta > 0$, $P(z)$ has three real roots,

(ii) if $\Delta < 0$, $P(z)$ has one simple real root only,

(iii) if $\Delta = 0$, $P(z)$ has multiple roots.

In case (iii), by using the higher-order derivatives of $K(z)$, it can be shown that

- if $q^2 > 3pr$, then $K(z)$ has an inflexion point and a minimum or a maximum (note that $P(z)$ would then have both one simple root and one double root);
- if $q^2 = 3pr$, then $K(z)$ has a maximum or a minimum (in this case, $P(z)$ would have a triple root).

Note that the inequality $q^2 < 3pr$ is not compatible with the previous conditions $m = 0$ and $\Delta = 0$ (as follows from Eqs. (17), (19) and (20)).

From this appendix the general classification scheme shown in section 2 can be inferred.

Appendix C

In this appendix we will show that the kurtosis reaches an extremal at the waist plane if and only if $s = 0$ (or, equivalently, when $\langle x^3 u \rangle = 0$).

Let us choose the plane $z = 0$ as the waist plane. At this plane Eq. (7) then becomes $s = 0$. Moreover, since, at the waist, $\langle xu \rangle = 0$ [3,9], then Eq. (12) implies $\langle x^3 u \rangle = 0$. The converse is also true (if $s = 0$ – or $\langle x^3 u \rangle = 0$ – then Eq. (7) is fulfilled).

Note that, if the beam is real at the waist plane, then $\langle x^3 u \rangle = 0$ at such a plane, and the kurtosis would reach an extremal at the waist. Moreover, in this case

$$K'' = 4r/\langle x^2 \rangle^3 \tag{C.1}$$

at the waist plane. Accordingly, the extremal of K would be a maximum or a minimum depending on the sign of r.

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