Regularity and Paracompactness: Relation in the Field of Fuzziness

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Abstract. The importance of paracompactness (and the concepts related to it) in the field of Topology is well known.

In this paper we obtain two characterizations of regular fuzzy topological spaces using Luo's and Abd El-Monsef and others' paracompact fuzzy topological spaces. Thus, it shows that regularity of fuzzy topological spaces can be considered as a paracompact-type property with several kinds of paracompact fuzzy topological spaces. Indeed, we prove that for a fuzzy Hausdorff fuzzy topological space (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness) there is a characterization using Luo's paracompact fuzzy topological spaces, and also another result with a characterization using definition due to Abd El-Monsef, Zeyada, El-Deeb, and Hanafy. This supposes a stimulus for further investigations, tending to obtain characterizations of other fuzzy separation properties (for example fuzzy normality, fuzzy complete regularity,..) as fuzzy covering properties.

Keywords: Topology, Fuzzy sets, Separation properties, Covering properties Fuzzy paracompactness

1 Introduction

Regularity is a very useful separation property of topological spaces. Some authors obtained characterizations of regularity as a covering property. In this paper we work on fuzziness, obtaining two characterizations of fuzzy regularity as a fuzzy covering property. Indeed, we show that one can characterize fuzzy regularity as a paracompact-type fuzzy property in Luo's and Abd El-Monsef and others' both senses.

In Section 3 we give previous definitions to provide better readership and make the material more accessible to reader, and main results of this paper.

Suggestions for future research are in the Conclusion section.

2 Literature review

There exist various different definitions of paracompactness of fuzzy topological spaces due to Luo [1] and Abd El-Monsef, Zeyada, El-Deeb, and Hanafy [2], On the other hand, various authors (Abdelhay [5], Boyte [6], Chew [7]) obtained characterization of regularity as a covering property of Hausdorff topological spaces. A notion of a Hausdorff fuzzy property that is a good extension of classical topological property can be seen in [3]. Many other definitions and results on topological fuzziness can be found in book [4].

3 Definition and main results

Definition 1 [1] Let μ be a set in a fts (X,τ) and let $r\in(0,1]$, $s\in[0,1)$; we define

$$\mu_{[r]}=\chi_{\{x\in X:\mu(x)\ddagger r\}}$$

$$\mu_{(s)}=\chi_{\{x\in X:\mu(x)>s\}}$$

$$\mu_{<r>}=r\mu_{[r]}$$

Definition 2 [1] Let $\mathscr{A}$ be a family of sets and μ be a set in a fts (X,τ). We say that $\mathscr{A}$ is locally finite (resp. $*$-locally finite) in μ for each point e in μ, there exists $\nu\in Q(e)$ such that ν is quasi-coincident (resp. intersects) with at most a finite number of sets of $\mathscr{A}$; we often omit the word "in μ" when $\mu=X$.

Definition 3 [1] A family of sets $\mathscr{A}$ is called a Q-cover of a set μ if for each $x\in supp(\mu)$, there exist a $\nu\in\mathscr{A}$ such that ν and μ are quasi-coincident at x. Let $r\in(0,1]$. $\mathscr{A}$ is called a $r-Q$ cover of μ if $\mathscr{A}$ is a Q-cover of $\mu_{\langle r\rangle.}$

Definition 4 [1] Let $r\in(0,1]$, μ be a set in a fts (X,τ). We say that μ is r-paracompact (resp. r^*-paracompact) if for each r-open Q-cover of μ there exists an open refinement of it which is both locally finite (resp.$*$-locally finite) in μ and a r-Q-cover of μ. The fuzzy set

μ is called S-paracompact (resp. S*-paracompact) if for every $r \in (0,1]$, μ is r-paracompact (resp. r^*-paracompact).

Definition 5 [2] A family of fuzzy sets $\mathscr{U}$ is called an L-cover of a fuzzy set μ if $\bigvee_{\upsilon \in \mathscr{U}} \upsilon \ddagger \mu$.

Definition 6 [2] Let μ be a fuzzy set in a fts (X,τ). We say that μ is fuzzy paracompact (resp. $*$-fuzzy paracompact) if for each open L-cover $\mathscr{B}$ of μ and for each $\xi \in (0,1]$, there exists an open refinement $\mathscr{B}^*$ of $\mathscr{B}$ which is both locally finite (resp. $*$-locally finite) in μ and L-cover of $\mu - \xi$. We say that a fts (X,τ) is fuzzy paracompact (resp. $*$-fuzzy paracompact) if each constant set in X is fuzzy paracompact (resp. $*$-fuzzy paracompact).

Theorem 1 Let (X,τ) a fuzzy Hausdorff fts (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness). Then (X,τ) is fuzzy regular if and only if for each $r \in (0,1]$, for each r-open Q-cover of (X,τ) and for each fuzzy point x_λ of X there exists an open refinement of it which is both $*$-locally finite in x_λ and a r-Q-cover of (X,τ).

Proof ($\Rightarrow$) For each $r \in (0,1]$, let $\mathscr{U}$ be a r-open Q-cover of (X,τ), and x_λ be a fuzzy point of X . Then, we have that the family of crisp sets $\{U_{(1-r)} \mid U \in \mathscr{U}\}$, is an open cover of $(X,[\tau])$, which is Hausdorff and regular ([3], [4]). Then ([5], [6], [7]), it has an open refinement $\mathscr{V}_x \subset [\tau]$ which is a cover of X, and is locally finite in x. For each $V \in \mathscr{V}_x$ we have an $U_V \in \mathscr{U}$ with $V \subset (U_V)_{(1-r)}$.

Let $\mathscr{W}_x = \{\chi_V \wedge U_V \mid V \in \mathscr{V}_x\}$. Then, $\mathscr{W}_x \subset \tau$, is both an open refinement of $\mathscr{U}$ and a $r-Q$-cover of (X,τ), and also is $*$-locally finite in x_λ, indeed, because $\mathscr{V}_x$ is locally finite in x, we have an open neigborhood G of x that G intersects with only finite number of members of $\mathscr{V}_x$. Then $\chi_G \in Q(x_\lambda)$ intersects with only a finite number of members of $\mathscr{W}_x$.

(⇐) Let $\mathscr{U} \subset [\tau]$ be an open cover of $(X,[\tau])$; then $\{\chi_U \mid U \in \mathscr{U}\}$ is an open Q-cover of 1_X, and, for each $x \in X$, it has an open refinement $\mathscr{V}_{x_r}$ which is a Q-cover of 1_X and also locally finite in x_{1-r}. Let $\mathscr{W} = \{V_{(1-r)} | V \in \mathscr{V}_{x_r}\}$; then $\mathscr{W} \subset [\tau]$ is both a refinement of $\mathscr{U}$ and a cover of $(X,[\tau])$. Also, $\mathscr{W}$ is locally finite in x. Indeed: we take $O_1 \in Q(x_{1-r})$ such that O_1, is quasi-coincident with only a finite number of members $V_1,...,V_n$, of $\mathscr{V}_{x_r}$. Let $\mathscr{O} = (O_1)_{(r)}$, then $x \in O \in [\tau]$. For each $V \in \mathscr{V}_{x_r}$, if $O \wedge V_{(1-r)} \neq \varnothing$, we have a crisp point $y \in X$, such that $O_1(y) > r,\ V(y) > 1-r,\ O_1(y) + V(y) > 1,$ then $O_1 q V$ and $V \in \{V_1,...,V_n\}$. Hence the neighborhood O of x intersects with only a finite number of members $(V_1)_{(1-r)},...,(V_n)_{(1-r)} \in \mathscr{W}$.

Theorem 2. Let (X,τ) be a fuzzy Hausdorff fts (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness [3]). Then (X,τ) is fuzzy regular if and only if for each $r \in I$, and for each open L-cover $\mathscr{B}$ of r, for each $\xi \in (0,1]$, and for each fuzzy point x_λ of X, there exists an open refinement $\mathscr{B}^*$ of it which is both $*$-locally finite in x_λ and L-cover of $r-\xi$.

Proof (⇒) For each $r \in I$, and for each open L-cover $\mathscr{B}$ of r, for each $\xi \in (0,1]$, and for each fuzzy point x_λ of X, we have that the family of crisp sets $\mathscr{U} = \{G^{-1}((r-\xi,1]) | G \in \mathscr{B}\} \subset [\tau]$ is an open cover of $(X,[\tau])$ which is Hausdorff and regular ([3, 4]). Then ([5],[6],[7]), it has an open refinement $\mathscr{U}_x^* \subset [\tau]$ which is a cover of X and is locally finite in x. For each $V \in \mathscr{U}_x^*$, there exists $G_V{}^{-1}((r-\xi,1]) \in \mathscr{U}$, such that $V \subset G_V{}^{-1}((r-\xi,1])$. So $\mathscr{B}^* = \{\chi_V \wedge G_V | V \in \mathscr{U}_x^*\} \subset \tau$ is refinement of $\mathscr{B}$. Then, there exists $V \in \mathscr{U}_x^*$ such that $x \in V$ and $G_V(x) > r-\xi$. So,

$(\chi_V \wedge G_V)(x) \geq r-\xi,$ and $\bigvee\left\{\chi_V \wedge G_V | V \in \mathscr{U}_x^*\right\} \geq r-\xi$. Since $\mathscr{U}_x^*$ is locally finite in x, there exists $A \in [\tau]$ with $x \in A$, such that intersects with at most a finite number of members of $\mathscr{U}_x^*$. Then, there exists $\chi_A \in \tau$ such that $x_\lambda q \chi_A$ and χ_A intersects with a finite number of fuzzy sets of $\mathscr{B}^*$.

($\Leftarrow$) Let $\mathscr{U} \subset [\tau]$ be an open cover of X and $x \in X$, then $\mathscr{B} = \{\chi_U \mid U \in \mathscr{U}\}$ is an open L-cover of (X, τ) and for each, $r \in I$ is $\bigvee\{\chi_U \mid U \in \mathscr{U}\} \geq r$. For each $\xi \in (0,1]$, there exists an open refinement $\mathscr{B}^*$ of $\mathscr{B}$ which is both locally finite in x_λ and L-cover of $r-\xi$. This implies that $\bigvee\{G \mid G \in \mathscr{B}^*\} \geq r-\xi_1$ for all $\xi_1 > \xi$. Let $\mathscr{U}^* = \left\{G^{-1}((r-\xi,1]) \mid G \in \mathscr{B}^*\right\}$, then $\mathscr{U}^* \subset [\tau]$ is an open refinement of $\mathscr{U}$ (indeed, for each $G^{-1}((r-\xi,1]) \in \mathscr{U}^*$ there exists $V_G \in \mathscr{U}$ such that $G^{-1}((r-\xi,1]) \subset V_G)$. Since $\bigvee\left\{G \mid G \in \mathscr{B}^*\right\} \geq r-\xi_1$, then $\mathscr{U}^*$ is an open refinement of $\mathscr{U}$. And, since $x \in X$ there exists $A \in Q(x_\lambda)$ which intersects with only $G_1,..,G_n \in \mathscr{B}^*$. Since $A(x)+\lambda > 1,$ we have $A(x) > 1-\lambda$, then $x \in A^{-1}((1-\lambda,1]) \in [\tau]$.

If $A^{-1}((1-\lambda,1]) \cap G^{-1}((r-\xi,1]) \neq \varnothing$, there exists some point z, such that $A(z) > 1-\lambda$ and $G(z) > r-\xi$, so $A \wedge G \neq \varnothing$. Then, if the neigborhood $A^{-1}((1-\lambda,1])$ of x intersects with infinite members of $\mathscr{U}^*$, A intersects with infinite members of $\mathscr{B}^*$.Thus $\mathscr{U}^*$ is locally finite in x.

This yields that the Hausdorff topological space $(X,[\tau])$ is regular ([5], [6], [7]) and (X, τ) is fuzzy regular ([4]).

4 Usability and applications

This a paper on fuzzy topology, therefore it is a theoretical paper. However, many papers on fuzzy mathematics are having applications to engineering, computing, artificial intelligence, medicine, and other sciences, in particular, some papers about fuzzy topology authored by ours.

5 Conclusion

In this paper, fuzzy regularity is characterized as a fuzzy covering property. Future research could obtain characterization of other fuzzy separation properties (fuzzy normality, fuzzy completely regularity) as fuzzy covering properties. Obviously, these are all theoretical problems, but this is not a limitation, because many theoretical findings have further practical applications

References

1 Luo, M.K.: Paracompactness in fuzzy topological spaces, J. Math. Anal. Appl. 130, 55-77, (1988).

2 Abd El-Monsef, M.E., Zeyada, F.M., El-Deeb, S.N., and Hanafy, I.M.: Good extensions of paracompactness, Math. Japonica 37, 195-200, (1992).

3 Wuyts, P. and Lowen, R.: On separation axioms in fuzzy topological spaces, Fuzzy neighborhood spaces, and fuzzy uniform spaces, J. Math. Anal. Appl. 93, 27-41, (1983).

4 Liu Y.-M. and Luo M.-K.: Fuzzy Topology, World Scientific Publishing, Singapore, (1997).

5 Abdelhay, J.: Characterizaçao dos espaços topológicos regulares e normais por meio de coberturas, Gaz. Mat. (Lisboa) 9, nº 37-38, 8-9, (1948) (in Portuguese).

6 Boyte, J.M.: Point (countable) paracompactness, J. Austral. Math. Soc. 15, 138-144, (1973).

7 Chew,J.: Regularity as a relaxation of paracompactness, Amer. Math. Monthly, 79, 630-632 (1972).